



HAL
open science

A compactness result for inhomogeneous nonlinear Schrödinger equations

Van Duong Dinh, Sahbi Keraani

► **To cite this version:**

Van Duong Dinh, Sahbi Keraani. A compactness result for inhomogeneous nonlinear Schrödinger equations. *Nonlinear Analysis: Theory, Methods and Applications*, 2022, 215, pp.112617. 10.1016/j.na.2021.112617 . hal-04505658

HAL Id: hal-04505658

<https://hal.science/hal-04505658v1>

Submitted on 18 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A COMPACTNESS RESULT FOR INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATIONS

VAN DUONG DINH AND SAHBI KERAANI

ABSTRACT. We establish a compactness property of the difference between nonlinear and linear operators (or the Duhamel operator) related to the inhomogeneous nonlinear Schrödinger equation. The proof is based on a refined profile decomposition for the equation. More precisely, we prove that any sequence $(\phi_n)_n$ of H^1 -functions which converges weakly in H^1 to a function ϕ , the corresponding solutions with initial data ϕ_n can be decomposed (up to a remainder term) as a sum of the corresponding solution with initial data ϕ and solutions to the linear equation.

1. INTRODUCTION

In this paper, we consider the Cauchy problem for the inhomogeneous nonlinear Schrödinger equations (INLS)

$$\begin{cases} i\partial_t u + \Delta u &= \nu|x|^{-b}|u|^\alpha u, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\ u|_{t=0} &= u_0 \in H^1(\mathbb{R}^N), \end{cases} \quad (1.1)$$

where $N \geq 3$, $u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $u_0 : \mathbb{R}^N \rightarrow \mathbb{C}$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. The parameter $\nu = 1$ (resp. $\nu = -1$) corresponds to the defocusing (resp. focusing) nonlinearity. This type of equations belongs to a class of inhomogeneous nonlinear Schrödinger equations of the form

$$i\partial_t u + \Delta u + K(x)|u|^\alpha u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (1.2)$$

The equation (1.2) arises as an effective model describing the laser beam that supports stable high-power propagation in a homogeneous bulk media in plasma. The stable high-power beam propagation can be achieved by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel (see e.g., [34, 40]).

In the last decade, there has been an increasing interest from the mathematical community devoted to the study of (1.2). When $K(x)$ is constant, it is the well-known nonlinear Schrödinger equation.

When $K(x)$ is a non-constant bounded function, the equation (1.2) has been studied in [29, 41, 42, 47]. In [42], Merle investigated the existence and non-existence of minimal mass blow-up solutions to (1.2) with $\alpha = \frac{4}{N}$ and $K_1 \leq K(x) \leq K_2$, where $K_1, K_2 > 0$ are constants. Afterwards, Raphaël and Szeftel [47] extended the result of [42] and showed the existence, uniqueness, and characterization of minimal mass blow-up solutions. Here by minimal mass blow-up solutions, we mean solutions to (1.2) having the mass equal to that of the ground state, which blow up in finite time. Later, Fibich and Wang [29] and Liu, Wang, and Wang [41] studied the stability and instability of solitary waves for (1.2) with $\alpha \geq \frac{4}{N}$ and $K(x) = K(\epsilon x)$, where $\epsilon > 0$ is a small parameter and $K \in C^4(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

When $K(x) = |x|^b$ with $b > 0$, there are several works devoted to (1.2) (see e.g., [13, 14, 22, 50]). In [13] and [14], Chen and Guo established sharp criteria for the existence of global and blow-up solutions. In [50], Zhu proved the existence and dynamical properties of blow-up solutions. See also [22].

When $K(x)$ behaves like $|x|^{-b}$ with $b > 0$, the equation (1.2) has been considered in [3, 31, 32, 35]. In [3], De Bouard and Fukuizumi studied the stability of standing waves when $\alpha < \frac{4-2b}{N}$. In [31], Fukuizumi and Ohta established the instability of standing waves when $\alpha > \frac{4-2b}{N}$. See also the works [32, 35] for other studies of standing waves related to this type of equation.

2010 *Mathematics Subject Classification.* 35Q55.

Key words and phrases. Inhomogeneous nonlinear Schrödinger equation; Compactness property; Linear profile decomposition; Nonlinear profile decomposition.

The equation (1.1) can be seen as a limiting equation of (1.2) with $K(x) \sim |x|^{-b}$ as $|x| \rightarrow \infty$, which plays an important role in the analysis of (1.2). Before stating our main contributions, let us recall some basic properties and known results related to (1.1). The equation (1.1) has the following scaling invariance

$$u_\lambda(t, x) := \lambda^{\frac{2-b}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

that is, if u is a solution to (1.1) with initial data $u|_{t=0} = u_0$, then u_λ is also a solution to (1.1) with data $u_\lambda|_{t=0} = \lambda^{\frac{2-b}{\alpha}} u_0(\lambda \cdot)$. A straightforward computation shows $\|u_\lambda(0)\|_{\dot{H}_x^\gamma} = \lambda^{\gamma + \frac{2-b}{\alpha} - \frac{N}{2}} \|u_0\|_{\dot{H}_x^\gamma}$, which implies that the above scaling leaves the \dot{H}^{γ_c} -norm of initial data invariant, where

$$\gamma_c := \frac{N}{2} - \frac{2-b}{\alpha}. \quad (1.3)$$

In the present paper, the condition $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$ with $N \geq 3$ is equivalent to $0 < \gamma_c < 1$, which corresponds to the mass-supercritical and energy-subcritical range.

The local well-posedness (LWP) for (1.1) was studied by Geneoud and Stuart [32, Appendix]. By making use of the energy method developed by Cazenave [10], they proved that (1.1) is locally well-posed for $N \geq 1, 0 < b < \min\{2, N\}, \alpha > 0$, and $\alpha < \frac{4-2b}{N-2}$ if $N \geq 3$. See also [19, 33] for other LWP results based on Strichartz estimates and the contraction mapping argument. There are two important quantities which are conserved along the flow of (1.1), namely

$$M(u(t)) = \int |u(t, x)|^2 dx = M(u_0), \quad (\text{Mass})$$

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{\nu}{\alpha+2} \int |x|^{-b} |u(t, x)|^{\alpha+2} dx = E(u_0). \quad (\text{Energy})$$

Thanks to these conservation laws, the blow-up alternative, and the small data global theory (see [33, Theorem 1.9]), we have the following global well-posedness results:

- $\nu = 1$;
- $\nu = -1$ and

$$\|u_0\|_{\dot{H}_x^{\gamma_c}} < \delta_c \quad (1.4)$$

for some constant $\delta_c > 0$.

In the defocusing case $\nu = 1$, we mention the works [19] for the scattering in a weighted L^2 space $\Sigma(\mathbb{R}^N) := H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$ and [18] for the scattering in the energy space $H^1(\mathbb{R}^N)$.

In the focusing case $\nu = -1$, (1.1) has been studied in many works [7, 9, 15–17, 21, 25–27, 43, 49]. In [25], Farah showed the global existence for (1.1) with $N \geq 1$ and $0 < b < \min\{2, N\}$ by assuming $u_0 \in H^1(\mathbb{R}^N)$ and

$$[E(u_0)]^{\gamma_c} [M(u_0)]^{1-\gamma_c} < [E(Q)]^{\gamma_c} [M(Q)]^{1-\gamma_c}, \quad (1.5)$$

$$\|\nabla u_0\|_{L_x^{\gamma_c}}^{\gamma_c} \|u_0\|_{L_x^2}^{1-\gamma_c} < \|\nabla Q\|_{L_x^{\gamma_c}}^{\gamma_c} \|Q\|_{L_x^2}^{1-\gamma_c}, \quad (1.6)$$

where Q is the unique positive radial solution to the elliptic equation

$$-\Delta Q + Q - |x|^{-b} |Q|^\alpha Q = 0. \quad (1.7)$$

He also proved the finite time blow-up for (1.1) with $u_0 \in \Sigma$ satisfying (1.5) and

$$\|\nabla u_0\|_{L_x^{\gamma_c}}^{\gamma_c} \|u_0\|_{L_x^2}^{1-\gamma_c} > \|\nabla Q\|_{L_x^{\gamma_c}}^{\gamma_c} \|Q\|_{L_x^2}^{1-\gamma_c}. \quad (1.8)$$

The latter result was extended to radial data by the first author in [17]. The energy scattering (or asymptotic completeness) for (1.1) was first established by Farah and Guzmán [26] with $0 < b < \frac{1}{2}, \alpha = 2, N = 3$, and radial data. This scattering result was later extended to dimensions $N \geq 2$ in [27]. The proofs of these results are based on the concentration/compactness and rigidity argument introduced by Kenig and Merle [38]. Campos [7] made use of a new idea of Dodson and Murphy [23] to give an alternative simple proof for the radial scattering results of Farah and Guzmán. He also extends the validity of b in dimensions $N \geq 3$. Note that the idea of Dodson and Murphy is a combination of a scattering criterion of Tao [48], localized virial estimates, and radial Sobolev embedding. Xu and Zhao [49], and the first author [20] have simultaneously showed the energy scattering for (1.1) with $0 < b < 1, N = 2$, and radial data. The proof relies on a new approach of Arora, Dodson, and Murphy [1], which is a refined version of the one in [23]. In [8], Campos and Cardoso studied long time dynamics such as global existence, energy scattering, and

finite time blow-up of H^1 -solutions to (1.1) with data in Σ lying above the ground state threshold. Miao, Murphy, and Zheng [43] showed a new nonlinear profile decomposition for non-radial solutions related to (1.1). In particular, they constructed nonlinear profiles with data living far away from the origin. This allows them to show the energy scattering of non-radial solution to (1.1) with $0 < b < \frac{1}{2}$, $\alpha = 2$, and $N = 3$. The result in [43] was extended to any dimensions $N \geq 2$ and $0 < b < \min\{2, \frac{N}{2}\}$ by Cardoso, Farah, Guzmán, and Murphy [9]. Recently, the authors [21] establish long time dynamics for (1.1) with data lying below, at, and above the ground state threshold. See also a new paper of Murphy [44] for an alternative simple proof of the scattering result in [43].

To state our result, let us introduce some notations. Let $\gamma \geq 0$. A pair (q, r) is called \dot{H}^γ -admissible if

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} - \gamma, \quad \frac{2N}{N-2\gamma} < r < \frac{2N}{N-2}. \quad (1.9)$$

The set of all \dot{H}^γ -admissible pairs is denoted by \mathcal{A}_γ . When $\gamma = 0$, we denote L^2 instead of \dot{H}^0 . In this case, the L^2 -admissible pair is also called Schrödinger admissible. For $I \subset \mathbb{R}$ an interval and $\gamma \geq 0$, we define the Strichartz norms

$$\|u\|_{S(I, \dot{H}^\gamma)} := \sup_{(q,r) \in \mathcal{B}_\gamma} \|u\|_{L_t^q(L_x^\infty)}$$

and

$$\|u\|_{S^1(I, \dot{H}^\gamma)} := \|\langle \nabla \rangle u\|_{S(I, \dot{H}^\gamma)},$$

where we have used the convention

$$\|\langle \nabla \rangle f\|_X := \|f\|_X + \|\nabla f\|_X.$$

Here \mathcal{B}_γ is a subset of \mathcal{A}_γ consisting of a finite number of \dot{H}^γ -admissible pairs used in our analysis (see the proofs of Lemmas 2.5 and 2.6). When $I = \mathbb{R}$, we omit the dependence on \mathbb{R} and simply denote $\|u\|_{S(\dot{H}^\gamma)}$ and $\|u\|_{S^1(\dot{H}^\gamma)}$. We also denote

$$\mathcal{X}(I) := L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2).$$

The main result of this paper reads as follows.

Theorem 1.1. *Let $N \geq 3$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. Denote $\mathcal{H} := H^1(\mathbb{R}^N)$ if $\nu = 1$ and $\mathcal{H} := \mathcal{H}_\delta$ if $\nu = -1$, where $\delta < \delta_c$ with δ_c as in (1.4) and*

$$\mathcal{H}_\delta := \{\phi \in H^1(\mathbb{R}^N) : \|\phi\|_{\dot{H}_x^{\gamma_c}} < \delta\}.$$

Then the mapping $\text{INLS} - \text{LS} : \mathcal{H} \rightarrow \mathcal{X}_{\text{loc}}(\mathbb{R})$ is compact in the sense that if $(\phi_n)_{n \geq 1} \subset \mathcal{H}$ satisfies $\phi_n \rightharpoonup \phi$ weakly in $H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$, then

$$u_n - e^{it\Delta} \phi_n \rightarrow u - e^{it\Delta} \phi \text{ strongly in } \mathcal{X}(I) \text{ as } n \rightarrow \infty$$

for all compact interval $I \subset \mathbb{R}$, where u_n and u are solutions to (1.1) with initial data $u_n|_{t=0} = \phi_n$ and $u|_{t=0} = \phi$. Here LS and INLS are respectively the linear and nonlinear Schrödinger maps related to (1.1).

Remark 1.1. The condition on \mathcal{H} in the case $\nu = -1$ ensures that the solution corresponding to (1.1) with data in \mathcal{H} exists globally in time.

Remark 1.2. Our compactness result only holds locally in time. The locality in time is needed in order to show that nonlinear profiles associated to the linear ones with divergent time shift are solutions to the underlying linear equation. It is not clear to us that this compactness property can be extended globally in time.

Remark 1.3. To our knowledge, this is a first result concerning the compactness property of the difference between the nonlinear and linear Schrödinger operators (or the Duhamel operator) related to nonlinear Schrödinger-type equations. This phenomenon can be compared to the so called nonlinear smoothing effect, i.e., the difference between nonlinear and linear evolutions for the same initial data, given by the integral term in Duhamel's formula, has higher regularity than the initial data. The later is known to hold for various type of nonlinear dispersive PDEs (see e.g.,

gKdV equations [4, 5], NLS [6, 37], fractional NLS [24], higher-order NLS [2], and [12] for a unified approach).

The proof of Theorem 1.1 is based on a refined nonlinear profile decomposition related to (1.1). It is well-known that the profile decomposition plays an essential role in the study of long time dynamics of solutions to (1.1). For instance, it is the core of the concentration-compactness and rigidity method showing the energy scattering (see e.g., [8, 9, 26, 27, 43]). It is also an important step to establish dynamical properties of blow-up solutions to (1.1) (see e.g., [45]). Before stating our nonlinear profile decomposition, let us recall the following linear profile decomposition related to (1.1) due to Farah and Guzmán [26, 27].

