



**HAL**  
open science

## Cox rings of blow-ups of multiprojective spaces

Michele Bolognesi, Alex Massarenti, Elena Poma

► **To cite this version:**

Michele Bolognesi, Alex Massarenti, Elena Poma. Cox rings of blow-ups of multiprojective spaces. Collectanea Mathematica, 2023, 10.1007/s13348-023-00428-2 . hal-04505973

**HAL Id: hal-04505973**

**<https://hal.science/hal-04505973>**

Submitted on 15 Mar 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# COX RINGS OF BLOW-UPS OF MULTIPROJECTIVE SPACES

MICHELE BOLOGNESI, ALEX MASSARENTI, AND ELENA POMA

ABSTRACT. Let  $X_r^{1,n}$  be the blow-up of  $\mathbb{P}^1 \times \mathbb{P}^n$  in  $r$  general points. We describe the Mori cone of  $X_r^{1,n}$  for  $r \leq n+2$  and for  $r = n+3$  when  $n \leq 4$ . Furthermore, we prove that  $X_{n+1}^{1,n}$  is log Fano and give an explicit presentation for its Cox ring.

## CONTENTS

1. Introduction	1
2. Cox rings, Mori dream spaces and log Fano varieties	3
3. Mori cones	4
4. The Cox ring of $X_{n+1}^{1,n}$	9
References	12

## 1. INTRODUCTION

*Mori dream spaces*, introduced by Y. Hu and S. Keel in [HK00], are varieties whose total coordinate ring, called the *Cox ring*, is finitely generated. The birational geometry of a Mori dream space is encoded in its cone of effective divisors together with a chamber decomposition on it, called *Mori chamber decomposition* whose chambers correspond to the nef cones of the birational models of the Mori dream space.

Finite generation of the Cox rings of blow-ups of projective spaces and products of projective spaces has been extensively investigated [Muk01], [CT06], [AC17], [LP17], [GPPS23]. In [CT06, Theorem 1.3] A. M. Castravet and J. Tevelev proved that the blow-up of  $(\mathbb{P}^n)^s$  in  $r$  general points is a Mori dream space if and only if

$$\frac{1}{s+1} + \frac{1}{r-n-1} + \frac{1}{n+1} > 1.$$

Indeed, when the above inequality is not satisfied the effective cones of these blow-ups are not finitely generated and the proof of this last fact relies on the symmetries of their Picard groups which carry a natural Weyl group action. For products of projective spaces with unbalanced dimensions this is not the case. In this paper we push a little further the investigation of the birational geometry of these varieties initiated by T. Grange, E. Postingshel and A. Prendergast-Smith in [GPPS23].

Let  $X_r^{1,n}$  be the blow-up of  $\mathbb{P}^1 \times \mathbb{P}^n$  in  $r$  general points  $p_1, \dots, p_r \in \mathbb{P}^1 \times \mathbb{P}^n$ . We will denote by  $\pi : X_r^{1,n} \rightarrow \mathbb{P}^1 \times \mathbb{P}^n$  the blow-down morphism and by  $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^n \rightarrow \mathbb{P}^1$ ,  $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^n \rightarrow \mathbb{P}^n$  the projections onto the factors. Moreover, let us denote by  $\tilde{\pi}_1, \tilde{\pi}_2$  the morphisms from  $X_r^{1,n}$  to  $\mathbb{P}^1$  and

---

*Date:* August 23, 2023.

*2020 Mathematics Subject Classification.* Primary 14E30; Secondary 14M99, 14N05.

*Key words and phrases.* Mori dream spaces; Cox rings; Fano varieties.

$\mathbb{P}^n$  induced by the projections. We summarize the situation in the following diagram

$$\begin{array}{ccc}
 & X_r^{1,n} & \\
 \tilde{\pi}_1 \swarrow & \downarrow \pi & \searrow \tilde{\pi}_2 \\
 & \mathbb{P}^1 \times \mathbb{P}^n & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 \mathbb{P}^1 & & \mathbb{P}^n
 \end{array}$$

Let  $\text{Pic}(X_r^{1,n})$  denote the Picard group of  $X_r^{1,n}$  and  $N_1(X_r^{1,n})$  the  $\mathbb{R}$ -vector space of 1-dimensional cycles modulo numerical equivalence. Throughout the paper we will denote by:

- $H_1$  the pull-back of a point of  $\mathbb{P}^1$  via  $\tilde{\pi}_1$ ;
- $H_2$  the pull-back of a hyperplane in  $\mathbb{P}^n$  via  $\tilde{\pi}_2$ ;
- $E_i$  the exceptional divisor over  $p_i$  for  $i = 1, \dots, r$ ;

and by

- $h_1$  the class of a general fiber of  $\tilde{\pi}_2$ ;
- $h_2$  the class of a line in a general fiber of  $\tilde{\pi}_1$ ;
- $e_i$  the class of a line in the exceptional divisor  $E_i$  for  $i = 1, \dots, r$ .

We have that  $\text{Pic}(X_r^{1,n}) = \mathbb{Z}[H_1, H_2, E_1, \dots, E_r]$  and  $N_1(X_r^{1,n}) = \mathbb{Z}[h_1, h_2, e_1, \dots, e_r]$ .

Our main results on the Mori cone of  $X_r^{1,n}$  in Propositions 3.1, 3.2, 3.6 can be summarized as follows:

**Theorem 1.1.** *The Mori cone of  $X_r^{1,n}$  is given by*

$$\text{NE}(X_r^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle$$

for all  $r \leq n + 1$ . Furthermore,

$$\text{NE}(X_{n+2}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + nh_2 - e_1 - \dots - e_{n+2} \rangle$$

and if  $n \leq 4$  then

$$\text{NE}(X_{n+3}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + nh_2 - e_{i_1} - \dots - e_{i_{n+2}} \rangle$$

for  $i, i_1, \dots, i_{n+2} \in \{1, \dots, n + 3\}$ .

Recall that a normal and  $\mathbb{Q}$ -factorial projective variety  $X$  is *log Fano* if there exists an effective divisor  $D \subset X$  such that  $-(K_X + D)$  is ample and the pair  $(X, D)$  is Kawamata log terminal. By [BCHM10, Corollary 1.3.2] log Fano varieties are Mori dream spaces. The converse does not hold in general, and there are several criteria for a Mori dream space to be log Fano [GOST15].

Our main results in the case  $r \leq n + 1$  in Proposition 4.2 and Theorem 4.3 can be summarized as follows:

**Theorem 1.2.** *For  $r \leq n + 1$  the variety  $X_r^{1,n}$  is log Fano and hence a Mori dream space. Furthermore, the Cox ring of  $X_{n+1}^{1,n}$  is given by*

$$\text{Cox}(X_{n+1}^{1,n}) \cong \frac{\mathbb{C}[S_0, \dots, S_n, T_{1,1}, T_{1,2}, \dots, T_{n+1,1}, T_{n+1,2}]}{\langle g_1, \dots, g_{n-1} \rangle}$$

where  $S_i$  is the section associated to  $H_2 - E_1 - \dots - E_{i-1} - E_{i+1} - \dots - E_{n+1}$ ,  $T_{i,1}$  is the section associated to  $H_1 - E_i$  and  $T_{i,2}$  is the section associated to  $E_i$ , and

$$g_i = (\beta_k \alpha_j - \beta_j \alpha_k) T_{i,1} T_{i,2} + (\beta_i \alpha_k - \beta_k \alpha_i) T_{j,1} T_{j,2} + (\beta_j \alpha_i - \beta_i \alpha_j) T_{k,1} T_{k,2}$$

where for  $1 \leq i \leq n - 1$  we set  $k = j + 1 = i + 2$ .

Furthermore, in Section 3 we explicitly describe the nef cones of  $X_r^{1,n}$  for  $r \leq n + 2$ , and of  $X_{n+3}^{1,n}$  for  $n \leq 4$ , and in Section 4 we give generators for the cones of movable divisors and of moving curves of  $X_{n+1}^{1,n}$ .

