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► **To cite this version:**

Sébastien Bossu, Stéphane Crépey, Hoang-Dung Nguyen. Spanning Multi-Asset Payoffs With ReLUs. 2024. hal-04505407v1

**HAL Id: hal-04505407**

**<https://hal.science/hal-04505407v1>**

Preprint submitted on 14 Mar 2024 (v1), last revised 20 Mar 2024 (v2)

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# Spanning Multi-Asset Payoffs With ReLUs\*

Sébastien Bossu<sup>†</sup>, Stéphane Crépey<sup>‡</sup>, Hoang-Dung Nguyen<sup>§</sup>

March 14, 2024

## Abstract

We propose a distributional formulation of the spanning problem of a multi-asset payoff by vanilla basket options. This problem is shown to have a unique solution if and only if the payoff function is even and absolutely homogeneous, and we establish a Fourier-based formula to calculate the solution. Financial payoffs are typically piecewise linear, resulting in a solution that may be derived explicitly, yet may also be hard to numerically exploit. One-hidden-layer feedforward neural networks instead provide a natural and efficient numerical alternative for discrete spanning. We test this approach for a selection of archetypal payoffs and obtain better hedging results with vanilla basket options compared to industry-favored approaches based on single-asset vanilla hedges.

**Keywords:** Carr-Madan spanning formula, basket options, Fourier transform, iterated integrals, measures, distributions, Cauchy principal value integral, one-hidden-layer feedforward ReLU neural network, dispersion call, static hedging.

**Mathematics Subject Classification:** 91G20, 62G08, 62M45, 42B10, 46A11.

## 1 Introduction

The popular Carr and Madan (1998) spanning formula shows that perfect replication may be achieved for single-asset options with twice differentiable payoffs, using an infinite number of vanilla calls whose strikes span a one-dimensional continuum. In honor of this formula, we refer to the practice of hedging a target payoff with a static

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\*Python code with documentation available at [https://github.com/hoangdungnguyen/Spanning\\_multi\\_asset\\_payoffs](https://github.com/hoangdungnguyen/Spanning_multi_asset_payoffs). We thank Nisrine Madhar for her contributions to the numerical portion of this paper.

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Target option *	Notation	Payoff $F(\mathbf{x}, k)$	AH variation <sup>†</sup>
Dispersion call/put	DC/DP	$(\sum_j  x_j  - k)^\pm$	$(\sum_j  x_j  -  k )^\pm$
Best-of call/put	BOC/BOP	$(\max_j x_j - k)^\pm$	$(\max_j  x_j  -  k )^\pm$
Worst-of call/put	WOC/WOP	$(\min_j x_j - k)^\pm$	$(\min_j  x_j  -  k )^\pm$
Best-of-binary call/put	BOBC/BOBP	$\mathbf{1}_{\pm(\max_j x_j - k) > 0}$	n/a
Worst-of-binary call/put	WOBC/WOBP	$\mathbf{1}_{\pm(\min_j x_j - k) > 0}$	n/a

Table 1: Option payoffs covered in our numerics.  $k$  is the strike price,  $\mathbf{x} = (x_1, x_2, \dots, x_d)$  is the vector of asset performances (price ratios or returns),  $z^\pm = \max(0, \pm z)$  is the positive or negative part of  $z$ .

Notes: \* All options trade on over-the-counter financial markets, either directly or as building blocks for structured products (Bossu, 2014; Schofield, 2017). † Absolutely homogeneous (AH) variation of the payoff formula, provided in view of Theorem 3.4.

portfolio of vanilla payoffs as “spanning” throughout this paper. Bossu, Carr, and Papanicolaou (2022) and Bossu (2022) spearheaded similar perfect spanning formulas for  $\ell^2$  dispersion options and more general multi-asset absolutely homogeneous payoffs, using an infinite number of vanilla basket calls whose weights span a multidimensional continuum. Specifically, given a target European payoff function  $F(x_1, \dots, x_d, k)$  of  $d$  underlying asset performances  $x_j$  (terminal price ratios or returns) and moneyness parameter  $k$ , one possible formulation of the continuum spanning problem in strong form is to find a combination of cash, underlying asset and vanilla basket options in respective quantities  $\alpha, \mu_j, \nu(\dots)$ , such that