Proposition 1.2 (Linear profile decomposition [26, Theorem 6.1] and [27, Theorem 5.1]). *Let $(\phi_n)_{n \geq 1}$ be a bounded sequence of H^1 -functions. Then for each integer $J \geq 1$, there exists a subsequence, still denoted by $(\phi_n)_{n \geq 1}$, and*

- for each $1 \leq j \leq J$, there exists a fixed profile $\psi^j \in H^1(\mathbb{R}^N)$;
- for each $1 \leq j \leq J$, there exists a sequence of time shifts $(t_n^j)_{n \geq 1} \subset \mathbb{R}$;
- for each $1 \leq j \leq J$, there exists a sequence of space shifts $(x_n^j)_{n \geq 1} \subset \mathbb{R}^N$;
- there exists a sequence of remainders $(W_n^J)_{n \geq 1} \subset H^1(\mathbb{R}^N)$;

such that

$$\phi_n(x) = \sum_{j=1}^J e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^J(x).$$

The time and space shifts have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq J$, we have

$$\lim_{n \rightarrow \infty} |t_n^j - t_n^k| + |x_n^j - x_n^k| = \infty. \quad (1.10)$$

The remainder has the following asymptotic smallness property

$$\lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|e^{it \Delta} W_n^J\|_{S(\dot{H}^{\gamma_c}) \cap L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma_c}})} \right] = 0, \quad (1.11)$$

where γ_c is as in (1.3). Moreover, for fixed $J \geq 1$ and $\gamma \in [0, 1]$, we have the asymptotic Pythagorean expansions

$$\|\phi_n\|_{\dot{H}_x^\gamma}^2 = \sum_{j=1}^J \|\psi^j\|_{\dot{H}_x^\gamma}^2 + \|W_n^J\|_{\dot{H}_x^\gamma}^2 + o_n(1). \quad (1.12)$$

Finally, we may assume either $t_n^j \equiv 0$ or $t_n^j \rightarrow \pm\infty$, and either $x_n^j \equiv 0$ or $|x_n^j| \rightarrow \infty$.

Remark 1.4. By the almost orthogonality (1.10), there exists at most one profile with zero space and time shifts. Without loss of generality, we assume ψ^1 is such a profile. That is

$$\phi_n(x) = \psi^1(x) + \sum_{j=2}^J e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^J(x) \quad (1.13)$$

with $|t_n^j| + |x_n^j| \rightarrow \infty$ for each $j = 2, \dots, J$.

Now we state our result on the refined nonlinear profile decomposition for (1.1).

Theorem 1.3. *Let $(\phi_n)_{n \geq 1}$ be a bounded sequence of H^1 -functions and consider the linear profile decomposition (1.13). Denote $u_n : (-T_{*,n}, T^{*,n}) \times \mathbb{R}^N \rightarrow \mathbb{C}$ and $u : (-T_*, T^*) \times \mathbb{R}^N \rightarrow \mathbb{C}$ the maximal solutions to (1.1) with initial data $u_n|_{t=0} = \phi_n$ and $u|_{t=0} = \psi^1$, respectively. Let $I \subset (-T_*, T^*)$ be a compact interval. Then for n sufficiently large, u_n is defined on $I \times \mathbb{R}^N$ and satisfies*

$$\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2)} < \infty. \quad (1.14)$$

Moreover, we have

$$u_n(t, x) = u(t, x) + \sum_{j=2}^J e^{i(t-t_n^j)\Delta} \psi^j(x - x_n^j) + e^{it\Delta} W_n^J(x) + r_n(t, x) \quad (1.15)$$

with

$$\lim_{n \rightarrow \infty} \|r_n\|_{L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma c}) \cap S^1(I, L^2)} = 0. \quad (1.16)$$

In addition, for fixed $J \geq 2$ and $\gamma \in [0, 1]$, we have for every $t \in I$,

$$\|u_n(t)\|_{\dot{H}_x^\gamma}^2 = \|u(t)\|_{\dot{H}_x^\gamma}^2 + \sum_{j=2}^J \|\psi^j\|_{\dot{H}_x^\gamma}^2 + \|W_n^J\|_{\dot{H}_x^\gamma}^2 + \epsilon_n(t) \quad (1.17)$$

with

$$\lim_{n \rightarrow \infty} \epsilon_n(t) = 0 \quad (1.18)$$

uniformly in $t \in I$.

To emphasize our contribution, let us recall the known nonlinear profile decomposition related to (1.1) due to Cardoso, Farah, Guzmán, and Murphy [9] (see also an earlier result of Farah and Guzmán [27]). Let $(\phi_n)_{n \geq 1}$ be a bounded sequence of H^1 -functions with the linear profile decomposition (1.13). We define the nonlinear profiles $v_n^j : I_n^j \times \mathbb{R}^N \rightarrow \mathbb{C}$ associated to ψ^j, t_n^j , and x_n^j as follows:

- If $x_n^j \equiv 0$ and $t_n^j \rightarrow -\infty$, then we take v^j the maximal lifespan solution to (1.1) that scatters to $e^{it\Delta}\psi^j$ as $t \rightarrow \infty$.
- If $x_n^j \equiv 0$ and $t_n^j \rightarrow \infty$, then we take v^j the maximal lifespan solution to (1.1) that scatters to $e^{it\Delta}\psi^j$ as $t \rightarrow -\infty$.

In both cases, we set $v_n^j(t) := v^j(t - t_n^j)$. On the other hand, if $|x_n^j| \rightarrow \infty$, then we take v_n^j the global scattering solution to (1.1) satisfying $v_n^j|_{t=0} = e^{-it_n^j\Delta}\psi^j(x - x_n^j)$. Such a solution exists due to [9, Proposition 3.2]. Let $I \subset (-T_*, T^*)$ be a compact interval, where $u : (-T_*, T^*) \times \mathbb{R}^N \rightarrow \mathbb{C}$ is the maximal solution to (1.1) with initial data $u|_{t=0} = \psi^1$. It was proved (see e.g., [27, Section 6] and [9, Section 3]) that for n sufficiently large, u_n is defined on $I \times \mathbb{R}^N$ and satisfies

$$\limsup_{n \rightarrow \infty} \|u_n\|_{S(I, \dot{H}^{\gamma c}) \cap S^1(I, L^2)} < \infty. \quad (1.19)$$

Moreover, we have

$$u_n(t, x) = u(t, x) + \sum_{j=2}^J v_n^j(t, x) + \tilde{r}_n(t, x) \quad (1.20)$$

with

$$\lim_{n \rightarrow \infty} \|\tilde{r}_n\|_{S(I, \dot{H}^{\gamma c})} = 0. \quad (1.21)$$

Comparing to the nonlinear profile decomposition (1.19) – (1.21), our nonlinear profile decomposition (1.14) – (1.18) contains the following refinements:

- The boundedness of $\|u_n\|_{L_t^\infty(I, H_x^1)}$ for n sufficiently large. This boundedness is not available in (1.19) as the endpoint $(\infty, 2)$ is not included in the definition of $S(I, L^2)$.
- All nonlinear profiles except the first one are solutions to the underlying linear equation instead of the general ones v_n^j as in [9]. Here we show that nonlinear profiles associated with the divergent space shift can be chosen to be the linear solutions. This improvement comes from the observation that the nonlinearity effect vanishes at spatial infinity.
- The remainder term is small for n sufficiently large not only in $S(I, \dot{H}^{\gamma c})$ -norm but also in $L_t^\infty(I, H_x^1)$ and H^1 -Strichartz norms. As a result, we prove the Pythagorean expansion of the \dot{H}^γ -norm of $u_n(t)$ with a remainder term being small uniformly in time.

Based on this refined nonlinear profile decomposition, we are able to show the compactness property of solutions to (1.1) given in Theorem 1.1.

This paper is organized as follows. In Section 2, we give some preliminary results which are needed in the sequel, including: dispersive, Strichartz, and local smoothing estimates; nonlinear estimates, the local well-posedness, and the stability results. Section 3 is devoted to a proof of the refined profile decomposition given in Theorem 1.3. Finally, Theorem 1.1 will be proved in Section 4.

2. PRELIMINARY

In this section, we present some preliminary results which are useful in our proofs. Let us start with the following dispersive estimates.

Lemma 2.1 (Dispersive estimates [10]). *For $r \in [2, \infty]$, we have*

$$\|e^{it\Delta} f\|_{L_x^r} \lesssim |t|^{-\frac{N}{2}(1-\frac{2}{r})} \|f\|_{L_x^{r'}}, \quad t \neq 0 \quad (2.1)$$

for all $f \in L^{r'}(\mathbb{R}^N)$.

A pair (q, r) is called $\dot{H}^{-\gamma}$ -admissible if

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} + \gamma, \quad \frac{2N}{N-2\gamma} < r < \frac{2N}{N-2}.$$

The set of all $\dot{H}^{-\gamma}$ -admissible pairs is denoted by $\mathcal{A}_{-\gamma}$. We also define the dual Strichartz norms

$$\|u\|_{S'(I, \dot{H}^{-\gamma})} := \inf_{(q, r) \in \mathcal{A}_{-\gamma}} \|u\|_{L_t^{q'}(I, L_x^r)}$$

When $I = \mathbb{R}$, we simply use $\|u\|_{S'(\dot{H}^{-\gamma})}$ instead of $\|u\|_{S'(\mathbb{R}, \dot{H}^{-\gamma})}$.

We have the following Strichartz estimates (see e.g., [10, 30, 36]).

Proposition 2.2 (Strichartz estimates [10, 30, 36]). *Let $\gamma \geq 0$. We have*

$$\|e^{it\Delta} f\|_{S(\dot{H}^\gamma)} \leq C \|f\|_{\dot{H}_x^\gamma}.$$

In addition, for any interval $I \subset \mathbb{R}$ containing 0, there exists a constant $C > 0$ independent of I such that

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{S(I, \dot{H}^\gamma)} \leq C \|F\|_{S'(I, \dot{H}^{-\gamma})}.$$

Moreover, the above estimates still hold with $L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma}})$ - and $L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma}})$ -norms in place of $S(\dot{H}^\gamma)$ - and $S(I, \dot{H}^\gamma)$ -norms, respectively.

Lemma 2.3 (Local smoothing estimate [11]). *Let $\psi \in C_0^\infty(\mathbb{R}^N)$. Then for all $f \in L^2(\mathbb{R}^N)$ and $R > 0$,*

$$\iint_{\mathbb{R} \times \mathbb{R}^N} \left| |\nabla|^{\frac{1}{2}} e^{it\Delta} f \right|^2 \psi(x/R) dx dt \lesssim R \|f\|_{L_x^2}^2. \quad (2.2)$$

Corollary 2.4. *Let $\gamma \in (0, 1)$, $(q, r) \in \mathcal{A}_\gamma$, and $\phi \in H^1(\mathbb{R}^N)$. Let $I \subset \mathbb{R}$ be a compact interval and $R > 0$. We have*

$$\|\nabla e^{it\Delta} \phi\|_{L_t^2(I, L_x^2(|x| \leq R))} \lesssim |I|^{\frac{1}{3}(\frac{1}{2} - \frac{1}{q})} R^{\frac{1}{3}(\frac{N+2}{2} - \frac{N}{r})} \|e^{it\Delta} \phi\|_{L_t^q(I, L_x^r)}^{\frac{1}{3}} \|\nabla \phi\|_{L_x^2}^{\frac{2}{3}}. \quad (2.3)$$

Proof. We follow an idea of Killip and Visan [39, Lemma 2.5]. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\text{supp}(\chi) \subset \{x \in \mathbb{R}^N : |x| \leq 2\}$. For $M > 0$ to be chosen later, we denote $\chi_M(\xi) = \chi(M^{-1}\xi)$. By Hölder's inequality and Bernstein's inequality, we have

$$\begin{aligned} \|\nabla e^{it\Delta} \chi_M(D)\phi\|_{L_t^2(I, L_x^2(|x| \leq R))} &= \|e^{it\Delta} |\nabla| \chi_M(D)\phi\|_{L_t^2(I, L_x^2(|x| \leq R))} \\ &\lesssim |I|^{\frac{1}{2} - \frac{1}{q}} R^{\frac{N}{2} - \frac{N}{r}} \|e^{it\Delta} |\nabla| \chi_M(D)\phi\|_{L_t^q(I, L_x^r)} \\ &= |I|^{\frac{1}{2} - \frac{1}{q}} R^{\frac{N}{2} - \frac{N}{r}} M \|e^{it\Delta} \tilde{\chi}_M(D)\phi\|_{L_t^q(I, L_x^r)} \\ &\lesssim |I|^{\frac{1}{2} - \frac{1}{q}} R^{\frac{N}{2} - \frac{N}{r}} M \|e^{it\Delta} \phi\|_{L_t^q(I, L_x^r)}, \end{aligned} \quad (2.4)$$

where $\chi_M(D)$ is the Fourier multiplier by $\chi_M(\xi)$, i.e.,

$$\widehat{\chi_M(D)f}(\xi) = \chi_M(\xi) \hat{f}(\xi)$$

and $\tilde{\chi}_M(\xi) = \tilde{\chi}(M^{-1}\xi)$ with $\tilde{\chi}(\xi) = |\xi| \chi(\xi)$. Here we have $q, r > 2$ as $(q, r) \in \mathcal{A}_\gamma$ with $\gamma > 0$.

On the other hand, we use the local smoothing estimate (2.2) to get

$$\begin{aligned} \|\nabla e^{it\Delta} (1 - \chi_M(D))\phi\|_{L_t^2(I, L_x^2(|x| \leq R))} &= \| |\nabla|^{\frac{1}{2}} e^{it\Delta} |\nabla|^{-\frac{1}{2}} (1 - \chi_M(D)) \nabla \phi \|_{L_t^2(I, L_x^2(|x| \leq R))} \\ &\lesssim R^{\frac{1}{2}} \| |\nabla|^{-\frac{1}{2}} (1 - \chi_M(D)) \nabla \phi \|_{L_x^2} \\ &\lesssim M^{-\frac{1}{2}} R^{\frac{1}{2}} \| \nabla \phi \|_{L_x^2}. \end{aligned} \quad (2.5)$$

Optimizing (2.4) and (2.5) with

$$M = |I|^{-\frac{2}{3}(\frac{1}{2}-\frac{1}{q})} R^{\frac{1}{3}-\frac{2}{3}(\frac{N}{2}-\frac{N}{r})} \|\nabla\phi\|_{L_x^2}^{\frac{2}{3}} \|e^{it\Delta}\phi\|_{L_t^q(I, L_x^r)}^{-\frac{2}{3}},$$

we obtain (2.3). \square

We also need the following nonlinear estimates due to [33, Lemma 4.1] and [7, Lemma 2.5].

Lemma 2.5 ([33, Lemma 4.1]). *Let $N \geq 3$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. Then there exists $\theta \in (0, \alpha)$ sufficiently small so that*

$$\| |x|^{-b} |u|^\alpha v \|_{S'(\dot{H}^{-\gamma_c})} \lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{\gamma_c})}^{\alpha-\theta} \|v\|_{S(\dot{H}^{\gamma_c})}.$$

When $b = 0$, we can take $\theta = 0$ in the above estimate.

