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Inertial Quasi-Newton Methods for Monotone Inclusions: Efficient Resolvent Calculus and Primal-Dual Algorithms

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Abstract

We introduce an inertial quasi-Newton Forward-Backward Splitting Algorithm to solve a class of monotone inclusion problems. While the inertial step is computationally cheap, in general, the bottleneck is the evaluation of the resolvent operator. A change of the metric makes its computation even harder, and this is even true for a simple operator whose resolvent is known for the standard metric. In order to fully exploit the advantage of adapting the metric, we develop a new efficient resolvent calculus for a low-rank perturbed standard metric, which accounts exactly for quasi-Newton metrics. Moreover, we prove the convergence of our algorithms, including linear convergence rates in case one of the two considered operators is strongly monotone. As a by-product of our general monotone inclusion framework, we instantiate a novel inertial quasi-Newton Primal-Dual Hybrid Gradient Method (PDHG) for solving saddle point problems. The favourable performance of our inertial quasi-Newton PDHG method is demonstrated on several numerical experiments in image processing.

1 Introduction

Nowadays, convex optimization appears in many modern disciplines, especially when dealing with datasets of large-scale. There is a strong need for efficient optimization schemes to solve such large-scale problems. Unfortunately, the high dimensionality of the problems at hand makes the use of second-order methods intractable. A promising alternative are quasi-Newton type methods, which aim for exploiting cheap and accurate first-order approximations of the second-order information. In particular, the so-called limited memory quasi-Newton method has proved very successful for solving unconstrained large-scale problems. However, many practical problems in machine learning, image processing or statistics naturally have constraints or are non-smooth by construction.

A problem structure that can cover a broad class of non-smooth problems in these applications is the following inclusion problem on a real Hilbert space \mathcal{H} :

$$\text{find } x \in \mathcal{H} \text{ such that } (A + B)x \ni 0, \quad (1)$$

where $A: \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator and $B: \mathcal{H} \rightarrow \mathcal{H}$ is a single-valued β -co-coercive operator with $\beta > 0$. As a special case, (1) comprises the setting of minimization problems of the form

$$\min_{x \in \mathcal{H}} f(x) + g(x) \quad (2)$$

with a proper lower semi-continuous convex function f and convex function g with Lipschitz continuous gradient by setting $A = \partial f$ and $B = \nabla g$.

A fundamental algorithmic scheme to tackle the problem class (1) is Forward-Backward Splitting (FBS). However, this algorithm may exhibit slow convergence for ill-conditioned problems, for which one would like to exploit the second-order information to adapt to the local geometry of the problem. As a computationally affordable approximation in this paper, we propose a quasi-Newton variant that takes advantage from a variable metric that is computed using first-order information only. We manage to remedy the main computational bottlenecks for this type of approaches by developing an efficient low-rank variable metric resolvent calculus.

Our approach is inspired by the proximal quasi-Newton method in [8]. We extend the framework proposed in [8] to the resolvent setting with a “ $M + \text{rank-}r$ ” or “ $M - \text{rank-}r$ ” symmetric positive-definite variable metric. We first study the convergence of two variants of FBS algorithm with respect to this type of metrics. One variant uses an inertial step which opens the door to acceleration. Although accelerated rates are not proved yet, numerical results show that the inertial variant yields significantly improved convergence rates. The second variant uses a relaxation technique which enables convergence under a weaker assumption on the variable metrics. In analogy to the proximal calculus proposed in [8], we develop a resolvent calculus that allows for an efficient evaluation of resolvent operators with respect to this type of metrics by splitting the evaluation into two computationally simple steps: calculating a resolvent operator with respect to a simple metric M and solving a low dimensional root-finding problem. This allows for the incorporation of popular quasi-Newton strategies, such as the limited memory SR1 or BFGS method, in our framework.

In order to exploit the power and to illustrate the variety of problems that can be solved via the framework in (1), the developed algorithms are instantiated for the following saddle point problem:

$$\min_{x \in \mathcal{H}_1} \max_{y \in \mathcal{H}_2} \langle Kx, y \rangle + g(x) + G(x) - f(y) - F(y), \quad (3)$$

where g and f are lower semi-continuous convex functions, G and F are convex, differentiable with Lipschitz-continuous gradients and K is a bounded linear operator between Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 . We develop a quasi-Newton Primal-Dual methods that has many potential applications in Imaging Processing, Machine Learning or Statistics [12, 19]. The numerical performance of our algorithms is tested on several experiments and demonstrates a clear improvement when our quasi-Newton methods are used.

1.1 Related Works

Smooth quasi-Newton. Quasi-Newton methods are widely studied and used for optimization with sufficiently smooth objective functions [42]. Their motivation is to build a cheap approximation to the Newton’s method. If the approximation of the second-order information (Hessian) is given by a positive-definite matrix, quasi-Newton methods can be interpreted as iteratively and locally adapting the metric of the space to the objective function. The success of these methods requires a good approximation of the second-order information of the objective by using the first-order information, which is still an active research area. Recently, [34] proposed to greedily select some basis vectors instead of using the difference of successive iterates for updating the Hessian approximations. Relying on [34], [28] constructed an approximation of the indefinite Hessian of a twice continuously differentiable function. All methods mentioned above require sufficient smoothness of the objective functions.

Non-smooth quasi-Newton. A broad class of optimization problems is interpreted as a composition of a smooth function f and a non-smooth function h . To deal with the non-smoothness of h efficiently, many authors consider a combination of FBS with the quasi-Newton methods. By using the forward-backward envelope, [31, 39] reinterpreted the FBS algorithm as a variable metric gradient method for a smooth optimization problem in order to apply the classical Newton or quasi-Newton method. For a non-smooth function g as simple as an indicator function of a non-empty convex set, [37, 36] proposed an elegant method named projected quasi-Newton algorithm (PQN) which, however, requires either solving a subproblem or using a diagonal metric. [26] extended PQN to a more general setting as long as the proximal operator of h is simple to compute. For a class of low-rank perturbed metrics, [7, 8] developed a proximal quasi-Newton method with a root-finding problem as the subproblem which can be solved easily and efficiently. This method can be extended to the nonconvex setting [22]. Based on [7, 8], [23] incorporated a limited-memory quasi-Newton update. The authors of [24] developed a different algorithm to evaluate the proximal operator of the separable l_1 norm with respect to a low-rank metric $V = M - UU^\top$. Recently, in [25], the authors proposed a generalized damped Newton type which is based on second-order generalized Hessians.

In this paper, we extend the quasi-Newton approach of [7, 8] from the nonsmooth convex minimization setting to monotone inclusion problems of type (1). Our framework opens the door to new problems (e.g. saddle-point problems) and algorithms (e.g. primal-dual algorithms) that are beyond the reach of the approach initiated in [7, 8]. It is generic to use a variable metric for solving a monotone inclusion problem (see [15, 16]). Its convergence relies on quasi-Fejér

monotonicity [15]. However, the efficient calculation of the resolvent remains an open problem. Our approaches use a variable metric to obtain a quasi-Newton method with an efficient resolvent calculus. Regularized Newton-type methods in continuous time, both for convex optimization and monotone inclusions have been studied in a series of papers by Attouch and his co-authors: [1, 3, 4, 2]. Time discretization of these dynamics gives algorithms providing insight into regularized Newton's method for solving monotone inclusions (see [1, 4, 2]). In [2], a relative error tolerance for the solution of the proximal subproblem is also allowed. However, in all these papers, the Hessian ends up being discretized.

PDHG. Primal-Dual Hybrid Gradient (PDHG) is widely used for solving saddle point problems of the form (3). PDHG can be interpreted as a proximal point algorithm [20] with a fixed metric applied to a monotone inclusion problem. Based on this idea, [30] proposed an inertial FBS method applied to the sum of set-valued operators from which a generalization of PDHG method is derived. Also based on similar ideas, [32] considers diagonal preconditioning to accelerate PDHG. Their method can be regarded as using a fixed blocked matrix as a metric. Later, [29] considers non-diagonal preconditioning and pointed out if a special preconditioner is chosen, that kind of preconditioned PDHG method will be a special form of the linearized ADMM. Their method requires an inner loop due to the non-diagonal preconditioning. [19] introduces an adaptive PDHG scheme which can also be understood as using a variable metric with step size tuned automatically. However, [19] focuses on changes to the diagonal of the metric. Our resolvent calculus in Section 4 provides another possibility to change metrics at elements off the diagonal to deal with resolvent operators. Additionally, in [27], the author investigated inexact inertial variable proximal point algorithm with a different condition on the inertial step and distinct assumptions on error terms compared to ours.

2 Preliminaries

Let us recall some essential notations and definitions. Let \mathcal{H} be a Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. The symbols \rightharpoonup and \rightarrow respectively denote weak and strong convergence. $\ell_+^1(\mathbb{N})$ is the set of all summable sequences in $[0, +\infty)$. An operator $K \in \mathcal{B}(\mathcal{D}, \mathcal{H})$ is a linear bounded mapping from a Hilbert space \mathcal{D} to \mathcal{H} . The adjoint of K is denoted by K^* . We abbreviate $\mathcal{B}(\mathcal{H}, \mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. We define $\mathcal{S}(\mathcal{H}) := \{M \in \mathcal{B}(\mathcal{H}) | M = M^*\}$ and the identity operator by $I \in \mathcal{S}(\mathcal{H})$. Without ambiguity, we also use the notation $\|M\|$ for the operator norm of $M \in \mathcal{S}(\mathcal{H})$ with respect to $\| \cdot \|$. The partial ordering on $\mathcal{S}(\mathcal{H})$ is given by

$$(\forall U \in \mathcal{S}(\mathcal{H}))(\forall V \in \mathcal{S}(\mathcal{H})) : U \succeq V \iff (\forall x \in \mathcal{H}) : \langle Ux, x \rangle \geq \langle Vx, x \rangle . \quad (4)$$

For $\sigma \in [0, +\infty)$, we introduce $\mathcal{S}_\sigma(\mathcal{H}) := \{U \in \mathcal{S}(\mathcal{H}) | U \succeq \sigma I\}$. Similarly, we introduce $\mathcal{S}_{++}(\mathcal{H}) := \{U \in \mathcal{S}(\mathcal{H}) | U \succ 0\}$. In particular, $\mathcal{S}_{++}(\mathbb{R}^n)$ denotes the set of $n \times n$ real symmetric positive definite matrices. The norm $\| \cdot \|_M$ is defined by $\sqrt{\langle M \cdot, \cdot \rangle}$ for $M \in \mathcal{S}_{++}(\mathcal{H})$. We say $Q \in \mathcal{S}_0(\mathcal{H})$ has finite rank r if $r = \dim(\text{im}(Q))$. Then there are linearly independent vectors u_i such that $Q : \mathcal{H} \rightarrow \mathcal{H}$, $x \mapsto \sum_{i=1}^r \langle x, u_i \rangle u_i$. As a consequence, $Q = UU^*$ where $U : \mathbb{R}^r \rightarrow \text{im}(Q)$, $\alpha \mapsto U\alpha := \sum_{i=1}^r \alpha_i u_i$ is an isomorphism defined by $(u_i)_{i=1, \dots, r}$.

A set valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is defined by its graph

$$\text{Graph } A := \{(x, y) \in \mathcal{H} | x \in \text{Dom}(A), y \in Ax\} ,$$

and has a domain given by

$$\text{Dom}(A) := \{x \in \mathcal{H} | Ax \neq \emptyset\} .$$

Given two set-valued operators $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$, we define $A + B : \mathcal{H} \rightrightarrows \mathcal{H}$ as follows:

$$\begin{aligned} \text{Dom}(A + B) &= \text{Dom}(A) \cap \text{Dom}(B) , \\ (A + B)x &= Ax + Bx := \{y \in \mathcal{H} | \exists y_1 \in Ax, \exists y_2 \in Bx \text{ such that } y = y_1 + y_2\} . \end{aligned}$$

The inverse of A is denoted by A^{-1} given by $A^{-1}(y) := \{x \in \mathcal{H} | y \in Ax\}$ and the zero set of A is denoted by $\text{zer}(A + B) := \{x \in \mathcal{H} | (A + B)x \ni 0\}$. We say that A is γ_A -strongly monotone with modulus $\gamma_A \geq 0$ with respect to norm $\| \cdot \|$, if $\langle x - y, u - v \rangle \geq \gamma_A \|x - y\|^2$ for any pair $(x, u), (y, v) \in \text{Graph } A$. We say that a single valued B is β -co-coercive with respect to norm $\| \cdot \|$,

if $\langle x - y, u - v \rangle \geq \beta \|u - v\|^2$ for any pair $(x, u), (y, v) \in \text{Graph } B$. The resolvent of $A: \mathcal{H} \rightrightarrows \mathcal{H}$ with respect to metric $M \in \mathcal{S}_{++}(\mathcal{H})$ is defined as

$$J_A^M := (I + M^{-1}A)^{-1} \text{ and we set } J_A := J_A^I \text{ for the identity mapping } I, \quad (5)$$

which, as shown for example in [6], enjoys the following properties.

Proposition 2.1. *Let $A: \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, $M \in \mathcal{S}_{++}(\mathcal{H})$ and $y \in \mathcal{H}$. Then, the following holds*

$$y = J_{\gamma A}^M(x) \iff x \in y + \gamma M^{-1}Ay \iff x - y \in \gamma M^{-1}Ay \iff (y, \gamma^{-1}M(x - y)) \in \text{Graph } A. \quad (6)$$

Proposition 2.2. *Let $A: \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone. Then for every sequence $(x_k, u_k)_{k \in \mathbb{N}}$ in $\text{Graph } A$ and every $(x, u) \in \mathcal{H} \times \mathcal{H}$, if $x_k \rightarrow x$ and $u_k \rightarrow u$, then $(x, u) \in \text{Graph } A$.*

Lemma 2.3. *Let $T: \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator and let $M \in \mathcal{S}_{++}(\mathcal{H})$. Then, for any $z \in \mathcal{H}$, we have $J_T^M(z) = M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z)$.*

Proof. See Appendix A.1. □

Lemma 2.4. *If A is strongly monotone with modulus $\gamma_A \geq 0$ and $M \in \mathcal{S}_{++}$, then J_A^M is Lipschitz continuous with respect to $\|\cdot\|_M$ with constant $1/(1 + \frac{\gamma_A}{C}) \in (0, 1]$ for any C satisfying $\|M\| \leq C < \infty$.*

Proof. See Appendix A.2. □

Proposition 2.5 (Variable Metric quasi-Fejér monotone sequence [15]). *Let $\sigma \in (0, +\infty)$, let $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ be strictly increasing and such that $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$, let $(M_k)_{k \in \mathbb{N}}$ be in $\mathcal{S}_\sigma(\mathcal{H})$, let C be a nonempty subset of \mathcal{H} , and let $(x_k)_{k \in \mathbb{N}}$ be a sequence in \mathcal{H} such that*

$$\begin{aligned} (\exists (\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})) (\forall z \in C) (\exists (\epsilon_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})) (\forall k \in \mathbb{N}): \\ \varphi(\|x_{k+1} - z\|_{M_{k+1}}) \leq (1 + \eta_k)\varphi(\|x_k - z\|_{M_k}) + \epsilon_k. \end{aligned} \quad (7)$$

(a) *Then $(x_k)_{k \in \mathbb{N}}$ is bounded and, for every $z \in C$, $(\|z_k - z\|_{M_k})_{k \in \mathbb{N}}$ converges.*

(b) *If additionally, there exists $M \in \mathcal{S}_\sigma(\mathcal{H})$ such that $M_k \rightarrow M$ pointwisely, as is the case when*

$$\sup_{k \in \mathbb{N}} \|M_k\| < +\infty \quad \text{and} \quad (\exists (\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})) (\forall k \in \mathbb{N}): (1 + \eta_k)M_k \succeq M_{k+1}, \quad (8)$$

then $(x_k)_{k \in \mathbb{N}}$ converges weakly to a point in C if and only if every weak sequential cluster point of $(x_k)_{k \in \mathbb{N}}$ lies in C .

A key result for our resolvent calculus in Section 4 is the following Attouch-Théra abstract duality principle.

Lemma 2.6 (A duality result for operators [5]). *Let $A: \mathcal{H} \rightrightarrows \mathcal{H}$ be an operator such that A^{-1} is single-valued and let $B: \mathcal{H} \rightarrow \mathcal{H}$ be a single-valued operator. Then, the following holds for $x, u \in \mathcal{H}$:*

$$\begin{cases} 0 \in Ax + Bx \\ 0 \in B^{-1}u - A^{-1}(-u) \end{cases} \iff \begin{cases} x \in B^{-1}u \\ -u \in Ax \end{cases} \iff \begin{cases} Bx = u \\ x = A^{-1}(-u) \end{cases}. \quad (9)$$

Moreover, if there exists $x \in \mathcal{H}$ such that $0 \in Ax + Bx$ or there exists $u \in \mathcal{H}$ such that $0 \in B^{-1}u - A^{-1}(-u)$, then there exists a unique primal-dual pair (x, u) that satisfies the equivalent conditions above.

We also need the following lemma which was stated as [33, Lemma 2.2.2].

