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COX RINGS OF BLOW-UPS OF MULTIPROJECTIVE SPACES

MICHELE BOLOGNESI, ALEX MASSARENTI, AND ELENA POMA

ABSTRACT. Let $X_r^{1,n}$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^n$ in r general points. We describe the Mori cone of $X_r^{1,n}$ for $r \leq n+2$ and for $r = n+3$ when $n \leq 4$. Furthermore, we prove that $X_{n+1}^{1,n}$ is log Fano and give an explicit presentation for its Cox ring.

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1. INTRODUCTION

Mori dream spaces, introduced by Y. Hu and S. Keel in [HK00], are varieties whose total coordinate ring, called the *Cox ring*, is finitely generated. The birational geometry of a Mori dream space is encoded in its cone of effective divisors together with a chamber decomposition on it, called *Mori chamber decomposition* whose chambers correspond to the nef cones of the birational models of the Mori dream space.

Finite generation of the Cox rings of blow-ups of projective spaces and products of projective spaces has been extensively investigated [Muk01], [CT06], [AC17], [LP17], [GPPS23]. In [CT06, Theorem 1.3] A. M. Castravet and J. Tevelev proved that the blow-up of $(\mathbb{P}^n)^s$ in r general points is a Mori dream space if and only if

$$\frac{1}{s+1} + \frac{1}{r-n-1} + \frac{1}{n+1} > 1.$$

Indeed, when the above inequality is not satisfied the effective cones of these blow-ups are not finitely generated and the proof of this last fact relies on the symmetries of their Picard groups which carry a natural Weyl group action. For products of projective spaces with unbalanced dimensions this is not the case. In this paper we push a little further the investigation of the birational geometry of these varieties initiated by T. Grange, E. Postingshel and A. Prendergast-Smith in [GPPS23].

Let $X_r^{1,n}$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^n$ in r general points $p_1, \dots, p_r \in \mathbb{P}^1 \times \mathbb{P}^n$. We will denote by $\pi : X_r^{1,n} \rightarrow \mathbb{P}^1 \times \mathbb{P}^n$ the blow-down morphism and by $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^n \rightarrow \mathbb{P}^1$, $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ the projections onto the factors. Moreover, let us denote by $\tilde{\pi}_1, \tilde{\pi}_2$ the morphisms from $X_r^{1,n}$ to \mathbb{P}^1 and

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\mathbb{P}^n induced by the projections. We summarize the situation in the following diagram

$$\begin{array}{ccc}
 & X_r^{1,n} & \\
 \tilde{\pi}_1 \swarrow & \downarrow \pi & \searrow \tilde{\pi}_2 \\
 & \mathbb{P}^1 \times \mathbb{P}^n & \\
 \pi_1 \swarrow & & \searrow \pi_2 \\
 \mathbb{P}^1 & & \mathbb{P}^n
 \end{array}$$

Let $\text{Pic}(X_r^{1,n})$ denote the Picard group of $X_r^{1,n}$ and $N_1(X_r^{1,n})$ the \mathbb{R} -vector space of 1-dimensional cycles modulo numerical equivalence. Throughout the paper we will denote by:

- H_1 the pull-back of a point of \mathbb{P}^1 via $\tilde{\pi}_1$;
- H_2 the pull-back of a hyperplane in \mathbb{P}^n via $\tilde{\pi}_2$;
- E_i the exceptional divisor over p_i for $i = 1, \dots, r$;

and by

- h_1 the class of a general fiber of $\tilde{\pi}_2$;
- h_2 the class of a line in a general fiber of $\tilde{\pi}_1$;
- e_i the class of a line in the exceptional divisor E_i for $i = 1, \dots, r$.

We have that $\text{Pic}(X_r^{1,n}) = \mathbb{Z}[H_1, H_2, E_1, \dots, E_r]$ and $N_1(X_r^{1,n}) = \mathbb{Z}[h_1, h_2, e_1, \dots, e_r]$.

Our main results on the Mori cone of $X_r^{1,n}$ in Propositions 3.1, 3.2, 3.6 can be summarized as follows:

Theorem 1.1. *The Mori cone of $X_r^{1,n}$ is given by*

$$\text{NE}(X_r^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle$$

for all $r \leq n + 1$. Furthermore,

$$\text{NE}(X_{n+2}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + nh_2 - e_1 - \dots - e_{n+2} \rangle$$

and if $n \leq 4$ then

$$\text{NE}(X_{n+3}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + nh_2 - e_{i_1} - \dots - e_{i_{n+2}} \rangle$$

for $i, i_1, \dots, i_{n+2} \in \{1, \dots, n + 3\}$.

Recall that a normal and \mathbb{Q} -factorial projective variety X is *log Fano* if there exists an effective divisor $D \subset X$ such that $-(K_X + D)$ is ample and the pair (X, D) is Kawamata log terminal. By [BCHM10, Corollary 1.3.2] log Fano varieties are Mori dream spaces. The converse does not hold in general, and there are several criteria for a Mori dream space to be log Fano [GOST15].

Our main results in the case $r \leq n + 1$ in Proposition 4.2 and Theorem 4.3 can be summarized as follows:

Theorem 1.2. *For $r \leq n + 1$ the variety $X_r^{1,n}$ is log Fano and hence a Mori dream space. Furthermore, the Cox ring of $X_{n+1}^{1,n}$ is given by*

$$\text{Cox}(X_{n+1}^{1,n}) \cong \frac{\mathbb{C}[S_0, \dots, S_n, T_{1,1}, T_{1,2}, \dots, T_{n+1,1}, T_{n+1,2}]}{\langle g_1, \dots, g_{n-1} \rangle}$$

where S_i is the section associated to $H_2 - E_1 - \dots - E_{i-1} - E_{i+1} - \dots - E_{n+1}$, $T_{i,1}$ is the section associated to $H_1 - E_i$ and $T_{i,2}$ is the section associated to E_i , and

$$g_i = (\beta_k \alpha_j - \beta_j \alpha_k) T_{i,1} T_{i,2} + (\beta_i \alpha_k - \beta_k \alpha_i) T_{j,1} T_{j,2} + (\beta_j \alpha_i - \beta_i \alpha_j) T_{k,1} T_{k,2}$$

where for $1 \leq i \leq n - 1$ we set $k = j + 1 = i + 2$.

Furthermore, in Section 3 we explicitly describe the nef cones of $X_r^{1,n}$ for $r \leq n + 2$, and of $X_{n+3}^{1,n}$ for $n \leq 4$, and in Section 4 we give generators for the cones of movable divisors and of moving curves of $X_{n+1}^{1,n}$.

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2. COX RINGS, MORI DREAM SPACES AND LOG FANO VARIETIES

Let X be a normal projective \mathbb{Q} -factorial variety. We denote by $N^1(X)$ the real vector space of \mathbb{R} -Cartier divisors modulo numerical equivalence. The *nef cone* of X is the closed convex cone $\text{Nef}(X) \subset N^1(X)$ generated by classes of nef divisors.