$$\begin{aligned}
F(x_1, \dots, x_d, k) &= \alpha + \sum_{j=1}^d \mu_j x_j \\
&+ \int \cdots \int \nu(w_1, \dots, w_d) \left( \sum_{j=1}^d w_j x_j - k \right)^+ dw_1 \dots dw_d, \quad \text{for all } x_1, \dots, x_d,
\end{aligned} \tag{1.1}$$

wherein  $w_1, \dots, w_d$  are the asset weights of the basket call payoff  $\left( \sum_{j=1}^d w_j x_j - k \right)^+$ . The primary aim of this paper is to deal with the delicate mathematical regularization aspects identified by the aforementioned authors. To this end, we introduce a weak formulation of the continuum spanning problem to allow for distributional solutions in a rigorous setting.

In practice, only a finite number of basket calls and puts may be traded and spanning is imperfect. On top of contributing to the above theory, we study how a European multi-asset payoff may be partially hedged with a finite combination of cash, underlying assets, and vanilla basket calls and puts, corresponding to a discretization of the right-hand side of (1.1). For the selection of archetypal payoffs listed in Table 1, we use neural network training techniques to empirically investigate how the discrete spanning error

$$F(x_1, \dots, x_d, k) - \alpha - \sum_{j=1}^d \mu_j x_j - \sum_{i=1}^n \nu_i \left( w_1^{(i)} x_1 + \cdots + w_d^{(i)} x_d - k_i \right)^+ \tag{1.2}$$

may be minimized with respect to some error metric such as MSE (mean squared

error) for optimal quantities  $\alpha$  of cash,  $\mu_j$  of underlying assets  $1 \leq j \leq d$ , and  $\nu_i$  of basket option payoffs  $1 \leq i \leq n$  with associated basket weights  $w_1^{(i)}, \dots, w_d^{(i)}$  and moneyness parameter  $k_i$ . The expressiveness power of the neural network family is well known in theory and in practice, and constitutes a particularly convincing paradigm for discrete spanning since (1.2) corresponds to the prediction error of a one-hidden-layer feedforward neural network. Incidentally, the Carr and Madan spanning formula can also be found in the neural network literature (see Savarese, Evron, Soudry, and Srebro, 2019, Eqns (15)–(17)).

## 1.1 Background and Review

The static hedging of a complex and illiquid payoff with more liquid instruments is useful to derivatives practitioners for price discovery, trading, and risk management applications. The topic of multi-asset payoffs has gained in popularity over the past decade in the academic literature. İlhan, Jonsson, and Sircar (2009) study the problem of optimally hedging exotic derivatives positions with dynamic and static trading strategies when the performance is quantified by a convex risk measure. Carr and Laurence (2011) express the joint implied distribution of several underlying assets  $x_1, \dots, x_d$  at time  $T$  as an inverse Radon transform of basket call prices with maturity  $T$  and derive a multi-asset version of Dupire’s formula. Alexander and Venkatramanan (2012) derive analytic pricing approximations for European basket and rainbow payoffs using a decomposition as sum of compound payoffs. Molchanov and Schmutz (2014) show how the joint implied distribution may be recovered from best-of option prices, and investigate various symmetries of multi-asset derivatives. Cui and Xu (2022) derive a multi-asset extension of the Carr and Madan (1998) static replication formula in the form of multiple integral of products of call payoffs. Chiu and Cont (2023) propose a model-free approach to determine the superhedging cost of path-dependent payoffs, including Asian options.

In the machine learning literature for finance, Lokeshwar, Bhardawaj, and Jain (2022) propose the use of neural network techniques for semi-static hedging of path-dependent exotics with short-term options. Lyons, Nejad, and Arribas (2020) use “signature payoffs” to approximate path-dependent exotic derivatives payoffs. Antonov and Piterbarg (2022) discuss alternatives to neural networks for financial function approximation with an emphasis on linear regression concepts.

