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41 uniqueness, and characterization of minimal blow-up solutions to the equation. Fibich
 42 and Wang [28], and Liu and Wang [42] investigated the stability and instability of
 43 solitary waves for (1.1) with $\alpha \geq \frac{4}{N}$ and $K(x) = K(\epsilon x)$, where $\epsilon > 0$ is a small
 44 parameter and $K \in C^4(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

45 When the potential $K(x)$ is unbounded, the problem becomes more subtle. The
 46 case $K(x) = |x|^b$, $b > 0$ was studied in several works, for instance, Chen and Guo [10],
 47 and Chen [9] established sharp criteria for the global existence and blow-up, and
 48 Zhu [56] studied the existence and dynamical properties of blow-up solutions. When
 49 $K(x)$ behaves like $|x|^{-b}$ with $b > 0$, De Bouard and Fukuizumi [12] studied the stability
 50 of standing waves for (1.1) with $\alpha < \frac{4-2b}{N}$. Fukuizumi and Ohta [30] established the
 51 instability of standing waves for (1.1) with $\alpha > \frac{4-2b}{N}$ (see also [33, 38] and references
 52 therein for other studies related to standing waves for this type of equation).

53 In this paper, we consider the Cauchy problem for a class of focusing inhomoge-
 54 neous nonlinear Schrödinger equations (INLS)

$$55 \quad (1.2) \quad \begin{cases} i\partial_t u + \Delta u &= -|x|^{-b}|u|^\alpha u, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\ u|_{t=0} &= u_0 \in H^1, \end{cases}$$

56 where $u : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, $u_0 : \mathbb{R}^N \rightarrow \mathbb{C}$, $N \geq 1$, $0 < b < \min\{2, N\}$, and $\frac{4-2b}{N} < \alpha <$
 57 $\alpha(N)$ with

$$58 \quad (1.3) \quad \alpha(N) := \begin{cases} \frac{4-2b}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2. \end{cases}$$

60 This equation plays an important role as a limiting equation in the analysis of (1.1)
 61 with $K(x) \sim |x|^{-b}$ as $|x| \rightarrow \infty$ (see e.g., [32, 33]).

62 The local well-posedness for (1.2) was studied by Geneoud and Stuart [33, Ap-
 63 pendix]. More precisely, they proved that (1.2) is locally well-posed in H^1 for $N \geq 1$,
 64 $0 < b < \min\{2, N\}$, and $0 < \alpha < \alpha(N)$. The proof of this result is based on the
 65 energy method developed by Cazenave [8], which does not use Strichartz estimates.
 66 See also [13, 37] for other proofs based on Strichartz estimates and the contraction
 67 mapping argument. Note that the local well-posedness in [13, 37] is more restrictive
 68 than the one in [33]. However, it provides more information on the local solutions,
 69 for instance, local solutions belong to $L^q_{\text{loc}}((-T_*, T^*), W^{1,r}(\mathbb{R}^N))$ for any Schrödinger
 70 admissible pair (q, r) (see Section 2 for the definition of L^2 admissibility), where
 71 $(-T_*, T^*)$ is the maximal time interval of existence. Note that the latter property
 72 plays an important role in the scattering theory.

73 It is well-known that solutions to (1.2) satisfy the conservation laws of mass and
 74 energy

$$75 \quad (\text{Mass}) \quad M(u(t)) = \|u(t)\|_{L^2}^2 = M(u_0),$$

$$76 \quad (\text{Energy}) \quad E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 - \frac{1}{\alpha + 2} \int |x|^{-b} |u(t, x)|^{\alpha+2} dx = E(u_0). \\ 77$$

78 The equation (1.2) also has the following scaling invariance

$$79 \quad (1.4) \quad u_\lambda(t, x) := \lambda^{\frac{2-b}{\alpha}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

81 A direct calculation gives

$$82 \quad \|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\gamma + \frac{2-b}{\alpha} - \frac{N}{2}} \|u_0\|_{\dot{H}^\gamma}$$

83 which shows that (1.4) leaves the \dot{H}^{γ_c} -norm of initial data invariant, where

$$84 \quad (1.5) \quad \gamma_c := \frac{N}{2} - \frac{2-b}{\alpha}.$$

86 The condition $\frac{4-2b}{N} < \alpha < \alpha(N)$ is equivalent to $0 < \gamma_c < 1$ which corresponds to the
87 mass-supercritical and energy-subcritical range (intercritical range, for short). For
88 later uses, it is convenient to introduce the following exponent

$$89 \quad (1.6) \quad \sigma_c := \frac{1-\gamma_c}{\gamma_c} = \frac{4-2b-(N-2)\alpha}{N\alpha-4+2b}.$$

91 The main purpose of the present paper is to study long time dynamics (global
92 existence, energy scattering, and finite time blow-up) of non-radial solutions to (1.2).
93 Before stating our contributions, let us recall known results related to dynamics of
94 (1.2) in the intercritical range.

95 In [25], Farah showed the global existence for (1.2) with $N \geq 1$ and $0 < b <$
96 $\min\{2, N\}$ by assuming $u_0 \in H^1$ and

$$97 \quad (1.7) \quad E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c},$$

$$98 \quad (1.8) \quad \|\nabla u_0\|_{L^2} \|u_0\|_{L^2}^{\sigma_c} < \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c},$$

100 where Q is the unique positive radial solution to the elliptic equation

$$101 \quad (1.9) \quad -\Delta Q + Q - |x|^{-b}|Q|^\alpha Q = 0.$$

103 He also proved the finite time blow-up for (1.2) with $u_0 \in \Sigma := H^1 \cap L^2(|x|^2 dx)$
104 satisfying (1.7) and

$$105 \quad (1.10) \quad \|\nabla u_0\|_{L^2} \|u_0\|_{L^2}^{\sigma_c} > \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c}.$$

107 The latter result was extended to radial data by the first author in [14]. Note that
108 the uniqueness of positive radial solution to (1.9) was established by Yanagida [55]
109 for $N \geq 3$, Genoud [32] for $N = 2$, and Toland [52] for $N = 1$.

110 The energy scattering (or asymptotic behavior) for (1.2) was first established
111 by Farah and Guzmán [26] with $0 < b < \frac{1}{2}, \alpha = 2, N = 3$, and radial data. The
112 proof of this result is based on the concentration/compactness and rigidity argument
113 introduced by Kenig and Merle [40]. This scattering result was later extended to
114 dimensions $N \geq 2$ in [27] by using the same concentration/compactness and rigidity
115 method.

116 Later, Campos [5] made use of a new idea of Dodson and Murphy [20] to give an
117 alternative simple proof for the radial scattering results of Farah and Guzmán. He
118 also extends the validity of b in dimensions $N \geq 3$. Note that the idea of Dodson
119 and Murphy is a combination of a scattering criterion of Tao [50], localized virial
120 estimates, and radial Sobolev embedding.

121 Afterwards, Xu and Zhao [54], and the first author [17] have simultaneously
122 showed the energy scattering for (1.2) with $0 < b < 1, N = 2$, and radial data.
123 The proof relies on a new approach of Arora, Dodson, and Murhpy [2], which is a
124 refined version of the one in [20].

125 In [6], Campos and Cardoso studied long time dynamics such as global existence,
126 energy scattering, and finite time blow-up of H^1 -solutions to (1.2) with data in Σ
127 lying above the ground state threshold.

128 Recently, Miao, Murphy, and Zheng [45] showed a new nonlinear profile for non-
 129 radial solutions related to (1.2). In particular, they constructed nonlinear profiles with
 130 data living far away from the origin. This allows them to show the energy scattering
 131 of non-radial solution to (1.2) with $0 < b < \frac{1}{2}$, $\alpha = 2$, and $N = 3$. This result was
 132 extended to any dimensions $N \geq 2$ and $0 < b < \min\{2, \frac{N}{2}\}$ by Cardoso, Farah,
 133 Guzmán, and Murphy [7].

134 We also mention the works [15, 17] for the energy scattering for the defocusing
 135 problem INLS and [11] for the energy scattering for the focusing energy-critical INLS.

136 Motivated by the aforementioned works, we study the global existence, energy
 137 scattering, and finite time blow-up of non-radial solutions to (1.2). To this end, let
 138 us start with the following scattering criterion for (1.2).

139 **THEOREM 1.1** (Scattering criterion). *Let $N \geq 1$, $0 < b < \min\{2, N\}$, and*
 140 *$\frac{4-2b}{N} < \alpha < \alpha(N)$. Let u be a solution to (1.2) defined on the maximal forward time*
 141 *interval of existence $[0, T^*)$. Assume that*

$$142 \quad (1.11) \quad \sup_{t \in [0, T^*)} P(u(t)) [M(u(t))]^{\sigma_c} < P(Q) [M(Q)]^{\sigma_c},$$

144 where

$$145 \quad (1.12) \quad P(f) := \int |x|^{-b} |f(x)|^{\alpha+2} dx.$$

146 Then $T^* = \infty$. Moreover, if we assume in addition that $N \geq 2$ and $0 < b <$
 148 $\min\{2, \frac{N}{2}\}$, then the solution scatters in H^1 forward in time, i.e., there exists $u_+ \in$
 149 H^1 such that

$$150 \quad (1.13) \quad \lim_{t \rightarrow \infty} \|u(t) - e^{it\Delta} u_+\|_{H^1} = 0.$$

152 A similar statement holds for negative times.

153 We note that a scattering condition similar to (1.11) was first introduced by
 154 Duyckaerts and Roudenko in [23, Theorem 3.7], where it was used to show the scat-
 155 tering beyond the ground state threshold for the focusing Schrödinger equation. The
 156 condition (1.11) was inspired by a recent work of Gao and Wang [31] (see also [16]).

157 The proof of Theorem 1.1 is based on the concentration/compactness and rigidity
 158 method. The main difficulty comes from the fact that the potential energy $P(u(t))$ is
 159 not conserved along the time evolution of (1.2). To overcome the difficulty, we estab-
 160 lish a Pythagorean expansion along bounded nonlinear flows. Since we are interested
 161 in non-radial solutions, we need to construct nonlinear profiles associated with the
 162 linear ones living far away from the origin. The latter was recently showed by Miao,
 163 Murphy, and Zheng [45] in three dimensions (see also [7] for dimensions $N \geq 2$). This
 164 type of nonlinear profiles is constructed by observing that in the regime $|x| \rightarrow \infty$, the
 165 nonlinearity becomes weak, and solutions to (1.2) can be approximated by solutions
 166 to the underlying linear Schrödinger equation. Thanks to an improved nonlinear esti-
 167 mate (see Lemma 2.2), we give a refined result with a simple proof of these results
 168 (see Lemma 2.8). For more details, we refer to Section 2.

169 Our next result is the following blow-up criterion for (1.2).

170 **THEOREM 1.2** (Blow-up criterion). *Let $N \geq 1$, $0 < b < \min\{2, N\}$, and $\frac{4-2b}{N} <$*
 171 *$\alpha < \alpha(N)$. Let u be a solution to (1.2) defined on the maximal forward time interval*
 172 *of existence $[0, T^*)$. Assume that*

$$173 \quad (1.14) \quad \sup_{t \in [0, T^*)} G(u(t)) \leq -\delta$$

174

175 for some $\delta > 0$, where

$$176 \quad (1.15) \quad G(f) := \|\nabla f\|_{L^2}^2 - \frac{N\alpha + 2b}{2(\alpha + 2)}P(f). \\ 177$$

178 Then either $T^* < \infty$, or $T^* = \infty$ and there exists a time sequence $t_n \rightarrow \infty$ such
179 that $\|\nabla u(t_n)\|_{L^2} \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, if we assume in addition that u has
180 finite variance, i.e., $|x|u(t) \in L^2(|x|^2 dx)$ for all $t \in [0, T^*)$, then $T^* < \infty$. A similar
181 statement holds for negative times.

182 The proof of this blow-up result is based on a contradiction argument using localized
183 virial estimates for general (non-radial and infinite variance) solutions to (1.2) (see
184 Lemma 3.3). We also take the advantage of the decay of the nonlinear term outside a
185 large ball. It is conjectured that if a general (not finite variance or radially symmetric)
186 solution to (1.2) satisfy (1.14), then it blows up in finite time. However, there is
187 no affirmative answer for this conjecture up to date even for the classical nonlinear
188 Schrödinger equation.

189 A first application of Theorems 1.1 and 1.2 is the following long time dynamics
190 below the ground state threshold.

191 THEOREM 1.3 (Dynamics below the ground state threshold). *Let $N \geq 1$, $0 <$
192 $b < \min\{2, N\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let $u_0 \in H^1$ satisfy (1.7).*

193 (1) *If u_0 satisfies (1.8), then the corresponding solution to (1.2) satisfies*

$$194 \quad (1.16) \quad \sup_{t \in (-T_*, T^*)} P(u(t))[M(u(t))]^{\sigma_c} < P(Q)[M(Q)]^{\sigma_c}. \\ 195$$

196 *In particular, the solution exists globally in time. Moreover, if we assume in addition*
197 *that $N \geq 2$ and $0 < b < \min\{2, \frac{N}{2}\}$, then the corresponding solution scatters in H^1*
198 *in both directions.*

199 (2) *If u_0 satisfies (1.10), then the corresponding solution to (1.2) satisfies*

$$200 \quad (1.17) \quad \sup_{t \in (-T_*, T^*)} G(u(t)) \leq -\delta \\ 201$$

202 for some $\delta > 0$. In particular, the solution either blows up in finite time, or there
203 exists a time sequence $(t_n)_{n \geq 1}$ satisfying $|t_n| \rightarrow \infty$ such that $\|\nabla u(t_n)\|_{L^2} \rightarrow \infty$ as
204 $n \rightarrow \infty$. Moreover, if we assume in addition that

- 205 • u_0 has finite variance,
- 206 • or $N \geq 2$, $\alpha \leq 4$, and u_0 is radially symmetric,
- 207 • or $N \geq 3$, $\alpha \leq 2$, and $u_0 \in \Sigma_N$, where

$$208 \quad (1.18) \quad \Sigma_N := \{f \in H^1 : f(y, x_N) = f(|y|, x_N), x_N f \in L^2\}$$

210 with $x = (y, x_N)$, $y = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$, and $x_N \in \mathbb{R}$,
211 then the corresponding solution blows up in finite time, i.e., $T_*, T^* < \infty$.

212 For the scattering part, Theorem 1.3 provides an alternative proof of a recent
213 result of Cardoso, Farah, Guzmán, and Murphy [7]. For the blow-up part, Theorem
214 1.3 extends earlier results of [25] (for finite variance data) and the first author [14]
215 (for radial data) to the case of cylindrically symmetric data. Note that the first work
216 addressed the finite time blow-up for NLS with cylindrically symmetric data is due
217 to Martel [43], where the blow-up was shown for data with negative energy. Recently,
218 Bellazzini and Forcella [3] extended Martel's result to the case of focusing cubic NLS

for data with non-negative energy data lying below the ground state threshold. Our result not only extends the ones of [3, 43] to the focusing inhomogeneous NLS but also provides an alternative simple proof for these results. In particular, our choice of cutoff function is simpler than that in [3, 43]. Our argument is robust and can be applied to show the existence of finite time blow-up solutions with cylindrically symmetric data for other Schrödinger-type equations. See [1, 4, 18].

Another application of Theorems (1.1) and (1.2) is the following long time dynamics at the ground state threshold.

THEOREM 1.4 (Dynamics at the ground state). *Let $N \geq 1$, $0 < b < \min\{2, N\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let $u_0 \in H^1$ be such that*

$$(1.19) \quad E(u_0)[M(u_0)]^{\sigma_c} = E(Q)[M(Q)]^{\sigma_c}.$$

(1) *If*

$$(1.20) \quad \|\nabla u_0\|_{L^2} \|u_0\|_{L^2}^{\sigma_c} < \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c},$$

then the corresponding solution to (1.2) exists globally in time. Moreover, the solution either satisfies

$$(1.21) \quad \sup_{t \in \mathbb{R}} P(u(t))[M(u(t))]^{\sigma_c} < P(Q)[M(Q)]^{\sigma_c}$$

or there exists a time sequence $(t_n)_{n \geq 1}$ satisfying $|t_n| \rightarrow \infty$ such that

$$(1.22) \quad u(t_n) \rightarrow e^{i\theta} Q \quad \text{strongly in } H^1$$

for some $\theta \in \mathbb{R}$ as $n \rightarrow \infty$. In particular, if we assume in addition that $N \geq 2$ and $0 < b < \min\{2, \frac{N}{2}\}$, then the solution either scatters in H^1 forward in time, or there exist a time sequence $t_n \rightarrow \infty$ and a sequence $(x_n)_{n \geq 1} \subset \mathbb{R}^N$ such that (1.22) holds.

(2) *If*

$$(1.23) \quad \|\nabla u_0\|_{L^2} \|u_0\|_{L^2}^{\sigma_c} = \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c},$$

then $u(t, x) = e^{it} e^{i\theta} Q(x)$ for some $\theta \in \mathbb{R}$.

(3) *If*

$$(1.24) \quad \|\nabla u_0\|_{L^2} \|u_0\|_{L^2}^{\sigma_c} > \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c},$$

then the corresponding solution to (1.2)

- i. either blows up forward in time, i.e., $T^* < \infty$,
- ii. or there exists a time sequence $t_n \rightarrow \infty$ such that $\|\nabla u(t_n)\|_{L^2} \rightarrow \infty$ as $n \rightarrow \infty$,
- iii. or there exists a time sequence $t_n \rightarrow \infty$ such that (1.22) holds.

Moreover, if we assume in addition that

- u_0 has finite variance,
- or $N \geq 2$, $\alpha \leq 4$, and u_0 is radially symmetric,
- or $N \geq 3$, $\alpha \leq 2$, and $u_0 \in \Sigma_N$,

then the possibility in Item ii. can be excluded.

To our knowledge, Theorem 1.4 is the first result addressing long time dynamics of solutions to (1.2) with data lying at the ground state threshold. For the classical NLS, dynamics at the ground state threshold was first studied by Duyckaerts and

264 Roudenko [22] for the 3D focusing cubic NLS. The proof in [22] relies on delicate
 265 spectral estimates which make it difficult to extend to higher dimensions. Recently,
 266 the first author in [16] gave a simple approach to study the dynamics at the threshold
 267 for the focusing NLS in any dimensions. Our result is an extension of the one in [16] to
 268 the focusing inhomogeneous NLS. The proof of Theorem 1.4 relies on the scattering
 269 and blow-up criteria given in Theorems 1.1, 1.2, and the compactness property of
 270 optimizing sequence for the Gagliardo-Nirenberg inequality (2.3) (see Lemma 4.2).
 271 We refer the reader to Section 4 for more details.

272 Finally, we study long time dynamics above the ground state threshold. Before
 273 stating our result, we introduce the virial quantity

$$274 \quad (1.25) \quad V(t) := \int |x|^2 |u(t, x)|^2 dx.$$

276 If $V(0) < \infty$, then $V(t) < \infty$ for all t in the existence time. Moreover, the following
 277 identities hold

$$278 \quad (1.26) \quad \begin{aligned} V'(t) &= 4 \operatorname{Im} \int \bar{u}(t, x) x \cdot \nabla u(t, x) dx, \\ V''(t) &= 8 \|\nabla u(t)\|_{L^2}^2 - \frac{4(N\alpha + 2b)}{\alpha + 2} P(u(t)). \end{aligned}$$

280 **THEOREM 1.5** (Dynamics above the ground state). *Let $N \geq 1$, $0 < b <$*
 281 *$\min\{2, N\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let $u_0 \in \Sigma$ satisfy*

$$282 \quad (1.27) \quad E(u_0)[M(u_0)]^{\sigma_c} \geq E(Q)[M(Q)]^{\sigma_c},$$

$$283 \quad (1.28) \quad \frac{E(u_0)[M(u_0)]^{\sigma_c}}{E(Q)[M(Q)]^{\sigma_c}} \left(1 - \frac{(V'(0))^2}{32E(u_0)V(0)} \right) \leq 1.$$

284
 285 (1) *If*

$$286 \quad (1.29) \quad P(u_0)[M(u_0)]^{\sigma_c} < P(Q)[M(Q)]^{\sigma_c},$$

$$287 \quad (1.30) \quad V'(0) \geq 0,$$

289 *then the corresponding solution to (1.2) satisfies (1.11). In particular, if $N \geq 2$ and*
 290 *$0 < b < \min\{2, \frac{N}{2}\}$, then the solution exists globally in time and scatters in H^1 in*
 291 *the sense of (1.13).*

292 (2) *If*

$$293 \quad (1.31) \quad P(u_0)[M(u_0)]^{\sigma_c} > P(Q)[M(Q)]^{\sigma_c},$$

$$294 \quad (1.32) \quad V'(0) \leq 0,$$

296 *then the corresponding solution to (1.2) blows up forward in time, i.e., $T^* < \infty$.*

297 For the scattering part, Theorem 1.5 improves a recent result of Campos and
 298 Cardoso [6] at two points: (1) removing the radial assumption and (2) extending the
 299 validity of b . For the blow-up part, we extend the one in [6] to any dimensions $N \geq 1$.
 300 The proof of Theorem 1.5 is based on virial identities and a continuity argument in
 301 the same spirit of Duyckaerts and Roudenko [23].

302 We finish the introduction by outlining the structure of the paper. In Section
 303 2, we give the proof of the scattering criterion given in Theorem 1.1. In Section 3,
 304 we prove the blow-up criterion given in Theorem 1.2. Finally, we study long time
 305 dynamics of H^1 -solutions lying below, at, and above the ground state threshold in
 306 Section 4.