Lemma 2.7. *Let $C_k \geq 0$ and let*

$$\begin{aligned} C_{k+1} &\leq (1 + \nu_k)C_k + \zeta_k, \quad \nu_k \geq 0, \quad \zeta_k \geq 0, \\ \sum_{k \in \mathbb{N}} \nu_k &< \infty, \quad \sum_{k \in \mathbb{N}} \zeta_k < \infty. \end{aligned} \quad (10)$$

Then, C_k converges to a non-negative limit.

3 Inertial Quasi-Newton FBS for Monotone Inclusions

3.1 Problem and Algorithms

We consider the monotone inclusion problem (1). In order to benefit from additional properties of A and B , whenever available, we assume that A and B are strongly monotone with modulus $\gamma_A \geq 0$ and $\gamma_B \geq 0$, respectively. Note that by setting $\gamma_A = 0$ or $\gamma_B = 0$, we include the general case of monotone operators that are not necessarily strongly monotone. In this paper, we propose two variants of an efficiently implementable quasi-Newton Forward-Backward Splitting (FBS) algorithm in Algorithm 1 and Algorithm 2 to solve (1). The main update step is a variable metric FBS step in both algorithms as shown in (12) and (13), i.e., a forward step with respect to the co-coercive operator B , followed by a proximal point step (computation of the resolvent) of the maximally monotone operator A , both evaluated with an iteration dependent metric M_k that is inspired by quasi-Newton methods and therefore adapts to the local geometry of the problem. In contrast to related works, as discussed in Section 1.1, we emphasize the importance of efficiently implementable resolvent operators (see Section 4). The two variants allow for more or less flexibility for the choice of the metric (see Section 5). Both variants account for potential errors in the evaluation of the forward-backward step. Algorithm 1 combines a FBS step with an additional inertial step which has the potential of accelerating the convergence, as we illuminate in our numerical experiments in Section 7. It is a generalization of the algorithms in [11, 30] to a quasi-Newton variant. In [30], Lorenz and Pock proposed an inertial Forward-Backward Splitting algorithm with a fixed metric which is different from our Algorithm 1. Algorithm 2 combines FBS with a relaxation step in (iii) which yields convergence under weak assumptions on the metric. It generalizes the correction step introduced in [20] which can be retrieved by setting $M_k \equiv M$ and $B \equiv 0$.

Algorithm 1 Inertial Quasi-Newton Forward-Backward Splitting

Require: $N \geq 0, (\|\epsilon_k\|)_{k \in \mathbb{N}}$

Initialization: $z_0 \in \mathcal{H}$

Update for $k = 0, \dots, N$:

- (i) Compute M_k according to a quasi-Newton framework (see Section 5).
- (ii) Compute the inertial step (extrapolation step):

$$\bar{z}_k \leftarrow z_k + \alpha_k(z_k - z_{k-1}), \quad (11)$$

- (iii) and the forward-backward step:

$$z_{k+1} \leftarrow J_A^{M_k}(\bar{z}_k - M_k^{-1}B\bar{z}_k) + \epsilon_k. \quad (12)$$

End

Algorithm 2 Quasi-Newton Forward-Backward Splitting with Relaxation

Require: $N \geq 0, (\|\epsilon_k\|)_{k \in \mathbb{N}}$

Initialization: $z_0 \in \mathcal{H}$

Update for $k = 0, \dots, N$:

- (i) Compute M_k according to a quasi-Newton framework (see Section 5).
- (ii) Compute the forward-backward step:

$$\tilde{z}_k \leftarrow J_A^{M_k}(z_k - M_k^{-1}Bz_k) + \epsilon_k, \quad (13)$$

- (iii) and relaxation step:

$$t_k = \frac{\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle}{2\|(M_k - B)(z_k - \tilde{z}_k)\|^2}, \quad (14)$$

$$z_{k+1} \leftarrow z_k - t_k[(M_k - B)(z_k - \tilde{z}_k)]. \quad (15)$$

End

3.2 Convergence Guarantees

In this subsection, we prove the convergence of Algorithm 1 and Algorithm 2. The implementation details for the specific quasi-Newton features are deferred to Section 4.

3.2.1 Algorithm 1: Inertial Quasi-Newton Forward-Backward Splitting

The following convergence result is a generalization of [13, Theorem 3.1] to an inertial version of variable metric Forward-Backward Splitting.

Assumption 1. Let $\sigma \in (0, +\infty)$. $(M_k)_{k \in \mathbb{N}}$ is a sequence in $\mathcal{S}_\sigma(\mathcal{H})$ such that

$$\begin{cases} C := \sup_{k \in \mathbb{N}} \|M_k\| < \infty, \\ (\exists (\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})) (\forall k \in \mathbb{N}): (1 + \eta_k)M_k \succeq M_{k+1}, \end{cases} \quad (16)$$

and $M_k - \frac{1}{2\beta}I \in \mathcal{S}_\kappa(\mathcal{H})$ for all $k \in \mathbb{N}$ and some $\kappa > 0$.

Theorem 3.1. Consider Problem (1) and let the sequence $(z_k)_{k \in \mathbb{N}}$ be generated by Algorithm 1 where Assumption 1 holds. The sequence $(\alpha_k)_{k \in \mathbb{N}}$ is selected such that $\alpha_k \in (0, \Lambda]$ with $\Lambda < \infty$ and

$$\sum_{k \in \mathbb{N}} \alpha_k \max\{\|z_k - z_{k-1}\|_{M_k}, \|z_k - z_{k-1}\|_{M_k}^2\} < +\infty,$$

while $(\epsilon_k)_{k \in \mathbb{N}}$ is a sequence in \mathcal{H} such that $\sum_{k \in \mathbb{N}} \|\epsilon_k\| < +\infty$. Suppose that $\text{zer}(A+B) \neq \emptyset$. Then $(z_k)_{k \in \mathbb{N}}$ is bounded and weakly converges to a point $z^* \in \text{zer}(A+B)$, i.e. $z_k \rightarrow z^*$ as $k \rightarrow \infty$.

Furthermore, if additionally we assume $\epsilon_k \equiv 0$ for any $k \in \mathbb{N}$, $\gamma_A > 0$ or $\gamma_B > 0$ and $M_k - \frac{1}{\beta}I \in \mathcal{S}_\kappa(\mathcal{H})$ for some $\kappa > 0$, then there exist some $\xi \in (0, 1)$, some $\Theta > 0$ and some $K_0 \in \mathbb{N}$ such that for any $k > K_0$,

$$\|z_k - z^*\|_{M_k}^2 \leq (1 - \xi)^{k-K_0} \|z_{K_0} - z^*\|_{M_{K_0}}^2 + \sum_{i=K_0}^{k-1} \Theta (1 - \xi)^{k-i} \alpha_i \max\{\|z_i - z_{i-1}\|_{M_i}, \|z_i - z_{i-1}\|_{M_i}^2\}. \quad (17)$$

Proof. See Appendix A.3. □

Remark 3.2. • Using Lemma 3.1 (iv) from [14], we deduce that the second term on the right hand of inequality (17) converges to 0.

- The linear convergence factor $1 - \xi$ is chosen such that there exists $K_0 \in \mathbb{N}$ with

$$(1 + \eta_k) \left(\frac{1 - \frac{\gamma_B}{C}}{1 + \frac{\gamma_A}{C}} \right) \leq 1 - \xi \quad \text{for all } k \geq K_0.$$

- The convergence rate for the strongly monotone setting can be influenced by the decay rate of α_k .

- (i) If $\alpha_k = O(q^k)$ for $q = 1 - \xi$, we have convergence rate of $O(kq^k)$ for $k > K_0$ where K_0 is sufficiently large.
- (ii) If $\alpha_k = O(\frac{1}{k^2})$, we have convergence rate of $O(\frac{1}{k})$ for $k > K_0$ where K_0 is sufficiently large.

Remark 3.3. In practice, Assumption 1 is hard to verify and restrictive, however, it is a common assumption for variable metric methods [16]. It can be avoided by relaxation methods which we propose in the next section for this reason.

3.2.2 Algorithm 2: Quasi-Newton Forward-Backward Splitting with Relaxation

At the expense of a relaxation instead of an inertial step, we achieve a substantial enhancement in the flexibility of selecting the metric. This method is inspired by [20]. It is worth noting that without loss of generality, we assume $\tilde{z}_k \neq z_k$ for all $k \in \mathbb{N}$, since otherwise \tilde{z}_k already solves the inclusion problem after a finite number of iterations.

Assumption 2. Let $\sigma \in (0, +\infty)$. $(M_k)_{k \in \mathbb{N}}$ is a sequence in $\mathcal{S}_\sigma(\mathcal{H})$ and $(\epsilon_k)_{k \in \mathbb{N}}$ is a sequence in \mathcal{H} such that:

(i) For all $k \in \mathbb{N}$, we have $(M_k - \frac{1}{\beta}I) \in \mathcal{S}_c(\mathcal{H})$, for some $c > 0$,

(ii) $C := \sup_{k \in \mathbb{N}} \|M_k\| < \infty$.

Theorem 3.4. Consider Problem (1), and let the sequence $(z_k)_{k \in \mathbb{N}}$ be generated by Algorithm 2 where Assumption 2 holds. The sequence $(\epsilon_k)_{k \in \mathbb{N}}$ in \mathcal{H} satisfies $\sum_{k \in \mathbb{N}} \|\epsilon_k\| < +\infty$. Then $(\|z_k - z^*\|)_{k \in \mathbb{N}}$ is bounded for any $z^* \in \text{zer}(A+B)$ and $(z_k)_{k \in \mathbb{N}}$ weakly converges to some $z^* \in \text{zer}(A+B)$, i.e. $z_k \rightharpoonup z^*$ as $k \rightarrow \infty$.

Moreover, if $\epsilon_k \equiv 0$, then $\|z_k - z^*\|$ decreases for any $z^* \in \text{zer}(A+B)$ as $k \rightarrow \infty$. Furthermore, if $\gamma_A > 0$ or $\gamma_B > 0$, then z_k converges linearly: there exist some $\xi \in (0, 1)$ such that

$$\|z_k - z^*\|^2 \leq (1 - \xi)^k \|z_0 - z^*\|^2. \quad (18)$$

Proof. See Appendix A.4. □

Remark 3.5. The linear convergence factor is given by

$$\xi = \frac{\delta}{2} \min\{2(\gamma_A + \gamma_B), c\}, \quad \text{where } \delta = \frac{c}{2(C+1/\beta)^2}.$$

Remark 3.6. In [20], authors studied a relaxed proximal point algorithm for primal-dual splitting with a fixed metric. Our setting and Algorithm 2 here are much broader.

Remark 3.7. It is noteworthy that Algorithm 2 can be interpreted as a closed loop system which uses the previous iterates (states) to update the relaxation parameter t_k and the quasi-Newton metric M_k , i.e., the update does not explicitly depend on k .

4 Resolvent Calculus for Low-Rank Perturbed Metric

In this section, we extend the proximal calculus of [8] to the setting of resolvent operators J_T^V with a symmetric positive definite metric $V = M + sQ$, where $s \in \{-1, +1\}$, M is symmetric positive definite and Q is symmetric positive semi-definite. This is a key result, as it enables an efficient application of quasi-Newton methods for solving monotone inclusion problems. Computing the resolvent operator $J_T^V(z)$ involves evaluating J_T^M at a shifted point $z - M^{-1}v^* \in \mathcal{H}$, where $v^* \in \mathcal{H}$ is derived from a root-finding problem, solvable by a semi-smooth Newton method (Algorithm 3) or bisection method (Algorithm 4). In conclusion, if J_T^M can be computed efficiently, the same is true for J_T^V . The result crucially relies on the abstract duality principle of Attouch-Théra [5] (see Lemma 2.6). We first state the abstract result in a Hilbert space \mathcal{H} in Theorem 4.1 and illustrate it in Corollary 4.3 with $\mathcal{H} = \mathbb{R}^n, n \in \mathbb{N}$.

Theorem 4.1. Let $T: \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator, $V := M + sQ \in \mathcal{S}_{++}(\mathcal{H})$ where $s \in \{-1, 1\}$, $M \in \mathcal{S}_{++}(\mathcal{H})$ and $Q \in \mathcal{S}_0(\mathcal{H})$ having finite rank r . Then, the resolvent operator J_T^V can be computed as follows:

$$x^* = J_T^V(z) \iff \begin{cases} x^* = J_T^M(z - sM^{-1}U\alpha^*) \text{ and} \\ \alpha^* \in \mathbb{R}^r \text{ solves } \ell(\alpha) = 0 \\ \text{where } \ell(\alpha) := U^*Q^{-1}U\alpha + U^*(z - J_T^M(z - sM^{-1}U\alpha)) = 0, \end{cases} \quad (19)$$

where $U: \mathbb{R}^r \rightarrow \text{im}(Q)$, $\alpha \mapsto U\alpha := \sum_{i=1}^r \alpha_i u_i$ is an isomorphism defined by any r linearly independent $u_1, \dots, u_r \in \text{im}(Q)$. The function $\ell: \mathbb{R}^r \rightarrow \mathbb{R}^r$ is Lipschitz continuous with constant $\|U^*Q^{-1}U\| + \|M^{-1/2}U\|^2$ and strictly monotone.

Proof. See Appendix A.5. □

Remark 4.2. *A priori Q^{-1} is set-valued, however it is easy to check that $U^*Q^{-1}U$ is single-valued (see Appendix A.5.1). We define $Q^+ : \text{im}(Q) \rightarrow \text{im}(Q)$ as the inverse of Q restricted to $\text{im}(Q)$ which is a single-valued mapping. It allows us to replace Q^{-1} by Q^+ in (19).*

In finite dimensions, Theorem 4.1 simplifies to the following corollary.

Corollary 4.3. *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximally monotone operator and consider $V := M + sQ \in \mathcal{S}_{++}(\mathbb{R}^n)$ where $s \in \{-1, +1\}$, $M \in \mathcal{S}_{++}(\mathbb{R}^n)$ and $Q \in \mathcal{S}_0(\mathbb{R}^n)$. Let $Q = UU^\top$ where $U : \mathbb{R}^r \rightarrow \mathbb{R}^n$ is a matrix of full rank r with $r \leq n$. Then, the resolvent operator J_T^V can be computed as follows:*

$$x^* = J_T^V(z) \iff \begin{cases} x^* = J_T^M(z - sM^{-1}U\alpha^*) \text{ and} \\ \alpha^* \in \mathbb{R}^r \text{ solves } \ell(\alpha) = 0 \\ \text{where } \ell(\alpha) := \alpha + U^\top(z - J_T^M(z - sM^{-1}U\alpha)). \end{cases} \quad (20)$$

The solution α^ is the unique root of $\ell : \mathbb{R}^r \rightarrow \mathbb{R}^r$ which is Lipschitz continuous with constant $1 + \|M^{-1/2}U\|^2$ and strictly monotone.*

Proof. See Appendix A.6. □

If $r \ll n$, then we have a so-called low rank perturbed metric $M + sUU^\top$, which leads to a root-finding problem of low dimension r . Together with the simple metric M with respect to which the resolvent operator is easy to evaluate, it leads to an efficient evaluation of a resolvent operator with respect to a low rank perturbed metric.

4.1 Solving the Root-Finding Problem

The efficiency of the reduction in Theorem 4.1 relies also on the solution of a root-finding problem which we discuss thoroughly in this subsection. We consider the space \mathbb{R}^r and the root-finding problem with $\ell : \mathbb{R}^r \rightarrow \mathbb{R}^r$. In several instances, the root-finding problem can be solved exactly, for instance, with $T = \partial g$ for special functions g as enumerated in [8, Table 3.1]. In such cases, the root-finding problem simplifies to one involving the proximal operator rather than the resolvent. Similarly, when J_T^M can be represented as a composition of proximal mappings with respect to these special functions, the associated root-finding problem can be exactly solved. In situations where this subproblem cannot be exactly solved, we employ a semi-smooth Newton approach which enjoys local super-linear convergence. Therefore, our Algorithms 1 and 2 have also modeled computational errors arising from inexactly solving the root-finding problem. To narrow down the neighborhood of the sought root for $r = 1$, we complement the semi-smooth Newton strategy by a bisection method in Section 4.1.2. For cases where $r \geq 1$, a globalization strategy is available as shown in [38].

4.1.1 Semi-smooth Newton Methods

In order to solve $\ell(\alpha) = 0$ in (19) efficiently, we employ a semi-smooth Newton method. A locally Lipschitz function is called semi-smooth if its Clarke Jacobian defines a Newton approximation scheme [18, Definition 7.4.2]. If $\ell(\alpha)$ is semi-smooth and any element of the Clarke Jacobian $\partial^C \ell(\alpha^*)$ is non-singular, then the inexact semi-smooth Newton method outlined in Algorithm 3 (analogous to [8]) can be applied. Semi-smoothness may seem restrictive. However, as shown in [9], the broad class of tame locally Lipschitz functions is semi-smooth. We refer to [9] for the definition of tameness. Therefore, it is sufficient to ensure $\ell(\alpha)$ is tame, which is asserted if the monotone operator T in $\ell(\alpha)$ is a tame map [21]. In this case, the convergence result for Algorithm 3 can be adapted from [8].

