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Abstract: In this paper, we derive a mass formula for the self-orthogonal codes and self-dual codes over a non-commutative non-unitary ring, namely, $E_p = \langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle$, where $a \neq b$ and p is any odd prime. We also give a classification of self-orthogonal codes and self-dual codes over E_p, where $p = 3, 5$, and 7, in short lengths.

Keywords: non-unitary non-commutative rings; left self-dual codes; self-dual codes; mass formula

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1. Introduction

Mass formulas serve as important tools for classifying self-dual codes over finite fields [1–5] and unitary rings [6–11]. When generating nonequivalent self-dual codes, mass formulas serve as terminating flags for the computing effort. Recently, this methodology was extended to the non-unitary ring I, in the terminology of [12], where quasi self-dual codes, a special class of self-orthogonal codes, serve as the central objects for classification [13,14]. Even more recently, a satisfying definition of self-dual codes for non-unitary non commutative rings was introduced in [15]. It is the intersection of the right dual with the left dual. Thus, a non-commutative ring leads to three types of self-dual codes: left self-dual, right self-dual and self-dual.

In the present note, we present mass formulas for self-orthogonal codes, self-dual codes, and left self-dual codes over the non-commutative non-unitary ring of order p^2, for p an odd prime, and type E in the classification of Fine [12]. This ring is henceforth denoted by E_p. We apply them to the classification of these codes when $p = 3, 5, 7$ in short lengths. The proof techniques combine self-dual mass formula over finite field, and linear algebra over the ring E_3 which is reminiscent of that on chain rings of depth 2 [9,10].

The content is organized in the following manner: Section 2 introduces essential preliminary concepts and notations necessary for comprehending the remainder of the paper. In Section 3, the framework of linear codes over E_p is established. Section 4 elaborates on the mass formulas. Section 5 is dedicated to the classification in short lengths for fixed types. Finally, Section 6 serves as the conclusion of the article

2. Preliminaries

2.1. Codes over $\mathbb{F}_p$

Let p be an odd prime number. A **linear code** $\mathcal{C}$ of length n and dimension k over a finite field $\mathbb{F}_p$ is an $\mathbb{F}_p$-subspace of the vector space $\mathbb{F}_p^n$ of dimension k. Compactly, we call $\mathcal{C}$ an $[n, k]$**-code**. The elements of a code are called **codewords**. Two codewords

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are orthogonal if their standard inner product $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i$ is zero, and the vector space consisting of all vectors in $\mathbb{F}_p^n$ that are orthogonal to every codeword in $\mathcal{C}$ is called the **dual** of $\mathcal{C}$, denoted by $\mathcal{C}^\perp$. $\mathcal{C}$ is said to be **self-orthogonal** (resp. **self-dual**) if $\mathcal{C} \subseteq \mathcal{C}^\perp$ (resp. $\mathcal{C} = \mathcal{C}^\perp$).

2.2. Additive Codes over $\mathbb{F}_{p^2}$

Let ω be a primitive element in $\mathbb{F}_{p^2}$, so that ω has order $r = p^2 - 1$ and $\omega^t \neq 1 \; \forall \; 0 \leq t \leq r$. Then $\mathbb{F}_{p^2} = \{0, 1, \omega, \omega^2, \cdots, \omega^{r-1}\}$. The **trace map**, $Tr : \mathbb{F}_{p^2} \mapsto \mathbb{F}_p$, is defined by $Tr(u) = u + u^p$. An **additive code** of length n over $\mathbb{F}_{p^2}$ is $\mathbb{F}_p$-additive subgroup of $\mathbb{F}_{p^2}^n$ containing p^k codewords for some integer k in the range $0 \leqslant k \leqslant 2n$.

2.3. The Ring E_p

Following [12], we define a ring on two distinct generators a, b by their relations

$$E_p = \Big\langle a, b \mid pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b \Big\rangle.$$

Thus , E_p consists of p^2 elements and has characteristic p. We also define a natural action of $\mathbb{F}_p$ on E_p by the rule $rx = xr$, for all $r \in E_p$ and for all $x \in \mathbb{F}_p$.

Lemma 1. *The ring E_p does not contain a unity element.*

Proof. Assume, by contradiction, that there is a unique element e in E_p such that $eu = ue = u$ for every $0 \neq u \in E_p$. Since E_p is generated by a and b, e can be written as

$$e = ai + bj, \text{ for some } 0 \leq i, j \leq p - 1.$$

Choose $u = a$. By assumption,

$$(ai + bj)a = a(ai + bj) = a,$$

which implies $e = a$. Also, choose $u = a + b$. Then

$$a(a + b) = a^2 + ab = a + a = 2a \neq a + b.$$

So a is not the unity of E_p, which leads to an inconsistency. Therefore, E_p is a non-unital ring. □

From the ring representation of E_p and Lemma 1, E_p is a non-commutative ring without multiplicative identity. Moreover, E_p contains a unique maximal ideal

$$J_p = \{e_{xy} \mid e_{xy} = ax + by \text{ with } x + y \equiv 0 \pmod{p} \text{ where } 0 \leqslant x, y < p \} = \langle e = a - b \rangle.$$

Thus, we can write E_p as

$$E_p = \{c_{xy} = ax + ey \mid x, y \in \mathbb{F}_p\}. \tag{1}$$

Define the reduction map modulo J_p as $\alpha : E_p \mapsto E_p / J_p \cong \mathbb{F}_p$ by $\alpha(ax + ey) = x$ where $0 \leqslant x < p$. This map can be extended in the natural way from E_p^n to $\mathbb{F}_p^n$.