307 **2. Scattering criterion.**

308 **2.1. Local theory.** In this subsection, we recall the well-posedness theory for
 309 (1.2) due to [26, 27, 37]. To this end, we introduce some notations. Let $\gamma \geq 0$. A pair
 310 (q, r) is called \dot{H}^γ -admissible if

$$311 \quad \frac{2}{q} + \frac{N}{r} = \frac{N}{2} - \gamma$$

312 and

$$313 \quad (2.1) \quad \begin{cases} \frac{2N}{N-2\gamma} < r < \frac{2N}{N-2} & \text{if } N \geq 3, \\ \frac{2}{1-\gamma} < r < \infty & \text{if } N = 2, \\ \frac{2}{1-2\gamma} < r < \infty & \text{if } N = 1. \end{cases}$$

315 The set of all \dot{H}^γ -admissible pairs is denoted by \mathcal{A}_γ . Similarly, a pair (q, r) is called
 316 $\dot{H}^{-\gamma}$ -admissible if

$$317 \quad \frac{2}{q} + \frac{N}{r} = \frac{N}{2} + \gamma$$

318 and r satisfies (2.1). The set of all $\dot{H}^{-\gamma}$ -admissible pairs is denoted by $\mathcal{A}_{-\gamma}$. Note
 319 that we do not consider the pair $(\infty, \frac{2N}{N-2\gamma})$ as a \dot{H}^γ -admissible pair. The reason for
 320 doing so will be clear in Subsection 2.3. When $\gamma = 0$, we denote L^2 instead of \dot{H}^0 . In
 321 this case, the L^2 -admissible pair is also called Schrödinger admissible.

322 Let $I \subset \mathbb{R}$ be an interval and $\gamma \geq 0$. We define the Strichartz norm

$$323 \quad \|u\|_{S(I, \dot{H}^\gamma)} := \sup_{(q,r) \in \mathcal{A}_\gamma} \|u\|_{L_t^q(I, L_x^r)}.$$

324 For a set $A \subset \mathbb{R}^N$, we denote

$$325 \quad \|u\|_{S(I, \dot{H}^\gamma(A))} := \sup_{(q,r) \in \mathcal{A}_\gamma} \|u\|_{L_t^q(I, L_x^r(A))}.$$

326 When $I = \mathbb{R}$, we omit the dependence on \mathbb{R} and denote $\|u\|_{S(\dot{H}^\gamma)}$ and $\|u\|_{S(\dot{H}^\gamma(A))}$.
 327 Similarly, we define

$$328 \quad \|u\|_{S'(I, \dot{H}^{-\gamma})} := \inf_{(q,r) \in \mathcal{A}_{-\gamma}} \|u\|_{L_t^{q'}(I, L_x^{r'})}$$

329 and for $A \subset \mathbb{R}^N$,

$$330 \quad \|u\|_{S'(I, \dot{H}^{-\gamma}(A))} := \inf_{(q,r) \in \mathcal{A}_{-\gamma}} \|u\|_{L_t^{q'}(I, L_x^{r'}(A))}.$$

331 As before, when $I = \mathbb{R}$, we simply use $\|u\|_{S'(\dot{H}^{-\gamma})}$ and $\|u\|_{S'(\dot{H}^{-\gamma}(A))}$.

332 We have the following Strichartz estimates (see e.g., [8, 29, 39]).

333 **PROPOSITION 2.1** (Strichartz estimates [8, 29, 39]). *Let $\gamma \geq 0$ and $I \subset \mathbb{R}$ be an*
 334 *interval containing 0. Then there exists a constant $C > 0$ independent of I such that*

$$335 \quad \|e^{it\Delta} f\|_{S(I, \dot{H}^\gamma)} \leq C \|f\|_{\dot{H}^\gamma}$$

336 and

$$337 \quad \left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{S(I, \dot{H}^\gamma)} \leq C \|F\|_{S'(I, \dot{H}^{-\gamma})}.$$

338 Moreover, the above estimates still hold with $L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma}})$ -norm in place of $S(I, \dot{H}^\gamma)$ -
 339 norm.

340 We also need the following nonlinear estimates due to [5, Lemma 2.5] and [7,
341 Lemma 2.1].

342 LEMMA 2.2 (Nonlinear estimates [5, 7]). *Let $N \geq 2$, $0 < b < \min\{2, \frac{N}{2}\}$, and
343 $\frac{4-2b}{N} < \alpha < \alpha(N)$. Then there exists $\theta \in (0, \alpha)$ sufficiently small so that*

$$\begin{aligned} 344 \quad & \| |x|^{-b} |u|^\alpha v \|_{S'(\dot{H}^{-\gamma_c})} \lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{\gamma_c})}^{\alpha-\theta} \|v\|_{S(\dot{H}^{\gamma_c})}, \\ 345 \quad & \| |x|^{-b} |u|^\alpha v \|_{S'(L^2)} \lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{\gamma_c})}^{\alpha-\theta} \|v\|_{S(L^2)}, \\ 346 \quad & \| \nabla (|x|^{-b} |u|^\alpha u) \|_{S'(L^2)} \lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{S(\dot{H}^{\gamma_c})}^{\alpha-\theta} \| \nabla u \|_{S(L^2)}. \end{aligned}$$

348 *Note that if $b = 0$, we can take $\theta = 0$ in the above estimates.*

349 *Proof.* The first two estimates were proved in [5, Lemma 2.5] (for $N \geq 3$) and [7,
350 Lemma 2.1] (for $N \geq 2$). An estimate similar to the last one was proved in [5, Lemma
351 2.5] for $N \geq 3$. However, the proof in [5] used the dual pair of the end-point $(2, \frac{2N}{N-2})$
352 which, however, is excluded in our definition of L^2 -admissible pair (see (2.1)). Thus
353 we need a different argument. Let $\theta > 0$ be a small parameter to be chosen later. We
354 denote

$$\begin{aligned} 355 \quad q' &= \frac{4}{2+\theta}, & r' &= \frac{2N}{N+2-\theta}, \\ 356 \quad \bar{a} &= \frac{4\alpha(\alpha+1-\theta)}{4-2b-(N-2)\alpha+\theta\alpha}, & \bar{r} &= \frac{2N\alpha(\alpha+1-\theta)}{(N+2-2b)\alpha-\theta(4-2b+\alpha)}, \\ 357 \quad \bar{q} &= \frac{4\alpha(\alpha+1-\theta)}{\alpha(N\alpha-2+2b)-\theta(N\alpha-4+2b-\alpha)}, & \bar{m}_\pm &= \frac{N\alpha}{2-b \mp N\alpha\theta}. \end{aligned}$$

359 Here (q', r') is the dual pair of $(\frac{4}{2-\theta}, \frac{2N}{N-2+\theta}) \in \mathcal{A}_0$. We can readily check that
360 $(\bar{q}, \bar{r}) \in \mathcal{A}_0$ and $(\bar{a}, \bar{r}) \in \mathcal{A}_{\gamma_c}$ provided that $\theta > 0$ is taken sufficiently small. Moreover,
361 as $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$, we have $2 < \bar{m}_\pm < \frac{2N}{N-2}$ for $\theta > 0$ sufficiently small.

362 We observe that

$$363 \quad (2.2) \quad \nabla (|x|^{-b} |u|^\alpha u) = |x|^{-b} \nabla (|u|^\alpha u) - b \frac{x}{|x|} |x|^{-b} (|x|^{-1} |u|^\alpha u)$$

365 and

$$366 \quad \| |x|^{-b} f \|_{L_x^{r'}(A)} \leq \| |x|^{-b} \|_{L_x^{r_1}(A)} \| f \|_{L_x^{r_2}},$$

367 where A stands for either $B = B(0, 1)$ or $B^c = \mathbb{R}^N \setminus B(0, 1)$. To ensure $\| |x|^{-b} \|_{L_x^{r_1}(A)} <$
368 ∞ , we take

$$369 \quad \frac{1}{r_1} = \frac{b}{N} \pm \theta^2,$$

370 where the plus sign is for $A = B$ and the minus one is for $A = B^c$. It follows that

$$371 \quad \frac{1}{r_2} = \frac{1}{r'} - \frac{1}{r_1} = \frac{N+2-2b-\theta}{2N} \mp \theta^2.$$

372 As $\frac{1}{N} < \frac{N+2-2b}{2N} < 1$ for $N \geq 2$ and $0 < b < \frac{N}{2}$, we choose $\theta > 0$ sufficiently small so
373 that $1 < r_2 < N$ which allows us to use the Hardy's inequality (see e.g., [46])

$$374 \quad \| |x|^{-1} f \|_{L_x^{r_2}} \leq \frac{r_2}{N-r_2} \| \nabla f \|_{L_x^{r_2}}.$$

375 Applying the above inequality to $f = |u|^\alpha u$ and using (2.2), we see that

$$376 \quad \|\nabla(|x|^{-b}|u|^\alpha u)\|_{L_x^{r'}} \lesssim \|\nabla(|u|^\alpha u)\|_{L_x^{r_2}}.$$

377 By Hölder's inequality and the fact that

$$378 \quad \frac{1}{r_2} = \frac{\theta}{\bar{m}_\pm} + \frac{\alpha + 1 - \theta}{\bar{r}},$$

379 we have

$$380 \quad \|\nabla(|x|^{-b}|u|^\alpha u)\|_{L_x^{r'}} \lesssim \|u\|_{L_x^{\bar{m}_\pm}}^\theta \|u\|_{L_x^{\bar{r}}}^{\alpha - \theta} \|\nabla u\|_{L_x^{\bar{r}}}.$$

381 By Hölder's inequality in time with

$$382 \quad \frac{1}{q'} = \frac{\alpha - \theta}{\bar{a}} + \frac{1}{\bar{q}},$$

383 we get

$$384 \quad \begin{aligned} \|\nabla(|x|^{-b}|u|^\alpha u)\|_{L_t^{q'} L_x^{r'}} &\lesssim \|u\|_{L_t^\infty L_x^{\bar{m}_\pm}}^\theta \|u\|_{L_t^{\bar{a}} L_x^{\bar{r}}}^{\alpha - \theta} \|\nabla u\|_{L_t^{\bar{q}} L_x^{\bar{r}}} \\ 385 \quad &\lesssim \|u\|_{L_t^\infty H_x^1}^\theta \|u\|_{L_t^{\bar{a}} L_x^{\bar{r}}}^{\alpha - \theta} \|\nabla u\|_{L_t^{\bar{q}} L_x^{\bar{r}}}, \end{aligned}$$

387 where the last inequality follows from the Sobolev embedding as $2 < \bar{m}_\pm < \frac{2N}{N-2}$. The
388 proof is complete. \square

389 Using Proposition 2.1 and Lemma 2.2, we have the following result.

390 PROPOSITION 2.3 (Local theory [26, 27, 37]). *Let $N \geq 2$, $0 < b < \min\{2, \frac{N}{2}\}$,
391 and $\frac{4-2b}{N} < \alpha < \alpha(N)$.*

392 (1) (Local well-posedness) *Let $u_0 \in H^1$. Then there exist $T_*, T^* \in (0, \infty]$, and a
393 unique local solution to (1.2) satisfying*

$$394 \quad u \in C((-T_*, T^*), H^1) \cap L_{\text{loc}}^q(-T_*, T^*), W^{1,r}$$

395 *for any $(q, r) \in \mathcal{A}_0$. If $T^* < \infty$ (resp. $T_* < \infty$), then $\lim_{t \nearrow T^*} \|\nabla u(t)\|_{L^2} = \infty$
396 (resp. $\lim_{t \searrow -T_*} \|\nabla u(t)\|_{L^2} = \infty$).*

397 (2) (Small data scattering) *Let $T > 0$ be such that $\|u(T)\|_{H^1} \leq A$ for some
398 constant $A > 0$. Then there exists $\delta = \delta(A) > 0$ such that if*

$$399 \quad \|e^{i(t-T)\Delta} u(T)\|_{S([T, \infty), \dot{H}^{\gamma_c})} < \delta,$$

400 *then the corresponding solution to (1.2) with initial data $u|_{t=T} = u(T)$ exists
401 globally in time and satisfies*

$$402 \quad \begin{aligned} \|u\|_{S([T, \infty), \dot{H}^{\gamma_c})} &\leq 2 \|e^{i(t-T)\Delta} u(T)\|_{S([T, \infty), \dot{H}^{\gamma_c})}, \\ 403 \quad \|\langle \nabla \rangle u\|_{S([T, \infty), L^2)} &\leq C \|u(T)\|_{H^1}. \end{aligned}$$

405 (3) (Scattering condition) *Let u be a global solution to (1.2). Assume that*

$$406 \quad \|u\|_{L_t^\infty(\mathbb{R}, H_x^1)} \leq A, \quad \|u\|_{S(\dot{H}^{\gamma_c})} < \infty.$$

407 *Then u scatters in H^1 in both directions.*

408 Here we have used the following convention

$$409 \quad \|\langle \nabla \rangle f\|_X := \|f\|_X + \|\nabla f\|_X, \quad f \in X.$$

410 We also recall the following stability result due to [26, 27].

411 LEMMA 2.4 (Stability). *Let $N \geq 2$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$.*
 412 *Let $0 \in I \subseteq \mathbb{R}$ and $\tilde{u} : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ be a solution to*

$$413 \quad i\partial_t \tilde{u} + \Delta \tilde{u} + |x|^{-b} |\tilde{u}|^\alpha \tilde{u} = e$$

414 *with $\tilde{u}|_{t=0} = \tilde{u}_0$ satisfying*

$$415 \quad \|\tilde{u}\|_{L_t^\infty(I, H_x^1)} \leq M, \quad \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} \leq L$$

416 *for some constants $M, L > 0$. Let $u_0 \in H^1$ be such that*

$$417 \quad \|u_0 - \tilde{u}_0\|_{H^1} \leq M', \quad \|e^{it\Delta}(u_0 - \tilde{u}_0)\|_{S(I, \dot{H}^{\gamma_c})} \leq \varepsilon$$

418 *for some $M' > 0$ and some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(M, M', L)$. Suppose that*

$$419 \quad \|\langle \nabla \rangle e\|_{S'(I, L^2)} + \|e\|_{S'(I, \dot{H}^{-\gamma_c})} \leq \varepsilon.$$

420 *Then there exists a unique solution $u : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ to (1.2) with $u|_{t=0} = u_0$ satisfying*

$$421 \quad \|u - \tilde{u}\|_{S(I, \dot{H}^{\gamma_c})} \leq C(M, M', L)\varepsilon,$$

$$422 \quad \|u\|_{L_t^\infty(I, H_x^1)} + \|\langle \nabla \rangle u\|_{S(I, L^2)} + \|u\|_{S(I, \dot{H}^{\gamma_c})} \leq C(M, M', L).$$

424 Remark 2.5. If we assume in addition that

$$425 \quad \|e^{it\Delta}(u_0 - \tilde{u}_0)\|_{L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma_c}})} \leq \varepsilon,$$

426 then

$$427 \quad \|u - \tilde{u}\|_{L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma_c}})} \leq C(M, M', L)\varepsilon.$$

428 In fact, by Duhamel's formula, we have

$$429 \quad u(t) - \tilde{u}(t) = e^{it\Delta}(u_0 - \tilde{u}_0) + i \int_0^t e^{i(t-s)\Delta} (|x|^{-b} |u(s)|^\alpha u(s) - |x|^{-b} |\tilde{u}(s)|^\alpha \tilde{u}(s)) ds$$

$$430 \quad + i \int_0^t e^{i(t-s)\Delta} e(s) ds.$$

432 By Strichartz estimates and Lemma 2.2, we have

$$433 \quad \|u - \tilde{u}\|_{L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma_c}})} \leq \|e^{it\Delta}(u_0 - \tilde{u}_0)\|_{L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma_c}})} + \|e\|_{S'(I, \dot{H}^{-\gamma_c})}$$

$$434 \quad + C\| |x|^{-b} |u|^\alpha u - |x|^{-b} |\tilde{u}|^\alpha \tilde{u} \|_{S'(I, \dot{H}^{\gamma_c})}$$

$$435 \quad \leq \|e^{it\Delta}(u_0 - \tilde{u}_0)\|_{L_t^\infty(I, L_x^{\frac{2N}{N-2\gamma_c}})} + \|e\|_{S'(I, \dot{H}^{-\gamma_c})}$$

$$436 \quad + C\left(\|u\|_{L_t^\infty(I, H_x^1)}^\theta \|u\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} + \|\tilde{u}\|_{L_t^\infty(I, H_x^1)}^\theta \|\tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}^{\alpha-\theta} \right) \|u - \tilde{u}\|_{S(I, \dot{H}^{\gamma_c})}$$

$$437 \quad \leq C(M, M', L)\varepsilon.$$

439 **2.2. Variational analysis.** We recall some properties of the ground state Q
 440 which is the unique positive radial solution to (1.9). The ground state Q optimizes
 441 the weighted Gagliardo-Nirenberg inequality: for $N \geq 1$ and $0 < b < \min\{2, N\}$,

$$443 \quad (2.3) \quad P(f) \leq C_{\text{opt}} \|\nabla f\|_{L^2}^{\frac{N\alpha+2b}{2}} \|f\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}}, \quad f \in H^1(\mathbb{R}^N),$$

444 that is

$$445 \quad C_{\text{opt}} = P(Q) \div \left[\|\nabla Q\|_{L^2}^{\frac{N\alpha+2b}{2}} \|Q\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}} \right],$$

446 where $P(f)$ is as in (1.12). We have the following Pohozaev's identities (see e.g., [25])

$$447 \quad (2.4) \quad \|Q\|_{L^2}^2 = \frac{4-2b-(N-2)\alpha}{N\alpha+2b} \|\nabla Q\|_{L^2}^2 = \frac{4-2b-(N-2)\alpha}{2(\alpha+2)} P(Q).$$

449 In particular, we have

$$450 \quad (2.5) \quad C_{\text{opt}} = \frac{2(\alpha+2)}{N\alpha+2b} (\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^{-\frac{N\alpha-4+2b}{2}}.$$

452 We also have

$$453 \quad (2.6) \quad E(Q) = \frac{N\alpha-4+2b}{2(N\alpha+2b)} \|\nabla Q\|_{L^2}^2 = \frac{N\alpha-4+2b}{4(\alpha+2)} P(Q)$$

455 hence

$$456 \quad (2.7) \quad E(Q)[M(Q)]^{\sigma_c} = \frac{N\alpha-4+2b}{2(N\alpha+2b)} (\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^2.$$

458 **2.3. Profile decompositions.** In this subsection, we recall the linear profile
 459 decomposition and construct some nonlinear profiles associated to (1.2). Let us start
 460 with the following result due to [24, 36] (see also [26, 27]).

461 **LEMMA 2.6** (Linear profile decomposition [24, 26, 27, 36]). *Let $N \geq 1$, $0 < b <$
 462 $\min\{2, N\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let $(\phi_n)_{n \geq 1}$ be a uniformly bounded sequence in
 463 H^1 . Then for each integer $J \geq 1$, there exists a subsequence, still denoted by ϕ_n , and*

- 464 • for each $1 \leq j \leq J$, there exists a fixed profile $\psi^j \in H^1$;
- 465 • for each $1 \leq j \leq J$, there exists a sequence of time shifts $(t_n^j)_{n \geq 1} \subset \mathbb{R}$;
- 466 • for each $1 \leq j \leq J$, there exists a sequence of space shifts $(x_n^j)_{n \geq 1} \subset \mathbb{R}^N$;
- 467 • there exists a sequence of remainders $(W_n^J)_{n \geq 1} \subset H^1$;

468 such that

$$469 \quad (2.8) \quad \phi_n(x) = \sum_{j=1}^J e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^J(x).$$

471 *The time and space shifts have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq J$,*
 472 *we have*

$$473 \quad (2.9) \quad \lim_{n \rightarrow \infty} |t_n^j - t_n^k| + |x_n^j - x_n^k| = \infty.$$

475 *The remainder has the following asymptotic smallness property*

$$476 \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|e^{it\Delta} W_n^J\|_{S(\dot{H}^{\gamma_c}) \cap L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma_c}})} \right] = 0,$$

477

478 where γ_c is as in (1.5). Moreover, for fixed J and $\gamma \in [0, 1]$, we have the asymptotic
479 Pythagorean expansions

$$480 \quad \|\phi_n\|_{\dot{H}^\gamma}^2 = \sum_{j=1}^J \|\psi^j\|_{\dot{H}^\gamma}^2 + \|W_n^J\|_{\dot{H}^\gamma}^2 + o_n(1).$$

481
482 Finally, we may assume either $t_n^j \equiv 0$ or $t_n^j \rightarrow \pm\infty$, and either $x_n^j \equiv 0$ or $|x_n^j| \rightarrow \infty$.

483 In the next lemmas, we will construct nonlinear profiles associated to the linear
484 ones with either divergent time or divergent space shifts.