Proposition 4.4. *Let $\ell(\alpha)$ be defined as in Theorem 4.1, where T is a set-valued tame mapping. Then $\ell(\alpha)$ is semi-smooth and all elements of $\partial^C \ell(\alpha^*)$ are non-singular where α^* is the unique root of $\ell(\alpha)$ from (19). In turn there exists $\bar{\eta}$ such that if $\eta_k \leq \bar{\eta}$ for every k , there exists a neighborhood of α^* such that for all α_0 in that neighborhood, the sequence generated by Algorithm 3 is well defined and converges to α^* linearly. If $\eta_k \rightarrow 0$, the convergence is superlinear.*

Algorithm 3 Semi-smooth Newton method to solve $\ell(\alpha) = 0$

Require: A point $\alpha_0 \in \mathbb{R}^r$. N is the maximal number of iterations.

Update for $k = 0, \dots, N$:

if $\ell(\alpha_k) = 0$ **then**

stop

else

Select $G_k \in \partial^C \ell(\alpha_k)$, compute α_{k+1} such that

$$\ell(\alpha_k) + G_k(\alpha_{k+1} - \alpha_k) = e_k,$$

where $e_k \in \mathbb{R}^r$ is an error term satisfying $\|e_k\| \leq \eta_k \|G_k\|$ and $\eta_k \geq 0$.

end if

End

Proof. See Appendix A.7. □

Example 4.5. If f is a tame function and locally Lipschitz, then by [21, Proposition 3.1], ∂f is a tame map.

Example 4.6. The assumption that T is a tame mapping is not restrictive. For example, in PDHG setting we have a set-valued operator $T = \begin{pmatrix} \partial g & K^* \\ -K & \partial f \end{pmatrix}$ as defined by (33). If g and f are both tame functions, then ∂g and ∂f are tame as well [21]. As a result, T is a tame mapping.

4.1.2 Bisection

In the case where $\mathcal{H} = \mathbb{R}^n$ and $r = 1$, we set $U = u \in \mathbb{R}^{n \times 1}$ and solve the root-finding problem $\ell(\alpha) = 0$ via the bisection method in Algorithm 4. A similar bound on the range of α^* as in [8] holds.

Proposition 4.7. Consider the setting of Corollary 4.3. For $r = 1$, the root α^* of $\ell(\alpha) = 0$ in Corollary 4.3 lies in the set $[-\zeta, \zeta]$, where

$$\zeta = \|u\|_{V^{-1}} (2\|z\|_V + \|J_T^V(0)\|_V). \quad (21)$$

Moreover, if $V \in \mathcal{S}_c(\mathbb{R}^n)$ and $\|V\| \leq C$, then

$$\zeta = \frac{C}{c} \|u\| (2\|z\| + \|J_T^V(0)\|). \quad (22)$$

Proof. See Appendix A.8. □

Algorithm 4 Bisection method to solve $\ell(\alpha) = 0$ when $r = 1$

Require: Tolerance $\epsilon \geq 0$, number of iterates N

Compute the bound ζ from (21), and set $k = 0$.

Set $\alpha_- = -\zeta$ and $\alpha_+ = \zeta$.

Update for $k = 0, \dots, N$:

Set $\alpha_k = \frac{1}{2}(\alpha_- + \alpha_+)$.

if $\ell(\alpha_k) > 0$ **then**

$\alpha_+ \leftarrow \alpha_k$

else

$\alpha_- \leftarrow \alpha_k$

end if

if $k > 1$ and $|\alpha_k - \alpha_{k-1}| < \epsilon$ **then**

return α_k

end if

End

Furthermore, we can combine the semi-smooth Newton method with the bisection method. Since the semi-smooth Newton method is locally convergent, it requires a starting point in a sufficiently near neighborhood of the solution. Using the bisection method, we can generate a sequence of points approaching the solution. When these points reach the neighborhood required for convergence of the semi-smooth Newton method, we transition to using the semi-smooth Newton method to achieve faster convergence. In Proposition 4.4, α_0 is required to belong to a neighborhood of α^* , which can be achieved by Algorithm 4, i.e., we can assert to find a point α such that $|\alpha - \alpha^*| < \delta$ in $\log_2((2\zeta/\delta))$ steps, where ζ is as in (21).

4.2 Implementation of the quasi-Newton Forward-Backward Step

In Algorithm 1 and 2, we evaluate the resolvent mappings with respect to M_k at \bar{z}_k and z_k respectively. Here, $M_k \in \mathcal{S}_{++}(\mathcal{H})$ is generated from quasi-Newton framework (see Section 5) and represents a low rank perturbed metric with $M_k = M_0 + sQ_k$ where Q_k has a finite rank r . In the quasi-Newton framework, there exists a connection between Q_k and U_k in the form of $Q_k = U_k U_k^*$ for some bounded linear isomorphism $U_k: \mathbb{R}^r \rightarrow \text{im}(Q_k)$. The computation of both formulations involves the same abstract type of calculation at some point z , as represented by:

$$J_A^V(z - V^{-1}Bz), \quad (23)$$

where $V = M + sQ \in \mathcal{S}_{++}(\mathcal{H})$ with $s \in \{-1, 1\}$, $M \in \mathcal{S}_{++}(\mathcal{H})$ and $Q = UU^*$ for some bounded linear isomorphism $U: \mathbb{R}^r \rightarrow \text{im}(Q)$. Here, we replace M_k by V , omit the subscript k and for Algorithm 1 and 2 we substitute \bar{z}_k with z and z_k with z respectively. While the computation of the resolvent (23) can be significantly simplified using Theorem 4.1, here, an efficient update formula incorporating the forward step is presented. Instead of evaluating $J_A^V(z - V^{-1}Bz)$ directly, we decompose the entire task into two subproblems: a root-finding problem and an evaluation of resolvent with respect to M which is assumed to be simple to compute.

Proposition 4.8. *Consider the setting of problem (1). Let $V = M + sQ \in \mathcal{S}_{++}(\mathcal{H})$ with $s \in \{-1, 1\}$, $M \in \mathcal{S}_{++}(\mathcal{H})$ and $Q \in \mathcal{S}_0(\mathcal{H})$ having $Q = UU^*$ for a bounded linear isomorphism $U: \mathbb{R}^r \rightarrow \text{im}(Q)$. Then (23) can be equivalently expressed by*

$$\hat{z} = J_A^V(z - V^{-1}Bz) = J_A^M(z - V^{-1}Bz - sM^{-1}U\alpha^*). \quad (24)$$

Here, $\alpha^* \in \mathbb{R}^r$ is the unique root of $\mathcal{L}: \mathbb{R}^r \rightarrow \mathbb{R}^r$,

$$\mathcal{L}(\alpha^*) := U^*(z - V^{-1}Bz - J_A^M(z - V^{-1}Bz - sM^{-1}U\alpha^*)) + \alpha^* = 0. \quad (25)$$

The function \mathcal{L} is Lipschitz continuous with constant $1 + \|M^{-1/2}U\|^2$ and strictly monotone.

Proof. Apply Theorem 4.1 at the shifted point $z - V^{-1}Bz$. \square

It remains now to compute the point $z - V^{-1}Bz$ which involves inverting V . Since V has the special structure $M + sQ$, we will show that the update amounts to inverting solely M . This is equivalent to applying the Sherman-Morrison formula.

Proposition 4.9. *Consider the setting of Proposition 4.8. Then (23) at a point z can be solved by*

$$\hat{z} = J_A^V(z - V^{-1}Bz) = J_A^M(z - M^{-1}Bz - sM^{-1}U\xi^*). \quad (26)$$

Here, $\xi^* \in \mathbb{R}^r$ is the unique zero of $\mathcal{J}: \mathbb{R}^r \rightarrow \mathbb{R}^r$,

$$\mathcal{J}(\xi^*) := U^*(z - J_A^M(z - M^{-1}Bz - sM^{-1}U\xi^*)) + \xi^* = 0. \quad (27)$$

The function \mathcal{J} is Lipschitz continuous with constant $1 + \|M^{-1/2}U\|^2$ and strictly monotone.

Proof. See Appendix A.9. \square

5 A General OSR1 Quasi-Newton Metric

We note that if we set in the inclusion problem (1) $A \equiv 0$, and $B = \nabla f$ of some convex smooth function f , the update step consisting of (12) and (11) reduces to Gradient Descent when $M_k \equiv I$ and to the classic Newton method when $M_k = \nabla^2 f(z_k)$. Motivated by classic Newton and quasi-Newton methods, we construct M_k as an approximation of the differential of Bz_k at z_k . We generalize the quasi-Newton method OSR1 from differentiable ∇f (SR1 method with 0-memory) to a co-coercive operator B . The approximation M_k shall satisfy the modified secant condition:

$$M_k s_k = y_k, \quad \text{where} \quad y_k = Bz_k - Bz_{k-1}, \quad s_k = z_k - z_{k-1}. \quad (28)$$

Choose $M_0 \in \mathcal{S}_{++}(\mathcal{H})$ positive-definite. If $\langle y_k - M_0 s_k, s_k \rangle = 0$, we skip the update of the metric. If $\langle y_k - M_0 s_k, s_k \rangle \neq 0$, the update is:

$$M_k = M_0 + s U_k U_k^*, \quad \hat{u}_k = (y_k - M_0 s_k) / \sqrt{|\langle y_k - M_0 s_k, s_k \rangle|}, \quad (29)$$

where $U_k: \mathbb{R} \rightarrow \mathcal{H}$, $\alpha \mapsto U_k \alpha := \alpha \sqrt{\gamma_k} \hat{u}_k$ is a bounded linear mapping, $\gamma_k \in [0, +\infty)$ needs to be selected such that M_k is positive-definite and $s \in \{-1, +1\}$ depends on the sign of $\langle y_k - M_0 s_k, s_k \rangle$. If $\langle y_k - M_0 s_k, s_k \rangle > 0$, we use

$$M_k = M_0 + U_k U_k^*, \quad (30)$$

and, if $\langle y_k - M_0 s_k, s_k \rangle < 0$, we use

$$M_k = M_0 - U_k U_k^*. \quad (31)$$

In the rest of this section, we provide conditions under which M_k defined in (29) meets Assumption 1 and Assumption 2. We introduce a bounded linear mapping $u_k: \mathbb{R} \rightarrow \mathcal{H}$, $\alpha \mapsto \alpha \hat{u}_k$ where $\hat{u}_k \in \mathcal{H}$ is defined by (29). For convenience, we will also equate $U_k U_k^*$ with $\gamma_k u_k u_k^*$. We start with Assumption 2.

Lemma 5.1. *Let M_0 be symmetric positive-definite and B is β -co-coercive. Suppose that $M_0 - \frac{1}{\beta} \in \mathcal{S}_c(\mathcal{H})$ for $c > 0$.*

(i) *If $M_k = M_0 + \gamma_k u_k u_k^*$, where $0 < \inf_{k \in \mathbb{N}} \gamma_k \leq \sup_{k \in \mathbb{N}} \gamma_k < +\infty$, then for all $k \in \mathbb{N}$, $M_k - \frac{1}{\beta} \in \mathcal{S}_c(\mathcal{H})$ and M_k verifies Assumption 2.*

(ii) *If $M_k = M_0 - \gamma_k u_k u_k^*$, where $0 < \inf_{k \in \mathbb{N}} \gamma_k \leq \sup_{k \in \mathbb{N}} \gamma_k < \frac{c^2}{(1/\beta + \|M_0\|)^2}$, then M_k is positive-definite and satisfies Assumption 2.*

Proof. See Appendix A.10. □

Lemma 5.2. *Let M_0 be symmetric positive-definite and B is β -co-coercive. Suppose that $M_0 - \frac{1}{\beta} \in \mathcal{S}_c(\mathcal{H})$ for $c > 0$. Let $(\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$.*

(i) *Case $M_k = M_0 + \gamma_k u_k u_k^*$: take $\gamma_k = \frac{\eta_k}{\|u_k\|^2} (1/\beta + c)$, and assume that $(\eta_k)_{k \in \mathbb{N}}$ is non-increasing. Then for all $k \in \mathbb{N}$, $M_k - \frac{1}{\beta} \in \mathcal{S}_c(\mathcal{H})$ and M_k verifies Assumption 1.*

(ii) *Case $M_k = M_0 - \gamma_k u_k u_k^*$: take $\gamma_k = \frac{\kappa \eta_k}{\|u_k\|^2} (1/\beta + c)$, where $\kappa \in]1/(1 + \beta c), 1[$, and assume that $\sup_{k \in \mathbb{N}} \eta_k \leq 1/\kappa - 1$. Then M_k is positive-definite and satisfies Assumption 1.*

Proof. See Appendix A.11. □

Remark 5.3. *The proof of Lemma 5.2 does not rely on co-coercivity of B unlike that of Lemma 5.1. If one uses that property, and more precisely the bounds in (105), then the choice of γ_k can be made independent of u_k in the form $\gamma_k = C \eta_k$, where C is a positive constant that depends only on $(\beta, c, \|M_0\|)$.*

6 Inertial Quasi-Newton PDHG for Saddle-point Problems

In this section, we consider a min-max problem as follows:

$$\min_{x \in \mathcal{H}_1} \max_{y \in \mathcal{H}_2} g(x) + G(x) + \langle Kx, y \rangle - f(y) - F(y) \quad (32)$$

with a linear mapping $K: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and proper lower semi-continuous convex functions $g: \mathcal{H}_1 \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and $f: \mathcal{H}_2 \rightarrow \overline{\mathbb{R}}$. Additionally, we consider continuously differentiable convex functions $G: \mathcal{H}_1 \rightarrow \mathbb{R}$ and $F: \mathcal{H}_2 \rightarrow \mathbb{R}$ with Lipschitz continuous gradients. This problem can be expressed as a special monotone inclusion problem from which we derive in Algorithm 5 an inertial quasi-Newton Primal-Dual Hybrid Gradient Method (PDHG) and in Algorithm 6 a quasi-Newton PDHG with relaxation step as special applications of Algorithm 1 and Algorithm 2 respectively (see Proposition 6.1). By Fermat's rule, the optimality condition for (32) is the following inclusion problem:

$$0 \in Az + Bz \quad \text{with } z = \begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } Az = \begin{pmatrix} \partial g(x) + K^*y \\ -Kx + \partial f(y) \end{pmatrix} \text{ and } Bz = \begin{pmatrix} \nabla G(x) \\ \nabla F(y) \end{pmatrix}. \quad (33)$$

Algorithm 5 Inertial quasi-Newton PDHG

Require: $N \geq 0$, $(\|\epsilon_k\|)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$

Update for $k = 0, \dots, N$:

- (i) Compute M_k according to a quasi-Newton framework.
- (ii) Compute the inertial step with parameter α_k :

$$\begin{aligned} \bar{x}_k &= x_k + \alpha_k(x_k - x_{k-1}), \\ \bar{y}_k &= y_k + \alpha_k(y_k - y_{k-1}). \end{aligned} \quad (34)$$

- (iii) Compute the main quasi-Newton PDHG step:

$$\begin{aligned} x_{k+1} &= \text{prox}_g^{\mathcal{T}}(\bar{x}_k - \mathcal{T}\nabla G(\bar{x}_k) - \mathcal{T}K^*\bar{y}_k - s\mathcal{T}U_{k,x}\xi_k) + \epsilon_{k,x}, \\ y_{k+1} &= \text{prox}_f^{\Sigma}(\bar{y}_k - \Sigma\nabla F(\bar{y}_k) + \Sigma K(2x_{k+1} - \bar{x}_k) - s\Sigma U_{k,y}\xi_k) + \epsilon_{k,y}, \end{aligned} \quad (35)$$

where ξ_k solves $\mathcal{J}(\xi_k) = 0$ (see (43)) and $\epsilon_k = \begin{pmatrix} \epsilon_{k,x} \\ \epsilon_{k,y} \end{pmatrix}$ is the error caused by computation at the k -th iterate.

End

Proposition 6.1 (PDHG method as a special proximal point algorithm [20]). *The update step of PDHG can be regarded as a proximal point algorithm with special metric M :*

$$M(z_{k+1} - z_k) + Az_{k+1} \ni -Bz_k, \quad (40)$$

where $M = \begin{pmatrix} \mathcal{T}^{-1} & -K^* \\ -K & \Sigma^{-1} \end{pmatrix}$ with two fixed operators $\mathcal{T} \in \mathcal{S}_{++}(\mathcal{H}_1)$ and $\Sigma \in \mathcal{S}_{++}(\mathcal{H}_2)$ such that $M \succ 0$. The latter is verified when $\|\Sigma^{1/2}K\mathcal{T}^{1/2}\| < 1$.

Remark 6.2. *It is straightforward to verify that*

$$M \succeq (1 - \|\Sigma^{1/2}K\mathcal{T}^{1/2}\|) \begin{pmatrix} \mathcal{T}^{-1} & 0 \\ 0 & \Sigma^{-1} \end{pmatrix}$$

If $\mathcal{T} = \tau I$ and $\Sigma = \sigma I$, we can retrieve a PDHG algorithm with constant stepsizes $\tau > 0$ and $\sigma > 0$. Moreover, $M \succeq (1 - \tau^{1/2}\sigma^{1/2}\|K\|) \min(\tau^{1/2}, \sigma^{1/2})$, and a sufficient condition for $M \succ 0$ is that $\tau\sigma\|K\|^2 < 1$.