The stable base locus $\mathbf{B}(D)$ of a \mathbb{Q} -divisor D is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers s such that sD is integral

$$(2.1) \quad \mathbf{B}(D) = \bigcap_{s>0} B(sD).$$

The *movable cone* of X is the convex cone $\text{Mov}(X) \subset N^1(X)$ generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in X . The *effective cone* of X is the convex cone $\text{Eff}(X) \subset N^1(X)$ generated by classes of *effective divisors*. We have inclusions $\text{Nef}(X) \subset \overline{\text{Mov}(X)} \subset \overline{\text{Eff}(X)}$. We refer to [Deb01, Chapter 1] for a comprehensive treatment of these topics.

We say that a birational map $f : X \dashrightarrow X'$ to a normal projective variety X' is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small \mathbb{Q} -factorial modification* if X' is \mathbb{Q} -factorial and f is an isomorphism in codimension one. If $f : X \dashrightarrow X'$ is a small \mathbb{Q} -factorial modification then the natural pullback map $f^* : N^1(X') \rightarrow N^1(X)$ sends $\text{Mov}(X')$ and $\overline{\text{Eff}(X')}$ isomorphically onto $\text{Mov}(X)$ and $\overline{\text{Eff}(X)}$ respectively. In particular, we have $f^*(\text{Nef}(X')) \subset \overline{\text{Mov}(X)}$.

Now, assume that the divisor class group $\text{Cl}(X)$ is free and finitely generated, and fix a subgroup G of the group of Weil divisors on X such that the canonical map $G \rightarrow \text{Cl}(X)$, mapping a divisor $D \in G$ to its class $[D]$, is an isomorphism. The *Cox ring* of X is defined as

$$\text{Cox}(X) = \bigoplus_{[D] \in \text{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

where $D \in G$ represents $[D] \in \text{Cl}(X)$, and the multiplication in $\text{Cox}(X)$ is defined by the standard multiplication of homogeneous sections in the field of rational functions on X .

Definition 2.2. A normal projective \mathbb{Q} -factorial variety X is called a *Mori dream space* if the following conditions hold:

- $\text{Pic}(X)$ is finitely generated, or equivalently $h^1(X, \mathcal{O}_X) = 0$,
- $\text{Nef}(X)$ is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small \mathbb{Q} -factorial modifications $f_i : X \dashrightarrow X_i$, such that each X_i satisfies the second condition above, and $\text{Mov}(X) = \bigcup_i f_i^*(\text{Nef}(X_i))$.

The collection of all faces of all cones $f_i^*(\text{Nef}(X_i))$ above forms a fan which is supported on $\text{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\text{Nef}(X_i))$ and $f_j^*(\text{Nef}(X_j))$, meet along a facet, then there exist a normal projective variety Y , a small modification $\varphi : X_i \dashrightarrow X_j$, and $h_i : X_i \rightarrow Y$, $h_j : X_j \rightarrow Y$ small birational morphisms of relative Picard number one such that $h_j \circ \varphi = h_i$. The fan structure on $\text{Mov}(X)$ can be extended to a fan supported on $\text{Eff}(X)$ as follows.

Definition 2.3. Let X be a Mori dream space. We describe a fan structure on the effective cone $\text{Eff}(X)$, called the *Mori chamber decomposition*. There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i : X \dashrightarrow Y_i$. The set $\text{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones \mathcal{C} of the Mori chamber decomposition of $\text{Eff}(X)$ are of the form $\mathcal{C}_i = \langle g_i^*(\text{Nef}(Y_i)), \text{Exc}(g_i) \rangle$. We call \mathcal{C}_i or its interior \mathcal{C}_i° a *maximal chamber* of $\text{Eff}(X)$. We refer to [HK00, Proposition 1.11] and [Oka16, Section 2.2] for details.

Definition 2.4. A normal and \mathbb{Q} -factorial projective variety X is *log Fano* if there exists an effective divisor $D \subset X$ such that $-(K_X + D)$ is ample and the pair (X, D) is Kawamata log terminal.

By [BCHM10, Corollary 1.3.2] if X is log Fano then it is a Mori dream space.

2.4. Varieties with a torus action. Let X be a normal projective variety and $T \times X \rightarrow X$ an effective algebraic torus action on X of complexity one that is such that its biggest T -orbits are of codimension one in X .

We will follow the treatment in [HS10]. For a given point $x \in X$, denote by $T_x \subset T$ its isotropy group

$$T_x = \{t \in T \mid t \cdot x = x\}$$

and consider the non-empty T -invariant open subset

$$X_0 = \{x \in X \mid \dim(T_x) = 0\} \subset X$$

of points in x with zero-dimensional isotropy group. There is a geometric quotient

$$q : X_0 \rightarrow X_0/T$$

where the orbit space X_0/T is of dimension one and has a separation that is a rational map $\pi : X_0/T \dashrightarrow \mathbb{P}^1$ which is a local isomorphism in codimension one.

Denote by $E_1, \dots, E_m \subseteq X$ the T -invariant prime divisors supported in $X \setminus X_0$ and by $D_1, \dots, D_n \subseteq X$ those T -invariant prime divisors who have a finite generic isotropy group of order $l_j > 1$. Moreover, let 1_{E_k} and 1_{D_j} denote the canonical sections of the divisors E_k and D_j respectively, and let $1_{q(D_j)}$ be the canonical section of $q(D_j)$.

Choose $r \geq 1$ and $a_0, \dots, a_r \in \mathbb{P}^1$, such that π is an isomorphism over $\mathbb{P}^1 \setminus \{a_0, \dots, a_r\}$ and all the divisors D_j occur among the

$$D_{i,j} = q^{-1}(y_{i,j})$$

where $\pi^{-1}(a_i) = \{y_{i,1}, \dots, y_{i,n_i}\}$. Let $l_{i,j} \in \mathbb{Z}_{\geq 1}$ denote the order of the generic isotropy group of $D_{i,j}$. For every $0 \leq i \leq r$ define a monomial

$$f_i = T_{i,1}^{l_{i,1}} \cdots T_{i,n_i}^{l_{i,n_i}} \in \mathbb{C}[T_{i,j}; 0 \leq i \leq r, 1 \leq j \leq n_i].$$

Moreover, write $a_i = [b_i, c_i]$ with $b_i, c_i \in \mathbb{C}$ and for every $0 \leq i \leq r-2$ set $k = j+1 = i+2$ and define a trinomial

$$g_i = (c_k b_j - c_j b_k) f_i + (c_i b_k - c_k b_i) f_j + (c_j b_i - c_i b_j) f_k.$$

In the previous notation we have the following:

Theorem 2.5. [HS10, Theorem 1.3] *Let $T \times X \rightarrow X$ be an algebraic torus action of complexity one. Then, in terms of the data defined above, the Cox ring of X is given as*

$$\text{Cox}(X) = \mathbb{C}[S_1, \dots, S_m, T_{i,j}; 0 \leq i \leq r, 1 \leq j \leq n_i] / \langle g_i; 0 \leq i \leq r-2 \rangle$$

where 1_{E_k} corresponds to S_k and $1_{D_{i,j}}$ to $T_{i,j}$, and the $\text{Cl}(X)$ -grading on the right hand side is defined by associating to S_k the class of E_k and to $T_{i,j}$ the class of $D_{i,j}$. In particular, $\text{Cox}(X)$ is finitely generated.