485 LEMMA 2.7 (Nonlinear profile with divergent time shift and no space translation).

486 Let $N \geq 2$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let $\psi \in H^1$ and $t_n \rightarrow \infty$.
487 Let $v_n : C((-T_*, T^*), H^1)$ denote the maximal solution to (1.2) with initial data

$$488 \quad (2.10) \quad v_n(0, x) = e^{-it_n\Delta}\psi(x).$$

490 Then for n sufficiently large, v_n exists globally backward in time, i.e., $T_* = \infty$. More-
491 over, we have for any $0 \leq T < T^*$,

$$492 \quad (2.11) \quad \lim_{n \rightarrow \infty} \|\langle \nabla \rangle (v_n - \psi_n)\|_{S((-\infty, T), L^2)} + \|v_n - \psi_n\|_{S((-\infty, T), \dot{H}^{\gamma_c})} = 0,$$

494 where

$$495 \quad (2.12) \quad \psi_n(t, x) := e^{i(t-t_n)\Delta}\psi(x).$$

497 In addition, we have

$$498 \quad (2.13) \quad \lim_{n \rightarrow \infty} \|v_n - \psi_n\|_{L_t^\infty((-\infty, T), H_x^1)} = 0.$$

500 Similarly, if $t_n \rightarrow -\infty$ and $v_n : C((-T_*, T^*), H^1)$ is the maximal solution to (1.2)
501 with initial data (2.10), then for n sufficiently large, v_n exists globally forward in time,
502 i.e., $T^* = \infty$. Moreover, we have for any $0 \leq T < T_*$,

$$503 \quad \lim_{n \rightarrow \infty} \|\langle \nabla \rangle (v_n - \psi_n)\|_{S((-T, \infty), L^2)} + \|v_n - \psi_n\|_{S((-T, \infty), \dot{H}^{\gamma_c})} = 0,$$

504 where ψ_n is as in (2.12). Moreover,

$$505 \quad \lim_{n \rightarrow \infty} \|v_n - \psi_n\|_{L_t^\infty((-T, \infty), H_x^1)} = 0.$$

506 *Proof.* We only treat the first point, the second point is similar. We see that ψ_n
507 satisfies

$$508 \quad i\partial_t \psi_n + \Delta \psi_n + |x|^{-b} |\psi_n|^\alpha \psi_n = e_n$$

509 with $e_n := |x|^{-b} |\psi_n|^\alpha \psi_n$. Since $v_n(0) = \psi_n(0)$, the result follows from the stability
510 given in Lemma 2.4 provided that

$$511 \quad (2.14) \quad \lim_{n \rightarrow \infty} \|\langle \nabla \rangle e_n\|_{S'((-\infty, T), L^2)} + \|e_n\|_{S'((-\infty, T), \dot{H}^{-\gamma_c})} = 0.$$

513 By Lemma 2.2, we have

$$\begin{aligned} 514 \quad & \|\langle \nabla \rangle e_n\|_{S'((-\infty, T), L^2)} = \|\langle \nabla \rangle (|x|^{-b} |\psi_n|^\alpha \psi_n)\|_{S'((-\infty, T), L^2)} \\ 515 \quad & = \|\langle \nabla \rangle (|x|^{-b} e^{it\Delta} \psi|^\alpha e^{it\Delta} \psi)\|_{S'((-\infty, T-t_n), L^2)} \\ 516 \quad & \lesssim \|e^{it\Delta} \psi\|_{L_t^\infty((-\infty, T-t_n), H_x^1)}^\theta \|e^{it\Delta} \psi\|_{S((-\infty, T-t_n), \dot{H}^{\gamma_c})}^{\alpha-\theta} \\ 517 \quad & \times \|\langle \nabla \rangle e^{it\Delta} \psi\|_{S((-\infty, T-t_n), L^2)} \rightarrow 0 \end{aligned}$$

519 as $n \rightarrow \infty$ as $\langle \nabla \rangle e^{it\Delta} \psi \in S(L^2)$ and $e^{it\Delta} \psi \in S(\dot{H}^{\gamma_c})$. Here we do not include the pairs
 520 $(\infty, 2)$ and $(\infty, \frac{2N}{N-2\gamma_c})$ into the definitions of L^2 and \dot{H}^{γ_c} admissibility, respectively.
 521 Similarly, we have

$$522 \quad \|e_n\|_{S'((-\infty, 0), \dot{H}^{-\gamma_c})} = \| |x|^{-b} |e^{it\Delta} \psi|^\alpha e^{it\Delta} \psi \|_{S'((-\infty, T-t_n), \dot{H}^{-\gamma_c})}$$

$$523 \quad \lesssim \|e^{it\Delta} \psi\|_{L_t^\infty((-\infty, T-t_n), H_x^1)}^\theta \|e^{it\Delta} \psi\|_{S((-\infty, T-t_n), \dot{H}^{\gamma_c})}^{\alpha+1-\theta} \rightarrow 0$$

525 as $n \rightarrow \infty$. This shows (2.14).

526 We next show (2.13). To see this, we have from (2.11),

$$527 \quad \|\langle \nabla \rangle \psi_n\|_{S((-\infty, T), L^2)} = \|\langle \nabla \rangle e^{it\Delta} \psi\|_{S((-\infty, T-t_n), L^2)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

528 and similarly for $\|\psi_n\|_{S((-\infty, T), \dot{H}^{\gamma_c})}$ that

$$529 \quad \lim_{n \rightarrow \infty} \|\langle \nabla \rangle v_n\|_{S((-\infty, T), L^2)} + \|v_n\|_{S((-\infty, T), \dot{H}^{\gamma_c})} = 0.$$

530 This together with Strichartz estimates, Lemma 2.2, and the fact that $\psi_n(t, x) =$
 531 $e^{it\Delta} v_n(0, x)$ imply $\|v_n\|_{L_t^\infty((-\infty, T), H_x^1)} \lesssim 1$. By Lemma 2.2, we have

$$532 \quad \|v_n - \psi_n\|_{L_t^\infty((-\infty, T), H_x^1)} \lesssim \|v_n\|_{L_t^\infty((-\infty, T), H_x^1)}^\theta \|v_n\|_{S((-\infty, T), \dot{H}^{\gamma_c})}^{\alpha-\theta} \|\langle \nabla \rangle v_n\|_{S((-\infty, T), L^2)}$$

533 which tends to zero as $n \rightarrow \infty$. The proof is complete. \square

534 **LEMMA 2.8** (Nonlinear profile with divergent space shift). *Let $N \geq 2$, $0 < b <$*
 535 *$\min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let $\psi \in H^1$ and $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^N$ satisfying*
 536 *$|x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Let $v_n : C((-T_*, T_*), H^1)$ denote the maximal solution to (1.2)*
 537 *with initial data*

$$538 \quad (2.15) \quad v_n(0, x) = e^{-it_n \Delta} \psi(x - x_n).$$

540 *Then for n sufficiently large, v_n exists globally in time, i.e., $T_* = T^* = \infty$. Moreover,*
 541 *we have*

$$542 \quad \lim_{n \rightarrow \infty} \|\langle \nabla \rangle (v_n - \psi_n)\|_{S(L^2)} + \|v_n - \psi_n\|_{S(\dot{H}^{\gamma_c})} = 0,$$

543 *where*

$$544 \quad (2.16) \quad \psi_n(t, x) := e^{i(t-t_n)\Delta} \psi(x - x_n).$$

546 *Remark 2.9.* The construction of nonlinear profiles with divergent space trans-
 547 *lations was first established by Miao, Murphy, and Zheng [45] for (1.2) with $\alpha = 2$*
 548 *and $N = 3$. This result was recently extended to (1.2) with $N \geq 2$ by Cardoso,*
 549 *Farah, Guzmán, and Murphy [7]. Here we give a refine result with a simple proof*
 550 *compared to the ones in [7, 45]. More precisely, for a linear profile with a divergent*
 551 *space shift, the associated nonlinear profile is close to the solution of the underlying*
 552 *linear Schrödinger equation.*

553 *Proof of Lemma 2.8.* As in the proof of Lemma 2.7, it suffices to show

$$554 \quad (2.17) \quad \lim_{n \rightarrow \infty} \|\langle \nabla \rangle e_n\|_{S'(L^2)} + \|e_n\|_{S'(\dot{H}^{-\gamma_c})} = 0.$$

555

556 To see this, we take $\varepsilon > 0$. We have

$$\begin{aligned}
557 \quad & \|\langle \nabla \rangle e_n\|_{S'(L^2)} = \|\langle \nabla \rangle (|x|^{-b} |\psi_n|^\alpha \psi_n)\|_{S'(L^2)} \\
558 \quad & = \|\langle \nabla \rangle (|x + x_n|^{-b} |e^{it\Delta} \psi|^\alpha e^{it\Delta} \psi)\|_{S'(L^2)} \\
559 \quad & \leq \|\langle \nabla \rangle (|x + x_n|^{-b} |e^{it\Delta} \psi|^\alpha e^{it\Delta} \psi)\|_{S'(L^2(B_R))} \\
560 \quad & \quad + \|\langle \nabla \rangle (|x + x_n|^{-b} |e^{it\Delta} \psi|^\alpha e^{it\Delta} \psi)\|_{S'(L^2(B_R^c))},
\end{aligned}$$

562 where $B_R := \{x \in \mathbb{R}^N : |x| \leq R\}$ and $B_R^c = \mathbb{R}^N \setminus B_R$ with $R > 0$ to be chosen later.

563 On B_R^c , by splitting $B_R^c = \Omega_1 \cup \Omega_2$ with

$$564 \quad \Omega_1 = \{x \in \mathbb{R}^N : |x| \geq R, |x + x_n| \leq 1\}, \quad \Omega_2 = \{x \in \mathbb{R}^N : |x| \geq R, |x + x_n| \geq 1\},$$

565 the same argument as in the proof of Lemma 2.2 implies that

$$566 \quad \|\langle \nabla \rangle (|x + x_n|^{-b} |\varphi|^\alpha \varphi)\|_{S'(L^2(B_R^c))} \lesssim \|\varphi\|_{L_t^\infty H_x^1(B_R^c)}^\theta \|\varphi\|_{S(\dot{H}^{\gamma_c}(B_R^c))}^{\alpha-\theta} \|\langle \nabla \rangle \varphi\|_{S(L^2(B_R^c))},$$

568 where

$$569 \quad \varphi(t, x) := e^{it\Delta} \psi(x).$$

570 As $\varphi \in S(\dot{H}^{\gamma_c})$ and $\langle \nabla \rangle \varphi \in S(L^2)$, we see that

$$571 \quad \|\varphi\|_{S(\dot{H}^{\gamma_c}(B_R^c))}, \|\langle \nabla \rangle \varphi\|_{S(L^2(B_R^c))} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

572 Note that it is crucial to exclude the pairs $(\infty, 2)$ and $(\infty, \frac{2N}{N-2\gamma_c})$ from the defini-
573 tions of L^2 and \dot{H}^{γ_c} admissible conditions, respectively. This shows that for $R_0 > 0$
574 sufficiently large,

$$575 \quad \|\langle \nabla \rangle (|x + x_n|^{-b} |\varphi|^\alpha \varphi)\|_{S'(L^2(B_{R_0}^c))} < \frac{\varepsilon}{4}$$

576 for all $n \geq 1$.

577 Next, for $x \in B_{R_0}$, as $|x_n| \rightarrow \infty$, we have $|x + x_n| \geq |x_n| - |x| \geq \frac{|x_n|}{2}$ for n
578 sufficiently large. It follows from Lemma 2.2 that

$$579 \quad \||x + x_n|^{-b} |\varphi|^\alpha \varphi\|_{S'(L^2(B_{R_0}))} \lesssim |x_n|^{-b} \|\varphi\|_{S'(L^2)}^\alpha \|\varphi\|_{S'(L^2)} \lesssim |x_n|^{-b} \|\varphi\|_{S(\dot{H}^{\gamma_c})}^\alpha \|\varphi\|_{S(L^2)} \rightarrow 0$$

581 as $n \rightarrow \infty$. Similarly, we have

$$\begin{aligned}
582 \quad & \|\nabla (|x + x_n|^{-b} |\varphi|^\alpha \varphi)\|_{S'(L^2(B_{R_0}))} \\
583 \quad & \lesssim \||x + x_n|^{-b} \nabla (|\varphi|^\alpha \varphi)\|_{S'(L^2(B_{R_0}))} + \||x + x_n|^{-b-1} |\varphi|^\alpha \varphi\|_{S'(L^2(B_{R_0}))} \\
584 \quad & \lesssim |x_n|^{-b} \|\nabla (|\varphi|^\alpha \varphi)\|_{S'(L^2)} + |x_n|^{-b-1} \|\varphi\|_{S'(L^2)}^\alpha \|\varphi\|_{S'(L^2)} \\
585 \quad & \lesssim |x_n|^{-b} \|\varphi\|_{S(\dot{H}^{\gamma_c})}^\alpha \|\nabla \varphi\|_{S(L^2)} + |x_n|^{-b-1} \|\varphi\|_{S(\dot{H}^{\gamma_c})}^\alpha \|\varphi\|_{S(L^2)} \rightarrow 0
\end{aligned}$$

587 as $n \rightarrow \infty$. Thus there exists $n_1 > 0$ sufficiently large such that for all $n \geq n_1$,

$$588 \quad \|\langle \nabla \rangle (|x + x_n|^{-b} |\varphi|^\alpha \varphi)\|_{S'(L^2(B_{R_0}))} < \frac{\varepsilon}{4},$$

589 hence

$$590 \quad \|\langle \nabla \rangle (|x + x_n|^{-b} |\varphi|^\alpha \varphi)\|_{S'(L^2)} < \frac{\varepsilon}{2}.$$

591 A similar argument show that for all $n \geq n_2$ with $n_2 > 0$ sufficiently large,

$$592 \quad \||x + x_n|^{-b} |\varphi|^\alpha \varphi\|_{S'(\dot{H}^{-\gamma_c})} < \frac{\varepsilon}{2}.$$

593 Therefore, we have for all $n \geq \max\{n_1, n_2\}$,

$$594 \quad \|\langle \nabla \rangle (|x + x_n|^{-b} |\varphi|^\alpha \varphi)\|_{S'(L^2)} + \||x + x_n|^{-b} |\varphi|^\alpha \varphi\|_{S'(\dot{H}^{-\gamma_c})} < \varepsilon$$

595 which proves (2.17). The proof is complete. \square

596 **2.4. Energy scattering.** In this section, we give the proof of the scattering
597 criterion given in Theorem 1.1. To this end, we need the following coercivity lemma.

598 LEMMA 2.10. *Let $N \geq 1$, $0 < b < \min\{2, N\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let*
599 *$f \in H^1$ satisfy*

$$600 \quad (2.18) \quad P(f)[M(f)]^{\sigma_c} \leq A < P(Q)[M(Q)]^{\sigma_c}$$

602 *for some constant $A > 0$. Then there exists $\nu = \nu(A, Q) > 0$ such that*

$$603 \quad (2.19) \quad G(f) \geq \nu \|\nabla f\|_{L^2}^2,$$

$$604 \quad (2.20) \quad E(f) \geq \frac{\nu}{2} \|\nabla f\|_{L^2}^2.$$

606 *Proof.* We write

$$607 \quad A = (1 - \rho)P(Q)[M(Q)]^{\sigma_c}$$

608 for some $\rho = \rho(A, Q) \in (0, 1)$. It follows from (2.3), (2.4), (2.5), and (2.18) that

$$\begin{aligned} 609 \quad & [P(f)]^{\frac{N\alpha+2b}{4}} \leq C_{\text{opt}} (P(f)[M(f)]^{\sigma_c})^{\frac{N\alpha-4+2b}{4}} \|\nabla f\|_{L^2}^{\frac{N\alpha+2b}{2}} \\ 610 \quad & = \frac{2(\alpha+2)}{N\alpha+2b} \left(\frac{P(f)[M(f)]^{\sigma_c}}{\|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma_c}} \right)^{\frac{N\alpha-4+2b}{4}} \|\nabla f\|_{L^2}^{\frac{N\alpha+2b}{2}} \\ 611 \quad & = \left(\frac{P(f)[M(f)]^{\sigma_c}}{P(Q)[M(Q)]^{\sigma_c}} \right)^{\frac{N\alpha-4+2b}{4}} \left(\frac{2(\alpha+2)}{N\alpha+2b} \|\nabla f\|_{L^2}^2 \right)^{\frac{N\alpha+2b}{4}} \\ 612 \quad & \leq (1 - \rho)^{\frac{N\alpha-4+2b}{4}} \left(\frac{2(\alpha+2)}{N\alpha+2b} \|\nabla f\|_{L^2}^2 \right)^{\frac{N\alpha+2b}{4}} \end{aligned}$$

614 which implies

$$615 \quad P(f) \leq \frac{2(\alpha+2)}{N\alpha+2b} (1 - \rho)^{\frac{N\alpha-4+2b}{N\alpha+2b}} \|\nabla f\|_{L^2}^2.$$

616 Thus we get

$$617 \quad G(f) = \|\nabla f\|_{L^2}^2 - \frac{N\alpha+2b}{2(\alpha+2)} P(f) \geq \left(1 - (1 - \rho)^{\frac{N\alpha-4+2b}{N\alpha+2b}} \right) \|\nabla f\|_{L^2}^2$$

618 which proves (2.19). As $N\alpha - 4 + 2b > 0$, we have

$$619 \quad E(f) = \frac{1}{2}G(f) + \frac{N\alpha-4+2b}{2(\alpha+2)}P(f) \geq \frac{1}{2}G(f)$$

620 which shows (2.20). The proof is complete. \square

621 We are now able to give the proof of Theorem 1.1.

622 *Proof of Theorem 1.1.* Let $u : [0, T^*) \times \mathbb{R}^N \rightarrow \mathbb{C}$ be a H^1 -solution to (1.2) satisfying
623 (1.11). By the conservation of mass and energy, we infer from (1.11) that

$$624 \quad \sup_{t \in [0, T^*)} \|\nabla u(t)\|_{L^2} \leq C(E, Q) < \infty.$$

625 By the local well-posedness given in Lemma 2.3, we have $T^* = \infty$.

626 Let $A > 0$ and $\delta > 0$. We define

$$627 \quad S(A, \delta) := \sup \left\{ \|u\|_{S([0, \infty), \dot{H}^{\gamma_c})} : u \text{ is a solution to (1.2) satisfying (2.21)} \right\},$$

628 where

$$629 \quad (2.21) \quad \sup_{t \in [0, \infty)} P(u(t))[M(u(t))]^{\sigma_c} \leq A, \quad E(u)[M(u)]^{\sigma_c} \leq \delta.$$

630 Thanks to the scattering condition (see again Lemma 2.3) and the definition of $S(A, \delta)$,
631 Theorem 1.1 is reduced to show the following proposition.

632 **PROPOSITION 2.11.** *Let $N \geq 2$, $0 < b < \min\{2, \frac{N}{2}\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. If
633 $A < P(Q)[M(Q)]^{\sigma_c}$, then for all ¹ $\delta > 0$, $S(A, \delta) < \infty$.*

634 The proof of Proposition 2.11 is based on the concentration/compactness and rigidity
635 argument introduced by Kenig and Merle [40] (see also [21]). The main difficulty
636 comes from the fact that the potential energy $P(u(t))$ is not conserved along the time
637 evolution of (1.2). To overcome the difficulty, we establish a Pythagorean decomposi-
638 tion along the bounded INLS flow (see Lemma 2.12). In the context of the standard
639 NLS, a similar result was shown by Guevara in [36, Lemma 3.9] (see also [19]).

640 The proof of Proposition 2.11 is done by several steps.

641 **Step 1. Small data scattering.** By (2.20), we have

$$642 \quad \|u_0\|_{\dot{H}^{\gamma_c}}^{\frac{2}{\nu}} \leq \|\nabla u_0\|_{L^2}^2 \|u_0\|_{L^2}^{2\sigma_c} \leq \frac{2}{\nu} E(u_0)[M(u_0)]^{\sigma_c} \leq \frac{2\delta}{\nu}.$$

643 By taking $\delta > 0$ sufficiently small, we see that $\|u_0\|_{\dot{H}^{\gamma_c}}$ is small which, by the small
644 data scattering given in Lemma 2.3, implies $S(A, \delta) < \infty$.

645 **Step 2. Existence of a critical solution.** Assume by contradiction that $S(A, \delta) =$
646 ∞ for some $\delta > 0$. By Step 1,

$$647 \quad (2.22) \quad \delta_c := \delta_c(A) := \inf \{ \delta > 0 : S(A, \delta) = \infty \}$$

648 is well-defined and positive. From the definition of δ_c , we have the following observa-
649 tions:

650 (1) If u is a solution to (1.2) satisfying

$$651 \quad \sup_{t \in [0, \infty)} P(u(t))[M(u(t))]^{\sigma_c} \leq A, \quad E(u)[M(u)]^{\sigma_c} < \delta_c,$$

652 then $\|u\|_{S([0, \infty), \dot{H}^{\sigma_c})} < \infty$ and the solution scatters in H^1 forward in time.

653 (2) There exists a sequence of solution u_n to (1.2) with initial data $u_{n,0}$ such that

$$654 \quad \sup_{t \in [0, \infty)} P(u_n(t))[M(u_n(t))]^{\sigma_c} \leq A \text{ for all } n,$$

$$655 \quad (2.23) \quad E(u_n)[M(u_n)]^{\sigma_c} \searrow \delta_c \text{ as } n \rightarrow \infty,$$

$$656 \quad \|u_n\|_{S([0, \infty), \dot{H}^{\gamma_c})} = \infty \text{ for all } n.$$

657 We will prove that there exists a critical solution u_c to (1.2) with initial data $u_{c,0}$
658 satisfying

$$659 \quad (2.24) \quad \begin{aligned} & M(u_c) = 1, \\ & \sup_{t \in [0, \infty)} P(u_c(t)) \leq A, \\ & E(u_c) = \delta_c, \\ & \|u_c\|_{S([0, \infty), \dot{H}^{\gamma_c})} = \infty. \end{aligned}$$

¹Note the energy is positive due to Lemma 2.10.