Proof. The proof of this result is given in [33, Lemam 4.1]. However, for our later purposes, we recall some details. Let $\theta > 0$ be a small parameter. We denote

$$\begin{aligned} \hat{a} &= \frac{2\alpha(\alpha+2-\theta)}{4-2b-(N-2)\alpha}, & \hat{r} &= \frac{N\alpha(\alpha+2-\theta)}{\alpha(N-b)-\theta(2-b)}, \\ \tilde{a} &= \frac{2\alpha(\alpha+2-\theta)}{\alpha(N(\alpha+1-\theta)-2+2b)-(4-2b)(1-\theta)}, & \bar{m}_\pm &= \frac{N\alpha}{2-b \mp N\alpha\theta}. \end{aligned}$$

We see that $(\hat{a}, \hat{r}) \in \mathcal{A}_{\gamma_c}$ and $(\tilde{a}, \hat{r}) \in \mathcal{A}_{-\gamma_c}$ provided that $\theta > 0$ is taken sufficiently small. By Hölder's inequality, we have

$$\| |x|^{-b} |u|^\alpha v \|_{L_x^{\hat{r}'}} \leq \| |x|^{-b} \|_{L_x^{\gamma(A)}} \| |u|^\alpha v \|_{L_x^{\hat{r}}},$$

where A is either the unit ball B_1 or $B_1^c = \mathbb{R}^N \setminus B_1$. To make $\| |x|^{-b} \|_{L_x^{\gamma(A)}} < \infty$, we take

$$\frac{1}{\gamma} = \frac{b}{N} \pm \theta^2,$$

where the plus sign is for $A = B_1$ and the minus sign is for $A = B_1^c$. It follows that

$$\frac{1}{\rho} = \frac{1}{\hat{r}'} - \frac{1}{\gamma} = \frac{(N\alpha + b\alpha + N + 3b)\alpha - \theta(N\alpha + b\alpha + b - 2)}{N\alpha(\alpha + 2 - \theta)} \mp \theta^2.$$

By Hölder's inequality with the fact that

$$\frac{1}{\rho} = \frac{\theta}{\bar{m}_\pm} + \frac{\alpha + 1 - \theta}{\hat{r}},$$

we have

$$\| |x|^{-b} |u|^\alpha v \|_{L_x^{\hat{r}'}} \lesssim \|u\|_{L_x^{\bar{m}_\pm}}^\theta \|u\|_{L_x^{\hat{r}}}^{\alpha-\theta} \|v\|_{L_x^{\hat{r}}}.$$

As

$$\frac{1}{\tilde{a}'} = \frac{\alpha + 1 - \theta}{\hat{a}},$$

we infer that

$$\begin{aligned} \| |x|^{-b} |u|^\alpha v \|_{L_t^{\tilde{a}'} L_x^{\hat{r}'}} &\lesssim \|u\|_{L_t^\infty L_x^{\bar{m}_\pm}}^\theta \|u\|_{L_t^{\hat{a}} L_x^{\hat{r}}}^{\alpha-\theta} \|v\|_{L_t^{\hat{a}} L_x^{\hat{r}}} \\ &\lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{\hat{a}} L_x^{\hat{r}}}^{\alpha-\theta} \|v\|_{L_t^{\hat{a}} L_x^{\hat{r}}}. \end{aligned}$$

Here in the second line, we have used the Sobolev embedding as $2 < \bar{m}_\pm < \frac{2N}{N-2}$ for $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$ provided that $\theta > 0$ is chosen sufficiently small. The proof is complete. \square

Lemma 2.6 ([7, 33]). *Let $N \geq 3$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. Then there exists $\theta \in (0, \alpha)$ sufficiently small so that*

$$\begin{aligned} \| |x|^{-b} |u|^\alpha v \|_{S'(L^2)} &\lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{\gamma_c})}^{\alpha-\theta} \|v\|_{S(L^2)}, \\ \|\nabla(|x|^{-b} |u|^\alpha u)\|_{S'(L^2)} &\lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla u\|_{S(L^2)}, \\ \| |x|^{-b-1} |u|^\alpha v \|_{S'(L^2)} &\lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla v\|_{S(L^2)}. \end{aligned}$$

Note that if $b = 0$, we can take $\theta = 0$ in the above estimates.

Proof. These estimates were essentially proved in [33, Lemma 4.2] and [7, Lemma 2.5]. However, the proofs presented in [33] and [7] used the dual pair of the end-point $\left(2, \frac{2N}{N-2}\right)$ which is excluded in our definition of L^2 -admissible pair. Thus we need a different argument. Let $\theta > 0$ be a small parameter. We denote

$$\begin{aligned} q' &= \frac{4}{2+\theta}, & r' &= \frac{2N}{N+2-\theta}, \\ \bar{a} &= \frac{4\alpha(\alpha+1-\theta)}{4-2b-(N-2)\alpha+\theta\alpha}, & \bar{r} &= \frac{2N\alpha(\alpha+1-\theta)}{(N+2-2b)\alpha-\theta(4-2b+\alpha)}, \\ \bar{q} &= \frac{4\alpha(\alpha+1-\theta)}{\alpha(N\alpha-2+2b)-\theta(N\alpha-4+2b-\alpha)}, & \bar{m}_{\pm} &= \frac{N\alpha}{2-b\mp N\alpha\theta}. \end{aligned}$$

Here (q', r') is the dual pair of $\left(\frac{4}{2-\theta}, \frac{2N}{N-2+\theta}\right) \in \mathcal{A}_0$. We can readily check that $(\bar{q}, \bar{r}) \in \mathcal{A}_0$ and $(\bar{a}, \bar{r}) \in \mathcal{A}_{\gamma_c}$ provided that $\theta > 0$ is taken sufficiently small.

For the first two estimates, it suffices to show the second one since the first is treated in a similar manner. We observe that

$$\nabla(|x|^{-b}|u|^{\alpha}u) = |x|^{-b}\nabla(|u|^{\alpha}u) - b\frac{x}{|x|}|x|^{-b}(|x|^{-1}|u|^{\alpha}u) \quad (2.6)$$

and

$$\||x|^{-b}f\|_{L_{x'}^{r'}(A)} \leq \||x|^{-b}\|_{L_{x'}^{r_1}(A)} \|f\|_{L_{x'}^{r_2}},$$

where A stands for either B_1 or B_1^c . To ensure $\||x|^{-b}\|_{L_{x'}^{r_1}(A)} < \infty$, we take

$$\frac{1}{r_1} = \frac{b}{N} \pm \theta^2,$$

where the plus sign is for $A = B_1$ and the minus one is for $A = B_1^c$. It follows that

$$\frac{1}{r_2} = \frac{1}{r'} - \frac{1}{r_1} = \frac{N+2-2b-\theta}{2N} \mp \theta^2.$$

As $\frac{1}{N} < \frac{N+2-2b}{2N} < 1$ for $0 < b < \frac{N}{2}$, we choose $\theta > 0$ sufficiently small so that $1 < r_2 < N$ which allows us to use the Hardy's inequality (see e.g., [46])

$$\||x|^{-1}f\|_{L_{x'}^{r_2}} \leq \frac{r_2}{N-r_2} \|\nabla f\|_{L_{x'}^{r_2}}.$$

Applying the above inequality to $f = |u|^{\alpha}u$ and using (2.6), we see that

$$\|\nabla(|x|^{-b}|u|^{\alpha}u)\|_{L_{x'}^{r'}} \lesssim \|\nabla(|u|^{\alpha}u)\|_{L_{x'}^{r_2}}.$$

By Hölder's inequality and the fact that

$$\frac{1}{r_2} = \frac{\theta}{\bar{m}_{\pm}} + \frac{\alpha+1-\theta}{\bar{r}},$$

we have

$$\|\nabla(|x|^{-b}|u|^{\alpha}u)\|_{L_{x'}^{r'}} \lesssim \|u\|_{L_x^{\bar{m}_{\pm}}}^{\theta} \|u\|_{L_x^{\bar{r}}}^{\alpha-\theta} \|\nabla u\|_{L_x^{\bar{r}}}.$$

By Hölder's inequality in time with

$$\frac{1}{q'} = \frac{\alpha-\theta}{\bar{a}} + \frac{1}{\bar{q}},$$

we get

$$\begin{aligned} \|\nabla(|x|^{-b}|u|^{\alpha}u)\|_{L_t^{q'} L_{x'}^{r'}} &\lesssim \|u\|_{L_t^{\infty} L_x^{\bar{m}_{\pm}}}^{\theta} \|u\|_{L_t^{\bar{q}} L_x^{\bar{r}}}^{\alpha-\theta} \|\nabla u\|_{L_t^{\bar{q}} L_x^{\bar{r}}} \\ &\lesssim \|u\|_{L_t^{\infty} H_x^1}^{\theta} \|u\|_{L_t^{\bar{q}} L_x^{\bar{r}}}^{\alpha-\theta} \|\nabla u\|_{L_t^{\bar{q}} L_x^{\bar{r}}} \end{aligned}$$

which proves the second estimate.

Let us show the last one. We have

$$\||x|^{-b-1}|u|^{\alpha}v\|_{L_{x'}^{r'}(A)} \leq \||x|^{-b-1}\|_{L_{x'}^{p_1}(A)} \| |u|^{\alpha}v \|_{L_{x'}^{p_2}},$$

where A is either B_1 or B_1^c . In order to make $\||x|^{-b-1}\|_{L_{x'}^{p_1}(A)} < \infty$, we take

$$\frac{1}{p_1} = \frac{b+1}{N} \pm \theta^2,$$

where the plus sign is for $A = B_1$ and the minus sign is for $A = B_1^c$. We see that

$$\frac{1}{p_2} = \frac{1}{r'} - \frac{1}{p_1} = \frac{N - 2b - \theta}{2N} \mp \theta^2.$$

By Hölder's inequality and the fact that

$$\frac{1}{p_2} = \frac{\theta}{\bar{m}_\pm} + \frac{\alpha - \theta}{\bar{r}} + \frac{1}{r_3}, \quad \frac{1}{r_3} = \frac{1}{\bar{r}} - \frac{1}{N},$$

we have

$$\begin{aligned} \| |x|^{-b-1} |u|^\alpha v \|_{L_x^{r'}} &\lesssim \|u\|_{L_x^{\bar{m}_\pm}}^\theta \|u\|_{L_x^{\bar{r}}}^{\alpha-\theta} \|v\|_{L_x^{r_3}} \\ &\lesssim \|u\|_{L_x^{\bar{m}_\pm}}^\theta \|u\|_{L_x^{\bar{r}}}^{\alpha-\theta} \|\nabla v\|_{L_x^{\bar{r}}}. \end{aligned}$$

Using Hölder's inequality in time, we get

$$\begin{aligned} \| |x|^{-b-1} |u|^\alpha v \|_{L_t^{q'} L_x^{r'}} &\lesssim \|u\|_{L_t^\infty L_x^{\bar{m}_\pm}}^\theta \|u\|_{L_t^{\bar{q}} L_x^{\bar{r}}}^{\alpha-\theta} \|\nabla v\|_{L_t^{\bar{q}} L_x^{\bar{r}}} \\ &\lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{\bar{q}} L_x^{\bar{r}}}^{\alpha-\theta} \|\nabla v\|_{L_t^{\bar{q}} L_x^{\bar{r}}}. \end{aligned}$$

The proof is complete. \square

Using Proposition 2.2, Lemma 2.5, and Lemma 2.6, we have the following result local well-posedness result.

Proposition 2.7 (Local well-posedness [33]). *Let $N \geq 3$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. Let $u_0 \in H^1(\mathbb{R}^N)$. Then there exist $T_*, T^* \in (0, \infty]$, and a unique local solution to (1.1) satisfying*

$$u \in C((-T_*, T^*), H^1(\mathbb{R}^N)) \cap L_{\text{loc}}^q(-T_*, T^*), W^{1,r}(\mathbb{R}^N)$$

for any $(q, r) \in \mathcal{A}_0$. If $T^* < \infty$ (resp. $T_* < \infty$), then

$$\lim_{t \nearrow T^*} \|\nabla u(t)\|_{L_x^2} = \infty \quad \left(\text{resp.} \quad \lim_{t \searrow -T_*} \|\nabla u(t)\|_{L_x^2} = \infty \right).$$

We also recall the following stability result which is needed in the sequel.

Lemma 2.8 (Short time perturbation). *Let $N \geq 3$, $0 < b < \min\{2, \frac{N}{2}\}$, $\nu \in \{\pm 1\}$, and $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. Let $0 \in I \subseteq \mathbb{R}$ and $\tilde{u} : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ be a solution to*

$$i\partial_t \tilde{u} + \Delta \tilde{u} - \nu |x|^{-b} |\tilde{u}|^\alpha \tilde{u} = e$$

with $\tilde{u}|_{t=0} = \tilde{u}_0$ satisfying

$$\|\tilde{u}\|_{L_t^\infty(I, H_x^1)} \leq M$$

for some positive constant $M > 0$. Let $u_0 \in H^1$ be such that

$$\|u_0 - \tilde{u}_0\|_{H_x^1} \leq M'$$

for some $M' > 0$. Assume the smallness conditions

$$\|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} \leq \varepsilon_0,$$

$$\|e^{it\Delta}(u_0 - \tilde{u}_0)\|_{S(I, \dot{H}^{\gamma_c})} \leq \varepsilon,$$

$$\|\langle \nabla \rangle e\|_{S'(I, L^2)} + \|e\|_{S'(I, \dot{H}^{-\gamma_c})} \leq \varepsilon,$$

for some $0 < \varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(M, M') > 0$ is a small constant. Then there exists a unique solution $u : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ to (1.1) with $u|_{t=0} = u_0$ satisfying

$$\|u - \tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} \lesssim \varepsilon,$$

$$\|u - \tilde{u}\|_{L_t^\infty(I, H_x^1) \cap S^1(I, L^2)} \lesssim M', \quad (2.7)$$

$$\|u\|_{L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2)} \leq C(M, M').$$

Proof. The proof is essentially given in [27, Lemma 4.12]. However, the version stated in [27, Lemma 4.12] does not provide enough information for our purpose. Thus we give some details. Let us start with the following observation.

Observation 2.1. If $\|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} \leq \varepsilon_0$ with $\varepsilon_0 > 0$ sufficiently small, then

$$\|\tilde{u}\|_{S^1(I, L^2)} \lesssim M. \quad (2.8)$$

Proof of Observation 2.1. By Duhamel formula, we have

$$\tilde{u}(t) = e^{it\Delta}\tilde{u}_0 - i \int_0^t e^{i(t-s)\Delta} (\nu|x|^{-b}|\tilde{u}(s)|^\alpha \tilde{u}(s) + e(s)) ds.$$

By Strichartz estimates and Lemma 2.6, we have

$$\begin{aligned} \|\tilde{u}\|_{S^1(I, L^2)} &\lesssim \|\tilde{u}_0\|_{H_x^1} + \|\langle \nabla \rangle (|x|^{-b}|\tilde{u}|^\alpha \tilde{u})\|_{S'(I, L^2)} + \|\langle \nabla \rangle e\|_{S'(I, L^2)} \\ &\lesssim \|\tilde{u}_0\|_{H_x^1} + \|\tilde{u}\|_{L_t^\infty(I, H_x^1)}^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\tilde{u}\|_{S^1(I, L^2)} + \|\langle \nabla \rangle e\|_{S'(I, L^2)} \\ &\lesssim \|\tilde{u}\|_{L_t^\infty(I, H_x^1)} + \|\tilde{u}\|_{L_t^\infty(I, H_x^1)}^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\tilde{u}\|_{S^1(I, L^2)} + \|\langle \nabla \rangle e\|_{S'(I, L^2)} \\ &\leq CM + C\varepsilon + M^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\tilde{u}\|_{S^1(I, L^2)}. \end{aligned}$$

If $\|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} \leq \varepsilon_0$ with $\varepsilon_0 > 0$ sufficiently small, then we get $\|\tilde{u}\|_{S^1(I, L^2)} \leq 2C(M + \varepsilon)$. \square