Therefore, we are able to apply Algorithm 1 and Algorithm 2 directly to a saddle point problem by regarding T as A in the inclusion problem (1), which will give us Algorithm 5 and Algorithm 6,

Algorithm 6 Quasi-Newton PDHG with relaxation

Require: $N \geq 0$, $(\|\epsilon_k\|)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$

Update for $k = 0, \dots, N$:

- (i) Compute M_k according to quasi-Newton framework.
- (ii) Compute the main quasi-Newton PDHG step:

$$\begin{aligned}\tilde{x}_k &= \text{prox}_g^{\mathcal{T}}(x_k - \mathcal{T}\nabla G(x_k) - \mathcal{T}K^*y_k - s\mathcal{T}U_{k,x}\xi_k) + \epsilon_{k,x}, \\ \tilde{y}_k &= \text{prox}_f^{\Sigma}(y_k - \Sigma\nabla F(y_k) + \Sigma K(2\tilde{x}_k - x_k) - s\Sigma U_{k,y}\xi_k) + \epsilon_{k,y},\end{aligned}\tag{36}$$

where ξ_k solves $\mathcal{J}(\xi_k) = 0$ and $\epsilon_k = \begin{pmatrix} \epsilon_{k,x} \\ \epsilon_{k,y} \end{pmatrix}$ is the error caused by computation at the k -th iterate.

- (iii) Relaxation step:

$$v_k := M_k \left(\begin{pmatrix} x_k \\ y_k \end{pmatrix} - \begin{pmatrix} \tilde{x}_k \\ \tilde{y}_k \end{pmatrix} \right) + \begin{pmatrix} \nabla G(\tilde{x}_k) \\ \nabla F(\tilde{y}_k) \end{pmatrix} - \begin{pmatrix} \nabla G(x_k) \\ \nabla F(y_k) \end{pmatrix}\tag{37}$$

to compute the relaxation parameter t_k

$$t_k = \frac{\left\langle \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \begin{pmatrix} \tilde{x}_k \\ \tilde{y}_k \end{pmatrix}, v_k \right\rangle}{2\|v_k\|^2}\tag{38}$$

and update x_k and y_k as follows

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} \leftarrow \begin{pmatrix} x_k \\ y_k \end{pmatrix} - t_k v_k.\tag{39}$$

End

respectively. We are going to use a variable metric $M_k = M + sQ_k \in \mathcal{S}(\mathcal{H}_1 \times \mathcal{H}_2)$ with $s \in \{1, -1\}$ instead of $M \in \mathcal{S}(\mathcal{H}_1 \times \mathcal{H}_2)$ in (40) for the k -th iterate. Here, we set $Q_k = U_k U_k^*$ where $U_k: \mathbb{R}^r \rightarrow \text{im}(Q)$, $\alpha \mapsto U_k(\alpha) := \sum_{i=1}^r \alpha_i u_{k,i}$ is an isomorphism defined by r linearly independent $u_{k,1}, \dots, u_{k,r} \in \mathcal{H}_1 \times \mathcal{H}_2$ for each k -th iterate. We obtain the update step: Find $z_{k+1} \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$M_k(z_{k+1} - z_k) + Az_{k+1} \ni -Bz_k.\tag{41}$$

In practice, Proposition 4.9 which is derived from our main result Theorem 4.1 turns out to be more tractable than other ways to evaluate variable metric proximal mapping, e.g. using coordinate descent to solve a subproblem which has the same dimension as the original problem [35]. To show how to calculate the update step, using Proposition 4.9 in quasi-Newton PDHG in Algorithm 5 and in Algorithm 6, we introduce Proposition 6.3. Let $U_k \in \mathcal{B}(\mathbb{R}^r, \mathcal{H}_1 \times \mathcal{H}_2)$. We set $U_k = \begin{pmatrix} U_{k,x} \\ U_{k,y} \end{pmatrix}$ with $U_{k,x} \in \mathcal{B}(\mathbb{R}^r, \mathcal{H}_1)$ and $U_{k,y} \in \mathcal{B}(\mathbb{R}^r, \mathcal{H}_2)$. The validity of the update step is verified in Proposition 6.3.

Proposition 6.3. *The update step from \bar{z}_k (z_k) to z_{k+1} (\bar{z}_k) in Algorithm 5 (Algorithm 6), which is the quasi-Newton PDHG update step, reduces to compute*

$$\begin{cases} x_{k+1} &= \text{prox}_g^{\mathcal{T}}(\bar{x}_k - \mathcal{T}\nabla G(\bar{x}_k) - \mathcal{T}K^*\bar{y}_k - s\mathcal{T}U_{k,x}\xi_k) \\ y_{k+1} &= \text{prox}_f^{\Sigma}(\bar{y}_k - \Sigma\nabla F(\bar{y}_k) + \Sigma K(2x_{k+1} - \bar{x}_k) - s\Sigma U_{k,y}\xi_k).\end{cases}\tag{42}$$

where, $\xi_k \in \mathbb{R}^r$ is the unique zero of $\mathcal{J}: \mathbb{R}^r \rightarrow \mathbb{R}^r$:

$$\begin{aligned}\mathcal{J}(\xi) &= (U_{k,x})^* \left[\underbrace{\bar{x}_k - \text{prox}_g^{\mathcal{T}}(\bar{x}_k - \mathcal{T}\nabla G(\bar{x}_k) - \mathcal{T}K^*\bar{y}_k - s\mathcal{T}U_{k,x}\xi)}_{x_{k+1}(\xi)} \right] \\ &+ (U_{k,y})^* [\bar{y}_k - \text{prox}_f^{\Sigma}(\bar{y}_k - \Sigma\nabla F(\bar{y}_k) + \Sigma K(2x_{k+1}(\xi) - \bar{x}_k) - s\Sigma U_{k,y}\xi)] + \xi.\end{aligned}\tag{43}$$

Proof. It is a direct consequence of Proposition 4.9 and Proposition 6.1. \square

Remark 6.4. By switching \bar{z}_k with z_k and z_{k+1} with \tilde{z}_k , we obtain the update step for Algorithm 6.

Remark 6.5. In special cases, the root-finding problem can be solved exactly. For instance, when $\mathcal{T} = \tau I$, $\Sigma = \sigma I$, $K = I$, $g \equiv 0$ and $f(y) = \|y\|_1$, according to [8], the root-finding problem with $r = 1$ is piece-wisely linear and can be solved exactly.

We would like to emphasize that using Proposition 6.3, we can avoid the computation of M_k^{-1} in the primal and dual setting, which is a computationally significant advantage. The convergence of Algorithms 5 and 6 is a direct consequence of Theorems 3.1 and 3.4.

Proposition 6.6 (Convergence of quasi-Newton PDHG method). *Let $M_0 = M$ as defined in (40) and $M_k = M_0 + sU_kU_k^*$.*

- (i) *If $(M_k)_{k \in \mathbb{N}}$ satisfies Assumption 1 (see e.g. Lemma 5.2), then (x_k, y_k) generated by Algorithm 5 converges weakly to some solution of (3). Furthermore, if g and f are both strongly convex (or G and F are both strongly convex), then we obtain the same convergence rate as in Theorem 3.1.*
- (ii) *If $(M_k)_{k \in \mathbb{N}}$ satisfies Assumption 2 (see e.g. Lemma 5.1), then (x_k, y_k) generated by Algorithm 6 converges weakly to some solution of (3). Furthermore, if g and f are both strongly convex (or G and F are both strongly convex), then we obtain linear convergence.*

7 Numerical experiments

The algorithms that we analyze in the experiments are summarized in Table 1.

Algorithm	Algorithm Name	Metric
FBS	Forward-Backward Primal-Dual Hybrid Gradient Method	M fixed as in (40)
IFBS	Inertial Primal-Dual Hybrid Gradient Method	M fixed as in (40)
QN-FBS	quasi-Newton Primal-Dual Hybrid Gradient Method Gradient	Variable metric as in (29) with $M_0 = M$ from (40)
RQN-FBS	Primal-Dual Hybrid Gradient Method with relaxation step	Variable metric as in (29) with $M_0 = M$ from (40)
IQN-FBS	Inertial quasi-Newton Primal-Dual Hybrid Gradient Method	Variable metric as in (29) with $M_0 = M$ from (40)

Table 1: Summary of algorithms used in the numerical experiments. Details are provided within each section.

Note that PDHG is used interchangeably as FBS in the later experiments since PDHG is a specialization of FBS.

7.1 TV- l_2 deconvolution

In this experiment, we solve a problem that is used for image deconvolution [10]. Given a blurry and noisy image $b \in \mathbb{R}^{MN}$ (interpreted as a vector by stacking the M columns of length N), we seek to find a clean image $x \in \mathbb{R}^{MN}$ by solving the following optimization problem:

$$\min_{0 \leq x \leq 255} \frac{1}{2} \|Lx - b\|_2^2 + \mu \|Dx\|_{2,1}, \quad (44)$$

where $L \in \mathbb{R}^{MN \times MN}$ is a linear operator that acts as a blurring operator and $\|Dx\|_{2,1}$ implements a discrete version of the isotropic total variation norm of x using simple forward differences in horizontal and vertical direction with $\|D\| \leq 2\sqrt{2}$. The parameter $\mu > 0$ stresses the influence of the regularization term $\|Dx\|_{2,1}$ versus the data fidelity term $\frac{1}{2} \|Lx - b\|_2^2$. In order to deal with the non-smoothness, we rewrite the problem as a saddle point problem:

$$\min_x \max_y \langle Dx, y \rangle + \delta_\Delta(x) + \frac{1}{2} \|Lx - b\|_2^2 - \delta_{\{\|\cdot\|_{2,\infty} \leq \mu\}}(y), \quad (45)$$

where $\Delta := \{x \in \mathbb{R}^{MN} | 0 \leq x_i \leq 255, \forall i \in \{1, \dots, MN\}\}$. We can cast this problem into the general class of problems (32) by setting $K = D$, $f = \delta_{\{\|\cdot\|_{2,\infty} \leq \lambda\}}$, $G(x) = \frac{1}{2} \|Lx - b\|_2^2$, $F(p) = 0$ and $g = \delta_\Delta$. Here, G is $1/\beta$ -smooth with $\beta = 2/3$ as we took $\|L\|^2 \leq 3/2$. Let $z_k = (x_k, y_k)$ be the primal-dual iterate sequence. Choosing $\mathcal{T} = \tau I = 0.05I$ and $\Sigma = \sigma I = 0.05I$, it follows from Proposition 6.1 and Remark 6.2 that $M - \frac{1}{\beta}I \succeq (16 - 3/2)I > 0$ as desired. We compute the low-rank part $Q_k = \gamma_k u_k u_k^\top$ by (29) with $Bz_k = \begin{pmatrix} L^\top Lx_k - L^\top b \\ 0 \end{pmatrix}$ which is of course β -co-coercive. This leads to a metric that affects only the primal update. In each iteration, we use the semi-smooth Newton method (Algorithm 3) to locate the root.

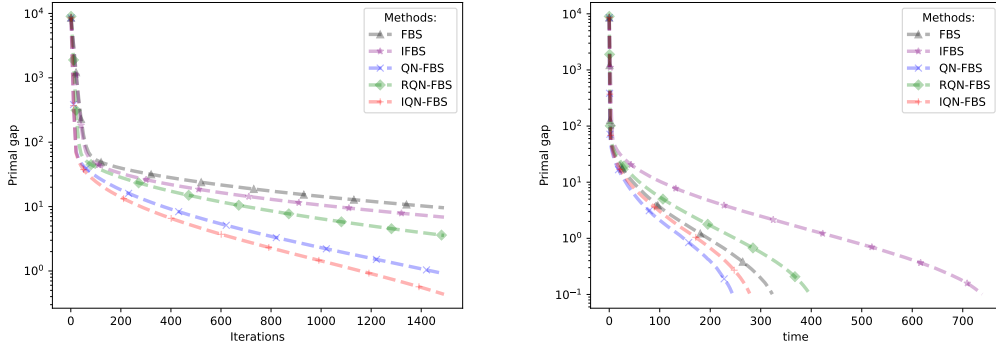


Figure 1: We compare convergence of the inertial quasi-Newton PDHG (IQN-FBS) to other algorithms in the Table 1 with $\mu = 0.0001$, $\tau = 0.05$, $\sigma = 0.05$ and inertial parameter $\alpha_0 = 10$, $\alpha_k = \frac{10}{k^{1.1}(\max\{\|z_k - z_{k-1}\|, \|z_k - z_{k-1}\|^2\})}$. The left plot depicts the convergence against the number of iterations, while the right plot shows the convergence with respect to time (seconds). We observe that our two quasi-Newton type algorithm QN-, and IQN-FBS clearly outperform the original FBS and IFBS algorithm.

Figure 1 shows the primal gap against the number of iterations and against the time (seconds), where the optimal primal value was computed by running the original PDHG method for 10000 iterations. For the variable metric at iteration k , we fixed $\gamma_k = \min(0.8, 15/\|u_k\|_2^2)$. Thus, Assumption 2 is satisfied and the convergence of RQN-FBS (Algorithm 2) is guaranteed. Although Assumption 1 can be guaranteed by appropriately defining $(\eta_k)_{k \in \mathbb{N}}$ in the view of Lemma 5.2, we choose not to include the condition that $(1 + \eta_k)M_k \succeq M_{k+1}$ with $(\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ in the numerical experiments. This decision comes from the challenge of setting $(\eta_k)_{k \in \mathbb{N}}$ in practice and the concern on imposing stringent conditions on $(\eta_k)_{k \in \mathbb{N}}$ in advance, which may undermine the advantages of utilizing a variable metric. In this practical problem, we still observe the convergence of IQN-FBS (Algorithm 1). We notice that our quasi-Newton type algorithms IQN-FBS, RQN-FBS and QN-FBS are much faster than original FBS algorithm and inertial FBS (IFBS) according to the left plot in Figure 1. This can be explained by the fact that the Hessian of $G(x)$ is not the identity and thus by applying our quasi-Newton SR1 methods, we can adapt the metric to the local geometry of the objective. Even though we are concerned about the cost of solving the root-finding problem, the right plot demonstrates the additional iterations can pay off by showing the convergence against the time. Quasi-Newton type algorithms (QN-FBS, IQN-FBS) achieve higher accuracy than FBS and IFBS after running the same time.

7.2 TV- l_2 deconvolution with infimal convolution type regularization

A source of optimization problems that fits (32) is derived from the following:

$$\min_{x \in \mathbb{R}^n} g(x) + G(x) + (f \square h)(Dx), \quad (46)$$

where $f \square h(\cdot) := \inf_{v \in \mathbb{R}^m} f(v) + h(\cdot - v)$ denotes the infimal convolution of f and h . As a prototypical image processing problem, we define a regularization term as infimal convolution between the total variation norm and a weighted squared norm, i.e. $g = 0$, $G(x) = \frac{1}{2} \|Lx - b\|^2$, $h(\cdot) = \frac{1}{2} \|W \cdot\|^2$ and

$f(\cdot) = \mu \|\cdot\|_{2,1}$. This yields the problem:

$$\min_x \frac{1}{2} \|Lx - b\|^2 + \mu R(Dx) \quad (47)$$

where $R(\cdot) := \inf_{v \in \mathbb{R}^m} \|v\|_{2,1} + \frac{1}{2\mu} \|W(\cdot - v)\|^2$, W is a diagonal matrix of weights which is given to favor discontinuities along image edges and L, b, D are defined as in the first experiment. In practice, W can be computed by additional edge finding steps or by extra information. Here, we select W such that $\frac{1}{6} \leq \|W\|^2 \leq 1$. The optimization problem (47) given in primal type can be converted into the saddle point problem:

$$\min_x \max_y \langle Dx, y \rangle + \frac{1}{2} \|Lx - b\|^2 - \delta_{\{\|\cdot\|_{2,\infty} \leq \mu\}}(y) - \frac{1}{2} \|W^{-1}y\|^2. \quad (48)$$

We compute the low-rank part $Q_k = \gamma_k u_k u_k^\top$ by (29) with $Bz_k = \begin{pmatrix} L^\top Lx_k - L^\top b \\ (W^{-1})^* W^{-1} y_k \end{pmatrix}$ which leads to a metric that affects both primal and dual update. Here, B is β -co-coercive with $\beta \geq 1/6$. In each iteration, we combine the bisection (Algorithm 4) and the semi-smooth Newton method (Algorithm 3) to locate the root.

Figure 2 also shows the primal gap where the optimal primal value was still computed by running original PDHG for 10000 iterates. For the variable metric at iterate k , we fixed $\gamma_k = 0.64$, $\mathcal{T} = \tau I = 0.01I$ and $\Sigma = \sigma I = 0.01I$. Thus, by Lemma 5.1, Assumption 2 is satisfied. As for Assumption 1, we follow the same strategy as in the previous experiment and drop the condition that $(1 + \eta_k)M_k \succeq M_{k+1}$ with $(\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ in this numerical experiment. We can observe from Figure 2: IQN-FBS is still the fastest one. Moreover, the two quasi-Newton type methods (IQN-FBS and QN-FBS) converge faster than IFBS and FBS. Inertial methods (IQN-FBS, IFBS) are slightly faster respectively than QN-FBS and FBS.