662 To see this, we consider the sequence $(u_{n,0})_{n \geq 1}$. Thanks to the scaling (1.4), we can
 663 assume that $M(u_{n,0}) = 1$ for all n . By the conservation of mass and energy, (2.23)
 664 becomes

$$\begin{aligned} & M(u_{n,0}) = 1 \text{ for all } n, \\ & \sup_{t \in [0, \infty)} P(u_n(t)) \leq A \text{ for all } n, \\ 665 \quad (2.25) \quad & E(u_{n,0}) \searrow \delta_c \text{ as } n \rightarrow \infty, \\ 666 \quad & \|u_n\|_{S([0, \infty), \dot{H}^{\gamma_c})} = \infty \text{ for all } n. \end{aligned}$$

667 Since $(u_{n,0})_{n \geq 1}$ is bounded in H^1 , we apply the linear profile decomposition to $u_{n,0}$
 668 and get

$$669 \quad (2.26) \quad u_{n,0}(x) = \sum_{j=1}^J e^{-it_n^j \Delta} \psi^j(x - x_n^j) + W_n^J(x)$$

670
 671 with the following properties:

$$672 \quad (2.27) \quad 1 \leq j \neq k \leq J, \quad \lim_{n \rightarrow \infty} |t_n^j - t_n^k| + |x_n^j - x_n^k| = \infty,$$

$$673 \quad (2.28) \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|e^{it \Delta} W_n^J\|_{S(\dot{H}^{\gamma_c}) \cap L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma_c}})} \right] = 0,$$

674
 675 and for fixed J and $\gamma \in [0, 1]$,

$$676 \quad (2.29) \quad \|u_{n,0}\|_{\dot{H}^\gamma}^2 = \sum_{j=1}^J \|\psi^j\|_{\dot{H}^\gamma}^2 + \|W_n^J\|_{\dot{H}^\gamma}^2 + o_n(1).$$

677
 678 Moreover, we also have the following Pythagorean expansions of the potential and
 679 total energies:

$$680 \quad (2.30) \quad P(u_{n,0}) = \sum_{j=1}^J P(e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j)) + P(W_n^J) + o_n(1),$$

$$681 \quad (2.31) \quad E(u_{n,0}) = \sum_{j=1}^J E(e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j)) + E(W_n^J) + o_n(1).$$

682
 683 For the proof of the above expansions, we refer to [26] (see also [27]). We now
 684 define the nonlinear profiles $v^j : I^j \times \mathbb{R}^N \rightarrow \mathbb{C}$ associated to ψ^j , t_n^j , and x_n^j as follows:

- 685 • If $x_n^j \equiv 0$ and $t_n^j \equiv 0$, then v^j is the maximal lifespan solution to (1.2) with
 686 initial data $v^j|_{t=0} = \psi^j$.
- 687 • If $x_n^j \equiv 0$ and $t_n^j \rightarrow -\infty$, then v^j is the maximal lifespan solution to (1.2)
 688 that scatters to $e^{it \Delta} \psi^j$ as $t \rightarrow \infty$ (Such a solution exists due to Lemma 2.7).
 689 In particular, $\|v^j\|_{S((0, \infty), \dot{H}^{\gamma_c})} < \infty$ and $\|v^j(-t_n^j) - e^{-it_n^j \Delta} \psi^j\|_{H^1} \rightarrow 0$ as
 690 $n \rightarrow \infty$.
- 691 • If $x_n^j \equiv 0$ and $t_n^j \rightarrow \infty$, then v^j is the maximal lifespan solution to (1.2) that
 692 scatters to $e^{it \Delta} \psi^j$ as $t \rightarrow -\infty$. In particular, $\|v^j\|_{S((-\infty, 0), \dot{H}^{\gamma_c})} < \infty$ and
 693 $\|v^j(-t_n^j) - e^{-it_n^j \Delta} \psi^j\|_{H^1} \rightarrow 0$ as $n \rightarrow \infty$.
- 694 • If $|x_n^j| \rightarrow \infty$, then we simply take $v^j(t) = e^{it \Delta} \psi^j$.

695 For each $j, n \geq 1$, we introduce $v_n^j : I_n^j \times \mathbb{R}^N \rightarrow \mathbb{C}$ defined by
 696 • if $x_n^j \equiv 0$, then $v_n^j(t) := v^j(t - t_n^j)$, where $I_n^j := \{t \in \mathbb{R} : t - t_n^j \in I^j\}$.
 697 • if $|x_n^j| \rightarrow \infty$, we define v_n^j a solution to (1.2) with initial data $v_n^j(0, x) =$
 698 $v^j(-t_n^j, x - x_n^j) = e^{-it_n^j \Delta} \psi^j(x - x_n^j)$. It follows from Lemma 2.8 that for
 699 n sufficiently large, v_n^j exists globally in time and scatters in H^1 in both
 700 directions.

701 We have from the definition of v_n^j and the continuity of the linear flow that

$$703 \quad (2.32) \quad \|v_n^j(0) - e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j)\|_{H^1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

704 Thus we rewrite (2.26) as

$$705 \quad (2.33) \quad u_{n,0}(x) = \sum_{j=1}^J v_n^j(0, x) + \tilde{W}_n^J(x),$$

706 where

$$708 \quad \tilde{W}_n^J(x) = \sum_{j=1}^J e^{-it_n^j \Delta} \psi^j(x - x_n^j) - v_n^j(0, x) + W_n^J(x).$$

709 By Strichartz estimates, we have

$$710 \quad \|e^{it\Delta} \tilde{W}_n^J\|_{S(\dot{H}^{\gamma_c}) \cap L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma_c}})}$$

$$711 \quad \lesssim \sum_{j=1}^J \|e^{-it_n^j \Delta} \psi^j(\cdot - x_n^j) - v_n^j(0)\|_{H^1} + \|e^{it\Delta} W_n^J\|_{S(\dot{H}^{\gamma_c}) \cap L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma_c}})}$$

712 which, by (2.28) and (2.32), implies that

$$714 \quad (2.34) \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|e^{it\Delta} \tilde{W}_n^J\|_{S(\dot{H}^{\gamma_c}) \cap L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma_c}})} \right] = 0.$$

716 Using the fact that

$$717 \quad \|\nabla f\|_{L^2}^2 - \|\nabla g\|_{L^2}^2 \lesssim \|\nabla f - \nabla g\|_{L^2} (\|\nabla f\|_{L^2} + \|\nabla g\|_{L^2})$$

718 and (see [27, Lemma 4.3])

$$719 \quad (2.35) \quad |P(f) - P(g)| \lesssim \|f - g\|_{L^{\alpha+2}} (\|f\|_{L^{\alpha+2}}^{\alpha+1} + \|g\|_{L^{\alpha+2}}^{\alpha+1}) + \|f - g\|_{L^r} (\|f\|_{L^r}^{\alpha+1} + \|g\|_{L^r}^{\alpha+1})$$

721 for some $\frac{2N\alpha}{N-b} < r < 2^*$, where

$$722 \quad (2.36) \quad 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2, \end{cases}$$

724 we infer from (2.31), Sobolev embedding, and (2.32) that

$$725 \quad (2.37) \quad E(u_{n,0}) = \sum_{j=1}^J E(v_n^j(0)) + E(\tilde{W}_n^J) + o_n(1).$$

727 Next, we show the following Pythagorean expansion along the bounded INLS flow
 728 (see [36, Lemma 3.9] for a similar result in the context of NLS).

729 LEMMA 2.12 (Pythagorean expansion along the bounded INLS flow). *Let $T \in$*
 730 *$(0, \infty)$ be a fixed time. Assume that for all $n \geq 1$, $u_n(t) := \text{INLS}(t)u_{n,0}$ exists up to*
 731 *time T and satisfies*

$$732 \quad (2.38) \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|\nabla u_n(t)\|_{L^2} < \infty,$$

734 *where $\text{INLS}(t)f$ denotes the solution to (1.2) with initial data f at time $t = 0$. We*
 735 *consider the nonlinear profile (2.33). Denote $\tilde{W}_n^J(t) := \text{INLS}(t)\tilde{W}_n^J$. Then for all*
 736 *$t \in [0, T]$,*

$$737 \quad (2.39) \quad \|\nabla u_n(t)\|_{L^2}^2 = \sum_{j=1}^J \|\nabla v_n^j(t)\|_{L^2}^2 + \|\nabla \tilde{W}_n^J(t)\|_{L^2}^2 + o_{J,n}(1),$$

739 *where $o_{J,n}(1) \rightarrow 0$ as $J, n \rightarrow \infty$ uniformly on $0 \leq t \leq T$. In particular, we have for*
 740 *all $t \in [0, T]$,*

$$741 \quad (2.40) \quad P(u_n(t)) = \sum_{j=1}^J P(v_n^j(t)) + P(\tilde{W}_n^J(t)) + o_{J,n}(1).$$

743 *Proof.* By (2.29), there exists J_0 large enough such that $\|\psi^j\|_{H^1}$ sufficiently small
 744 for all $j \geq J_0 + 1$. By the triangle inequality using (2.32), we see that for n large,
 745 $\|v_n^j(0)\|_{H^1}$ is small which, by the small data theory, implies that v_n^j exists globally
 746 in time and scatters in H^1 in both directions. Moreover, we can assume that for all
 747 $1 \leq j \leq J_0$, $x_n^j \equiv 0$ since otherwise, if $|x_n^j| \rightarrow \infty$, then by Lemma 2.8, we have for n
 748 large, v_n^j exists globally in time and scatters in H^1 in both directions. In particular,
 749 we have for all $j \geq J_0 + 1$,

$$750 \quad (2.41) \quad \|v_n^j\|_{S(\dot{H}^{\gamma_c})} < \infty$$

752 for n large. We reorder the first J_0 profiles and let $0 \leq J_2 \leq J_0$ such that

- 753 • for any $1 \leq j \leq J_2$, the time shifts $t_n^j \equiv 0$ for all n . Here $J_2 \equiv 0$ means
 754 that there is no j in this case. Note that by the pairwise divergence property
 755 (2.9), we have $J_2 \leq 1$.
- 756 • for any $J_2 + 1 \leq j \leq J_0$, the time shifts $|t_n^j| \rightarrow \infty$ as $n \rightarrow \infty$. Here $J_2 = J_0$
 757 means that there is no j in this case.

758 In the following, we only consider the case $J_2 = 1$. The one for $J_2 = 0$ is treated
 759 similarly (even simpler). Fix $T \in (0, \infty)$ and assume that $u_n(t) = \text{INLS}(t)u_{n,0}$ exists
 760 up to time T and satisfies (2.38). We observe that for $2 \leq j \leq J_0$,

$$761 \quad (2.42) \quad \|v_n^j\|_{S([0, T], \dot{H}^{\gamma_c})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

763 Indeed, if $t_n^j \rightarrow \infty$, then as $\|v^j\|_{S((-\infty, 0), \dot{H}^{\gamma_c})} < \infty$, we have

$$764 \quad \|v_n^j\|_{S([0, T], \dot{H}^{\gamma_c})} = \|v^j\|_{S([-t_n^j, T-t_n^j], \dot{H}^{\gamma_c})} \rightarrow 0$$

765 as $n \rightarrow \infty$. Note that we do not consider $(\infty, \frac{2N}{N-2\gamma_c})$ as a \dot{H}^{γ_c} -admissible pair. A
 766 similar argument goes for $t_n^j \rightarrow -\infty$.

767 Moreover, for $2 \leq j \leq J_0$, we have for all $2 < r \leq 2^*$,

$$768 \quad (2.43) \quad \|v_n^j\|_{L_t^\infty([0, T], L_x^r)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

770 In fact, we have

$$\begin{aligned}
771 \quad & \|v_n^j\|_{L_t^\infty([0,T],L_x^r)} \\
772 \quad & \leq \|e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j)\|_{L_t^\infty([0,T],L_x^r)} + \|v_n^j - e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j)\|_{L_t^\infty([0,T],L_x^r)} \\
773 \quad & \leq \|e^{i(t-t_n^j)\Delta}\psi^j\|_{L_t^\infty([0,T],L_x^r)} + C\|v_n^j - e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j)\|_{L_t^\infty([0,T],H_x^1)}.
\end{aligned}$$

775 By the decay of the linear flow, the first term tends to zero as n tends to infinity due
776 to $|t_n^j| \rightarrow \infty$. For the second term, we use the Duhamel formula

$$777 \quad v_n^j(t) = e^{it\Delta}v_n^j(0) + i \int_0^t e^{i(t-s)\Delta}|x|^{-b}|v_n^j(s)|^\alpha v_n^j(s)ds,$$

778 Strichartz estimates, and Lemma 2.2 to have

$$\begin{aligned}
779 \quad & \|v_n^j\|_{L_t^\infty([0,T],H_x^1)} + \|\langle \nabla \rangle v_n^j\|_{S([0,T],L^2)} \\
780 \quad & \lesssim \|v_n^j(0)\|_{H^1} + \|v_n^j\|_{L_t^\infty([0,T],H_x^1)}^\theta \|v_n^j\|_{S([0,T],\dot{H}^{\gamma_c})}^{\alpha-\theta} \|\langle \nabla \rangle v_n^j\|_{S([0,T],L^2)} \\
781 \quad & \lesssim \|e^{-it_n^j\Delta}\psi^j\|_{H^1} + 1 \\
782 \quad & + \left(\|v_n^j\|_{L_t^\infty([0,T],H_x^1)} + \|\langle \nabla \rangle v_n^j\|_{S([0,T],L^2)}\right)^{1+\theta} \|v_n^j\|_{S([0,T],\dot{H}^{\gamma_c})}^{\alpha-\theta}.
\end{aligned}$$

784 It follows from (2.42) that

$$785 \quad (2.44) \quad \|v_n^j\|_{L_t^\infty([0,T],H_x^1)} + \|\langle \nabla \rangle v_n^j\|_{S([0,T],L^2)} \lesssim 1.$$

787 Similarly, we have

$$\begin{aligned}
788 \quad & \|v_n^j - e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j)\|_{L_t^\infty([0,T],H_x^1)} \\
789 \quad & \lesssim \|e^{it\Delta}v_n^j(0) - e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j)\|_{L_t^\infty([0,T],H_x^1)} \\
790 \quad & \quad + \|v_n^j\|_{L_t^\infty([0,T],H_x^1)}^\theta \|v_n^j\|_{S([0,T],\dot{H}^{\gamma_c})}^{\alpha-\theta} \|\langle \nabla \rangle v_n^j\|_{S([0,T],L^2)} \\
791 \quad & \lesssim \|v_n^j(0) - e^{-it_n^j\Delta}\psi^j(\cdot - x_n^j)\|_{H^1} \\
792 \quad & \quad + \|v_n^j\|_{L_t^\infty([0,T],H_x^1)}^\theta \|v_n^j\|_{S([0,T],\dot{H}^{\gamma_c})}^{\alpha-\theta} \|\langle \nabla \rangle v_n^j\|_{S([0,T],L^2)}
\end{aligned}$$

794 which, by (2.32), (2.42), and (2.44), implies

$$795 \quad \|v_n^j - e^{i(t-t_n^j)\Delta}\psi^j(\cdot - x_n^j)\|_{L_t^\infty([0,T],H_x^1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

796 We thus prove (2.43).

797 Denote

$$798 \quad B := \max \left\{ 1, \limsup_{n \rightarrow \infty} \sup_{t \in [0,T]} \|\nabla u_n(t)\|_{L^2} \right\} < \infty.$$

799 and let T^1 the maximal forward time such that

$$800 \quad \sup_{t \in [0,T^1]} \|\nabla v^1(t)\|_{L^2} \leq 2B.$$

801 In what follows, we will show that for all $t \in [0, T^1]$,

$$802 \quad (2.45) \quad \|\nabla u_n(t)\|_{L^2}^2 = \sum_{j=1}^J \|\nabla v_n^j(t)\|_{L^2}^2 + \|\nabla \tilde{W}_n^J(t)\|_{L^2}^2 + o_{J,n}(1),$$

803

804 where $o_{J,n}(1) \rightarrow 0$ as $J, n \rightarrow \infty$ uniformly on $0 \leq t \leq T^1$. We see that (2.45) implies
 805 (2.39) as $T^1 \geq T$. In fact, if $T^1 < T$, then by (2.45),

$$\begin{aligned} 806 \quad \sup_{t \in [0, T^1]} \|\nabla v^1(t)\|_{L^2} &= \sup_{t \in [0, T^1]} \|\nabla v_n^1(t)\|_{L^2} \leq \sup_{t \in [0, T^1]} \|\nabla u_n(t)\|_{L^2} \\ 807 &\leq \sup_{t \in [0, T]} \|\nabla u_n(t)\|_{L^2} \leq B. \end{aligned}$$

808
 809 Note that $t_n^1 \equiv 0$. By the continuity, it contradicts the maximality of T^1 .

810 We estimate $\|v_n^1\|_{S([0, T^1], \dot{H}^{\gamma_c})}$ as follows. For $N \geq 3$, by interpolation between
 811 endpoints and Sobolev embedding, we have

$$\begin{aligned} 812 \quad \|v_n^1\|_{S([0, T^1], \dot{H}^{\gamma_c})} &= \|v^1\|_{S([0, T^1], \dot{H}^{\gamma_c})} \\ 813 &\lesssim \|v^1\|_{L_t^{\frac{2}{1-\gamma_c}}([0, T^1], L_x^{\frac{2N}{N-2}})} + \|v^1\|_{L_t^\infty([0, T^1], L_x^{\frac{2N}{N-2}\gamma_c})} \\ 814 &\lesssim \|v^1\|_{L_t^{\frac{2}{1-\gamma_c}}([0, T^1], L_x^{\frac{2N}{N-2}})} + \|v^1\|_{L_t^\infty([0, T^1], L_x^2)}^{\frac{1-\gamma_c}{2}} \|v^1\|_{L_t^\infty([0, T^1], L_x^{\frac{2N}{N-2}})}^{\gamma_c} \\ 815 &\lesssim (T^1)^{\frac{1-\gamma_c}{2}} \|\nabla v^1\|_{L_t^\infty([0, T^1], L_x^2)} + C \|\nabla v^1\|_{L_t^\infty([0, T^1], L_x^{\frac{2N}{N-2}})}^{\gamma_c} \\ 816 &\lesssim (T^1)^{\frac{1-\gamma_c}{2}} B + CB^{\gamma_c}. \end{aligned}$$

818 Here we have use the conservation of mass and the choice of v^1 to have that for all
 819 $t \in [0, T^1]$,

$$820 \quad \|v^1(t)\|_{L^2} = \lim_{n \rightarrow \infty} \|v^1(-t_n^1)\|_{L^2} = \lim_{n \rightarrow \infty} \|e^{-it_n^1 \Delta} \psi^1\|_{L^2} = \|\psi^1\|_{L^2} \leq \|u_{n,0}\|_{L^2} \leq 1.$$

821 When $N = 2$, a similar estimate holds by interpolating between $(\infty, \frac{2}{1-\gamma_c})$ and
 822 $(\frac{2}{1-\gamma_c}, r)$ with r sufficiently large and using Sobolev embedding. This shows that

$$823 \quad (2.46) \quad \|v_n^1\|_{S([0, T^1], \dot{H}^{\gamma_c})} \leq C(T^1, B).$$

825 Now we define the approximation

$$826 \quad \tilde{u}_n^J(t, x) := \sum_{j=1}^J v_n^j(t, x).$$

827 We have

$$828 \quad u_{n,0}(x) - \tilde{u}_n^J(0, x) = \tilde{W}_n^J(x).$$

829 By (2.34), we have

$$830 \quad (2.47) \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|e^{it\Delta}(u_{n,0} - \tilde{u}_n^J(0))\|_{S(\dot{H}^{\gamma_c}) \cap L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2}\gamma_c})} \right] = 0.$$

832 We also have

$$833 \quad i\partial_t \tilde{u}_n^J + \Delta \tilde{u}_n^J + |x|^{-b} |\tilde{u}_n^J|^\alpha \tilde{u}_n^J = \tilde{e}_n^J,$$

834 where

$$835 \quad \tilde{e}_n^J = \sum_{j=1}^J F(v_n^j) - F\left(\sum_{j=1}^J v_n^j\right)$$

836 with $F(u) := |x|^{-b}|u|^\alpha u$. We also have the following properties of the approximate
 837 solutions.