Without loss of generality, we assume that $0 = \inf I$. We will prove the existence of a solution to the following Cauchy problem

$$\begin{cases} i\partial_t w + \Delta w &= -H(x, \tilde{u}, w) - e, \\ w|_{t=0} &= w_0 := u_0 - \tilde{u}_0, \end{cases} \quad (2.9)$$

where $H(x, \tilde{u}, w) := \nu|x|^{-b}(G(\tilde{u} + w) - G(\tilde{u}))$ with

$$G(z) := |z|^\alpha z. \quad (2.10)$$

To this end, we will show that the functional

$$\Phi(w)(t) := e^{it\Delta}w_0 + i \int_0^t e^{i(t-s)\Delta} (H(x, \tilde{u}, w) + e)(s) ds$$

is a contraction mapping on

$$Y := \left\{ w \in C(I, H^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2) : \|w\|_{S(I, \dot{H}^{\gamma_c})} \leq \rho, \quad \|w\|_{L_t^\infty(I, H_x^1) \cap S^1(I, L^2)} \leq K \right\}$$

equipped with the distance

$$d(u, v) := \|u - v\|_{S(I, L^2)} + \|u - v\|_{S(I, \dot{H}^{\gamma_c})},$$

where $\rho \sim \varepsilon$ and $K \sim M'$ will be chosen later. By Strichartz estimates, we have

$$\begin{aligned} \|\Phi(w)\|_{S(I, \dot{H}^{\gamma_c})} &\lesssim \|e^{it\Delta}w_0\|_{S(I, \dot{H}^{\gamma_c})} + \|H(\cdot, \tilde{u}, w)\|_{S'(I, \dot{H}^{-\gamma_c})} + \|e\|_{S'(I, \dot{H}^{-\gamma_c})}, \\ \|\Phi(w)\|_{L_t^\infty(I, L_x^2) \cap S(I, L^2)} &\lesssim \|w_0\|_{L_x^2} + \|H(\cdot, \tilde{u}, w)\|_{S'(I, L^2)} + \|e\|_{S'(I, L^2)}, \end{aligned}$$

and

$$\|\nabla \Phi(w)\|_{L_t^\infty(I, L_x^2) \cap S(I, L^2)} \lesssim \|\nabla w_0\|_{L_x^2} + \|\nabla H(\cdot, \tilde{u}, w)\|_{S'(I, L^2)} + \|\nabla e\|_{S'(I, L^2)}.$$

Using the fact that

$$|G(\tilde{u} + w) - G(\tilde{u})| \lesssim (|\tilde{u}|^\alpha + |w|^\alpha)|w|,$$

we infer from Lemma 2.5 and Lemma 2.6 that

$$\begin{aligned} \|H(\cdot, \tilde{u}, w)\|_{S'(I, \dot{H}^{-\gamma_c})} &\lesssim \left(\|\tilde{u}\|_{L_t^\infty(I, H_x^1)}^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} + \|w\|_{L_t^\infty(I, H_x^1)}^\theta \|w\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \right) \|w\|_{S(I, \dot{H}^{\gamma_c})}, \\ \|H(\cdot, \tilde{u}, w)\|_{S'(I, L^2)} &\lesssim \left(\|\tilde{u}\|_{L_t^\infty(I, H_x^1)}^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} + \|w\|_{L_t^\infty(I, H_x^1)}^\theta \|w\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \right) \|w\|_{S(I, L^2)}. \end{aligned}$$

We also have (see e.g., [27])

$$|\nabla(G(\tilde{u} + w) - G(\tilde{u}))| \lesssim (|\tilde{u}|^\alpha + |w|^\alpha)|\nabla w| + \begin{cases} |w|^\alpha |\nabla \tilde{u}| & \text{if } 0 < \alpha \leq 1, \\ (|\tilde{u}|^{\alpha-1} + |w|^{\alpha-1})|w| |\nabla \tilde{u}| & \text{if } \alpha > 1. \end{cases}$$

By Lemma 2.6, we have

$$\|\nabla H(\cdot, \tilde{u}, w)\|_{S'(I, L^2)} \lesssim \left(\|\tilde{u}\|_{L_t^\infty(I, H_x^1)}^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} + \|w\|_{L_t^\infty(I, H_x^1)}^\theta \|w\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \right) \|\nabla w\|_{S(I, L^2)} + E,$$

where

$$E = \|w\|_{L_t^\infty(I, H_x^1)}^\theta \|w\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla \tilde{u}\|_{S(I, L^2)} \text{ if } 0 < \alpha \leq 1$$

and

$$E = \left(\|\tilde{u}\|_{L_t^\infty(I, H_x^1)}^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-1-\theta} + \|w\|_{L_t^\infty(I, H_x^1)}^\theta \|w\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-1-\theta} \right) \|w\|_{S(I, \dot{H}^{\gamma_c})} \|\nabla \tilde{u}\|_{S(I, L^2)} \text{ if } \alpha > 1.$$

It follows that for $w \in Y$, we have from (2.8) that

$$\begin{aligned} \|H(\cdot, \tilde{u}, w)\|_{S'(I, \dot{H}^{\gamma_c})} &\lesssim (M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) \rho, \\ \|H(\cdot, \tilde{u}, w)\|_{S'(I, L^2)} &\lesssim (M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) K, \end{aligned}$$

and

$$\begin{aligned} \|\nabla H(\cdot, \tilde{u}, w)\|_{S'(I, L^2)} &\lesssim (M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) K \\ &+ \begin{cases} K^\theta \rho^{\alpha-\theta} M & \text{if } 0 < \alpha \leq 1, \\ (M^\theta \varepsilon_0^{\alpha-1-\theta} + K^\theta \rho^{\alpha-1-\theta}) \rho M & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

Thus there exists $C > 0$ such that

$$\begin{aligned} \|\Phi(w)\|_{S(I, \dot{H}^{\gamma_c})} &\leq C\varepsilon + C(M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) \rho, \\ \|\Phi(w)\|_{L_t^\infty(I, L_x^2) \cap S(I, L^2)} &\leq C\|w_0\|_{L_x^2} + C\varepsilon + C(M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) K, \end{aligned}$$

and

$$\begin{aligned} \|\nabla \Phi(w)\|_{L_t^\infty(I, L_x^2) \cap S(I, L^2)} &\leq C\|\nabla w_0\|_{L_x^2} + C\varepsilon + C(M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) K \\ &+ \begin{cases} CK^\theta \rho^{\alpha-\theta} M & \text{if } 0 < \alpha \leq 1, \\ C(M^\theta \varepsilon_0^{\alpha-1-\theta} + K^\theta \rho^{\alpha-1-\theta}) \rho M & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} \|\Phi(w)\|_{S(I, \dot{H}^{\gamma_c})} &\leq C\varepsilon + C(M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) \rho, \\ \|\Phi(w)\|_{L_t^\infty(I, H_x^1) \cap S^1(I, L^2)} &\leq CM' + C\varepsilon + C(M^\theta \varepsilon_0^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}) K \\ &+ \begin{cases} CK^\theta \rho^{\alpha-\theta} M & \text{if } 0 < \alpha \leq 1, \\ C(M^\theta \varepsilon_0^{\alpha-1-\theta} + K^\theta \rho^{\alpha-1-\theta}) \rho M & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

By taking $\rho = 2C\varepsilon$ and $K = 2CM'$, we see that

$$\|\Phi(w)\|_{S(I, \dot{H}^{\gamma_c})} \leq C\varepsilon + C(M^\theta \varepsilon_0^{\alpha-\theta} + (2CM')^\theta (2C\varepsilon)^{\alpha-\theta}) 2C\varepsilon$$

and

$$\begin{aligned} \|\Phi(w)\|_{L_t^\infty(I, H_x^1) \cap S^1(I, L^2)} &\leq CM' + C\varepsilon + C(M^\theta \varepsilon_0^{\alpha-\theta} + (2CM')^\theta (2C\varepsilon)^{\alpha-\theta}) 2CM' \\ &+ \begin{cases} C(2CM')^\theta (2C\varepsilon)^{\alpha-\theta} M & \text{if } 0 < \alpha \leq 1, \\ C(M^\theta \varepsilon_0^{\alpha-1-\theta} + (2CM')^\theta (2C\varepsilon)^{\alpha-1-\theta}) \rho M & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

By choosing $\varepsilon_0 = \varepsilon_0(M, M') > 0$ sufficiently small, we have for all $0 < \varepsilon \leq \varepsilon_0$,

$$\|\Phi(w)\|_{S(I, \dot{H}^{\gamma_c})} \leq \rho, \quad \|\Phi(w)\|_{L_t^\infty(I, H_x^1) \cap S^1(I, L^2)} \leq K$$

or Φ maps Y into itself. By the same argument, we prove as well that Φ is a contraction on Y . Thus the fixed point theorem implies that there exists a unique solution w on $I \times \mathbb{R}^N$ satisfying

$$\|w\|_{S(I, \dot{H}^{\gamma_c})} \lesssim \varepsilon, \quad \|w\|_{L_t^\infty(I, H_x^1) \cap S^1(I, L^2)} \lesssim M'.$$

Finally, we see that $u = \tilde{u} + w$ is a solution to (1.1) satisfying (2.7) (see also (2.8)). The proof is complete. \square

By an iteration argument as in [27, Lemma 4.14], we have the following long time perturbation result.

Lemma 2.9 (Long time perturbation [27, Lemma 4.14]). *Let $N \geq 3$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$. Let $0 \in I \subseteq \mathbb{R}$ and $\tilde{u} : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ be a solution to*

$$i\partial_t \tilde{u} + \Delta \tilde{u} + |x|^{-b} |\tilde{u}|^\alpha \tilde{u} = e$$

with $\tilde{u}|_{t=0} = \tilde{u}_0$ satisfying

$$\|\tilde{u}\|_{L_t^\infty(I, H_x^1)} \leq M, \quad \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} \leq L$$

for some constants $M, L > 0$. Let $u_0 \in H^1$ be such that

$$\|u_0 - \tilde{u}_0\|_{H_x^1} \leq M', \quad \|e^{it\Delta}(u_0 - \tilde{u}_0)\|_{S(I, \dot{H}^{\gamma_c})} \leq \varepsilon$$

for some constant $M' > 0$ and some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(M, M', L)$. Suppose that

$$\|\langle \nabla \rangle e\|_{S'(I, L^2)} + \|e\|_{S'(I, \dot{H}^{-\gamma_c})} \leq \varepsilon.$$

Then there exists a unique solution $u : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ to (1.1) with $u|_{t=0} = u_0$ satisfying

$$\begin{aligned} \|u - \tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} &\leq C(M, M', L)\varepsilon, \\ \|u - \tilde{u}\|_{L_t^\infty(I, H_x^1) \cap S^1(I, L^2)} &\leq C(M, M', L)M', \\ \|u\|_{L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2)} &\leq C(M, M', L). \end{aligned} \quad (2.11)$$

We also need the following property due to [28, Lemma 5.3].

Lemma 2.10 ([28, Lemma 5.3]). *Let $\psi \in H^1(\mathbb{R}^N)$, $(t_n)_{n \geq 1} \subset \mathbb{R}$, and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$. If $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$, then*

$$e^{it_n \Delta} \psi(\cdot + x_n) \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^N)$$

as $n \rightarrow \infty$. Moreover, if $(\psi_n)_{n \geq 1} \subset H^1(\mathbb{R}^N)$ satisfies

$$\psi_n \rightharpoonup 0, \quad e^{it_n \Delta} \psi_n(\cdot + x_n) \rightharpoonup \psi \text{ weakly in } H^1(\mathbb{R}^N)$$

as $n \rightarrow \infty$ for some $\psi \in H^1(\mathbb{R}^N) \setminus \{0\}$, then $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$.

3. REFINED PROFILE DECOMPOSITION

This section is devoted to the proof of the refined profile decomposition given in Theorem 1.3. Let $(\phi_n)_{n \geq 1}$ be a bounded sequence of H^1 -functions and consider the linear profile decomposition

$$\phi_n(x) = \psi^1(x) + \sum_{j=2}^J e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^J(x)$$

with $|t_n^j| + |x_n^j| \rightarrow \infty$ for each $j = 2, \dots, J$. Keeping in mind the properties (1.11) and (1.12). Denote $u_n : (-T_{*,n}, T_{*,n}) \times \mathbb{R}^N \rightarrow \mathbb{C}$ and $u : (-T_*, T_*) \times \mathbb{R}^N \rightarrow \mathbb{C}$ the maximal solutions to (1.1) with initial data $u_n|_{t=0} = \phi_n$ and $u|_{t=0} = \psi^1$, respectively.

Let $I \subset (-T_*, T_*)$ be a compact interval. We will approximate u_n over $I \times \mathbb{R}^N$ by

$$u_n^J(t, x) = u(t, x) + \sum_{j=2}^J e^{i(t-t_n^j)\Delta} \psi^j(x - x_n^j) + e^{it\Delta} W_n^J(x).$$

To this end, we make use of the stability theory given in Lemma 2.9. We first have

$$u_n^J(0, x) = \phi_n(x)$$

and

$$i\partial_t u_n^J + \Delta u_n^J + |x|^{-b} |u_n^J|^\alpha u_n^J = e_n^J,$$

where

$$e_n^J = |x|^{-b} (G(u_n^J) - G(u))$$

with $G(z)$ as in (2.10). We will check the following conditions:

$$\sup_{J \geq 2} \limsup_{n \rightarrow \infty} \|u_n^J\|_{L_t^\infty(I, H_x^1)} \leq M, \quad (3.1)$$

$$\sup_{J \geq 2} \limsup_{n \rightarrow \infty} \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})} \leq L, \quad (3.2)$$

for some constant $M, L > 0$ and

$$\|e_n^J\|_{S'(I, L^2)} + \|\nabla e_n^J\|_{S'(I, L^2)} + \|e_n^J\|_{S'(I, \dot{H}^{-\gamma_c})} \rightarrow 0 \text{ as } J, n \rightarrow \infty. \quad (3.3)$$

Towards this goal, let us prepare some lemmas.