**Acknowledgments.** The first named author is supported by and member of the ANR project *FanoHK* ANR-20-CE40-0023. The first named author is member of the GDR GAGC. The first and second named authors are members of the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of the Istituto Nazionale di Alta Matematica "F. Severi" (GNSAGA-INDAM). The mobility of the third named author is supported by a UFI Scholarship VINCI. We thank Antonio Laface and Elisa Postinghel for their helpful comments on a preliminary version of the paper.

## 2. COX RINGS, MORI DREAM SPACES AND LOG FANO VARIETIES

Let  $X$  be a normal projective  $\mathbb{Q}$ -factorial variety. We denote by  $N^1(X)$  the real vector space of  $\mathbb{R}$ -Cartier divisors modulo numerical equivalence. The *nef cone* of  $X$  is the closed convex cone  $\text{Nef}(X) \subset N^1(X)$  generated by classes of nef divisors.

The stable base locus  $\mathbf{B}(D)$  of a  $\mathbb{Q}$ -divisor  $D$  is the set-theoretic intersection of the base loci of the complete linear systems  $|sD|$  for all positive integers  $s$  such that  $sD$  is integral

$$(2.1) \quad \mathbf{B}(D) = \bigcap_{s>0} B(sD).$$

The *movable cone* of  $X$  is the convex cone  $\text{Mov}(X) \subset N^1(X)$  generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in  $X$ . The *effective cone* of  $X$  is the convex cone  $\text{Eff}(X) \subset N^1(X)$  generated by classes of *effective divisors*. We have inclusions  $\text{Nef}(X) \subset \overline{\text{Mov}(X)} \subset \overline{\text{Eff}(X)}$ . We refer to [Deb01, Chapter 1] for a comprehensive treatment of these topics.

We say that a birational map  $f : X \dashrightarrow X'$  to a normal projective variety  $X'$  is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small  $\mathbb{Q}$ -factorial modification* if  $X'$  is  $\mathbb{Q}$ -factorial and  $f$  is an isomorphism in codimension one. If  $f : X \dashrightarrow X'$  is a small  $\mathbb{Q}$ -factorial modification then the natural pullback map  $f^* : N^1(X') \rightarrow N^1(X)$  sends  $\text{Mov}(X')$  and  $\overline{\text{Eff}(X')}$  isomorphically onto  $\text{Mov}(X)$  and  $\overline{\text{Eff}(X)}$  respectively. In particular, we have  $f^*(\text{Nef}(X')) \subset \overline{\text{Mov}(X)}$ .

Now, assume that the divisor class group  $\text{Cl}(X)$  is free and finitely generated, and fix a subgroup  $G$  of the group of Weil divisors on  $X$  such that the canonical map  $G \rightarrow \text{Cl}(X)$ , mapping a divisor  $D \in G$  to its class  $[D]$ , is an isomorphism. The *Cox ring* of  $X$  is defined as

$$\text{Cox}(X) = \bigoplus_{[D] \in \text{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

where  $D \in G$  represents  $[D] \in \text{Cl}(X)$ , and the multiplication in  $\text{Cox}(X)$  is defined by the standard multiplication of homogeneous sections in the field of rational functions on  $X$ .

**Definition 2.2.** A normal projective  $\mathbb{Q}$ -factorial variety  $X$  is called a *Mori dream space* if the following conditions hold:

- $\text{Pic}(X)$  is finitely generated, or equivalently  $h^1(X, \mathcal{O}_X) = 0$ ,
- $\text{Nef}(X)$  is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small  $\mathbb{Q}$ -factorial modifications  $f_i : X \dashrightarrow X_i$ , such that each  $X_i$  satisfies the second condition above, and  $\text{Mov}(X) = \bigcup_i f_i^*(\text{Nef}(X_i))$ .

The collection of all faces of all cones  $f_i^*(\text{Nef}(X_i))$  above forms a fan which is supported on  $\text{Mov}(X)$ . If two maximal cones of this fan, say  $f_i^*(\text{Nef}(X_i))$  and  $f_j^*(\text{Nef}(X_j))$ , meet along a facet, then there exist a normal projective variety  $Y$ , a small modification  $\varphi : X_i \dashrightarrow X_j$ , and  $h_i : X_i \rightarrow Y$ ,  $h_j : X_j \rightarrow Y$  small birational morphisms of relative Picard number one such that  $h_j \circ \varphi = h_i$ . The fan structure on  $\text{Mov}(X)$  can be extended to a fan supported on  $\text{Eff}(X)$  as follows.

**Definition 2.3.** Let  $X$  be a Mori dream space. We describe a fan structure on the effective cone  $\text{Eff}(X)$ , called the *Mori chamber decomposition*. There are finitely many birational contractions from  $X$  to Mori dream spaces, denoted by  $g_i : X \dashrightarrow Y_i$ . The set  $\text{Exc}(g_i)$  of exceptional prime divisors of  $g_i$  has cardinality  $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$ . The maximal cones  $\mathcal{C}$  of the Mori chamber decomposition of  $\text{Eff}(X)$  are of the form  $\mathcal{C}_i = \langle g_i^*(\text{Nef}(Y_i)), \text{Exc}(g_i) \rangle$ . We call  $\mathcal{C}_i$  or its interior  $\mathcal{C}_i^\circ$  a *maximal chamber* of  $\text{Eff}(X)$ . We refer to [HK00, Proposition 1.11] and [Oka16, Section 2.2] for details.

**Definition 2.4.** A normal and  $\mathbb{Q}$ -factorial projective variety  $X$  is *log Fano* if there exists an effective divisor  $D \subset X$  such that  $-(K_X + D)$  is ample and the pair  $(X, D)$  is Kawamata log terminal.

By [BCHM10, Corollary 1.3.2] if  $X$  is log Fano then it is a Mori dream space.

**2.4. Varieties with a torus action.** Let  $X$  be a normal projective variety and  $T \times X \rightarrow X$  an effective algebraic torus action on  $X$  of complexity one that is such that its biggest  $T$ -orbits are of codimension one in  $X$ .

We will follow the treatment in [HS10]. For a given point  $x \in X$ , denote by  $T_x \subset T$  its isotropy group

$$T_x = \{t \in T \mid t \cdot x = x\}$$

and consider the non-empty  $T$ -invariant open subset

$$X_0 = \{x \in X \mid \dim(T_x) = 0\} \subset X$$

of points in  $x$  with zero-dimensional isotropy group. There is a geometric quotient

$$q : X_0 \rightarrow X_0/T$$

where the orbit space  $X_0/T$  is of dimension one and has a separation that is a rational map  $\pi : X_0/T \dashrightarrow \mathbb{P}^1$  which is a local isomorphism in codimension one.

Denote by  $E_1, \dots, E_m \subseteq X$  the  $T$ -invariant prime divisors supported in  $X \setminus X_0$  and by  $D_1, \dots, D_n \subseteq X$  those  $T$ -invariant prime divisors who have a finite generic isotropy group of order  $l_j > 1$ . Moreover, let  $1_{E_k}$  and  $1_{D_j}$  denote the canonical sections of the divisors  $E_k$  and  $D_j$  respectively, and let  $1_{q(D_j)}$  be the canonical section of  $q(D_j)$ .