3. Codes over E_p

A **linear code** $\mathcal{C}$ of length n over E_p, or simply an E_p-code, is a one-sided E_p-submodule of E_p^n. We denote the (Hamming) weight of $\mathbf{x} \in E_p^n$ by $wt(\mathbf{x})$ and adapt the notation

$$[< 0, 1 >, \cdots, < i, A_i >, \cdots, < n, A_n >],$$

where A_i is the number of codewords of Hamming weight i, for the weight distribution of a code over E_p as in Magma [16]. The inner product between two vectors $\mathbf{x} =$

$(x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in E_p^n$ is defined as $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i$. For an E_p-code C, the **right dual** of $\mathcal{C}$, defined as

$$\mathcal{C}^{\perp_R} = \{\mathbf{y} \in E_p^n | \ \forall \ \mathbf{x} \in \mathcal{C}, (\mathbf{x}, \mathbf{y}) = 0\},$$

is a right module of E_p^n, while the **left dual** of $\mathcal{C}$, defined as

$$\mathcal{C}^{\perp_L} = \{\mathbf{y} \in E_p^n | \ \forall \ \mathbf{x} \in \mathcal{C}, (\mathbf{y}, \mathbf{x}) = 0\},$$

is a left module of E_p^n. The two-sided dual of $\mathcal{C}$ is $\mathcal{C}^{\perp} = \mathcal{C}^{\perp_R} \cap \mathcal{C}^{\perp_L}$. The code $\mathcal{C}$ is said to be **self-orthogonal** if every a codeword in $\mathcal{C}$ is orthogonal to every codeword in $\mathcal{C}$, that is, $\mathcal{C} \subseteq \mathcal{C}^{\perp}$. If $\mathcal{C} = \mathcal{C}^{\perp_R}$ (resp. $\mathcal{C} = \mathcal{C}^{\perp_L}$) then $\mathcal{C}$ is **right self-dual (RSD)** (resp. **left self-dual (LSD)**). If $\mathcal{C} = \mathcal{C}^{\perp}$, then $\mathcal{C}$ is **self-dual (SD)**.

Using the map $\phi : E_p \mapsto \mathbb{F}_{p^2}$ defined as

$$\phi(0) = 0, \phi(a) = \omega, \text{and } \phi(b) = -\omega + 1, \tag{2}$$

we can attach to an E_p-code $\mathcal{C}$ an additive $\mathbb{F}_{p^2}$-code $\phi(\mathcal{C}) = \{(\phi(c_1), \phi(c_2), \ldots, \phi(c)_n) \mid (c_1, c_2, \ldots, c_n) \in \mathcal{C}\}$. It can be easily seen that $Tr(\phi(u)) = \alpha(u)$ for all $u \in E_p$.

Lemma 2.

(i) For any positive integer n, there exist a self-orthogonal code over E_p of length n.
(ii) For any odd prime p, there is a left self-orthogonal code ($\mathcal{C} \subseteq \mathcal{C}^{\perp_L}$) over E_p.

Proof.

(i) Consider J_p as linear code over E_p of length 1. To prove the self-orthogonality of J_p, for all $(ai + bj), (ai' + bj') \in J_p$, we have that

$$(ai + bj, ai' + bj') = a((i' + j')i) + b((i' + j')j),$$

$$(ai' + bj', ai + bj) = a((i + j)i') + b((i + j)j').$$

Since $(ai + bj), (ai' + bj') \in J_p$, it follows that, $i + j \equiv 0$ and $i' + j' \equiv 0 \pmod{p}$. Then, $J_p \subseteq J_p^{\perp_R} \cap J_p^{\perp_L} = J_p^{\perp}$. Taking the direct sum of n copies of this code yields a self-orthogonal code of any length n.

(ii) Let p be a prime and let $\mathbf{1}_p$ denote the all-one codeword of length p. The repetition code of length p is then defined by $R_p = \{u(\mathbf{1}_p) | \ u \in E_p\}$. Clearly R_p is a linear code over E_p. Since E_p has characteristic p, then we have that $R_p \subseteq {R_p}^{\perp_L}$.

□

Let $\mathcal{C}$ be linear code over E_p. We define the **residue** code of $\mathcal{C}$ as

$$\mathbf{res}(\mathcal{C}) = \{\mathbf{x} \in \mathbb{F}_p^n | \ \exists \ \mathbf{y} \in \mathbb{F}_p^n \text{ such that } a\mathbf{x} + e\mathbf{y} \in \mathcal{C}\}.$$

and the **torsion** code of $\mathcal{C}$ as

$$\mathbf{tor}(\mathcal{C}) = \{\mathbf{y} \in \mathbb{F}_p^n | \ e\mathbf{y} \in \mathcal{C}\}.$$

From Equation (2), we have $\mathbf{res}(\mathcal{C}) = Tr(\phi(\mathcal{C}))$ and $\mathbf{tor}(\mathcal{C})$ is the subfield subcode of $\phi(\mathcal{C})$ defined by $\phi(\mathcal{C}) \cap \mathbb{F}_p^n$. Let $\alpha_{\mathcal{C}}$ be the restriction of α to $\mathcal{C}$. We have that $e\,\mathbf{tor}(\mathcal{C}) = Ker\,\alpha_{\mathcal{C}}$, and that $\mathbf{res}(\mathcal{C}) = Im\,\alpha_{\mathcal{C}}$. Let $\mathbf{dim}(\mathbf{res}(\mathcal{C})) = k_1$ and $k_2 = \mathbf{dim}(\mathbf{tor}(\mathcal{C})) - k_1$. We say that $\mathcal{C}$ is linear code of **type** $\{k_1, k_2\}$. It can be seen that $\mathcal{C}$ is **free** as an E_p-module if and only if $\mathbf{res}(\mathcal{C}) = \mathbf{tor}(\mathcal{C})$. By the first isomorphism theorem applied to $\alpha_{\mathcal{C}}$ we have $|\mathcal{C}| = p^{2k_1 + k_2}$.