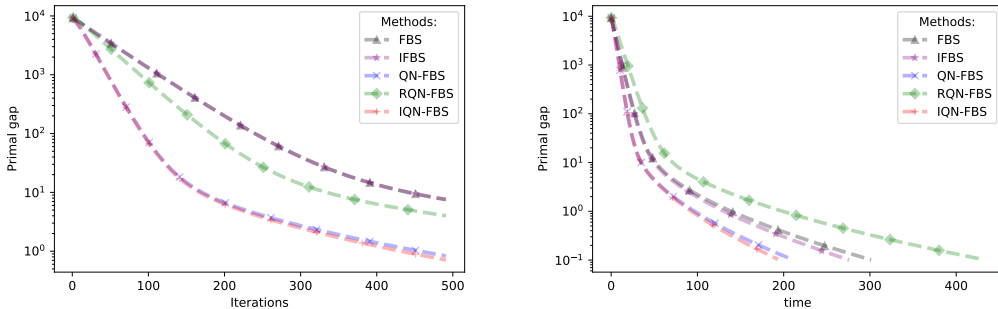


Figure 2: We compare convergence of the inertial quasi-Newton PDHG (IQN-FBS) to other algorithms in the Table 1 with $\mu = 0.5$, $\tau = 0.01I$, $\sigma = 0.01$ and the extrapolation parameter $\alpha_0 = 1$, $\alpha_k = \min\left\{\frac{10}{k^{1.1}(\max\{\|z_k - z_{k-1}\|, \|z_k - z_{k-1}\|^2\})}, 1\right\}$. Our quasi-Newton type algorithm IQN-FBS can converge faster than the original FBS and IFBS algorithm.

7.2.1 Image denoising

We consider the same setting (48). Let $L = I$ in (48). We then obtain an image denoising problem with special norm defined by infimal convolution of total variation and weighted norm, which has strong convexity for both primal part and dual part. Besides, due to the simple formula, we obtain the dual problem explicitly which means that we can calculate primal and dual gap. The dual problem reads

$$\max_{\|y\|_{2,\infty} \leq 1} -\frac{1}{2} \|D^*y - b\|^2 - \frac{1}{2} \|W^{-1}y\|^2. \quad (49)$$

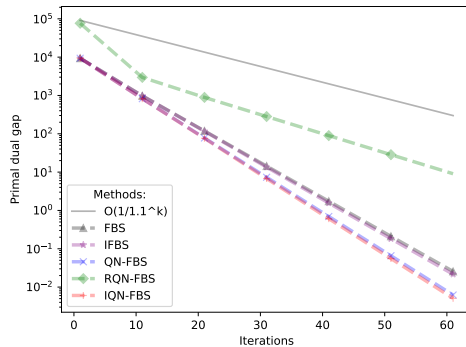


Figure 3: We compare convergence of the inertial quasi-Newton PDHG (IQN-FBS) to other algorithms in the Table 1 with $\mu = 0.1$, $\tau = 0.1$, $\sigma = 0.1$ and extrapolation parameter $\alpha_0 = 10$, $\alpha_k = \frac{10}{\max\{k^{1.1}, k^{1.1}\|z_k - z_{k-1}\|^2\}}$. The plot shows that all algorithms converge linearly and faster than $O(\frac{1}{1.1^k})$.

Figure 3 shows the primal dual gap and it will decrease to zero by using any algorithm from Table 1. To construct Q_k , we use $\gamma_k = \frac{1}{\|u_k\|^2}$, $\mathcal{T} = \tau I = 0.1I$ and $\Sigma = \sigma I = 0.1I$. Assumption 2 is satisfied. However, Assumption 1 is not satisfied since we still not include the condition that $(1 + \eta_k)M_k \succeq M_{k+1}$ with $(\eta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ in this numerical experiment. As we can observe from Figure 3, we have linear convergence for quasi-Newton type methods as what we expected in Theorem 3.1 and 3.4. Figure 3 shows that in this experiment quasi-Newton type algorithms are in fact not more efficient compared to FBS or IFBS, which is plausible due to the well-conditioned $H = I$.

7.3 Conclusion

In this paper, we extended the framework of [8] for variable metrics to the setting of resolvent operators, solving efficiently the monotone inclusion problem (1) consisting of a set-valued operator A and a co-coercive operator B . We proposed two variants of quasi-Newton Forward-Backward Splitting. We develop a general efficient resolvent calculus that applies to this quasi-Newton setting. The convergence of the variant with relaxation requires mild assumptions on the metric which are easy to satisfy, whereas the other variant implements an inertial feature and is therefore often fast. As a special case of this framework, we developed an inertial quasi-Newton primal-dual algorithm that can be flexibly applied to a large class of saddle point problems. Throughout the paper, we employ a rank-1 perturbed variable metric denoted as $M_k = M_0 + sU_kU_k^*$ with $r = 1$ and $U_k: \mathbb{R} \rightarrow \mathcal{H}$ which is generated using the 0-memory SR1 method. Alternatively, one can generate the variable metric using m -memory quasi-Newton method (refer to [23, 41]), wherein $U_k: \mathbb{R}^m \rightarrow \mathcal{H}$. Consequently, we are able to derive an m -memory quasi-Newton primal-dual method.

Another potential application of our resolvent calculus in Theorem 4.1 lies in non-diagonal preconditioning of the primal-dual method (PDHG). Moreover, there are many directions to improve our methods. Further investigation is needed to develop an optimal sequence of variable metric $(M_k)_{k \in \mathbb{N}}$. Given that our variable metrics are currently designed solely based on the geometry of the single valued operator B , it is reasonable to explore a metric that adapts to both operators A and B . Additionally, there remains an open question regarding how to eliminate the condition that the growth of M_k is controlled by a summable sequence $(\eta_k)_{k \in \mathbb{N}}$ while ensuring fast convergence.

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Appendices

A Appendix

A.1 Proof of Lemma 2.3

Proof. We assume $y = M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z)$. Then, we obtain

$$\begin{aligned}
y &= M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z) \\
M^{1/2}y &= J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z) \\
M^{1/2}y &= (I + M^{-1/2}TM^{-1/2})^{-1}(M^{1/2}z) \\
(M^{1/2}z) &\in (I + M^{-1/2}TM^{-1/2})M^{1/2}y \\
(M^{1/2}z) &\in (M^{1/2} + M^{-1/2}T)y \\
z &\in (I + M^{-1}T)y \\
y &= (I + M^{-1}T)^{-1}(z) \\
y &= J_T^M(z).
\end{aligned} \tag{50}$$

□

A.2 Proof of Lemma 2.4

Proof. This proof is adapted from [6]. Let $(u, v) = (J_A^M(x), J_A^M(y))$ for some $x, y \in \mathcal{H}$. By the definition of resolvent operator J_A^M , we obtain

$$u = J_A^M(x) \iff M(x - u) \in Au.$$

Similarly, we obtain $M(y - v) \in Av$. Then γ_A -strong monotonicity of A yields

$$\begin{aligned}
\langle M(x - u) - M(y - v), u - v \rangle &\geq \gamma_A \|u - v\|^2 \\
\langle M(x - y), u - v \rangle - \langle M(u - v), u - v \rangle &\geq \gamma_A \|u - v\|^2 \\
\langle M(x - y), u - v \rangle &\geq \gamma_A \|u - v\|^2 + \|u - v\|_M^2.
\end{aligned}$$

Since $\|M\|$ is bounded by C , we obtain

$$\langle M(x - y), u - v \rangle \geq \gamma_A \|u - v\|^2 + \|u - v\|_M^2 \geq (1 + \frac{\gamma_A}{C}) \|u - v\|_M^2. \tag{51}$$

Consequently, J_A^M is $(1 + \frac{\gamma_A}{C})$ -co-coercive in the metric M and Lipschitz continuous with constant $1/(1 + \frac{\gamma_A}{C})$ with respect to the norm $\|\cdot\|_M$. □

A.3 Proof of Theorem 3.1

Proof. For convenience, we set $B_k := M_k^{-1}B$. Fix $z \in \text{zer}(A + B)$ which is equivalent to $z = J_A^{M_k}(z - B_k z)$. We set $\hat{z}_{k+1} := J_A^{M_k}(\bar{z}_k - B_k \bar{z}_k)$, i.e., $z_{k+1} = \hat{z}_{k+1} + \epsilon_k$.

Boundedness:

First, we are going to show that $\|z_k - z\|_{M_k}$ is bounded. The following are several useful estimates we will use later. Assumption 1 yields

$$\|z_{k+1} - z\|_{M_{k+1}}^2 \leq (1 + \eta_k) \|z_{k+1} - z\|_{M_k}^2. \tag{52}$$

Since $z_{k+1} = \hat{z}_{k+1} + \epsilon_k$, it follows that

$$\|z_{k+1} - z\|_{M_k}^2 = \|\hat{z}_{k+1} - z + \epsilon_k\|_{M_k}^2 \leq \|\hat{z}_{k+1} - z\|_{M_k}^2 + 2\|\epsilon_k\|_{M_k} \|\hat{z}_{k+1} - z\|_{M_k} + \|\epsilon_k\|_{M_k}^2. \tag{53}$$

The assumption that B is β -co-coercive yields that

$$\langle \bar{z}_k - z, B_k \bar{z}_k - B_k z \rangle_{M_k} = \langle \bar{z}_k - z, B \bar{z}_k - Bz \rangle \geq \beta \|B \bar{z}_k - Bz\|^2. \tag{54}$$

The assumption that $M_k - \frac{1}{2\beta}I \in \mathcal{S}_\kappa(\mathcal{H})$ yields that $2\beta I - M_k^{-1} \in \mathcal{S}_{++}(\mathcal{H})$.

The co-coercivity of B and monotonicity of B yield the following estimates (I) and (II) respectively:

$$\begin{aligned} \|(\bar{z}_k - B_k \bar{z}_k) - (z - B_k z)\|_{M_k}^2 &= \|\bar{z}_k - z\|_{M_k}^2 - 2 \langle \bar{z}_k - z, B_k \bar{z}_k - B_k z \rangle_{M_k} + \|B_k \bar{z}_k - B_k z\|_{M_k}^2 \\ &\stackrel{(i)}{=} \|\bar{z}_k - z\|_{M_k}^2 - 2 \langle \bar{z}_k - z, B \bar{z}_k - Bz \rangle + \|B \bar{z}_k - Bz\|_{M_k^{-1}}^2 \\ &\stackrel{(ii)}{\leq} \|\bar{z}_k - z\|_{M_k}^2 - \|B \bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 \end{aligned} \quad (\text{I})$$

or

$$\stackrel{(iii)}{\leq} \|\bar{z}_k - z\|_{M_k}^2 - \|B \bar{z}_k - Bz\|_{\beta - M_k^{-1}}^2 - \gamma_B \|\bar{z}_k - z\|^2, \quad (\text{II})$$

where (i) uses $\|B_k \bar{z}_k - B_k z\|_{M_k}^2 = \|B \bar{z}_k - Bz\|_{M_k^{-1}}^2$, (ii) uses (54) and (iii) uses strong monotonicity of B . Note that we use the shorthand $\beta - M_k^{-1}$ for $\beta I - M_k^{-1}$. The fact that $J_A^{M_k}$ is firmly non-expansive since A is maximally monotone with respect to M_k implies that

$$\begin{aligned} \|\hat{z}_{k+1} - z\|_{M_k}^2 &= \|J_A^{M_k}(\bar{z}_k - B_k \bar{z}_k) - J_A^{M_k}(z - B_k z)\|_{M_k}^2 \\ &\leq \|(\bar{z}_k - B_k \bar{z}_k) - (z - B_k z)\|_{M_k}^2 \\ &\quad - \|(I - J_A^{M_k})(\bar{z}_k - B_k \bar{z}_k) - (I - J_A^{M_k})(z - B_k z)\|_{M_k}^2 \\ &\leq \|\bar{z}_k - z\|_{M_k}^2 - \|B \bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 - \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k \bar{z}_k - B_k z)\|_{M_k}^2, \end{aligned} \quad (55)$$

where the last inequality uses (I). It follows from Assumption 1 that the term $\|B \bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 \geq 0$. We continue to bound the first term on the right hand side of (55). Using [6, Lemma 2.14] and the definition of \bar{z}_k , we obtain the following:

$$\|\bar{z}_k - z\|_{M_k}^2 = (1 + \alpha_k) \|z_k - z\|_{M_k}^2 - \alpha_k \|z_{k-1} - z\|_{M_k}^2 + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2, \quad (56)$$

and by using the triangle inequality, we also obtain another estimation:

$$\|\bar{z}_k - z\|_{M_k} \leq (1 + \alpha_k) \|z_k - z\|_{M_k} + \alpha_k \|z_{k-1} - z\|_{M_k}. \quad (57)$$

In order to address complete update step, we make the following estimation:

$$\begin{aligned} \|z_{k+1} - z\|_{M_{k+1}}^2 &\leq (1 + \eta_k) \|z_{k+1} - z\|_{M_k}^2 \\ &\stackrel{(i)}{\leq} (1 + \eta_k) (\|\hat{z}_{k+1} - z\|_{M_k}^2 + 2\|\epsilon_k\|_{M_k} \|\hat{z}_{k+1} - z\|_{M_k} + \|\epsilon_k\|_{M_k}^2) \\ &\stackrel{(ii)}{\leq} (1 + \eta_k) (\|\bar{z}_k - z\|_{M_k}^2 + 2\|\epsilon_k\|_{M_k} \|\bar{z}_k - z\|_{M_k} + \|\epsilon_k\|_{M_k}^2 \\ &\quad - \|B \bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 - \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k \bar{z}_k - B_k z)\|_{M_k}^2) \\ &\stackrel{(iii)}{\leq} (1 + \eta_k) ((1 + \alpha_k) \|z_k - z\|_{M_k}^2 - \alpha_k \|z_{k-1} - z\|_{M_k}^2 \\ &\quad + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2 \\ &\quad + 2\|\epsilon_k\|_{M_k} ((1 + \alpha_k) \|z_k - z\|_{M_k} + \alpha_k \|z_{k-1} - z\|_{M_k}) \\ &\quad + \|\epsilon_k\|_{M_k}^2 - \|B \bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 - \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k \bar{z}_k - B_k z)\|_{M_k}^2) \\ &\stackrel{(iv)}{\leq} (1 + \eta_k) (\|z_k - z\|_{M_k}^2 \\ &\quad + \alpha_k (\|z_k - z\|_{M_k} - \|z_{k-1} - z\|_{M_k}) (\|z_k - z\|_{M_k} + \|z_{k-1} - z\|_{M_k}) \\ &\quad + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2 \\ &\quad + 2\|\epsilon_k\|_{M_k} ((1 + \alpha_k) \|z_k - z\|_{M_k} + \alpha_k \|z_{k-1} - z\|_{M_k}) \\ &\quad + \|\epsilon_k\|_{M_k}^2 - \|B \bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 - \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k \bar{z}_k - B_k z)\|_{M_k}^2) \\ &\stackrel{(v)}{\leq} (1 + \eta_k) (\|z_k - z\|_{M_k}^2 + \alpha_k \|z_k - z_{k-1}\|_{M_k} (\|z_k - z\|_{M_k} + \|z_{k-1} - z\|_{M_k}) \\ &\quad + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2 + 2\|\epsilon_k\|_{M_k} ((1 + \alpha_k) \|z_k - z\|_{M_k} + \alpha_k \|z_{k-1} - z\|_{M_k}) \\ &\quad + \|\epsilon_k\|_{M_k}^2 - \|B \bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 - \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k \bar{z}_k - B_k z)\|_{M_k}^2). \end{aligned} \quad (58)$$

where (i) uses (53), (ii) uses (55), (iii) uses (56) and (57), (iv) uses factorization of the quadratic, and (v) uses the triangle inequality to obtain the bound $\|z_k - z\|_{M_k} - \|z_{k-1} - z\|_{M_k} \leq \|z_k - z_{k-1}\|_{M_k}$.