Choose  $r \geq 1$  and  $a_0, \dots, a_r \in \mathbb{P}^1$ , such that  $\pi$  is an isomorphism over  $\mathbb{P}^1 \setminus \{a_0, \dots, a_r\}$  and all the divisors  $D_j$  occur among the

$$D_{i,j} = q^{-1}(y_{i,j})$$

where  $\pi^{-1}(a_i) = \{y_{i,1}, \dots, y_{i,n_i}\}$ . Let  $l_{i,j} \in \mathbb{Z}_{\geq 1}$  denote the order of the generic isotropy group of  $D_{i,j}$ . For every  $0 \leq i \leq r$  define a monomial

$$f_i = T_{i,1}^{l_{i,1}} \cdots T_{i,n_i}^{l_{i,n_i}} \in \mathbb{C}[T_{i,j}; 0 \leq i \leq r, 1 \leq j \leq n_i].$$

Moreover, write  $a_i = [b_i, c_i]$  with  $b_i, c_i \in \mathbb{C}$  and for every  $0 \leq i \leq r-2$  set  $k = j+1 = i+2$  and define a trinomial

$$g_i = (c_k b_j - c_j b_k) f_i + (c_i b_k - c_k b_i) f_j + (c_j b_i - c_i b_j) f_k.$$

In the previous notation we have the following:

**Theorem 2.5.** [HS10, Theorem 1.3] *Let  $T \times X \rightarrow X$  be an algebraic torus action of complexity one. Then, in terms of the data defined above, the Cox ring of  $X$  is given as*

$$\text{Cox}(X) = \mathbb{C}[S_1, \dots, S_m, T_{i,j}; 0 \leq i \leq r, 1 \leq j \leq n_i] / \langle g_i; 0 \leq i \leq r-2 \rangle$$

where  $1_{E_k}$  corresponds to  $S_k$  and  $1_{D_{i,j}}$  to  $T_{i,j}$ , and the  $\text{Cl}(X)$ -grading on the right hand side is defined by associating to  $S_k$  the class of  $E_k$  and to  $T_{i,j}$  the class of  $D_{i,j}$ . In particular,  $\text{Cox}(X)$  is finitely generated.

Theorem 2.5 will be crucial in the study of the Cox ring of  $X_{n+1}^{1,n}$ .

### 3. MORI CONES

In this section we compute the cone of effective curves of  $X_r^{1,n}$  for  $r \leq n+2$ , and for  $r = n+3$  when  $n \leq 4$ .

**Proposition 3.1.** *The Mori cone of  $X_r^{1,n}$  is given by*

$$\text{NE}(X_r^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle$$

for all  $r \leq n+1$ .

*Proof.* It is enough to prove the claim for  $r = n + 1$ . The case  $n = 1$  is well-known. We will assume that the claim is proved for  $X_n^{1,n-1}$  and prove it for  $X_{n+1}^{1,n}$ .

Let  $C \subset X_{n+1}^{1,n}$  be an irreducible curve. If  $C$  is contracted by  $\pi$  then  $C \subset E_i$  for some  $i = 1, \dots, n+1$  and hence  $C$  is a multiple of  $e_i$ . Assume that  $\pi(C)$  is a curve of bidegree  $(d_1, d_2)$  and such that  $\text{mult}_{p_i} C = m_i$ . Then we may write

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \dots - m_n e_n - m_{n+1} e_{n+1}.$$

Note that  $C \cdot (H_1 - E_i) = d_1 - m_i$ . If  $d_1 - m_i < 0$  then  $C$  is contained in the strict transform of the fiber of  $\pi_1$  passing through  $p_i$ . Such fiber is  $\mathbb{P}^n$  blown-up in a point and hence  $C$  may be written as a linear combination with non negative coefficients of  $e_i$  and  $h_2 - e_i$ . Hence, we may assume that  $d_1 - m_i \geq 0$  for all  $i = 1, \dots, n+1$ .

Now, consider the projection  $\tilde{\pi}_2(C)$ . If  $\tilde{\pi}_2(C)$  is a point then  $C$  is a linear combination with non negative coefficients of  $h_1 - e_i$  and  $e_i$  for some  $i = 1, \dots, n+1$ . Assume that  $\tilde{\pi}_2(C)$  is a curve. Then this curve has degree  $d_2$  and  $\text{mult}_{\pi_2(p_i)} \tilde{\pi}_2(C) = m_i$ . Let  $\Pi_{1,\dots,n} \subset \mathbb{P}^n$  be the hyperplane generated by  $p_1, \dots, p_n$ . If

$$m_1 + \dots + m_n > d_2$$

then  $\tilde{\pi}_2(C) \subset \Pi_{1,\dots,n}$  and hence  $C$  is contained in the strict transform of  $\mathbb{P}^1 \times \Pi_{1,\dots,n}$  which is isomorphic to  $X_n^{1,n-1}$ . Therefore, we can conclude by induction on  $n$ . On the other hand, if

$$m_1 + \dots + m_n \leq d_2$$

we may write

$$C \sim \sum_{i=1}^n m_i (h_2 - e_i) + m_{n+1} (h_1 - e_{n+1}) + (d_1 - m_{n+1}) h_1 + (d_2 - m_1 - \dots - m_n) h_2$$

where all the coefficients are non negative, concluding the proof.  $\square$

Consider the Segre embedding

$$\begin{aligned} \sigma_{1,n} : \quad \mathbb{P}^1 \times \mathbb{P}^n &\longrightarrow \mathbb{P}^{2n+1} \\ ([u_0, u_1], [v_0, \dots, v_n]) &\mapsto [u_0 v_0, u_0 v_1, \dots, u_1 v_n] \end{aligned}$$

and set  $\Sigma^{1,n} = \sigma_{1,n}(\mathbb{P}^1 \times \mathbb{P}^n)$ . Recall that  $\deg(\Sigma^{1,n}) = n + 1$ .

**Proposition 3.2.** *The Mori cone of  $X_{n+2}^{1,n}$  is given by*

$$\text{NE}(X_{n+2}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + n h_2 - e_1 - \dots - e_{n+2} \rangle.$$

*Proof.* The case  $n = 1$  is well-known. Let  $C$  be an irreducible curve in  $X_{n+2}^{1,n}$ . If  $C$  is contracted by  $\pi$  then  $C \subset E_i$  for some  $i = 1, \dots, n+2$  and hence  $C$  is a multiple of  $e_i$ . Assume that  $\pi(C)$  is a curve of bidegree  $(d_1, d_2)$  and such that  $\text{mult}_{p_i} C = m_i$ . Then we may write

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \dots - m_{n+1} e_{n+1} - m_{n+2} e_{n+2}.$$

If

$$d_1 + d_2 \geq m_1 + \dots + m_{n+2}$$

then  $C$  may be written as a linear combination with non negative coefficients of  $e_i$ ,  $h_2 - e_i$  and  $h_1 - e_i$ . Recall that  $\sigma_{1,n}(\pi(C))$  is a curve of degree  $d_1 + d_2$  passing through the points  $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$  with multiplicities  $m_i$ .

Let us denote by  $\Pi_{1,\dots,n+2} \subset \mathbb{P}^{2n+1}$  the  $(n+1)$ -plane spanned by  $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$ . If

$$d_1 + d_2 < m_1 + \dots + m_{n+2}$$

then  $\sigma_{1,n}(\pi(C)) \subset \Pi_{1,\dots,n+2}$ . Since  $\Sigma^{1,n}$  has degree  $n + 1$ , the intersection  $\Pi_{1,\dots,n+2} \cap \Sigma^{1,n}$  is a rational normal curve of degree  $n + 1$  and with class  $h_1 + n h_2 - e_1 - \dots - e_{n+2}$ .  $\square$

Let  $a, b \geq 1$  be two integers, fix two complementary subspace  $\Lambda^a, \Lambda^b \subset \mathbb{P}^{a+b+1}$ , two rational normal curves  $C_a \subset \Lambda^a, C_b \subset \Lambda^b$ , and an isomorphism  $\phi : C_a \rightarrow C_b$ . The surface

$$S_{(a,b)} = \bigcup_{x \in C_a} \langle x, \phi(x) \rangle \subset \mathbb{P}^{2n+1}$$

where  $\langle x, \phi(x) \rangle$  denotes the line through  $x, \phi(x)$ , is a rational normal scroll of type  $(a, b)$ . This is a smooth rational surface of degree  $\deg(S_{(a,b)}) = a + b$ .