Theorem 2.5 will be crucial in the study of the Cox ring of $X_{n+1}^{1,n}$.

3. MORI CONES

In this section we compute the cone of effective curves of $X_r^{1,n}$ for $r \leq n+2$, and for $r = n+3$ when $n \leq 4$.

Proposition 3.1. *The Mori cone of $X_r^{1,n}$ is given by*

$$\text{NE}(X_r^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle$$

for all $r \leq n+1$.

Proof. It is enough to prove the claim for $r = n + 1$. The case $n = 1$ is well-known. We will assume that the claim is proved for $X_n^{1,n-1}$ and prove it for $X_{n+1}^{1,n}$.

Let $C \subset X_{n+1}^{1,n}$ be an irreducible curve. If C is contracted by π then $C \subset E_i$ for some $i = 1, \dots, n+1$ and hence C is a multiple of e_i . Assume that $\pi(C)$ is a curve of bidegree (d_1, d_2) and such that $\text{mult}_{p_i} C = m_i$. Then we may write

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \dots - m_n e_n - m_{n+1} e_{n+1}.$$

Note that $C \cdot (H_1 - E_i) = d_1 - m_i$. If $d_1 - m_i < 0$ then C is contained in the strict transform of the fiber of π_1 passing through p_i . Such fiber is \mathbb{P}^n blown-up in a point and hence C may be written as a linear combination with non negative coefficients of e_i and $h_2 - e_i$. Hence, we may assume that $d_1 - m_i \geq 0$ for all $i = 1, \dots, n+1$.

Now, consider the projection $\tilde{\pi}_2(C)$. If $\tilde{\pi}_2(C)$ is a point then C is a linear combination with non negative coefficients of $h_1 - e_i$ and e_i for some $i = 1, \dots, n+1$. Assume that $\tilde{\pi}_2(C)$ is a curve. Then this curve has degree d_2 and $\text{mult}_{\pi_2(p_i)} \tilde{\pi}_2(C) = m_i$. Let $\Pi_{1,\dots,n} \subset \mathbb{P}^n$ be the hyperplane generated by p_1, \dots, p_n . If

$$m_1 + \dots + m_n > d_2$$

then $\tilde{\pi}_2(C) \subset \Pi_{1,\dots,n}$ and hence C is contained in the strict transform of $\mathbb{P}^1 \times \Pi_{1,\dots,n}$ which is isomorphic to $X_n^{1,n-1}$. Therefore, we can conclude by induction on n . On the other hand, if

$$m_1 + \dots + m_n \leq d_2$$

we may write

$$C \sim \sum_{i=1}^n m_i (h_2 - e_i) + m_{n+1} (h_1 - e_{n+1}) + (d_1 - m_{n+1}) h_1 + (d_2 - m_1 - \dots - m_n) h_2$$

where all the coefficients are non negative, concluding the proof. \square

Consider the Segre embedding

$$\begin{aligned} \sigma_{1,n} : \quad \mathbb{P}^1 \times \mathbb{P}^n &\longrightarrow \mathbb{P}^{2n+1} \\ ([u_0, u_1], [v_0, \dots, v_n]) &\mapsto [u_0 v_0, u_0 v_1, \dots, u_1 v_n] \end{aligned}$$

and set $\Sigma^{1,n} = \sigma_{1,n}(\mathbb{P}^1 \times \mathbb{P}^n)$. Recall that $\deg(\Sigma^{1,n}) = n + 1$.

Proposition 3.2. *The Mori cone of $X_{n+2}^{1,n}$ is given by*

$$\text{NE}(X_{n+2}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + n h_2 - e_1 - \dots - e_{n+2} \rangle.$$

Proof. The case $n = 1$ is well-known. Let C be an irreducible curve in $X_{n+2}^{1,n}$. If C is contracted by π then $C \subset E_i$ for some $i = 1, \dots, n+2$ and hence C is a multiple of e_i . Assume that $\pi(C)$ is a curve of bidegree (d_1, d_2) and such that $\text{mult}_{p_i} C = m_i$. Then we may write

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \dots - m_{n+1} e_{n+1} - m_{n+2} e_{n+2}.$$

If

$$d_1 + d_2 \geq m_1 + \dots + m_{n+2}$$

then C may be written as a linear combination with non negative coefficients of e_i , $h_2 - e_i$ and $h_1 - e_i$. Recall that $\sigma_{1,n}(\pi(C))$ is a curve of degree $d_1 + d_2$ passing through the points $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$ with multiplicities m_i .

Let us denote by $\Pi_{1,\dots,n+2} \subset \mathbb{P}^{2n+1}$ the $(n+1)$ -plane spanned by $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$. If

$$d_1 + d_2 < m_1 + \dots + m_{n+2}$$

then $\sigma_{1,n}(\pi(C)) \subset \Pi_{1,\dots,n+2}$. Since $\Sigma^{1,n}$ has degree $n + 1$, the intersection $\Pi_{1,\dots,n+2} \cap \Sigma^{1,n}$ is a rational normal curve of degree $n + 1$ and with class $h_1 + n h_2 - e_1 - \dots - e_{n+2}$. \square

Let $a, b \geq 1$ be two integers, fix two complementary subspace $\Lambda^a, \Lambda^b \subset \mathbb{P}^{a+b+1}$, two rational normal curves $C_a \subset \Lambda^a, C_b \subset \Lambda^b$, and an isomorphism $\phi : C_a \rightarrow C_b$. The surface

$$S_{(a,b)} = \bigcup_{x \in C_a} \langle x, \phi(x) \rangle \subset \mathbb{P}^{2n+1}$$

where $\langle x, \phi(x) \rangle$ denotes the line through $x, \phi(x)$, is a rational normal scroll of type (a, b) . This is a smooth rational surface of degree $\deg(S_{(a,b)}) = a + b$.