Now, our goal is to conclude boundedness using Lemma 2.7. For simplicity, we set:

$$\begin{cases} e_k & := \alpha_k \|z_k - z_{k-1}\|_{M_k}^2 \\ r_k & := \alpha_k \|z_k - z_{k-1}\|_{M_k} \\ \theta_k & := \|z_k - z\|_{M_k} \\ m_k & := \|z_{k-1} - z\|_{M_k} \\ p_k & := \|B\bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 \\ q_k & := \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k\bar{z}_k - B_kz)\|_{M_k}^2. \end{cases}$$

By Assumption 1, we have $m_k \leq (1 + \eta_{k-1})\theta_{k-1}$. Without loss of generality, we can assume $0 < \eta_k < 1$ for any $k \in \mathbb{N}$. Replacing each term in (58) with new corresponding notations, we obtain:

$$\begin{aligned} \theta_{k+1}^2 &\leq (1 + \eta_k)(\theta_k^2 + r_k(\theta_k + m_k)) \\ &\quad + (1 + \alpha_k)e_k + 2\|\epsilon_k\|_{M_k}((1 + \alpha_k)\theta_k + \alpha_k m_k) + \|\epsilon_k\|_{M_k}^2 - p_k - q_k \\ &\leq (1 + \eta_k)(\theta_k^2 + r_k(\theta_k + (1 + \eta_{k-1})\theta_{k-1})) \\ &\quad + (1 + \alpha_k)e_k + 2\|\epsilon_k\|_{M_k}((1 + \alpha_k)\theta_k + \alpha_k(1 + \eta_{k-1})\theta_{k-1}) + \|\epsilon_k\|_{M_k}^2 \\ &\leq (1 + \eta_k)(\theta_k^2 + r_k(\theta_k + 2\theta_{k-1})) + (1 + \Lambda)e_k + 2\|\epsilon_k\|_{M_k}((1 + \Lambda)\theta_k + 2\Lambda\theta_{k-1}) + \|\epsilon_k\|_{M_k}^2, \end{aligned} \tag{59}$$

where the last inequality uses $0 < \alpha_k \leq \Lambda$ and $0 < \eta_k < 1$. Now, we claim that θ_k is bounded in two steps. We introduce an auxiliary bounded sequence $(C_k)_{k \in \mathbb{N}}$ (step 1) such that $\theta_k \leq C_k$ for any $k \in \mathbb{N}$ (step 2). The boundedness of θ_k follows from that of C_k .

Step1: We construct a sequence C_k as the following:

$$\begin{cases} C_0 & = \max\{\theta_0, 1\}, \\ C_{k+1} & = (1 + \eta_k)C_k + \nu_k, \end{cases} \tag{60}$$

where $\nu_k = (1 + \eta_k)((1 + \Lambda)e_k + 2r_k + (1 + 3\Lambda)\|\epsilon_k\|_{M_k})$. From our assumptions, it holds that $(M_k)_{k \in \mathbb{N}}$ is bounded from above, $(r_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$, $(e_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$ and $(\|\epsilon_k\|)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$, which implies $(\nu_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$. Using Lemma 2.7, we obtain the convergence of C_k to some $C_\infty < +\infty$. Step2: From the update step of (60), we observe that $(C_k)_k$ is a non-decreasing sequence and $C_k \geq 1$ for any $k \in \mathbb{N}$. We claim that for each k , $\theta_k \leq C_k$. We argument by induction. Clearly, we have $\theta_0 \leq C_0$. Assume $\theta_i \leq C_i$ holds true for $i \leq k$. Then, (59) yields that

$$\begin{aligned} \theta_{k+1}^2 &\leq (1 + \eta_k)(C_k^2 + r_k(C_k + 2C_{k-1})) + (1 + \Lambda)e_k + 2\|\epsilon_k\|_{M_k}((1 + \Lambda)C_k + 2\Lambda C_{k-1}) + \|\epsilon_k\|_{M_k}^2 \\ &\stackrel{(*)}{\leq} (1 + \eta_k)(C_k^2 + 4r_k C_k + 2(1 + \Lambda)e_k C_k + 2(1 + 3\Lambda)\|\epsilon_k\|_{M_k} C_k + \|\epsilon_k\|_{M_k}^2) \\ &\leq (1 + \eta_k)(C_k + 2r_k + (1 + \Lambda)e_k + (1 + 3\Lambda)\|\epsilon_k\|_{M_k})^2, \end{aligned} \tag{61}$$

where (*) uses $C_k \geq C_{k-1} \geq 1$ and $r_k > 0$. By the definition of C_{k+1} , we obtain

$$\begin{aligned} \theta_{k+1} &\stackrel{(i)}{\leq} \sqrt{1 + \eta_k}(C_k + 2r_k + (1 + \Lambda)e_k + (1 + 3\Lambda)\|\epsilon_k\|_{M_k}) \\ &\stackrel{(ii)}{\leq} (1 + \eta_k)C_k + (1 + \eta_k)(2r_k + (1 + \Lambda)e_k + (1 + 3\Lambda)\|\epsilon_k\|_{M_k}) \\ &\stackrel{(iii)}{\leq} (1 + \eta_k)C_k + \nu_k \\ &= C_{k+1}, \end{aligned} \tag{62}$$

where (i) uses (61), (ii) holds true since $(1 + \eta_k) > 1$ and (iii) uses definition of ν_k . This concludes the induction, and we deduce that θ_k is bounded and therefore, z_k and \bar{z}_k are both bounded.

Weak convergence:

This part of the proof is adapted from the one for [13, Theorem 4.1]. Since θ_k is bounded, we set $\zeta := \sup_{k \in \mathbb{N}} \theta_k$. The last inequality in (58) implies that

$$\begin{aligned} \theta_{k+1}^2 &\leq (1 + \eta_k)(\theta_k^2 + r_k(\zeta + 2\zeta) + (1 + \Lambda)e_k + 2(1 + 3\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|_{M_k}^2 - p_k - q_k) \\ &\leq (1 + \eta_k)(\theta_k^2 + 3r_k\zeta + (1 + \Lambda)e_k + 2(1 + 3\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|_{M_k}^2 - p_k - q_k) \\ &\leq \theta_k^2 + \eta_k\theta_k^2 + (1 + \eta_k)(3r_k\zeta + (1 + \Lambda)e_k + (2 + 6\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|_{M_k}^2) - p_k - q_k \\ &\leq \theta_k^2 + \underbrace{\eta_k\zeta^2 + 2(3r_k\zeta + (1 + \Lambda)e_k + (2 + 6\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|_{M_k}^2)}_{\delta_k} - p_k - q_k. \end{aligned} \quad (63)$$

We set $\delta_k := \eta_k\zeta^2 + 2(3r_k\zeta + (1 + \Lambda)e_k + (2 + 6\Lambda)\|\epsilon_k\|_{M_k}\zeta + \|\epsilon_k\|_{M_k}^2)$ and observe that $(\delta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$. Now, (63) yields that

$$\theta_{k+1}^2 \leq \theta_k^2 + \delta_k. \quad (64)$$

Using (64) and Lemma 2.7, we obtain the convergence of $\theta_k^2 = \|z_k - z\|_{M_k}^2$ for any $z \in \text{zer}(A + B)$. Rearranging (63) to $p_k \leq \theta_k^2 - \theta_{k+1}^2 + \delta_k$, using Assumption 1 and summing it for $k = 0, \dots, N$, there exists some $\varepsilon := \rho_{\min}(2\beta I - M_k^{-1}) > 0$ s.t.

$$\varepsilon \sum_{k=0}^N \|B\bar{z}_k - Bz\|^2 \leq \sum_{k=0}^N \|B\bar{z}_k - Bz\|_{2\beta - M_k^{-1}}^2 = \sum_{k=0}^N p_k \leq \theta_0^2 - \theta_N^2 + \sum_{k=0}^N \delta_k \leq \zeta^2 + \sum_{k=0}^N \delta_k. \quad (65)$$

Since $(\delta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$, by taking limit as $N \rightarrow +\infty$, we obtain

$$\sum_{k \in \mathbb{N}} \|B\bar{z}_k - Bz\|^2 \leq \frac{1}{\varepsilon^2}(\zeta^2 + \sum_{k \in \mathbb{N}} \delta_k) < +\infty. \quad (66)$$

Similarly, we obtain from (63) using $q_k \leq \theta_k^2 - \theta_{k+1}^2 + \delta_k$ that

$$\sum_{k \in \mathbb{N}} \|(\bar{z}_k - \hat{z}_{k+1}) - (B_k\bar{z}_k - B_k z)\|_{M_k}^2 < +\infty. \quad (67)$$

Set z^* as an arbitrary weak sequential cluster point of $(z_k)_{k \in \mathbb{N}}$, namely, a subsequence $z_{k_n} \rightharpoonup z^*$ as $n \rightarrow \infty$.

In order to obtain weak convergence of z_k , by Proposition 2.5 with $\varphi(t) = t^2$, (64) and Assumption 1, it suffices to show that $z^* \in \text{zer}(A + B)$. It follows from the selection of α_k that:

$$\|\bar{z}_k - z_k\| \leq \alpha_k \|z_k - z_{k-1}\| \rightarrow 0. \quad (68)$$

Thus, (68) yields $\bar{z}_{k_n} \rightharpoonup z^*$. From (66), we obtain that $B\bar{z}_{k_n} \rightarrow Bz$ as $n \rightarrow \infty$. Since B is co-coercive, it is maximally monotone and we can use the weak strong graph closedness of B in Proposition 2.2 to infer that $(z^*, Bz) \in \text{Graph } B$, i.e. $Bz \in Bz^*$. However, since B is single valued, we obtain $Bz^* = Bz$ and hence $B\bar{z}_{k_n} \rightarrow Bz^*$. Setting $u_k := M_k(\bar{z}_k - \hat{z}_{k+1}) - B\bar{z}_k$, by definition of the resolvent $J_A^{M_k}$, we have $u_k \in A(\hat{z}_{k+1})$ for all $k \in \mathbb{N}$. From (67), we obtain as $k \rightarrow +\infty$,

$$\begin{aligned} \|u_k + Bz^*\| &= \|M_k(\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*)\| \\ &\leq C\|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\| \\ &\leq \frac{C}{\sqrt{\sigma}}\|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\|_{M_k} \rightarrow 0. \end{aligned} \quad (69)$$

Furthermore, from (66) and (67), we have

$$\begin{aligned} \|\bar{z}_k - \hat{z}_{k+1}\| &\leq \|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\| + \|B_k\bar{z}_k - B_k z^*\| \\ &\leq \|\bar{z}_k - \hat{z}_{k+1} - B_k\bar{z}_k + B_k z^*\| + \frac{1}{\sqrt{\sigma}}\|B\bar{z}_k - Bz^*\| \rightarrow 0. \end{aligned} \quad (70)$$

Therefore, together with (70), $\bar{z}_{k_n} \rightharpoonup z^*$ implies $\hat{z}_{k_{n+1}} \rightharpoonup z^*$ as $n \rightarrow \infty$. Now we already have $u_{k_n} \rightarrow -Bz^*$ as $n \rightarrow \infty$ and

$$(\forall k \in \mathbb{N}): \quad (\hat{z}_{k_{n+1}}, u_{k_n}) \in \text{Graph } A. \quad (71)$$

Since A is maximally monotone and using Proposition 2.2, we infer that $-Bz^* \in Az^*$, hence $z^* \in \text{zer}(A+B)$. As mentioned above, the result follows from Proposition 2.5 with $\varphi(t) = t^2$.

Convergence rate:

In the following part, we are going to show the convergence rate of Algorithm 1: Assume $\epsilon_k \equiv 0$ for $k \in \mathbb{N}$ and either $\gamma_A > 0$ or $\gamma_B > 0$. Because of (52), Assumption 1 and Lipschitz continuity of $J_A^{M_k}$, we obtain for any $z \in \text{zer}(A+B)$ that

$$\begin{aligned}
\|z_{k+1} - z\|_{M_{k+1}}^2 &\leq (1 + \eta_k) \|z_{k+1} - z\|_{M_k}^2 \\
&\stackrel{\text{Lemma 2.4}}{\leq} (1 + \eta_k) \left(\frac{1}{1 + \frac{\gamma_A}{C}} \right)^2 \|(\bar{z}_k - B_k \bar{z}_k) - (z - B_k z)\|_{M_k}^2 \\
&\stackrel{(i)}{\leq} (1 + \eta_k) \left(\frac{1}{1 + \frac{\gamma_A}{C}} \right)^2 (\|\bar{z}_k - z\|_{M_k}^2 - \|B\bar{z}_k - Bz\|_{\beta - M_k^{-1}}^2 - \gamma_B \|\bar{z}_k - z\|^2) \\
&\stackrel{(ii)}{\leq} (1 + \eta_k) \left(\frac{1}{1 + \frac{\gamma_A}{C}} \right)^2 \left(1 - \frac{\gamma_B}{C}\right) (\|\bar{z}_k - z\|_{M_k}^2) \\
&\stackrel{(iii)}{\leq} (1 + \eta_k) \frac{(1 - \frac{\gamma_B}{C})}{(1 + \frac{\gamma_A}{C})^2} ((1 + \alpha_k) \|z_k - z\|_{M_k}^2 - \alpha_k \|z_{k-1} - z\|_{M_k}^2 \\
&\quad + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2) \\
&= (1 + \eta_k) \frac{(1 - \frac{\gamma_B}{C})}{(1 + \frac{\gamma_A}{C})^2} (\|z_k - z\|_{M_k}^2 + \alpha_k (\|z_k - z\|_{M_k}^2 - \|z_{k-1} - z\|_{M_k}^2) \\
&\quad + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2) \\
&\stackrel{(iv)}{\leq} (1 + \eta_k) \frac{(1 - \frac{\gamma_B}{C})}{(1 + \frac{\gamma_A}{C})^2} (\|z_k - z\|_{M_k}^2 + \alpha_k \|z_k - z_{k-1}\|_{M_k} (\|z_k - z\|_{M_k} \\
&\quad + \|z_{k-1} - z\|_{M_k}) + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2) \\
&\stackrel{(v)}{\leq} (1 + \eta_k) \frac{(1 - \frac{\gamma_B}{C})}{(1 + \frac{\gamma_A}{C})^2} (\|z_k - z\|_{M_k}^2 + \alpha_k \|z_k - z_{k-1}\|_{M_k} (\|z_k - z\|_{M_k} \\
&\quad + (1 + \eta_{k-1}) \|z_{k-1} - z\|_{M_{k-1}}) + (1 + \alpha_k) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2) \\
&\stackrel{(vi)}{\leq} (1 + \eta_k) \frac{(1 - \frac{\gamma_B}{C})}{(1 + \frac{\gamma_A}{C})^2} (\|z_k - z\|_{M_k}^2 + 3\alpha_k \zeta \|z_k - z_{k-1}\|_{M_k} \\
&\quad + (1 + \Lambda) \alpha_k \|z_k - z_{k-1}\|_{M_k}^2)
\end{aligned} \tag{72}$$

where (i) uses (II), (ii) uses the fact M_k is bounded uniformly and the assumption that $M_k - \frac{1}{\beta}I \in \mathcal{S}_\kappa(\mathcal{H})$, (iii) uses (56), (iv) uses factorization of the quadratic and uses the triangle inequality to obtain the bound $\|z_k - z\|_{M_k} - \|z_{k-1} - z\|_{M_k} \leq \|z_k - z_{k-1}\|_{M_k}$, (v) uses Assumption 1 and (vi) uses boundedness of α_k and $\|z_k - z\|_{M_k}$. Since either $\gamma_A > 0$ or $\gamma_B > 0$, $\frac{1 - \frac{\gamma_B}{C}}{(1 + \frac{\gamma_A}{C})^2} < 1$. Then there exists sufficient large $K_0 > 0$ such that for any $k > K_0$, $(1 + \eta_k) \left(\frac{1 - \frac{\gamma_B}{C}}{(1 + \frac{\gamma_A}{C})^2} \right) < 1 - \xi < 1$ for some $\xi \in (0, 1)$. Thus, we infer that for any $k > K_0$:

$$\|z_k - z\|_{M_k}^2 \leq (1 - \xi)^{k - K_0} \|z_{K_0} - z\|_{M_{K_0}}^2 + \sum_{i=K_0}^{k-1} (1 - \xi)^{k-i} \alpha_i (3\zeta \|z_i - z_{i-1}\|_{M_i} + (1 + \Lambda) \|z_i - z_{i-1}\|_{M_i}^2). \tag{73}$$

Let $\Theta = 3\zeta + (1 + \Lambda)$. Therefore (73) can be simplified as the following:

$$\|z_k - z\|_{M_k}^2 \leq (1 - \xi)^{k - K_0} \|z_{K_0} - z\|_{M_{K_0}}^2 + \sum_{i=K_0}^{k-1} \Theta (1 - \xi)^{k-i} \alpha_i \max\{\|z_i - z_{i-1}\|_{M_i}, \|z_i - z_{i-1}\|_{M_i}^2\}. \tag{74}$$

Since z_k is bounded and $M_k \in \mathcal{S}_\sigma(\mathcal{H})$, it follows that

$$\|z_k - z\|^2 \leq \frac{1}{\sigma} (1 - \xi)^{k - K_0} \|z_{K_0} - z\|_{M_{K_0}}^2 + O\left(\sum_{i=K_0}^{k-1} (1 - \xi)^{k-i} \alpha_i\right). \tag{75}$$

Furthermore, if $\alpha_i \equiv 0$, for $k > K_0$ and for any $z \in \text{zer}(A+B)$, we obtain linear convergence:

$$\|z_{k+1} - z\|^2 \leq \frac{1}{\sigma} (1 - \xi)^{k - K_0} \|z_{K_0} - z\|_{M_{K_0}}^2. \tag{76}$$

If $\alpha_k \neq 0$, $\alpha_k = O(\frac{1}{k^2})$ and K_0 large enough, then $\|z_{k+1} - z\|^2$ converges in the rate of $O(\frac{1}{k})$ for $k > K_0$ according to [33, Lemma 2.2.4 (Chung)]; if $\alpha_k \neq 0$ and $\alpha_k = O(q^k)$ for $q = 1 - \xi$ and $k > K_0$, then $\|z_{k+1} - z\|^2$ converges in the rate of $O(kq^k)$ for $k > K_0$ since (75). \square

A.4 Proof of Theorem 3.4

Proof. For simplicity, we set

$$\begin{cases} \hat{z}_k := J_A^{M_k}(z_k - M_k^{-1}Bz_k), \text{ and, } \tilde{z}_k = \hat{z}_k + \epsilon_k, \\ \delta_k := \max\{1, \rho\}\|\epsilon_k\|, \end{cases} \quad (77)$$

where $\rho = \frac{1}{2}\sqrt{\frac{C}{\sigma c}}(C + \frac{1}{\beta})$ and $(\delta_k)_{k \in \mathbb{N}} \in \ell_+^1(\mathbb{N})$. We set z^* such that $-Bz^* \in Az^*$.