In Theorem 1, we will extend a few results from [15,17] by simply substituting codes over $\mathbb{F}_p$ for binary codes in the proofs.

Theorem 1. *Suppose $\mathcal{C}$ is a linear code over E_p. Let k_1, k_2, n be non-negative integers with $k_1 + k_2 \leq n$. Then the following hold:*

(i) Every linear code $\mathcal{C}$ over E_p of length n and type $\{k_1, k_2\}$ is equivalent to a code with generator matrix in standard form

$$\begin{bmatrix} aI_{k_1} & aA + eB \\ 0 & eZ \end{bmatrix} \tag{3}$$

where I_{k_1} is the identity matrix, the matrices A, B and Z have entries from $\mathbb{F}_p$.

(ii) $a\,\boldsymbol{res}(\mathcal{C}) \subseteq \mathcal{C}$ *and* $\boldsymbol{res}(\mathcal{C}) \subseteq \boldsymbol{tor}(\mathcal{C})$.

(iii) $\mathcal{C} = a\,\boldsymbol{res}(\mathcal{C}) \oplus e\,\boldsymbol{tor}(\mathcal{C})$

(iv) $\mathcal{C}^{\perp_R} = a\,\boldsymbol{tor}(\mathcal{C})^{\perp} \oplus e\,\mathbb{F}_p^n$.

(v) $\mathcal{C}^{\perp_L} = a\,\boldsymbol{res}(\mathcal{C})^{\perp} \oplus e\,\boldsymbol{res}(\mathcal{C})^{\perp}$.

(vi) $\mathcal{C}^{\perp} = a\,\boldsymbol{tor}(\mathcal{C})^{\perp} \oplus e\,\boldsymbol{res}(\mathcal{C})^{\perp}$.

If G is a $k \times n$ matrix over E_p, we denote by $E_p^k G$ the code of length n over E_p with generator matrix G.

Next, we make a modification on the construction of self-orthogonal codes, (left or right) self-dual codes, and self-dual codes in [15], to be suitable for an odd prime p.

Theorem 2. *If $\mathcal{C} = a\mathcal{C}_1 \oplus e\mathcal{C}_2$ is a linear code over E_p such that $\mathcal{C}_1$ and $\mathcal{C}_2$ be arbitrary linear codes over $\mathbb{F}_p$, then*

(i) $\mathcal{C}$ is a self-orthogonal code if and only if $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_1^{\perp}$;

(ii) $\mathcal{C}$ is a self-dual if and only if $\mathcal{C}_1 \subseteq \mathcal{C}_2 = \mathcal{C}_1^{\perp}$;

(iii) $\mathcal{C}$ is a left self-dual if and only if $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_1^{\perp}$;

(iv) $\mathcal{C}$ is a right self-dual if and only if $\mathcal{C} = e\,\mathbb{F}_p^n$.

Proof. First, we will prove that $\mathcal{C}_1 = \mathbf{res}(\mathcal{C})$ and $\mathcal{C}_2 = \mathbf{tor}(\mathcal{C})$. Observe that $\mathcal{C}_1 = \alpha(a\,\mathcal{C}_1) \subseteq \mathbf{res}(\mathcal{C})$. Let $\mathbf{x} \in \mathbf{res}(\mathcal{C})$. Then $a\mathbf{x} + e\mathbf{0} \in \mathcal{C}$, so $\mathbf{x} \in \mathcal{C}_1$, thus, we have $\mathbf{res}(\mathcal{C}) = \mathcal{C}_1$. Now, $e\mathcal{C}_2 \subseteq \mathcal{C}$, so by definition of torsion code, $\mathcal{C}_2$ is a subset of $\mathbf{tor}(\mathcal{C})$. Let $\mathbf{y} \in \mathbf{tor}(\mathcal{C})$. Since the zero vector is in $\mathcal{C}_1$, $e\mathbf{y} = a\mathbf{0} + e\mathbf{y} \in \mathcal{C}$. Therefore, $\mathbf{y} \in \mathcal{C}_2$.

(i) Let $\mathcal{C}$ be a self-orthogonal code. Note that $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Suppose that $\mathbf{z} \in \mathcal{C}_2$. Since $\mathcal{C}$ is self-orthogonal, for all $a\mathbf{x} + e\mathbf{y} \in \mathcal{C}$, we have

$$0 = (a\mathbf{x} + e\mathbf{y}, e\mathbf{z}) = e(\mathbf{x}, \mathbf{z}).$$

Hence $\mathcal{C}_2 \subseteq \mathcal{C}_1^{\perp}$. Conversely, to prove the self-orthogonality of $\mathcal{C}$, for all $\mathbf{x}, \mathbf{x}' \in \mathcal{C}_1$ and for all $\mathbf{y}, \mathbf{y}' \in \mathcal{C}_2$ we have

$$\begin{aligned} (\mathbf{c}, \mathbf{c}') &= (a\mathbf{x} + e\mathbf{y}, a\mathbf{x}' + e\mathbf{y}') \\ &= a(\mathbf{x}, \mathbf{x}') + 0(\mathbf{x}, \mathbf{y}') + e(\mathbf{y}, \mathbf{x}') + 0(\mathbf{y}, \mathbf{y}') \\ &= a(0) + 0 + e(0) + 0 = 0, \end{aligned}$$

since $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_1^{\perp}$.