Lemma 3.3. *Let $H \subset \mathbb{P}^{2n+1}$ be a general $(n+2)$ -plane. Then the intersection $H \cap \Sigma^{1,n}$ is a rational normal scroll $S_{(a,b)}$ with*

$$(a, b) = \begin{cases} \left(\frac{n+1}{2}, \frac{n+1}{2}\right) & \text{if } n \text{ is odd;} \\ \left(\frac{n}{2}, \frac{n+2}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The Segre variety $\Sigma^{1,n}$ is the projectivization over \mathbb{P}^1 of the rank $n+1$ vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{n+1}$. A codimension $n-1$ linear section corresponds to the projectivization of the kernel of a morphism

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{n+1} \rightarrow \mathcal{O}_{\mathbb{P}^1}^{n-1}$$

which is a rank two vector bundle $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$. To conclude it is enough to note that for H general the splitting type $(-a, -b)$ is the one given in the statement. \square

Lemma 3.4. *Let $f : X \rightarrow Y$ be a morphism of projective varieties, and $\text{NE}(f)$ the cone of curves contracted by f . Then $\text{NE}(f)$ is extremal in $\text{NE}(X)$.*

Proof. Let Γ be the class of an irreducible curve in $\text{NE}(f)$, and assume that $\Gamma = \Gamma_1 + \Gamma_2$ for $\Gamma_1, \Gamma_2 \in \text{NE}(X)$. Applying f_* we get $f_*\Gamma_1 + f_*\Gamma_2 = f_*\Gamma = 0$ since Γ is contracted by f . Therefore, $f_*\Gamma_1 = f_*\Gamma_2 = 0$ and hence $\Gamma_1, \Gamma_2 \in \text{NE}(f)$. \square

Proposition 3.5. *Fix $p_1, \dots, p_{n+3} \in \Sigma^{1,n}$ general points. Set $H = \langle p_1, \dots, p_{n+3} \rangle$. Let $\tilde{S}_{a,b}$ be the blow-ups of $S_{a,b} = H \cap \Sigma^{1,n}$ at the p_i . Then*

$$\text{NE}(X_{n+3}^{1,n}) = \left\langle \text{NE}(\tilde{S}_{a,b}), h_1 - e_1, \dots, h_1 - e_{n+3} \right\rangle.$$

Proof. Let $C \sim ah_1 + bh_2 - \sum_{i=1}^{n+3} m_i e_i$ be an irreducible curve in $X_{n+3}^{1,n}$, and $\Gamma \subset \Sigma^{1,n}$ its image in \mathbb{P}^{2n+1} . Then $\deg(\Gamma) = a + b$ and $\text{mult}_{p_i} \Gamma = m_i$ for $i = 1, \dots, n+3$.

If $a + b < \sum_{i=1}^{n+3} m_i$ then Γ is contained in all the hyperplanes containing H and hence $\Gamma \subset H \cap \Sigma^{1,n}$. By Lemma 3.3 $S_{a,b} = H \cap \Sigma^{1,n}$ is a scroll.

If $a + b \geq \sum_{i=1}^{n+3} m_i$ we can pair each e_i with one among h_1 and h_2 , and write C as a linear combination with non negative coefficients of $h_1 - e_i, h_2 - e_j, e_k$.

The curves of class $h_2 - e_i$ are numerically equivalent to the strict transform of the line through p_i in the ruling of $S_{a,b}$. The curves of class $h_1 - e_i$ are contracted by $\tilde{\pi}_2$ and the curves of class $h_2 - e_j$ are contracted by $\tilde{\pi}_1$. Hence, by Lemma 3.4 $h_1 - e_i$ and $h_2 - e_j$ generate extremal rays of $\text{NE}(X_{n+3}^{1,n})$.

Summing-up, we have showed that an irreducible curve $C \subset X_{n+3}^{1,n}$ can be written as a linear combination with non negative coefficients of a curve $\Gamma \subset \tilde{S}_{a,b}$ and of the $h_1 - e_i$. \square

3.5. Mori cones of del Pezzo surfaces. Let S_r be the blow-up of \mathbb{P}^2 at $p_1, \dots, p_r \in \mathbb{P}^2$ general points. The surface S_r is del Pezzo if and only if $0 \leq r \leq 8$. We recall the structure of the Mori cone $\text{NE}(S_r)$ for $r = 6, 7, 8$. We will denote by \bar{h} the pull-back of a line in \mathbb{P}^2 and by \bar{e}_i the exceptional divisor over the point p_i for $i = 1, \dots, r$.

3.5.1. del Pezzo of degree 3. The Mori cone of S_3 is generated by the following classes

Divisor class	Number of extremal rays
\bar{e}_i	6
$\bar{h} - \bar{e}_i - \bar{e}_j$	15
$2\bar{h} - \bar{e}_{i_1} - \dots - \bar{e}_{i_5}$	6

for a total of 27 extremal rays.

3.5.2. *del Pezzo of degree 2.* The Mori cone of S_2 is generated by the following classes

<i>Divisor class</i>	<i>Number of extremal rays</i>
\bar{e}_i	7
$\bar{h} - \bar{e}_i - \bar{e}_j$	21
$2\bar{h} - \bar{e}_{i_1} - \cdots - \bar{e}_{i_5}$	21
$3\bar{h} - 2\bar{e}_i - \bar{e}_{j_1} - \cdots - \bar{e}_{j_6}$	7

for a total of 56 extremal rays.

3.5.3. *del Pezzo of degree 1.* The Mori cone of S_1 is generated by the following classes

<i>Divisor class</i>	<i>Number of extremal rays</i>
\bar{e}_i	8
$\bar{h} - \bar{e}_i - \bar{e}_j$	28
$2\bar{h} - \bar{e}_{i_1} - \cdots - \bar{e}_{i_5}$	56
$3\bar{h} - 2\bar{e}_i - \bar{e}_{j_1} - \cdots - \bar{e}_{j_6}$	56
$4\bar{h} - 2\bar{e}_{i_1} - \cdots - 2\bar{e}_{i_3} - \bar{e}_{j_1} - \cdots - \bar{e}_{j_5}$	56
$5\bar{h} - 2\bar{e}_{i_1} - \cdots - 2\bar{e}_{i_6} - \bar{e}_{j_1} - \bar{e}_{j_2}$	28
$6\bar{h} - 3\bar{e}_i - 2\bar{e}_{i_1} - \cdots - 2\bar{e}_{j_7}$	8

for a total of 240 extremal rays.

Proposition 3.6. *If $n \leq 4$, the Mori cone of $X_{n+3}^{1,n}$ is given by*

$$\text{NE}(X_{n+3}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i, h_1 + nh_2 - e_{i_1} - \cdots - e_{i_{n+2}} \rangle$$

for $i, i_1, \dots, i_{n+2} \in \{1, \dots, n+3\}$.

Proof. The case $n = 1$ is well-known. Let C be an irreducible curve in $X_{n+3}^{1,n}$. If C is contracted by π then $C \subset E_i$ for some $i = 1, \dots, n+3$ and hence C is a multiple of e_i . Now, assume that $\pi(C)$ is a curve of bidegree (d_1, d_2) in $\mathbb{P}^1 \times \mathbb{P}^n$ and such that $\text{mult}_{p_i} C = m_i$. This means that we can write

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \cdots - m_{n+2} e_{n+2} - m_{n+3} e_{n+3}.$$

First, we suppose that

$$d_1 + d_2 \geq m_1 + \cdots + m_{n+3}.$$

Then C may be written as a linear combination with non negative coefficients of e_i , $h_2 - e_i$ and $h_1 - e_i$.