838 LEMMA 2.13. *The functions \tilde{u}_n^J and \tilde{e}_n^J satisfy*

$$839 \quad (2.48) \quad \limsup_{n \rightarrow \infty} \left(\|\tilde{u}_n^J\|_{L_t^\infty([0, T^1], H_x^1)} + \|\tilde{u}_n^J\|_{S([0, T^1], \dot{H}^{\gamma_c})} \right) \lesssim 1$$

840 uniformly in J and

$$842 \quad (2.49) \quad \lim_{J \rightarrow \infty} \lim_{n \rightarrow \infty} \|\langle \nabla \rangle \tilde{e}_n^J\|_{S'([0, T^1], L^2)} + \|\tilde{e}_n^J\|_{S'([0, T^1], \dot{H}^{-\gamma_c})} = 0.$$

844 *Proof.* The boundedness of $\|\tilde{u}_n^J\|_{S([0, T^1], \dot{H}^{\gamma_c})}$ follows from (2.41), (2.42), and (2.46). \blacksquare

845 The boundedness of $\|\tilde{u}_n^J\|_{L_t^\infty([0, T^1], L_x^2)}$ follows from (2.29) and the fact that

$$846 \quad \|v_n^j(t)\|_{L^2} = \|v^j(t - t_n^j)\|_{L^2} = \lim_{n \rightarrow \infty} \|v(-t_n^j)\|_{L^2} = \lim_{n \rightarrow \infty} \|e^{-it_n^j \Delta} \psi^j\|_{L^2} = \|\psi^j\|_{L^2}.$$

847 To see the boundedness of $\|\nabla \tilde{u}_n^J\|_{L_t^\infty([0, T^1], L_x^2)}$, we proceed as follows. For $j \geq J_0$, by
848 (2.41), we split $[0, T^1]$ into finite subintervals I_k , $k = 1, \dots, M$ such that $\|v_n^j\|_{S(I_k, \dot{H}^{\gamma_c})}$
849 is small. By Duhamel's formula, Strichartz estimates, and Lemma 2.2, we have

$$850 \quad \|\nabla v_n^j\|_{L_t^\infty(I_k, L_x^2)} \lesssim \|\nabla v_n^j(t_k)\|_{L^2}, \quad I_k = [t_k, t_{k+1}], \quad k = 1, \dots, M.$$

851 Summing over these finite intervals, we get

$$852 \quad \|\nabla v_n^j\|_{L_t^\infty([0, T^1], L_x^2)} \lesssim \|\nabla v_n^j(0)\|_{L^2}.$$

853 For $2 \leq j \leq J_0$, we have from the Duhamel formula, Strichartz estimates, Lemma 2.2,
854 and (2.42), we have

$$855 \quad \|\nabla v_n^j\|_{L_t^\infty([0, T^1], L_x^2)} \lesssim \|\nabla v_n^j(0)\|_{L^2}$$

856 for n sufficiently large. Thus we have

$$857 \quad \begin{aligned} \|\nabla \tilde{u}_n^J\|_{L_t^\infty([0, T^1], L_x^2)}^2 &\leq \|\nabla v^1\|_{L_t^\infty([0, T^1], L_x^2)}^2 + \sum_{j=2}^J \|\nabla v_n^j\|_{L_t^\infty([0, T^1], L_x^2)}^2 \\ 858 \quad &\lesssim B^2 + \sum_{j=2}^J \|\nabla v_n^j(0)\|_{L^2}^2 \\ 859 \quad &\lesssim B^2 + \sum_{j=2}^J \|\nabla \psi^j\|_{L^2}^2 + o_n(1) \\ 860 \quad &\lesssim B^2 + \|\nabla u_{n,0}\|_{L^2}^2 + o_n(1) \\ 861 \quad &\lesssim B^2 + o_n(1). \end{aligned}$$

863 This shows the boundedness of $\|\nabla \tilde{u}_n^J\|_{L_t^\infty([0, T^1], L_x^2)}$ and we prove (2.48). To see (2.49),
864 we follow from the same argument as in [27, Claim 1 (6.23)]. We thus omit the details. \square

865 Thanks to (2.47) and Lemma 2.13, the stability given in Lemma 2.4 (see also
866 Remark 2.5) implies

$$867 \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|u_n - \tilde{u}_n^J\|_{S([0, T^1], \dot{H}^{\gamma_c}) \cap L_t^\infty([0, T^1], L_x^{\frac{2N}{N-2\gamma_c}})} \right] = 0.$$

868 By interpolating between endpoints and using Sobolev embedding, we infer that

$$870 \quad \begin{aligned} \|u_n - \tilde{u}_n^J\|_{L_t^\infty([0, T^1], L_x^{\alpha+2}) \cap L_t^\infty([0, T^1], L_x^r)} \\ 871 \quad \lesssim \|u_n - \tilde{u}_n^J\|_{L_t^\infty([0, T^1], L_x^{\frac{2N}{N-2\gamma_c}})} \|\langle \nabla \rangle (u_n - \tilde{u}_n^J)\|_{L_t^\infty([0, T^1], L_x^2)} \rightarrow 0 \\ 872 \end{aligned}$$

873 as $J, n \rightarrow \infty$, where r is an exponent satisfying $\frac{2N}{N-2\gamma_c} < \frac{N(\alpha+2)}{N-b} < r < 2^*$. This
 874 estimate together with (2.35) yield

$$875 \quad (2.50) \quad |P(u_n(t)) - P(\tilde{u}_n^J(t))| \rightarrow 0$$

877 as $J, n \rightarrow \infty$ uniformly on $0 \leq t \leq T^1$. On the other hand, we have from the same
 878 argument as in [27, Proposition 5.3] using (2.43) that for all $t \in [0, T^1]$,

$$879 \quad (2.51) \quad P(\tilde{u}_n^J(t)) = \sum_{j=1}^J P(v_n^j(t)) + o_{J,n}(1) = \sum_{j=1}^J P(v_n^j(t)) + P(\tilde{W}_n^J(t)) + o_{J,n}(1).$$

881 Here we have used the fact that $P(\tilde{W}_n^J(t)) = o_{J,n}(1)$ uniformly on $0 \leq t \leq T^1$. In
 882 fact, by the Duhamel formula and Lemma 2.2, we have

$$883 \quad \|\tilde{W}_n^J(t)\|_{S(\dot{H}^{\gamma_c})} \leq \|e^{it\Delta}\tilde{W}_n^J\|_{S(\dot{H}^{\gamma_c})} + C\|\tilde{W}_n^J(t)\|_{L_t^\infty(\mathbb{R}, H_x^1)}^\theta \|\tilde{W}_n^J(t)\|_{S(\dot{H}^{\gamma_c})}^{\alpha+1-\theta}$$

884 for some $\theta > 0$ sufficiently small. Since $\|\tilde{W}_n^J(t)\|_{L_t^\infty(\mathbb{R}, H_x^1)} \lesssim 1$ (by the small data
 885 theory), the continuity argument together with (2.28) imply

$$886 \quad (2.52) \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|\tilde{W}_n^J(t)\|_{S(\dot{H}^{\gamma_c})} \right] = 0.$$

888 Thanks to (2.52), Strichartz estimates, and (2.34), we have

$$889 \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|\tilde{W}_n^J(t)\|_{L_t^\infty(\mathbb{R}, L_x^{\frac{2N}{N-2\gamma_c}})} \right] = 0$$

890 which together with (2.35) yield

$$891 \quad \lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \sup_{t \in \mathbb{R}} P(\tilde{W}_n^J(t)) \right] = 0.$$

892 Moreover, by the conservation of energy, we have

$$893 \quad E(u_n(t)) = E(u_{n,0}) = \sum_{j=1}^J E(v_n^j(0)) + E(\tilde{W}_n^J) + o_n(1)$$

$$894 \quad (2.53) \quad = \sum_{j=1}^J E(v_n^j(t)) + E(\tilde{W}_n^J(t)) + o_{J,n}(1).$$

896 Collecting (2.50), (2.51), and (2.53), we prove (2.45). The proof is complete. \square

897 We come back to the proof of Proposition 2.11. We will consider two cases.

898 **Case 1. More than one non-zero profiles.** We have

$$899 \quad M(v_n^j(t)) = M(v_n^j(0)) = M(e^{-it_n^j \Delta} \psi^j) = M(\psi^j) < 1, \quad \forall j \geq 1.$$

900 By (2.23) and (2.40), we have

$$901 \quad \sup_{t \in [0, \infty)} P(v_n^j(t)) [M(v_n^j(t))]^{\sigma_c} < A, \quad \forall j \geq 1.$$

902 Here we note that by (2.39), $\|\nabla v_n^j(t)\|_{L^2}$ is bounded uniformly which implies v_n^j exists
 903 globally in time. By Lemma 2.10, we have $E(v_n^j(t)) \geq 0$, hence

$$904 \quad E(v_n^j(t))[M(v_n^j(t))]^{\sigma_c} < \delta_c, \quad \forall j \geq 1.$$

905 By Item (1) (see after (2.22)), we have

$$906 \quad \|v_n^j\|_{S([0,\infty),\dot{H}^{\gamma_c})} < \infty, \quad \forall j \geq 1.$$

907 We can approximate u_n by

$$908 \quad u_n^J(t, x) := \sum_{j=1}^J v_n^j(t)$$

909 and get for J sufficiently large that

$$910 \quad \|u_n\|_{S([0,\infty),\dot{H}^{\gamma_c})} < \infty$$

911 which is a contradiction.

912 **Case 2. Only one non-zero profile.** We must have only one non-zero profile, i.e.,

$$913 \quad u_{n,0}(x) = e^{-it_n\Delta}\psi^1(x - x_n^1) + W_n(x), \quad \lim_{n \rightarrow \infty} \|e^{it\Delta}W_n\|_{S([0,\infty),\dot{H}^{\gamma_c})} = 0.$$

914 We note that t_n^1 cannot tend to $-\infty$. Indeed, if $t_n^1 \rightarrow -\infty$, then we have

$$915 \quad \|e^{it\Delta}u_{n,0}\|_{S([0,\infty),\dot{H}^{\gamma_c})} \leq \|e^{it\Delta}\psi^1\|_{S([-t_n^1,\infty),\dot{H}^{\gamma_c})} + \|e^{it\Delta}W_n\|_{S([0,\infty),\dot{H}^{\gamma_c})} \rightarrow 0$$

916 as $n \rightarrow \infty$. By the Duhamel formula, Lemma 2.2, and the continuity argument,
 917 $\|u_n\|_{S([0,\infty),\dot{H}^{\gamma_c})} < \infty$ for n sufficiently large which is a contradiction.

918 We claim that $x_n^1 \equiv 0$. Otherwise, if $|x_n^1| \rightarrow \infty$, then, by Lemma 2.8, for n
 919 large, there exist global solutions v_n to (1.2) satisfying $v_n(0, x) = e^{-it_n\Delta}\psi^1(x - x_n^1)$.
 920 Moreover, v_n scatters in H^1 in both directions. In particular, $\|v_n\|_{S(\dot{H}^{\gamma_c})} < \infty$. Again,
 921 by the long time perturbation, we show that $\|u_n\|_{S([0,\infty),\dot{H}^{\gamma_c})} < \infty$ for n sufficiently
 922 large which is a contradiction.

923 Let v^1 be the nonlinear profile associated to ψ^1 and t_n^1 , we have

$$924 \quad u_{n,0}(x) = v^1(-t_n^1, x) + \tilde{W}_n(x).$$

925 Set $v_n^1(t) = v^1(t - t_n^1)$. Arguing as above, we have

$$926 \quad M(v_n^1(t)) \leq 1, \quad \sup_{t \in [0,\infty)} P(v_n^1(t)) \leq A, \quad E(v_n^1(t)) \leq \delta_c, \quad \lim_{n \rightarrow \infty} \|\tilde{W}_n(t)\|_{S(\dot{H}^{\gamma_c})} = 0.$$

927 We infer that $M(v_n^1(t)) = 1$ and $E(v_n^1(t)) = \delta_c$. Otherwise, if $M(v_n^1(t)) < 1$, then

$$928 \quad \sup_{t \in [0,\infty)} P(v_n^1(t))[M(v_n^1(t))]^{\sigma_c} < A, \quad E(v_n^1)[M(v_n^1)]^{\sigma_c} < \delta_c.$$

929 By Item (1) (see again after (2.22)), we have $\|v_n^1\|_{S([0,\infty),\dot{H}^{\gamma_c})} < \infty$. Thus we get a
 930 contradiction by the long time perturbation argument.

931 Now we define u_c the solution to (1.2) with initial data $u_c|_{t=0} = v^1(0)$. We have

$$932 \quad M(u_c) = M(v^1(0)) = M(v^1(t - t_n^1)) = M(v_n^1(t)) = 1,$$

$$933 \quad E(u_c) = E(v^1(0)) = E(v^1(t - t_n^1)) = E(v_n^1(t)) = \delta_c.$$

935 Moreover,

$$936 \quad \sup_{t \in [0, \infty)} P(u_c(t)) = \sup_{t \in [0, \infty)} P(v^1(t)) = \sup_{t \in [t_n^1, \infty)} P(v^1(t - t_n^1)) = \sup_{t \in [t_n^1, \infty)} P(v_n^1(t)) \leq A.$$

937 By the definition of δ_c , we must have $\|u_c\|_{S([0, \infty), \dot{H}^{\gamma_c})} = \infty$. This shows (2.24).

938 By the same argument as in the proof of [27, Proposition 6.3], we show that the
939 set

$$940 \quad \mathcal{K} := \{u_c(t) : t \in [0, \infty)\}$$

941 is precompact in H^1 .

942 **Step 3. Exclusion of the critical solution.** Thanks to the above compactness
943 result, the standard rigidity argument using localized virial estimates and Lemma
944 2.10 shows that $u_c \equiv 0$ which contradicts (2.24). We refer the reader to [27, Section
945 7] for more details. The proof of Proposition 2.11 is now complete. This also ends
946 the proof of Theorem 1.1. \square

947 **3. Blow-up criterion.** In this section, we give the proof of the blow-up criterion
948 given in Theorem 1.2. Let us recall the following virial identity (see e.g., [14]).

949 LEMMA 3.1. *Let $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a sufficiently smooth and decaying function. Let*
950 *u be a solution to (1.2) defined on the maximal forward time interval of existence*
951 *$[0, T^*)$. Define*

$$952 \quad (3.1) \quad V_\varphi(t) := \int \varphi(x) |u(t, x)|^2 dx.$$

953
954 Then we have for all $t \in [0, T^*)$,

$$955 \quad V'_\varphi(t) = 2 \operatorname{Im} \int \nabla \varphi(x) \cdot \nabla u(t, x) \bar{u}(t, x) dx$$

956 and

$$957 \quad V''_\varphi(t) \\ 958 \quad = - \int \Delta^2 \varphi(x) |u(t, x)|^2 dx + 4 \sum_{j, k=1}^N \operatorname{Re} \int \partial_{jk}^2 \varphi(x) \partial_j \bar{u}(t, x) \partial_k u(t, x) dx \\ 959 \quad - \frac{2\alpha}{\alpha + 2} \int |x|^{-b} \Delta \varphi(x) |u(t, x)|^{\alpha+2} dx + \frac{4}{\alpha + 2} \int \nabla \varphi(x) \cdot \nabla (|x|^{-b}) |u(t, x)|^{\alpha+2} dx.$$

960
961 Remark 3.2. (1) In the case $\varphi(x) = |x|^2$, we have

$$962 \quad \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8G(u(t)),$$

963 where $G(f)$ is as in (1.15).

964 (2) In the case φ is radially symmetric, it follows from

$$965 \quad \partial_j = \frac{x_j}{r} \partial_r, \quad \partial_{jk}^2 = \left(\frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_r + \frac{x_j x_k}{r^2} \partial_r^2$$

966 that

$$\begin{aligned}
 967 \quad & \sum_{j,k=1}^N \operatorname{Re} \int \partial_{jk}^2 \varphi(x) \partial_j \bar{u}(t, x) \partial_k u(t, x) dx \\
 & = \int \frac{\varphi'(r)}{r} |\nabla u(t, x)|^2 dx + \int \left(\frac{\varphi''(r)}{r^2} - \frac{\varphi'(r)}{r^3} \right) |x \cdot \nabla u(t, x)|^2 dx.
 \end{aligned}$$

970 In particular, we have

$$\begin{aligned}
 971 \quad V_\varphi''(t) & = - \int \Delta^2 \varphi(x) |u(t, x)|^2 dx + 4 \int \frac{\varphi'(r)}{r} |\nabla u(t, x)|^2 dx \\
 972 \quad (3.2) \quad & + 4 \int \left(\frac{\varphi''(r)}{r^2} - \frac{\varphi'(r)}{r^3} \right) |x \cdot \nabla u(t, x)|^2 dx \\
 & - \frac{2\alpha}{\alpha+2} \int |x|^{-b} \Delta \varphi(x) |u(t, x)|^{\alpha+2} - \frac{4b}{\alpha+2} \int |x|^{-b} \frac{\varphi'(r)}{r} |u(t, x)|^{\alpha+2} dx.
 \end{aligned}$$

975 (3) Denote $x = (y, x_N)$ with $y = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. Let $\psi : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ be a sufficiently smooth decaying function. Set $\varphi(x) = \varphi(y, x_N) = \psi(y) + x_N^2$. We have

$$978 \quad V_\varphi'(t) = 2 \operatorname{Im} \int (\nabla_y \psi(y) \cdot \nabla_y u(t, x) + 2x_N \partial_N u(t, x)) \bar{u}(t, x) dx$$

980 and

$$\begin{aligned}
 981 \quad V_\varphi''(t) & = - \int \Delta_y^2 \psi(y) |u(t, x)|^2 dx + 4 \sum_{j,k=1}^{N-1} \operatorname{Re} \int \partial_{jk}^2 \psi(y) \partial_j \bar{u}(t, x) \partial_k u(t, x) dx \\
 982 \quad & - \frac{2\alpha}{\alpha+2} \int |x|^{-b} \Delta_y \psi(y) |u(t, x)|^{\alpha+2} dx - \frac{4b}{\alpha+2} \int \nabla_y \psi(y) \cdot y |x|^{-b-2} |u(t, x)|^{\alpha+2} dx \\
 983 \quad & + 8 \|\partial_N u(t)\|_{L^2}^2 - \frac{4\alpha}{\alpha+2} \int |x|^{-b} |u(t, x)|^{\alpha+2} dx - \frac{8b}{\alpha+2} \int x_N^2 |x|^{-b-2} |u(t, x)|^{\alpha+2} dx. \blacksquare
 \end{aligned}$$

986 Let χ be a smooth radial function satisfying

$$987 \quad \chi(x) = \chi(r) = \begin{cases} r^2 & \text{if } r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases} \quad \chi''(r) \leq 2 \quad \forall r = |x| \geq 0.$$

989 Given $R > 1$, we define the radial function

$$990 \quad (3.3) \quad \varphi_R(x) := R^2 \chi(x/R).$$

992 We have the following localized virial estimate.

993 **PROPOSITION 3.3.** *Let $N \geq 1$, $0 < b < \min\{2, N\}$, and $\frac{4-2b}{N} < \alpha < \alpha(N)$. Let*
 994 *u be a solution to (1.2) defined on the maximal forward time interval of existence*
 995 *$[0, T^*)$. Let φ_R be as in (3.3) and define $V_{\varphi_R}(t)$ as in (3.1). Then we have for all*
 996 *$t \in [0, T^*)$,*

$$997 \quad V_{\varphi_R}'(t) = 2 \operatorname{Im} \int \nabla \varphi_R(x) \cdot \nabla u(t, x) \bar{u}(t, x) dx$$

998 and

$$1000 \quad V_{\varphi_R}''(t) \leq 8G(u(t)) + CR^{-2} + CR^{-b} \|u(t)\|_{H^1}^{\alpha+2},$$

1001 where G is as in (1.15) and some constant $C > 0$ independent of R .

1002 *Proof.* It follows from (3.2) that

$$\begin{aligned}
1003 \quad V_{\varphi_R}''(t) &= 8G(u(t)) - 8\|\nabla u(t)\|_{L^2}^2 + \frac{4(N\alpha + 2b)}{\alpha + 2} \int |x|^{-b} |u(t, x)|^{\alpha+2} dx \\
1004 \quad &\quad - \int \Delta^2 \varphi_R(x) |u(t, x)|^2 dx + 4 \int \frac{\varphi_R'(r)}{r} |\nabla u(t, x)|^2 dx \\
1005 \quad &\quad + 4 \int \left(\frac{\varphi_R''(r)}{r^2} - \frac{\varphi_R'(r)}{r^3} \right) |x \cdot \nabla u(t, x)|^2 dx \\
1006 \quad &\quad - \frac{2\alpha}{\alpha + 2} \int |x|^{-b} \Delta \varphi_R(x) |u(t, x)|^{\alpha+2} dx - \frac{4b}{\alpha + 2} \int |x|^{-b} \frac{\varphi_R'(r)}{r} |u(t, x)|^{\alpha+2} dx.
\end{aligned}$$

1008 As $\|\Delta^2 \varphi_R\|_{L^\infty} \lesssim R^{-2}$, the conservation of mass implies that

$$1009 \quad \left| \int \Delta^2 \varphi_R(x) |u(t, x)|^2 dx \right| \lesssim R^{-2} \|u(t)\|_{L^2}^2 \lesssim R^{-2}.$$

1010 By the Cauchy-Schwarz inequality $|x \cdot \nabla u| \leq |x| |\nabla u| = r |\nabla u|$ and the fact $\varphi_R''(r) \leq 2$,
1011 we see that

$$\begin{aligned}
1012 \quad &4 \int \frac{\varphi_R'(r)}{r} |\nabla u(t, x)|^2 dx + 4 \int \left(\frac{\varphi_R''(r)}{r^2} - \frac{\varphi_R'(r)}{r^3} \right) |x \cdot \nabla u(t, x)|^2 dx - 8\|\nabla u(t)\|_{L^2}^2 \\
1013 \quad &\leq 4 \int \left(\frac{\varphi_R'(r)}{r} - 2 \right) |\nabla u(t, x)|^2 dx + 4 \int \frac{1}{r^2} \left(2 - \frac{\varphi_R'(r)}{r} \right) |x \cdot \nabla u(t, x)|^2 dx \leq 0.
\end{aligned}$$

1015 Moreover,

$$\begin{aligned}
1016 \quad &\frac{4(N\alpha + 2b)}{\alpha + 2} \int |x|^{-b} |u(t, x)|^{\alpha+2} dx - \frac{2\alpha}{\alpha + 2} \int |x|^{-b} \Delta \varphi_R(x) |u(t, x)|^{\alpha+2} dx \\
1017 \quad &\quad - \frac{4b}{\alpha + 2} \int |x|^{-b} \frac{\varphi_R'(r)}{r} |u(t, x)|^{\alpha+2} dx \\
1018 \quad &= \frac{2\alpha}{\alpha + 2} \int |x|^{-b} (2N - \Delta \varphi_R(x)) |u(t, x)|^{\alpha+2} dx \\
1019 \quad &\quad + \frac{4b}{\alpha + 2} \int |x|^{-b} \left(2 - \frac{\varphi_R'(r)}{r} \right) |u(t, x)|^{\alpha+2} dx.
\end{aligned}$$

1021 Since $\Delta \varphi_R \leq 2N$, $\frac{\varphi_R'(r)}{r} \leq 2$, $\Delta \varphi_R(x) = 2N$, and $\frac{\varphi_R'(r)}{r} = 2$ for $r = |x| \leq R$, the
1022 above quantity is bounded by

$$1023 \quad C \int_{|x| \geq R} |x|^{-b} |u(t)|^{\alpha+2} dx \leq CR^{-b} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \leq CR^{-b} \|u(t)\|_{H^1}^{\alpha+2},$$

1024 where the last inequality follows from the Sobolev embedding as $\alpha < \alpha(N)$. Collecting
1025 the above estimates, we end the proof. \square

1026 *Proof of Theorem 1.2.* Let $u : [0, T^*) \times \mathbb{R}^N \rightarrow \mathbb{C}$ be a solution to (1.2) satisfying
1027 (1.14). If $T^* < \infty$, then we are done. If $T^* = \infty$, then we show that there exists
1028 $t_n \rightarrow \infty$ such that $\|\nabla u(t_n)\|_{L^2} \rightarrow \infty$ as $n \rightarrow \infty$. Assume by contradiction that it
1029 does not hold, i.e., $\sup_{t \in [0, \infty)} \|\nabla u(t)\|_{L^2} \leq C_0$ for some $C_0 > 0$. By the conservation
1030 of mass, we have

$$1031 \quad (3.4) \quad \sup_{t \in [0, \infty)} \|u(t)\|_{H^1} \leq C_1$$

1033 for some $C_1 > 0$.

