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THE SOBOLEV-MORAWETZ APPROACH FOR THE ENERGY SCATTERING OF NONLINEAR SCHRÖDINGER-TYPE EQUATIONS WITH RADIAL DATA

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Abstract. Based on recent works of Dodson-Murphy [12] and Arora-Dodson-Murphy [3], we give a unified approach for the energy scattering with radially symmetric initial data for nonlinear Schrödinger equations and nonlinear Choquard equations in any dimensions $N \geq 2$. We also discuss its applications for other Schrödinger-type equations.

1. **Introduction.** We first consider the Cauchy problem for the focusing intercritical nonlinear Schrödinger equation

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u &=& -|u|^\alpha u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) &=& u_0(x), \end{array} \right. \tag{NLS}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $u_0 : \mathbb{R}^N \to \mathbb{C}$ and $\alpha_* < \alpha < \alpha^*$ with

$$\alpha_* := \frac{4}{N}, \quad \alpha^* := \left\{ \begin{array}{ll} \frac{4}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2. \end{array} \right.$$

It is well-known that the equation (NLS) is locally well-posed in H^1 (see e.g. [6]). Moreover, local solutions satisfy the conservation of mass and energy

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u_0), \tag{Mass}$$

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx - \frac{1}{\alpha+2} \int |u(t,x)|^{\alpha+2} dx = E(u_0), \tag{Energy}$$

for all t in the existence time. The equation (NLS) also enjoys the following scaling invariance

$$u_\lambda(t,x) := \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

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Note that this scaling leaves the $\dot{H}^{\gamma_c}$-norm of the initial data invariant, i.e.

$$\|u_\lambda(0)\|_{\dot{H}^{\gamma_c}} = \|u_0\|_{\dot{H}^{\gamma_c}}, \quad \gamma_c := \frac{N}{2} - \frac{2}{\alpha}. \tag{1.1}$$

In this paper, we are interested in the asymptotic completeness or energy scattering for (NLS).

Definition 1.1 (Energy scattering)**.** A global solution $u \in C(\mathbb{R}, H^1)$ to (NLS) is said to be scattering in H^1 forward in time (resp. backward in time) if there exists $u_+ \in H^1$ (resp. $u_- \in H^1$) such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_+\|_{H^1} = 0 \quad \left(\text{resp. } \lim_{t\to-\infty} \|u(t) - e^{it\Delta}u_-\|_{H^1} = 0\right).$$

The energy scattering for small data can be proved easily using Strichartz estimates and the contraction mapping argument (see e.g. [6]). More precisely, it is known that there exists $\delta > 0$ sufficiently small such that if $\|u_0\|_{H^1} < \delta$, then the corresponding solution to (NLS) exists globally in time and scatters in H^1 in both directions. A natural question is: What happens for large data? A simple observation is that there exists a global but non-scattering solution to (NLS) of the form $u(t,x) = e^{it}Q(x)$, where Q is the ground state, i.e. the unique positive, radially symmetric, radially decreasing solution to the elliptic equation

$$-\Delta Q + Q - |Q|^\alpha Q = 0. \tag{1.2}$$

Note that the existence of radial solution to (1.2) was proved by Weinstein [39] and the uniqueness of positive radial solution was proved by Coffman [9] and Kwong [27].

Holmer-Roudenko [22] proved the energy scattering with radially symmetric initial data for the focusing cubic nonlinear Schrödinger equation in dimension three. More precisely, they proved the following result.

Theorem 1.2 ([22])**.** *Let $N = 3$ and $\alpha = 2$. Let $u_0 \in H^1$ be radially symmetric and satisfy*

$$E(u_0)M(u_0) < E(Q)M(Q), \quad \|\nabla u_0\|_{L^2}\|u_0\|_{L^2} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}.$$

Then the corresponding solution to (NLS) *exists globally in time and scatters in H^1 in both directions.*

We note that the global existence for data below the ground state goes back to the work of Stubbe [34]. Theorem 1.2 was later extended to the non-radial case by Duyckaerts-Holmer-Roudenko [14] and to the general case by Cazenave-Fang-Xie [7], Akahori-Nawa [1] and Guevara [18].

Theorem 1.3 ([1, 7, 14, 18])**.** *Let $N \geq 1$, $\alpha_* < \alpha < \alpha^*$ and $u_0 \in H^1$ satisfy*

$$E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c}, \tag{1.3}$$

$$\|\nabla u_0\|_{L^2}\|u_0\|_{L^2}^{\sigma_c} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}, \tag{1.4}$$

where $\sigma_c := \frac{1-\gamma_c}{\gamma_c} = \frac{4-(N-2)\alpha}{N\alpha-4}$. Then the corresponding solution to (NLS) *exists globally in time and scatters in H^1 in both directions.*

We next consider the Cauchy problem for the focusing intercritical nonlinear Choquard equation

$$\begin{cases} i\partial_t u + \Delta u &= -(I_\gamma * |u|^p)|u|^{p-2}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) &= u_0(x), \end{cases} \tag{NLC}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $u_0 : \mathbb{R}^N \to \mathbb{C}$, $N \geq 1$, $p \geq 2$ and $p_* < p < p^*$ with

$$p_* := 1 + \frac{\gamma+2}{N}, \quad p^* := \begin{cases} 1 + \frac{\gamma+2}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2. \end{cases} \tag{1.5}$$

Here I_γ is the Riesz potential defined by

$$I_\gamma(x) := \frac{A(\gamma)}{|x|^{N-\gamma}}, \quad A(\gamma) := \frac{\Gamma\left(\frac{N-\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\pi^{\frac{N}{2}}2^\gamma}, \quad x \neq 0 \tag{1.6}$$

with $0 < \gamma < N$ and Γ is the Gamma function.

It was known (see [4, 29]) that the equation (NLC) is locally well-posed in H^1. Moreover, the following conservation laws hold:

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u_0), \tag{Mass}$$

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{1}{2p}\int (I_\gamma * |u|^p)(t,x)|u(t,x)|^p dx = E(u_0). \tag{Energy}$$

The equation (NLC) is also invariant under the scaling

$$u_\lambda(t,x) := \lambda^{\frac{\gamma+2}{2(p-1)}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Note that this scaling leaves the $\dot{H}^{\gamma_c}$-norm of the initial data invariant, i.e.

$$\|u_\lambda(0)\|_{\dot{H}^{\gamma_c}} = \|u_0\|_{\dot{H}^{\gamma_c}}, \quad \gamma_c := \frac{N}{2} - \frac{\gamma+2}{2(p-1)}. \tag{1.7}$$

As for (NLS), the energy scattering for (NLC) with small data can be proved by using Strichartz estimates and the fixed point argument. Moreover, the equation (NLC) admits a global but non-scattering solution $u(t,x) = e^{it}Q(x)$, where Q is a ground state related to the elliptic equation

$$-\Delta Q + Q - (I_\gamma * |Q|^p)|Q|^{p-2}Q = 0. \tag{1.8}$$

Recall that a non-zero, non-negative H^1 function to (1.8) is called a ground state related to (1.8) if it minimizes the Weinstein's functional

$$W(f) := \left[\|\nabla f\|_{L^2}^{N(p-1)-\gamma}\|f\|_{L^2}^{\gamma+2-(N-2)(p-1)}\right] \div \int (I_\gamma * |f|^p)|f|^p dx,$$

that is,

$$W(Q) = \inf\left\{W(f) \ : \ f \in H^1\backslash\{0\}\right\}.$$

The existence of positive solutions along with the regularity and radial symmetric solutions to (1.8) were studied by Moroz-Schaftingen [30]. The uniqueness of positive solutions to (1.2) is still an open problem except for $p = 2, \gamma = 2$ and $N = 3, 4, 5$ (see [28, 26, 4]) and for $p > 2$ close to 2 and $N = 3$ (see [40]).

The large data energy scattering for (NLC) was established recently by Arora-Roudenko [4]. More precisely, we have the following result.

Theorem 1.4 ([4]). *Let $N \geq 1$, $0 < \gamma < N$, $p \geq 2$ and $p_* < p < p^*$. Let $u_0 \in H^1$ satisfy*

$$E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c} \tag{1.9}$$

and

$$\|\nabla u_0\|_{L^2}\|u_0\|_{L^2}^{\sigma_c} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}, \tag{1.10}$$

where $\sigma_c := \frac{1-\gamma_c}{\gamma_c} = \frac{\gamma+2-(N-2)(p-1)}{N(p-1)-\gamma-2}$. *Then the corresponding solution to* (NLC) *exists globally in time and scatters in* H^1 *in both directions.*

The proofs of Theorem 1.2, Theorem 1.3 and Theorem 1.4 are based on the concentration-compactness-rigidity argument of Kenig-Merle [25]. It is done by three main steps: scattering criteria, construction of the critical solution and rigidity argument.

Step 1: Scattering criteria. Thanks to Strichartz estimates, one proves that if u is a global solution to (NLS) or (NLC) satisfying

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} < \infty, \quad \|u\|_{S(\mathbb{R},\dot{H}^{\gamma_c})} < \infty, \tag{1.11}$$

then u scatters in H^1 in both directions. Here

$$\|u\|_{S(I,\dot{H}^{\gamma_c})} := \sup_{(q,r)\in S_\gamma} \|u\|_{L^q(I,L^r)},$$

where S_γ is the set of $\dot{H}^\gamma$ admissible pairs, i.e.

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} - \gamma$$

and

$$\begin{cases} \frac{2N}{N-2\gamma} \le r \le \left(\frac{2N}{N-2}\right)^- & \text{if } N \ge 3, \\ \frac{2}{1-\gamma} \le r \le \left(\left(\frac{2}{1-\gamma}\right)^+\right)^* & \text{if } N = 2, \\ \frac{2}{1-2\gamma} \le r \le \infty & \text{if } N = 1. \end{cases}$$

We have used the notation a^+ for a fixed number slightly greater than a, a^- for a fixed number slightly smaller than a and

$$\frac{1}{a} = \frac{1}{a^+} + \frac{1}{(a^+)^*}.$$

Step 2: Construction of the critical solution. Denote

$$\mathcal{A}_\delta := \left\{ u_0 \in H^1 : \begin{cases} E(u_0)[M(u_0)]^{\sigma_c} & < & \delta \\ \|\nabla u_0\|_{L^2}\|u_0\|_{L^2}^{\sigma_c} & < & \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c} \end{cases} \right\}$$

and

$$\delta_c := \sup\left\{\delta > 0 \ : \ \text{if } u_0 \in \mathcal{A}_\delta, \text{ then the solution satisfies (1.11)}\right\}.$$

By the small data scattering, one knows that $\delta_c > 0$. If $\delta_c \ge E(Q)[M(Q)]^{\sigma_c}$, then it is done. Assuming that $\delta_c < E(Q)[M(Q)]^{\sigma_c}$, one will derive a contradiction. By the definition of δ_c, there exists a sequence of solutions u_n to (NLS) or (NLC) with initial data u_n^0 satisfying

$$E(u_n^0)[M(u_n^0)]^{\sigma_c} \searrow \delta_c, \quad \|\nabla u_n^0\|_{L^2}\|u_n^0\|_{L^2}^{\sigma_c} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}$$

for which (1.11) does not hold for all $n \ge 1$. In particular, $\|u_n\|_{S(\mathbb{R},\dot{H}^{\gamma_c})} = \infty$ for all $n \ge 1$.

Applying the profile decomposition to $(u_n)_{n\ge1}$, we can construct a critical solution, denoted by u_c, that lies exactly at the threshold δ_c. Moreover, the critical solution satisfies $\|u_c\|_{S(\mathbb{R},\dot{H}^{\gamma_c})} = \infty$ and

$$\mathcal{K} := \{u_c(t) \ : \ t \in [0,\infty)\}$$

is a precompact set in H^1.

Step 3: Rigidity argument. Using localized virial estimates, we show that such a critical solution is identically zero which is a contradiction.

Recently, Dodson-Murphy [12], Arora-Dodson-Murphy [3] and Arora [2] gave alternative simple proofs for the large data scattering with radially symmetric initial data that avoids the concentration-compactness-rigidity argument. See also [13] for a new proof of scattering for (NLS) with general data that avoids the concentration-compactness-rigidity argument.

In the case $N \geq 3$, the proofs of these results are based on the following two main ingredients. The first ingredient is the scattering criterion of Tao [36].

Lemma 1.5 (Scattering criterion [36]). *Suppose u is a radially symmetric global solution to* (NLS) *or* (NLC) *satisfying*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} \leq A$$

for some constant $A > 0$. Then there exist $\varepsilon = \varepsilon(A) > 0$ sufficiently small and $R = R(A) > 0$ sufficiently large such that if

$$\lim_{t\to\infty} \int_{|x|\leq R} |u(t,x)|^2 dx \leq \varepsilon,$$

then u scatters in H^1 forward in time.

This scattering criterion was proved in [36] (see also [12]) for the focusing cubic NLS in three dimensions. It was later extended to (NLS) and (NLC) by Arora [2]. However, the proof presented in [2] contains a flaw. More precisely, the author in [2] used the following inhomogeneous Strichartz estimate

$$\left\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \right\|_{L^{\frac{2(N+2)}{N-2s}}(I\times\mathbb{R}^N)} \lesssim \|F\|_{L^2(I, L^{\frac{2N}{N+2(1-s)}})} \tag{1.12}$$

which is not clear to hold for all $0 < s < 1$. In fact, to our knowledge, the best known inhomogeneous Strichartz estimates were proved independently by Foschi [17] and Vilela [38]. According to their results, the estimate (1.12) holds true provided that

$$(q,r) := \left(\frac{2(N+2)}{N-2s}, \frac{2(N+2)}{N-2s}\right), \quad (m,n) := \left(2, \frac{2N}{N-2+2s}\right)$$

are $\frac{N}{2}$-acceptable, i.e.

$$1 \leq q,r,m,n \leq \infty, \quad \frac{1}{q} < \frac{N}{2} - \frac{N}{r}, \quad \frac{1}{m} < \frac{N}{2} - \frac{N}{n} \tag{1.13}$$

satisfying

$$\frac{2}{q} + \frac{2}{m} = N - \frac{N}{r} - \frac{N}{n} \tag{1.14}$$

and

$$\frac{N-2}{N} \leq \frac{r}{n} \leq \frac{N}{N-2}. \tag{1.15}$$

It is easy to check that the last inequality in (1.13) requires $s < \frac{1}{2}$, and the second inequality in (1.15) requires $s \leq \frac{N^2+2N-4}{2N^2-4} < 1$.

The second ingredient is the evacuation of the potential energy.

Lemma 1.6 (Energy evacuation (NLS) [12, 2]). *There exist a sequence of times $t_n \to \infty$ and a sequence of radii $R_n \to \infty$ such that*

$$\lim_{n\to\infty} \int_{|x|\le R_n} |u(t_n,x)|^{\alpha+2} dx = 0.$$

Lemma 1.7 (Energy evacuation (NLC) [2]). *There exist a sequence of times $t_n \to \infty$ and a sequence of radii $R_n \to \infty$ such that*

$$\lim_{n\to\infty} \int_{|x|\le R_n} |u(t_n,x)|^{\frac{2Np}{N+\gamma}} dx = 0.$$

The proofs of these energy evacuations are based on the coercivity property of global solutions below the ground state and localized Morawetz estimates using the radial Sobolev embedding.

In the case $N = 2$, the scattering criterion of Tao is not sufficient to show the energy scattering due to the logarithmic divergence of an integral appearing in using dispersive estimates. To overcome this difficulty, Arora-Dodson-Murphy [3] made use of the space-time estimate

$$\int_0^T \int_{\mathbb{R}^2} |u(t,x)|^{\alpha+2} dxdt \lesssim T^\beta, \quad \beta := \max\left\{\frac{1}{3}, \frac{2}{\alpha+2}\right\}$$

to show the global space-time bound

$$\|u\|_{L^{2\alpha}(\mathbb{R}\times\mathbb{R}^2)} < \infty$$

which implies the scattering.

The purpose of this paper is to give a unified proof for the energy scattering with radially symmetric initial data for both (NLS) and (NLC) in any dimensions $N \ge 2$. Let us give a brief description of the proof for (NLS), the one for (NLC) is similar. The proof is divided into three steps.