Note that $\sigma_{1,n}(\pi(C))$ is a curve of degree $d_1 + d_2$ passing through the points $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+3})$ with multiplicities m_i . Let $\Pi_{1,\dots,n+3} \subset \mathbb{P}^{2n+1}$ be the $(n+2)$ -plane generated by $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+3})$. If

$$d_1 + d_2 < m_1 + \cdots + m_{n+3}$$

then $\sigma_{1,n}(\pi(C)) \subset \Pi_{1,\dots,n+3}$. By Lemma 3.3 the intersection $X = \Pi_{1,\dots,n+3} \cap \Sigma^{1,n}$ is a degree $n+1$ scroll in \mathbb{P}^{2n+1} , and by Proposition 3.5 the Mori cone of $X_{n+3}^{1,n}$ is generated by the Mori cone of X and the $h_1 - e_i$ for $i = 1, \dots, n+3$. Note that $\sigma_{1,n}(\pi(C)) \subset X$. Let us consider separately the cases $n = 2, 3, 4$.

If $n = 2$ then X is a cubic scroll and the projection $\pi_2 : X \rightarrow \mathbb{P}^2$ is the blow-down of the exceptional divisor. Hence, $C \subset \text{Bl}_5 X \cong \text{Bl}_6 \mathbb{P}^2$, which is a del Pezzo surface of degree three. Let \bar{h} be the pull-back of a line in \mathbb{P}^2 and \bar{e}_i the classes of the exceptional divisors. The Mori cone of $\text{Bl}_6 \mathbb{P}^2$ is described in 3.5.1. The isomorphism $\text{Bl}_6 \mathbb{P}^2 \rightarrow \text{Bl}_5 X \subset X_5^{1,2}$ induces a map $N_1(\text{Bl}_6(\mathbb{P}^2)) \rightarrow N_1(X_5^{1,2})$ such that

$$\bar{e}_1 \mapsto h_1, \bar{h} \mapsto h_1 + h_2, \bar{e}_i \mapsto e_{i-1}, \text{ for } i = 2, \dots, 5.$$

To conclude it is enough to show that the generators of $\text{NE}(\text{Bl}_6 \mathbb{P}^2)$ can be written as linear combinations with non negative coefficients of the effective classes listed in the statement.

First, note that $\bar{e}_i = e_{i-1}$ for $i = 2, \dots, 6$, and $\bar{e}_1 = h_1 = (h_1 - e_i) + e_i$. Furthermore

$$\begin{aligned} \bar{h} - \bar{e}_i - \bar{e}_j &= (h_1 - e_{i-1}) + (h_2 - e_{j-1}) \text{ for } i, j = 2, \dots, 6; \\ \bar{h} - \bar{e}_1 - \bar{e}_j &= h_2 - e_{j-1} \text{ for } j = 2, \dots, 6; \\ 2\bar{h} - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= (h_1 + 2h_2 - e_{i_1-1} - \dots - e_{i_4-1}) + (h_1 - e_{i_5-1}) \text{ for } 2 \leq i_1, \dots, i_5 \leq 6; \\ 2\bar{h} - \bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_4} &= h_1 + 2h_2 - e_{j_1-1} - \dots - e_{j_4-1}. \end{aligned}$$

If $n = 3$ then X is a quartic scroll given by the intersection in $\mathbb{P}^1 \times \mathbb{P}^3$ of two hypersurfaces of bidegree $(1, 1)$

$$X = \{x_0 f_1 + x_1 f_2 = x_0 g_1 + x_1 g_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3$$

with $f_i, g_i \in k[y_0, \dots, y_3]_1$.

The image of the projection $X \rightarrow \mathbb{P}^3$ is the quadric surface $\{f_1 g_2 - f_2 g_1 = 0\} \subset \mathbb{P}^3$. Note that since f_1, f_2, g_1, g_2 do not vanish simultaneously at a point such projection is an isomorphism.

Hence, we have that $C \subset \text{Bl}_6 X \cong \text{Bl}_6(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{Bl}_7(\mathbb{P}^2)$, which is a del Pezzo surface of degree two. The Mori cone of $\text{Bl}_7(\mathbb{P}^2)$ is described in 3.5.2. The inclusion $\text{Bl}_7(\mathbb{P}^2) \rightarrow X_6^{1,3}$ maps

$$\bar{h} \mapsto h_1 + 2h_2 - e_1, \bar{e}_1 \mapsto h_2 - e_1, \bar{e}_2 \mapsto h_1 + h_2 - e_1, \bar{e}_i \mapsto e_{i-1}, \text{ for } i = 3, \dots, 7.$$

For the \bar{e}_i there is nothing to prove. We have

$$\begin{aligned} \bar{h} - \bar{e}_1 - \bar{e}_2 &= e_1; \\ \bar{h} - \bar{e}_1 - \bar{e}_i &= (h_1 - e_1) + (h_2 - e_{i-1}) + e_1; \\ \bar{h} - \bar{e}_2 - \bar{e}_i &= h_2 - e_{i-1}; \\ \bar{h} - \bar{e}_i - \bar{e}_j &= (h_1 - e_1) + (h_2 - e_{i-1}) + (h_2 - e_{j-1}); \\ 2\bar{h} - \bar{e}_1 - \bar{e}_2 - \bar{e}_{i_1} - \bar{e}_{i_2} - \bar{e}_{i_3} &= (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + (h_2 - e_{i_3-1}); \\ 2\bar{h} - \bar{e}_1 - \bar{e}_{i_1} - \bar{e}_{i_2} - \bar{e}_{i_3} - \bar{e}_{i_4} &= (h_1 - e_1) + (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + (h_2 - e_{i_3-1}) + (h_2 - e_{i_4-1}); \\ 2\bar{h} - \bar{e}_2 - \bar{e}_{i_1} - \bar{e}_{i_2} - \bar{e}_{i_3} - \bar{e}_{i_4} &= h_1 + 3h_2 - e_1 - e_{i_1-1} - e_{i_2-1} - e_{i_3-1} - e_{i_4-1}; \\ 3\bar{h} - 2\bar{e}_1 - \bar{e}_2 - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= h_1 + (h_1 + 3h_2 - e_{i_1-1} - \dots - e_{i_5-1}); \\ 3\bar{h} - \bar{e}_1 - 2\bar{e}_2 - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= h_1 + 3h_2 - e_{i_1-1} - \dots - e_{i_5-1}; \\ 3\bar{h} - \bar{e}_1 - \bar{e}_2 - 2\bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= (h_1 - e_1) + (h_2 - e_{i_1-1}) + (h_1 + 3h_2 - e_{i_1-1} - \dots - e_{i_5-1}). \end{aligned}$$