1034 By Proposition 3.3, (1.14), and (3.4), we have for all $t \in [0, \infty)$,

$$1035 \quad V''_{\varphi_R}(t) \leq 8G(u(t)) + CR^{-2} + CR^{-b}\|u(t)\|_{L^2}^{\alpha+2} \leq -8\delta + CR^{-2} + CR^{-b}C_1^{\alpha+2}.$$

1037 By taking $R > 1$ sufficiently large, we have for all $t \in [0, \infty)$,

$$1038 \quad V''_{\varphi_R}(t) \leq -4\delta.$$

1039 Integrating this estimate, there exists $t_0 > 0$ sufficiently large such that $V_{\varphi_R}(t_0) < 0$
1040 which is impossible. This finishes the first part of Theorem 1.2.

1041 If we assume in addition that u has finite variance, i.e., $u(t) \in L^2(|x|^2 dx)$ for all
1042 $t \in [0, T^*)$, then we have $T^* < \infty$. In fact, it follows from Remark 3.2 and (1.14) that

$$1043 \quad \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8G(u(t)) \leq -8\delta$$

1044 for all $t \in [0, T^*)$. The convexity argument of Glassey [35] implies $T^* < \infty$. \square

1045 **4. Long time dynamics.** In this section, we give the proofs of long time dy-
1046 namics of H^1 -solutions given in Theorems 1.3, 1.4 and 1.5.

1047 *Proof of Theorem 1.3.* We will consider separately two cases.

1048 **Case 1. Global existence and energy scattering.** Let $u_0 \in H^1$ satisfy (1.7) and
1049 (1.8). Let us prove (1.16). To see this, we first claim that there exists $\rho = \rho(u_0, Q) > 0$
1050 such that

$$1051 \quad (4.1) \quad \|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}^{\sigma_c} \leq (1 - \rho) \|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c}$$

1053 for all $t \in (-T_*, T^*)$. We assume (4.1) for the moment and prove (1.16). By (2.3)
1054 and (4.1), we have

$$1055 \quad \begin{aligned} P(u(t))[M(u(t))]^{\sigma_c} &\leq C_{\text{opt}} \|\nabla u(t)\|_{L^2}^{\frac{N\alpha+2b}{2}} \|u(t)\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2} + 2\sigma_c} \\ 1056 \quad &= C_{\text{opt}} (\|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}^{\sigma_c})^{\frac{N\alpha+2b}{2}} \\ 1057 \quad &\leq C_{\text{opt}} (1 - \rho)^{\frac{N\alpha+2b}{2}} (\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^{\frac{N\alpha+2b}{2}} \end{aligned}$$

1059 for all $t \in (-T_*, T^*)$. By (2.4) and (2.5), we get

$$1060 \quad \begin{aligned} P(u(t))[M(u(t))]^{\sigma_c} &\leq \frac{2(\alpha+2)}{N\alpha+2b} (1 - \rho)^{\frac{N\alpha+2b}{2}} (\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^2 \\ 1061 \quad &= (1 - \rho)^{\frac{N\alpha+2b}{2}} P(Q)[M(Q)]^{\sigma_c} \end{aligned}$$

1063 for all $t \in (-T_*, T^*)$ which shows (1.16). By Theorem 1.1, the solution exists globally
1064 in time. Moreover, if $N \geq 2$ and $0 < b < \min\{2, \frac{N}{2}\}$, then the solution scatters in
1065 H^1 in both directions.

1066 Let us now prove the claim (4.1). By the definition of energy and (2.3), we have

$$1067 \quad \begin{aligned} E(u(t))[M(u(t))]^{\sigma_c} &\geq \frac{1}{2} (\|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}^{\sigma_c})^2 - \frac{C_{\text{opt}}}{\alpha+2} \|\nabla u(t)\|_{L^2}^{\frac{N\alpha+2b}{2}} \|u(t)\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2} + 2\sigma_c} \\ 1068 \quad &= F(\|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}^{\sigma_c}), \end{aligned}$$

1069 (4.2)

1071 where

$$1072 \quad F(\lambda) := \frac{1}{2}\lambda^2 - \frac{C_{\text{opt}}}{\alpha + 2}\lambda^{\frac{N\alpha+2b}{2}}.$$

1073 Using (2.4), (2.5) and (2.7), we see that

$$1074 \quad F(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}) = \frac{N\alpha - 4 + 2b}{2(N\alpha + 2b)} (\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c})^2 = E(Q)[M(Q)]^{\sigma_c}.$$

1075 It follows from (1.7), (4.2) and the conservation of mass and energy that

$$1076 \quad F(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}) \leq E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c} = F(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c})$$

1077 for all $t \in (-T_*, T^*)$. By (1.8), the continuity argument implies

$$1078 \quad (4.3) \quad \|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}$$

1080 for all $t \in (-T_*, T^*)$. Next, using (1.7), we take $\vartheta = \vartheta(u_0, Q) > 0$ such that

$$1081 \quad (4.4) \quad E(u_0)[M(u_0)]^{\sigma_c} \leq (1 - \vartheta)E(Q)[M(Q)]^{\sigma_c}.$$

1083 Using

$$1084 \quad \begin{aligned} E(Q)[M(Q)]^{\sigma_c} &= \frac{N\alpha - 4 + 2b}{2(N\alpha + 2b)} (\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c})^2 \\ &= \frac{N\alpha - 4 + 2b}{4(\alpha + 2)} (\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c})^{\frac{N\alpha+2b}{2}}, \end{aligned}$$

1087 we we infer from (4.2) and (4.4) that

$$1088 \quad \frac{N\alpha + 2b}{N\alpha - 4 + 2b} \left(\frac{\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}}{\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}} \right)^2 - \frac{4}{N\alpha - 4 + 2b} \left(\frac{\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c}}{\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}} \right)^{\frac{N\alpha+2b}{2}} \\ 1089 \quad (4.5) \quad \leq 1 - \vartheta$$

1091 for all $t \in (-T_*, T^*)$. Let us consider the function

$$1092 \quad (4.6) \quad G(\lambda) := \frac{N\alpha + 2b}{N\alpha - 4 + 2b}\lambda^2 - \frac{4}{N\alpha - 4 + 2b}\lambda^{\frac{N\alpha+2b}{2}}$$

1094 with $0 < \lambda < 1$ due to (4.3). We see that G is strictly increasing on $(0, 1)$ with
1095 $G(0) = 0$ and $G(1) = 1$. It follows from (4.6) that there exists $\rho > 0$ depending on ϑ
1096 such that $\lambda \leq 1 - \rho$ which is (4.1). This finishes the first part of Theorem 1.3.

1097 **Case 2. Blow-up.** Let $u_0 \in H^1$ satisfy (1.7) and (1.10). Let us prove (1.17). By
1098 the same argument as above using (1.10) instead of (1.8), we have

$$1099 \quad (4.7) \quad \|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c} > \|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c}$$

1101 for all $t \in (-T_*, T^*)$. Let ϑ be as in (4.4). By the conservation laws of mass and
1102 energy together with (4.7) and (2.7), we have

$$1103 \quad \begin{aligned} &G(u(t))[M(u(t))]^{\sigma_c} \\ 1104 \quad &= \|\nabla u(t)\|_{L^2}^2\|u(t)\|_{L^2}^{2\sigma_c} - \frac{N\alpha + 2b}{2(\alpha + 2)}P(u(t))[M(u(t))]^{\sigma_c} \\ 1105 \quad &= \frac{N\alpha + 2b}{2}E(u(t))[M(u(t))]^{\sigma_c} - \frac{N\alpha - 4 + 2b}{4}(\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2}^{\sigma_c})^2 \\ 1106 \quad &\leq \frac{N\alpha + 2b}{2}(1 - \vartheta)E(Q)[M(Q)]^{\sigma_c} - \frac{N\alpha - 4 + 2b}{4}(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c})^2 \\ 1107 \quad &= -\frac{N\alpha - 4 + 2b}{4}\vartheta(\|\nabla Q\|_{L^2}\|Q\|_{L^2}^{\sigma_c})^2 \end{aligned}$$

1109 for all $t \in (-T_*, T^*)$. This shows (1.17) with

$$1110 \quad \delta := \frac{N\alpha - 4 + 2b}{4} \vartheta \|\nabla Q\|_{L^2}^2 \left(\frac{M(Q)}{M(u_0)} \right)^{\sigma_c} > 0.$$

1111 By Theorem 1.2, the corresponding solution either blows up in finite time, or there
1112 exists a time sequence $(t_n)_{n \geq 1}$ satisfying $|t_n| \rightarrow \infty$ such that $\|\nabla u(t_n)\|_{L^2} \rightarrow \infty$ as
1113 $n \rightarrow \infty$.

1114 • **Finite variance data.** If we assume in addition that $u_0 \in \Sigma$, then the correspond-
1115 ing solution blows up in finite time. It directly follows from Theorem 1.2.

1116 • **Radially symmetric data.** If we assume in addition that $N \geq 2$, $\alpha \leq 4$, and u_0
1117 is radially symmetric, then the corresponding solution blows up in finite time. This
1118 result was shown in [14]. Note that in [14], α is assumed to be strictly smaller than
1119 4. However, a closer look at the proof of [14], we see that $\alpha = 4$ is allowed.

1120 • **Cylindrically symmetric data.** If we assume in addition that $N \geq 3$, $\alpha \leq 2$, and
1121 $u_0 \in \Sigma_N$ (see (1.18)), then the corresponding solution blows up in finite time. To this
1122 end, let η be a smooth radial function satisfying

$$1123 \quad \eta(y) = \eta(\tau) = \begin{cases} \tau^2 & \text{if } \tau \leq 1, \\ 0 & \text{if } \tau \geq 2, \end{cases} \quad \eta''(\tau) \leq 0, \quad \forall \tau = |y| \geq 0.$$

1124 Given $R > 1$, we define the radial function

$$1125 \quad (4.8) \quad \psi_R(y) := R^2 \eta(y/R).$$

1127 Set

$$1128 \quad (4.9) \quad \varphi_R(x) := \psi_R(y) + x_N^2.$$

1130 Applying Remark 3.2, we have

$$1131 \quad V'_{\varphi_R}(t) = 2 \operatorname{Im} \int (\nabla_y \psi_R(y) \cdot \nabla_y u(t, x) + 2x_N \partial_N u(t, x)) \bar{u}(t, x) dx$$

1133 and

$$1134 \quad V''_{\varphi_R}(t) = - \int \Delta_y^2 \psi_R(y) |u(t, x)|^2 dx + 4 \sum_{j,k=1}^{N-1} \operatorname{Re} \int \partial_{jk}^2 \psi_R(y) \partial_j \bar{u}(t, x) \partial_k u(t, x) dx$$

$$1135 \quad - \frac{2\alpha}{\alpha+2} \int |x|^{-b} \Delta_y \psi_R(y) |u(t, x)|^{\alpha+2} dx - \frac{4b}{\alpha+2} \int |y|^2 \frac{\psi'_R(\tau)}{\tau} |x|^{-b-2} |u(t, x)|^{\alpha+2} dx$$

$$1136 \quad + 8 \|\partial_N u(t)\|_{L^2}^2 - \frac{4\alpha}{\alpha+2} \int |x|^{-b} |u(t, x)|^{\alpha+2} dx - \frac{8b}{\alpha+2} \int x_N^2 |x|^{-b-2} |u(t, x)|^{\alpha+2} dx.$$

1138 We can rewrite it as

$$1139 \quad V''_{\varphi_R}(t) = 8G(u(t)) - 8\|\nabla_y u(t)\|_{L^2}^2 + \frac{4((N-1)\alpha + 2b)}{\alpha+2} P(u(t))$$

$$1140 \quad - \int \Delta_y^2 \psi_R(y) |u(t, x)|^2 dx + 4 \sum_{j,k=1}^{N-1} \operatorname{Re} \int \partial_{jk}^2 \psi_R(y) \partial_j \bar{u}(t, x) \partial_k u(t, x) dx$$

$$1141 \quad - \frac{2\alpha}{\alpha+2} \int \Delta_y \psi_R(y) |x|^{-b} |u(t, x)|^{\alpha+2} dx - \frac{4b}{\alpha+2} \int |y|^2 \frac{\psi'_R(\tau)}{\tau} |x|^{-b-2} |u(t, x)|^{\alpha+2} dx$$

$$1142 \quad - \frac{8b}{\alpha+2} \int x_N^2 |x|^{-b-2} |u(t, x)|^{\alpha+2} dx.$$

1143

1144 Rewriting it further, we get

$$\begin{aligned}
1145 \quad V''_{\varphi_R}(t) &= 8G(u(t)) - 8\|\nabla_y u(t)\|_{L^2}^2 + 4 \sum_{j,k=1}^{N-1} \operatorname{Re} \int \partial_{jk}^2 \psi_R(y) \partial_j \bar{u}(t, x) \partial_k u(t, x) dx \\
1146 \quad &- \int \Delta_y^2 \psi_R(y) |u(t, x)|^2 dx + \frac{2\alpha}{\alpha+2} \int (2(N-1) - \Delta_y \psi_R(y)) |x|^{-b} |u(t, x)|^{\alpha+2} dx \\
1147 \quad &+ \frac{4b}{\alpha+2} \int \left(2|x|^2 - \frac{\psi'_R(\tau)}{\tau} |y|^2 - 2x_N^2 \right) |x|^{-b-2} |u(t, x)|^{\alpha+2} dx. \\
1148
\end{aligned}$$

1149 Since u is radially symmetric with respect to the first $N-1$ variables, we use the fact
1150 that

$$1151 \quad \partial_j = \frac{y_j}{\tau} \partial_\tau, \quad \partial_{jk}^2 = \left(\frac{\delta_{jk}}{\tau} - \frac{y_j y_k}{\tau^3} \right) \partial_\tau + \frac{y_j y_k}{\tau^2} \partial_\tau^2, \quad \tau = |y|, \quad j, k = 1, \dots, N-1$$

1152 to have

$$1153 \quad \sum_{j,k=1}^{N-1} \partial_{jk}^2 \psi_R(y) \partial_j \bar{u}(t, x) \partial_k u(t, x) = \psi''_R(\tau) |\partial_\tau u(t, x)|^2 \leq 2 |\partial_\tau u(t, x)|^2 = 2 |\nabla_y u(t, x)|^2.$$

1154 Thus we get

$$1155 \quad 4 \sum_{j,k=1}^{N-1} \operatorname{Re} \int \partial_{jk}^2 \psi_R(y) \partial_j \bar{u}(t, x) \partial_k u(t, x) dx - 8\|\nabla_y u(t)\|_{L^2}^2 \leq 0.$$

1156 By the conservation of mass and the fact $\|\Delta_y \psi_R\|_{L^\infty} \lesssim R^{-2}$, we have

$$1157 \quad \left| \int \Delta_y^2 \psi_R(y) |u(t, x)|^2 dx \right| \lesssim R^{-2}.$$

1158 Moreover, since $\psi_R(y) = |y|^2$ for $|y| \leq R$ and $\|\Delta_y \psi_R\|_{L^\infty} \lesssim 1$, we see that

$$1159 \quad \left| \int (2(N-1) - \Delta_y \psi_R(y)) |x|^{-b} |u(t, x)|^{\alpha+2} dx \right| \lesssim \int_{|y| \geq R} |x|^{-b} |u(t, x)|^{\alpha+2} dx. \\ 1160$$

1161 Similarly, we have

$$1162 \quad \left| \int \left(2|x|^2 - \frac{\psi'_R(\tau)}{\tau} |y|^2 - 2x_N^2 \right) |x|^{-b-2} |u(t, x)|^{\alpha+2} dx \right| \lesssim \int_{|y| \geq R} |x|^{-b} |u(t, x)|^{\alpha+2} dx. \\ 1163$$

1164 We thus obtain

$$1165 \quad (4.10) \quad V''_{\varphi_R}(t) \leq 8G(u(t)) + CR^{-2} + CR^{-b} \int_{|y| \geq R} |u(t, x)|^{\alpha+2} dx. \\ 1166$$

1167 To estimate the last term in the right hand side of (4.10), we recall the following
1168 radial Sobolev embedding due to Strauss [48]: for any radial function $f : \mathbb{R}^{N-1} \rightarrow \mathbb{C}$,
1169 it holds that

$$1170 \quad (4.11) \quad \sup_{y \neq 0} |y|^{\frac{N-2}{2}} |f(y)| \leq C(N) \|f\|_{L_y^2}^{\frac{1}{2}} \|\nabla_y f\|_{L_y^2}^{\frac{1}{2}}. \\ 1171$$

1172 We estimate

$$1173 \quad \int_{\mathbb{R}} \int_{|y| \geq R} |u(t, y, x_N)|^{\alpha+2} dy dx_N \leq \int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^\infty(|y| \geq R)}^\alpha \|u(t, x_N)\|_{L_y^2}^2 dx_N.$$

1175 We consider separately two subcases: $\alpha = 2$ and $\alpha < 2$.

1176 **Subcase 1.** $\alpha = 2$. We have

$$1177 \quad \int_{\mathbb{R}} \int_{|y| \geq R} |u(t, y, x_N)|^{\alpha+2} dy dx_N \leq \left(\sup_{x_N \in \mathbb{R}} \|u(t, x_N)\|_{L_y^2}^2 \right) \int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^\infty(|y| \geq R)}^2 dx_N.$$

1179 By the radial Sobolev embedding (4.11) and the conservation of mass, we have

$$1180 \quad \int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^\infty(|y| \geq R)}^2 dx_N \lesssim R^{-\frac{N-2}{2}} \int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^2} \|\nabla_y u(t, x_N)\|_{L_y^2} dx_N$$

$$1181 \quad \lesssim R^{-\frac{N-2}{2}} \left(\int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^2}^2 dx_N \right)^{1/2} \left(\int_{\mathbb{R}} \|\nabla_y u(t, x_N)\|_{L_y^2}^2 dx_N \right)^{1/2}$$

$$1182 \quad = R^{-\frac{N-2}{2}} \|u(t)\|_{L_x^2} \|\nabla_y u(t)\|_{L_x^2}$$

$$1183 \quad \lesssim R^{-\frac{N-2}{2}} \|\nabla_y u(t)\|_{L_x^2}.$$

1185 Set $g(x_N) := \|u(t, x_N)\|_{L_y^2}^2$. We have

$$1186 \quad g(x_N) = \int_{-\infty}^{x_N} \partial_s g(s) ds = 2 \int_{-\infty}^{x_N} \operatorname{Re} \int_{\mathbb{R}^{N-1}} \bar{u}(t, y, s) \partial_s u(t, y, s) dy ds$$

$$1187 \quad \leq 2 \|u(t)\|_{L_x^2} \|\partial_N u(t)\|_{L_x^2}.$$

1189 Thus we get

$$1190 \quad \sup_{x_N \in \mathbb{R}} \|u(t, x_N)\|_{L_y^2}^2 \leq C \|\partial_N u(t)\|_{L_x^2}.$$

1192 This shows that

$$1193 \quad \int_{\mathbb{R}} \int_{|y| \geq R} |u(t, y, x_N)|^{\alpha+2} dy dx_N \lesssim R^{-\frac{N-2}{2}} \|\nabla_y u(t)\|_{L_x^2} \|\partial_N u(t)\|_{L_x^2}$$

$$1194 \quad \lesssim R^{-\frac{N-2}{2}} \|\nabla u(t)\|_{L_x^2}^2.$$

1196 **Subcase 2.** $\alpha < 2$. We have

$$1197 \quad \int_{\mathbb{R}} \int_{|y| \geq R} |u(t, y, x_N)|^{\alpha+2} dy dx_N$$

$$1198 \quad \leq \left(\int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^\infty(|y| \geq R)}^2 dx_N \right)^{\frac{\alpha}{2}} \left(\int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^2}^{\frac{4}{2-\alpha}} dx_N \right)^{\frac{2-\alpha}{2}}.$$

1200 By the Gagliardo-Nirenberg inequality, we have

$$1201 \quad \int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^2}^{\frac{4}{2-\alpha}} dx_N \lesssim \left\| \partial_N \left(\|u(t, x_N)\|_{L_y^2} \right) \right\|_{L_{x_N}^2}^{\frac{\alpha}{2-\alpha}} \left\| \|u(t, x_N)\|_{L_y^2} \right\|_{L_{x_N}^2}^{\frac{4-\alpha}{2-\alpha}}.$$

1203 By the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
1204 \quad & 2 \left| \partial_N \left(\|u(t, x_N)\|_{L_y^2} \right) \right| \|u(t, x_N)\|_{L_y^2} = |\partial_N \left(\|u(t, x_N)\|_{L_y^2}^2 \right)| \\
1205 \quad & = 2 \left| \operatorname{Re} \int_{\mathbb{R}^{N-1}} \bar{u}(t, y, x_N) \partial_N u(t, y, x_N) dy \right| \\
1206 \quad & \leq 2 \|u(t, x_N)\|_{L_y^2} \|\partial_N u(t, x_N)\|_{L_y^2}
\end{aligned}$$