Step 1: Scattering criteria. Using a suitable inhomogeneous Strichartz estimate, we prove that if u is a global solution to (NLS) satisfying $\|u\|_{L^\infty(\mathbb{R},H^1)} < \infty$, then there exists $\delta > 0$ such that if

$$\|e^{i(t-T)\Delta} u(T)\|_{L^k([T,\infty),L^r)} < \delta \tag{1.16}$$

for some $T > 0$, then u scatters in H^1 forward in time, where

$$k := \frac{2\alpha(\alpha+2)}{4-(N-2)\alpha}, \quad r := \alpha+2.$$

Step 2: Localized Morawetz estimates. By using some variational analysis, we prove that under the assumptions (1.3) and (1.4), the corresponding solution to (NLS) exists globally in time, and there exist $\nu = \nu(u_0, Q) > 0$ and $R_0 = R_0(u_0, Q) > 0$ such that for any $R \ge R_0$,

$$H(\chi_R u(t)) \ge \nu \|\chi_R u(t)\|_{L^{\alpha+2}}^{\alpha+2} \tag{1.17}$$

for all $t \in \mathbb{R}$. Here

$$H(u) := \|\nabla u\|_{L^2}^2 - \frac{N\alpha}{2(\alpha+2)} \|u\|_{L^{\alpha+2}}^{\alpha+2}$$

is nothing but the virial functional

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8H(u(t))$$

and $\chi_R(x) = \chi(x/R)$ with $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \le \chi \le 1$ and

$$\chi(x) = \begin{cases} 1 & \text{if} \quad |x| \le 1/2, \\ 0 & \text{if} \quad |x| \ge 1. \end{cases}$$

Using the coercivity property (1.17), localized Morawetz estimates and the radial Sobolev embedding, we prove that the solution to (NLS) satisfies for any time interval $I \subset \mathbb{R}$,

$$\int_I \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} dt \lesssim |I|^\beta, \quad \beta := \max\left\{\frac{1}{3}, \frac{2}{(N-1)\alpha+2}\right\}. \tag{1.18}$$

Step 3: Energy scattering. By Step 1, it suffices to find $T > 0$ so that (1.16) holds. To this end, let $\varepsilon > 0$ be a small parameter. For $T > \varepsilon^{-\sigma}$, we use the Duhamel formula to write

$$e^{i(t-T)\Delta}u(T) = e^{it\Delta}u_0 + F_1(t) + F_2(t),$$

where

$$F_1(t) := i\int_I e^{i(t-s)\Delta}|u(s)|^\alpha u(s)ds, \quad F_2(t) := i\int_J e^{i(t-s)\Delta}|u(s)|^\alpha u(s)ds$$

with $I := [T-\varepsilon^{-\sigma}, T]$ and $J := [0, T-\varepsilon^{-\sigma}]$.

Thanks to Strichartz estimates, the linear part can be made small by taking $T > \varepsilon^{-\sigma}$ sufficiently large. Combining Strichartz estimates, (1.18) and the radial Sobolev embedding, the term F_1 becomes small by taking a suitable value of σ. Finally, to treat the term F_2, we make use of dispersive estimates and the space-time estimate (1.18). We refer the reader to Section 3 for more details.

Comparing to previous works [12, 3, 2], the main contributions of the paper are the followings:

- We give a unified simple proof for the radial data energy scattering for both (NLS) and (NLC) in any dimensions $N \ge 2$. In particular, we give the proof of the energy scattering with radially symmetric initial data for (NLC) in two dimensions which, to our knowledge, is new.
- We have fixed some flaws in the proofs of [2].
- Finally, we discuss some possible extensions of this method for other Schrödinger-type equations.

This paper is organized as follows. In Section 2, we recall some Strichartz estimates which are needed in the sequel. In Section 3, we prove the energy scattering for nonlinear Schrödinger equations with radial data. The proof of the radial data energy scattering for Choquard equations will be given in Section 4. Finally, we discuss some possible extensions of the radial Sobolev-Morawetz method to other Schrödinger-type equations in Section 5.

2. Strichartz estimates.

Let $e^{it\Delta}$ be the propagator for the free Schrödinger equation $i\partial_t u + \Delta u = 0$. We have from the explicit formula

$$e^{it\Delta}f(x) = Ct^{-\frac{N}{2}}\int e^{i\frac{|x-y|^2}{4t}}f(y)dy$$

the standard dispersive estimate

$$\|e^{it\Delta}f\|_{L^\infty} \lesssim |t|^{-\frac{N}{2}}\|f\|_{L^1}.$$

By interpolating this inequality with the L^2-isometry $\|e^{it\Delta}f\|_{L^2} = \|f\|_{L^2}$, we have the following dispersive estimates: for any $r \in [2, \infty]$,

$$\|e^{it\Delta}f\|_{L^r} \lesssim |t|^{-\frac{N}{2}\left(1-\frac{2}{r}\right)}\|f\|_{L^{r'}}, \tag{2.1}$$

where (r, r') is the Hölder's conjugate pair.

Let $I \subset \mathbb{R}$ be an interval and $q, r \in [1, \infty]$. We define the mixed norm

$$\|u\|_{L^q(I,L^r)} := \left(\int_I \left(\int_{\mathbb{R}^N} |u(t,x)|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}$$

with a usual modification when either q or r are infinity. When $q = r$, we use the notation $L^q(I \times \mathbb{R}^N)$ instead of $L^q(I, L^q)$.

Definition 2.1. A pair (q, r) is said to be Schrödinger admissible if

$$\begin{cases} r \in \left[2, \frac{2N}{N-2}\right] & \text{if } N \geq 3, \\ r \in [2, \infty) & \text{if } N = 2, \\ r \in [2, \infty] & \text{if } N = 1, \end{cases} \qquad \frac{2}{q} + \frac{N}{r} = \frac{N}{2}.$$

Proposition 1 (Strichartz estimates [6, 24])**.** *Let $N \geq 1$ and $I \subset \mathbb{R}$ be an interval. There exists a constant $C > 0$ independent of I such that the following estimates hold:*

- *(Homogeneous estimates)*

$$\|e^{it\Delta}f\|_{L^q(I,L^r)} \leq C\|f\|_{L^2}$$

for any $f \in L^2$ and any Schrödinger admissible pair (q, r).

- *(Inhomogeneous estimates)*

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s)ds\right\|_{L^q(I,L^r)} \leq C\|F\|_{L^{m'}(I,L^{n'})}$$

for any $F \in L^{m'}(I, L^{n'})$ and any Schrödinger admissible pairs $(q, r), (m, n)$.

We also have the following inhomogeneous Strichartz estimates for non Schrödinger admissible pairs.

Lemma 2.2 ([8])**.** *Let $N \geq 1$, $I \subset \mathbb{R}$ be an interval. Let (q, r) be a Schrödinger admissible pair with $r > 2$. Fix $k > \frac{q}{2}$ and define m by*

$$\frac{1}{k} + \frac{1}{m} = \frac{2}{q}. \tag{2.2}$$

Then there exists $C > 0$, depending only on N, r and k, such that

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s)ds\right\|_{L^k(I,L^r)} \leq C\|F\|_{L^{m'}(I,L^{r'})} \tag{2.3}$$

for any $F \in L^{m'}(I, L^{r'})$.

We refer the reader to [8, Lemma 2.1] for the proof of this result.

3. Nonlinear Schrödinger equations.

3.1. **Small data theory.** We have the following nonlinear estimates which follow directly from Hölder's inequality.

Lemma 3.1 (Nonlinear estimates)**.** *Let $N \geq 1$, $\alpha_* < \alpha < \alpha^*$ and $I \subset \mathbb{R}$. Denote*

$$q := \frac{4(\alpha+2)}{N\alpha}, \quad r := \alpha + 2, \quad k := \frac{2\alpha(\alpha+2)}{4-(N-2)\alpha}, \quad m := \frac{2\alpha(\alpha+2)}{N\alpha^2+(N-2)\alpha-4}. \tag{3.1}$$

Then the following estimates hold:

$$\| |u|^\alpha u \|_{L^{m'}(I,L^{r'})} \lesssim \|u\|^{\alpha+1}_{L^k(I,L^r)}, \tag{3.2}$$

$$\| \langle\nabla\rangle (|u|^\alpha u)\|_{L^{q'}(I,L^{r'})} \lesssim \|u\|^{\alpha}_{L^k(I,L^r)} \| \langle\nabla\rangle u\|_{L^q(I,L^r)}. \tag{3.3}$$

Remark 1. Let q, r, k and m be as in (3.1). It is easy to check that (q, r) is a Schrödinger admissible pair. Moreover, k, m and q satisfy (2.2), hence (2.3) holds for such choice of exponents.

Lemma 3.2 (Small data global well-posedness)**.** *Let $N \geq 1$ and $\alpha_* < \alpha < \alpha^*$. Let $T > 0$ be such that $u(T) \in H^1$. Then there exists $\delta > 0$ such that if*

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} < \delta,$$

then there exists a unique global solution to (NLS) *with initial data $u(T)$ satisfying*

$$\|u\|_{L^k([T,+\infty),L^r)} \leq 2\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)}$$

and

$$\| \langle\nabla\rangle u\|_{L^q([T,+\infty),L^r)} \leq 2C\|u(T)\|_{H^1}.$$

Here k, q and r are as in (3.1).

Proof. Since the proof is nowadays standard, we only sketch it. Let q, r, k and m be as in (3.1). By Remark 1, (2.3) and Lemma 3.1, it is easy to show that the functional

$$\Phi(u(t)) := e^{i(t-T)\Delta}u(T) + i\int_T^t e^{i(t-s)\Delta}|u(s)|^\alpha u(s)ds$$

is a contraction on (X, d), where

$$X := \left\{u \ : \ \|u\|_{L^k([T,+\infty),L^r)} \leq M, \quad \| \langle\nabla\rangle u\|_{L^q([T,+\infty),L^r)} \leq L\right\}$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^k([T,+\infty),L^r)} + \|u - v\|_{L^q([T,+\infty),L^r)}$$

with $L = 2C\|u(T)\|_{H^1}$ and $M = 2\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)}$ sufficiently small. □

Lemma 3.3 (Small data scattering)**.** *Let $N \geq 1$ and $\alpha_* < \alpha < \alpha^*$. Suppose that u is a global solution to* (NLS) *satisfying*

$$\|u\|_{L^\infty(\mathbb{R},H^1)} < \infty.$$

Then there exists $\delta > 0$ such that if

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} < \delta$$

for some $T > 0$, where k and r are as in (3.1)*, then u scatters in H^1 forward in time.*

Proof. Let $\delta > 0$ be as in Lemma 3.2. It follows from Lemma 3.2 that the solution satisfies

$$\begin{aligned} \|u\|_{L^k([T,+\infty),L^r)} &\leq 2\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)}, \\ \|\langle\nabla\rangle u\|_{L^q([T,+\infty),L^r)} &\leq 2C\|u(T)\|_{H^1}. \end{aligned}$$

Now let $0 < \tau < t < +\infty$. By Strichartz estimates, we see that

$$\begin{aligned} \|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} &= \left\| \int_\tau^t e^{-is\Delta}|u(s)|^\alpha u(s)ds \right\|_{H^1} \\ &\lesssim \|\langle\nabla\rangle (|u|^\alpha u)\|_{L^{q'}((\tau,t),L^{r'})} \\ &\lesssim \|u\|^\alpha_{L^k((\tau,t),L^r)} \|\langle\nabla\rangle u\|_{L^q((\tau,t),L^r)} \to 0 \end{aligned}$$

as $\tau, t \to +\infty$. This shows that $(e^{-it\Delta}u(t))_t$ is a Cauchy sequence in H^1 as $t \to \infty$. Thus the limit

$$u_+ := u_0 + i\int_t^{+\infty} e^{-is\Delta}|u(s)|^\alpha u(s)ds$$

exists in H^1. By the same reasoning as above, we prove as well that

$$\|u(t) - e^{it\Delta}u_+\|_{H^1} \to 0$$

as $t \to +\infty$. The proof is complete. □

3.2. **Variational analysis.** We recall some properties of the ground state Q related to (1.2). The ground state Q optimizes the sharp Gagliardo-Nirenberg inequality

$$\|f\|^{\alpha+2}_{L^{\alpha+2}} \leq C_{\rm opt}\|\nabla f\|^{\frac{N\alpha}{2}}_{L^2}\|f\|^{\frac{4-(N-2)\alpha}{2}}_{L^2}$$

that is

$$C_{\rm opt} = \|Q\|^{\alpha+2}_{L^{\alpha+2}} \div \left[\|\nabla Q\|^{\frac{N\alpha}{2}}_{L^2}\|Q\|^{\frac{4-(N-2)\alpha}{2}}_{L^2}\right].$$

Recall that Q satisfies the following Pohozaev's identities

$$\|Q\|^2_{L^2} = \frac{4-(N-2)\alpha}{N\alpha}\|\nabla Q\|^2_{L^2} = \frac{4-(N-2)\alpha}{2(\alpha+2)}\|Q\|^{\alpha+2}_{L^{\alpha+2}}.$$

It follows that

$$E(Q) = \frac{N\alpha-4}{2N\alpha}\|\nabla Q\|^2_{L^2} = \frac{N\alpha-4}{4(\alpha+2)}\|Q\|^{\alpha+2}_{L^{\alpha+2}}$$

and

$$C_{\rm opt} = \frac{2(\alpha+2)}{N\alpha}\left(\|\nabla Q\|_{L^2}\|Q\|^{\sigma_c}_{L^2}\right)^{-\frac{N\alpha-4}{2}}.$$

Lemma 3.4 (Coercivity I [2])**.** *Let $N \geq 1$, $\alpha_* < \alpha < \alpha^*$ and $u_0 \in H^1$ satisfy (1.3) and (1.4). It follows that the corresponding solution exists globally in time. Moreover, there exists $\rho = \rho(u_0, Q) > 0$ such that*

$$\|\nabla u(t)\|_{L^2}\|u(t)\|^{\sigma_c}_{L^2} < (1-2\rho)\|\nabla Q\|_{L^2}\|Q\|^{\sigma_c}_{L^2} \tag{3.4}$$

for all $t \in \mathbb{R}$.

The proof of this result follows from the sharp Gagliardo-Nirenberg inequality and the continuity argument. We refer the reader to [2, Lemma 3.1] for a detailed proof.

Lemma 3.5 (Coercivity II [2]). *Let $N \geq 1$ and $\alpha_* < \alpha < \alpha^*$. Let ρ be as in (3.4). There exists $R_0 = R_0(\rho, \|u_0\|_{L^2}) > 0$ such that for any $R \geq R_0$,*

$$\|\nabla(\chi_R u(t))\|_{L^2}\|\chi_R u(t)\|_{L^2}^{\sigma_c} < (1-\rho)\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c} \tag{3.5}$$

for all $t \in \mathbb{R}$, where $\chi_R(x) = \chi(x/R)$ with $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \leq \chi \leq 1$,

$$\chi(x) = \begin{cases} 1 & \text{if} \quad |x| \leq 1/2, \\ 0 & \text{if} \quad |x| \geq 1. \end{cases} \tag{3.6}$$

In particular, there exists $\delta = \delta(\rho) > 0$ such that for any $R \geq R_0$,

$$\|\nabla(\chi_R u(t))\|_{L^2}^2 - \frac{N\alpha}{2(\alpha+2)}\|\chi_R u(t)\|_{L^{\alpha+2}}^{\alpha+2} \geq \delta\|\chi_R u(t)\|_{L^{\alpha+2}}^{\alpha+2} \tag{3.7}$$

for all $t \in \mathbb{R}$.

For the proof of this result, we refer the reader to [2, Lemma 3.2].

We also have the following Morawetz identity.

Lemma 3.6 (Morawetz identity). *Let $N \geq 1$ and $0 < \alpha < \alpha^*$. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a sufficiently smooth and decaying function. Let u be a H^1 solution to* (NLS). *Define*

$$M_\varphi(t) := 2\int \nabla\varphi \cdot \mathrm{Im}(\overline{u}(t)\nabla u(t))dx.$$

Then

$$\frac{d}{dt}M_\varphi(t) = -\int \Delta^2\varphi|u(t)|^2dx + 4\sum_{j,k=1}^N \int \partial_{jk}^2\varphi \mathrm{Re}(\partial_j\overline{u}(t)\partial_k u(t))dx - \frac{2\alpha}{\alpha+2}\int \Delta\varphi|u(t)|^{\alpha+2}dx.$$

Let $\zeta : [0,\infty) \to [0,2]$ be a smooth function satisfying

$$\zeta(r) = \begin{cases} 2 & \text{if} \quad 0 \leq r \leq 1, \\ 0 & \text{if} \quad r \geq 2. \end{cases} \tag{3.8}$$

We define the function $\theta : [0,\infty) \to [0,\infty)$ by

$$\theta(r) := \int_0^r \int_0^s \zeta(z)dzds.$$

Given $R > 0$, we define a radial function

$$\varphi_R(x) = \varphi_R(r) := R^2\theta(r/R), \quad r = |x|. \tag{3.9}$$

It is easy to check that

$$2 \geq \varphi_R''(r) \geq 0, \quad 2 - \frac{\varphi_R'(r)}{r} \geq 0, \quad 2N - \Delta\varphi_R(x) \geq 0, \quad \forall r \geq 0, \quad \forall x \in \mathbb{R}^N.$$

Lemma 3.7 (Morawetz estimate). *Let $N \geq 2$ and $\alpha_* < \alpha < \alpha^*$. Let $u_0 \in H^1$ be radially symmetric satisfying (1.3) and (1.4). Then the corresponding solution to* (NLS) *satisfies for any time interval $I \subset \mathbb{R}$,*

$$\int_I \|u(t)\|_{L^{\alpha+2}}^{\alpha+2}dt \leq C(u_0,Q)|I|^\beta, \quad \beta := \max\left\{\frac{1}{3}, \frac{2}{(N-1)\alpha+2}\right\} \tag{3.10}$$

for some constant $C(u_0,Q) > 0$ depending only on u_0 and Q.