(ii) Let $\mathcal{C}$ be a self-dual code. Then from the preceding case, $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_1^{\perp}$. Now, let $\mathbf{x} \in \mathcal{C}_1^{\perp}$ be arbitrary. From Theorem 1 (vi), we have $e\mathbf{x} \in \mathcal{C}$. Hence, $\mathbf{x} \in \mathbf{tor}(\mathcal{C}) = \mathcal{C}_2$. It follows that $\mathcal{C}_2 = \mathcal{C}_1^{\perp}$. For the converse, suppose that $\mathcal{C}_1 \subseteq \mathcal{C}_2 = \mathcal{C}_1^{\perp}$. From the preceding case, we have $\mathcal{C} \subseteq \mathcal{C}^{\perp}$. Since $|\mathcal{C}| = p^{2k_1+k_2}$ and $\mathcal{C}_2 = \mathcal{C}_1^{\perp}$, $p^{k_1+(n-k_1)} = p^n$. It follows that $\mathcal{C} = \mathcal{C}^{\perp}$.

(iii) $\mathcal{C}$ is left self-dual code if and only if $\mathcal{C}_1 = \mathcal{C}_1^{\perp} = \mathcal{C}_2$, by Theorem 1 (iii) and (v).

(iv) $\mathcal{C}$ is right self-dual code if and only if $\mathcal{C}_1 = \mathcal{C}_2^{\perp}$ (by Theorem 1 (iii) and (iv)). Equivalently, we have $\mathcal{C}_1^{\perp} = (\mathcal{C}_2^{\perp})^{\perp} = \mathcal{C}_2 = \mathbb{F}_p^n$ if and only if $\mathcal{C}_1 = \{0\}$ if and only if $\mathcal{C} = e\,\mathbb{F}_p^n$.

□

4. Computation of the Mass Formula

At the start, we define the notion of equivalence of codes. Two codes $\mathcal{C}$ and $\mathcal{C}'$ over E_p are **monomially equivalent** if there is an $n \times n$ monomial matrix M (with exactly one entry $\in \{1, -1\}$ in each row and column and all other entries are zero) such that $\mathcal{C}' = \{cM : c \in \mathcal{C}\}$. The **monomial automorphism** group $Aut(\mathcal{C})$ of code $\mathcal{C}$ consists of all M such that $\mathcal{C} = \mathcal{C}M$. Let M_n be the signed symmetric group of order $|M_n| = 2^n n!$. The number of codes equivalent to a code $\mathcal{C}$ of length n is

$$\frac{|M_n|}{|Aut(\mathcal{C})|}.$$

The mass formula for self-orthogonal codes is given by

$$\sum_{\mathcal{C}} \frac{|M_n|}{|Aut(\mathcal{C})|},$$

where the sum runs through all inequivalent self-orthogonal codes $\mathcal{C}$ over E_p of length n.

We apply a similar approach to that used for the computation of a mass formula in [7]. Let C_1 be a code over $\mathbb{F}_p$ of length n with dimension k_1 and generator matrix

$$\begin{bmatrix} I_{k_1} & A \end{bmatrix}, \tag{4}$$

and C_2 be a code over $\mathbb{F}_p$ of length n with dimension $k_1 + k_2$ and generator matrix

$$\begin{bmatrix} I_{k_1} & A \\ 0 & Z \end{bmatrix}, \tag{5}$$

where $A \in M_{k_1 \times (n-k_1)}(\mathbb{F}_p)$, and $Z \in M_{k_2 \times (n-k_1)}(\mathbb{F}_p)$ is of full row rank. Observe that $C_1 \subseteq C_2$, and the code with generator matrix (3) has residue code C_1 and torsion code C_2.

We need the following lemmas to count the number free self-orthogonal E_p-codes.

Lemma 3. *If $\mathcal{C}$ is a free E_p-code of length n, then the matrix B in Theorem 1 (i) is unique.*

Proof. Suppose $\mathcal{C}$ is a free code and there exist $B_1, B_2 \in M_{k_1 \times (n-k_1)}(\mathbb{F}_p)$ such that

$$E_p^{k_1}[aI_{k_1}\ aA + eB_1] \quad = \quad E_p^{k_1}[aI_{k_1}\ aA + eB_2]. \tag{6}$$

Then $aA + eB_1 = aA + eB_2$. Hence, $B_1 = B_2$. □

For the remainder of this section, assume that $C_1 \subseteq C_1^{\perp}$. Then

$$I_{k_1} + AA^T = 0, \tag{7}$$

and

$$ZA^T = 0. \tag{8}$$

It follows from (7) that A is of full row rank.

Lemma 4. *The map f_A defined by*

$$f_A : M_{k_1 \times m}(\mathbb{F}_p) \to M_{k_1 \times k_1}(\mathbb{F}_p)$$

$$X \mapsto AX^T$$

is a surjective linear map.