Boundedness:

We claim that by choosing proper t_k for each $k \in \mathbb{N}$, we have

$$\|z_{k+1} - z^*\| \leq \|z_k - z^*\| + \delta_k. \quad (78)$$

Note that if (78) is satisfied, it follows from Lemma 2.7 that $\|z_k - z^*\|$ is bounded and converges.

To stress the relation between z_{k+1} and t_k , we define $z(t) := z_k - t[(M_k - B)(z_k - \tilde{z}_k)]$ and we will use z_{k+1} and $z(t_k)$ interchangeably. We also set $\gamma_k(t) := (\delta_k + \|z_k - z^*\|)^2 - \|z(t) - z^*\|^2$.

In order to prove (78), it is sufficient to show that for proper t_k at each iterate, $\gamma_k(t_k) > 0$. It results from the definition of $\gamma_k(t_k)$ that

$$\begin{aligned} \gamma_k(t) &= (\|z_k - z^*\| + \delta_k)^2 - \|z(t) - z^*\|^2 \\ &= \langle z_k - z^* + z(t) - z^*, z_k - z(t) \rangle + 2\delta_k\|z_k - z^*\| + \delta_k^2 \\ &= \underbrace{2t \langle z_k - z^*, (M_k - B)(z_k - \tilde{z}_k) \rangle}_{(I)} - \underbrace{t^2 \|(M_k - B)(z_k - \tilde{z}_k)\|^2}_{(II)} \\ &\quad + 2\delta_k\|z_k - z^*\| + \delta_k^2. \end{aligned} \quad (79)$$

We need several useful properties to estimate (I) and (II).

It follows from the definition of \hat{z}_k that $-M_k(\hat{z}_k - z_k) - Bz_k \in A\hat{z}_k$, and γ_A -strong monotonicity of A implies that

$$\begin{aligned} \gamma_A\|\hat{z}_k - z^*\|^2 &\leq \langle \hat{z}_k - z^*, -M_k(\hat{z}_k - z_k) - Bz_k + Bz^* \rangle \\ &= \langle \hat{z}_k - z^*, M_k(z_k - \hat{z}_k) - Bz_k + B\hat{z}_k - B\hat{z}_k + Bz^* \rangle \\ &= \langle \hat{z}_k - z^*, M_k(z_k - \hat{z}_k) - Bz_k + B\hat{z}_k \rangle - \langle \hat{z}_k - z^*, B\hat{z}_k - Bz^* \rangle. \end{aligned} \quad (80)$$

We deduce by (80) and γ_B -strong monotonicity of B that

$$\begin{aligned} \langle \hat{z}_k - z^*, M_k(z_k - \hat{z}_k) - Bz_k + B\hat{z}_k \rangle &\geq \langle \hat{z}_k - z^*, B\hat{z}_k - Bz^* \rangle + \gamma_A\|\hat{z}_k - z^*\|^2 \\ &\geq \gamma_B\|\hat{z}_k - z^*\|^2 + \gamma_A\|\hat{z}_k - z^*\|^2. \end{aligned} \quad (81)$$

$J_A^{M_k}$ is Lipschitz since A is monotone. Therefore,

$$\begin{aligned} \|\hat{z}_k - z^*\|_{M_k}^2 &\leq \|(z_k - B_k z_k) - (z^* - B_k z^*)\|_{M_k}^2 \\ &= \|z_k - z^*\|_{M_k}^2 - 2 \langle z_k - z^*, B_k z_k - B_k z^* \rangle_{M_k} + \|B_k z_k - B_k z^*\|_{M_k}^2 \\ &\leq \|z_k - z^*\|_{M_k}^2 - \|Bz_k - Bz^*\|_{2\beta - M_k^{-1}} \\ &\stackrel{(*)}{\leq} \|z_k - z^*\|_{M_k}^2, \end{aligned} \quad (82)$$

where (*) uses Assumption 2 that $M_k - \frac{1}{\beta}I \in \mathcal{S}_c(\mathcal{H})$. Using the assumption that $M_k - \frac{1}{\beta}I \in \mathcal{S}_c(\mathcal{H})$ again, we obtain

$$\langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle \geq \|z_k - \tilde{z}_k\|_{M_k - \frac{1}{\beta}I}^2 > 0. \quad (83)$$

Combining (81), (82), the first term (I) in (79) can be estimated by the following:

$$\begin{aligned}
\text{(I)} &= 2t \langle z_k - z^*, (M_k - B)(z_k - \tilde{z}_k) \rangle \\
&= 2t \langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle + 2t \langle \tilde{z}_k - \hat{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle \\
&\quad + 2t \langle \hat{z}_k - z^*, (M_k - B)(z_k - \hat{z}_k) \rangle + 2t \langle \hat{z}_k - z^*, (M_k - B)(\hat{z}_k - \tilde{z}_k) \rangle \\
&\stackrel{\text{(i)}}{\geq} 2t \langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle + 2t \langle \tilde{z}_k - \hat{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle \\
&\quad + 2t(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 + 2t \langle \hat{z}_k - z^*, (M_k - B)(\hat{z}_k - \tilde{z}_k) \rangle \\
&\stackrel{\text{(ii)}}{\geq} 2t \langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle - 2t \|\epsilon_k\| \|(M_k - B)(z_k - \tilde{z}_k)\| \\
&\quad + 2t(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 - 2t \|(M_k - B)\epsilon_k\|_{M_k^{-1}} \|\hat{z}_k - z^*\|_{M_k} \\
&\stackrel{\text{(iii)}}{\geq} 2t \langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle - \|\epsilon_k\|^2 - t^2 \|(M_k - B)(z_k - \tilde{z}_k)\|^2 \\
&\quad + 2t(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 - 2\frac{1}{\sqrt{\sigma}}(C + \frac{1}{\beta})t \|\epsilon_k\| \|z_k - z^*\|_{M_k} \\
&\stackrel{\text{(iv)}}{\geq} 2t \langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle - \|\epsilon_k\|^2 - t^2 \|(M_k - B)(z_k - \tilde{z}_k)\|^2 \\
&\quad + 2t(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 - 2\sqrt{\frac{C}{\sigma}}(C + \frac{1}{\beta})t \|\epsilon_k\| \|z_k - z^*\|,
\end{aligned} \tag{84}$$

where (i) uses (81), (ii) uses Cauchy inequality and (iii) uses (82), $2ab \leq a^2 + b^2$ and Assumption 2. We set $b_k := \langle z_k - \tilde{z}_k, (M_k - B)(z_k - \tilde{z}_k) \rangle$ and $a_k := \|(M_k - B)(z_k - \tilde{z}_k)\|^2$. The definition of δ_k yields that $\delta_k \geq \|\epsilon_k\|^2$ and it follows from (84) that:

$$\begin{aligned}
\gamma_k(t) &\geq 2tb_k - 2t^2a_k + 2\delta_k \|z_k - z^*\| + \delta_k^2 - \|\epsilon_k\|^2 \\
&\quad - 2\sqrt{\frac{C}{\sigma}}(C + \frac{1}{\beta})t \|\epsilon_k\| \|z_k - z^*\| + 2t(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 \\
&\geq \underbrace{2tb_k - 2t^2a_k}_{\text{(III)}} + 2t(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 + \underbrace{2(\delta_k - \sqrt{\frac{C}{\sigma}}(C + \frac{1}{\beta})t \|\epsilon_k\|) \|z_k - z^*\|}_{\text{(IV)}}.
\end{aligned} \tag{85}$$

We continue to find a proper t_k such that $\gamma_k(t_k) > 0$. This goal boils down to ensuring both (III) and (IV) are positive.

Let $t_k = \frac{b_k}{2a_k}$. We first show $t_k = \frac{b_k}{2a_k}$ is the proper value to make sure (III) is positive. From (83), we observe that $b_k > 0$. Since $a_k > 0$ and $b_k > 0$, the quadratic term (III) in (85) will be zero for $t_k = 0$ or $t_k = \frac{b_k}{a_k}$ and will be strictly positive for any $t_k \in (0, b_k/a_k)$ with the maximum value obtained at $t_k = \frac{b_k}{2a_k}$. As a result, (III) is strictly positive.

Second, we will show (IV) is positive when $t_k = \frac{b_k}{2a_k}$. We observe that $0 < t_k = \frac{b_k}{2a_k} < \frac{1}{2\sqrt{c}}$ for Assumption 2. Thus, the definition of δ_k and $t_k = \frac{b_k}{2a_k}$ imply that

$$\begin{aligned}
\text{(IV)} &= 2(\delta_k - \sqrt{\frac{C}{\sigma}}(C + \frac{1}{\beta})t_k \|\epsilon_k\|) \|z_k - z^*\| \\
&\geq (2\delta_k - \sqrt{\frac{C}{\sigma c}}(C + \frac{1}{\beta})\|\epsilon_k\|) \|z_k - z^*\| \\
&\geq 0.
\end{aligned} \tag{86}$$

Since (III) and (IV) both are positive when $t_k = \frac{b_k}{2a_k}$, (85) yields:

$$\begin{aligned}
\gamma_k(t_k) &\geq 2t_k b_k - 2t_k^2 a_k + 2t_k(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 + (2\delta_k - \sqrt{\frac{C}{\sigma c}}(C + \frac{1}{\beta})\|\epsilon_k\|) \|z_k - z^*\| \\
&\geq \frac{b_k^2}{2a_k} + 2t_k(\gamma_A + \gamma_B) \|\hat{z}_k - z^*\|^2 \\
&> 0,
\end{aligned} \tag{87}$$

It results from (87) and the definition of $\gamma_k(t_k)$ that for each $k \in \mathbb{N}$, $(\|z_k - z^*\| + \delta_k) \geq \|z(t_k) - z^*\| = \|z_{k+1} - z^*\|$. We conclude that if $t_k = \frac{b_k}{2a_k}$, then $\gamma_k(t_k) > 0$ for all $k \in \mathbb{N}$ and the sequence $\|z_k - z^*\|$ is bounded and converges as $k \rightarrow +\infty$ by using Lemma 2.7.

Weak convergence:

The sequence $(z_k)_{k \in \mathbb{N}}$ generated by Algorithm 2 is bounded and $\|z_k - z\|$ converges as $k \rightarrow \infty$ and

$\gamma_k(t_k)$ converges to zero as $k \rightarrow \infty$ for all $z \in \text{zer}(A + B)$. Set z^* as an arbitrary weak sequential cluster point of $(z_k)_{k \in \mathbb{N}}$ and there exists a subsequence $(z_{k_n})_{n \in \mathbb{N}}$ such that $z_{k_n} \rightharpoonup z^*$.

In order to obtain weak convergence of z_k , by Proposition 2.5 with $\varphi(t) = t$ and fixed metric $M_k = I$ and (78), it suffices to show that $z^* \in \text{zer}(A + B)$.

Using Assumption 2, $(M_k)_{k \in \mathbb{N}}$ is bounded uniformly by C . Together with boundedness of operator B , we obtain a_k is bounded by $(C + \frac{1}{\beta})^2 \|z_k - \tilde{z}_k\|^2$ for each $k \in \mathbb{N}$. Using Assumption 2, we obtain that $b_k \geq \|z_k - \tilde{z}_k\|_{M_k - \frac{1}{\beta}I}^2 \geq c \|z_k - \tilde{z}_k\|^2$. By definition of t_k and (87), we have

$\gamma_k(t_k) \geq \frac{b_k^2}{2a_k} \geq \frac{c^2}{2(C + \frac{1}{\beta})^2} \|z_k - \tilde{z}_k\|^2$. Since $\gamma_k(t_k) \rightarrow 0$, $\|z_k - \tilde{z}_k\| \rightarrow 0$ as $k \rightarrow \infty$. Moreover, since $\epsilon_k \rightarrow 0$, $\|z_k - \hat{z}_k\| \rightarrow 0$. We set $u_k := M_k(z_k - \hat{z}_k) + B\hat{z}_k - Bz_k$. Therefore, we obtain that $u_k \rightarrow 0$ as $k \rightarrow +\infty$ and $\hat{z}_{k_n} \rightarrow z^*$ as $n \rightarrow +\infty$. We observe that $u_k \in A\hat{z}_k + B\hat{z}_k$. Then, by using Proposition 2.2 and the fact that $A + B$ is maximally monotone, we conclude that $0 \in Az^* + Bz^*$. Besides, $\|z_k - z^*\|$ decreases since $z^* \in \text{zer}(A + B)$. As mentioned above, the result follows from Proposition 2.5 with $\varphi(t) = t$.

Linear convergence rate:

If we assume $\epsilon_k \equiv 0$, then $\hat{z}_k = \tilde{z}_k$ and $\delta_k \equiv 0$. Therefore, from (87) we can obtain an estimation for $\gamma_k(t_k)$ when $t_k = \frac{b_k}{2a_k}$:

$$\gamma_k(t_k) = \|z_k - z^*\|^2 - \|z_{k+1} - z^*\|^2 \geq \frac{b_k^2}{2a_k} + 2t_k(\gamma_A + \gamma_B)\|\hat{z}_k - z^*\|^2. \quad (88)$$

The following part is to derive linear convergence for the case that either $\gamma_A > 0$ or $\gamma_B > 0$. The definition of b_k and that of a_k yield the following estimation for t_k :

$$t_k = \frac{b_k}{2a_k} = \frac{\langle z_k - \hat{z}_k, (M_k - B)(z_k - \hat{z}_k) \rangle}{2\|(M_k - B)(z_k - \hat{z}_k)\|^2} \stackrel{(i)}{\geq} \frac{\|z_k - \hat{z}_k\|_{M_k - \frac{1}{\beta}I}^2}{2(C + \frac{1}{\beta})^2 \|z_k - \hat{z}_k\|^2} \stackrel{(ii)}{>} \frac{c}{2(C + \frac{1}{\beta})^2}, \quad (89)$$

where both (i) and (ii) use Assumption 2. For convenience, we denote $\frac{c}{2(C + \frac{1}{\beta})^2}$ by δ . Using

Assumption 2 again, we have the estimation for the first term at the right hand side of (88):

$$\frac{b_k^2}{2a_k} \geq \delta \|z_k - \hat{z}_k\|_{M_k - \frac{1}{\beta}I}^2 > c\delta \|z_k - \hat{z}_k\|^2. \quad (90)$$

Furthermore, combining (88) with (89), (90) and the definition $\gamma_k(t_k) := \|z_k - z^*\|^2 - \|z_{k+1} - z^*\|^2$, we obtain

$$\begin{aligned} \|z_{k+1} - z^*\|^2 &\leq \|z_k - z^*\|^2 - 2(\gamma_A + \gamma_B)\delta\|\hat{z}_k - z^*\|^2 - c\delta\|z_k - \hat{z}_k\|^2 \\ &\leq \|z_k - z^*\|^2 - \frac{1}{2} \min\{2(\gamma_A + \gamma_B)\delta, c\delta\}(2\|\hat{z}_k - z^*\|^2 + 2\|z_k - \hat{z}_k\|^2) \\ &\stackrel{(i)}{\leq} \|z_k - z^*\|^2 - \frac{1}{2} \min\{2(\gamma_A + \gamma_B)\delta, c\delta\}(\|\hat{z}_k - z^*\| + \|z_k - \hat{z}_k\|)^2 \\ &\stackrel{(ii)}{\leq} \|z_k - z^*\|^2 - \frac{1}{2} \min\{2(\gamma_A + \gamma_B)\delta, c\delta\}\|z_k - z^*\|^2 \\ &\leq (1 - \frac{1}{2} \min\{2(\gamma_A + \gamma_B)\delta, c\delta\})\|z_k - z^*\|^2, \end{aligned} \quad (91)$$

where (i) uses inequality $2x^2 + 2y^2 \geq (x + y)^2$ and (ii) uses triangle inequality. Consequently, we obtain linear convergence if $(\gamma_A + \gamma_B) > 0$:

$$\|z_k - z^*\|^2 \leq (1 - \xi)^k \|z_0 - z^*\|^2, \quad (92)$$

where $\xi = \frac{1}{2} \min\{2(\gamma_A + \gamma_B)\delta, c\delta\} > 0$. □

A.5 Proof of Theorem 4.1

Now, we give the proof of Theorem 4.1. For convenience, we define translation operator $\tau_p: \mathcal{H} \rightarrow \mathcal{H}$ by $\tau_p(x) = x - p$ with inverse $\tau_p^{-1} = \tau_{-p}$.