Proof. Let $\rho = \rho(u_0, Q)$ be as in (3.4), and $R = R(\rho, \|u_0\|_{L^2})$ be as in Lemma 3.5. We define φ_R as in (3.9). By the Cauchy-Schwarz inequality and (3.4), we see that

$$|M_{\varphi_R}(t)| \lesssim \|\nabla \varphi_R\|_{L^\infty} \|u(t)\|_{L^2} \|\nabla u(t)\|_{L^2} \lesssim R \tag{3.11}$$

for all $t \in \mathbb{R}$. Here the implicit constant depends only on u_0 and Q. By Lemma 3.6,

$$\begin{aligned}\frac{d}{dt} M_{\varphi_R}(t) &= -\int \Delta^2 \varphi_R |u(t)|^2 dx + 4 \sum_{j,k=1}^N \int \partial_{jk}^2 \varphi_R \mathrm{Re}(\partial_j \overline{u}(t) \partial_k u(t)) dx \\ &\qquad - \frac{2\alpha}{\alpha+2} \int \Delta \varphi_R |u(t)|^{\alpha+2} dx \\ &= 8 \left(\int_{|x|\le R} |\nabla u(t)|^2 dx - \frac{N\alpha}{2(\alpha+2)} \int_{|x|\le R} |u(t)|^{\alpha+2} dx \right) \\ &\quad - \int \Delta^2 \varphi_R |u(t)|^2 dx + 4 \sum_{j,k=1}^N \int_{|x|>R} \partial_{jk}^2 \varphi_R \mathrm{Re}(\partial_j \overline{u}(t) \partial_k u(t)) dx \\ &\quad - \frac{2\alpha}{\alpha+2} \int_{|x|>R} \Delta \varphi_R |u(t)|^{\alpha+2} dx.\end{aligned}$$

Since $\|\Delta^2 \varphi_R\|_{L^\infty} \lesssim R^{-2}$, the conservation of mass implies

$$\left| \int \Delta^2 \varphi_R |u(t)|^2 dx \right| \lesssim R^{-2}.$$

Since u is radial, we use the fact

$$\partial_{jk}^2 = \left(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_r + \frac{x_j x_k}{r^2} \partial_r^2$$

to get

$$\sum_{j,k=1}^N \partial_{jk}^2 \varphi_R \partial_j \overline{u} \partial_k u = \varphi_R'' |\partial_r u|^2 \ge 0$$

which implies

$$4 \sum_{j,k=1}^N \int_{|x|>R} \partial_{jk}^2 \varphi_R \mathrm{Re}(\partial_j \overline{u}(t) \partial_k u(t)) dx \ge 0.$$

Moreover, since $\|\Delta \varphi_R\|_{L^\infty} \lesssim 1$, we have from the radial Sobolev embedding (see e.g. [35]): for $N \ge 2$,

$$\sup_{x \ne 0} |x|^{\frac{N-1}{2}} |f(x)| \le C(N) \|f\|_{H^1}, \quad \forall f \in H^1_{\mathrm{rad}}, \tag{3.12}$$

that

$$\begin{aligned}\int_{|x|>R} |u(t)|^{\alpha+2} dx &\le \|u(t)\|^\alpha_{L^\infty(|x|>R)} \|u(t)\|^2_{L^2} \\ &\lesssim R^{-\frac{(N-1)\alpha}{2}} \|u(t)\|^\alpha_{H^1} \|u(t)\|^2_{L^2} \lesssim R^{-\frac{(N-1)\alpha}{2}}.\end{aligned}$$

We thus have

$$\begin{aligned}\frac{d}{dt} M_{\varphi_R}(t) \ge 8 \left(\int_{|x|\le R} |\nabla u(t)|^2 dx - \frac{N\alpha}{2(\alpha+2)} \int_{|x|\le R} |u(t)|^{\alpha+2} dx \right) \\ + O\left(R^{-2} + R^{-\frac{(N-1)\alpha}{2}} \right).\end{aligned}$$

Now let χ_R be as in Lemma 3.5. We have

$$\begin{aligned}\int |\nabla(\chi_R u(t))|^2 dx &= \int \chi_R^2 |\nabla u(t)|^2 dx - \int \chi_R \Delta(\chi_R)|u(t)|^2 dx \\ &= \int_{|x|\le R} |\nabla u(t)|^2 dx - \int_{R/2\le|x|\le R} (1-\chi_R^2)|\nabla u(t)|^2 dx \\ &\qquad - \int \chi_R \Delta(\chi_R)|u(t)|^2 dx \\ \int |\chi_R u(t)|^{\alpha+2} dx &= \int_{|x|\le R} |u(t)|^{\alpha+2} dx - \int_{R/2\le|x|\le R} (1-\chi_R^{\alpha+2})|u(t)|^{\alpha+2} dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{|x|\le R} |\nabla u(t)|^2 dx - \frac{N\alpha}{2(\alpha+2)} \int_{|x|\le R} |u(t)|^{\alpha+2} dx \\ &\quad = \int |\nabla(\chi_R u(t))|^2 dx - \frac{N\alpha}{2(\alpha+2)} \int |\chi_R u(t)|^{\alpha+2} dx \\ &\qquad + \int_{R/2\le|x|\le R} (1-\chi_R^2)|\nabla u(t)|^2 dx + \int \chi_R \Delta(\chi_R)|u(t)|^2 dx \\ &\qquad - \frac{N\alpha}{2(\alpha+2)} \int_{R/2\le|x|\le R} (1-\chi_R^{\alpha+2})|u(t)|^{\alpha+2} dx.\end{aligned}$$

Thanks to the fact that $0 \le \chi_R \le 1$, $\|\Delta(\chi_R)\|_{L^\infty} \lesssim R^{-2}$ and the radial Sobolev embedding, we get

$$\begin{aligned}&\int_{|x|\le R} |\nabla u(t)|^2 dx - \frac{N\alpha}{2(\alpha+2)} \int_{|x|\le R} |u(t)|^{\alpha+2} dx \\ &\quad \ge \int |\nabla(\chi_R u(t))|^2 dx - \frac{N\alpha}{2(\alpha+2)} \int |\chi_R u(t)|^{\alpha+2} dx + O\left(R^{-2} + R^{-\frac{(N-1)\alpha}{2}}\right).\end{aligned}$$

We thus obtain

$$\frac{d}{dt} M_{\varphi_R}(t) \ge 8\left(\|\nabla(\chi_R u(t))\|_{L^2}^2 - \frac{N\alpha}{2(\alpha+2)} \|\chi_R u(t)\|_{L^{\alpha+2}}^{\alpha+2}\right) + O\left(R^{-2} + R^{-\frac{(N-1)\alpha}{2}}\right).$$

By Lemma 3.5 and (3.11), there exists $\delta = \delta(\rho) > 0$ such that for any $R \ge R_0$,

$$8\delta \|\chi_R u(t)\|_{L^{\alpha+2}}^{\alpha+2} \le \frac{d}{dt} M_{\varphi_R}(t) + O\left(R^{-2} + R^{-\frac{(N-1)\alpha}{2}}\right)$$

which implies for any time interval $I \subset \mathbb{R}$,

$$8\delta \int_I \int |\chi_R u(t)|^{\alpha+2} dx dt \le \sup_{t\in I} |M_{\varphi_R}(t)| + O\left(R^{-2} + R^{-\frac{(N-1)\alpha}{2}}\right)|I|.$$

It follows from the definition of χ_R and (3.11) that

$$\int_I \int_{|x|\le R/2} |u(t,x)|^{\alpha+2} dx dt \lesssim R + \left(R^{-2} + R^{-\frac{(N-1)\alpha}{2}}\right)|I|.$$

On the other hand, by radial Sobolev embeddings,

$$\int_{|x|\ge R/2} |u(t,x)|^{\alpha+2} dx \le \left(\sup_{|x|\ge R/2} |u(t,x)|^\alpha\right) \|u(t)\|_{L^2}^2 \lesssim R^{-\frac{(N-1)\alpha}{2}}.$$

We thus get

$$\int_I \int |u(t,x)|^{\alpha+2} dxdt \lesssim R + \left(R^{-2} + R^{-\frac{(N-1)\alpha}{2}}\right)|I| \lesssim R + R^{-\sigma}|I|,$$

where

$$\sigma := \min\left\{2, \frac{(N-1)\alpha}{2}\right\}.$$

Taking $R = |I|^{\frac{1}{1+\sigma}}$, we get for $|I|$ sufficiently large,

$$\int_I \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} dt \lesssim |I|^{\frac{1}{1+\sigma}} = |I|^{\beta},$$

where β is as in (3.10). In the case $|I|$ is sufficiently small, it follows from Sobolev embeddings and (3.4) that

$$\int_I \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} dt \lesssim \int_I \|u(t)\|_{H^1}^{\alpha+2} dt \lesssim |I| \lesssim |I|^{\beta}$$

since $\beta < 1$. This proves (3.10) and the proof is complete. □

Lemma 3.8. *Let $N \geq 2$, $\alpha_* < \alpha < \alpha^*$ and $u_0 \in H^1$ be radially symmetric satisfying (1.3) and (1.4). Then the corresponding solution to* (NLS) *satisfies*

$$\liminf_{t\to+\infty} \|u(t)\|_{L^{\alpha+2}} = 0. \tag{3.13}$$

Proof. Assume by contradiction that (3.13) does not hold. Then there exist $t_0 > 0$ and $\varrho > 0$ such that

$$\|u(t)\|_{L^{\alpha+2}} \geq \varrho$$

for all $t \geq t_0$. This implies in particular that for every $I \subset [t_0, +\infty)$,

$$\int_I \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \geq \varrho^{\alpha+2}|I|$$

which contradicts (3.10) for $|I|$ large since $\beta < 1$. □

Corollary 1. *Let $N \geq 2$, $\alpha_* < \alpha < \alpha^*$ and $u_0 \in H^1$ be radially symmetric satisfying (1.3) and (1.4). Then there exists $t_n \to +\infty$ such that the corresponding solution to* (NLS) *satisfies for any $R > 0$,*

$$\lim_{n\to\infty} \int_{|x|\leq R} |u(t_n, x)|^2 dx = 0. \tag{3.14}$$

Proof. By (3.13), there exists $t_n \to +\infty$ such that

$$\lim_{n\to\infty} \|u(t_n)\|_{L^{\alpha+2}} = 0.$$

Let $R > 0$. By Hölder's inequality, we see that

$$\begin{aligned}\int_{|x|\leq R} |u(t_n,x)|^2 dx &\leq \left(\int_{|x|\leq R} dx\right)^{\frac{\alpha}{\alpha+2}} \left(\int_{|x|\leq R} |u(t_n,x)|^{\alpha+2} dx\right)^{\frac{2}{\alpha+2}} \\ &\lesssim R^{\frac{N\alpha}{\alpha+2}} \left(\int |u(t_n,x)|^{\alpha+2} dx\right)^{\frac{2}{\alpha+2}} \to 0\end{aligned}$$

as $n \to \infty$. □

3.3. **Scattering below the ground state.**

Proposition 2. *Let $N \geq 2$, $\alpha_* < \alpha < \alpha^*$ and $u_0 \in H^1$ be radially symmetric satisfying (1.3) and (1.4). Then for $\varepsilon > 0$ sufficiently small, there exists $T = T(\varepsilon, u_0, Q)$ sufficiently large such that the corresponding solution to* (NLS) *satisfies*

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{\mu} \tag{3.15}$$

for some $\mu > 0$, where k and r are as in (3.1).

Proof. We will consider separately two cases: $N \geq 3$ and $N = 2$.

Case 1. $N \geq 3$.

Let $T > 0$ be a large parameter depending on ε, u_0 and Q to be chosen later. For $T > \varepsilon^{-\sigma}$ with some $\sigma > 0$ to be chosen later, we use the Duhamel formula to write

$$\begin{aligned} e^{i(t-T)\Delta}u(T) &= e^{it\Delta}u_0 + i\int_0^T e^{i(t-s)\Delta}|u(s)|^{\alpha}u(s)ds \\ &= e^{it\Delta}u_0 + F_1(t) + F_2(t), \end{aligned} \tag{3.16}$$

where

$$F_1(t) := i\int_I e^{i(t-s)\Delta}|u(s)|^{\alpha}u(s)ds, \quad F_2(t) := i\int_J e^{i(t-s)\Delta}|u(s)|^{\alpha}u(s)ds$$

with $I := [T - \varepsilon^{-\sigma}, T]$ and $J := [0, T - \varepsilon^{-\sigma}]$.

Step 1. Estimate the linear part. By Strichartz estimates, Sobolev embeddings, (1.3) and (1.4),

$$\|e^{it\Delta}u_0\|_{L^k(\mathbb{R},L^r)} \lesssim \||\nabla|^{\gamma_c}e^{it\Delta}u_0\|_{L^k(\mathbb{R},L^l)} \lesssim \|u_0\|_{\dot{H}^{\gamma_c}} \lesssim \|u_0\|_{H^1} \leq C(u_0, Q) < \infty,$$

where

$$l = \frac{2N\alpha(\alpha+2)}{N\alpha^2 + 4(N-1)\alpha - 8}. \tag{3.17}$$

Note that (k, l) is a Schrödinger admissible pair. By the monotone convergence, we may find $T > \varepsilon^{-\sigma}$ so that

$$\|e^{it\Delta}u_0\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon. \tag{3.18}$$

Step 2. Estimate F_1. By Remark 1, (2.3), (3.2) and Sobolev embedding, we have

$$\|F_1\|_{L^k([T,+\infty),L^r)} \lesssim \||u|^{\alpha}u\|_{L^{m'}(I,L^{r'})} \lesssim \|u\|^{\alpha+1}_{L^k(I,L^r)} \lesssim |I|^{\frac{\alpha+1}{k}}\|u\|^{\alpha+1}_{L^{\infty}(I,L^r)}.$$

We estimate $\|u\|_{L^{\infty}(I,L^r)}$ as follows. Fix $R = \max\left\{\varepsilon^{-2-\sigma}, \varepsilon^{-\frac{4-(N-2)\alpha}{(N-1)\alpha}}\right\}$, we have from (3.14) (by enlarging T if necessary) that

$$\int_{|x|\leq R} |u(T,x)|^2 dx \lesssim \varepsilon^2.$$

By the definition of χ_R,

$$\int \chi_R(x)|u(T,x)|^2 dx \lesssim \varepsilon^2.$$

Using the fact that

$$\begin{aligned} \left|\frac{d}{dt}\int \chi_R(x)|u(t,x)|^2 dx\right| &= \left|2\int \nabla\chi_R(x)\cdot \mathrm{Im}(\overline{u}(t,x)\nabla u(t,x))dx\right| \\ &\leq 2\|\nabla\chi_R\|_{L^{\infty}}\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \lesssim R^{-1} \end{aligned}$$

for all $t \in \mathbb{R}$, we have for any $t \in I$,

$$\begin{aligned}\int \chi_R(x)|u(t,x)|^2dx &= \int \chi_R(x)|u(T,x)|^2dx - \int_t^T \left(\frac{d}{ds}\int \chi_R(x)|u(s,x)|^2dx\right)ds\\ &\le \int \chi_R(x)|u(T,x)|^2dx + CR^{-1}(T-t)\\ &\le C\varepsilon^2 + CR^{-1}\varepsilon^{-\sigma} \le 2C\varepsilon^2\end{aligned}$$

for some constant $C = C(u_0, Q) > 0$. This shows that

$$\|\chi_R u\|_{L^\infty(I,L^2)} \lesssim \varepsilon,$$

where we have used the fact $\chi_R^2 \le \chi_R$ since $0 \le \chi_R \le 1$. By Hölder's inequality, the radial Sobolev embedding and the fact $\|u\|_{L^\infty(\mathbb{R},H^1)} \le C(u_0, Q)$,

$$\begin{aligned}\|u\|_{L^\infty(I,L^r)} &\le \|\chi_R u\|_{L^\infty(I,L^r)} + \|(1-\chi_R)u\|_{L^\infty(I,L^r)}\\ &\le \|\chi_R u\|_{L^\infty(I,L^2)}^{\frac{4-(N-2)\alpha}{2(\alpha+2)}}\|\chi_R u\|_{L^\infty(I,L^{\frac{2N}{N-2}})}^{\frac{N\alpha}{2(\alpha+2)}}\\ &\quad + \|(1-\chi_R)u\|_{L^\infty(I,L^\infty)}^{\frac{\alpha}{\alpha+2}}\|(1-\chi_R)u\|_{L^\infty(I,L^2)}^{\frac{2}{\alpha+2}}\\ &\lesssim \varepsilon^{\frac{4-(N-2)\alpha}{2(\alpha+2)}} + R^{-\frac{(N-1)\alpha}{2(\alpha+2)}} \lesssim \varepsilon^{\frac{4-(N-2)\alpha}{2(\alpha+2)}}.\end{aligned}$$