Lemma 3.1. *We have*

$$\sup_{J \geq 2} \limsup_{n \rightarrow \infty} \|u_n^J\|_{L_t^\infty(I, H_x^1)} \leq C(I, B),$$

where

$$B := \sup_{n \geq 1} \|\phi_n\|_{H_x^1}. \quad (3.4)$$

Proof. We have for any $t \in I$ and $J \geq 2$,

$$\begin{aligned} \|u_n^J(t)\|_{H_x^1}^2 &\lesssim \|u(t)\|_{H_x^1}^2 + \left\| \sum_{j=2}^J e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j) + e^{it\Delta} W_n^J \right\|_{H_x^1}^2 \\ &= \|u(t)\|_{H_x^1}^2 + \left\| \sum_{j=2}^J e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j) + W_n^J \right\|_{H_x^1}^2 \\ &= \|u(t)\|_{H_x^1}^2 + \sum_{j=2}^J \|e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j)\|_{H_x^1}^2 + \|W_n^J\|_{H_x^1}^2 + o_n(1). \end{aligned}$$

Here we have used the pairwise divergence property (1.10) to get the second inequality. It follows that

$$\|u_n^J\|_{L_t^\infty(I, H_x^1)}^2 \leq C(I, \|\psi^1\|_{H_x^1}) + \sum_{j=2}^J \|\psi^j\|_{H_x^1}^2 + \|W_n^J\|_{H_x^1}^2 + o_n(1)$$

which implies

$$\limsup_{n \rightarrow \infty} \|u_n^J\|_{L_t^\infty(I, H_x^1)} \leq C(I, B)$$

for some constant $C(I, B)$ depending on I and B . \square

Lemma 3.2. *We have*

$$\sup_{J \geq 2} \limsup_{n \rightarrow \infty} \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})} \leq C(I, B).$$

Proof. By Strichartz estimates and the pairwise divergence property, we have for any $J \geq 2$,

$$\begin{aligned} \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^2 &\lesssim \|u\|_{S(I, \dot{H}^{\gamma_c})}^2 + \left\| \sum_{j=2}^J e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j) + e^{it\Delta} W_n^J \right\|_{S(I, \dot{H}^{\gamma_c})}^2 \\ &\lesssim \|u\|_{S(I, \dot{H}^{\gamma_c})}^2 + \left\| \sum_{j=2}^J e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j) + W_n^J \right\|_{H_x^1}^2 \\ &\leq C(I, \|\psi^1\|_{H_x^1}) + \sum_{j=2}^J \|\psi^j\|_{H_x^1}^2 + \|W_n^J\|_{H_x^1}^2 + o_n(1). \end{aligned}$$

This shows that

$$\limsup_{n \rightarrow \infty} \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})} \leq C(I, B). \quad \square$$

Lemma 3.3. *We have*

$$\|e_n^J\|_{S'(I, L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof. Using the fact that

$$|G(z) - G(w)| \lesssim (|z|^\alpha + |w|^\alpha) |z - w|, \quad (3.5)$$

we have

$$\begin{aligned} \|e_n^J\|_{S'(I,L^2)} &\lesssim \| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) (u_n^J - u) \|_{S'(I,L^2)} \\ &\lesssim \sum_{j=2}^J \| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j) \|_{S'(I,L^2)} \\ &\quad + \| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) e^{it\Delta} W_n^J \|_{S'(I,L^2)}. \end{aligned} \quad (3.6)$$

Observation 3.1. We have

$$\| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) e^{it\Delta} W_n^J \|_{S'(I,L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof of Observation 3.1. In fact, by Lemmas 2.6, 3.1, and 3.2, we have

$$\begin{aligned} \| |x|^{-b} |u_n^J|^\alpha e^{it\Delta} W_n^J \|_{S'(I,L^2)} &\lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|e^{it\Delta} W_n^J\|_{S(I,L^2)} \\ &\lesssim C(I, B) \|e^{it\Delta} W_n^J\|_{S(I,L^2)}. \end{aligned}$$

Let $(q, r) \in \mathcal{A}_0$. As $2 < \frac{2N}{N-2\gamma_c} < \frac{2N}{N-2}$ due to $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$, we interpolate between L^2 and $L^{\frac{2N}{N-2\gamma_c}}$, or $L^{\frac{2N}{N-2\gamma_c}}$ and $L^{\frac{2N}{N-2}}$, and use Sobolev embedding to get

$$\begin{aligned} \|e^{it\Delta} W_n^J\|_{L_t^q(I, L_x^r)} &\lesssim |I|^{\frac{1}{q}} \|e^{it\Delta} W_n^J\|_{L_t^\infty(I, L_x^r)} \\ &\lesssim |I|^{\frac{1}{q}} \|e^{it\Delta} W_n^J\|_{L_t^\infty(I, H_x^1)}^\vartheta \|e^{it\Delta} W_n^J\|_{L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma_c}}}^{1-\vartheta} \\ &\lesssim |I|^{\frac{1}{q}} \|W_n^J\|_{H_x^1}^\vartheta \|e^{it\Delta} W_n^J\|_{L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma_c}}}^{1-\vartheta} \rightarrow 0 \end{aligned}$$

as $J, n \rightarrow \infty$, where $\vartheta \in (0, 1)$ is a constant depending on r . It follows that

$$\|e^{it\Delta} W_n^J\|_{S(I,L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

This shows that

$$\| |x|^{-b} |u_n^J|^\alpha e^{it\Delta} W_n^J \|_{S'(I,L^2)} \rightarrow \infty \text{ as } J, n \rightarrow \infty$$

and similarly for $\| |x|^{-b} |u|^\alpha e^{it\Delta} W_n^J \|_{S'(I,L^2)}$. Thus we prove Observation 3.1. \square

Observation 3.2. Let $(t_n)_{n \geq 1} \subset \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ satisfy $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then we have for any $\psi \in H^1(\mathbb{R}^N)$,

$$\| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I,L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of Observation 3.2. To see this, we consider two cases: $x_n \equiv 0$ and $|x_n| \rightarrow \infty$.

Case 1. $x_n \equiv 0$. We must have $|t_n| \rightarrow \infty$. By Lemmas 2.6, 3.1, and 3.2, we have

$$\begin{aligned} \| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I,L^2)} &\lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|e^{i(t-t_n)\Delta} \psi\|_{S(I,L^2)} \\ &\lesssim \|e^{i(t-t_n)\Delta} \psi\|_{S(I,L^2)}. \end{aligned}$$

For $(q, r) \in \mathcal{A}_0$, we have

$$\|e^{i(t-t_n)\Delta} \psi\|_{L_t^q(I, L_x^r)} \lesssim |I|^{\frac{1}{q}} \|e^{i(t-t_n)\Delta} \psi\|_{L_t^\infty(I, L_x^r)}.$$

As I is compact, we see that $|t - t_n| \rightarrow \infty$ for each $t \in I$ as $n \rightarrow \infty$ which, by dispersive estimates (2.1), implies

$$\|e^{i(t-t_n)\Delta} \psi\|_{L_t^\infty(I, L_x^r)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus we get

$$\| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I,L^2)} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Case 2. $|x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Take $\varepsilon > 0$. We have

$$\begin{aligned} \| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I,L^2)} &= \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I,L^2)} \\ &\leq \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I,L^2(B_R))} \\ &\quad + \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I,L^2(B_R^c))}, \end{aligned}$$

where $B_R := \{x \in \mathbb{R}^N : |x| \leq R\}$ and $B_R^c := \mathbb{R}^N \setminus B_R$ with $R > 0$ to be chosen later.

On B_R^c , we split $B_R^c = \Omega_1 \cup \Omega_2$ with

$$\Omega_1 = \{x \in \mathbb{R}^N : |x| \geq R, |x + x_n| \leq 1\}, \quad \Omega_2 = \{x \in \mathbb{R}^N : |x| \geq R, |x + x_n| \geq 1\}. \quad (3.7)$$

Using the same notation as in the proof of Lemma 2.6, we see that

$$\begin{aligned} & \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L^{r'}(A)} \\ & \leq \| |x + x_n|^{-b} \|_{L^{r_1}(A)} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{r_2}(B_R^c)} \\ & \lesssim \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{r_2}(B_R^c)}, \end{aligned}$$

where A is either Ω_1 or Ω_2 and $\frac{1}{r_1} = \frac{b}{N} \pm \theta^2$ with the plus sign for $A = \Omega_1$ and the minus sign for $A = \Omega_2$. By Hölder's inequality, we have

$$\begin{aligned} & \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{r'}(B_R^c)} \\ & \lesssim \| |u_n^J(t, \cdot + x_n)|^\alpha \|_{L_x^{\bar{m}^\pm}(B_R^c)} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{\bar{r}}(B_R^c)} \end{aligned}$$

By Hölder's inequality in time (see again the proof of Lemma 2.6), we get

$$\begin{aligned} & \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_t^{q'}(I, L_x^{r'}(B_R^c))} \\ & \lesssim \| |u_n^J(t, \cdot + x_n)|^\alpha \|_{L_t^\infty(I, H_x^1)} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_t^{\bar{q}}(I, L_x^{\bar{r}}(B_R^c))} \\ & \lesssim \| |u_n^J(t, \cdot + x_n)|^\alpha \|_{L_t^\infty(I, H_x^1)} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{it\Delta} \psi \|_{L_t^{\bar{q}}(\mathbb{R}, L_x^{\bar{r}}(B_R^c))}. \end{aligned}$$

As $(\bar{q}, \bar{r}) \in \mathcal{A}_0$, we see that $e^{it\Delta} \psi \in L_t^{\bar{q}}(\mathbb{R}, L_x^{\bar{r}})$, which, by the dominated convergence theorem, yields

$$\| e^{it\Delta} \psi \|_{L_t^{\bar{q}}(\mathbb{R}, L_x^{\bar{r}}(B_R^c))} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

This shows that for $R_0 > 0$ sufficiently large, we have

$$\| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2(B_{R_0}))} < \frac{\varepsilon}{2}$$

for all $n \geq 1$.

Next, for $x \in B_{R_0}$, as $|x_n| \rightarrow \infty$, we have $|x + x_n| \geq |x_n| - |x| \geq \frac{|x_n|}{2}$ for n sufficiently large. It follows from Lemma 2.6 that

$$\begin{aligned} & \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2(B_{R_0}))} \\ & \lesssim |x_n|^{-b} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2)} \\ & \lesssim |x_n|^{-b} \| |u_n^J(t, \cdot + x_n)|^\alpha \|_{S(I, \dot{H}^{\gamma_c})} \| e^{i(t-t_n)\Delta} \psi \|_{S(I, L^2)} \\ & \lesssim |x_n|^{-b} \| |u_n^J(t, \cdot + x_n)|^\alpha \|_{S(I, \dot{H}^{\gamma_c})} \| e^{it\Delta} \psi \|_{S(L^2)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus there exists $n_0 > 0$ sufficiently large such that for all $n \geq n_0$,

$$\| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2(B_{R_0}))} < \frac{\varepsilon}{2}.$$

We obtain for all $n \geq n_0$,

$$\| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, L^2)} < \varepsilon$$

which shows that

$$\| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, in both cases, we have

$$\| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty$$

and similarly for $\| |x|^{-b} |u|^\alpha e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, L^2)}$. Thus Observation 3.2 is proved. \square

Collecting (3.6), Observations 3.1, and 3.2, we finish the proof of Lemma 3.3. \square

Lemma 3.4. *We have*

$$\| \nabla e_n^J \|_{S'(I, L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof. Using the fact

$$|\nabla(|x|^{-b}(G(z) - G(w)))| \lesssim |x|^{-b}|\nabla(G(z) - G(w))| + |x|^{-b-1}|G(z) - G(w)|$$

we have

$$\begin{aligned} \|\nabla e_n^J\|_{S'(I, L^2)} &\lesssim \| |x|^{-b}|\nabla(G(u_n^J) - G(u))\|_{S'(I, L^2)} + \| |x|^{-b-1}|G(u_n^J) - G(u)|\|_{S'(I, L^2)} \\ &=: \text{(I)} + \text{(II)}. \end{aligned} \quad (3.8)$$

Term (I). Let us first treat the term (I). From the fact that

$$|\nabla(G(z) - G(w))| \lesssim |z|^\alpha |\nabla(z - w)| + \begin{cases} |\nabla w||z - w|^\alpha & \text{if } 0 < \alpha \leq 1, \\ (|z|^{\alpha-1} + |w|^{\alpha-1})|\nabla w||z - w| & \text{if } \alpha > 1, \end{cases}$$

we have

$$\begin{aligned} \text{(I)} &\lesssim \| |x|^{-b}|u_n^J|^\alpha |\nabla(u_n^J - u)|\|_{S'(I, L^2)} \\ &\quad + \begin{cases} \| |x|^{-b}|\nabla u||u_n^J - u|^\alpha\|_{S'(I, L^2)} & \text{if } 0 < \alpha \leq 1 \\ \| |x|^{-b}(|u_n^J|^{\alpha-1} + |u|^{\alpha-1})|\nabla u||u_n^J - u|\|_{S'(I, L^2)} & \text{if } \alpha > 1 \end{cases} \\ &=: (a) + \begin{cases} (b_1) & \text{if } 0 < \alpha \leq 1, \\ (b_2) & \text{if } \alpha > 1. \end{cases} \end{aligned}$$

For the term (b_1) , we have

$$\begin{aligned} \| |x|^{-b}|\nabla u||u_n^J - u|^\alpha\|_{S'(I, L^2)} &\lesssim \sum_{j=2}^J \| |x|^{-b}|\nabla u||e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j)|^\alpha\|_{S'(I, L^2)} \\ &\quad + \| |x|^{-b}|\nabla u||e^{it\Delta}W_n^J|^\alpha\|_{S'(I, L^2)}. \end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned} \| |x|^{-b}|\nabla u||e^{it\Delta}W_n^J|^\alpha\|_{S'(I, L^2)} &\lesssim \|e^{it\Delta}W_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|e^{it\Delta}W_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla u\|_{S(I, L^2)} \\ &\lesssim C(I, B) \|e^{it\Delta}W_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \rightarrow 0 \text{ as } J, n \rightarrow \infty. \end{aligned}$$

Observation 3.3. Let $(t_n)_{n \geq 1} \subset \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ satisfy $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then we have for any $\psi \in H^1(\mathbb{R}^N)$,

$$\| |x|^{-b}|\nabla u||e^{i(t-t_n)\Delta}\psi(\cdot - x_n)|^\alpha\|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of Observation 3.3. The proof follows by the same argument as in the proof of Observation 3.2. In the case $x_n \equiv 0$, we use Lemma 2.6 to have

$$\begin{aligned} \| |x|^{-b}|\nabla u||e^{i(t-t_n)\Delta}\psi|^\alpha\|_{S'(I, L^2)} &\lesssim \|e^{i(t-t_n)\Delta}\psi\|_{L_t^\infty(I, H_x^1)}^\theta \|e^{i(t-t_n)\Delta}\psi\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla u\|_{S(I, L^2)} \\ &\lesssim \|e^{i(t-t_n)\Delta}\psi\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

In the case $|x_n| \rightarrow \infty$, we estimate

$$\begin{aligned} \| |x|^{-b}|\nabla u||e^{i(t-t_n)\Delta}\psi(\cdot - x_n)|^\alpha\|_{S'(I, L^2)} &= \| |x + x_n|^{-b}|\nabla u(t, \cdot + x_n)||e^{i(t-t_n)\Delta}\psi|^\alpha\|_{S'(I, L^2)} \\ &\leq \| |x + x_n|^{-b}|\nabla u(t, \cdot + x_n)||e^{i(t-t_n)\Delta}\psi|^\alpha\|_{S'(I, L^2(B_R))} \\ &\quad + \| |x + x_n|^{-b}|\nabla u(t, \cdot + x_n)||e^{i(t-t_n)\Delta}\psi|^\alpha\|_{S'(I, L^2(B_R^c))}. \end{aligned}$$

On B_R^c , we estimate as in the proof of Lemma 2.6 and have

$$\begin{aligned} \| |x + x_n|^{-b}|\nabla u(t, \cdot + x_n)||e^{i(t-t_n)\Delta}\psi|^\alpha\|_{S'(I, L^2(B_R^c))} &\lesssim \|e^{i(t-t_n)\Delta}\psi\|_{L_t^\infty(I, H_x^1)}^\theta \|e^{i(t-t_n)\Delta}\psi\|_{L_t^{\bar{a}}(I, L_x^{\bar{r}}(B_R^c))}^{\alpha-\theta} \|\nabla u\|_{S(I, L^2)} \\ &\lesssim \|e^{it\Delta}\psi\|_{L_t^{\bar{a}}(\mathbb{R}, L_x^{\bar{r}}(B_R^c))} \rightarrow 0 \text{ as } R \rightarrow \infty, \end{aligned}$$

where \bar{a} and \bar{r} are as in the proof of Lemma 2.6. On B_R with R fixed, we can use the decay of $|x|^{-b}$ for $|x|$ large. We omit the details. \square

The term (b_2) is treated in a similar manner. More precisely, we have

$$\begin{aligned} & \| |x|^{-b} (|u_n^J|^{\alpha-1} + |u|^{\alpha-1}) |\nabla u| |u_n^J - u| \|_{S'(I, L^2)} \\ & \lesssim \sum_{j=2}^N \| |x|^{-b} (|u_n^J|^{\alpha-1} + |u|^{\alpha-1}) |\nabla u| |e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j)| \|_{S'(I, L^2)} \\ & \quad + \| |x|^{-b} (|u_n^J|^{\alpha-1} + |u|^{\alpha-1}) |\nabla u| |e^{it\Delta} W_n^J| \|_{S'(I, L^2)}. \end{aligned}$$

As $\alpha > 1$, by Lemma 2.6, we see that

$$\begin{aligned} \| |x|^{-b} |u_n^J|^{\alpha-1} |\nabla u| |e^{it\Delta} W_n^J| \|_{S'(I, L^2)} & \lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-1-\theta} \|e^{it\Delta} W_n^J\|_{S(I, \dot{H}^{\gamma_c})} \|\nabla u\|_{S(I, L^2)} \\ & \lesssim C(I, B) \|e^{it\Delta} W_n^J\|_{S(I, \dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } J, n \rightarrow \infty. \end{aligned}$$

A similar estimate goes for $\| |x|^{-b} |u|^{\alpha-1} |\nabla u| |e^{it\Delta} W_n^J| \|_{S'(I, L^2)}$. The terms

$$\| |x|^{-b} (|u_n^J|^{\alpha-1} + |u|^{\alpha-1}) |\nabla u| |e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j)| \|_{S'(I, L^2)}$$

are treated as in Observation 3.3.