Proof. Note that f_A is a linear map by properties of matrix. To prove the surjective condition, choose an arbitrary matrix G in $M_{k_1 \times k_1}(\mathbb{F}_p)$. Consider the matrix $-G^T A$ in $M_{k_1 \times m}(\mathbb{F}_p)$. We have

$$f_A(-G^TA) = A(-G^TA)^T = -AA^TG = I_{k_1}(G) = G.$$

□

Now, let us consider the sets

$$\begin{aligned} \mathbf{X} &= \{\mathcal{C} \mid \mathcal{C} \text{ be a self-orthogonal } E_p\text{-code, with } \mathbf{res}(\mathcal{C}) = C_1 = \mathbf{tor}(\mathcal{C})\}, \\ \mathbf{X}' &= \{\mathcal{C}' \mid \mathcal{C}' \text{ be a self-orthogonal } E_p\text{-code, with } \mathbf{res}(\mathcal{C}') = C_1,\ \mathbf{tor}(\mathcal{C}') = C_2\}. \end{aligned}$$

Lemma 5. *If $\mathcal{C}' \in \mathbf{X}'$, then $|\{\mathcal{C} \in \mathbf{X} \mid \mathcal{C} \subseteq \mathcal{C}'\}| = p^{k_1k_2}$.*

Proof. By Theorem 1 (i), $\mathcal{C}'$ has a generator matrix (3). Consider the map

$$\mu : M_{k_1\times k_2}(\mathbb{F}_p) \to \{\mathcal{C} \in \mathbf{X} \mid \mathcal{C} \subseteq \mathcal{C}'\}$$

$$M \mapsto E_p^{k_1}[a\, I_{k_1}\ \ aA + e(B + MZ)].$$

Clearly, μ is well defined. Now, we will show that μ is bijective. Suppose $M_1, M_2 \in M_{k_1\times k_2}(\mathbb{F}_p)$ such that $\mu(M_1) = \mu(M_2)$. Then $aA + e(B + M_1Z) = aA + e(B + M_2Z)$. Since Z is of full row rank, we conclude $M_1 = M_2$, which shows that μ is injective. Now, suppose that $\mathcal{C} \in \mathbf{X}$ such that $\mathcal{C} \subseteq \mathcal{C}'$. By Theorem 1 (i), $\mathcal{C} = E_p^{k_1}[a\, I_{k_1}\ aA + eF]$, for some matrix F. The inclusion $\mathcal{C} \subseteq \mathcal{C}'$ implies that

$$aA + eF = aA + e(B + MZ)$$

for some matrix M of size $k_1 \times k_2$. So $F = B + MZ$, which shows that μ is surjective, and hence, μ is bijective. Therefore,

$$|\{\mathcal{C} \in \mathbf{X} \mid \mathcal{C} \subseteq \mathcal{C}'\}| = |M_{k_1\times k_2}(\mathbb{F}_p)| = p^{k_1k_2}.$$

□

Lemma 6. *If $\mathcal{C} \in \mathbf{X}$, then there is a unique code $\mathcal{C}' \in \mathbf{X}'$, such that $\mathcal{C} \subseteq \mathcal{C}'$.*

Proof. By Theorem 1 (i) and Lemma 3, $\mathcal{C}$ has a generator matrix $[aI_{k_1}\ \ aA + eB]$ for some unique matrix B. Then the code $\mathcal{C}_0'$ with a generator matrix

$$\begin{bmatrix} aI_{k_1} & aA + eB \\ 0 & eZ \end{bmatrix} \tag{9}$$

satisfies $\mathbf{res}(\mathcal{C}_0') = C_1$, and $\mathbf{tor}(\mathcal{C}_0') = C_2$. Since $\mathcal{C} \in \mathbf{X}$, (8) implies that $\mathcal{C}_0'$ is a self-orthogonal code, hence $\mathcal{C}_0' \in \mathbf{X}'$. Now, suppose that $\mathcal{C}' \in \mathbf{X}'$ and $\mathcal{C} \subseteq \mathcal{C}'$. By Theorem 1 (i), we have that $E_p^{k_2}\ [0\ eZ] \subseteq \mathcal{C}'$. This, together with $\mathcal{C} \subseteq \mathcal{C}'$ forces $\mathcal{C}_0' \subseteq \mathcal{C}'$. Since $|\mathcal{C}_0'| = |\mathcal{C}'|$, we have $\mathcal{C}_0' = \mathcal{C}'$. □

Let $\sigma_p(n, k_1)$ be the number of self-orthogonal codes of length n and dimension k_1 over $\mathbb{F}_p$, which is found in [18–20].

In [21], the number of subspaces of dimension k contained in an n-dimensional vector space over $\mathbb{F}_p$ is given by the Gaussian coefficient $\begin{bmatrix} j \\ i \end{bmatrix}_p$ for $i \leqslant j$, where

$$\begin{bmatrix} j \\ i \end{bmatrix}_p = \frac{(p^j - 1)(p^j - p)\dots(p^j - p^{i-1})}{(p^i - 1)(p^i - p)\dots(p^i - p^{i-1})}.$$

Let $\sigma_{E_p}(n, k_1, k_2)$ be the number of distinct self-orthogonal codes over E_p of length n. Mass formulas are useful for finding all inequivalent codes of given length. Our goal now is to compute $\sigma_{E_p}(n, k_1, k_2)$.