It follows that

$$\|F_1\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{-\frac{(\alpha+1)\sigma}{k}}\varepsilon^{\frac{(4-(N-2)\alpha)(\alpha+1)}{2(\alpha+2)}} = \varepsilon^{(\alpha+1)\left[-\frac{\sigma}{k}+\frac{4-(N-2)\alpha}{2(\alpha+2)}\right]}.$$

By the definition of k, we see that

$$\|F_1\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{\frac{(\alpha+1)(4-(N-2)\alpha)(\alpha-\sigma)}{2\alpha(\alpha+2)}}. \tag{3.19}$$

Step 3. Estimate F_2. We estimate

$$\|F_2\|_{L^k([T,+\infty),L^r)} \le \|F_2\|_{L^k([T,+\infty),L^l)}^{\theta}\|F_2\|_{L^k([T,+\infty),L^n)}^{1-\theta}$$

where l is as in (3.17), $\theta \in (0,1)$ and $n > r$ satisfy

$$\frac{1}{r} = \frac{\theta}{l} + \frac{1-\theta}{n}.$$

Using the fact (k,l) is a Schrödinger admissible pair and

$$F_2(t) = e^{i(t-T+\varepsilon^{-\sigma})\Delta}u(T-\varepsilon^{-\sigma}) - e^{it\Delta}u_0,$$

Strichartz estimates imply

$$\|F_2\|_{L^k([T,+\infty),L^l)} \lesssim 1.$$

On the other hand, by the dispersive estimates (2.1) and Sobolev embeddings with the fact that $\|u\|_{L^\infty(\mathbb{R},H^1)} \le C(u_0,Q)$, we have for any $t \ge T$,

$$\begin{aligned}\|F_2(t)\|_{L^n} &\lesssim \int_J (t-s)^{-\frac{N}{2}\left(1-\frac{2}{n}\right)}\||u(s)|^\alpha u(s)\|_{L^{n'}}ds\\ &= \int_0^{T-\varepsilon^{-\sigma}} (t-s)^{-\frac{N}{2}\left(1-\frac{2}{n}\right)}\|u(s)\|_{L^{n'(\alpha+1)}}^{\alpha+1}ds\\ &\lesssim (t-T+\varepsilon^{-\sigma})^{-\frac{N}{2}\left(1-\frac{2}{n}\right)+1}\end{aligned}$$

provided

$$n'(\alpha+1) \in \left[2, \frac{2N}{N-2}\right], \quad \frac{N}{2}\left(1-\frac{2}{n}\right) - 1 > 0.$$

It follows that

$$\|F_2\|_{L^k([T,+\infty),L^n)} \lesssim \left(\int_T^{+\infty} (t-T+\varepsilon^{-\sigma})^{-\left[\frac{N}{2}\left(1-\frac{2}{n}\right)-1\right]k} dt\right)^{\frac{1}{k}} \lesssim \varepsilon^{\sigma\left[\frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}\right]}$$

provided

$$\frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}>0.$$

We thus obtain

$$\|F_2\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{\sigma\left[\frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}\right](1-\theta)}. \tag{3.20}$$

The above estimate holds true provided

$$n>r, \quad n'(\alpha+1)\in\left[2,\frac{2N}{N-2}\right], \quad \frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}>0.$$

We will choose a suitable n satisfying the above conditions. By the choice of r and k, the above conditions become

$$0\le\frac{1}{n}<\frac{1}{\alpha+2}, \quad \frac{1}{n}\in\left[\frac{1-\alpha}{2},\frac{N+2-(N-2)\alpha}{2N}\right], \quad \frac{1}{n}<\frac{(N-2)(\alpha^2+3\alpha)-4}{2N\alpha(\alpha+2)}. \tag{3.21}$$

In the case $\alpha>1$, we take $\frac{1}{n}=0$ or $n=\infty$.

In the case $\alpha\le 1$, which together with $\frac{4}{N}<\alpha<\frac{4}{N-2}$ imply $N\ge 5$, we take $\frac{1}{n}=\frac{1-\alpha}{2}$ or $n=\frac{2}{1-\alpha}$. It is not hard to check that the conditions in (3.21) are satisfied with this choice of n.

Step 4. Conclusion. By (3.16), we get from (3.18), (3.19) and (3.20) that for $\sigma>0$ sufficiently small, there exists $\mu=\mu(\sigma)>0$ such that

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{\mu}.$$

Case 2. $N=2$.

Recall that we are considering $\alpha>2$ for $N=2$. In this case, the last condition in (3.21) does not work. To overcome this difficulty, we use the space time estimate (3.10) as follows. By the dispersive estimate and Hölder's inequality, we see that for $t\ge T$,

$$\begin{aligned}
\|F_2(t)\|_{L^\infty} &\lesssim \int_J (t-s)^{-1}\|u(s)\|_{L^{\alpha+1}}^{\alpha+1} ds \\
&\lesssim \int_J (t-s)^{-1}\|u(s)\|_{L^{\alpha+2}}^{\frac{(\alpha-1)(\alpha+2)}{\alpha}}\|u(s)\|_{L^2}^{\frac{2}{\alpha}} ds \\
&\lesssim \int_J (t-s)^{-1}\|u(s)\|_{L^{\alpha+2}}^{\frac{(\alpha-1)(\alpha+2)}{\alpha}} ds \\
&\lesssim \|(t-s)^{-1}\|_{L_s^{\alpha}(J)}\left\|\|u(s)\|_{L^{\alpha+2}}^{\frac{(\alpha-1)(\alpha+2)}{\alpha}}\right\|_{L_s^{\frac{\alpha}{\alpha-1}}(J)} \\
&\lesssim \|(t-s)^{-1}\|_{L_s^{\alpha}(J)}\left(\|u\|_{L^{\alpha+2}(J\times\mathbb{R}^2)}^{\alpha+2}\right)^{\frac{\alpha-1}{\alpha}}.
\end{aligned}$$

We see that for $t \geq T$,

$$\begin{aligned}\|(t-s)^{-1}\|_{L^{\alpha}_{s}(J)} &= \left(\int_0^{T-\varepsilon^{-\sigma}} (t-s)^{-\alpha} ds\right)^{\frac{1}{\alpha}} \\ &= \left(\frac{(t-s)^{-\alpha+1}}{\alpha-1}\Big|_{s=0}^{s=T-\varepsilon^{-\sigma}}\right)^{\frac{1}{\alpha}} \\ &\approx \left((t-T+\varepsilon^{-\sigma})^{-\alpha+1} - t^{-\alpha+1}\right)^{\frac{1}{\alpha}} \\ &\lesssim (t-T+\varepsilon^{-\sigma})^{-\frac{\alpha-1}{\alpha}},\end{aligned}$$

where we have used $t \geq t-T+\varepsilon^{-\sigma}$ since $T > \varepsilon^{-\sigma}$. On the other hand, by (3.10),

$$\|u\|^{\alpha+2}_{L^{\alpha+2}(J\times\mathbb{R}^2)} \lesssim |J|^{\beta} \lesssim T^{\beta},$$

where $\beta = \max\left\{\frac{1}{3}, \frac{2}{\alpha+2}\right\}$. It yields that for $t \geq T$,

$$\|F_2(t)\|_{L^\infty} \lesssim (t-T+\varepsilon^{-\sigma})^{-\frac{\alpha-1}{\alpha}} T^{\frac{(\alpha-1)\beta}{\alpha}}.$$

It follows that

$$\begin{aligned}\|F_2\|_{L^k([T,+\infty),L^\infty)} &\lesssim T^{\frac{(\alpha-1)\beta}{\alpha}} \left(\int_T^{+\infty} (t-T+\varepsilon^{-\sigma})^{-\frac{(\alpha-1)k}{\alpha}} dt\right)^{\frac{1}{k}} \\ &\lesssim T^{\frac{(\alpha-1)\beta}{\alpha}} \left((t-T+\varepsilon^{-\sigma})^{-\frac{(\alpha-1)k}{\alpha}+1}\Big|_{t=T}^{t=+\infty}\right)^{\frac{1}{k}} \\ &\lesssim T^{\frac{(\alpha-1)\beta}{\alpha}} \varepsilon^{\sigma\left(\frac{\alpha-1}{\alpha}-\frac{1}{k}\right)}.\end{aligned}$$

We thus get

$$\|F_2\|_{L^k([T,+\infty),L^r)} \lesssim \left[T^{\frac{(\alpha-1)\beta}{\alpha}} \varepsilon^{\sigma\left(\frac{\alpha-1}{\alpha}-\frac{1}{k}\right)}\right]^{1-\frac{l}{r}} = \left(T^{\frac{(\alpha-1)\beta}{\alpha}} \varepsilon^{\frac{(\alpha^2+\alpha-4)\sigma}{\alpha(\alpha+2)}}\right)^{\frac{\alpha^2-4}{\alpha^2+2\alpha-4}}. \tag{3.22}$$

Collecting (3.16), (3.18), (3.19) and (3.22) that

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon + \varepsilon^{\frac{2(\alpha+1)(\alpha-\sigma)}{\alpha(\alpha+2)}} + \left(T^{\frac{(\alpha-1)\beta}{\alpha}} \varepsilon^{\frac{(\alpha^2+\alpha-4)\sigma}{\alpha(\alpha+2)}}\right)^{\frac{\alpha^2-4}{\alpha^2+2\alpha-4}}.$$

By taking $T = \varepsilon^{-a\sigma}$ with some $a > 1$ to be chosen shortly (it ensures $T > \varepsilon^{-\sigma}$) and choosing $\sigma > 0$ small enough, we obtain

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{\mu} \tag{3.23}$$

for some $\mu > 0$. The above estimate requires

$$\frac{\alpha^2+\alpha-4}{\alpha(\alpha+2)} - \frac{a(\alpha-1)\beta}{\alpha} > 0 \quad \text{or} \quad a < \frac{\alpha^2+\alpha-4}{\beta(\alpha+2)(\alpha-1)}.$$

It remains to show that

$$\frac{\alpha^2+\alpha-4}{\beta(\alpha+2)(\alpha-1)} > 1. \tag{3.24}$$

In the case $\beta = \frac{1}{3}$ or $\alpha \geq 4$, we see that (3.24) is equivalent to

$$\frac{2\alpha^2+2\alpha-10}{(\alpha+2)(\alpha-1)} > 0$$

which is satisfied for $\alpha \geq 4$. In the case $\beta = \frac{2}{\alpha+2}$ or $2 < \alpha \leq 4$, (3.24) is equivalent to

$$\frac{\alpha^2 - \alpha - 2}{2(\alpha - 1)} > 0$$

which is also satisfied for $2 < \alpha \leq 4$. Therefore, (3.24) is satisfied for all $\alpha > 2$, and we can choose $a > 1$ so that (3.23) holds. The proof is complete. □

The proof of energy scattering for (NLS) with radial data follows immediately from (3.3), Lemma 3.4 and Proposition 2.

4. **Choquard equations.**

4.1. **Small data theory.**

Let us start with the following Hardy-Littlewood-Sobolev inequality is useful for our purpose.

Lemma 4.1 (Hardy-Littlewood-Sobolev inequality [33]). *Let $N \geq 1, 0 < \gamma < N$ and $1 < q < r < \infty$ be such that*

$$\frac{1}{q} = \frac{1}{r} - \frac{\gamma}{N}.$$

Then there exists $C = C(N, \gamma, q, r) > 0$ such that

$$\|I_\gamma * f\|_{L^q} \leq C\|f\|_{L^r}$$

for all $f \in L^r$.

We also have the following nonlinear estimates.

Lemma 4.2 (Nonlinear estimates). *Let $N \geq 1, 0 < \gamma < N$, $p \geq 2$, $p_* < p < p^*$ and $I \subset \mathbb{R}$. Denote*

$$\begin{aligned} q &:= \frac{4p}{N(p-1) - \gamma}, & r &:= \frac{2Np}{N + \gamma}, \\ k &:= \frac{4p(p-1)}{\gamma + 2 - (N-2)(p-1)}, & m &:= \frac{4p(p-1)}{2N(p-1)^2 + (N - 2\gamma - 2)(p-1) - \gamma - 2}. \end{aligned} \tag{4.1}$$

Then the following estimates hold:

$$\|(I_\gamma * |u|^p)|u|^{p-2}u\|_{L^{m'}(I,L^{r'})} \lesssim \|u\|^{2p-1}_{L^k(I,L^r)}, \tag{4.2}$$

$$\|\langle\nabla\rangle\left[(I_\gamma * |u|^p)|u|^{p-2}u\right]\|_{L^{q'}(I,L^{r'})} \lesssim \|u\|^{2(p-1)}_{L^k(I,L^r)}\|\langle\nabla\rangle u\|_{L^q(I,L^r)}. \tag{4.3}$$

Proof. By Hölder's inequality and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \|(I_\gamma * |u|^p)|u|^{p-2}u\|_{L^{m'}(I,L^{r'})} &\leq \|I_\gamma * |u|^p\|_{L^a(I,L^b)}\||u|^{p-2}u\|_{L^c(I,L^d)} \\ &\lesssim \||u|^p\|_{L^a(I,L^e)}\||u|^{p-2}u\|_{L^c(I,L^d)} \\ &\lesssim \|u\|^{2p-1}_{L^k(I,L^r)}, \end{aligned}$$

where

$$\begin{aligned} a &= \frac{4(p-1)}{\gamma + 2 - (N-2)(p-1)}, & b &= \frac{2N}{N - \gamma}, \\ c &= \frac{4p}{\gamma + 2 - (N-2)(p-1)}, & d &= \frac{2Np}{(N+\gamma)(p-1)}, & e &= \frac{2N}{N+\gamma}. \end{aligned}$$

To see (4.3), it suffices to show

$$\|\nabla[(I_\gamma * |u|^p)|u|^{p-2}u]\|_{L^{q'}(I,L^{r'})} \lesssim \|u\|^{2(p-1)}_{L^k(I,L^r)}\|\nabla u\|_{L^q(I,L^r)}. \tag{4.4}$$

We estimate

$$\begin{aligned}\|\nabla[(I_\gamma * |u|^p)|u|^{p-2}u]\|_{L^{q'}(I,L^{r'})} &\le \|\nabla(I_\gamma * |u|^p)|u|^{p-2}u\|_{L^{q'}(I,L^{r'})} \\ &\quad + \|(I_\gamma * |u|^p)\nabla(|u|^{p-2}u)\|_{L^{q'}(I,L^{r'})}.\end{aligned}$$

By Hölder's inequality and the Hardy-Littlewood-Sobolev inequality, we see that

$$\begin{aligned}\|\nabla(I_\gamma * |u|^p)|u|^{p-2}u\|_{L^{q'}(I,L^{r'})} &\le \|\nabla(I_\gamma * |u|^p)\|_{L^{a_1}(I,L^{b_1})}\||u|^{p-2}u\|_{L^{c_1}(I,L^{d_1})} \\ &\lesssim \|\nabla(|u|^p)\|_{L^{a_1}(I,L^{e_1})}\||u|^{p-2}u\|_{L^{c_1}(I,L^{d_1})} \\ &\lesssim \|u\|^{2(p-1)}_{L^k(I,L^r)}\|\nabla u\|_{L^q(I,L^r)},\end{aligned} \tag{4.5}$$

where

$$a_1 = 2, \qquad b_1 = \frac{2N}{N-\gamma},$$
$$c_1 = \frac{4p}{\gamma+2-(N-2)(p-1)}, \qquad d_1 = \frac{2Np}{(N+\gamma)(p-1)}, \qquad e_1 = \frac{2N}{N+\gamma}.$$

Similarly, we have

$$\begin{aligned}\|(I_\gamma * |u|^p)\nabla(|u|^{p-2}u)\|_{L^{q'}(I,L^{r'})} &\le \|I_\gamma * |u|^p\|_{L^{a_2}(I,L^{b_2})}\|\nabla(|u|^{p-2}u)\|_{L^{c_2}(I,L^{d_2})} \\ &\lesssim \||u|^p\|_{L^{a_2}(I,L^{e_2})}\|\nabla(|u|^{p-2}u)\|_{L^{c_2}(I,L^{d_2})} \\ &\lesssim \|u\|^{2(p-1)}_{L^k(I,L^r)}\|\nabla u\|_{L^q(I,L^r)},\end{aligned} \tag{4.6}$$

where

$$a_2 = \frac{4(p-1)}{\gamma+2-(N-2)(p-1)}, \qquad b_2 = \frac{2N}{N-\gamma},$$
$$c_2 = \frac{4p(p-1)}{N(p-1)+2p(p-2)-\gamma}, \qquad d_2 = \frac{2Np}{(N+\gamma)(p-1)}, \qquad e_2 = \frac{2N}{N+\gamma}.$$

Collecting (4.5) and (4.6), we prove (4.4). The proof is complete. □

Remark 2. Let q, r, k and m be as in (4.1). It is easy to see that (q, r) is a Schrödinger admissible pair. Moreover, k, m and q satisfy (2.2), hence (2.3) holds for such choice of exponents.