Thus we have proved that

$$(b_1), (b_2) \rightarrow 0 \text{ as } J, n \rightarrow \infty. \quad (3.9)$$

For the term (a) , we have

$$\begin{aligned} (a) & \lesssim \sum_{j=2}^J \| |x|^{-b} |u_n^J|^\alpha |\nabla e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j)| \|_{S'(I, L^2)} + \| |x|^{-b} |u_n^J|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \\ & \lesssim \sum_{j=2}^J \| |x|^{-b} |u_n^J|^\alpha |\nabla e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j)| \|_{S'(I, L^2)} \\ & \quad + \sum_{j=2}^J \| |x|^{-b} |e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j)|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \\ & \quad + \| |x|^{-b} |u|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} + \| |x|^{-b} |e^{it\Delta} W_n^J|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)}. \end{aligned} \quad (3.10)$$

Observation 3.4. Let $(t_n)_{n \geq 1} \subset \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ satisfy $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then we have for any $\psi \in H^1$,

$$\| |x|^{-b} |u_n^J|^\alpha |\nabla e^{i(t-t_n)\Delta} \psi(\cdot - x_n)| \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of Observation 3.4. This result follows from the same arguments as in the proof Observation 3.2 using $\nabla e^{it\Delta} \psi \in S(I, L^2)$. In particular,

$$\| \nabla e^{it\Delta} \psi \|_{L_t^\infty(\mathbb{R}, L_x^\infty(B_R^c))} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

We omit the details. \square

Observation 3.5. Let $(t_n)_{n \geq 1} \subset \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ satisfy $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then we have for any $\psi \in H^1$,

$$\| |x|^{-b} |e^{i(t-t_n)\Delta} \psi(\cdot - x_n)|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of Observation 3.5. As in the proof of Observation 3.2, we consider two cases: $x_n \equiv 0$ and $|x_n| \rightarrow \infty$.

Case 1. $x_n \equiv 0$. We have $|t_n| \rightarrow \infty$. By Lemma 2.6 and the fact that

$$\sup_{J \geq 2} \sup_{n \geq 1} \| \nabla e^{it\Delta} W_n^J \|_{S(I, L^2)} \leq B,$$

we have

$$\begin{aligned} & \| |x|^{-b} |e^{i(t-t_n)\Delta} \psi|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \\ & \lesssim \| e^{i(t-t_n)\Delta} \psi \|_{L_t^\infty(I, H_x^1)}^\theta \| e^{i(t-t_n)\Delta} \psi \|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \| \nabla e^{it\Delta} W_n^J \|_{S(I, L^2)} \\ & \lesssim C(B) \| e^{i(t-t_n)\Delta} \psi \|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta}, \end{aligned}$$

where B is as in (3.4). By the same argument as in the proof of Observation 3.2 using dispersive estimates (2.1), we have

$$\|e^{i(t-t_n)\Delta}\psi\|_{S(I,\dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Note that for each $(q, r) \in \mathcal{A}_{\gamma_c}$, we have $r \in \left(\frac{6}{3-2\gamma_c}, 6\right) \subset (2, 6)$.

Case 2. $|x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Take $\varepsilon > 0$. We have

$$\begin{aligned} & \| |x|^{-b} |e^{i(t-t_n)\Delta}\psi(\cdot - x_n)|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \\ &= \| |x + x_n|^{-b} |e^{i(t-t_n)\Delta}\psi|^\alpha |\nabla e^{it\Delta} W_n^J(\cdot + x_n)| \|_{S'(I, L^2)} \\ &\leq \| |x + x_n|^{-b} |e^{i(t-t_n)\Delta}\psi|^\alpha |\nabla e^{it\Delta} W_n^J(\cdot + x_n)| \|_{S'(I, L^2(B_R))} \\ &\quad + \| |x + x_n|^{-b} |e^{i(t-t_n)\Delta}\psi|^\alpha |\nabla e^{it\Delta} W_n^J(\cdot + x_n)| \|_{S'(I, L^2(B_R^c))}. \end{aligned}$$

Estimating as in the proof of Observation 3.2 (see also the proof of Lemma 2.6), we see that

$$\begin{aligned} & \| |x + x_n|^{-b} |e^{i(t-t_n)\Delta}\psi|^\alpha |\nabla e^{it\Delta} W_n^J(\cdot + x_n)| \|_{S'(I, L^2(B_R^c))} \\ &\lesssim \|e^{i(t-t_n)\Delta}\psi\|_{L_t^\infty(I, H_x^1)}^\theta \|e^{i(t-t_n)\Delta}\psi\|_{L_t^{\frac{\alpha-\theta}{\alpha}}(I, L_x^{\bar{r}}(B_{\bar{r}}))}^{\alpha-\theta} \|\nabla e^{it\Delta} W_n^J\|_{S(I, L^2)} \\ &\lesssim \|e^{it\Delta}\psi\|_{L_t^{\bar{\alpha}}(\mathbb{R}, L_x^{\bar{r}}(B_{\bar{r}}^c))} \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

Here we have used the fact that $e^{it\Delta}\psi \in L_t^{\bar{\alpha}}(\mathbb{R}, L_x^{\bar{r}})$ as $(\bar{\alpha}, \bar{r}) \in \mathcal{A}_{\gamma_c}$. Thus there exists $R_0 > 0$ sufficiently large such that

$$\| |x + x_n|^{-b} |e^{i(t-t_n)\Delta}\psi|^\alpha |\nabla e^{it\Delta} W_n^J(\cdot + x_n)| \|_{S'(I, L^2(B_{R_0}^c))} < \frac{\varepsilon}{2}$$

for all $n \geq 1$.

Next, for $x \in B_{R_0}$, we have $|x + x_n|^{-b} \lesssim |x_n|^{-b}$ for n sufficiently large. It follows that

$$\begin{aligned} & \| |x + x_n|^{-b} |e^{i(t-t_n)\Delta}\psi|^\alpha |\nabla e^{it\Delta} W_n^J(\cdot + x_n)| \|_{S'(I, L^2(B_{R_0}))} \\ &\lesssim |x_n|^{-b} \|e^{i(t-t_n)\Delta}\psi\|_{S(I, \dot{H}^{\gamma_c})}^\alpha \|\nabla e^{it\Delta} W_n^J\|_{S(I, L^2)} \\ &\lesssim |x_n|^{-b} \|e^{it\Delta}\psi\|_{S(\dot{H}^{\gamma_c})}^\alpha \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence there exists $n_0 > 0$ sufficiently large such that for all $n \geq n_0$,

$$\| |x + x_n|^{-b} |e^{i(t-t_n)\Delta}\psi|^\alpha |\nabla e^{it\Delta} W_n^J(\cdot + x_n)| \|_{S'(I, L^2(B_{R_0}))} < \frac{\varepsilon}{2}.$$

We obtain for all $n \geq n_0$,

$$\| |x|^{-b} |e^{i(t-t_n)\Delta}\psi(\cdot - x_n)|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} < \varepsilon$$

or

$$\| |x|^{-b} |e^{i(t-t_n)\Delta}\psi(\cdot - x_n)|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The proof of Observation 3.5 is complete. \square

Observation 3.6. We have

$$\| |x|^{-b} |e^{it\Delta} W_n^J|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof of Observation 3.6. It follows directly from Lemma 2.6 and the fact that $\|e^{it\Delta} W_n^J\|_{S(\dot{H}^{\gamma_c})} \rightarrow 0$ as $J, n \rightarrow \infty$. \square

Observation 3.7. Let $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^N)$. We have

$$\| |x|^{-b} |\varphi|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof of Observation 3.7. Without a loss of generality, we assume that $\text{supp}(\varphi) \subset K \times B_R$, where K is a compact interval of \mathbb{R} with a non-empty intersection with I and B_R is the ball center at the origin and of radius R . Let (q', r') be as in the proof of Lemma 2.6. We have

$$\begin{aligned} \| |x|^{-b} |\varphi|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} &\leq \| |x|^{-b} |\varphi|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{L_t^{q'}(I, L_x^{r'})} \\ &\leq \| |x|^{-b} \|_{L_x^{r_1}(B_R)} \|\varphi|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{L_t^{q'}(I \cap K, L_x^{r_2}(B_R))}, \end{aligned}$$

where

$$\frac{1}{r_1} = \frac{b}{N} + \theta^2, \quad \frac{1}{r_2} = \frac{1}{r'} - \frac{1}{r_1} = \frac{N + 2 - 2b - \theta}{2N} - \theta^2$$

with $\theta > 0$ sufficiently small. By Hölder's inequality, we have

$$\begin{aligned} \| |x|^{-b} |\varphi|^\alpha |\nabla e^{it\Delta} W_n^J| \|_{S'(I, L^2)} &\lesssim \|\varphi\|_{L_t^{\frac{2}{\delta}}(I \cap K, L_x^{\alpha\mu}(B_R))}^\alpha \|\nabla e^{it\Delta} W_n^J\|_{L_t^2(I \cap K, L_x^2(B_R))} \\ &\lesssim \|\nabla e^{it\Delta} W_n^J\|_{L_t^2(I \cap K, L_x^2(B_R))}, \end{aligned}$$

where

$$\delta := \frac{4}{\theta}, \quad \mu := \frac{2N}{2 - 2b - \theta - 2N\theta^2}.$$

Let $(\bar{a}, \bar{r}) \in \mathcal{A}_{\gamma_c}$ be as in the proof of Lemma 2.6. By (2.3), we have

$$\begin{aligned} \|\nabla e^{it\Delta} W_n^J\|_{L_t^2(I \cap K, L_x^2(B_R))} &\lesssim |I|^{\frac{1}{3}(\frac{1}{2} - \frac{1}{\alpha})} R^{\frac{1}{3}(\frac{N+2}{2} - \frac{N}{\alpha})} \|e^{it\Delta} W_n^J\|_{L_t^{\frac{1}{3}}(I, L_x^{\frac{2}{3}})}^{\frac{1}{3}} \|\nabla W_n^J\|_{L^2}^{\frac{2}{3}} \\ &\lesssim C(|I|, R, B) \|e^{it\Delta} W_n^J\|_{S(\dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } J, n \rightarrow \infty \end{aligned}$$

which proves Observation 3.7. \square

It follows from Observations 3.4–3.7 and (3.10), we have

$$(a) \rightarrow 0 \text{ as } J, n \rightarrow \infty. \quad (3.11)$$

Combining (3.9) and (3.11), we have

$$(I) \rightarrow 0 \text{ as } J, n \rightarrow \infty. \quad (3.12)$$

Term (II). Let us now estimate the term (II). By (3.5), we have

$$\begin{aligned} (II) &\lesssim \sum_{j=2}^J \| |x|^{-b-1} (|u_n^J|^\alpha + |u|^\alpha) e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j) \|_{S'(I, L^2)} \\ &\quad + \| |x|^{-b-1} (|u_n^J|^\alpha + |u|^\alpha) e^{it\Delta} W_n^J \|_{S'(I, L^2)} \end{aligned} \quad (3.13)$$

Observation 3.8. Let $(t_n)_{n \geq 1} \subset \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ satisfy $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then we have for any $\psi \in H^1$,

$$\| |x|^{-b-1} (|u_n^J|^\alpha + |u|^\alpha) e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of Observation 3.8. As in the proof of Observation 3.2, we consider two cases: $x_n \equiv 0$ and $|x_n| \rightarrow \infty$.

Case 1. $x_n \equiv 0$. By Lemmas 2.6, (3.1), and (3.2), we have

$$\begin{aligned} \| |x|^{-b-1} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2)} &\lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla e^{i(t-t_n)\Delta} \psi\|_{S(I, L^2)} \\ &\lesssim \|\nabla e^{i(t-t_n)\Delta} \psi\|_{S(I, L^2)}. \end{aligned}$$

Since $t \in I$ and $|t_n| \rightarrow \infty$, we have $|t - t_n| \rightarrow \infty$ as $n \rightarrow \infty$. From dispersive estimates (2.1) and Sobolev embedding, we infer that

$$\|\nabla e^{i(t-t_n)\Delta} \psi\|_{S(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Case 2. $|x_n| \rightarrow \infty$. We proceed as in the proof of Observation 3.2. Let $\varepsilon > 0$. We have

$$\begin{aligned} \| |x|^{-b-1} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, L^2)} &= \| |x + x_n|^{-b-1} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2)} \\ &\leq \| |x + x_n|^{-b-1} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2(B_R))} \\ &\quad + \| |x + x_n|^{-b-1} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, L^2(B_R^c))}, \end{aligned}$$

where $R > 0$ to be chosen later.

On B_R^c , we have

$$\begin{aligned} \| |x + x_n|^{-b-1} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_t^{q'}(I, L_x^{r'}(B_R^c))} &\lesssim \|u_n^J(t, \cdot + x_n)\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J(t, \cdot + x_n)\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla e^{i(t-t_n)\Delta} \psi\|_{L_t^{\frac{q'}{\alpha}}(I, L_x^{\frac{r'}{\alpha}}(B_R^c))} \\ &\lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\nabla e^{it\Delta} \psi\|_{L_t^{\frac{q'}{\alpha}}(\mathbb{R}, L_x^{\frac{r'}{\alpha}}(B_R^c))}, \end{aligned}$$

where the exponents q' , r' , \bar{q} , and \bar{r} are as in the proof of Lemma 2.6. Since $(\bar{q}, \bar{r}) \in \mathcal{A}_0$, we see that

$$\|\nabla e^{it\Delta}\psi\|_{L_t^{\bar{q}}(\mathbb{R}, L_x^{\bar{r}}(B_R^c))} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Thus there exists $R_0 > 0$ sufficiently large such that

$$\| |x + x_n|^{-b-1} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta}\psi \|_{S'(I, L^2(B_{R_0}^c))} < \frac{\varepsilon}{2}$$

for all $n \geq 1$.