**Lemma 3.3.** *Let  $H \subset \mathbb{P}^{2n+1}$  be a general  $(n+2)$ -plane. Then the intersection  $H \cap \Sigma^{1,n}$  is a rational normal scroll  $S_{(a,b)}$  with*

$$(a, b) = \begin{cases} \left(\frac{n+1}{2}, \frac{n+1}{2}\right) & \text{if } n \text{ is odd;} \\ \left(\frac{n}{2}, \frac{n+2}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* The Segre variety  $\Sigma^{1,n}$  is the projectivization over  $\mathbb{P}^1$  of the rank  $n+1$  vector bundle  $\mathcal{O}_{\mathbb{P}^1}(-1)^{n+1}$ . A codimension  $n-1$  linear section corresponds to the projectivization of the kernel of a morphism

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}^1}^{n-1}$$

which is a rank two vector bundle  $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$ . To conclude it is enough to note that for  $H$  general the splitting type  $(-a, -b)$  is the one given in the statement.  $\square$

**Lemma 3.4.** *Let  $f : X \rightarrow Y$  be a morphism of projective varieties, and  $\text{NE}(f)$  the cone of curves contracted by  $f$ . Then  $\text{NE}(f)$  is extremal in  $\text{NE}(X)$ .*

*Proof.* Let  $\Gamma$  be the class of an irreducible curve in  $\text{NE}(f)$ , and assume that  $\Gamma = \Gamma_1 + \Gamma_2$  for  $\Gamma_1, \Gamma_2 \in \text{NE}(X)$ . Applying  $f_*$  we get  $f_*\Gamma_1 + f_*\Gamma_2 = f_*\Gamma = 0$  since  $\Gamma$  is contracted by  $f$ . Therefore,  $f_*\Gamma_1 = f_*\Gamma_2 = 0$  and hence  $\Gamma_1, \Gamma_2 \in \text{NE}(f)$ .  $\square$

**Proposition 3.5.** *Fix  $p_1, \dots, p_{n+3} \in \Sigma^{1,n}$  general points. Set  $H = \langle p_1, \dots, p_{n+3} \rangle$ . Let  $\tilde{S}_{a,b}$  be the blow-ups of  $S_{a,b} = H \cap \Sigma^{1,n}$  at the  $p_i$ . Then*

$$\text{NE}(X_{n+3}^{1,n}) = \left\langle \text{NE}(\tilde{S}_{a,b}), h_1 - e_1, \dots, h_1 - e_{n+3} \right\rangle.$$

*Proof.* Let  $C \sim ah_1 + bh_2 - \sum_{i=1}^{n+3} m_i e_i$  be an irreducible curve in  $X_{n+3}^{1,n}$ , and  $\Gamma \subset \Sigma^{1,n}$  its image in  $\mathbb{P}^{2n+1}$ . Then  $\deg(\Gamma) = a + b$  and  $\text{mult}_{p_i} \Gamma = m_i$  for  $i = 1, \dots, n+3$ .

If  $a + b < \sum_{i=1}^{n+3} m_i$  then  $\Gamma$  is contained in all the hyperplanes containing  $H$  and hence  $\Gamma \subset H \cap \Sigma^{1,n}$ . By Lemma 3.3  $S_{a,b} = H \cap \Sigma^{1,n}$  is a scroll.

If  $a + b \geq \sum_{i=1}^{n+3} m_i$  we can pair each  $e_i$  with one among  $h_1$  and  $h_2$ , and write  $C$  as a linear combination with non negative coefficients of  $h_1 - e_i, h_2 - e_j, e_k$ .

The curves of class  $h_2 - e_i$  are numerically equivalent to the strict transform of the line through  $p_i$  in the ruling of  $S_{a,b}$ . The curves of class  $h_1 - e_i$  are contracted by  $\tilde{\pi}_2$  and the curves of class  $h_2 - e_j$  are contracted by  $\tilde{\pi}_1$ . Hence, by Lemma 3.4  $h_1 - e_i$  and  $h_2 - e_j$  generate extremal rays of  $\text{NE}(X_{n+3}^{1,n})$ .

Summing-up, we have showed that an irreducible curve  $C \subset X_{n+3}^{1,n}$  can be written as a linear combination with non negative coefficients of a curve  $\Gamma \subset \tilde{S}_{a,b}$  and of the  $h_1 - e_i$ .  $\square$

**3.5. Mori cones of del Pezzo surfaces.** Let  $S_r$  be the blow-up of  $\mathbb{P}^2$  at  $p_1, \dots, p_r \in \mathbb{P}^2$  general points. The surface  $S_r$  is del Pezzo if and only if  $0 \leq r \leq 8$ . We recall the structure of the Mori cone  $\text{NE}(S_r)$  for  $r = 6, 7, 8$ . We will denote by  $\bar{h}$  the pull-back of a line in  $\mathbb{P}^2$  and by  $\bar{e}_i$  the exceptional divisor over the point  $p_i$  for  $i = 1, \dots, r$ .

**3.5.1. del Pezzo of degree 3.** The Mori cone of  $S_3$  is generated by the following classes

Divisor class	Number of extremal rays
$\bar{e}_i$	6
$\bar{h} - \bar{e}_i - \bar{e}_j$	15
$2\bar{h} - \bar{e}_{i_1} - \dots - \bar{e}_{i_5}$	6

for a total of 27 extremal rays.

3.5.2. *del Pezzo of degree 2.* The Mori cone of  $S_2$  is generated by the following classes

<i>Divisor class</i>	<i>Number of extremal rays</i>
$\bar{e}_i$	7
$\bar{h} - \bar{e}_i - \bar{e}_j$	21
$2\bar{h} - \bar{e}_{i_1} - \cdots - \bar{e}_{i_5}$	21
$3\bar{h} - 2\bar{e}_i - \bar{e}_{j_1} - \cdots - \bar{e}_{j_6}$	7

for a total of 56 extremal rays.

3.5.3. *del Pezzo of degree 1.* The Mori cone of  $S_1$  is generated by the following classes

<i>Divisor class</i>	<i>Number of extremal rays</i>
$\bar{e}_i$	8
$\bar{h} - \bar{e}_i - \bar{e}_j$	28
$2\bar{h} - \bar{e}_{i_1} - \cdots - \bar{e}_{i_5}$	56
$3\bar{h} - 2\bar{e}_i - \bar{e}_{j_1} - \cdots - \bar{e}_{j_6}$	56
$4\bar{h} - 2\bar{e}_{i_1} - \cdots - 2\bar{e}_{i_3} - \bar{e}_{j_1} - \cdots - \bar{e}_{j_5}$	56
$5\bar{h} - 2\bar{e}_{i_1} - \cdots - 2\bar{e}_{i_6} - \bar{e}_{j_1} - \bar{e}_{j_2}$	28
$6\bar{h} - 3\bar{e}_i - 2\bar{e}_{i_1} - \cdots - 2\bar{e}_{j_7}$	8

for a total of 240 extremal rays.

**Proposition 3.6.** *If  $n \leq 4$ , the Mori cone of  $X_{n+3}^{1,n}$  is given by*

$$\text{NE}(X_{n+3}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + nh_2 - e_{i_1} - \cdots - e_{i_{n+2}} \rangle$$

for  $i, i_1, \dots, i_{n+2} \in \{1, \dots, n+3\}$ .

*Proof.* The case  $n = 1$  is well-known. Let  $C$  be an irreducible curve in  $X_{n+3}^{1,n}$ . If  $C$  is contracted by  $\pi$  then  $C \subset E_i$  for some  $i = 1, \dots, n+3$  and hence  $C$  is a multiple of  $e_i$ . Now, assume that  $\pi(C)$  is a curve of bidegree  $(d_1, d_2)$  in  $\mathbb{P}^1 \times \mathbb{P}^n$  and such that  $\text{mult}_{p_i} C = m_i$ . This means that we can write

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \cdots - m_{n+2} e_{n+2} - m_{n+3} e_{n+3}.$$

First, we suppose that

$$d_1 + d_2 \geq m_1 + \cdots + m_{n+3}.$$

Then  $C$  may be written as a linear combination with non negative coefficients of  $e_i$ ,  $h_2 - e_i$  and  $h_1 - e_i$ .

Note that  $\sigma_{1,n}(\pi(C))$  is a curve of degree  $d_1 + d_2$  passing through the points  $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+3})$  with multiplicities  $m_i$ . Let  $\Pi_{1,\dots,n+3} \subset \mathbb{P}^{2n+1}$  be the  $(n+2)$ -plane generated by  $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+3})$ . If

$$d_1 + d_2 < m_1 + \cdots + m_{n+3}$$

then  $\sigma_{1,n}(\pi(C)) \subset \Pi_{1,\dots,n+3}$ . By Lemma 3.3 the intersection  $X = \Pi_{1,\dots,n+3} \cap \Sigma^{1,n}$  is a degree  $n+1$  scroll in  $\mathbb{P}^{2n+1}$ , and by Proposition 3.5 the Mori cone of  $X_{n+3}^{1,n}$  is generated by the Mori cone of  $X$  and the  $h_1 - e_i$  for  $i = 1, \dots, n+3$ . Note that  $\sigma_{1,n}(\pi(C)) \subset X$ . Let us consider separately the cases  $n = 2, 3, 4$ .