Lemma 4.3 (Small data global well-posedness)**.** *Let $N \ge 1$, $0 < \gamma < N$, $p \ge 2$ and $p_* < p < p^*$. Let $T > 0$ be such that $u(T) \in H^1$. Then there exists $\delta > 0$ such that if*

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} < \delta,$$

then there exists a unique global solution to (NLC) *with initial data $u(T)$ satisfying*

$$\|u\|_{L^k([T,+\infty),L^r)} \le 2\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)}$$

and

$$\|\langle\nabla\rangle u\|_{L^q([T,+\infty),L^r)} \le 2C\|u(T)\|_{H^1},$$

where k, r and q are as in (4.1).

Proof. Let q, r, k and m be as in (4.1). Consider

$$Y := \left\{ u \ : \ \|u\|_{L^k(I,L^r)} \le M, \quad \|\langle\nabla\rangle u\|_{L^q(I,L^r)} \le L \right\}$$

equipped with the distance

$$d(u,v) := \|u-v\|_{L^k(I,L^r)} + \|u-v\|_{L^q(I,L^r)},$$

where $I = [T, +\infty)$ and $M, L > 0$ will be chosen later. We will show that the functional

$$\Phi(u(t)) := e^{i(t-T)\Delta}u(T) + i\int_T^t e^{i(t-s)\Delta}(I_\gamma * |u(s)|^p)|u(s)|^{p-2}u(s)ds$$

is a contraction on (Y,d). By Remark 2, (2.3) and Lemma 4.2,

$$\begin{aligned}\|\Phi(u)\|_{L^k(I,L^r)} &\le \|e^{i(t-T)\Delta}u(T)\|_{L^k(I,L^r)} + \|(I_\gamma * |u|^p)|u|^{p-2}u\|_{L^{m'}(I,L^{r'})} \\ &\le \|e^{i(t-T)\Delta}u(T)\|_{L^k(I,L^r)} + \|u\|^{2p-1}_{L^k(I,L^r)}.\end{aligned}$$

By Strichartz estimates and Lemma 4.2,

$$\begin{aligned}\|\langle\nabla\rangle \Phi(u)\|_{L^q(I,L^r)} &\lesssim \|\langle\nabla\rangle e^{i(t-T)\Delta}u(T)\|_{L^q(I,L^r)} + \|\langle\nabla\rangle [(I_\gamma * |u|^p)|u|^{p-2}u]\|_{L^{q'}(I,L^{r'})} \\ &\lesssim \|u(T)\|_{H^1} + \|u\|^{2(p-1)}_{L^k(I,L^r)}\|\langle\nabla\rangle u\|_{L^q(I,L^r)}.\end{aligned}$$

We also have

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^k(I,L^r)} &\lesssim \|(I_\gamma * |u|^p)|u|^{p-2}u - (I_\gamma * |v|^p)|v|^{p-2}v\|_{L^{m'}(I,L^{r'})} \\ &\le \|(I_\gamma * |u|^p)|u|^{p-2}u - (I_\gamma * |v|^p)|u|^{p-2}u\|_{L^{m'}(I,L^{r'})} \\ &\quad + \|(I_\gamma * |v|^p)|u|^{p-2}u - (I_\gamma * |v|^p)|v|^{p-2}v\|_{L^{m'}(I,L^{r'})} \\ &\lesssim \left(\|u\|^{2(p-1)}_{L^k(I,L^r)} + \|v\|^{2(p-1)}_{L^k(I,L^r)}\right)\|u-v\|_{L^k(I,L^r)}\end{aligned}$$

and

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^q(I,L^r)} &\lesssim \|(I_\gamma * |u|^p)|u|^{p-2}u - (I_\gamma * |v|^p)|v|^{p-2}v\|_{L^{q'}(I,L^{r'})} \\ &\lesssim \left(\|u\|^{2(p-1)}_{L^k(I,L^r)} + \|v\|^{2(p-1)}_{L^k(I,L^r)}\right)\|u-v\|_{L^q(I,L^r)}.\end{aligned}$$

Thus, there exists $C > 0$ independent of T such that for any $u, v \in Y$,

$$\begin{aligned}\|\Phi(u)\|_{L^k(I,L^r)} &\le \|e^{i(t-T)\Delta}u(T)\|_{L^k(I,L^r)} + CM^{2p-1}, \\ \|\langle\nabla\rangle \Phi(u)\|_{L^q(I,L^r)} &\le C\|u(T)\|_{H^1} + CM^{2(p-1)}L\end{aligned}$$

and

$$d(\Phi(u),\Phi(v)) \le CM^{2(p-1)}d(u,v).$$

By choosing $M = 2\|e^{i(t-T)\Delta}u(T)\|_{L^k(I,L^r)}$, $L = 2C\|u(T)\|_{H^1}$ and taking M sufficiently small so that $CM^{2(p-1)} \le \frac{1}{2}$, we see that Φ is a contraction on (Y,d). The proof is complete. □

Lemma 4.4 (Small data scattering)**.** *Let $N \ge 1$, $0 < \gamma < N$, $p \ge 2$ and $p_* < p < p^*$. Suppose that u is a global solution to* (NLC) *satisfying*

$$\|u\|_{L^\infty(\mathbb{R},H^1)} < \infty.$$

Then there exists $\delta > 0$ such that if

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} < \delta$$

for some $T > 0$, where k and r are as in (3.1), then u scatters in H^1 forward in time.

Proof. Let $\delta > 0$ be as in Lemma 4.3. It follows from Lemma 4.3 that the solution satisfies

$$\begin{aligned} \|u\|_{L^k([T,+\infty),L^r)} &\leq 2\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)}, \\ \|\langle\nabla\rangle u\|_{L^q([T,+\infty),L^r)} &\leq 2C\|u(T)\|_{H^1}. \end{aligned}$$

Now let $0 < \tau < t < +\infty$. By Strichartz estimates, we see that

$$\begin{aligned} \|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} &= \left\| \int_\tau^t e^{-is\Delta}(I_\gamma * |u(s)|^p)|u(s)|^{p-2}u(s)ds \right\|_{H^1} \\ &\lesssim \|\langle\nabla\rangle [(I_\gamma * |u|^p)|u|^{p-2}u]\|_{L^{q'}((\tau,t),L^{r'})} \\ &\lesssim \|u\|^{2(p-1)}_{L^k((\tau,t),L^r)} \|\langle\nabla\rangle u\|_{L^q((\tau,t),L^r)} \to 0 \end{aligned}$$

as $\tau, t \to +\infty$. This shows that $(e^{-it\Delta}u(t))_t$ is a Cauchy sequence in H^1 as $t \to \infty$. Thus the limit

$$u_+ := u_0 + i\int_t^{+\infty} e^{-is\Delta}(I_\gamma * |u(s)|^p)|u(s)|^{p-2}u(s)ds$$

exists in H^1. By the same reasoning as above, we prove as well that

$$\|u(t) - e^{it\Delta}u_+\|_{H^1} \to 0$$

as $t \to +\infty$. The proof is complete. □

4.2. **Variational analysis.** We recall some properties of the ground state Q related to (1.8). The ground state Q optimizes the sharp Gagliardo-Nirenberg inequality

$$\int (I_\gamma * |f|^p)|f|^p dx \leq C_{\rm opt}\|\nabla f\|_{L^2}^{N(p-1)-\gamma}\|f\|_{L^2}^{\gamma+2-(N-2)(p-1)}, \quad \forall f \in H^1\backslash\{0\},$$

that is

$$C_{\rm opt} = \int (I_\gamma * |Q|^p)|Q|^p dx \div \left[\|\nabla Q\|_{L^2}^{N(p-1)-\gamma}\|Q\|_{L^2}^{\gamma+2-(N-2)(p-1)}\right].$$

Recall that Q satisfies the following Pohozaev's identities

$$\|Q\|_{L^2}^2 = \frac{2+\gamma-(N-2)(p-1)}{N(p-1)-\gamma}\|\nabla Q\|_{L^2}^2 = \frac{2+\gamma-(N-2)(p-1)}{2p}\int (I_\gamma * |Q|^p)|Q|^p dx.$$

It follows that

$$E(Q) = \frac{N(p-1)-\gamma-2}{2[N(p-1)-\gamma]}\|\nabla Q\|_{L^2}^2 = \frac{N(p-1)-\gamma-2}{4p}\int (I_\gamma * |Q|^p)|Q|^p dx$$

and

$$C_{\rm opt} = \frac{2p}{N(p-1)-\gamma}\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}\right)^{-[N(p-1)-\gamma-2]}. \tag{4.7}$$

Lemma 4.5 (Coercivity I)**.** *Let $N \geq 1$, $0 < \gamma < N$, $p \geq 2$ and $p_* < p < p^*$. Let $u_0 \in H^1$ satisfy (1.9) and (1.10). It follows that the corresponding solution exists globally in time. Moreover, there exists $\rho = \rho(u_0, Q) > 0$ such that*

$$\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c} < (1-2\rho)\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c} \tag{4.8}$$

for all $t \in \mathbb{R}$.

Proof. By the sharp Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&E(u(t))[M(u(t))]^{\sigma_c} \\
&= \frac{1}{2}\left(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}\right)^2 - \frac{1}{2p}\left(\int (I_\gamma * |u(t)|^p)|u(t)|^p dx\right)\|u(t)\|_{L^2}^{2\sigma_c} \\
&\geq \frac{1}{2}\left(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}\right)^2 - \frac{C_{\text{opt}}}{2p}\|\nabla u(t)\|_{L^2}^{N(p-1)-\gamma}\|u(t)\|_{L^2}^{\gamma+2-(N-2)(p-1)+2\sigma_c} \\
&= f\left(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}\right), \qquad (4.9)
\end{aligned}$$

where

$$f(x) = \frac{1}{2}x^2 - \frac{C_{\text{opt}}}{2p}x^{N(p-1)-\gamma}.$$

Using Pohozaev's identities,

$$f\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}\right) = \frac{N(p-1)-\gamma-2}{2[N(p-1)-\gamma]}\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}\right)^2 = E(Q)[M(Q)]^{\sigma_c}.$$

We have from (1.9), the conservation of mass and energy that

$$f\left(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}\right) \leq E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c} = f\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}\right)$$

for all t in the existence time. By (1.10), the continuity argument implies that

$$\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}$$

for all t in the existence time. The conservation of mass then implies that $\|\nabla u(t)\|_{L^2}$ is uniformly bounded. The blow-up alternative then shows that the solution exists globally in time.

To see (4.8), we use (1.9) to take $\vartheta = \vartheta(u_0, Q) > 0$ such that

$$E(u_0)[M(u_0)]^{\sigma_c} < (1-\vartheta)E(Q)[M(Q)]^{\sigma_c}. \qquad (4.10)$$

Using the fact

$$\begin{aligned}
E(Q)[M(Q)]^{\sigma_c} &= \frac{N(p-1)-\gamma-2}{2[N(p-1)-\gamma]}\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}\right)^2 \\
&= \frac{N(p-1)-\gamma-2}{4p}C_{\text{opt}}\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}\right)^{N(p-1)-\gamma},
\end{aligned}$$

we infer from (4.9) and (4.10) that

$$\begin{aligned}
&\frac{N(p-1)-\gamma}{N(p-1)-\gamma-2}\left(\frac{\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}}{\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}}\right)^2 \\
&\qquad - \frac{2}{N(p-1)-\gamma-2}\left(\frac{\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}}{\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}}\right)^{N(p-1)-\gamma} \leq 1-\vartheta. \qquad (4.11)
\end{aligned}$$

Consider the function

$$g(y) = \frac{N(p-1)-\gamma}{N(p-1)-\gamma-2}y^2 - \frac{2}{N(p-1)-\gamma-2}y^{N(p-1)-\gamma}, \quad 0 < y < 1.$$

It is easy to see that g is strictly increasing in $(0,1)$ with $g(0) = 0$ and $g(1) = 1$. It follows from (4.11) that there exists $\rho > 0$ depending on ϑ such that $y < 1 - 2\rho$ which shows (4.8). The proof is complete. □

Remark 3. If $u_0 \in H^1$ satisfies (1.9) and (1.10), then $E(u_0) > 0$. Indeed, by (4.9),

$$E(u_0)[M(u_0)]_{L^2}^{\sigma_c} \geq f\left(\|\nabla u_0\|_{L^2}\|u_0\|_{L^2}^{\sigma_c}\right).$$

It is easy to see that f is strictly increasing on $(0, \zeta_0)$ and strictly decreasing on (ζ_0, ∞), where

$$\zeta_0 := \left(\frac{2p}{[N(p-1)-\gamma]C_{\rm opt}}\right)^{\frac{1}{N(p-1)-\gamma-2}} = \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}.$$

Here the last equality follows from (4.7). It follows that $f(\zeta) > f(0) = 0$ or $E(u_0) > 0$.

Lemma 4.6 (Coercivity II)**.** *Let $N \geq 1$, $0 < \gamma < N$, $p \geq 2$ and $p_* < p < p^*$. Let ρ be as in (4.8). There exists $R_0 = R_0(\rho, \|u_0\|_{L^2}) > 0$ such that for any $R \geq R_0$,*

$$\|\nabla(\chi_R u(t))\|_{L^2}\|\chi_R u(t)\|_{L^2}^{\sigma_c} < (1-\rho)\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c} \tag{4.12}$$

for all $t \in \mathbb{R}$, where $\chi_R(x) = \chi(x/R)$ with $\chi \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \leq \chi \leq 1$ and (3.6). In particular, there exists $\nu = \nu(\rho) > 0$ such that for any $R \geq R_0$,

$$\|\nabla(\chi_R u(t))\|_{L^2}^2 - \frac{N(p-1)-\gamma}{2p}\int (I_\gamma * |\chi_R u(t)|^p)|\chi_R u(t)|^p dx \geq \nu\|\chi_R u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}} \tag{4.13}$$

for all $t \in \mathbb{R}$.

Proof. By the definition of χ_R, we have that $\|\chi_R u(t)\|_{L^2} \leq \|u(t)\|_{L^2}$. On the other hand, using the fact

$$\int |\nabla(\chi f)|^2 dx = \int \chi^2 |\nabla f|^2 dx - \int \chi \Delta\chi |f|^2 dx, \tag{4.14}$$

we have

$$\|\nabla(\chi_R u(t))\|_{L^2}^2 \leq \|\nabla u(t)\|_{L^2}^2 + O\left(R^{-2}\|u(t)\|_{L^2}^2\right).$$

Thus

$$\begin{aligned}
\|\nabla(\chi_R u(t))\|_{L^2}\|\chi_R u(t)\|_{L^2}^{\sigma_c} &\leq \left(\|\nabla u(t)\|_{L^2}^2 + O\left(R^{-2}\|u(t)\|_{L^2}^2\right)\right)^{\frac{1}{2}}\|u(t)\|_{L^2}^{\sigma_c} \\
&\leq \|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c} + O\left(R^{-1}\|u(t)\|_{L^2}^{\sigma_c+1}\right) \\
&< (1-2\rho)\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c} + O\left(R^{-1}\|u_0\|_{L^2}^{\sigma_c+1}\right) \\
&< (1-\rho)\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}
\end{aligned}$$

provided $R > 0$ is taken sufficiently large depending on $\rho, \|u_0\|_{L^2}$.