If $n = 4$ then X is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^4$ of three hypersurfaces of bidegree $(1, 1)$

$$X = \{x_0 f_1 + x_1 f_2 = x_0 g_1 + x_1 g_2 = x_0 h_1 + x_1 h_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^4$$

with $f_i, g_i, h_i \in k[y_0, \dots, y_4]_1$. Note that X is projected onto a cubic 3-fold $S \subset \mathbb{P}^4$. Recall that S is isomorphic to the blow-up of \mathbb{P}^2 at a point that we can assume to be $[1 : 0 : 0]$. The rational map

$$\begin{aligned} \tau : \quad \mathbb{P}^2 &\dashrightarrow \quad \mathbb{P}^1 \times \mathbb{P}^4 \\ [x, y, z] &\longmapsto \quad ([y, z], [xy, xz, y^2, yz, z^2]) \end{aligned}$$

yields an embedding $\tilde{\tau} : S \rightarrow \mathbb{P}^1 \times \mathbb{P}^4$. Denote by \bar{h} the pull-back to S of a line in \mathbb{P}^2 and by $\bar{e}_1 \subset S$ the exceptional divisor over $[1 : 0 : 0]$. Hence, X is isomorphic to \mathbb{P}^2 blown-up at eight points that is to a del Pezzo surface of degree one whose generators of the Mori cone are listed in 3.5.3. Then $\tilde{\tau}$ lifts to an embedding $X \rightarrow X_7^{1,4}$ which in turn maps

$$\bar{h} \mapsto h_1 + 2h_2, \bar{e}_1 \mapsto h_1 + h_2, \bar{e}_i \mapsto e_{i-1}, \text{ for } i = 2, \dots, 8.$$

We have

$$\begin{aligned} \bar{h} - \bar{e}_1 - \bar{e}_j &= h_2 - e_{j-1}; \\ \bar{h} - \bar{e}_i - \bar{e}_j &= (h_1 - e_1) + (h_2 - e_{i-1}) + (h_2 - e_{j-1}) + e_1; \\ 2\bar{h} - \bar{e}_1 - \bar{e}_{i_1} - \dots - \bar{e}_{i_4} &= (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + (h_2 - e_{i_3-1}) + (h_2 - e_{i_4-1}); \\ 2\bar{h} - \bar{e}_{i_1} - \dots - \bar{e}_{i_5} &= \sum_{k=1}^2 (h_1 - e_{i_k-1}) + \sum_{t=3}^5 (h_2 - e_{i_t-1}) + (h_2 - e_1) + e_1; \\ 3\bar{h} - 2\bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_6} &= 2(h_1 - e_{i_1-1}) + (h_1 - e_{j_1-1}) + \sum_{k=1}^6 (h_2 - e_{j_k-1}) + e_{j_1-1}; \\ 3\bar{h} - 2\bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_6} &= h_1 + 4h_2 - e_{j_1-1} - \dots - e_{j_6-1}; \\ 3\bar{h} - 2\bar{e}_i - \bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_5} &= 2(h_1 - e_{i-1}) + (h_2 - e_{j_1-1}) + \dots + (h_2 - e_{j_5-1}); \\ 4\bar{h} - 2\bar{e}_1 - 2\bar{e}_{i_1} - 2\bar{e}_{i_2} - \bar{e}_{j_1} - \dots - \bar{e}_{j_5} &= 2(h_2 - e_{i_1-1}) + (h_1 - e_{i_2-1}) + (h_1 + 4h_2 - e_{i_2-1} - \sum_{k=1}^5 e_{j_k-1}); \\ 4\bar{h} - 2\bar{e}_{i_1} - \dots - 2\bar{e}_{i_3} - \bar{e}_1 - \bar{e}_{j_1} - \dots - \bar{e}_{j_4} &= \sum_{k=1}^3 2(h_2 - e_{i_k-1}) + (h_2 - e_{j_1-1}) + \sum_{t=2}^4 (h_1 - e_{j_t-1}); \\ 5\bar{h} - 2\bar{e}_1 - 2\bar{e}_{i_1} - \dots - 2\bar{e}_{i_5} - \bar{e}_{j_1} - \bar{e}_{j_2} &= (h_1 + 4h_2 - 2 \sum_{k=1}^3 e_{i_k-1}) + 2 \sum_{t=1}^2 (h_t - e_{i_{t+3}-1}) + \sum_{k=1}^2 (h_2 - e_{j_k-1}); \\ 5\bar{h} - 2\bar{e}_{i_1} - \dots - 2\bar{e}_{i_6} - \bar{e}_1 - \bar{e}_j &= \sum_{k=1}^4 2(h_2 - e_{i_k-1}) + \sum_{t=5}^6 2(h_1 - e_{i_t-1}) + (h_2 - e_{j-1}); \\ 6\bar{h} - 2\bar{e}_1 - 3\bar{e}_2 - 2\bar{e}_3 - \dots - 2\bar{e}_8 &= 2(h_1 + 4h_2 - e_3 - \dots - e_8) + (h_1 - e_2) + 2(h_2 - e_2); \\ 6\bar{h} - 3\bar{e}_1 - 2\bar{e}_2 - \dots - 2\bar{e}_8 &= 2(h_1 + 4h_2 - e_2 - \dots - e_7) + (h_1 - e_8) + (h_2 - e_8); \end{aligned}$$

concluding the proof. \square

Corollary 3.7. *If $r \leq n + 1$ the nef cone of $X_r^{1,n}$ is given by*

$$\text{Nef}(X_r^{1,n}) = \langle H_1, H_2, H_1 + H_2 - E_{i_1} - \cdots - E_{i_r} \rangle$$

with $r = 1, \dots, n + 1$. In particular, $\text{Nef}(X_{n+1}^{1,n})$ has $2^{n+1} + 1$ extremal rays.

Proof. The claim follows by duality from Proposition 3.1. \square

Corollary 3.8. *The nef cone of $X_{n+2}^{1,n}$ is given by*

$$\text{Nef}(X_{n+2}^{1,n}) = \left\langle H_1, H_2, H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}, 2H_1 + H_2 - \sum_{i=1}^{n+2} E_i, nH_1 + (n+1)H_2 - n \sum_{i=1}^{n+2} E_i \right\rangle$$

with $r = 1, \dots, n + 1$. In particular, $\text{Nef}(X_{n+2}^{1,n})$ has $2^{n+2} + 2$ extremal rays.

Proof. The claim follows by duality from Proposition 3.2. \square

Corollary 3.9. *Consider the following divisor classes on $X_{n+3}^{1,n}$:*

$$\begin{aligned} D_t &= H_1 + H_2 - E_{i_1} - \cdots - E_{i_t}; \\ D_r &= 2H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}; \\ D'_r &= nH_1 + (n+1)H_2 - nE_{i_1} - \cdots - nE_{i_r}; \\ D_{k,s} &= kH_1 + kH_2 - kE_{i_1} - \cdots - kE_{i_2} - (k-1)E_{i_{s+1}} - \cdots - (k-1)E_{i_{n+3}}. \end{aligned}$$

If $n \leq 4$ then the nef cone of $X_{n+3}^{1,n}$ is given by

$$\text{Nef}(X_{n+3}^{1,n}) = \langle H_1, H_2, D_t, D_r, D'_r, D_{k,s} \rangle$$

with $t = 1, \dots, n + 1$, $r \in \{n + 2, n + 3\}$, $k = 2, \dots, n + 1$ and $s = n - k + 2$. In particular, if $n \leq 4$ then $\text{Nef}(X_{n+3}^{1,n})$ has $2^{n+4} - \frac{(n+3)(n+2)}{2}$ extremal rays.