If  $n = 2$  then  $X$  is a cubic scroll and the projection  $\pi_2 : X \rightarrow \mathbb{P}^2$  is the blow-down of the exceptional divisor. Hence,  $C \subset \text{Bl}_5 X \cong \text{Bl}_6 \mathbb{P}^2$ , which is a del Pezzo surface of degree three. Let  $\bar{h}$  be the pull-back of a line in  $\mathbb{P}^2$  and  $\bar{e}_i$  the classes of the exceptional divisors. The Mori cone of  $\text{Bl}_6 \mathbb{P}^2$  is described in 3.5.1. The isomorphism  $\text{Bl}_6 \mathbb{P}^2 \rightarrow \text{Bl}_5 X \subset X_5^{1,2}$  induces a map  $N_1(\text{Bl}_6(\mathbb{P}^2)) \rightarrow N_1(X_5^{1,2})$  such that

$$\bar{e}_1 \mapsto h_1, \bar{h} \mapsto h_1 + h_2, \bar{e}_i \mapsto e_{i-1}, \text{ for } i = 2, \dots, 5.$$

To conclude it is enough to show that the generators of  $\text{NE}(\text{Bl}_6 \mathbb{P}^2)$  can be written as linear combinations with non negative coefficients of the effective classes listed in the statement.



First, note that  $\bar{e}_i = e_{i-1}$  for  $i = 2, \dots, 6$ , and  $\bar{e}_1 = h_1 = (h_1 - e_i) + e_i$ . Furthermore

$$\begin{aligned} \bar{h} - \bar{e}_i - \bar{e}_j &= (h_1 - e_{i-1}) + (h_2 - e_{j-1}) \text{ for } i, j = 2, \dots, 6; \\ \bar{h} - \bar{e}_1 - \bar{e}_j &= h_2 - e_{j-1} \text{ for } j = 2, \dots, 6; \\ 2\bar{h} - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= (h_1 + 2h_2 - e_{i_1-1} - \dots - e_{i_4-1}) + (h_1 - e_{i_5-1}) \text{ for } 2 \leq i_1, \dots, i_5 \leq 6; \\ 2\bar{h} - \bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_4} &= h_1 + 2h_2 - e_{j_1-1} - \dots - e_{j_4-1}. \end{aligned}$$

If  $n = 3$  then  $X$  is a quartic scroll given by the intersection in  $\mathbb{P}^1 \times \mathbb{P}^3$  of two hypersurfaces of bidegree  $(1, 1)$

$$X = \{x_0 f_1 + x_1 f_2 = x_0 g_1 + x_1 g_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3$$

with  $f_i, g_i \in k[y_0, \dots, y_3]_1$ .

The image of the projection  $X \rightarrow \mathbb{P}^3$  is the quadric surface  $\{f_1 g_2 - f_2 g_1 = 0\} \subset \mathbb{P}^3$ . Note that since  $f_1, f_2, g_1, g_2$  do not vanish simultaneously at a point such projection is an isomorphism.

Hence, we have that  $C \subset \text{Bl}_6 X \cong \text{Bl}_6(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{Bl}_7(\mathbb{P}^2)$ , which is a del Pezzo surface of degree two. The Mori cone of  $\text{Bl}_7(\mathbb{P}^2)$  is described in 3.5.2. The inclusion  $\text{Bl}_7(\mathbb{P}^2) \rightarrow X_6^{1,3}$  maps

$$\bar{h} \mapsto h_1 + 2h_2 - e_1, \bar{e}_1 \mapsto h_2 - e_1, \bar{e}_2 \mapsto h_1 + h_2 - e_1, \bar{e}_i \mapsto e_{i-1}, \text{ for } i = 3, \dots, 7.$$

For the  $\bar{e}_i$  there is nothing to prove. We have

$$\begin{aligned} \bar{h} - \bar{e}_1 - \bar{e}_2 &= e_1; \\ \bar{h} - \bar{e}_1 - \bar{e}_i &= (h_1 - e_1) + (h_2 - e_{i-1}) + e_1; \\ \bar{h} - \bar{e}_2 - \bar{e}_i &= h_2 - e_{i-1}; \\ \bar{h} - \bar{e}_i - \bar{e}_j &= (h_1 - e_1) + (h_2 - e_{i-1}) + (h_2 - e_{j-1}); \\ 2\bar{h} - \bar{e}_1 - \bar{e}_2 - \bar{e}_{i_1} - \bar{e}_{i_2} - \bar{e}_{i_3} &= (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + (h_2 - e_{i_3-1}); \\ 2\bar{h} - \bar{e}_1 - \bar{e}_{i_1} - \bar{e}_{i_2} - \bar{e}_{i_3} - \bar{e}_{i_4} &= (h_1 - e_1) + (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + (h_2 - e_{i_3-1}) + (h_2 - e_{i_4-1}); \\ 2\bar{h} - \bar{e}_2 - \bar{e}_{i_1} - \bar{e}_{i_2} - \bar{e}_{i_3} - \bar{e}_{i_4} &= h_1 + 3h_2 - e_1 - e_{i_1-1} - e_{i_2-1} - e_{i_3-1} - e_{i_4-1}; \\ 3\bar{h} - 2\bar{e}_1 - \bar{e}_2 - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= h_1 + (h_1 + 3h_2 - e_{i_1-1} - \dots - e_{i_5-1}); \\ 3\bar{h} - \bar{e}_1 - 2\bar{e}_2 - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= h_1 + 3h_2 - e_{i_1-1} - \dots - e_{i_5-1}; \\ 3\bar{h} - \bar{e}_1 - \bar{e}_2 - 2\bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= (h_1 - e_1) + (h_2 - e_{i_1-1}) + (h_1 + 3h_2 - e_{i_1-1} - \dots - e_{i_5-1}). \end{aligned}$$

If  $n = 4$  then  $X$  is a complete intersection in  $\mathbb{P}^1 \times \mathbb{P}^4$  of three hypersurfaces of bidegree  $(1, 1)$

$$X = \{x_0 f_1 + x_1 f_2 = x_0 g_1 + x_1 g_2 = x_0 h_1 + x_1 h_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^4$$

with  $f_i, g_i, h_i \in k[y_0, \dots, y_4]_1$ . Note that  $X$  is projected onto a cubic 3-fold  $S \subset \mathbb{P}^4$ . Recall that  $S$  is isomorphic to the blow-up of  $\mathbb{P}^2$  at a point that we can assume to be  $[1 : 0 : 0]$ . The rational map

$$\begin{aligned} \tau : \quad \mathbb{P}^2 &\dashrightarrow \quad \mathbb{P}^1 \times \mathbb{P}^4 \\ [x, y, z] &\longmapsto \quad ([y, z], [xy, xz, y^2, yz, z^2]) \end{aligned}$$

yields an embedding  $\tilde{\tau} : S \rightarrow \mathbb{P}^1 \times \mathbb{P}^4$ . Denote by  $\bar{h}$  the pull-back to  $S$  of a line in  $\mathbb{P}^2$  and by  $\bar{e}_1 \subset S$  the exceptional divisor over  $[1 : 0 : 0]$ . Hence,  $X$  is isomorphic to  $\mathbb{P}^2$  blown-up at eight points that is to a del Pezzo surface of degree one whose generators of the Mori cone are listed in 3.5.3. Then  $\tilde{\tau}$  lifts to an embedding  $X \rightarrow X_7^{1,4}$  which in turn maps

$$\bar{h} \mapsto h_1 + 2h_2, \bar{e}_1 \mapsto h_1 + h_2, \bar{e}_i \mapsto e_{i-1}, \text{ for } i = 2, \dots, 8.$$