To show (4.13), we use the following fact: if

$$\|\nabla f\|_{L^2}\|f\|_{L^2}^{\sigma_c} < (1-\rho)\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}, \tag{4.15}$$

then there exists $\delta = \delta(\rho) > 0$ such that

$$\|\nabla f\|_{L^2}^2 - \frac{N(p-1)-\gamma}{2p}\int (I_\gamma * |f|^p)|f|^p dx \geq \delta\|\nabla f\|_{L^2}^2. \tag{4.16}$$

To see (4.16), we first have from the Gagliardo-Nirenberg inequality and (4.15) that

$$\begin{aligned}
E(f) &\geq \frac{1}{2}\|\nabla f\|_{L^2}^2 - \frac{C_{\text{opt}}}{2p}\|\nabla f\|_{L^2}^{N(p-1)-\gamma}\|f\|_{L^2}^{\gamma+2-(N-2)(p-1)} \\
&= \frac{1}{2}\|\nabla f\|_{L^2}^2\left(1 - \frac{C_{\text{opt}}}{p}\|\nabla f\|_{L^2}^{N(p-1)-\gamma-2}\|f\|_{L^2}^{\gamma+2-(N-2)(p-1)}\right) \\
&= \frac{1}{2}\|\nabla f\|_{L^2}^2\left(1 - \frac{C_{\text{opt}}}{p}\left(\|\nabla f\|_{L^2}\|f\|_{L^2}^{\sigma_c}\right)^{N(p-1)-\gamma-2}\right) \\
&> \frac{1}{2}\|\nabla f\|_{L^2}^2\left(1 - \frac{C_{\text{opt}}}{p}(1-\rho)^{N(p-1)-\gamma}\left(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}\right)^{N(p-1)-\gamma-2}\right) \\
&= \frac{1}{2}\|\nabla f\|_{L^2}^2\left(1 - \frac{2}{N(p-1)-\gamma}(1-\rho)^{N(p-1)-\gamma}\right).
\end{aligned}$$

We now set $K(f)$ the left hand side of (4.16). We have that

$$\begin{aligned}
K(f) &= [N(p-1)-\gamma]E(f) - \frac{N(p-1)-\gamma-2}{2}\|\nabla f\|_{L^2}^2 \\
&\geq \frac{N(p-1)-\gamma}{2}\|\nabla f\|_{L^2}^2\left(1 - \frac{2}{N(p-1)-\gamma}(1-\rho)^{N(p-1)-\gamma}\right) \\
&\qquad - \frac{N(p-1)-\gamma-2}{2}\|\nabla f\|_{L^2}^2 \\
&= \left(1-(1-\rho)^{N(p-1)-\gamma}\right)\|\nabla f\|_{L^2}^2
\end{aligned}$$

which proves (4.16).

Next, by enlarging R_0 if necessary, we have for any $R \geq R_0$,

$$\inf_{t\in\mathbb{R}}\|\nabla(\chi_R u(t))\|_{L^2} \geq C > 0. \tag{4.17}$$

In fact, by Remark 3, we have $E(u_0) > 0$, hence

$$\inf_{t\in\mathbb{R}}\|\nabla u(t)\|_{L^2} \geq 2E(u_0) > 0. \tag{4.18}$$

Assume that (4.17) is not true, then there exist $(t_n)_n \subset \mathbb{R}$ and $R_n \to \infty$ such that

$$\lim_{n\to\infty}\|\nabla(\chi_{R_n} u(t_n))\|_{L^2} = 0.$$

Using the identity

$$\|\nabla(\chi_{R_n}u(t_n))\|_{L^2}^2 = \int \chi_{R_n}^2|\nabla u(t_n)|^2dx - \int \chi_{R_n}\Delta(\chi_{R_n})|u(t_n)|^2dx$$

and the fact $|\Delta(\chi_{R_n})| \lesssim R_n^{-2}$, we obtain

$$\int \chi_{R_n}^2|\nabla u(t_n)|^2dx = \|\nabla(\chi_{R_n}u(t_n))\|_{L^2}^2 + O(R_n^{-2}) \to 0$$

as $n \to \infty$ which contradicts (4.18). Combining (4.16), (4.17), the Sobolev embedding

$$H^1 \hookrightarrow L^{\frac{2Np}{N+\gamma}} \tag{4.19}$$

and the conservation of mass, we prove (4.13). The proof is complete. □

Remark 4. The estimate (4.13) was proved in [2, Lemma 3.4] by a similar argument using the Sobolev embedding $\|u\|_{L^{\frac{2Np}{N+\gamma}}} \lesssim \|\nabla u\|_{L^2}$. However, this type of Sobolev embedding does not hold in general without (4.17).

We also have the following Morawetz identity.

Lemma 4.7 (Morawetz identity)**.** *Let $N \geq 1$, $0 < \gamma < N$ and $2 \leq p < p^*$. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a sufficiently smooth and decaying function. Let u be a H^1 solution to* (NLC)*. Define*

$$M_\varphi(t) := 2 \int \nabla\varphi \cdot \mathrm{Im}(\overline{u}(t)\nabla u(t))dx.$$

Then

$$\begin{aligned}\frac{d}{dt}M_\varphi(t) = &-\int \Delta^2\varphi|u(t)|^2dx + 4\sum_{j,k=1}^N \int \partial_{jk}^2\varphi\,\mathrm{Re}(\partial_j\overline{u}(t)\partial_k u(t))dx\\ &-\frac{2(p-2)}{p}\int \Delta\varphi(I_\gamma * |u(t)|^p)|u(t)|^p dx\\ &-\frac{2(N-\gamma)}{p}A(\gamma)\iint (\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)\frac{|u(t,x)|^p|u(t,y)|^p}{|x-y|^{N-\gamma+2}}dxdy,\end{aligned} \tag{4.20}$$

where $A(\gamma)$ be as in (1.6)*.*

Proof. It follows from [37, Lemma 5.3] that if u is a solution to $i\partial_t u + \Delta u = N(u)$ with $N(u)$ satisfying $\mathrm{Im}(N(u)\overline{u}) = 0$, then

$$\begin{aligned}\frac{d}{dt}M_\varphi(t) = -\int \Delta^2\varphi|u(t)|^2dx + 4\sum_{j,k=1}^N \int \partial_{jk}^2\varphi\partial_j\,\mathrm{Re}(\overline{u}(t)\partial_k u(t))dx\\ + 2\int \nabla\varphi\cdot\{N(u),u\}_m(t)dx,\end{aligned}$$

where $\{f,g\}_m := \mathrm{Re}(f\nabla\overline{g} - g\nabla\overline{f})$ is the momentum bracket. Applying this result to $N(u) = -(I_\gamma * |u|^p)|u|^{p-2}u$. Observe that

$$\begin{aligned}\nabla[(I_\gamma * |u|^p)|u|^p] &= \nabla(I_\gamma * |u|^p)|u|^p + (I_\gamma * |u|^p)\nabla(|u|^p)\\ &= \nabla(I_\gamma * |u|^p)|u|^p + p(I_\gamma * |u|^p)|u|^{p-2}\,\mathrm{Re}(u\nabla\overline{u}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\nabla[(I_\gamma * |u|^p)|u|^p] &= \nabla[(I_\gamma * |u|^p)|u|^{p-2}\overline{u}u]\\ &= \nabla[(I_\gamma * |u|^p)|u|^{p-2}\overline{u}]u + (I_\gamma * |u|^p)|u|^{p-2}\overline{u}\nabla u.\end{aligned}$$

Therefore,

$$\begin{aligned}\{N(u),u\}_m &= \mathrm{Re}\left[\nabla[(I_\gamma * |u|^p)|u|^{p-2}\overline{u})u\right] - \mathrm{Re}\left[(I_\gamma * |u|^p)|u|^{p-2}u\nabla\overline{u}\right]\\ &= \nabla[(I_\gamma * |u|^p)|u|^p] - 2\,\mathrm{Re}[(I_\gamma * |u|^p)|u|^{p-2}u\nabla\overline{u}].\end{aligned}$$

Note that $\nabla(|u|^p) = p|u|^{p-2}\,\mathrm{Re}(u\nabla\overline{u})$, hence

$$\{N(u),u\}_m = \nabla[(I_\gamma * |u|^p)|u|^p] - \frac{2}{p}(I_\gamma * |u|^p)\nabla(|u|^p).$$

We thus get

$$\begin{aligned}
&\int \nabla\varphi\cdot\{N(u),u\}_m dx\\
&\quad= \int \nabla\varphi\cdot\nabla[(I_\gamma * |u|^p)|u|^p]dx - \frac{2}{p}\int \nabla\varphi\cdot\nabla(|u|^p)(I_\gamma * |u|^p)dx\\
&\quad= -\int \Delta\varphi(I_\gamma * |u|^p)|u|^p dx + \frac{2}{p}\int \Delta\varphi(I_\varphi * |u|^p)|u|^p dx + \frac{2}{p}\int \nabla\varphi\cdot\nabla(I_\gamma * |u|^p)|u|^p dx\\
&\quad= -\frac{p-2}{p}\int \Delta\varphi(I_\gamma * |u|^p)|u|^p dx + \frac{2}{p}\int \nabla\varphi\cdot\nabla(I_\gamma * |u|^p)|u|^p dx.
\end{aligned}$$

Note also that

$$\begin{aligned}
\nabla(I_\gamma * |u|^p) &= \nabla_x \int A(\gamma)|x-y|^{-(N-\gamma)}|u(y)|^p dy\\
&= -(N-\gamma)\int A(\gamma)(x-y)|x-y|^{-(N-\gamma)-2}|u(y)|^p dy.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int \nabla\varphi\cdot\nabla(I_\gamma * |u|^p)|u|^p dx\\
&\qquad= -(N-\gamma)A(\gamma)\iint \nabla\varphi(x)\cdot(x-y)\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\gamma+2}}dxdy\\
&\qquad= -\frac{N-\gamma}{2}A(\gamma)\iint (\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\gamma+2}}dxdy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int \nabla\varphi\cdot\{N(u),u\}_m dx\\
&\qquad= -\frac{p-2}{p}\int \Delta\varphi(I_\gamma * |u|^p)|u|^p dx\\
&\qquad\quad -\frac{N-\gamma}{p}A(\gamma)\iint (\nabla\varphi(x)-\nabla\varphi(y))\cdot(x-y)\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\gamma+2}}dxdy.
\end{aligned}$$

Combining the above calculations, we prove (4.20). The proof is complete. □

Lemma 4.8 (Morawetz estimate)**.** *Let $N \geq 2$, $0 < \gamma < N$, $p \geq 2$ and $p_* < p < p^*$. Let $u_0 \in H^1$ be radially symmetric satisfying (1.9) and (1.10). Then the corresponding solution global solution to* (NLC) *satisfies for any time interval $I \subset \mathbb{R}$,*

$$\int_I \|u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}} dt \leq C(u_0,Q)|I|^\beta, \quad \beta := \max\left\{\frac{1}{3}, \frac{Np}{Np+(N-1)[N(p-1)-\gamma]}\right\} \tag{4.21}$$

for some constant $C(u_0,Q) > 0$ depending only on u_0 and Q.

Proof. Let $\rho = \rho(u_0,Q)$ be as in (4.8) and $R = R(\rho, \|u_0\|_{L^2})$ be as in Lemma 4.6. We define φ_R as in (3.8)–(3.9). By the Cauchy-Schwarz inequality and (4.8), we see that

$$|M_{\varphi_R}(t)| \lesssim \|\nabla\varphi_R\|_{L^\infty}\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \lesssim R \tag{4.22}$$

for all $t \in \mathbb{R}$. Here the implicit constant depends only on u_0 and Q. By Lemma 4.7,

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(t) =& -\int \Delta^2\varphi_R|u(t)|^2dx + 4\sum_{j,k=1}^{N}\int \partial_{jk}^2\varphi_R \,\mathrm{Re}(\partial_j\overline{u}(t)\partial_k u(t))dx\\ &- \frac{2(p-2)}{p}\int \Delta\varphi_R(I_\gamma * |u(t)|^p)|u(t)|^p dx\\ &- \frac{2(N-\gamma)}{p}A(\gamma)\iint (\nabla\varphi_R(x)-\nabla\varphi_R(y))\cdot(x-y)\frac{|u(t,x)|^p|u(t,y)|^p}{|x-y|^{N-\gamma+2}}dxdy.\end{aligned}$$

Using the fact $\varphi_R(x) = |x|^2$ for $|x| \le R$, we see that

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(t) =& 8\left[\int_{|x|\le R}|\nabla u(t)|^2dx - \frac{N(p-1)-\gamma}{2p}\int_{|x|\le R}(I_\gamma * |u(t)|^p)|u(t)|^p dx\right]\\ &- \int \Delta^2\varphi_R|u(t)|^2dx + 4\sum_{j,k=1}^{N}\int_{|x|>R} \partial_{jk}^2\varphi_R \,\mathrm{Re}(\partial_j\overline{u}(t)\partial_k u(t))dx\\ &- \frac{2(p-2)}{p}\int_{|x|>R} \Delta\varphi_R(I_\gamma * |u(t)|^p)|u(t)|^p dx\\ &- \frac{2(N-\gamma)}{p}A(\gamma)\iint_{\Omega} (\nabla\varphi_R(x)-\nabla\varphi_R(y))\cdot(x-y)\frac{|u(t,x)|^p|u(t,y)|^p}{|x-y|^{N-\gamma+2}}dxdy,\end{aligned} \tag{4.23}$$

where

$$\Omega := \left\{(x,y)\in\mathbb{R}^{2N} \ : \ R\le|x|\le 2R \quad \text{or} \quad R\le|y|\le 2R\right\}.$$

Since $\|\Delta^2\varphi_R\|_{L^\infty} \lesssim R^{-2}$, the conservation of mass implies

$$\left|\int \Delta^2\varphi_R|u(t)|^2dx\right| \lesssim R^{-2}.$$

Since u is radial, we use the fact

$$\partial_j = \frac{x_j}{r}\partial_r, \quad \partial_{jk}^2 = \left(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\right)\partial_r + \frac{x_jx_k}{r^2}\partial_r^2$$

to get

$$\sum_{j,k=1}^{N}\partial_{jk}^2\varphi_R\partial_j \,\mathrm{Re}(\overline{u}(t)\partial_k u(t)) = \varphi_R''|\partial_r u(t)|^2 \ge 0$$

which implies

$$4\sum_{j,k=1}^{N}\int_{|x|>R}\partial_{jk}^2\varphi_R \,\mathrm{Re}(\partial_j\overline{u}(t)\partial_k u(t))dx \ge 0.$$

By Hölder's inequality and the Hardy-Littlewood-Sobolev inequality, we see that

$$\begin{aligned}\left|\int_{|x|>R}(I_\gamma * |u(t)|^p)|u(t)|^p dx\right| &\lesssim \|I_\gamma * |u(t)|^p\|_{L^{\frac{2N}{N-\gamma}}(|x|>R)}\||u(t)|^p\|_{L^{\frac{2N}{N+\gamma}}(|x|>R)}\\ &\lesssim \|u(t)\|^p_{L^{\frac{2Np}{N+\gamma}}}\|u(t)\|^p_{L^{\frac{2Np}{N+\gamma}}(|x|>R)}.\end{aligned} \tag{4.24}$$

For the first term in (4.24), we use the Sobolev embedding (4.19), the conservation of mass and (4.8) to get

$$\|u(t)\|_{L^{\frac{2Np}{N+\gamma}}} \lesssim \|u(t)\|_{H^1} \le C(u_0,Q).$$

For the second term in (4.24), we use the radial Sobolev embedding (3.12) to have

$$\begin{aligned}\|u(t)\|^{\frac{2Np}{N+\gamma}}_{L^{\frac{2Np}{N+\gamma}}(|x|>R)} &= \int_{|x|>R} |u(t,x)|^{\frac{2Np}{N+\gamma}}dx \\ &\leq \left(\sup_{|x|>R}|u(t,x)|\right)^{\frac{2[N(p-1)-\gamma]}{N+\gamma}} \|u(t)\|^2_{L^2} \\ &\lesssim R^{-\frac{(N-1)[N(p-1)-\gamma]}{N+\gamma}} \|u(t)\|^{\frac{2[N(p-1)-\gamma]}{N+\gamma}}_{H^1} \|u(t)\|^2_{L^2} \\ &\lesssim R^{-\frac{(N-1)[N(p-1)-\gamma]}{N+\gamma}}.\end{aligned}$$

This shows that

$$\left|\int_{|x|>R} (I_\gamma * |u(t)|^p)|u(t)|^p dx\right| \lesssim R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}. \tag{4.25}$$

Using (4.25) and the fact $\|\Delta\varphi_R\|_{L^\infty} \lesssim 1$, we have

$$\left|\int_{|x|>R} \Delta\varphi_R (I_\gamma * |u(t)|^p)|u(t)|^p dx\right| \lesssim R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}.$$

For the term in (4.23), we use the fact

$$|(\nabla\varphi_R(x) - \nabla\varphi_R(y))\cdot(x-y)| \lesssim |x-y|^2$$

for all $(x-y)\in\Omega$ and (4.24) to get

$$\begin{aligned}\left|A(\gamma)\iint_\Omega (\nabla\varphi_R(x) - \nabla\varphi_R(y))\cdot(x-y)\frac{|u(t,x)|^p|u(t,y)|^p}{|x-y|^{N-\gamma+2}}dxdy\right| &\lesssim \int_{|x|>R}(I_\gamma * |u(t)|^p)|u(t)|^p dx \\ &\lesssim R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}.\end{aligned}$$

We thus obtain

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(t) \geq & 8\left[\int_{|x|\leq R}|\nabla u(t)|^2dx - \frac{N(p-1)-\gamma}{2p}\int_{|x|\leq R}(I_\gamma * |u(t)|^p)|u(t)|^p dx\right] \\ & + O\left(R^{-2} + R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}\right).\end{aligned} \tag{4.26}$$