Proof. Computing the resolvent operator shows the following equivalences

$$\begin{aligned}
x^* &= J_T^V(z) = (I + V^{-1}T)^{-1}(z) \\
&\iff Vz \in (V + T)(x^*) \\
&\iff Mz \in (M + T)(x^*) + sQ(x^* - z) \\
[y^* = M^{1/2}x^*] &\iff Mz \in (M + T)(M^{-1/2}y^*) + sQ(M^{-1/2}y^* - z) \\
&\iff Mz \in (M + T)(M^{-1/2}y^*) + sQM^{-1/2}(y^* - M^{1/2}z) \\
&\iff Mz \in (M^{1/2} + TM^{-1/2})(y^*) + sQM^{-1/2}(y^* - M^{1/2}z) \\
[W = M^{-1/2}QM^{-1/2}] &\iff M^{1/2}z \in (I + M^{-1/2}TM^{-1/2})(y^*) + sW(y^* - M^{1/2}z).
\end{aligned} \tag{93}$$

Since uniqueness and existence of x^* is guaranteed by the properties of J_T^V , Lemma 2.6 yields the existence of a unique primal-dual pair (x^*, u^*) that satisfies the equivalent relations in Lemma 2.6 with $B := +sW \circ \tau_{M^{1/2}z}$ and $A := \tau_{M^{1/2}z} \circ (I + M^{-1/2}TM^{-1/2})$. The mapping B is single-valued and, as T is a maximally monotone operator and M is positive-definite, $A^{-1} = J_{M^{-1/2}TM^{-1/2}} \circ \tau_{-M^{1/2}z}$ is single valued. Therefore, the solution of J_T^V can be computed by finding $u^* \in \text{im}(W)$, namely, $u^* \in \text{im}(M^{-1/2}Q)$, such that

$$0 \in B^{-1}u^* - A^{-1}(-u^*) = [(sW)^{-1}u^* + M^{1/2}z] - J_{M^{-1/2}TM^{-1/2}}(M^{1/2}z - u^*), \tag{94}$$

and the using

$$x^* = M^{-1/2}y^* \quad \text{and} \quad y^* = A^{-1}(-u^*) = J_{M^{-1/2}TM^{-1/2}}(M^{1/2}z - u^*). \tag{95}$$

Substituting $u^* = sM^{-1/2}v^* \in \text{im}(M^{-1/2}Q)$ in both problems, multiplying the former one from left with $M^{-1/2}$, and using $M^{-1/2}W^{-1}M^{-1/2} = Q^{-1}$ where Q^{-1} is a set-valued inverse operator of Q defined by the graph $\{(v, w) \in \text{im}(Q) \times \mathcal{H} \mid Qv = w\}$ leads to

$$\begin{cases} 0 \in Q^{-1}v^* + z - M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z - sM^{-1}v^*) \\ x^* = M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}(z - sM^{-1}v^*). \end{cases} \tag{96}$$

Since a unique solution to J_T^V exists, there exists $v^* \in \text{im}(Q)$ that satisfies the inclusion. We notice that $\text{im}(Q^{-1}) = \text{im}(Q^+) + \ker Q$ where $Q^+ : \text{im}(Q) \rightarrow \text{im}(Q)$ is the inverse of Q restricted on $\text{im}(Q)$ and $\ker(Q)$ denotes the kernel of Q . Given the linear mapping $U : \mathbb{R}^r \rightarrow \text{im}(Q)$, which can be realized using r linearly independent $u_1, \dots, u_r \in \mathcal{H}$ by $\alpha \rightarrow \sum_{i=1}^r \alpha_i u_i$, the inclusion problem is equivalent to finding the unique root $\alpha^* \in \mathbb{R}^r$ of $\ell(\alpha)$, namely,

$$\ell(\alpha) := U^*Q^+U\alpha + U^*(z - J_T^M(z - sM^{-1}U\alpha)), \tag{97}$$

where U^* denotes the adjoint of U and J_T^M is an abbreviation of the mapping $M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}$. The following shows that $\ell(\alpha)$ is Lipschitz continuous with constant $\|U^*Q^+U\| + \|M^{-1/2}U\|^2$: (We abbreviate $J_{M^{-1/2}TM^{-1/2}}$ by J in the following)

$$\begin{aligned}
&\langle \ell(\alpha) - \ell(\beta), \alpha - \beta \rangle \\
&= \|\alpha - \beta\|_{U^*Q^+U}^2 - \left\langle J(M^{1/2}z - sM^{-1/2}U\alpha) - J(M^{1/2}z - sM^{-1/2}U\beta), M^{-1/2}U(\alpha - \beta) \right\rangle \\
&\leq \|U^*Q^+U\| \|\alpha - \beta\|^2 + \|M^{-1/2}U\|^2 \|\alpha - \beta\|^2,
\end{aligned} \tag{98}$$

where, in the last line, we use the 1-Lipschitz continuity (non-expansive) of J .

The following shows strict monotonicity of l . We rewrite $\ell(\alpha)$ as follows:

$$\begin{aligned}
\ell(\alpha) &= U^*Q^+U\alpha + U^*M^{-1/2}(M^{1/2}z - J_{M^{-1/2}TM^{-1/2}}(M^{1/2}z - sM^{-1/2}U\alpha)) \\
&= U^*Q^+U\alpha + sU^*M^{-1}U\alpha + U^*M^{-1/2}(I - J_{M^{-1/2}TM^{-1/2}})(M^{1/2}z - sM^{-1/2}U\alpha) \\
&= U^*(Q^+ + sM^{-1})U\alpha + U^*M^{-1/2}J_{M^{-1/2}T^{-1}M^{-1/2}}(M^{1/2}z - sM^{-1/2}U\alpha).
\end{aligned} \tag{99}$$

Using the 1-co-coercivity of $J_{M^{1/2}T^{-1}M^{1/2}}$, the function $\ell(\alpha)$ can be seen to be strictly monotone if $\alpha \mapsto U^*(Q^+ + sM^{-1})U\alpha$ is strictly monotone. This fact is clear for the case $s = 1$. Therefore, in the remainder, we show strictly monotonicity of $\alpha \mapsto U^*(Q^+ - M^{-1})U\alpha = U^*M^{-1/2}(M^{1/2}Q^+M^{1/2} - I)M^{-1/2}U\alpha$. We observe $M - Q \in \mathcal{S}_0(\mathcal{H})$ implies that $\|M^{-1/2}QM^{-1/2}\| < 1$ and by $1 \leq \|AA^{-1}\| \leq \|A\|\|A^{-1}\|$, we conclude that $\|M^{1/2}Q^+M^{1/2}\|_{\text{im}(M^{-1/2}Q)} > 1$ for the restriction of the operator norm to $\text{im}(M^{-1/2}Q)$, hence, $Q^+ - M^{-1} \in \mathcal{S}_{++}(\mathcal{H})$.

According to Lemma 2.3, we can replace $M^{-1/2} \circ J_{M^{-1/2}TM^{-1/2}} \circ M^{1/2}$ with J_T^M . Then we obtain the formula in the statement of Theorem 4.1. \square

A.5.1 Proof of Remark 4.2

Proof. Given $y \in \text{im}(Q)$, assume there exist $U^*x_1, U^*x_2 \in U^*Q^{-1}y$ with $x_1, x_2 \in Q^{-1}y$. For arbitrary $\beta \in \mathbb{R}^r$, we have $\langle \beta, U^*x_1 - U^*x_2 \rangle = \langle U\beta, x_1 - x_2 \rangle$. Since $U\beta \in \text{im}Q$, then there exists some z such that $U\beta = Qz$. As a result, $\langle \beta, U^*x_1 - U^*x_2 \rangle = \langle U\beta, x_1 - x_2 \rangle = \langle Qz, x_1 - x_2 \rangle = \langle z, Qx_1 - Qx_2 \rangle = 0$. We notice that $\langle \beta, U^*x_1 - U^*x_2 \rangle$ holds for arbitrary $\beta \in \mathbb{R}^r$. It implies $U^*x_1 = U^*x_2$ and $U^*Q^{-1}y$ is single-valued. \square

A.6 Proof of Corollary 4.3

Proof. Let $\mathcal{H} = \mathbb{R}^n$. In this case, we can identify a linear mapping $U: \mathbb{R}^r \rightarrow \mathcal{H}$ with a low rank matrix $U \in \mathbb{R}^{n \times r}$. Similarly, we can also identify $U^*: \mathcal{H} \rightarrow \mathbb{R}^r$ with $U^\top \in \mathbb{R}^{n \times r}$. Since $Q = UU^\top$ for some $U \in \mathcal{B}(\mathbb{R}^r, \mathbb{R}^n)$, we have

$$\text{im}(Q) = \{UU^\top v | v \in \mathbb{R}^n\} = \{U\alpha | \alpha \in \mathbb{R}^r\}. \quad (100)$$

Since Q^+ is inverse of Q on $\text{im}(Q)$, the following holds for arbitrary $v \in \mathbb{R}^n$:

$$QQ^+Qv = Qv \iff UU^\top Q^+UU^\top v = UU^\top v \iff UU^\top Q^+U\alpha = U\alpha. \quad (101)$$

Since the column vectors $\{u_i\}_{i=1, \dots, r}$ of U are independent with each other, $UU^\top Q^+U\alpha = U\alpha$ yields that $U^\top Q^+U\alpha = \alpha$. Therefore, the root-finding problem in Theorem 4.1 simplifies to (20). \square

A.7 Proof of Proposition 4.4

Proof. Our proof relies on the convergence result [18, Theorem 7.5.5]. By the same argument as the one in Appendix B.5 paper [8], we obtain $\partial^C l(\alpha^*)$ is non-singular. If $l(\alpha)$ is tame, then by [9, Theorem 1], $l(\alpha)$ is semi-smooth. In order to apply this result it remains to show that $l(\alpha)$ is tame. The property of definable functions is preserved by operations including the sum, composition by a linear operator, derivation and canonical projection ([40], [17]). Since T is a tame mapping, $I + M^{-1/2}TM^{-1/2}$ is tame as well as its graph. Here I is identity. Then the resolvent $J_{M^{-1/2}TM^{-1/2}} = (I + M^{-1/2}TM^{-1/2})^{-1}$ which is defined by the inverse of the same graph is tame ([21]) and single-valued. By the stability of the sum and composition by linear operator, we obtain that $l(\alpha)$ is tame. \square

A.8 Proof of Proposition 4.7

Proof. Let $p = J_T^V(z)$. Since resolvent operator is non-expansive with respect to V , $\|p\|_V = \|J_T^V(z)\|_V \leq \|z\|_V + \|J_T^V(0)\|_V$. By duality, optimal α^* will satisfy $\alpha^* = u^\top(p - z)$. Then,

$$\begin{aligned} |\alpha^*| &= |u^*(p - z)| \\ &\leq \|u\|_{V^{-1}}(2\|z\|_V + \|J_T^V(0)\|_V). \end{aligned} \quad (102)$$

If $V \in \mathcal{S}_c(\mathbb{R}^n)$ is bounded by a constant C , then

$$|\alpha^*| \leq \|u\|_{V^{-1}}(2\|z\|_V + \|J_T^V(0)\|_V) \leq \frac{C}{c} \|u\|(2\|z\| + \|J_T^V(0)\|). \quad \square \quad (103)$$

A.9 Proof of Proposition 4.9

Proof. $\mathcal{L}(\alpha)$ is as defined in Proposition 4.8. Substituting $\alpha = \xi + U^*V^{-1}Bz$ in $\mathcal{L}(\alpha)$, we obtain $J(\xi) = \mathcal{L}(\alpha)$. Then, there exists ξ^* such that $\alpha^* = \xi^* + U^*V^{-1}Bz$. In (24), we do the same

substitution.

$$\begin{aligned}
\hat{z} &= J_T^M(\check{z} - sM^{-1}U\alpha^*) \\
&= J_A^M(\check{z} - sM^{-1}U\xi^* - sM^{-1}UU^*V^{-1}Bz) \\
&= J_A^M(z - M^{-1}MV^{-1}Bz - sM^{-1}UU^*V^{-1}Bz - sM^{-1}U\xi^*) \\
&= J_A^M(z - M^{-1}\underbrace{(M + sUU^*)}_{=V}V^{-1}Bz - sM^{-1}U\xi^*) \\
&= J_A^M(z - M^{-1}Bz - sM^{-1}U\xi^*).
\end{aligned} \tag{104}$$

Due to Proposition 4.8, $\mathcal{L}(\alpha)$ is Lipschitz with constant $1 + \|M^{-1/2}U\|^2$ and strongly monotone. Since $\mathcal{J}(\xi)$ is obtained by translation, it enjoys the same properties. \square

A.10 Proof of Lemma 5.1

Proof. (i) M_0 is symmetric positive-definite and $\gamma_k u_k u_k^*$ is symmetric positive semi-definite since $\inf_{k \in \mathbb{N}} \gamma_k > 0$. Thus, $M_k \succeq M_0$ showing that M_k is symmetric positive-definite. Moreover, $M_k - \frac{1}{\beta}I \succeq M_0 - \frac{1}{\beta}I \succeq cI$, which shows part (i) of Assumption 2. We now show that M_k obeys (ii) of the assumption. Since B is β -co-coercive, we have in view of [6, Remark 4.34 and Proposition 4.35] that

$$\beta \|y_k\|^2 \leq \langle y_k, s_k \rangle \leq \|s_k\|^2 / \beta. \tag{105}$$

Assume that $s_k \neq 0$ (otherwise, there is nothing to prove). Thus

$$\langle M_0 s_k - y_k, s_k \rangle = \|s_k\|_{M_0}^2 - \langle y_k, s_k \rangle \geq \|s_k\|_{M_0}^2 - \|s_k\|^2 / \beta = \|s_k\|_{M_0 - \beta I}^2 \geq c \|s_k\|^2.$$

Combining this with (105), we get

$$\|u_k\| = \frac{\|y_k - M_0 s_k\|}{\sqrt{\langle M_0 s_k - y_k, s_k \rangle}} \leq \frac{\|y_k\| + \|M_0\| \|s_k\|}{\sqrt{c} \|s_k\|} \leq \frac{1/\beta + \|M_0\|}{\sqrt{c}}. \tag{106}$$

This entails that

$$\sup_{k \in \mathbb{N}} \|M_k\| \leq \|M_0\| + \sup_{k \in \mathbb{N}} \gamma_k \|u_k\|^2 \leq \|M_0\| + \frac{(1/\beta + \|M_0\|)^2}{c} \sup_{k \in \mathbb{N}} \gamma_k < +\infty.$$

(ii) Let us focus on part (i) of Assumption 2. We have, using (106),

$$M_k - \frac{1}{\beta}I \succeq M_0 - \frac{1}{\beta}I - \gamma_k \|u_k u_k^*\| I = (c - \gamma_k \|u_k\|^2) I \succeq \left(c - \frac{(1/\beta + \|M_0\|)^2}{c} \gamma_k \right)$$

and the last term is positive under the prescribed choice of γ_k . To verify part (ii), it is sufficient to observe that $\|M_k\| \leq \|M_0\|$. \square

A.11 Proof of Lemma 5.2

Proof. (i) As in the proof of Lemma 5.1, $M_k \succeq M_0 \succeq (1/\beta + c)I > 0$. Moreover

$$\begin{aligned}
(1 + \eta_k)M_k - M_{k+1} &= \eta_k M_0 + (1 + \eta_k)\gamma_k u_k u_k^* - \gamma_{k+1} u_{k+1} u_{k+1}^* \\
&\succeq \eta_k (1/\beta + c)I - \gamma_{k+1} \|u_{k+1}\|^2 \\
&= \eta_k (1/\beta + c)I - \eta_{k+1} (1/\beta + c)I \\
&= (1/\beta + c)(\eta_k - \eta_{k+1})I \succeq 0,
\end{aligned}$$

since η_k is non-increasing. The uniform boundedness of M_k is straightforward as $\|M_k\| \leq \|M_0\| + \eta_k(1/\beta + c)$.

(ii) We have in this case

$$M_k - \frac{1}{\beta}I \succeq M_0 - \frac{1}{\beta}I - \gamma_k \|u_k u_k^*\| I \succeq (c - \eta_k \kappa(1/\beta + c)) I \succeq (c - (1 - \kappa)(1/\beta + c)) I$$

Under our condition on κ , we have $c - (1 - \kappa)(1/\beta + c) > 0$. In addition,

$$\begin{aligned} (1 + \eta_k)M_k - M_{k+1} &= \eta_k M_0 - (1 + \eta_k)\gamma_k u_k u_k^* + \gamma_{k+1} u_{k+1} u_{k+1}^* \\ &\succeq \eta_k(1/\beta + c)I - (1 + \eta_k)\eta_k \kappa(1/\beta + c)I \\ &= \eta_k(1/\beta + c)(1 - (1 + \eta_k)\kappa)I \succeq 0 \end{aligned}$$

since $\eta_k \leq (1 - \kappa)/\kappa$. M_k is also uniformly bounded with the same argument as above. This completes the proof. \square

Data of availability

In order to record the exact algorithmic details for our experiments, all code for experiments from this paper is available at https://github.com/wsdxiaohao/quasi_Newton_FBS.git.

Conflict of interest statements

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