We have

$$\begin{aligned} \bar{h} - \bar{e}_1 - \bar{e}_j &= h_2 - e_{j-1}; \\ \bar{h} - \bar{e}_i - \bar{e}_j &= (h_1 - e_1) + (h_2 - e_{i-1}) + (h_2 - e_{j-1}) + e_1; \\ 2\bar{h} - \bar{e}_1 - \bar{e}_{i_1} - \dots - \bar{e}_{i_4} &= (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + (h_2 - e_{i_3-1}) + (h_2 - e_{i_4-1}); \\ 2\bar{h} - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= \sum_{k=1}^2 (h_1 - e_{i_k-1}) + \sum_{t=3}^5 (h_2 - e_{i_t-1}) + (h_2 - e_1) + e_1; \\ 3\bar{h} - 2\bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_6} &= 2(h_1 - e_{i_1-1}) + (h_1 - e_{j_1-1}) + \sum_{k=1}^6 (h_2 - e_{j_k-1}) + e_{j_1-1}; \\ 3\bar{h} - 2\bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_6} &= h_1 + 4h_2 - e_{j_1-1} - \dots - e_{j_6-1}; \\ 3\bar{h} - 2\bar{e}_i - \bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_5} &= 2(h_1 - e_{i-1}) + (h_2 - e_{j_1-1}) + \dots + (h_2 - e_{j_5-1}); \\ 4\bar{h} - 2\bar{e}_1 - 2\bar{e}_{i_1} - 2\bar{e}_{i_2} - \bar{e}_{j_1} - \dots - \bar{e}_{j_5} &= 2(h_2 - e_{i_1-1}) + (h_1 - e_{i_2-1}) + (h_1 + 4h_2 - e_{i_2-1} - \sum_{k=1}^5 e_{j_k-1}); \\ 4\bar{h} - 2\bar{e}_{i_1} - \dots - 2\bar{e}_{i_3} - \bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_4} &= \sum_{k=1}^3 2(h_2 - e_{i_k-1}) + (h_2 - e_{j_1-1}) + \sum_{t=2}^4 (h_1 - e_{j_t-1}); \\ 5\bar{h} - 2\bar{e}_1 - 2\bar{e}_{i_1} - \dots - 2\bar{e}_{i_5} - \bar{e}_{j_1} - \bar{e}_{j_2} &= (h_1 + 4h_2 - 2 \sum_{k=1}^3 e_{i_k-1}) + 2 \sum_{t=1}^2 (h_t - e_{i_{t+3}-1}) + \sum_{k=1}^2 (h_2 - e_{j_k-1}); \\ 5\bar{h} - 2\bar{e}_{i_1} - \dots - 2\bar{e}_{i_6} - \bar{e}_1 - \bar{e}_j &= \sum_{k=1}^4 2(h_2 - e_{i_k-1}) + \sum_{t=5}^6 2(h_1 - e_{i_t-1}) + (h_2 - e_{j-1}); \\ 6\bar{h} - 2\bar{e}_1 - 3\bar{e}_2 - 2\bar{e}_3 - \dots - 2\bar{e}_8 &= 2(h_1 + 4h_2 - e_3 - \dots - e_8) + (h_1 - e_2) + 2(h_2 - e_2); \\ 6\bar{h} - 3\bar{e}_1 - 2\bar{e}_2 - \dots - 2\bar{e}_8 &= 2(h_1 + 4h_2 - e_2 - \dots - e_7) + (h_1 - e_8) + (h_2 - e_8); \end{aligned}$$

concluding the proof.  $\square$

**Corollary 3.7.** *If  $r \leq n + 1$  the nef cone of  $X_r^{1,n}$  is given by*

$$\text{Nef}(X_r^{1,n}) = \langle H_1, H_2, H_1 + H_2 - E_{i_1} - \cdots - E_{i_r} \rangle$$

with  $r = 1, \dots, n + 1$ . In particular,  $\text{Nef}(X_{n+1}^{1,n})$  has  $2^{n+1} + 1$  extremal rays.

*Proof.* The claim follows by duality from Proposition 3.1.  $\square$

**Corollary 3.8.** *The nef cone of  $X_{n+2}^{1,n}$  is given by*

$$\text{Nef}(X_{n+2}^{1,n}) = \left\langle H_1, H_2, H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}, 2H_1 + H_2 - \sum_{i=1}^{n+2} E_i, nH_1 + (n+1)H_2 - n \sum_{i=1}^{n+2} E_i \right\rangle$$

with  $r = 1, \dots, n + 1$ . In particular,  $\text{Nef}(X_{n+2}^{1,n})$  has  $2^{n+2} + 2$  extremal rays.

*Proof.* The claim follows by duality from Proposition 3.2.  $\square$

**Corollary 3.9.** *Consider the following divisor classes on  $X_{n+3}^{1,n}$ :*

$$\begin{aligned} D_t &= H_1 + H_2 - E_{i_1} - \cdots - E_{i_t}; \\ D_r &= 2H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}; \\ D'_r &= nH_1 + (n+1)H_2 - nE_{i_1} - \cdots - nE_{i_r}; \\ D_{k,s} &= kH_1 + kH_2 - kE_{i_1} - \cdots - kE_{i_2} - (k-1)E_{i_{s+1}} - \cdots - (k-1)E_{i_{n+3}}. \end{aligned}$$

If  $n \leq 4$  then the nef cone of  $X_{n+3}^{1,n}$  is given by

$$\text{Nef}(X_{n+3}^{1,n}) = \langle H_1, H_2, D_t, D_r, D'_r, D_{k,s} \rangle$$

with  $t = 1, \dots, n + 1$ ,  $r \in \{n + 2, n + 3\}$ ,  $k = 2, \dots, n + 1$  and  $s = n - k + 2$ . In particular, if  $n \leq 4$  then  $\text{Nef}(X_{n+3}^{1,n})$  has  $2^{n+4} - \frac{(n+3)(n+2)}{2}$  extremal rays.

*Proof.* The claim follows by duality from Proposition 3.6.  $\square$

#### 4. THE COX RING OF $X_{n+1}^{1,n}$

The  $n$ -dimensional complex torus  $T = (\mathbb{C}^*)^n$  acts on  $\mathbb{P}^1 \times \mathbb{P}^n$  as follows:

$$(4.1) \quad \begin{array}{ccc} T \times (\mathbb{P}^1 \times \mathbb{P}^n) & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^n \\ ((t_1, \dots, t_n), ([x_0, x_1], [y_0, \dots, y_n])) & \longmapsto & ([x_0, x_1], [y_0, t_1 y_1, t_2 y_2, \dots, t_n y_n]). \end{array}$$

Let us write the  $n + 1$  general points  $p_1, \dots, p_{n+1} \in \mathbb{P}^1 \times \mathbb{P}^n$  as follows:

$$\begin{aligned} p_1 &= ([\alpha_1, \beta_1], [1, 0, \dots, 0]); \\ p_2 &= ([\alpha_2, \beta_2], [0, 1, 0, \dots, 0]); \\ p_3 &= ([\alpha_3, \beta_3], [0, 0, 1, 0, \dots, 0]); \\ p_4 &= ([\alpha_4, \beta_4], [0, 0, 0, 1, 0, \dots, 0]); \\ &\vdots \\ p_{n+1} &= ([\alpha_{n+1}, \beta_{n+1}], [0, \dots, 0, 1]); \end{aligned}$$

where  $[\alpha_1, \beta_1] = [0, 1]$ ,  $[\alpha_2, \beta_2] = [1, 0]$  and  $[\alpha_3, \beta_3] = [1, 1]$ . Note that (4.1) lifts to an action of  $T$  on  $X_{n+1}^{1,n}$  whose orbits of maximal dimension have codimension one in  $X_{n+1}^{1,n}$ . Hence,  $X_{n+1}^{1,n}$  has complexity one and Theorem 2.5 implies that  $X_{n+1}^{1,n}$  is a Mori dream space. We begin by proving the following stronger statement:

**Proposition 4.2.** *The variety  $X_r^{1,n}$  is log Fano for  $r \leq n + 1$ .*

*Proof.* Consider the case  $r = n + 1$ . The result for  $r < n + 1$  will then follow from [PS09, Theorem 2.9]. The claim amounts to find an effective  $\mathbb{Q}$ -divisor  $\Delta$  on  $X_{n+1}^{1,n}$  such that  $-K_{X_{n+1}^{1,n}} - \Delta$  is ample, and the pair  $(X_{n+1}^{1,n}, \Delta)$  is Kawamata log terminal. Recall that

$$-K_{X_{n+1}^{1,n}} = 2H_1 + (n+1)H_2 - nE_1 - \cdots - nE_{n+1}$$

and consider the divisor

$$D = (n+1)H_2 - nE_1 - \cdots - nE_{n+1}.$$

The divisor  $D$  is effective since it is the pull-back of the effective divisor on  $\mathbb{P}^n$  consisting in the union of the  $(n+1)$  hyperplanes passing through  $n$  among the  $n+1$  projections of the blown-up points. Set  $\Delta_\epsilon = \epsilon D$  with  $\epsilon \in \mathbb{Q}_{>0}$ , and consider the divisor

$$-K_{X_{n+1}^{1,n}} - \Delta_\epsilon = 2H_1 + ((n+1) - \epsilon(n+1))H_2 + \sum_{i=1}^{n+1} (\epsilon n - n)E_i.$$

By Proposition 3.1 we have that  $\text{NE}(X_{n+1}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle$ . Since

$$\begin{aligned} (-K_{X_{n+1}^{1,n}} - \Delta_\epsilon) \cdot e_i &= n - \epsilon n > 0 && \text{if and only if } \epsilon < 1; \\ (-K_{X_{n+1}^{1,n}} - \Delta_\epsilon) \cdot (h_1 - e_i) &= 2 - n + \epsilon n > 0 && \text{if and only if } \epsilon > (n-2)/n; \\ (-K_{X_{n+1}^{1,n}} - \Delta_\epsilon) \cdot (h_2 - e_i) &= n+1 - \epsilon(n+1) - n + \epsilon n > 0 && \text{if and only if } \epsilon < 1; \end{aligned}$$

and hence the divisor  $-K_{X_{n+1}^{1,n}} - \Delta_\epsilon$  is ample if and only if  $(n-2)/n < \epsilon < 1$ .

Next, we show that the pair  $(X_{n+1}^{1,n}, \Delta_\epsilon)$  is Kawamata log terminal. Consider the images  $q_i = \pi_2(p_i)$  of the blown-up points via the second projection. There exist  $n+1$  hyperplanes in  $\mathbb{P}^n$  through each subset of  $n$  points among the  $q_i$ , and through each  $q_i$  there pass  $n$  of them. Let us denote by  $\tilde{H}_i \subset X_{n+1}^{1,n}$  the strict transforms of the inverse images via the second projection of these hyperplanes. The divisor  $D$  has multiplicity  $n$  along the strict transforms of the curves  $\pi_2^{-1}(q_i)$  for all  $i$ . Hence, any  $n$  among the divisors  $\tilde{H}_i$  intersect transversally along one of the strict transforms of these curves.

Generalizing this argument, fix a set of  $m$  points among the  $q_i$ , with  $1 \leq m \leq (n-1)$ , and denote by  $\Lambda_m$  their linear span. The divisor  $D$  has multiplicity  $n+1-m$  along the strict transform  $\tilde{\Lambda}_m$  of  $\pi_2^{-1}(\Lambda_m)$ . Any  $n+1-m$  among the  $\tilde{H}_i$  intersect transversally along a subvariety of type  $\tilde{\Lambda}_m \subset X_{n+1}^{1,n}$ , and hence  $\Delta_\epsilon$  is a simple normal crossing effective divisor for all  $\epsilon > 0$ . Summing-up, we proved that for all  $(n-2)/n < \epsilon < 1$  the divisor  $-K_{X_{n+1}^{1,n}} - \Delta_\epsilon$  is ample and the pair  $(X_{n+1}^{1,n}, \Delta_\epsilon)$  is Kawamata log terminal concluding the proof.  $\square$

Let  $\tilde{H}_i \subset X_{n+1}^{1,n}$  be the strict transform of  $\pi_2^{-1}(\{y_i = 0\})$ . For  $p \in \mathbb{P}^1 \setminus \{\pi_1(p_1), \dots, \pi_1(p_{n+1})\}$  the fiber  $F_p = \tilde{\pi}_1^{-1}(p)$  is isomorphic to  $\mathbb{P}^n$  while  $F_{p_i} = \tilde{\pi}_1^{-1}(p_i)$  is the union of two irreducible components  $D_{i,1} \cup D_{i,2}$  where  $D_{i,1}$  is isomorphic to  $\mathbb{P}^n$  blown-up in a point with exceptional divisor  $\overline{E}_i \cong \mathbb{P}^{n-1}$ ,  $D_{i,2} = E_i$  and  $D_{i,1} \cap D_{i,2} = \overline{E}_i$ . Recall that  $E_i \cong \mathbb{P}^n$  and denote by  $R_i \subset E_i$  the union of the  $T$ -invariant divisors of the lifting of (4.1) restricted to  $E_i$ . Then

$$X_0 = X_{n+1}^{1,n} \setminus \left( \bigcup_{i=0}^n \tilde{H}_i \cup \bigcup_{j=1}^{n+1} R_j \right) \subset X_{n+1}^{1,n}$$

is the  $T$ -invariant open subset of points of  $X_{n+1}^{1,n}$  having isotropy group of dimension zero. Hence, the  $\tilde{H}_i$  are the  $T$ -invariant prime divisors supported in  $X_{n+1}^{1,n} \setminus X_0$ . Moreover, note that the isotropy group of a general point of both  $D_{i,1}, D_{i,2}$ , with respect to the lifting of (4.1), is trivial. In particular, it is finite of order one.

Now, associate to the  $\tilde{H}_i$  variables  $S_i$ , to the  $D_{i,1}$  variables  $T_{i,1}$  and to the  $D_{i,2}$  variables  $T_{i,2}$ . For  $1 \leq i \leq n-1$  set  $k = j+1 = i+2$  and

$$g_i = (\beta_k \alpha_j - \beta_j \alpha_k) T_{i,1} T_{i,2} + (\beta_i \alpha_k - \beta_k \alpha_i) T_{j,1} T_{j,2} + (\beta_j \alpha_i - \beta_i \alpha_j) T_{k,1} T_{k,2}.$$

Then we have the following result:

**Theorem 4.3.** *For the Cox ring of  $X_{n+1}^{1,n}$  we have the following explicit presentation*

$$\text{Cox}(X_{n+1}^{1,n}) \cong \frac{\mathbb{C}[S_0, \dots, S_n, T_{1,1}, T_{1,2}, \dots, T_{n+1,1}, T_{n+1,2}]}{\langle g_1, \dots, g_{n-1} \rangle}$$

where  $S_i$  is the section associated to  $H_2 - E_1 - \dots - E_{i-1} - E_{i+1} - \dots - E_{n+1}$ ,  $T_{i,1}$  is the section associated to  $H_1 - E_i$  and  $T_{i,2}$  is the section associated to  $E_i$ .

*Proof.* In the previous set up it is enough to apply Theorem 2.5.  $\square$

In the rest of this section we will derive some of consequences of Theorem 4.3.

**Proposition 4.4.** *Consider the following divisor classes on  $X_{n+1}^{1,n}$ :*

$$\begin{aligned} D_1 &= H_1; \\ D_h &= H_2 - E_{i_1} - \dots - E_{i_h}; \\ D_{i_1, \dots, i_n} &= H_1 + H_2 - E_{i_1} - \dots - E_{i_n}; \\ D_{1, \dots, n+1} &= H_1 + H_2 - E_1 - \dots - E_{n+1}; \\ D_k &= kH_2 - kE_{i_1} - \dots - kE_{i_{(n-k)}} - (k-1)E_{i_{(n-k+1)}} - \dots - (k-1)E_{i_{n+1}}; \\ D_n &= nH_2 - (n-1)E_1 - \dots - (n-1)E_{(n+1)}. \end{aligned}$$

The movable cone of  $X_{n+1}^{1,n}$  is given by

$$\text{Mov}(X_{n+1}^{1,n}) = \langle D_1, D_h, D_{i_1, \dots, i_n}, D_{1, \dots, n+1}, D_k, D_n \rangle$$

for  $2 \leq k \leq n-1$ ,  $0 \leq h \leq n-1$  and  $\{i_1, \dots, i_n\} \subset \{1, \dots, n+1\}$ .