Proof. The claim follows by duality from Proposition 3.6. \square

4. THE COX RING OF $X_{n+1}^{1,n}$

The n -dimensional complex torus $T = (\mathbb{C}^*)^n$ acts on $\mathbb{P}^1 \times \mathbb{P}^n$ as follows:

$$(4.1) \quad \begin{array}{ccc} T \times (\mathbb{P}^1 \times \mathbb{P}^n) & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^n \\ ((t_1, \dots, t_n), ([x_0, x_1], [y_0, \dots, y_n])) & \longmapsto & ([x_0, x_1], [y_0, t_1 y_1, t_2 y_2, \dots, t_n y_n]). \end{array}$$

Let us write the $n + 1$ general points $p_1, \dots, p_{n+1} \in \mathbb{P}^1 \times \mathbb{P}^n$ as follows:

$$\begin{aligned} p_1 &= ([\alpha_1, \beta_1], [1, 0, \dots, 0]); \\ p_2 &= ([\alpha_2, \beta_2], [0, 1, 0, \dots, 0]); \\ p_3 &= ([\alpha_3, \beta_3], [0, 0, 1, 0, \dots, 0]); \\ p_4 &= ([\alpha_4, \beta_4], [0, 0, 0, 1, 0, \dots, 0]); \\ &\vdots \\ p_{n+1} &= ([\alpha_{n+1}, \beta_{n+1}], [0, \dots, 0, 1]); \end{aligned}$$

where $[\alpha_1, \beta_1] = [0, 1]$, $[\alpha_2, \beta_2] = [1, 0]$ and $[\alpha_3, \beta_3] = [1, 1]$. Note that (4.1) lifts to an action of T on $X_{n+1}^{1,n}$ whose orbits of maximal dimension have codimension one in $X_{n+1}^{1,n}$. Hence, $X_{n+1}^{1,n}$ has complexity one and Theorem 2.5 implies that $X_{n+1}^{1,n}$ is a Mori dream space. We begin by proving the following stronger statement:

Proposition 4.2. *The variety $X_r^{1,n}$ is log Fano for $r \leq n + 1$.*

Proof. Consider the case $r = n + 1$. The result for $r < n + 1$ will then follow from [PS09, Theorem 2.9]. The claim amounts to find an effective \mathbb{Q} -divisor Δ on $X_{n+1}^{1,n}$ such that $-K_{X_{n+1}^{1,n}} - \Delta$ is ample, and the pair $(X_{n+1}^{1,n}, \Delta)$ is Kawamata log terminal. Recall that

$$-K_{X_{n+1}^{1,n}} = 2H_1 + (n+1)H_2 - nE_1 - \cdots - nE_{n+1}$$

and consider the divisor

$$D = (n+1)H_2 - nE_1 - \cdots - nE_{n+1}.$$

The divisor D is effective since it is the pull-back of the effective divisor on \mathbb{P}^n consisting in the union of the $(n+1)$ hyperplanes passing through n among the $n+1$ projections of the blown-up points. Set $\Delta_\epsilon = \epsilon D$ with $\epsilon \in \mathbb{Q}_{>0}$, and consider the divisor

$$-K_{X_{n+1}^{1,n}} - \Delta_\epsilon = 2H_1 + ((n+1) - \epsilon(n+1))H_2 + \sum_{i=1}^{n+1} (\epsilon n - n)E_i.$$

By Proposition 3.1 we have that $\text{NE}(X_{n+1}^{1,n}) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle$. Since

$$\begin{aligned} (-K_{X_{n+1}^{1,n}} - \Delta_\epsilon) \cdot e_i &= n - \epsilon n > 0 && \text{if and only if } \epsilon < 1; \\ (-K_{X_{n+1}^{1,n}} - \Delta_\epsilon) \cdot (h_1 - e_i) &= 2 - n + \epsilon n > 0 && \text{if and only if } \epsilon > (n-2)/n; \\ (-K_{X_{n+1}^{1,n}} - \Delta_\epsilon) \cdot (h_2 - e_i) &= n+1 - \epsilon(n+1) - n + \epsilon n > 0 && \text{if and only if } \epsilon < 1; \end{aligned}$$

and hence the divisor $-K_{X_{n+1}^{1,n}} - \Delta_\epsilon$ is ample if and only if $(n-2)/n < \epsilon < 1$.

Next, we show that the pair $(X_{n+1}^{1,n}, \Delta_\epsilon)$ is Kawamata log terminal. Consider the images $q_i = \pi_2(p_i)$ of the blown-up points via the second projection. There exist $n+1$ hyperplanes in \mathbb{P}^n through each subset of n points among the q_i , and through each q_i there pass n of them. Let us denote by $\tilde{H}_i \subset X_{n+1}^{1,n}$ the strict transforms of the inverse images via the second projection of these hyperplanes. The divisor D has multiplicity n along the strict transforms of the curves $\pi_2^{-1}(q_i)$ for all i . Hence, any n among the divisors \tilde{H}_i intersect transversally along one of the strict transforms of these curves.

Generalizing this argument, fix a set of m points among the q_i , with $1 \leq m \leq (n-1)$, and denote by Λ_m their linear span. The divisor D has multiplicity $n+1-m$ along the strict transform $\tilde{\Lambda}_m$ of $\pi_2^{-1}(\Lambda_m)$. Any $n+1-m$ among the \tilde{H}_i intersect transversally along a subvariety of type $\tilde{\Lambda}_m \subset X_{n+1}^{1,n}$, and hence Δ_ϵ is a simple normal crossing effective divisor for all $\epsilon > 0$. Summing-up, we proved that for all $(n-2)/n < \epsilon < 1$ the divisor $-K_{X_{n+1}^{1,n}} - \Delta_\epsilon$ is ample and the pair $(X_{n+1}^{1,n}, \Delta_\epsilon)$ is Kawamata log terminal concluding the proof. \square

Let $\tilde{H}_i \subset X_{n+1}^{1,n}$ be the strict transform of $\pi_2^{-1}(\{y_i = 0\})$. For $p \in \mathbb{P}^1 \setminus \{\pi_1(p_1), \dots, \pi_1(p_{n+1})\}$ the fiber $F_p = \tilde{\pi}_1^{-1}(p)$ is isomorphic to \mathbb{P}^n while $F_{p_i} = \tilde{\pi}_1^{-1}(p_i)$ is the union of two irreducible components $D_{i,1} \cup D_{i,2}$ where $D_{i,1}$ is isomorphic to \mathbb{P}^n blown-up in a point with exceptional divisor $\overline{E}_i \cong \mathbb{P}^{n-1}$, $D_{i,2} = E_i$ and $D_{i,1} \cap D_{i,2} = \overline{E}_i$. Recall that $E_i \cong \mathbb{P}^n$ and denote by $R_i \subset E_i$ the union of the T -invariant divisors of the lifting of (4.1) restricted to E_i . Then

$$X_0 = X_{n+1}^{1,n} \setminus \left(\bigcup_{i=0}^n \tilde{H}_i \cup \bigcup_{j=1}^{n+1} R_j \right) \subset X_{n+1}^{1,n}$$

is the T -invariant open subset of points of $X_{n+1}^{1,n}$ having isotropy group of dimension zero. Hence, the \tilde{H}_i are the T -invariant prime divisors supported in $X_{n+1}^{1,n} \setminus X_0$. Moreover, note that the isotropy group of a general point of both $D_{i,1}, D_{i,2}$, with respect to the lifting of (4.1), is trivial. In particular, it is finite of order one.