On B_{R_0} , we have

$$\begin{aligned} & \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta}\psi \|_{S'(I, L^2(B_{R_0}^c))} \\ & \lesssim |x_n|^{-b-1} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta}\psi \|_{S'(I, L^2)} \\ & \lesssim |x_n|^{-b-1} \| |u_n^J(t, \cdot + x_n)|^\alpha \|_{S(I, \dot{H}^{\gamma_c})} \| e^{i(t-t_n)\Delta}\psi \|_{S(I, L^2)} \\ & \lesssim |x_n|^{-b-1} \| |u_n^J|^\alpha \|_{S(I, \dot{H}^{\gamma_c})} \| e^{it\Delta}\psi \|_{S(L^2)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

Collecting the above cases, we prove that

$$\| |x|^{-b-1} |u_n^J|^\alpha e^{i(t-t_n)\Delta}\psi(\cdot - x_n) \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

A similar argument goes for $\| |x|^{-b-1} |u|^\alpha e^{i(t-t_n)\Delta}\psi(\cdot - x_n) \|_{S'(I, L^2)}$ and the proof is complete. \square

Observation 3.9. Let $(t_n)_{n \geq 1} \subset \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ satisfy $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then we have for any $\psi \in H^1$,

$$\| |x|^{-b-1} |e^{i(t-t_n)\Delta}\psi(\cdot - x_n)|^\alpha |e^{it\Delta}W_n^J| \|_{S'(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of Observation 3.9. The proof is similar to that of Observation 3.5 using Lemma 2.6. We thus omit the details. \square

Observation 3.10. Let $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^N)$. We have

$$\| |x|^{-b-1} |\varphi|^\alpha |e^{it\Delta}W_n^J| \|_{S'(I, L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof of Observation 3.10. As in the proof of Observation 3.7, we can assume that $\text{supp}(\varphi) \subset K \times B_R$, where K is a compact interval of \mathbb{R} with a non-empty intersection with I and B_R is the ball center at the origin and of radius R . Let (q', r') be as in the proof of Lemma 2.6. We have

$$\begin{aligned} \| |x|^{-b-1} |\varphi|^\alpha |e^{it\Delta}W_n^J| \|_{S'(I, L^2)} & \leq \| |x|^{-b-1} |\varphi|^\alpha |e^{it\Delta}W_n^J| \|_{L_t^{q'}(I, L_x^{r'})} \\ & \leq \| |x|^{-b-1} \|_{L_x^{p_1}(B_R)} \| |\varphi|^\alpha |e^{it\Delta}W_n^J| \|_{L_t^{q'}(I \cap K, L_x^{p_2}(B_R))}, \end{aligned}$$

where

$$\frac{1}{p_1} = \frac{b+1}{N} + \theta^2, \quad \frac{1}{p_2} = \frac{1}{r'} - \frac{1}{p_1} = \frac{N-2b-\theta}{2N} - \theta^2$$

with $\theta > 0$ sufficiently small. By Hölder's inequality and Sobolev embedding, we have

$$\begin{aligned} \| |x|^{-b-1} |\varphi|^\alpha |e^{it\Delta}W_n^J| \|_{S'(I, L^2)} & \lesssim \| \varphi \|_{L_t^{\alpha\delta}(I \cap K, L_x^{\alpha\mu}(B_R))} \| e^{it\Delta}W_n^J \|_{L_t^2(I \cap K, L_x^{\frac{2N}{N-2}}(B_R))} \\ & \lesssim \| \nabla e^{it\Delta}W_n^J \|_{L_t^2(I \cap K, L_x^2(B_R))}, \end{aligned}$$

where δ and μ are as in the proof of Observation 3.7. Arguing as in the proof of Observation 3.7, we have

$$\begin{aligned} \| \nabla e^{it\Delta}W_n^J \|_{L_t^2(I \cap K, L_x^2(B_R))} & \lesssim |I|^{\frac{1}{3}(\frac{1}{2}-\frac{1}{\alpha})} R^{\frac{1}{3}(\frac{N+2}{2}-\frac{N}{r'})} \| e^{it\Delta}W_n^J \|_{L_t^{\frac{1}{3}}(I, L_x^{\frac{1}{3}})} \| \nabla W_n^J \|_{L^{\frac{2}{3}}} \\ & \lesssim C(|I|, R, B) \| e^{it\Delta}W_n^J \|_{S(\dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } J, n \rightarrow \infty. \end{aligned}$$

This shows that

$$\| |x|^{-b-1} |\varphi|^\alpha |e^{it\Delta}W_n^J| \|_{S'(I, L^2)} \lesssim C(|I|, R, B) \| \varphi \|_{L_t^{\alpha\delta}(I \cap K, L_x^{\alpha\mu}(B_R))} \| e^{it\Delta}W_n^J \|_{S(\dot{H}^{\gamma_c})} \rightarrow 0$$

as $J, n \rightarrow \infty$. The proof is complete. \square

Using (3.13) and Observations 3.8, (3.9), and (3.10), we prove that

$$(II) \rightarrow 0 \text{ as } J, n \rightarrow \infty. \quad (3.14)$$

Finally, the proof of Lemma 3.4 is complete by combining (3.8), (3.12), and (3.14). \square

Lemma 3.5. *We have*

$$\|e_n^J\|_{S'(I, \dot{H}^{-\gamma_c})} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof. By (3.5), we have

$$\begin{aligned} \|e_n^J\|_{S'(I, \dot{H}^{-\gamma_c})} &\lesssim \| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) (u_n^J - u) \|_{S'(I, \dot{H}^{-\gamma_c})} \\ &\lesssim \sum_{j=2}^J \| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j) \|_{S'(I, \dot{H}^{-\gamma_c})} \\ &\quad + \| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) e^{it\Delta} W_n^J \|_{S'(I, \dot{H}^{-\gamma_c})}. \end{aligned} \quad (3.15)$$

Observation 3.11. We have

$$\| |x|^{-b} (|u_n^J|^\alpha + |u|^\alpha) e^{it\Delta} W_n^J \|_{S'(I, \dot{H}^{-\gamma_c})} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Proof of Observation 3.11. By Lemma 2.5, 3.1, and (3.2), we have

$$\begin{aligned} \| |x|^{-b} |u_n^J|^\alpha e^{it\Delta} W_n^J \|_{S'(I, \dot{H}^{-\gamma_c})} &\lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|e^{it\Delta} W_n^J\|_{S(I, \dot{H}^{\gamma_c})} \\ &\lesssim C(I, B) \|e^{it\Delta} W_n^J\|_{S(I, \dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } J, n \rightarrow \infty. \end{aligned}$$

A similar estimate goes for $\| |x|^{-b} |u|^\alpha e^{it\Delta} W_n^J \|_{S'(I, \dot{H}^{-\gamma_c})}$. \square

Observation 3.12. Let $(t_n)_{n \geq 1} \subset \mathbb{R}$ and $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ satisfy $|t_n| + |x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then for any $\psi \in H^1$,

$$\| |x|^{-b} (|u^J|^\alpha + |u|^\alpha) e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, \dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof of Observation 3.12. The proof of this result follows from the same argument as in Observation 3.2. In the case $x_n \equiv 0$, hence $|t_n| \rightarrow \infty$, we have

$$\begin{aligned} \| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, \dot{H}^{-\gamma_c})} &\lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|e^{i(t-t_n)\Delta} \psi\|_{S(I, \dot{H}^{\gamma_c})} \\ &\lesssim C(I, B) \|e^{i(t-t_n)\Delta} \psi\|_{S(I, \dot{H}^{\gamma_c})}. \end{aligned}$$

By Sobolev embedding and the fact that $|t - t_n| \rightarrow \infty$ for each $t \in I$ as $n \rightarrow \infty$, we infer from dispersive estimates (2.1) that

$$\|e^{i(t-t_n)\Delta} \psi\|_{S(I, \dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In the case $|x_n| \rightarrow \infty$, we take $\varepsilon > 0$ and estimate

$$\begin{aligned} \| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta} \psi(\cdot - x_n) \|_{S'(I, \dot{H}^{-\gamma_c})} &= \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, \dot{H}^{-\gamma_c})} \\ &\leq \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, \dot{H}^{-\gamma_c}(B_R))} \\ &\quad + \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{S'(I, \dot{H}^{-\gamma_c}(B_R^c))} \end{aligned}$$

for some $R > 0$ to be chosen later.

On B_R^c , using the same notation as in the proof of Lemma 2.5, we have

$$\begin{aligned} \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{\hat{\gamma}'}(A)} &\leq \| |x + x_n|^{-b} \|_{L_x^{\hat{\gamma}}(A)} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{\hat{\gamma}}(B_R^c)} \\ &\lesssim \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{\hat{\gamma}}(B_R^c)}, \end{aligned}$$

where A is either Ω_1 or Ω_2 (see (3.7)) and $\frac{1}{\hat{\gamma}} = \frac{b}{N} \pm \theta^2$ with the plus sign for $A = \Omega_1$ and the minus sign for $A = \Omega_2$. By Hölder's inequality, we have

$$\begin{aligned} \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta} \psi \|_{L_x^{\hat{\gamma}'}(B_R^c)} &\lesssim \|u_n^J(t, \cdot + x_n)\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J(t, \cdot + x_n)\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|e^{i(t-t_n)\Delta} \psi\|_{L_t^{\hat{\gamma}}(I, L_x^{\hat{\gamma}}(B_R^c))} \\ &\lesssim \|u_n^J\|_{L_t^\infty(I, H_x^1)}^\theta \|u_n^J\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \|e^{it\Delta} \psi\|_{L_t^{\hat{\gamma}}(\mathbb{R}, L_x^{\hat{\gamma}}(B_R^c))}. \end{aligned}$$

Since $(\hat{a}, \hat{r}) \in \mathcal{A}_{\gamma_c}$, we have $e^{it\Delta}\psi \in L_t^{\hat{a}}(\mathbb{R}, L_x^{\hat{r}})$, which, by the dominated convergence theorem, implies

$$\|e^{it\Delta}\psi\|_{L_t^{\hat{a}}(\mathbb{R}, L_x^{\hat{r}}(B_{R^c}^c))} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Thus there exists $R_0 > 0$ sufficiently large such that

$$\| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta}\psi \|_{S'(I, \dot{H}^{-\gamma_c}(B_{R_0^c}^c))} < \frac{\varepsilon}{2}$$

for all $n \geq 1$.

On B_{R_0} , we make use of the fact that $|x + x_n|^{-b} \lesssim |x_n|^{-b}$ for n sufficiently large to get

$$\begin{aligned} & \| |x + x_n|^{-b} |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta}\psi \|_{S'(I, \dot{H}^{-\gamma_c}(B_{R_0}))} \\ & \lesssim |x_n|^{-b} \| |u_n^J(t, \cdot + x_n)|^\alpha e^{i(t-t_n)\Delta}\psi \|_{S'(I, \dot{H}^{-\gamma_c})} \\ & \lesssim |x_n|^{-b} \| u_n^J(t, \cdot + x_n) \|_{S(I, \dot{H}^{\gamma_c})}^\alpha \| e^{i(t-t_n)\Delta}\psi \|_{S(I, \dot{H}^{\gamma_c})} \\ & \lesssim |x_n|^{-b} \| u_n^J \|_{S(I, \dot{H}^{\gamma_c})}^\alpha \| e^{it\Delta}\psi \|_{S(\dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus we prove that there exists $n_0 > 0$ sufficiently large such that for all $n \geq n_0$,

$$\| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta}\psi(\cdot - x_n) \|_{S'(I, \dot{H}^{-\gamma_c})} < \varepsilon$$

hence

$$\| |x|^{-b} |u_n^J|^\alpha e^{i(t-t_n)\Delta}\psi(\cdot - x_n) \|_{S'(I, \dot{H}^{-\gamma_c})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Collecting the above two cases, we prove Observation 3.12. \square

Finally, Lemma 3.5 follows directly from (3.15), Observations (3.11), and (3.12). \square

From Lemmas 3.1, 3.2, 3.3, 3.4, and 3.5, we infer from the stability theory Lemma 2.9 that for n sufficiently large, u_n is defined on $I \times \mathbb{R}^N$ and satisfies

$$\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2)} < \infty$$

and

$$u_n(t, x) = u_n^J(t, x) + r_n^J(t, x)$$

with

$$\|r_n^J\|_{L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2)} \rightarrow 0 \text{ as } J, n \rightarrow \infty.$$

Since

$$r_n^J(t, x) = u_n(t, x) - u_n^J(t, x) = (u_n(t, x) - e^{it\Delta}\phi_n(x)) - (u(t, x) - e^{it\Delta}\psi^1(x)), \quad (3.16)$$

we see that r_n^J is independent of J , hence $r_n^J = r_n$ with

$$\|r_n\|_{L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.17)$$

Let $\gamma \in [0, 1]$ and fix $J \geq 2$ and $t \in I$. As $\|r_n\|_{L_t^\infty(I, H_x^1)} \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\begin{aligned} \|u_n(t)\|_{\dot{H}_x^\gamma}^2 &= \left\| u(t) + \sum_{j=2}^J e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j) + e^{it\Delta}W_n^J \right\|_{\dot{H}_x^\gamma}^2 + o_n(t) \\ &= \|u(t)\|_{\dot{H}_x^\gamma}^2 + \left\| \sum_{j=2}^J e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j) + e^{it\Delta}W_n^J \right\|_{\dot{H}_x^\gamma}^2 + o_n(t) \\ &\quad + \sum_{j=2}^J \left\langle u(t), e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j) \right\rangle + \langle u(t), e^{it\Delta}W_n^J \rangle \\ &= \|u(t)\|_{\dot{H}_x^\gamma}^2 + \sum_{j=2}^J \|\psi^j\|_{\dot{H}_x^\gamma}^2 + \|W_n^J\|_{\dot{H}_x^\gamma}^2 + o_n(t) \\ &\quad + \sum_{j=2}^J \left\langle u(t), e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j) \right\rangle + \langle u(t), e^{it\Delta}W_n^J \rangle, \end{aligned} \quad (3.18)$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\dot{H}^\gamma(\mathbb{R}^N)$. Since $|t_n^j| + |x_n^j| \rightarrow \infty$, it follows from Lemma 2.10 that for each $j \in \{2, \dots, J\}$,

$$\left\langle u(t), e^{i(t-t_n^j)\Delta} \psi^j(\cdot - x_n^j) \right\rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Here we note that the time t is fixed. Similarly, as $W_n^J \rightharpoonup 0$ weakly in H^1 as $J, n \rightarrow \infty$, we have $\langle u(t), e^{it\Delta} W_n^J \rangle \rightarrow 0$ as $J, n \rightarrow \infty$. Thus for $\varepsilon > 0$, there exist $J_1, n_1 \in \mathbb{N}$ sufficiently large such that for $n \geq n_1$,

$$\left| \|u_n(t)\|_{\dot{H}_x^\gamma}^2 - \|u(t)\|_{\dot{H}_x^\gamma}^2 - \sum_{j=2}^{J_1} \|\psi^j\|_{\dot{H}_x^\gamma}^2 - \|W_n^{J_1}\|_{\dot{H}_x^\gamma}^2 \right| < \frac{\varepsilon}{2}.$$

It follows that

$$\left| \|u_n(t)\|_{\dot{H}_x^\gamma}^2 - \|u(t)\|_{\dot{H}_x^\gamma}^2 - \sum_{j=2}^J \|\psi^j\|_{\dot{H}_x^\gamma}^2 - \|W_n^J\|_{\dot{H}_x^\gamma}^2 - \sum_{j=J+1}^{J_1} \|\psi^j\|_{\dot{H}_x^\gamma}^2 + \|W_n^J\|_{\dot{H}_x^\gamma}^2 - \|W_n^{J_1}\|_{\dot{H}_x^\gamma}^2 \right| < \frac{\varepsilon}{2}.$$

As

$$W_n^J(x) = \sum_{j=J+1}^{J_1} e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^{J_1}(x),$$

we see that

$$\|W_n^J\|_{\dot{H}_x^\gamma}^2 = \sum_{j=J+1}^{J_1} \|\psi^j\|_{\dot{H}_x^\gamma}^2 + \|W_n^{J_1}\|_{\dot{H}_x^\gamma}^2 + o_n(1).$$