Now let χ_R be as in Lemma 4.6. We have

$$\begin{aligned}\int|\nabla(\chi_R u(t))|^2dx &= \int\chi_R^2|\nabla u(t)|^2dx - \int\chi_R\Delta(\chi_R)|u(t)|^2dx \\ &= \int_{|x|\leq R}|\nabla u(t)|^2dx - \int_{R/2\leq|x|\leq R}(1-\chi_R^2)|\nabla u(t)|^2dx \\ &\quad - \int\chi_R\Delta(\chi_R)|u(t)|^2dx\end{aligned}$$

and

$$\begin{aligned}
&\int (I_\gamma * |\chi_R u(t)|^p)|\chi_R u(t)|^p dx \\
&= \int_{|x|\le R} (I_\gamma * |\chi_R u(t)|^p)|u(t)|^p dx - \int_{R/2\le|x|\le R} (1-\chi_R^p)(I_\gamma * |\chi_R u(t)|^p)|u(t)|^p dx \\
&= \int_{|x|\le R} (I_\gamma * |u(t)|^p)|u(t)|^p dx - \int_{|x|\le R} \left(I_\gamma * [(1-\chi_R^p)|u(t)|^p]\right)|u(t)|^p dx \\
&\qquad - \int_{R/2\le|x|\le R} (1-\chi_R^p)(I_\gamma * |\chi_R u(t)|^p)|u(t)|^p dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{|x|\le R} |\nabla u(t)|^2 dx - \frac{N(p-1)-\gamma}{2p} \int_{|x|\le R} (I_\gamma * |u(t)|^p)|u(t)|^p dx \\
&\qquad = \int |\nabla(\chi_R u(t))|^2 dx - \frac{N(p-1)-\gamma}{2p} \int (I_\gamma * |\chi_R u(t)|^p)|\chi_R u(t)|^p dx \\
&\qquad\quad + \int_{R/2\le|x|\le R} (1-\chi_R^2)|\nabla u(t)|^2 dx + \int \chi_R \Delta(\chi_R)|u(t)|^2 dx \\
&\qquad\quad - \frac{N(p-1)-\gamma}{2p} \int_{|x|\le R} \left(I_\gamma * [(1-\chi_R^p)|u(t)|^p]\right)|u(t)|^p dx \\
&\qquad\quad - \frac{N(p-1)-\gamma}{2p} \int_{R/2\le|x|\le R} (1-\chi_R^p)(I_\gamma * |\chi_R u(t)|^p)|u(t)|^p dx.
\end{aligned}$$

Thanks to the fact that $0 \le \chi_R \le 1$, $\mathrm{supp}(1-\chi_R^p) \subset \{|x| \ge R/2\}$, $\|\Delta(\chi_R)\|_{L^\infty} \lesssim R^{-2}$ and estimating as in (4.24), we get

$$\begin{aligned}
&\int_{|x|\le R} |\nabla u(t)|^2 dx - \frac{N(p-1)-\gamma}{2p} \int_{|x|\le R} (I_\gamma * |u(t)|^p)|u(t)|^p dx \\
&\qquad \ge \int |\nabla(\chi_R u(t))|^2 dx - \frac{N(p-1)-\gamma}{2p} \int (I_\gamma * |\chi_R u(t)|^p)|\chi_R u(t)|^p dx \\
&\qquad\quad + O\left(R^{-2} + R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}\right).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\frac{d}{dt} M_{\varphi_R}(t) \ge 8\Big[\|\nabla(\chi_R u(t))\|_{L^2}^2 - \frac{N(p-1)-\gamma}{2p} \int (I_\gamma * |\chi_R u(t)|^p)|\chi_R u(t)|^p dx\Big] \\
+ O\left(R^{-2} + R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}\right).
\end{aligned}$$

By Lemma 4.6, there exists $\nu = \nu(\rho) > 0$ such that for any $R \ge R_0$,

$$8\nu \|\chi_R u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}} \le \frac{d}{dt} M_{\varphi_R}(t) + O\left(R^{-2} + R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}\right)$$

which implies for any time interval $I \subset \mathbb{R}$,

$$8\nu \int_I \|\chi_R u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}} dt \le \sup_{t\in I} |M_{\varphi_R}(t)| + O\left(R^{-2} + R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}\right)|I|. \quad (4.27)$$

By the definition of χ_R and (4.22),

$$\int_I \|u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}(|x|\le R/2)} dt \lesssim R + O\left(R^{-2} + R^{-\frac{(N-1)[N(p-1)-\gamma]}{2N}}\right)|I|.$$

On the other hand, by radial Sobolev embeddings,

$$\int_{|x|\geq R/2} |u(t,x)|^{\frac{2Np}{N+\gamma}} dx \leq \left(\sup_{|x|\geq R/2} |u(t,x)| \right)^{\frac{2[N(p-1)-\gamma]}{N+\gamma}} \|u(t)\|_{L^2}^2 \lesssim R^{-\frac{(N-1)[N(p-1)-\gamma]}{N+\gamma}}.$$

We thus get

$$\int_I \|u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}} dt \lesssim R + \left(R^{-2} + R^{-\frac{(N-1)[N(p-1)-\gamma]}{Np}} \right) |I| \lesssim R + R^{-\sigma}|I|,$$

where

$$\sigma := \min\left\{ 2, \frac{(N-1)[N(p-1)-\gamma]}{Np} \right\}.$$

Here we have used the fact $p \geq 2$ to get the first estimate. Taking $R = |I|^{\frac{1}{1+\sigma}}$, we get for $|I|$ sufficiently large,

$$\int_I \|u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}} dt \lesssim |I|^{\frac{1}{1+\sigma}} = |I|^{\beta},$$

where β is as in (4.21). In the case $|I|$ is sufficiently small, it follows from the Sobolev embedding (4.19) that

$$\int_I \|u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}} dt \lesssim \int_I \|u(t)\|^2_{H^1} dt \lesssim |I| \lesssim |I|^{\beta}$$

since $\beta < 1$. This proves (4.21) and the proof is complete. □

Lemma 4.9. *Let $N \geq 2$, $0 < \gamma < N$, $p \geq 2$ and $p_* < p < p^*$. Let $u_0 \in H^1$ be radially symmetric satisfying (1.9) and (1.10). Then the corresponding global solution to* (NLC) *satisfies*

$$\liminf_{t\to+\infty} \|u(t)\|_{L^{\frac{2Np}{N+\gamma}}(\mathbb{R}^N)} = 0. \tag{4.28}$$

Proof. Assume by contradiction that (4.28) is not true. There exist $t_0 > 0$ and $\varrho > 0$ such that

$$\|u(t)\|_{L^{\frac{2Np}{N+\gamma}}(\mathbb{R}^N)} > \varrho, \quad \forall t \geq t_0.$$

This implies in particular that, for every $I \subset [t_0, +\infty)$,

$$\int_I \|u(t)\|^2_{L^{\frac{2Np}{N+\gamma}}(\mathbb{R}^N)} dt \geq \varrho^2 |I|,$$

which contradicts (4.21) for $|I|$ large since $\beta < 1$. □

Corollary 2. *Let $N \geq 2$, $0 < \gamma < N$, $p \geq 2$ and $p_* < p < p^*$. Let $u_0 \in H^1$ be radially symmetric satisfying (1.9) and (1.10). Then there exists $t_n \to +\infty$ such that the corresponding solution to* (NLC) *satisfies for any $R > 0$,*

$$\lim_{n\to\infty} \int_{|x|\leq R} |u(t_n,x)|^2 dx = 0. \tag{4.29}$$

Proof. By (4.28), there exists $t_n \to +\infty$ such that

$$\lim_{n\to\infty} \|u(t_n)\|_{L^{\frac{2Np}{N+\gamma}}(\mathbb{R}^N)} = 0.$$

Let $R>0$. By Hölder's inequality, we see that

$$\begin{aligned}\int_{|x|\le R}|u(t_n,x)|^2dx &\le \left(\int_{|x|\le R}dx\right)^{\frac{N(p-1)-\gamma}{Np}}\left(\int_{|x|\le R}|u(t_n,x)|^{\frac{2Np}{N+\gamma}}dx\right)^{\frac{N+\gamma}{Np}}\\ &\lesssim R^{\frac{N(p-1)-\gamma}{p}}\left(\int|u(t_n,x)|^{\frac{2Np}{N+\gamma}}dx\right)^{\frac{N+\gamma}{Np}}\to 0\end{aligned}$$

as $n\to\infty$. □

4.3. **Scattering below the ground state.**

Proposition 3. *Let $N\ge 2$, $0<\gamma<N$, $p\ge 2$ and $p_*<p<p^*$. Let $u_0\in H^1$ be radially symmetric satisfying (1.9) and (1.10). Then for $\varepsilon>0$ sufficiently small, there exists $T=T(\varepsilon,u_0,Q)$ sufficiently large such that the corresponding solution to* (NLC) *satisfies*

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)}\lesssim\varepsilon^\mu \tag{4.30}$$

for some $\mu>0$, where k and r are as in (4.1).

Proof. We will consider separately two cases: $N\ge 3$ and $N=2$.

Case 1. $N\ge 3$.

Let $T>0$ be a large parameter depending on ε,u_0 and Q to be chosen later. For $T>\varepsilon^{-\sigma}$ with some $\sigma>0$ to be chosen later, we use the Duhamel formula to write

$$\begin{aligned}e^{i(t-T)\Delta}u(T)&=e^{it\Delta}u_0+i\int_0^T e^{i(t-s)\Delta}(I_\gamma*|u(s)|^p)|u(s)|^{p-2}u(s)ds\\&=e^{it\Delta}u_0+F_1(t)+F_2(t),\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}F_1(t)&:=i\int_I e^{i(t-s)\Delta}(I_\gamma*|u(s)|^p)|u(s)|^{p-2}u(s)ds,\\F_2(t)&:=i\int_J e^{i(t-s)\Delta}(I_\gamma*|u(s)|^p)|u(s)|^{p-2}u(s)ds\end{aligned}$$

with $I:=[T-\varepsilon^{-\sigma},T]$ and $J:=[0,T-\varepsilon^{-\sigma}]$.

Step 1. Estimate the linear part. By Strichartz estimates, Sobolev embeddings, (1.9) and (1.10),

$$\|e^{it\Delta}u_0\|_{L^k(\mathbb{R},L^r)}\lesssim\||\nabla|^{\gamma_c}e^{it\Delta}u_0\|_{L^k(\mathbb{R},L^l)}\lesssim\|u_0\|_{\dot H^{\gamma_c}}\lesssim\|u_0\|_{H^1}\le C(u_0,Q)<\infty,$$

where

$$l=\frac{2Np(p-1)}{(Np+N-2)(p-1)-\gamma-2}. \tag{4.32}$$

Note that $(k,l)\in S_0$. By the monotone convergence, we may find $T>\varepsilon^{-\sigma}$ so that

$$\|e^{it\Delta}u_0\|_{L^k([T,+\infty),L^r)}\lesssim\varepsilon. \tag{4.33}$$

Step 2. Estimate F_1. By Remark 2, (2.3), (4.24) and Sobolev embedding, we have

$$\|F_1\|_{L^k([T,+\infty),L^r)}\lesssim\|(I_\gamma*|u|^p)|u|^{p-2}u\|_{L^{m'}(I,L^{r'})}\lesssim\|u\|^{2p-1}_{L^k(I,L^r)}\lesssim|I|^{\frac{2p-1}{k}}\|u\|^{2p-1}_{L^\infty(I,L^r)}.$$

To estimate $\|u\|_{L^\infty(I,L^r)}$, we take $R = \max\left\{\varepsilon^{-2-\sigma}, \varepsilon^{-\frac{N[2+\gamma-(N-2)(p-1)]}{(N-1)(N+\gamma)}}\right\}$ and get from (4.29) (by enlarging T if necessary) that

$$\int_{|x|\le R} |u(T,x)|^2 dx \lesssim \varepsilon^2.$$

By the definition of χ_R,

$$\int \chi_R(x)|u(T,x)|^2 dx \lesssim \varepsilon^2.$$

Using the fact that

$$\begin{aligned}\left|\frac{d}{dt}\int \chi_R(x)|u(t,x)|^2 dx\right| &= \left|2\int \nabla\chi_R(x)\cdot \mathrm{Im}(\overline{u}(t,x)\nabla u(t,x))dx\right| \\ &\le 2\|\nabla\chi_R\|_{L^\infty}\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \lesssim R^{-1}\end{aligned}$$

for all $t\in\mathbb{R}$, we have for any $t\in I$,

$$\begin{aligned}\int \chi_R(x)|u(t,x)|^2 dx &= \int \chi_R(x)|u(T,x)|^2 dx - \int_t^T \left(\frac{d}{ds}\int \chi_R(x)|u(s,x)|^2 dx\right) ds \\ &\le \int \chi_R(x)|u(T,x)|^2 dx + CR^{-1}(T-t) \\ &\le C\varepsilon^2 + CR^{-1}\varepsilon^{-\sigma} \le 2C\varepsilon^2\end{aligned}$$

for some constant $C = C(u_0, Q) > 0$. This implies

$$\|\chi_R u\|_{L^\infty(I,L^2)} \lesssim \varepsilon,$$

where we have used the fact $\chi_R^2 \le \chi_R$ since $0\le\chi_R\le 1$. By Hölder's inequality, the radial Sobolev embedding and the fact $\|u\|_{L^\infty(\mathbb{R},H^1)} \le C(u_0,Q)$,

$$\begin{aligned}\|u\|_{L^\infty(I,L^r)} &\le \|\chi_R u\|_{L^\infty(I,L^r)} + \|(1-\chi_R)u\|_{L^\infty(I,L^r)} \\ &\le \|\chi_R u\|_{L^\infty(I,L^2)}^{\frac{2+\gamma-(N-2)(p-1)}{2p}} \|\chi_R u\|_{L^\infty(I,L^{\frac{2N}{N-2}})}^{\frac{N(p-1)-\gamma}{2p}} \\ &\quad + \|(1-\chi_R)u\|_{L^\infty(I,L^\infty)}^{\frac{N+\gamma}{Np}} \|(1-\chi_R)u\|_{L^\infty(I,L^2)}^{\frac{N(p-1)-\gamma}{Np}} \\ &\lesssim \varepsilon^{\frac{2+\gamma-(N-2)(p-1)}{2p}} + R^{-\frac{(N-1)(N+\gamma)}{2Np}} \lesssim \varepsilon^{\frac{2+\gamma-(N-2)(p-1)}{2p}}.\end{aligned}$$

It follows that

$$\begin{aligned}\|F_1\|_{L^k([T,+\infty),L^r)} &\lesssim \varepsilon^{-\frac{(2p-1)\sigma}{k}} \varepsilon^{\frac{[2+\gamma-(N-2)(p-1)](2p-1)}{2p}} \\ &= \varepsilon^{(2p-1)\left[-\frac{\sigma}{k}+\frac{2+\gamma-(N-2)(p-1)}{2p}\right]}.\end{aligned}$$

By the definition of k, we see that

$$\|F_1\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{\frac{[2+\gamma-(N-2)(p-1)][2(p-1)-\sigma]}{4p(p-1)}}. \tag{4.34}$$

Step 3. Estimate F_2. We estimate

$$\|F_2\|_{L^k([T,+\infty),L^r)} \le \|F_2\|_{L^k([T,+\infty),L^l)}^{\theta} \|F_2\|_{L^k([T,+\infty),L^n)}^{1-\theta}$$

where l is as in (4.32), $\theta\in(0,1)$ and $n>r$ satisfy

$$\frac{1}{r} = \frac{\theta}{l} + \frac{1-\theta}{n}.$$

Using the fact $(k,l)\in S_0$ and

$$F_2(t)=e^{i(t-T+\varepsilon^{-\sigma})\Delta}u(T-\varepsilon^{-\sigma})-e^{it\Delta}u_0,$$

Strichartz estimates imply

$$\|F_2\|_{L^k([T,+\infty),L^l)}\lesssim 1.$$

On the other hand, by the dispersive estimates (2.1), we have for any $t\geq T$,

$$\|F_2(t)\|_{L^n}\lesssim\int_J(t-s)^{-\frac{N}{2}\left(1-\frac{2}{n}\right)}\|(I_\gamma*|u(s)|^p)|u(s)|^{p-2}u(s)\|_{L^{n'}}ds.$$

By the Hardy-Littlewood-Sobolev inequality, we see that

$$\begin{aligned}\|(I_\gamma*|u|^p)|u|^{p-2}u\|_{L^{n'}}&\leq\|I_\gamma*|u|^p\|_{L^{\frac{Np(N+\gamma n')}{(2p-1)N^2n'-\gamma p(N+\gamma n')}}}\||u|^{p-2}u\|_{L^{\frac{(2p-1)Nn'}{(p-1)(N+\gamma n')}}}\\&\lesssim\|u\|^{2p-1}_{L^{\frac{(2p-1)Nn'}{N+\gamma n'}}}\lesssim 1\end{aligned}\tag{4.35}$$

provided

$$\frac{(2p-1)Nn'}{N+\gamma n'}\in\left[2,\frac{2N}{N-2}\right].$$

It follows that for any $t\geq T$,

$$\|F_2(t)\|_{L^n}\lesssim\int_0^{T-\varepsilon^{-\sigma}}(t-s)^{-\frac{N}{2}\left(1-\frac{2}{n}\right)}ds\lesssim(t-T+\varepsilon^{-\sigma})^{-\frac{N}{2}\left(1-\frac{2}{n}\right)+1}$$

provided

$$\frac{N}{2}\left(1-\frac{2}{n}\right)-1>0.$$

We infer that

$$\|F_2\|_{L^k([T,+\infty),L^n)}\lesssim\left(\int_T^{+\infty}(t-T+\varepsilon^{-\sigma})^{-\left[\frac{N}{2}\left(1-\frac{2}{n}\right)-1\right]k}dt\right)^{\frac{1}{k}}\lesssim\varepsilon^{\sigma\left[\frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}\right]}$$

provided

$$\frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}>0.$$

We thus obtain

$$\|F_2\|_{L^k([T,+\infty),L^r)}\lesssim\varepsilon^{\sigma\left[\frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}\right](1-\theta)}.\tag{4.36}$$

The above estimate holds true provided

$$n>r,\quad\frac{(2p-1)Nn'}{N+\gamma n'}\in\left[2,\frac{2N}{N-2}\right],\quad\frac{N}{2}\left(1-\frac{2}{n}\right)-1-\frac{1}{k}>0.$$

We will choose a suitable n satisfying the above conditions. By the choice of r and k, the above conditions become

$$\begin{aligned}&\frac{1}{n}<\frac{N+\gamma}{2Np},\quad\frac{1}{n}\in\left[\frac{N+2\gamma-2(p-1)N}{2N},\frac{N+2+2\gamma-2(p-1)(N-2)}{2N}\right],\\&\frac{1}{n}<\frac{(N-2)(p-1)(2p+1)-\gamma-2}{4Np(p-1)}.\end{aligned}\tag{4.37}$$

Note that it is easy to check that $(N-2)(p-1)(2p+1)-\gamma-2>0$ for any $\frac{2+\gamma}{N}<p-1<\frac{2+\gamma}{N-2}$.