1208 which implies that $\left| \partial_N \left(\|u(t, x_N)\|_{L_y^2} \right) \right| \leq \|\partial_N u(t, x_N)\|_{L_y^2}$. It follows that

$$\begin{aligned}
1209 \quad & \int_{\mathbb{R}} \|u(t, x_N)\|_{L_y^2}^{\frac{4}{2-\alpha}} dx_N \lesssim \|\partial_N u(t, x_N)\|_{L_y^2}^{\frac{\alpha}{2-\alpha}} \|u(t)\|_{L_x^2}^{\frac{4-\alpha}{2-\alpha}} \\
1210 \quad & = \|\partial_N u(t)\|_{L_x^2}^{\frac{\alpha}{2-\alpha}} \|u(t)\|_{L_x^2}^{\frac{4-\alpha}{2-\alpha}} \\
1211 \quad & \lesssim \|\partial_N u(t)\|_{L_x^2}^{\frac{\alpha}{2-\alpha}}.
\end{aligned}$$

1213 Thus, by the Young inequality, we get

$$\begin{aligned}
1214 \quad & \int_{\mathbb{R}} \int_{|y| \geq R} |u(t, y, x_N)|^{\alpha+2} dy dx_N \lesssim R^{-\frac{(N-2)\alpha}{4}} \|\nabla_y u(t)\|_{L_x^2}^{\frac{\alpha}{2}} \|\partial_N u(t)\|_{L_x^2}^{\frac{\alpha}{2}} \\
1215 \quad & \lesssim R^{-\frac{(N-2)\alpha}{4}} (\|\nabla_y u(t)\|_{L_x^2} \|\partial_N u(t)\|_{L_x^2} + 1) \\
1216 \quad & \lesssim R^{-\frac{(N-2)\alpha}{4}} \|\nabla u(t)\|_{L_x^2}^2 + CR^{-\frac{(N-2)\alpha}{4}}.
\end{aligned}$$

1218 Collecting the above subcases and using (4.10), we obtain

$$\begin{aligned}
1219 \quad & V''_{\varphi_R}(t) \leq 8G(u(t)) + CR^{-2} \\
1220 \quad (4.12) \quad & + \begin{cases} CR^{-\frac{N-2}{2}-b} \|\nabla u(t)\|_{L^2}^2 & \text{if } \alpha = 2, \\ CR^{-\frac{(N-2)\alpha}{4}-b} \|\nabla u(t)\|_{L^2}^2 + CR^{-\frac{(N-2)\alpha}{4}-b} & \text{if } \alpha < 2, \end{cases} \\
1221 \quad &
\end{aligned}$$

1222 for all $t \in (-T_*, T^*)$. Under the assumptions (1.7) and (1.10), we have the following
1223 estimate due to [14, (5.8)]: for $\varepsilon > 0$ small enough, there exists a constant $\delta = \delta(\varepsilon) > 0$
1224 such that

$$1225 \quad (4.13) \quad 8G(u(t)) + \varepsilon \|\nabla u(t)\|_{L^2}^2 \leq -\delta$$

1227 for all $t \in (-T_*, T^*)$. Thanks to (4.12), we take $R > 1$ sufficiently large to get

$$1228 \quad V''_{\psi_R}(t) \leq -\frac{\delta}{2} < 0$$

1229 for all $t \in (-T_*, T^*)$. The standard convexity argument yields $T_*, T^* < \infty$. The proof
1230 is complete. \square

1231 We are next interested in long time dynamics of H^1 -solutions for (1.2) with data
1232 at the ground state threshold. To this end, we need the following lemmas.

1233 **LEMMA 4.1.** *Let $N \geq 1$, $0 < b < \min\{2, N\}$, and $0 < \alpha < \alpha(N)$. Let $(f_n)_{n \geq 1}$ be
1234 a bounded sequence in H^1 . Then, there exist a subsequence still denoted by $(f_n)_{n \geq 1}$
1235 and a function $f \in H^1$ such that:*

- 1236 • $f_n \rightarrow f$ weakly in H^1 .
- 1237 • $f_n \rightarrow f$ strongly in L^r_{loc} for all $1 \leq r < 2^*$.

1238 • $\lim_{n \rightarrow \infty} P(f_n) = P(f)$ as $n \rightarrow \infty$, where P is as in (1.12).

1239 *Proof.* The first two items are well-known. Let us prove the last one. Let $\varepsilon > 0$.
 1240 Since $(f_n)_{n \geq 1}$ is bounded in H^1 , we have for any $R > 0$,

$$\begin{aligned} 1241 \quad \left| \int_{|x| \geq R} |x|^{-b} (|f_n(x)|^{\alpha+2} - |f(x)|^{\alpha+2}) dx \right| &\leq R^{-b} (\|f_n\|_{L^{\alpha+2}}^{\alpha+2} + \|f\|_{L^{\alpha+2}}^{\alpha+2}) \\ 1242 &\leq CR^{-b} (\|f_n\|_{H^1}^{\alpha+2} + \|f\|_{H^1}^{\alpha+2}) \\ 1243 &\leq CR^{-b}. \end{aligned}$$

1245 By choosing $R > 0$ sufficiently large, we have

$$1246 \quad (4.14) \quad \left| \int_{|x| \geq R} |x|^{-b} (|f_n(x)|^{\alpha+2} - |f(x)|^{\alpha+2}) dx \right| < \frac{\varepsilon}{2}.$$

1248 On the other hand, we have

$$\begin{aligned} 1249 \quad \left| \int_{|x| \leq R} |x|^{-b} (|f_n(x)|^{\alpha+2} - |f(x)|^{\alpha+2}) dx \right| \\ 1250 \quad \leq \| |x|^{-b} \|_{L^\delta(|x| \leq R)} \| |f_n|^{\alpha+2} - |f|^{\alpha+2} \|_{L^\mu(|x| \leq R)} \end{aligned}$$

1252 provided that $\delta, \mu \geq 1, 1 = \frac{1}{\delta} + \frac{1}{\mu}$. The term $\| |x|^{-b} \|_{L^\delta(|x| \leq R)}$ is finite provided that
 1253 $\frac{N}{\delta} > b$. Thus $\frac{1}{\delta} > \frac{b}{N}$ and $\frac{1}{\mu} = 1 - \frac{1}{\delta} < \frac{N-b}{N}$. We next bound

$$1254 \quad \| |f_n|^{\alpha+2} - |f|^{\alpha+2} \|_{L^\mu(|x| \leq R)} \lesssim (\|f_n\|_{L^\sigma}^{\alpha+1} + \|f\|_{L^\sigma}^{\alpha+1}) \|f_n - f\|_{L^\sigma(|x| \leq R)}$$

1255 provided that

$$1256 \quad (4.15) \quad \frac{\alpha+2}{\sigma} = \frac{1}{\mu} < \frac{N-b}{N}.$$

1258 By the Sobolev embedding $H^1 \hookrightarrow L^r$ for any $2 \leq r < 2^*$ and the fact that $f_n \rightarrow f$
 1259 strongly in $L^r(|x| \leq R)$ for any $1 \leq r < 2^*$, we are able to choose $\sigma \in (2, 2^*)$ so that
 1260 (4.15) holds. Indeed, in the case $N \geq 3$, we choose σ smaller but close to $\frac{2N}{N-2}$. We
 1261 see that (4.15) is satisfied provided that

$$1262 \quad \frac{(\alpha+2)(N-2)}{2N} < \frac{N-b}{N}.$$

1263 This condition is fulfilled since $\alpha < \frac{4-2b}{N-2}$. In the case $N = 1, 2$, we see that (4.15) is
 1264 satisfied by choosing σ sufficiently large. As a consequence, we get

$$1265 \quad (4.16) \quad \left| \int_{|x| \leq R} |x|^{-b} (|f_n(x)|^{\alpha+2} - |f(x)|^{\alpha+2}) dx \right| \leq C \|f_n - f\|_{L^\sigma(|x| \leq R)} < \frac{\varepsilon}{2}$$

1267 for n sufficiently large. Collecting (4.14) and (4.16), we prove the result. \square

1268 **LEMMA 4.2.** *Let $N \geq 1, 0 < b < \min\{2, N\}$, and $0 < \alpha < \alpha(N)$. Let Q be the*
 1269 *unique positive radial solution to (1.9). Let $(f_n)_{n \geq 1}$ be a sequence of H^1 -functions*
 1270 *satisfying*

$$1271 \quad M(f_n) = M(Q), \quad E(f_n) = E(Q), \quad \forall n \geq 1$$

1272 and

$$1273 \quad \lim_{n \rightarrow \infty} \|\nabla f_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

1274 Then there exists a subsequence still denoted by $(f_n)_{n \geq 1}$ such that

$$1275 \quad f_n \rightarrow e^{i\theta} Q \quad \text{strongly in } H^1$$

1276 for some $\theta \in \mathbb{R}$ as $n \rightarrow \infty$.

1277 *Proof.* Since (f_n) is a bounded sequence in H^1 , by Lemma 4.1, there exist a
1278 subsequence still denoted by $(f_n)_{n \geq 1}$ and a function $f \in H^1$ such that $f_n \rightarrow f$ weakly
1279 in H^1 and $P(f_n) \rightarrow P(f)$ as $n \rightarrow \infty$. We first observe that

$$\begin{aligned} 1280 \quad P(f) &= \lim_{n \rightarrow \infty} P(f_n) = \lim_{n \rightarrow \infty} (\alpha + 2) \left(\frac{1}{2} \|\nabla f_n\|_{L^2}^2 - E(f_n) \right) \\ 1281 \quad &= (\alpha + 2) \left(\frac{1}{2} \|\nabla Q\|_{L^2}^2 - E(Q) \right) \\ 1282 \quad &= \frac{2(\alpha + 2)}{N\alpha + 2b} \|\nabla Q\|_{L^2}^2 = P(Q). \end{aligned}$$

1284 This shows that $f \neq 0$. Moreover, by the Gagliardo-Nirenberg inequality (2.3), we
1285 have

$$1286 \quad P(f) - C_{\text{opt}} \|\nabla f\|_{L^2}^{\frac{N\alpha+2b}{2}} \|f\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}} \leq 0.$$

1287 By the lower continuity of weak convergence, we have

$$1288 \quad \|\nabla f\|_{L^2} \leq \liminf_{n \rightarrow \infty} \|\nabla f_n\|_{L^2}^2$$

1289 which implies that

$$\begin{aligned} 1290 \quad P(f) - C_{\text{opt}} \|\nabla f\|_{L^2}^{\frac{N\alpha+2b}{2}} \|f\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}} \\ 1291 \quad &\geq \liminf_{n \rightarrow \infty} P(f_n) - C_{\text{opt}} \|\nabla f_n\|_{L^2}^{\frac{N\alpha+2b}{2}} \|f_n\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}} \\ 1292 \quad &= P(Q) - C_{\text{opt}} \|\nabla Q\|_{L^2}^{\frac{N\alpha+2b}{2}} \|Q\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}} = 0. \end{aligned}$$

1294 This shows that f is an optimizer for the Gagliardo-Nirenberg inequality (2.3). We
1295 also have

$$1296 \quad \|\nabla f\|_{L^2} = \lim_{n \rightarrow \infty} \|\nabla f_n\|_{L^2}^2,$$

1297 hence $f_n \rightarrow f$ strongly in H^1 . We claim that there exists $\theta \in \mathbb{R}$ such that $f(x) =$
1298 $e^{i\theta} g(x)$, where g is a non-negative radial optimizer for (2.3). Indeed, since $\|\nabla(|f|)\|_{L^2}$
1299 $\leq \|\nabla f\|_{L^2}$, it is clear that $|f|$ is also an optimizer for (2.3) and

$$1300 \quad (4.17) \quad \|\nabla(|f|)\|_{L^2} = \|\nabla f\|_{L^2}.$$

1302 Set $w(x) := \frac{f(x)}{|f(x)|}$. Since $|w(x)|^2 = 1$, it follows that $\text{Re}(\bar{w} \nabla w(x)) = 0$ and

$$1303 \quad \nabla f(x) = \nabla(|f(x)|)w(x) + |f(x)|\nabla w(x) = w(x)(\nabla(|f(x)|) + |f(x)|\bar{w}(x)\nabla w(x))$$

1304 which implies $|\nabla f(x)|^2 = |\nabla(|f(x)|)|^2 + |f(x)|^2|\nabla w(x)|^2$ for all $x \in \mathbb{R}^3$. From (4.17),
1305 we get

$$1306 \quad \int_{\mathbb{R}^3} |f(x)|^2 |\nabla w(x)|^2 dx = 0$$

1307 which shows $|\nabla w(x)| = 0$, hence $w(x)$ is a constant, and the claim follows with
 1308 $g(x) = |f(x)|$. Moreover, by replacing g with its symmetric rearrangement, we can
 1309 assume that g is radially symmetric. Since g is an optimizer for (2.3), g must satisfy
 1310 the Euler-Lagrange equation

$$1311 \quad \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} W(g + \varepsilon\phi) = 0,$$

1312 where W is the Weinstein functional

$$1313 \quad W(f) := P(f) \div \left[\|\nabla f\|_{L^2}^{\frac{N\alpha+2b}{2}} \|f\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}} \right].$$

1314 A direct computation shows

$$1315 \quad -m\Delta g + ng - \frac{\alpha+2}{C_{\text{opt}}} |x|^{-b} |g|^\alpha g = 0,$$

1316 where

$$1318 \quad m := \frac{N\alpha+2b}{2} \|\nabla f\|_{L^2}^{\frac{N\alpha+2b-4}{2}} \|f\|_{L^2}^{\frac{4-2b-(N-2)\alpha}{2}},$$

$$1319 \quad n := \frac{4-2b-(N-2)\alpha}{2} \|\nabla f\|_{L^2}^{\frac{N\alpha+2b}{2}} \|f\|_{L^2}^{\frac{2b+(N-2)\alpha}{2}}.$$

1321 By a change of variable $g(x) = \lambda\phi(\mu x)$ with $\lambda, \mu > 0$ satisfying

$$1322 \quad \mu^2 = \frac{n}{m}, \quad \lambda^\alpha = \frac{nC_{\text{opt}}}{\alpha+2} \mu^{-b},$$

1323 we see that ϕ solves (1.9) and $W(g) = W(\phi) = C_{\text{opt}}$. By the uniqueness of positive
 1324 radial solution to (1.9) due to [32, 52, 55], we have $\phi \equiv Q$. As $\|g\|_{L^2} = \|Q\|_{L^2}$ and
 1325 $\|\nabla g\|_{L^2} = \|\nabla Q\|_{L^2}$, we infer that $\lambda = \mu = 1$. This shows that $f(x) = e^{i\theta}Q(x)$ for
 1326 some $\theta \in \mathbb{R}$. The proof is complete. \square

1327 *Proof of Theorem 1.4.* We consider separately three cases.

1328 **Case 1.** Let $u_0 \in H^1$ satisfy (1.19) and (1.20). We first note that (1.19) and (1.20)
 1329 are invariant under the scaling

$$1330 \quad (4.18) \quad u_0^\lambda(x) := \lambda^{\frac{2-b}{\alpha}} u_0(\lambda x), \quad \lambda > 0.$$

1332 By choosing a suitable scaling, we can assume that

$$1333 \quad (4.19) \quad M(u_0) = M(Q), \quad E(u_0) = E(Q).$$

1335 Thus (1.20) becomes $\|\nabla u_0\|_{L^2} < \|\nabla Q\|_{L^2}$. We first claim that

$$1336 \quad (4.20) \quad \|\nabla u(t)\|_{L^2} < \|\nabla Q\|_{L^2}$$

1338 for all $t \in (-T_*, T^*)$. Assume by contradiction that there exists $t_0 \in (-T_*, T^*)$ such
 1339 that $\|\nabla u(t_0)\|_{L^2} \geq \|\nabla Q\|_{L^2}$. By continuity, there exists $t_1 \in (-T_*, T^*)$ such that
 1340 $\|\nabla u(t_1)\|_{L^2} = \|\nabla Q\|_{L^2}$. By the conservation of energy and (2.6), we see that

$$1341 \quad P(u(t_1)) = (\alpha+2) \left(\frac{1}{2} \|\nabla u(t_1)\|_{L^2}^2 - E(u(t_1)) \right)$$

$$1342 \quad = (\alpha+2) \left(\frac{1}{2} \|\nabla Q\|_{L^2}^2 - E(Q) \right)$$

$$1343 \quad = \frac{2(\alpha+2)}{N\alpha+2b} \|\nabla Q\|_{L^2}^2.$$

1344

1345 This shows that $u(t_1)$ is an optimizer for the Gagliardo-Nirenberg inequality (2.3).
 1346 Arguing as in the proof of Lemma 4.2, we have $u(t_1) = e^{i\theta}Q$ for some $\theta \in \mathbb{R}$. Moreover,
 1347 by the uniqueness of solution to (1.2), we infer that $u(t) = e^{it}e^{i\theta}Q$ which contradicts
 1348 (1.20). This shows (4.20). In particular, the solution exists globally in time. We now
 1349 have two possibilities.

1350 **First possibility.** If

$$1351 \quad \sup_{t \in \mathbb{R}} \|\nabla u(t)\|_{L^2} < \|\nabla Q\|_{L^2},$$

1352 then there exists $\rho > 0$ such that

$$1353 \quad \|\nabla u(t)\|_{L^2} \leq (1 - \rho)\|\nabla Q\|_{L^2}$$

1354 which, by (4.19), implies that (4.1) holds for all $t \in \mathbb{R}$. By the same argument as in the
 1355 proof of Theorem 1.3, we prove (1.21). In particular, if $N \geq 2$ and $0 < b < \min\{2, \frac{N}{2}\}$,
 1356 then by Theorem 1.1, the solution scatters in H^1 in both directions.

1357 **Second possibility.** If

$$1358 \quad \sup_{t \in \mathbb{R}} \|\nabla u(t)\|_{L^2} = \|\nabla Q\|_{L^2},$$

1359 then there exists a time sequence $(t_n)_{n \geq 1} \subset \mathbb{R}$ such that

$$1360 \quad M(u(t_n)) = M(Q), \quad E(u(t_n)) = E(Q), \quad \lim_{n \rightarrow \infty} \|\nabla u(t_n)\|_{L^2} = \|\nabla Q\|_{L^2}.$$

1361 We notice that $|t_n| \rightarrow \infty$. Otherwise, passing to a subsequence if necessary, we have
 1362 $t_n \rightarrow t_0$ as $n \rightarrow \infty$. By continuity of the solution, we have $u(t_n) \rightarrow u(t_0)$ strongly in
 1363 H^1 . This implies that $u(t_0)$ is an optimizer for (2.3) which is a contradiction.

1364 Applying Lemma 4.2 with $f_n = u(t_n)$, we prove that up to a subsequence,

$$1365 \quad u(t_n) \rightarrow e^{i\theta}Q \text{ strongly in } H^1$$

1366 for some $\theta \in \mathbb{R}$ as $n \rightarrow \infty$.

1367 **Case 2.** Let $u_0 \in H^1$ satisfy (1.19) and (1.23). By the scaling (4.18), we can assume
 1368 that

$$1369 \quad M(u_0) = M(Q), \quad \|\nabla u_0\|_{L^2} = \|\nabla Q\|_{L^2}, \quad E(u_0) = E(Q).$$

1370 In particular, u_0 is an optimizer for (2.3) which implies $u_0(x) = e^{i\theta}Q(x)$ for some
 1371 $\theta \in \mathbb{R}$. By the uniqueness of solution to (1.2), we have $u(t, x) = e^{it}e^{i\theta}Q(x)$.

1372 **Case 3.** Let $u_0 \in H^1$ satisfy (1.19) and (1.24). As in Case 1, we can assume that

$$1373 \quad (4.21) \quad M(u_0) = M(Q), \quad E(u_0) = E(Q), \quad \|\nabla u_0\|_{L^2} > \|\nabla Q\|_{L^2}.$$

1375 Arguing as above, we prove that

$$1376 \quad \|\nabla u(t)\|_{L^2} > \|\nabla Q\|_{L^2}$$

1377 for all $t \in (-T_*, T^*)$. Let us consider only positive times. The one for negative times
 1378 is similar. If $T^* < \infty$, then we are done. Otherwise, if $T^* = \infty$, then we consider two
 1379 possibilities.

1380 **First possibility.** If

$$1381 \quad \sup_{t \in [0, \infty)} \|\nabla u(t)\|_{L^2} > \|\nabla Q\|_{L^2},$$

1382 then there exists $\rho > 0$ such that

$$1383 \quad (4.22) \quad \|\nabla u(t)\|_{L^2} \geq (1 + \rho)\|\nabla Q\|_{L^2}$$

1385 for all $t \in [0, \infty)$. By (4.21) and the conservation laws of mass and energy, we have

$$\begin{aligned} 1386 \quad & G(u(t))[M(u(t))]^{\sigma_c} \\ 1387 \quad &= \frac{N\alpha + 2b}{2} E(u(t))[M(u(t))]^{\sigma_c} - \frac{N\alpha - 4 + 2b}{4} (\|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2}^{\sigma_c})^2 \\ 1388 \quad &\leq \frac{N\alpha + 2b}{2} E(Q)[M(Q)]^{\sigma_c} - \frac{N\alpha - 4 + 2b}{4} ((1 + \rho)\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^2 \\ 1389 \quad &= -\frac{N\alpha - 4 + 2b}{4} ((1 + \rho)^2 - 1) (\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^2 \\ 1390 \end{aligned}$$

1391 for all $t \in [0, \infty)$. By Theorem 1.2, there exists a time sequence $t_n \rightarrow \infty$ such that
1392 $\|\nabla u(t_n)\|_{L^2} \rightarrow \infty$ as $n \rightarrow \infty$.