Theorem 3. *For all codes of length n with type* $\{k_1, 0\}$*, the number of free self-orthogonal codes over* E_p *is*

$$\sigma_{E_p}(n, k_1, 0) = \sigma_p(n, k_1) p^{k_1(n-2k_1)}. \tag{10}$$

Proof. We may assume without loss of generality that C_1 is a code with generator matrix (4). If $\mathcal{C}$ is a self-orthogonal code of length n over E_p of type $\{k_1, 0\}$, then by setting $C_1 = \mathbf{res}(\mathcal{C}) = \mathbf{tor}(\mathcal{C})$, C_1 satisfies Theorem 2 (i). We have $\sigma_p(n, k_1)$ codes C_1, which is the number of self-orthogonal codes over $\mathbb{F}_p$. By Theorem 1 (i), $\mathcal{C}$ has generator matrix $[a\ I_{k_1}\ \ aA + eB]$. Finally, we have

$$a^2 I_{k_1} + a^2 AA^T + aeAB^T + eaBA^T + e^2 BB^T = 0,$$

so,

$$a(I_{k_1} + AA^T) + e(BA^T) = 0.$$

Since $C_1 \subseteq C_1^{\perp}$, then $I_{k_1} + AA^T = 0$. Now, we have find the number of the matrix B which satisfies $AB^T = 0$. Setting $m = n - k_1$ in the linear map f_A in Lemma 4,

$$|\{B \in M_{k_1 \times n-k_1}(\mathbb{F}_p) \mid AB^T = 0\}| = |Ker\, f_A| = \frac{|M_{k_1 \times n-k_1}(\mathbb{F}_p)|}{|Im\, f_A|} = \frac{p^{k_1(n-k_1)}}{p^{k_1^2}}. \tag{11}$$

□

Example 1. *We consider the case* $n = 3$*, and* $p = 3$*. In Table 1, we give the list of inequivalent self-orthogonal codes over the ring* E_3 *of type* $\{1, 0\}$*. Using the mass formula in Theorem 3, we have the following computations:*

$$\sum_{i=1}^{3} \frac{1}{|Aut(\mathcal{C}_i)|} = \frac{1}{12} + \frac{1}{6} = \frac{4 \cdot 3^{1(3-2)}}{48} = \frac{\sigma_{E_3}(3, 1, 0)}{2^3 \cdot 3!}. \tag{12}$$

In the next theorem, we give the number of self-orthogonal codes over E_p.

Theorem 4. *The number of self-orthogonal codes over* E_p *of length n with type* $\{k_1, k_2\}$ *is given by*

$$\sigma_{E_p}(n, k_1, k_2) = \sigma_p(n, k_1) \begin{bmatrix} n - 2k_1 \\ k_2 \end{bmatrix}_p p^{k_1(n-2k_1-k_2)}. \tag{13}$$

Proof. We may assume without loss of generality that C_1 and C_2 are codes with generator matrices (4) and (5), respectively. Let $\mathcal{C}$ be a self-orthogonal code of length n over E_p of type $\{k_1, k_2\}$. By setting $C_1 = \mathbf{res}(\mathcal{C})$ and $C_2 = \mathbf{tor}(\mathcal{C})$, C_1 and C_2 satisfy Theorem 2 (i). Then, there are $\sigma_p(n, k_1)$ self-orthogonal codes C_1 and we have $\begin{bmatrix} n - 2k_1 \\ k_2 \end{bmatrix}_p$ codes C_2 such that $C_1 \subseteq C_2 \subseteq C_1^{\perp}$. Now, we have to compute $|\mathbf{X}'|$. By Lemmas 5 and 6 , we have

$$\begin{aligned} p^{k_1 k_2} |\mathbf{X}'| &= \sum_{\mathcal{C}' \in \mathbf{X}'} |\{\mathcal{C} \in \mathbf{X} | \mathcal{C} \subseteq \mathcal{C}'\}| \\ &= \sum_{\mathcal{C} \in \mathbf{X}} |\{\mathcal{C}' \in \mathbf{X}' | \mathcal{C} \subseteq \mathcal{C}'\}| \\ &= \sum_{\mathcal{C} \in \mathbf{X}} 1 \\ &= |\mathbf{X}|. \end{aligned}$$

From Theorem 3, we have that $|\mathbf{X}| = p^{k_1(n-2k_1)}$. Therefore, $|\mathbf{X}'| = p^{k_1(n-2k_1-k_2)}$. □

Remark 1. *Let* $\mathcal{C}$ *be an* E_p*-code of length n and of type* $\{0, m\}$*, where* $m \leq n$*. Then* $\mathcal{C}$ *will be a self-orthogonal code. Furthermore, if* $m = n$*, then* $\mathcal{C}$ *will be self-dual.*

The following results derive the mass formula for the SD codes and the LSD codes over E_p, respectively.

Theorem 5. *For a given integer $n \geq 2$ we have the identity*

$$\sum_{\mathcal{C}} \frac{1}{|Aut(\mathcal{C})|} = \frac{\sigma_p(n,k)}{2^n n!},$$

where $\mathcal{C}$ runs over distinct representatives of equivalence classes under monomial action of SD E_p-codes of length n and type $\{k, n-2k\}$.