• If $p-1 \geq \frac{N+2\gamma}{2N}$, then we take $\frac{1}{n}=0$ or $n=\infty$.

• If $p-1 < \frac{N+2\gamma}{2N}$, then we take $\frac{1}{n} = \frac{N+2\gamma-2(p-1)N}{2N}$. We need to check the following conditions

$$\begin{aligned} &\frac{N+2\gamma-2(p-1)N}{2N} < \frac{N+\gamma}{2Np}, \\ &\frac{N+2\gamma-2(p-1)N}{2N} < \frac{(N-2)(p-1)(2p+1)-\gamma-2}{4Np(p-1)}. \end{aligned} \tag{4.38}$$

Note that the condition $p-1 < \frac{N+2\gamma}{2N}$ combining with $\frac{2+\gamma}{N} < p-1 < \frac{2+\gamma}{N-2}$ require $N \geq 5$.

The first condition in (4.38) is equivalent to

$$2Np(p-1) - (N+2\gamma)p + N + \gamma > 0. \tag{4.39}$$

The left hand side of (4.39) is written as $f(p-1)$, where

$$f(x) := 2Nx^2 + (N-2\gamma)x - \gamma.$$

It is easy to check that f is increasing on $x > \frac{2+\gamma}{N}$ and $f\left(\frac{2+\gamma}{N}\right) > 0$. This shows that the first condition in (4.38) is satisfied for $\frac{2+\gamma}{N} < p-1 < \frac{2+\gamma}{N-2}$.

The second condition in (4.38) is in turn equivalent to

$$4Np(p-1)^2 + (N-2)(p-1)(2p+1) - 2(N+2\gamma)p(p-1) - \gamma - 2 > 0. \tag{4.40}$$

The left hand side of (4.40) is written as $g(p-1)$, where

$$g(x) := 4Nx^3 + 4(N-1-\gamma)x^2 + (N-6-4\gamma)x - \gamma - 2.$$

We have $g'(x) = 12Nx^2 + 8(N-1-\gamma)x + N - 6 - 4\gamma$. We see that g' is increasing on $x > \frac{2+\gamma}{N}$ and $g'\left(\frac{2+\gamma}{N}\right) > 0$. It follows that g is increasing on $x > \frac{2+\gamma}{N}$ which together with the fact $g\left(\frac{2+\gamma}{N}\right) > 0$ imply that $g(x) > 0$ for any $x > \frac{2+\gamma}{N}$. Therefore, the second condition in (4.38) is also satisfied.

Step 4. Conclusion. By (4.31), we get from (4.33), (4.34) and (4.36) that for $\sigma > 0$ sufficiently small, there exists $\mu = \mu(\sigma) > 0$ such that

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^\mu.$$

Case 2. $N = 2$.

Recall that we are considering $2p > 4+\gamma$ here. In this case, the last estimate in (4.37) does not work. To overcome this difficulty, we use the space time estimate (4.21) as follows. By the dispersive estimate and the Hardy-Littlewood-Sobolev inequality (4.35), we see that for $t \geq T$,

$$\begin{aligned} \|F_2(t)\|_{L^\infty} &\lesssim \int_J (t-s)^{-1} \|(I_\gamma * |u(s)|^p)|u(s)|^{p-2}u(s)\|_{L^1} ds \\ &\lesssim \int_J (t-s)^{-1} \|u(s)\|^{2p-1}_{L^{\frac{2(2p-1)}{2+\gamma}}} ds. \end{aligned}$$

By the interpolation inequality, we have

$$\|u\|_{L^{\frac{2(2p-1)}{2+\gamma}}} \leq \|u\|^{\frac{2p(2p-3-\gamma)}{(2p-1)(2p-2-\gamma)}}_{L^{\frac{4p}{2+\gamma}}} \|u\|^{\frac{2p+2+\gamma}{(2p-1)(2p-2-\gamma)}}_{L^2}$$

which together with the conservation of mass imply

$$\begin{aligned}\|F_2(t)\|_{L^\infty} &\lesssim \int_J (t-s)^{-1}\|u(s)\|_{L^{\frac{4p}{2+\gamma}}}^{\frac{2p(2p-3-\gamma)}{2p-2-\gamma}} ds\\ &\lesssim \|(t-s)^{-1}\|_{L^2_s(J)} \left\| \|u(s)\|_{L^{\frac{4p}{2+\gamma}}}^{\frac{2p(2p-3-\gamma)}{2p-2-\gamma}} \right\|_{L^2_s(J)}\\ &\lesssim \|(t-s)^{-1}\|_{L^2_s(J)} \|u\|_{L^{\frac{4p(2p-3-\gamma)}{2p-2-\gamma}}(J,L^{\frac{4p}{2+\gamma}})}^{\frac{2p(2p-3-\gamma)}{2p-2-\gamma}}.\end{aligned}$$

We see that for $t \geq T$,

$$\begin{aligned}\|(t-s)^{-1}\|_{L^2_s(J)} &= \left(\int_0^{T-\varepsilon^{-\sigma}} (t-s)^{-2} ds\right)^{\frac{1}{2}}\\ &= \left(-(t-s)^{-1}\big|_{s=0}^{s=T-\varepsilon^{-\sigma}}\right)^{\frac{1}{2}}\\ &\approx \left((t-T+\varepsilon^{-\sigma})^{-1} - t^{-1}\right)^{\frac{1}{2}}\\ &\lesssim (t-T+\varepsilon^{-\sigma})^{-\frac{1}{2}},\end{aligned}$$

where we have used $t \geq t-T+\varepsilon^{-\sigma}$ since $T > \varepsilon^{-\sigma}$. On the other hand, by (4.21) and the Sobolev embedding $H^1 \hookrightarrow L^{\frac{4p}{2+\gamma}}$ with $\|u\|_{L^\infty(\mathbb{R},H^1)} \leq C(u_0,Q)$, we see that

$$\begin{aligned}\|u\|_{L^{\frac{4p(2p-3-\gamma)}{2p-2-\gamma}}(J,L^{\frac{4p}{2+\gamma}})} &\leq \|u\|_{L^2(J,L^{\frac{4p}{2+\gamma}})}^{\frac{2p-2-\gamma}{2p(2p-3-\gamma)}} \|u\|_{L^\infty(J,L^{\frac{4p}{2+\gamma}})}^{\frac{(2p-1)(2p-3-\gamma)-1}{2p(2p-3-\gamma)}}\\ &\lesssim \|u\|_{L^2(J,L^{\frac{4p}{2+\gamma}})}^{\frac{2p-2-\gamma}{2p(2p-3-\gamma)}}\\ &\lesssim |J|^{\frac{(2p-2-\gamma)\beta}{4p(2p-3-\gamma)}}\\ &\lesssim T^{\frac{(2p-2-\gamma)\beta}{4p(2p-3-\gamma)}},\end{aligned}$$

where

$$\beta := \max\left\{\frac{1}{3}, \frac{2p}{4p-2-\gamma}\right\} = \frac{2p}{4p-2-\gamma}. \tag{4.41}$$

It yields that for $t \geq T$,

$$\|F_2(t)\|_{L^\infty} \lesssim (t-T+\varepsilon^{-\sigma})^{-\frac{1}{2}} T^{\frac{\beta}{2}}.$$

It follows that

$$\begin{aligned}\|F_2\|_{L^k([T,+\infty),L^\infty)} &\lesssim T^{\frac{\beta}{2}} \left(\int_T^{+\infty} (t-T+\varepsilon^{-\sigma})^{-\frac{k}{2}} dt\right)^{\frac{1}{k}}\\ &\lesssim T^{\frac{\beta}{2}} \left((t-T+\varepsilon^{-\sigma})^{-\frac{k}{2}+1}\Big|_{t=T}^{t=+\infty}\right)^{\frac{1}{k}}\\ &\lesssim T^{\frac{\beta}{2}} \varepsilon^{\sigma\left(\frac{1}{2}-\frac{1}{k}\right)}.\end{aligned}$$

We thus get

$$\|F_2\|_{L^k([T,+\infty),L^r)} \lesssim \left[T^{\frac{\beta}{2}} \varepsilon^{\sigma\left(\frac{1}{2}-\frac{1}{k}\right)}\right]^{1-\frac{l}{r}} = \left(T^{\frac{\beta}{2}} \varepsilon^{\frac{[2p(p-1)-2-\gamma]\sigma}{4p(p-1)}}\right)^{\frac{(p-1)(2p-2-\gamma)-\gamma-2}{2p(p-1)-\gamma-2}}. \tag{4.42}$$

Collecting (4.31), (4.33), (4.34) and (4.42) that

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon + \varepsilon^{\frac{(2+\gamma)(2p-2-\gamma)}{4p(p-1)}} + \Big(T^{\frac{\beta}{2}}\varepsilon^{\frac{[2p(p-1)-2-\gamma]\sigma}{4p(p-1)}}\Big)^{\frac{(p-1)(2p-2-\gamma)-\gamma-2}{2p(p-1)-\gamma-2}}.$$

By taking $T=\varepsilon^{-a\sigma}$ with some $a>1$ to be chosen shortly and choosing $\sigma>0$ small enough, we obtain

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T,+\infty),L^r)} \lesssim \varepsilon^{\mu} \tag{4.43}$$

for some $\mu>0$. The above estimate requires

$$\frac{2p(p-1)-2-\gamma}{4p(p-1)} - \frac{a\beta}{2} > 0 \quad \text{or} \quad a < \frac{2p(p-1)-2-\gamma}{2p(p-1)\beta}.$$

Using (4.41) and the fact $2p>4+\gamma$, it is easy to check that

$$\frac{2p(p-1)-2-\gamma}{2p(p-1)\beta} > 1.$$

Therefore, we can choose $a>1$ so that (4.43) holds. The proof is complete. □

The proof of the energy scattering for (NLC) with radial data follows immediately from (4.4), Lemma 4.5 and Proposition 3.

5. **Some possible extensions.**

The radial Sobolev-Morawetz method introduced by Dodson-Murphy [12] has been applied to show the energy scattering with radially symmetric initial data for other Schrödinger-type equations. It turns out to be useful for equations which do not have the conservation of momentum. This method gives alternative simple proofs to similar results proved via the concentration-compactness-rigidity argument.

5.1. **Nonlinear Schrödinger equations with potential.**

Consider the Cauchy problem for the focusing intercritical nonlinear Schrödinger equation with potential

$$\begin{cases} i\partial_t u + \Delta u - Vu &= -|u|^\alpha u, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3, \\ u(0,x) &= u_0(x), \end{cases} \tag{5.1}$$

where $u:\mathbb{R}\times\mathbb{R}^3\to\mathbb{C}$, $u_0:\mathbb{R}^3\to\mathbb{C}$ and $\frac{4}{3}<\alpha<4$. Here $V:\mathbb{R}^3\to\mathbb{R}$ is a real-valued potential satisfying

$$V\in\mathcal{K}\cap L^{\frac{3}{2}}, \quad \|V_-\|_{\mathcal{K}}<4\pi,$$

where $\mathcal{K}$ is the class of Kato potential,

$$\|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}dy$$

and $V_-(x):=\min\{V(x),0\}$.

The energy scattering for (5.1) with $\alpha=2$ was studied by Hong [23] using the concentration-compactness-rigidity argument of Kenig-Merle [25]. Recently, Hamano-Ikeda [21] adapted the Dodson-Murphy's method to extend the result in [23] to the whole range of the intercritical case.

5.2. **Inhomogenous nonlinear Schrödinger equations.** Consider the Cauchy problem for the focusing intercritical inhomogeneous nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u &= -|x|^{-b}|u|^{\alpha}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) &= u_0(x), \end{cases} \tag{5.2}$$

where $u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $u_0 : \mathbb{R}^N \to \mathbb{C}$, $0 < b < \min\{2, N\}$ and $\alpha_* < \alpha < \alpha^*$ with

$$\alpha_* := \frac{4-2b}{N}, \quad \alpha^* := \begin{cases} \frac{4-2b}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2. \end{cases}$$

The energy scattering for (5.2) was first established by Farah-Guzman [15] with $0 < b < 1$, $\alpha = 2$ and $N = 3$. This result was later extended to higher dimensions in [16]. The proofs of these results are based on the concentration-compactness-rigidity argument of Kenig-Merle [25]. Campos [5] used the Dodson-Murphy's method to give an alternative simple proof for the results of Farah-Guzman. He also extended the validity of b in dimensions $N \geq 3$. Recently, Xu-Zhao [41] and the first author in [10] has simultaneously proved the energy scattering for (5.2) in the two dimensions.

5.3. **Fractional nonlinear Schrödinger equations.** Consider the Cauchy problem for the focusing intercritical fractional nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - (-\Delta)^s u &= -|u|^{\alpha}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) &= u_0(x), \end{cases} \tag{5.3}$$

where $u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $u_0 : \mathbb{R}^N \to \mathbb{C}$, $N \geq 2$, $\frac{1}{2} < s < 1$ and $\frac{4s}{N} < \alpha < \frac{4s}{N-2s}$. The operator $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u := \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}(u)],$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform respectively.

The energy scattering for (5.3) was first established by Sun-Wang-Yao-Zheng [32] using a modified argument of Dodson-Murphy [12]. However, due to dispersive estimates with a loss of derivatives, there are some restrictions on the validity of s and α. In particular, the result in [32] is only available for $3 \leq N \leq 5$. Later, Guo-Zhu [20] extended the result of [32] to any dimensions $N \geq 2$ and the whole range of the intercritical case by using the concentration-compactness-rigidity argument of Kenig-Merle [25]. However, due to the fact that there is no dispersive estimates without loss of derivatives for the fractional Schrödinger operator available even in the radial case, the inhomogeneous Strichartz estimates stated in [20] require some restrictions on the validity of s. Thus, the claimed result in [20] is doubtful.

5.4. **Biharmonic nonlinear Schrödinger equations.** Consider the Cauchy problem for the focusing intercritical biharmonic nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta^2 u &= -|u|^{\alpha}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) &= u_0(x), \end{cases} \tag{5.4}$$

where $u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $u_0 : \mathbb{R}^N \to \mathbb{C}$, $N \geq 4$ and $\alpha_* < \alpha < \alpha^*$ with

$$\alpha_* := \frac{8}{N}, \quad \alpha^* := \begin{cases} \frac{8}{N-4} & \text{if } N \geq 5, \\ \infty & \text{if } N = 4. \end{cases}$$

The energy scattering for the equation (5.4) has been established by Guo [19] using the concentration-compactness argument of Kenig-Merle [25]. In [11], the first author makes use of the radial Sobolev-Morawetz method to give an alternative simple proof for the result in [19].

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