Thus there exists $n_2 \in \mathbb{N}$ sufficiently large such that for all $n \geq n_2$, we have

$$\left| \|W_n^J\|_{\dot{H}_x^\gamma}^2 - \sum_{j=J+1}^{J_1} \|\psi^j\|_{\dot{H}_x^\gamma}^2 - \|W_n^{J_1}\|_{\dot{H}_x^\gamma}^2 \right| < \frac{\varepsilon}{2}.$$

It follows that for $n \geq \max\{n_1, n_2\}$, we have

$$\left| \|u_n(t)\|_{\dot{H}_x^\gamma}^2 - \|u(t)\|_{\dot{H}_x^\gamma}^2 - \sum_{j=2}^J \|\psi^j\|_{\dot{H}_x^\gamma}^2 - \|W_n^J\|_{\dot{H}_x^\gamma}^2 \right| < \varepsilon.$$

Therefore, for each $J \geq 2$, $t \in I$, and $\gamma \in [0, 1]$, we have

$$\|u_n(t)\|_{\dot{H}_x^\gamma}^2 = \|u(t)\|_{\dot{H}_x^\gamma}^2 + \sum_{j=2}^J \|\psi^j\|_{\dot{H}_x^\gamma}^2 + \|W_n^J\|_{\dot{H}_x^\gamma}^2 + \epsilon_n(t). \quad (3.19)$$

Finally we will show that the Pythagorean expansion (3.19) is uniformly in $t \in I$, that is,

$$\lim_{n \rightarrow \infty} \epsilon_n(t) = 0$$

uniformly in $t \in I$. From (3.17) and (3.18), it suffices to show that if $\varphi \in C(I, H^1)$ and ψ_n converges weakly to zero in $H^1(\mathbb{R}^N)$, then

$$\lim_{n \rightarrow \infty} \langle \varphi(t), e^{it\Delta} \psi_n \rangle = 0 \quad (3.20)$$

uniformly in $t \in I$. Assume by contradiction that (3.20) does not hold. Then there exist $C > 0$ and $t_n \in I$ such that, up to extraction,

$$|\langle \varphi(t_n), e^{it_n \Delta} \psi_n \rangle| \geq C > 0.$$

Since I is compact, passing to a subsequence if necessary, one can assume that $t_n \rightarrow \beta \in I$ and

$$|\langle \varphi(t_n), e^{it_n \Delta} \psi_n \rangle| = |\langle e^{-it_n \Delta} \varphi(t_n), \psi_n \rangle| \geq C > 0.$$

As $e^{-it_n \Delta} \varphi(t_n) \rightarrow e^{-i\beta \Delta} \varphi(\beta)$ and $\psi_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$, we get

$$\langle e^{-it_n \Delta} \varphi(t_n), \psi_n \rangle \rightarrow 0 \text{ as } n \rightarrow \infty$$

which is a contradiction. The proof of Theorem 1.3 is now complete.

4. COMPACTNESS PROPERTY

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Let $(\phi_n)_{n \geq 1} \subset \mathcal{H}$ satisfy $\phi_n \rightharpoonup \phi$ weakly in $H^1(\mathbb{R}^N)$. Denote $u_n : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$ and $u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$ the global solutions to (1.1) with initial data $u_n|_{t=0} = \phi_n$ and $u|_{t=0} = \phi$, respectively. Note that our assumptions on \mathcal{H} ensures that these solutions exist globally in time.

Since $(\phi_n)_{n \geq 1}$ is bounded in $H^1(\mathbb{R}^N)$, we have: for each integer $J \geq 2$, there exists a subsequence, still denoted by $(\phi_n)_{n \geq 1}$, such that

$$\phi_n(x) = \psi^1(x) + \sum_{j=2}^J e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^J(x)$$

with $|t_n^j| + |x_n^j| \rightarrow \infty$ for each $j = 2, \dots, J$ and the properties (1.11), (1.12). Thanks to Lemma 2.10 and the fact that $W_n^J \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$, we see that $\phi_n \rightharpoonup \psi^1$ weakly in $H^1(\mathbb{R}^N)$. By the uniqueness of weak convergence, we have $\psi^1 \equiv \phi$.

Let $I \subset \mathbb{R}$ be a compact interval. It follows from (3.16) that

$$(u_n(t) - e^{it\Delta} \phi_n) - (u(t) - e^{it\Delta} \phi) = r_n(t)$$

which, by (3.17), implies

$$u_n - e^{it\Delta} \phi_n \rightarrow u - e^{it\Delta} \phi \text{ strongly in } L_t^\infty(I, H_x^1) \cap S(I, \dot{H}^{\gamma_c}) \cap S^1(I, L^2)$$

as $n \rightarrow \infty$.

As the limit does not depend on the choice of subsequence of $(\phi_n)_{n \geq 1}$, the above convergence holds for the whole sequence. The proof is complete. \square

ACKNOWLEDGMENT

This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01). V. D. D. would like to express his deep gratitude to his wife - Uyen Cong for her encouragement and support. The authors would like to thank the reviewer for his/her helpful comments and suggestions.

REFERENCES

- [1] A. K. Arora, B. Dodson, and J. Murphy, Scattering below the ground state for the $2d$ radial nonlinear Schrödinger equation, Proc. Amer. Math. Soc. 148 (2020), no. 4, 1653–1663. 2
- [2] A. Bensouilah and S. Keraani, Smoothing property for the L^2 -critical high-order NLS II, Discrete Contin. Dyn. Syst. 39 (2019), no. 5, 2961–2976. 4
- [3] A. De Bouard and R. Fukuizumi, Stability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, Ann. Henri Poincaré 6 (2005), no. 6, 1157–1177. 1
- [4] J. L. Bona and J.-C. Saut, Dispersive blow-up for solutions of generalized KdV equations, J. Differential Equations 103 (1993), 3–57. 4
- [5] J. L. Bona, G. Ponce, J.-C. Saut, and C. Sparber, Dispersive blow-up for nonlinear Schrödinger equations revisited, J. Math. Pures Appl. 102 (2014), no. 4, 782–811. 4
- [6] J. Bourgain, Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity, Int. Math. Res. Notices 5 (1998), 253–283. 4
- [7] L. Campos, Scattering of radial solutions to the inhomogeneous nonlinear Schrödinger equation, Nonlinear Anal. 202 (2021), 112118. 2, 7, 8
- [8] L. Campos and M. Cardoso, Blow up and scattering criteria above the threshold for the focusing inhomogeneous nonlinear Schrödinger equation, preprint, available at <http://arxiv.org/abs/2001.11613>. 2, 4
- [9] M. Cardoso, L. G. Farah, C. M. Guzmán, and J. Murphy, Scattering below the ground state for the intercritical non-radial inhomogeneous NLS, preprint, available at <http://arxiv.org/abs/2007.06165>. 2, 3, 4, 5
- [10] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. 2, 6
- [11] P. Constantin and J.-C. Saut, Local smoothing properties of dispersive equations, J. Amer. Math. Soc. 1 (1988), no. 2, 413–439. 6
- [12] S. Correia and J. D. Silva, Nonlinear smoothing for dispersive PDE: a unified approach, J. Differential Equations 269 (2020), no. 5, 4253–4285. 4
- [13] J. Chen, On a class of nonlinear inhomogeneous Schrödinger equation, J. Appl. Math. Comput. 32 (2010), no. 1, 237–253. 1
- [14] J. Chen and B. Guo, Sharp global existence and blowing up results for inhomogeneous Schrödinger equations, Discrete Contin. Dyn. Syst. Ser. B 8 (2007), no. 2, 357–367. 1

- [15] Y. Cho, S. Hong, and K. Lee, On the global well-posedness of focusing energy-critical inhomogeneous NLS, *J. Evol. Equ.* 20 (2020), no. 4, 1349–1380. [2](#)
- [16] Y. Cho and K. Lee, On the focusing energy-critical inhomogeneous NLS: Weighted space approach, *Nonlinear Anal.* 205 (2021), 112261. [2](#)
- [17] V. D. Dinh, Blowup of H^1 solutions for a class of the focusing inhomogeneous nonlinear Schrödinger equation, *Nonlinear Anal.* 174 (2018), 169–188. [2](#)
- [18] V. D. Dinh, Energy scattering for a class of the defocusing inhomogeneous nonlinear Schrödinger equation, *J. Evol. Equ.* 19 (2019), no. 2, 411–434. [2](#)
- [19] V. D. Dinh, Scattering theory in weighted L^2 space for a class of the defocusing inhomogeneous nonlinear Schrödinger equation, *Advances in Pure and Applied Mathematics* (in press), 2021, available at <https://arxiv.org/abs/1710.01392>. [2](#)
- [20] V. D. Dinh, Energy scattering for a class of inhomogeneous nonlinear Schrödinger equation in two dimensions, *J. Hyperbolic Differ. Equ.* 18 (2021), 1–28. [2](#)
- [21] V. D. Dinh and S. Keraani, Long time dynamics of non-radial solutions to inhomogeneous nonlinear Schrödinger equations, *SIAM Journal on Mathematical Analysis* (in press), 2021, available at <https://arxiv.org/abs/2105.04941>. [2](#), [3](#)
- [22] V. D. Dinh, M. Majdoub, and T. Saanouni, Long time dynamics and blow-up for the focusing inhomogeneous nonlinear Schrödinger equation with spatial growing nonlinearity, preprint, available at <https://arxiv.org/abs/2107.01479>. [1](#)
- [23] B. Dodson and J. Murphy, A new proof of scattering below the ground state for the 3D radial focusing cubic NLS, *Proc. Amer. Math. Soc.* 145 (2017), no. 11, 4859–4867. [2](#)
- [24] M. B. Erdoğan, T. B. Gürel, and N. Tzirakis, Smoothing for the fractional Schrödinger equation on the torus and the real line, *Indiana Univ. Math. J.* 68 (2019), no. 2, 369–392. [4](#)
- [25] L. G. Farah, Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation, *J. Evol. Equ.* 16 (2016), no. 1, 193–208. [2](#)
- [26] L. G. Farah and C. M. Guzmán, Scattering for the radial 3D cubic focusing inhomogeneous nonlinear Schrödinger equation, *J. Differential Equations* 262 (2017), no. 8, 4175–4231. [2](#), [4](#)
- [27] L. G. Farah and C. M. Guzmán, Scattering for the radial focusing inhomogeneous NLS equation in higher dimensions, *Bull. Braz. Math. Soc. (N.S.)* 51 (2020), no. 2, 449–512. [2](#), [4](#), [5](#), [9](#), [10](#), [11](#)
- [28] D. Fang, J. Xie, and T. Cazenave, Scattering for the focusing energy-subcritical nonlinear Schrödinger equation, *Sci. China Math.* 54 (2011), no. 10, 2037–2062. [12](#)
- [29] G. Fibich and X. Wang, Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, *Phys. D* 175 (2003), no. 1-2, 96–108. [1](#)
- [30] D. Foschi, Inhomogeneous Strichartz estimates, *J. Hyperbolic Differ. Equ.* 2 (2005), no. 1, 1–24. [6](#)
- [31] R. Fukuzumi and M. Ohta, Instability of standing waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities, *J. Math. Kyoto Univ.* 45 (2005), no. 1, 145–158. [1](#)
- [32] F. Genoud and C. A. Stuart, Schrödinger equations with a spatially decaying nonlinearity: existence and stability of standing waves, *Discrete Contin. Dyn. Syst.* 21 (2008), no. 1, 137–186. [1](#), [2](#)
- [33] C. M. Guzmán, On well posedness for the inhomogeneous nonlinear Schrödinger equation, *Nonlinear Anal. Real World Appl.* 37 (2017), 249–286. [2](#), [7](#), [8](#), [9](#)
- [34] T. S. Gill, Optical guiding of laser beam in nonuniform plasma, *Pramana* 55 (2000), 835–842. [1](#)
- [35] L. Jeanjean and S. Le Coz, An existence and stability result for standing waves of nonlinear Schrödinger equations, *Adv. Differential Equations* 11 (2006), no. 7, 813–840. [1](#)
- [36] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.* 120 (1998), no. 5, 955–980. [6](#)
- [37] S. Keraani and A. Vargas, A smoothing property for the L^2 -critical NLS equations and an application to blowup theory, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009), no. 3, 745–762. [4](#)
- [38] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, *Invent. Math.* 166 (2006), no. 3, 645–675. [2](#)
- [39] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, *Amer. J. Math.* 132 (2010), no. 2, 361–424. [6](#)
- [40] C. S. Liu and V. K. Tripathi, Laser guiding in an axially nonuniform plasma channel, *Physics of Plasmas* 1 (1994), no. 9, 3100–3103. [1](#)
- [41] Y. Liu, X. Wang, and K. Wang, Instability of standing waves of the Schrödinger equation with inhomogeneous nonlinearity, *Trans. Amer. Math. Soc.* 358 (2006), no. 5, 2105–2122. [1](#)
- [42] F. Merle, Nonexistence of minimal blow-up solutions of equations $iu_t = -\Delta u - k(x)|u|^{4/N}u$ in \mathbb{R}^N , *Ann. Inst. H. Poincaré Phys. Théor.* 64 (1996), no. 1, 33–85. [1](#)
- [43] C. Miao, J. Murphy, and J. Zheng, Scattering for the non-radial inhomogeneous NLS, *Mathematical Research Letters* (in press), available at <http://arxiv.org/abs/1912.01318>. [2](#), [3](#), [4](#)
- [44] J. Murphy, A simple proof of scattering for the intercritical inhomogeneous NLS, *Proceedings of the American Mathematical Society* (in press), 2021, available at <https://doi.org/10.1090/proc/15717>. [3](#)
- [45] C. Peng and D. Zhao, Blow-up dynamics of L^2 -critical inhomogeneous nonlinear Schrödinger equation, *Math. Meth. Appl. Sci.* 41, no. 18, 9408–9421. [4](#)
- [46] B. Opic and A. Kufner, Hardy-type inequalities, *Pitman Research Notes in Mathematics Series*, vol. 219, Longman Scientific & Technical, Harlow, 1990. [8](#)
- [47] P. Raphaël and J. Szeftel, Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS, *J. Amer. Math. Soc.* 24 (2011), no. 2, 471–546. [1](#)

- [48] T. Tao, On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation, Dyn. Partial Differ. Equ. 1 (2004), no. 1, 1–48. [2](#)
- [49] C. Xu and T. Zhao, A remark on the scattering theory for the 2D radial focusing INLS, preprint, available at <https://arxiv.org/abs/1908.00743>. [2](#)
- [50] S. Zhu, Blow-up solutions for the inhomogeneous Schrödinger equation with L^2 supercritical nonlinearity, J. Math. Anal. Appl. 409 (2014), no. 2, 760–776. [1](#)

(V. D. Dinh) LABORATOIRE PAUL PAINLEVÉ UMR 8524, UNIVERSITÉ DE LILLE CNRS, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE AND DEPARTMENT OF MATHEMATICS, HCMC UNIVERSITY OF EDUCATION, 280 AN DUONG VUONG, HO CHI MINH CITY, VIETNAM

Email address: contact@duongdinh.com

(S. Keraani) LABORATOIRE PAUL PAINLEVÉ UMR 8524, UNIVERSITÉ DE LILLE CNRS, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE

Email address: sahbi.keraani@univ-lille.fr