1393 **Second possibility.** If

$$1394 \quad \sup_{t \in [0, \infty)} \|\nabla u(t)\|_{L^2} = \|\nabla Q\|_{L^2},$$

1395 then there exists a time sequence $(t_n)_{n \geq 1}$ such that $\|\nabla u(t_n)\|_{L^2} \rightarrow \|\nabla Q\|_{L^2}$ as $n \rightarrow \infty$.
1396 Arguing as in Case 1, we show that $t_n \rightarrow \infty$ and

$$1397 \quad u(t_n) \rightarrow e^{i\theta} Q \text{ strongly in } H^1$$

1398 for some $\theta \in \mathbb{R}$ as $n \rightarrow \infty$. This completes the first part of Item (3) of Theorem 1.4.

1399 Let us prove the second part of Item (3) of Theorem 1.4.

1400 • **Finite variance data.** If we assume in addition that $u_0 \in \Sigma$, then the first
1401 possibility cannot occur. In fact, if it occurs, then there exists $\delta > 0$ such that

$$1402 \quad G(u(t)) \leq -\delta$$

1403 for all $t \in [0, \infty)$. This is impossible by the convexity argument as

$$1404 \quad \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8G(u(t)).$$

1405 • **Radially symmetric data.** If we assume in addition that $N \geq 2$, $\alpha \leq 4$, and u_0 is
1406 radially symmetric, then the first possibility cannot occur. In fact, suppose that the
1407 first possibility occurs, so (4.22) holds. It follows from (4.21) and (2.7) that

$$\begin{aligned} 1408 \quad & 8G(u(t)) + \varepsilon \|\nabla u(t)\|_{L^2}^2 \\ 1409 \quad &= 4(N\alpha + 2b)E(u(t))[M(u(t))]^{\sigma_c} - (2N\alpha - 4b + 8 - \varepsilon)\|\nabla u(t)\|_{L^2}^2 [M(u(t))]^{\sigma_c} \\ 1410 \quad &\leq 4(N\alpha + 2b)E(Q)[M(Q)]^{\sigma_c} - (2N\alpha - 4b + 8 - \varepsilon)(1 + \rho)^2 (\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^2 \\ 1411 \quad &= -2(N\alpha - 4 + 2b) (\|\nabla Q\|_{L^2} \|Q\|_{L^2}^{\sigma_c})^2 (1 + \rho)^2 \left[\frac{(1 + \rho)^2 - 1}{(1 + \rho)^2} - \frac{\varepsilon}{2(N\alpha - 4 + 2b)} \right] \\ 1412 \end{aligned}$$

1413 for all $t \in [0, \infty)$. Taking $\varepsilon > 0$ sufficiently small, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$1414 \quad (4.23) \quad 8G(u(t)) + \varepsilon \|\nabla u(t)\|_{L^2}^2 \leq -\delta$$

1416 for all $t \in [0, \infty)$. We recall the following estimate due to [14, Lemma 3.4]: for any
1417 $R > 1$ and any $\varepsilon > 0$,

$$1418 \quad V''_{\varphi_R}(t) \leq 8G(u(t)) + \begin{cases} CR^{-2} + CR^{-[2(N-1)+b]} \|\nabla u(t)\|_{L^2}^2 & \text{if } \alpha = 4, \\ CR^{-2} + C\varepsilon^{-\frac{\alpha}{4-\alpha}} R^{-\frac{2[(N-1)\alpha+2b]}{4-\alpha}} + \varepsilon \|\nabla u(t)\|_{L^2}^2 & \text{if } \alpha < 4. \end{cases}$$

1419 Thanks to (4.23), we take $R > 1$ sufficiently large if $\alpha = 4$, and $\varepsilon > 0$ sufficiently
1420 small and $R > 1$ sufficiently large depending on ε , we obtain

$$1421 \quad V''_{\varphi_R}(t) \leq -\frac{\delta}{2}$$

1422 for all $t \in [0, \infty)$. This is impossible.

1423 **• Cylindrically symmetric data.** If we assume in addition that $N \geq 3$, $\alpha \leq 2$, and
1424 $u_0 \in \Sigma_N$, then the first possibility cannot occur. This is done by the same argument
1425 as above using (4.12) and (4.23). The proof of Theorem 1.4 is now complete. \square

1426 Finally, we study long time dynamics of H^1 -solutions for (1.2) with data above
1427 the ground state threshold.

1428 *Proof of Theorem 1.5.* Let us consider two cases.

1429 **Case 1.** Let $u_0 \in \Sigma$ satisfy (1.27), (1.28), (1.29), and (1.30). We will show that
1430 (1.11) holds. To this end, let us start with the following estimate: for $f \in \Sigma$,

$$1431 \quad \left(\operatorname{Im} \int \bar{f}x \cdot \nabla f dx \right)^2 \\ 1432 \quad (4.24) \quad \leq \|xf\|_{L^2}^2 \left(\|\nabla f\|_{L^2}^2 - [C_{\text{opt}}]^{-\frac{4}{N\alpha+2b}} [M(f)]^{-\frac{4-2b-(N-2)\alpha}{N\alpha+2b}} [P(f)]^{\frac{4}{N\alpha+2b}} \right). \\ 1433$$

1434 In fact, let $\lambda > 0$. We have

$$1435 \quad \int |\nabla(e^{i\lambda|x|^2} f)|^2 dx = 4\lambda^2 \|xf\|_{L^2}^2 + 4\lambda \operatorname{Im} \int \bar{f}x \cdot \nabla f dx + \|\nabla f\|_{L^2}^2. \\ 1436$$

1437 By the Gagliardo-Nirenberg inequality (2.3), we have

$$1438 \quad [P(f)]^{\frac{4}{N\alpha+2b}} = [P(e^{i\lambda|x|^2} f)]^{\frac{4}{N\alpha+2b}} \leq [C_{\text{opt}}]^{-\frac{4}{N\alpha+2b}} \|\nabla(e^{i\lambda|x|^2} f)\|_{L^2}^2 \|f\|_{L^2}^{\frac{2[4-2b-(N-2)\alpha]}{N\alpha+2b}}$$

1439 or

$$1440 \quad \|\nabla(e^{i\lambda|x|^2} f)\|_{L^2}^2 \geq [C_{\text{opt}}]^{-\frac{4}{N\alpha+2b}} M(f)^{-\frac{4-2b-(N-2)\alpha}{N\alpha+2b}} [P(f)]^{\frac{4}{N\alpha+2b}}.$$

1441 It follows that

$$1442 \quad 4\lambda^2 \|xf\|_{L^2}^2 + 4\lambda \operatorname{Im} \int \bar{f}x \cdot \nabla f dx + \|\nabla f\|_{L^2}^2 \\ 1443 \quad - [C_{\text{opt}}]^{-\frac{4}{N\alpha+2b}} [M(f)]^{-\frac{4-2b-(N-2)\alpha}{N\alpha+2b}} [P(f)]^{\frac{4}{N\alpha+2b}} \geq 0$$

1445 for all $\lambda > 0$. Since the left hand side is a quadratic polynomial in λ , its discriminant
1446 must be non-positive which proves (4.24).

1447 We also have

$$1448 \quad V''(t) = 8\|\nabla u(t)\|_{L^2}^2 - \frac{4(N\alpha + 2b)}{\alpha + 2} P(u(t)) \\ 1449 \quad = 16E(u(t)) - \frac{4(N\alpha - 4 + 2b)}{\alpha + 2} P(u(t)) \\ 1450 \quad = 4(N\alpha + 2b)E(u(t)) - 2(N\alpha - 4 + 2b)\|\nabla u(t)\|_{L^2}^2$$

1452 which implies that

$$1453 \quad P(u(t)) = \frac{\alpha + 2}{4(N\alpha - 4 + 2b)} (16E(u(t)) - V''(t)),$$

$$1454 \quad \|\nabla u(t)\|_{L^2}^2 = \frac{1}{2(N\alpha - 4 + 2b)} (4(N\alpha + 2b)E(u(t)) - V''(t)).$$

1456 Since $P(u(t)) \geq 0$, we have $V''(t) \leq 16E(u(t)) = 16E(u_0)$. Inserting the above
1457 identities to (4.24), we get

$$1458 \quad (V'(t))^2 \leq 16V(t) \left[\frac{1}{2(N\alpha - 4 + 2b)} (4(N\alpha + 2b)E(u(t)) - V''(t)) \right. \\ 1459 \quad \left. - [C_{\text{opt}}]^{-\frac{4}{N\alpha+2b}} [M(u(t))]^{-\frac{4-2b-(N-2)\alpha}{N\alpha+2b}} \left(\frac{\alpha + 2}{4(N\alpha - 4 + 2b)} (16E(u(t)) - V''(t)) \right)^{\frac{4}{N\alpha+2b}} \right]$$

1461 which implies

$$1462 \quad (4.25) \quad (z'(t))^2 \leq 4g(V''(t)),$$

1464 where

$$1465 \quad z(t) := \sqrt{V(t)}$$

1466 and

$$1468 \quad g(\lambda) := \frac{1}{2(N\alpha - 4 + 2b)} (4(N\alpha + 2b)E - \lambda) \\ 1469 \quad - [C_{\text{opt}}]^{-\frac{4}{N\alpha+2b}} M^{-\frac{4-2b-(N-2)\alpha}{N\alpha+2b}} \left(\frac{\alpha + 2}{4(N\alpha - 4 + 2b)} (16E - \lambda) \right)^{\frac{4}{N\alpha+2b}}$$

1471 with $\lambda \leq 16E$. Here we have used the notation $E(u(t)) = E$, $M(u(t)) = M$ due to the
1472 conservation of mass and energy. Since $N\alpha + 2b > 4$, we see that $g(\lambda)$ is decreasing
1473 on $(-\infty, \lambda_0)$ and increasing on $(\lambda_0, 16E)$, where λ_0 satisfies

$$1474 \quad (4.26) \quad \frac{N\alpha + 2b}{2(\alpha + 2)} = [C_{\text{opt}}]^{-\frac{4}{N\alpha+2b}} M^{-\frac{4-2b-(N-2)\alpha}{N\alpha+2b}} \left(\frac{\alpha + 2}{4(N\alpha - 4 + 2b)} (16E - \lambda_0) \right)^{\frac{4-N\alpha-2b}{N\alpha+2b}}.$$

1476 A direct calculation shows

$$1477 \quad g(\lambda_0) = \frac{1}{2(N\alpha - 4 + 2b)} (4(N\alpha + 2b)E - \lambda_0) - \frac{N\alpha + 2b}{8(N\alpha - 4 + 2b)} (16E - \lambda_0) = \frac{\lambda_0}{8}.$$

1479 Using the fact that

$$1480 \quad C_{\text{opt}} = \frac{2(\alpha + 2)}{N\alpha + 2b} \left(\frac{2(N\alpha + 2b)}{N\alpha - 4 + 2b} E(Q)[M(Q)]^{\sigma_c} \right)^{-\frac{N\alpha-4+2b}{4}},$$

1481 we infer from (4.26) that

$$1482 \quad 1 = \frac{16E(Q)[M(Q)]^{\sigma_c}}{(16E - \lambda_0)M^{\sigma_c}}$$

1483 or

$$1484 \quad (4.27) \quad \frac{EM^{\sigma_c}}{E(Q)[M(Q)]^{\sigma_c}} \left(1 - \frac{\lambda_0}{16E} \right) = 1.$$

1485

1486 Thus the assumption (1.27) is equivalent to

$$1487 \quad (4.28) \quad \lambda_0 \geq 0.$$

1489 Moreover, the assumption (1.28) is equivalent to

$$1490 \quad (V'(0))^2 \geq 2V(0)\lambda_0$$

1491 or

$$1492 \quad (4.29) \quad (z'(0))^2 \geq \frac{\lambda_0}{2} = 4g(\lambda_0).$$

1494 Similarly, the assumption (1.30) is equivalent to

$$1495 \quad (4.30) \quad z'(0) \geq 0.$$

1497 Finally, the assumption (1.29) is equivalent to

$$1498 \quad (4.31) \quad V''(0) > \lambda_0.$$

1500 Indeed, from (1.29), we have

$$\begin{aligned} 1501 \quad V''(0) &= 16E - \frac{4(N\alpha - 4 + 2b)}{\alpha + 2} P(u_0) \\ 1502 \quad &> 16E - \frac{4(N\alpha - 4 + 2b)}{\alpha + 2} \frac{P(Q)[M(Q)]^{\sigma_c}}{M^{\sigma_c}} \\ 1503 \quad &= 16 \left(E - \frac{E(Q)[M(Q)]^{\sigma_c}}{M^{\sigma_c}} \right) \\ 1504 \quad &= 16E \left(1 - \frac{E(Q)[M(Q)]^{\sigma_c}}{EM^{\sigma_c}} \right) \\ 1505 \quad &= \lambda_0, \end{aligned}$$

1507 where we have used (4.27) to get the last equality.

1508 Next, we claim that there exists $\delta_0 > 0$ small such that for all $t \in [0, T^*)$,

$$1509 \quad (4.32) \quad V''(t) \geq \lambda_0 + \delta_0.$$

1511 Assume (4.32) for the moment, we prove (1.11). We have

$$\begin{aligned} 1512 \quad P(u(t))[M(u(t))]^{\sigma_c} &= \frac{\alpha + 2}{4(N\alpha - 4 + 2b)} (16E - V''(t)) M^{\sigma_c} \\ 1513 \quad &\leq \frac{\alpha + 2}{4(N\alpha - 4 + 2b)} (16E - \lambda_0 - \delta_0) M^{\sigma_c} \\ 1514 \quad &= \frac{4(\alpha + 2)}{N\alpha - 4 + 2b} E(Q)[M(Q)]^{\sigma_c} - \frac{\alpha + 2}{4(N\alpha - 4 + 2b)} \delta_0 M^{\sigma_c} \\ 1515 \quad &= (1 - \rho) P(Q)[M(Q)]^{\sigma_c} \end{aligned}$$

1517 for all $t \in [0, T^*)$, where $\rho := \frac{\alpha + 2}{4(N\alpha - 4 + 2b)} \delta_0 \frac{M^{\sigma_c}}{P(Q)[M(Q)]^{\sigma_c}} > 0$. Here we have used
 1518 (4.27) to get the third line. This shows (1.11). In particular, if $N \geq 2$ and $0 < b <$
 1519 $\min\{2, \frac{N}{2}\}$, then the solution scatters in H^1 forward in time.

1520 It remains to show (4.32). By (4.31), we take $\delta_1 > 0$ so that

$$1521 \quad V''(0) \geq \lambda_0 + 2\delta_1.$$

1522 By continuity, we have

$$1523 \quad (4.33) \quad V''(t) > \lambda_0 + \delta_1, \quad \forall t \in [0, t_0].$$

1525 for $t_0 > 0$ sufficiently small. By reducing t_0 if necessary, we can assume that

$$1526 \quad (4.34) \quad z'(t_0) > 2\sqrt{g(\lambda_0)}.$$

1528 In fact, if $z'(0) > 2\sqrt{g(\lambda_0)}$, then (4.34) follows from the continuity argument. Other-
1529 wise, if $z'(0) = 2\sqrt{g(\lambda_0)}$, then using the fact that

$$1530 \quad (4.35) \quad z''(t) = \frac{1}{z(t)} \left(\frac{V''(t)}{2} - (z'(t))^2 \right)$$

1532 and (4.31), we have $z''(0) > 0$. This shows (4.34) by taking $t_0 > 0$ sufficiently small.
1533 Thanks to (4.34), we take $\epsilon_0 > 0$ be a small constant so that

$$1534 \quad (4.36) \quad z'(t_0) \geq 2\sqrt{g(\lambda_0)} + 2\epsilon_0.$$

1536 We will prove by contradiction that

$$1537 \quad (4.37) \quad z'(t) > 2\sqrt{g(\lambda_0)} + \epsilon_0, \quad \forall t \geq t_0.$$

1539 Suppose that it is not true and set

$$1540 \quad t_1 := \inf \left\{ t \geq t_0 : z'(t) \leq 2\sqrt{g(\lambda_0)} + \epsilon_0 \right\}.$$

1541 By (4.36), we have $t_1 > t_0$. By continuity, we have

$$1542 \quad (4.38) \quad z'(t_1) = 2\sqrt{g(\lambda_0)} + \epsilon_0$$

1544 and

$$1545 \quad (4.39) \quad z'(t) \geq 2\sqrt{g(\lambda_0)} + \epsilon_0, \quad \forall t \in [t_0, t_1].$$

1547 By (4.25), we see that

$$1548 \quad (4.40) \quad \left(2\sqrt{g(\lambda_0)} + \epsilon_0 \right)^2 \leq (z'(t))^2 \leq 4g(V''(t)), \quad \forall t \in [t_0, t_1].$$

1550 It follows that $g(V''(t)) > g(\lambda_0)$ for all $t \in [t_0, t_1]$, thus $V''(t) \neq \lambda_0$ and by continuity,
1551 $V''(t) > \lambda_0$ for all $t \in [t_0, t_1]$.

1552 We will prove that there exists a constant $C > 0$ such that

$$1553 \quad (4.41) \quad V''(t) \geq \lambda_0 + \frac{\sqrt{\epsilon_0}}{C}, \quad \forall t \in [t_0, t_1].$$

1555 Indeed, by the Taylor expansion of g near λ_0 with the fact $g'(\lambda_0) = 0$, there exists
1556 $a > 0$ such that

$$1557 \quad (4.42) \quad g(\lambda) \leq g(\lambda_0) + a(\lambda - \lambda_0)^2, \quad \forall \lambda : |\lambda - \lambda_0| \leq 1.$$

1559 If $V''(t) \geq \lambda_0 + 1$, then (4.41) holds by taking C large. If $\lambda_0 < V''(t) \leq \lambda_0 + 1$, then
 1560 by (4.40) and (4.42), we get

$$1561 \quad \left(2\sqrt{g(\lambda_0)} + \epsilon_0\right)^2 \leq (z'(t))^2 \leq 4g(V''(t)) \leq 4g(\lambda_0) + 4a(V''(t) - \lambda_0)^2$$

1562 thus

$$1563 \quad 4\epsilon_0\sqrt{g(\lambda_0)} + \epsilon_0^2 \leq 4a(V''(t) - \lambda_0)^2.$$

1564 This shows (4.41) with $C = \sqrt{a}[g(\lambda_0)]^{-\frac{1}{4}}$.

1565 However, by (4.35), (4.38) and (4.41), we have

$$1566 \quad z''(t_1) = \frac{1}{z(t_1)} \left(\frac{V''(t_1)}{2} - (z'(t_1))^2 \right)$$

$$1567 \quad \geq \frac{1}{z(t_1)} \left(\frac{\lambda_0}{2} + \frac{\sqrt{\epsilon_0}}{2C} - \left(2\sqrt{g(\lambda_0)} + \epsilon_0\right)^2 \right)$$

$$1568 \quad \geq \frac{1}{z(t_1)} \left(\frac{\sqrt{\epsilon_0}}{2C} - 4\epsilon_0\sqrt{g(\lambda_0)} - \epsilon_0^2 \right) > 0$$

1570 provided that ϵ_0 is taken small enough. This however contradicts (4.38) and (4.39).
 1571 This proves (4.37). Note that we have also proved (4.41) for all $t \in [t_0, T^*)$. This
 1572 together with (4.33) imply (4.32) with $\delta_0 = \min \left\{ \delta_1, \frac{\sqrt{\epsilon_0}}{C} \right\}$.

1573 **Case 2.** Let $u_0 \in \Sigma$ satisfy (1.27), (1.28), (1.31) and (1.32). As in Step 1, we see
 1574 that the conditions (1.27), (1.28), (1.31) and (1.32) are respectively equivalent to

$$1575 \quad (4.43) \quad \lambda_0 \geq 0, \quad (z'(0))^2 \geq 4g(\lambda_0) = \frac{\lambda_0}{2}, \quad V''(0) < \lambda_0, \quad z'(0) \leq 0.$$

1577 We claim that

$$1578 \quad (4.44) \quad z''(t) < 0, \quad \forall t \in [0, T^*).$$

1580 Note that by (4.35), we have $z''(0) < 0$. Assume by contraction that (4.44) does not
 1581 hold. Then there exists $t_0 \in (0, T^*)$ such that

$$1582 \quad z''(t) < 0, \quad \forall t \in [0, t_0)$$

1583 and $z''(t_0) = 0$. By (4.43), we have

$$1584 \quad z'(t) < z'(0) \leq -2\sqrt{g(\lambda_0)}, \quad \forall t \in (0, t_0].$$

1585 Hence $(z'(t))^2 > 2g(\lambda_0)$ which combined with (4.25) imply that

$$1586 \quad g(V''(t)) > g(\lambda_0), \quad \forall t \in (0, t_0].$$

1587 It follows that $V''(t) \neq \lambda_0$ for all $t \in (0, t_0]$, and by continuity, we have

$$1588 \quad V''(t) < \lambda_0, \quad \forall t \in [0, t_0].$$

1589 By (4.35), we obtain

$$1590 \quad z''(t_0) = \frac{1}{z(t_0)} \left(\frac{V''(t_0)}{2} - (z'(t_0))^2 \right) < \frac{1}{z(t_0)} \left(\frac{\lambda_0}{2} - \frac{\lambda_0}{2} \right) = 0$$

1591 which is absurd. Now, assume by contradiction that the solution exists globally
 1592 forward in time, i.e., $T^* = \infty$. By (4.44), we see that

$$1593 \quad z'(t) \leq z'(1) < z'(0) \leq 0, \quad \forall t \in [1, \infty).$$

1594 This contradicts with the fact that $z(t)$ is positive. The proof is complete. \square

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