Proof. From Theorem 2 (ii), the number of SD E_p-codes depends on the number of self-orthogonal codes over $\mathbb{F}_p$, and $\mathbf{tor}(\mathcal{C}) = \mathbf{res}(\mathcal{C})^{\perp}$. □

Example 2. *We consider the case $n = 3$, and $p = 3$. In Table 1, we give the list of inequivalent self- orthogonal codes over E_3 of type $\{1,1\}$. Using the mass formula in Theorem 5, we make the following computations,*

$$\sum_{i=1} \frac{1}{|Aut(\mathcal{C}_i)|} = \frac{1}{12} = \frac{\sigma_{E_3}(3,1,1)}{48} = \frac{\sigma_3(3,1)}{48}. \tag{14}$$

Corollary 1. *For a given even integer $n \geq 2$, we have the identity*

$$\sum_{\mathcal{C}} \frac{1}{|Aut(\mathcal{C})|} = \frac{\sigma_p(n,n/2)}{2^n n!},$$

where $\mathcal{C}$ runs over distinct representatives of equivalence classes under monomial action of left SD E_p-codes of length n and type $\{n/2, 0\}$.

Proof. From Theorem 2 (iii), the number of left self-dual E_p-codes depends on the number of self-dual codes over $\mathbb{F}_p$, with $\mathbf{tor}(\mathcal{C}) = \mathbf{res}(\mathcal{C})$. Thus, the result follows. □

Proposition 1. *For all codes $\mathcal{C}$ of length n, there is a unique right self-dual E_p- code.*

Proof. The result follows from Theorem 2 (iv), where $\mathcal{C} = e\,\mathbb{F}_p^n$ is the unique code. □

Table 1. Self-orthogonal codes of length $n \leq 4$ over E_3.

Length	Type	Generator Matrices	$\lvert Aut(\mathcal{C})\rvert$	Weight Distribution	RSD Code	LSD Code	SD Code
1	$\{0,1\}$	(e)	2	[<0, 1>, <1, 2>]	✓		✓
2	$\{0,1\}$	$\begin{pmatrix} e & 0 \end{pmatrix}$	4	[<0, 1>, <1, 2>]			
	$\{0,2\}$	eI_2	8	[<0, 1>, <1, 4>, <2, 4>]	✓		✓
3	$\{1,0\}$	$\begin{pmatrix} a & a & a \end{pmatrix}$	12	[<0, 1>, <3, 8>]			
		$\begin{pmatrix} a & c_{11} & c_{12} \end{pmatrix}$	6	[<0, 1>, <3, 8>]			
	$\{0,1\}$	$\begin{pmatrix} e & 0 & 0 \end{pmatrix}$	16	[<0, 1>, <1, 2>]			
	$\{0,2\}$	$\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \end{pmatrix}$	16	[<0, 1>, <1, 4>, <2, 4>]			
	$\{0,3\}$	eI_3	48	[<0, 1>, <1, 6>, <2, 12>, <3, 8>]	✓		✓
	$\{1,1\}$	$\begin{pmatrix} a & a & a \\ 0 & 2e & e \end{pmatrix}$	12	[<0, 1>, <2, 6>, <3, 20>]			✓

Table 1. *Cont.*

Length	Type	Generator Matrices	$\lvert Aut(\mathcal{C})\rvert$	Weight Distribution	RSD Code	LSD Code	SD Code
4	$\{1,0\}$	$\begin{pmatrix} a & a & a & 0 \end{pmatrix}$	24	[<0, 1>, <3, 8>]			
		$\begin{pmatrix} a & c_{11} & c_{12} & 0 \end{pmatrix}$	12	[<0, 1>, <3, 8>]			
		$\begin{pmatrix} a & a & a & e \end{pmatrix}$	12	[<0, 1>, <3, 8>]			
		$\begin{pmatrix} a & c_{11} & c_{12} & e \end{pmatrix}$	6	[<0, 1>, <3, 2>, <4, 6>]			
	$\{2,0\}$	$\begin{pmatrix} a & 0 & 2a & 2a \\ 0 & a & 2a & a \end{pmatrix}$	48	[<0, 1>, <3, 32>, <4, 48>]		✓	
	$\{0,1\}$	$\begin{pmatrix} e & 0 & 0 & 0 \end{pmatrix}$	96	[<0, 1>, <1, 2>]			
	$\{0,2\}$	$\begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \end{pmatrix}$	64	[<0, 1>, <1, 4>, <2, 4>]			
	$\{0,3\}$	$\begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \end{pmatrix}$	96	[<0, 1>, <1, 6>, <2, 12>, <3, 8>]			
	$\{0,4\}$	$e\,I_4$	384	[<0, 1>, <1, 8>, <2, 24>, <3, 32>, <4, 16>]	✓		✓

5. Classification

We classify self-orthogonal codes and self-dual codes of length $n \leq 4$ with given residue of dimension $k_1 = 0, 1, 2$, where $p = 3, 5$. Also, we classify self-orthogonal codes and self-dual codes of length $n \leq 3$ with given residue of dimension $k_1 = 0, 1$, where $p = 7$ using the building method discussed in Theorem 2. To carry out the the classification, we represent codes over E_p by their associated additive codes over $\mathbb{F}_{p^2}$ under the mapping ϕ defined in (2), and considered the action of the group of monomial matrices (with 1 and -1 as nonzero entries) to directly calculate the automorphism group. These calculations are performed using MAGMA [16]. See Tables 1–3 for a summary of our results for $p = 3, 5, 7$, respectively.

Table 2. Self-orthogonal codes of length $n \leq 4$ over E_5.