Now, associate to the \tilde{H}_i variables S_i , to the $D_{i,1}$ variables $T_{i,1}$ and to the $D_{i,2}$ variables $T_{i,2}$. For $1 \leq i \leq n-1$ set $k = j+1 = i+2$ and

$$g_i = (\beta_k \alpha_j - \beta_j \alpha_k) T_{i,1} T_{i,2} + (\beta_i \alpha_k - \beta_k \alpha_i) T_{j,1} T_{j,2} + (\beta_j \alpha_i - \beta_i \alpha_j) T_{k,1} T_{k,2}.$$

Then we have the following result:

Theorem 4.3. *For the Cox ring of $X_{n+1}^{1,n}$ we have the following explicit presentation*

$$\text{Cox}(X_{n+1}^{1,n}) \cong \frac{\mathbb{C}[S_0, \dots, S_n, T_{1,1}, T_{1,2}, \dots, T_{n+1,1}, T_{n+1,2}]}{\langle g_1, \dots, g_{n-1} \rangle}$$

where S_i is the section associated to $H_2 - E_1 - \dots - E_{i-1} - E_{i+1} - \dots - E_{n+1}$, $T_{i,1}$ is the section associated to $H_1 - E_i$ and $T_{i,2}$ is the section associated to E_i .

Proof. In the previous set up it is enough to apply Theorem 2.5. \square

In the rest of this section we will derive some of consequences of Theorem 4.3.

Proposition 4.4. *Consider the following divisor classes on $X_{n+1}^{1,n}$:*

$$\begin{aligned} D_1 &= H_1; \\ D_h &= H_2 - E_{i_1} - \dots - E_{i_h}; \\ D_{i_1, \dots, i_n} &= H_1 + H_2 - E_{i_1} - \dots - E_{i_n}; \\ D_{1, \dots, n+1} &= H_1 + H_2 - E_1 - \dots - E_{n+1}; \\ D_k &= kH_2 - kE_{i_1} - \dots - kE_{i_{(n-k)}} - (k-1)E_{i_{(n-k+1)}} - \dots - (k-1)E_{i_{n+1}}; \\ D_n &= nH_2 - (n-1)E_1 - \dots - (n-1)E_{(n+1)}. \end{aligned}$$

The movable cone of $X_{n+1}^{1,n}$ is given by

$$\text{Mov}(X_{n+1}^{1,n}) = \langle D_1, D_h, D_{i_1, \dots, i_n}, D_{1, \dots, n+1}, D_k, D_n \rangle$$

for $2 \leq k \leq n-1$, $0 \leq h \leq n-1$ and $\{i_1, \dots, i_n\} \subset \{1, \dots, n+1\}$.

Furthermore, let $\text{Mov}_1(X_{n+1}^{1,n})$ be the cone of curves covering a divisor in $X_{n+1}^{1,n}$. Then

$$\text{Mov}_1(X_{n+1}^{1,n}) = \langle h_1, h_2 - e_i, h_1 + (n-1)h_2 - e_{i_1} - \dots - e_{i_n}, e_i \rangle$$

for $1 \leq i \leq n+1$, $\{i_1, \dots, i_n\} \subset \{1, \dots, n+1\}$.

Proof. Set $\mathcal{C} = \langle D_1, D_h, D_{i_1, \dots, i_n}, D_{1, \dots, n+1}, D_k, D_n \rangle$. Since any divisor appearing in \mathcal{C} is movable we have that $\mathcal{C} \subset \text{Mov}(X_{n+1}^{1,n})$. Then taking the dual cones we get $\text{Mov}(X_{n+1}^{1,n})^* \subset \mathcal{C}^*$.

Now, $\text{Mov}_1(X_{n+1}^{1,n}) \subset \text{Mov}(X_{n+1}^{1,n})^* \subset \mathcal{C}^*$. If we prove that any extremal ray of \mathcal{C}^* corresponds to a curve that covers a divisor in $X_{n+1}^{1,n}$ we would have that $\text{Mov}_1(X_{n+1}^{1,n}) = \text{Mov}(X_{n+1}^{1,n})^* = \mathcal{C}^*$ and hence $\text{Mov}(X_{n+1}^{1,n}) = \mathcal{C}$ concluding the proof.

A straightforward computation shows that

$$\mathcal{C} = \langle h_1, h_2 - e_i, h_1 + (n-1)h_2 - e_{i_1} - \dots - e_{i_n}, e_i \rangle$$

for $1 \leq i \leq n+1$, $\{i_1, \dots, i_n\} \subset \{1, \dots, n+1\}$. Clearly, the curves with classes $h_1, h_2 - e_i, e_i$ cover divisors in $X_{n+1}^{1,n}$. Now, consider the projections in \mathbb{P}^n of the blown-up points p_{i_1}, \dots, p_{i_n} and let $H \subset \mathbb{P}^n$ be the hyperplane spanned by them. Let $X_n^{1,n-1} \subset X_{n+1}^{1,n}$ be the strict transform of $\mathbb{P}^1 \times H \subset \mathbb{P}^1 \times \mathbb{P}^n$. To conclude it is enough to note that the curves of class $h_1 + (n-1)h_2 - e_{i_1} - \dots - e_{i_n}$ cover $X_n^{1,n-1}$. \square

Remark 4.5. Let X be a \mathbb{Q} -factorial projective variety. In general $\text{Mov}_1(X) \subsetneq \text{Mov}(X)^*$. However, by Proposition 3.1 we have that

$$\text{Mov}_1(X_{n+1}^{1,n}) = \text{Mov}(X_{n+1}^{1,n})^*.$$

For further information on the behavior of these cones for general projective varieties we refer to [Pay06], [BDPS23].

Remark 4.6. We developed Magma [BCP97] scripts computing the nef cones of $X_{n+1}^{1,n}$, $X_{n+2}^{1,n}$ and of $X_{n+3}^{1,n}$ for $n \leq 4$, as well as the Mori chamber decomposition of $X_{n+1}^{1,n}$. A Magma library containing the scripts can be downloaded at the following link:

<https://github.com/msslxa/Cox-rings-of-blow-ups-of-multiprojective-spaces>

We managed to compute the Mori chamber decomposition of $X_{n+1}^{1,n}$ for $n = 2, 3, 4$, and we got 92, 550 and 6307 chambers respectively. Finally, we would like to mention that the Mori chamber decomposition of $X_6^{1,2}$ has been fully computed by T. Grange in [Gra22, Chapter 3].

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