Furthermore, let  $\text{Mov}_1(X_{n+1}^{1,n})$  be the cone of curves covering a divisor in  $X_{n+1}^{1,n}$ . Then

$$\text{Mov}_1(X_{n+1}^{1,n}) = \langle h_1, h_2 - e_i, h_1 + (n-1)h_2 - e_{i_1} - \dots - e_{i_n}, e_i \rangle$$

for  $1 \leq i \leq n+1$ ,  $\{i_1, \dots, i_n\} \subset \{1, \dots, n+1\}$ .

*Proof.* Set  $\mathcal{C} = \langle D_1, D_h, D_{i_1, \dots, i_n}, D_{1, \dots, n+1}, D_k, D_n \rangle$ . Since any divisor appearing in  $\mathcal{C}$  is movable we have that  $\mathcal{C} \subset \text{Mov}(X_{n+1}^{1,n})$ . Then taking the dual cones we get  $\text{Mov}(X_{n+1}^{1,n})^* \subset \mathcal{C}^*$ .

Now,  $\text{Mov}_1(X_{n+1}^{1,n}) \subset \text{Mov}(X_{n+1}^{1,n})^* \subset \mathcal{C}^*$ . If we prove that any extremal ray of  $\mathcal{C}^*$  corresponds to a curve that covers a divisor in  $X_{n+1}^{1,n}$  we would have that  $\text{Mov}_1(X_{n+1}^{1,n}) = \text{Mov}(X_{n+1}^{1,n})^* = \mathcal{C}^*$  and hence  $\text{Mov}(X_{n+1}^{1,n}) = \mathcal{C}$  concluding the proof.

A straightforward computation shows that

$$\mathcal{C} = \langle h_1, h_2 - e_i, h_1 + (n-1)h_2 - e_{i_1} - \dots - e_{i_n}, e_i \rangle$$

for  $1 \leq i \leq n+1$ ,  $\{i_1, \dots, i_n\} \subset \{1, \dots, n+1\}$ . Clearly, the curves with classes  $h_1, h_2 - e_i, e_i$  cover divisors in  $X_{n+1}^{1,n}$ . Now, consider the projections in  $\mathbb{P}^n$  of the blown-up points  $p_{i_1}, \dots, p_{i_n}$  and let  $H \subset \mathbb{P}^n$  be the hyperplane spanned by them. Let  $X_n^{1,n-1} \subset X_{n+1}^{1,n}$  be the strict transform of  $\mathbb{P}^1 \times H \subset \mathbb{P}^1 \times \mathbb{P}^n$ . To conclude it is enough to note that the curves of class  $h_1 + (n-1)h_2 - e_{i_1} - \dots - e_{i_n}$  cover  $X_n^{1,n-1}$ .  $\square$

**Remark 4.5.** Let  $X$  be a  $\mathbb{Q}$ -factorial projective variety. In general  $\text{Mov}_1(X) \subsetneq \text{Mov}(X)^*$ . However, by Proposition 3.1 we have that

$$\text{Mov}_1(X_{n+1}^{1,n}) = \text{Mov}(X_{n+1}^{1,n})^*.$$

For further information on the behavior of these cones for general projective varieties we refer to [Pay06], [BDPS23].

**Remark 4.6.** We developed Magma [BCP97] scripts computing the nef cones of  $X_{n+1}^{1,n}$ ,  $X_{n+2}^{1,n}$  and of  $X_{n+3}^{1,n}$  for  $n \leq 4$ , as well as the Mori chamber decomposition of  $X_{n+1}^{1,n}$ . A Magma library containing the scripts can be downloaded at the following link:

<https://github.com/msslxa/Cox-rings-of-blow-ups-of-multiprojective-spaces>

We managed to compute the Mori chamber decomposition of  $X_{n+1}^{1,n}$  for  $n = 2, 3, 4$ , and we got 92, 550 and 6307 chambers respectively. Finally, we would like to mention that the Mori chamber decomposition of  $X_6^{1,2}$  has been fully computed by T. Grange in [Gra22, Chapter 3].

## REFERENCES

- [AC17] C. Araujo and C. Casagrande, *On the Fano variety of linear spaces contained in two odd-dimensional quadrics*, *Geom. Topol.* **21** (2017), no. 5, 3009–3045. MR 3687113
- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, *J. Amer. Math. Soc.* **23** (2010), no. 2, 405–468. MR 2601039
- [BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, *J. Symbolic Comput.* **24** (1997), no. 3-4, 235–265, *Computational algebra and number theory* (London, 1993). MR MR1484478
- [BDPS23] M. C. Brambilla, O. Dumitrescu, E. Postinghel, and L. J. Santana Sánchez, *Duality and polyhedrality of cones for mori dream spaces*, <https://arxiv.org/abs/2305.18536>, 2023.
- [CT06] A-M. Castravet and J. Tevelev, *Hilbert’s 14th problem and Cox rings*, *Compos. Math.* **142** (2006), no. 6, 1479–1498. MR 2278756
- [Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001. MR 1841091
- [GOST15] Y. Gongyo, S. Okawa, A. Sannai, and S. Takagi, *Characterization of varieties of Fano type via singularities of Cox rings*, *J. Algebraic Geom.* **24** (2015), no. 1, 159–182. MR 3275656
- [GPPS23] T. Grange, E. Postinghel, and A. Prendergast-Smith, *Log Fano blowups of mixed products of projective spaces and their effective cones*, *Rev. Mat. Complut.* **36** (2023), no. 2, 393–442. MR 4581756
- [Gra22] T. Grange, *Birational geometry and positivity of blow-ups of products of projective spaces*, 10.26174/thesis.lboro.19103108.v1 (2022).
- [HK00] Y. Hu and S. Keel, *Mori dream spaces and GIT*, *Michigan Math. J.* **48** (2000), 331–348, Dedicated to William Fulton on the occasion of his 60th birthday. MR 1786494
- [HS10] Jürgen Hausen and Hendrik Süß, *The Cox ring of an algebraic variety with torus action*, *Adv. Math.* **225** (2010), no. 2, 977–1012. MR 2671185
- [LP17] J. Lesieutre and J. Park, *Log Fano structures and Cox rings of blow-ups of products of projective spaces*, *Proc. Amer. Math. Soc.* **145** (2017), no. 10, 4201–4209. MR 3690606
- [Muk01] S. Mukai, *Counterexample to hilbert’s fourteenth problem for the 3-dimensional additive group*, Technical report, Kyoto University, Research Institute for Mathematical Sciences, 2001.
- [Oka16] S. Okawa, *On images of Mori dream spaces*, *Math. Ann.* **364** (2016), no. 3-4, 1315–1342. MR 3466868
- [Pay06] S. Payne, *Stable base loci, movable curves, and small modifications, for toric varieties*, *Math. Z.* **253** (2006), no. 2, 421–431. MR 2218709
- [PS09] Y. G. Prokhorov and V. V. Shokurov, *Towards the second main theorem on complements*, *J. Algebraic Geom.* **18** (2009), no. 1, 151–199. MR 2448282

MICHELE BOLOGNESI, IMAG - UNIVERSITÉ DE MONTPELLIER, CNRS, PLACE EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 5, FRANCE

*Email address:* `michele.bolognesi@umontpellier.fr`

ALEX MASSARENTI, DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI FERRARA, VIA MACHIAVELLI 30, 44121 FERRARA, ITALY

*Email address:* `msslxa@unife.it`

ELENA POMA, IMAG - UNIVERSITÉ DE MONTPELLIER, PLACE EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 5, FRANCE AND DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI FERRARA, VIA MACHIAVELLI 30, 44121 FERRARA, ITALY

*Email address:* `elena.poma@umontpellier.fr`, `elena.poma@unife.it`