Length	Type	Generator Matrices	$\lvert Aut(\mathcal{C})\rvert$	Weight Distribution	RSD Code	LSD Code	SD Code
1	$\{0,1\}$	(e)	2	[<0, 1>, <1, 4>]	✓		✓
2	$\{0,1\}$	$(e\,0)$	4	[<0, 1>, <1, 4>]			
	$\{0,2\}$	eI_2	8	[<0, 1>, <1, 8>, <2, 16>]	✓		✓
	$\{1,0\}$	$\begin{pmatrix} a & 2a \end{pmatrix}$	4	[<0, 1>, <2, 24>]		✓	✓
3	$\{0,1\}$	$\begin{pmatrix} e & 0 & 0 \end{pmatrix}$	16	[<0, 1>, <1, 4>]			
	$\{0,2\}$	$\begin{pmatrix} eI_2 & 0 \end{pmatrix}$	16	[<0, 1>, <1, 8>, <2, 16>]			
	$\{0,3\}$	eI_3	48	[<0, 1>, <1, 12>, <2, 48>, <3, 64>]	✓		✓
	$\{1,0\}$	$\begin{pmatrix} a & 2a & 0 \end{pmatrix}$	8	[<0, 1>, <2, 24>]			
		$\begin{pmatrix} a & 2a & e \end{pmatrix}$	2	[<0, 1>, <2, 4>, <3, 20>]			
	$\{1,1\}$	$\begin{pmatrix} a & 0 & 2a \\ 0 & e & 0 \end{pmatrix}$	8	[<0, 1>, <1, 4>, <2, 24>, <3, 96>]			✓
4	$\{1,0\}$	$\begin{pmatrix} a & 2a & 0 & 0 \end{pmatrix}$	32	[<0, 1>, <2, 24>]			
		$\begin{pmatrix} a & a & 2a & 2a \end{pmatrix}$	16	[<0, 1>, <4, 24>]			
		$\begin{pmatrix} a & 2a & e & 0 \end{pmatrix}$	4	[<0, 1>, <2, 4>, <3, 20>]			
		$\begin{pmatrix} a & 2a & e & e \end{pmatrix}$	4	[<0, 1>, <2, 4>, <3, 20>]			
		$\begin{pmatrix} a & c_{11} & c_{21} & c_{21} \end{pmatrix}$	4	[<0, 1>, <4, 24>]			
		$\begin{pmatrix} a & c_{14} & c_{22} & c_{21} \end{pmatrix}$	2	[<0, 1>, <4, 24>]			
		$\begin{pmatrix} a & c_{12} & c_{22} & c_{22} \end{pmatrix}$	4	[<0, 1>, <4, 24>]			
		$\begin{pmatrix} a & c_{13} & 2a & c_{21} \end{pmatrix}$	4	[<0, 1>, <4, 24>]			
		$\begin{pmatrix} a & c_{11} & 2a & c_{22} \end{pmatrix}$	4	[<0, 1>, <4, 24>]			
		$\begin{pmatrix} a & 2a & e & 2e \end{pmatrix}$	4	[<0, 1>, <2, 4>, <4, 20>]			
	$\{2,0\}$	$\begin{pmatrix} a & 0 & 2a & 0 \\ 0 & a & 0 & 2a \end{pmatrix}$	32	[<0, 1>, <2, 48>, <4, 576>]		✓	✓

Table 3. Self-orthogonal codes of length $n \leq 3$ over E_7.

Length	Type	Generator Matrices	$\lvert Aut(\mathcal{C})\rvert$	Weight Distribution	RSD Code	LSD Code	SD Code
1	$\{0,1\}$	(e)	2	[<0, 1>, <1, 6>]	✓		✓
2	$\{0,1\}$	$(e\ 0)$	4	[<0, 1>, <1, 6>]			
	$\{0,2\}$	eI_2	8	[<0, 1>, <1, 12>, <2, 36>]	✓		✓
3	$\{0,1\}$	$\begin{pmatrix} e & 0 & 0 \end{pmatrix}$	16	[<0, 1>, <1, 6>]			
	$\{0,2\}$	$\begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \end{pmatrix}$	16	[<0, 1>, <1, 12>, <2, 36>]			
	$\{0,3\}$	eI_3	48	[<0, 1>, <1, 18>, <2, 108>, <3, 216>]	✓		✓
	$\{1,0\}$	$\begin{pmatrix} a & 2a & 3a \end{pmatrix}$	6	[<0, 1>, <3, 48>]			
		$\begin{pmatrix} 2a & b & c_{35} \end{pmatrix}$	2	[<0, 1>, <3, 48>]			
		$\begin{pmatrix} 4a & 6b & c_{53} \end{pmatrix}$	2	[<0, 1>, <3, 48>]			
	$\{1,1\}$	$\begin{pmatrix} a & 2a & 3a \\ 0 & e & 4e \end{pmatrix}$	6	[<0, 1>, <2, 18>, <3, 324>]			✓

6. Conclusions and Open Problems

In this paper, we have given a mass formula to classify certain self-orthogonal codes over the non-unitary non-commutative ring E_p, with p an odd prime. Particularly, we were considering the two main cases of classification self-orthogonal codes, and SD codes under monomial action. In the previous section, concrete classifications in short lengths are given. Extension of these results to higher lengths would require more programming or more computing power. Similar theoretical and experimental questions remain open for other non-unitary non-commutative rings in the Rhagavandran list [12,22] in an odd characteristic.

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