## 1.2 Standing Notation and Organization of the Paper

We write  $\mathbf{1}_A \times (\cdot)$  for  $(\cdot)$  if the Boolean expression  $A$  is true, and 0 otherwise (even if  $(\cdot)$  is not defined when  $A$  is false, in accordance with the Iverson bracket definition). We use boldface for vectors. We denote by  $\mathbb{R}_+^*$ ,  $\mathbb{R}_+$ , and  $\mathbb{R}^*$  the sets of positive, nonnegative and nonzero real numbers, respectively. We write  $\lim_{x \rightarrow 0^+}$  for the limit when  $x$  goes to 0 in  $\mathbb{R}_+^*$ . For any positive integer  $q$ , vector  $\mathbf{z} \in \mathbb{R}^q$  and index  $j \in 1..q$ ,  $\mathbf{z}_{\neq j}$  is the subvector excluding the  $j$ th coefficient of  $\mathbf{z}$ ,  $\mathbf{z}_{\leq j} \in \mathbb{R}^j$  is the subvector of the first  $j$  coefficients of  $\mathbf{z}$ , and  $\mathbf{z}_{> j} \in \mathbb{R}^{q-j}$  is the subvector of the last  $q - j$  coefficients of  $\mathbf{z}$  (with  $\mathbf{z}_{> q} = \emptyset$ );  $\|\mathbf{z}\|_1$ ,  $\|\mathbf{z}\|_2 = |\mathbf{z}|$ , and  $\|\mathbf{z}\|_\infty$  are respectively the  $\ell^1$ ,  $\ell^2$  (Euclidean) and  $\ell^\infty$  (maximum) norms, and for any other vector  $\mathbf{z}' \in \mathbb{R}^q$ ,  $\mathbf{z} \cdot \mathbf{z}'$  is the dot product. We

denote by  $\mathcal{F}$  the Fourier transform operator, with  $i^2 = -1$  (see Section B for further notation and details). Unless indicated otherwise, we work with real-valued functions, measures, and distributions, and their corresponding vector spaces and linear forms; in particular:

- $L^p(\mathbb{R}^q)$ , for  $p \in [1, +\infty)$ , the space of Lebesgue-measurable functions on  $\mathbb{R}^q$  such that  $\|f\|_{L^p} := (\int_{\mathbb{R}^q} |f(\mathbf{x})|^p d\mathbf{x})^{1/p} < \infty$ ;
- $\mathcal{C}^\infty(\mathbb{R}^q)$ , the space of infinitely differentiable functions over  $\mathbb{R}^q$ ;
- $\mathcal{S}(\mathbb{R}^q) \subset \mathcal{C}^\infty(\mathbb{R}^q)$ , the space of Schwartz functions over  $\mathbb{R}^q$ , i.e.  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^q)$  such that  $\varphi(\mathbf{z})$  and each of its derivatives vanish faster than any inverse power of  $|\mathbf{z}|$  as  $|\mathbf{z}| \rightarrow \infty$  (Ramm and Katsevich, 1996, page 395), (King, 2009, page 480);
- $\mathcal{D}(\mathbb{R}^q) \subset \mathcal{S}(\mathbb{R}^q)$ , the Schwartz subspace of functions with compact support;
- $\langle T, \varphi \rangle$ , or  $\langle T_{d\mathbf{z}}, \varphi(\mathbf{z}) \rangle_{\mathbf{z}}$  when there may be ambiguity about the variable, the action of a distribution  $T$  over a Schwartz function  $\varphi \in \mathcal{S}(\mathbb{R}^q)$  (generally, the action of a linear form on a test function in an appropriate functional space);
- $\langle \nu(d\mathbf{z}), \varphi(\mathbf{z}) \rangle_{\mathbf{z}} := \int \nu(d\mathbf{z})\varphi(\mathbf{z})$ , the integral of  $\varphi \in \mathcal{S}(\mathbb{R}^q)$  with respect to a measure  $\nu$  on  $\mathbb{R}^q$ , whenever this integral is well defined (e.g. for  $\nu(d\mathbf{z}) = f(\mathbf{z}) d\mathbf{z}$  for  $f$  locally integrable), and  $\nu(d\mathbf{z})$ , the corresponding linear form  $\mathcal{S}(\mathbb{R}^q) \ni \varphi \mapsto \langle \nu(d\mathbf{z}), \varphi(\mathbf{z}) \rangle_{\mathbf{z}}$ ;
- $\langle \delta_{\mathbf{a}}(d\mathbf{z}), f(\mathbf{z}) \rangle_{\mathbf{z}} := f(\mathbf{a})$ , the Dirac mass at point  $\mathbf{a} \in \mathbb{R}^q$  of any function  $f : \mathbb{R}^q \rightarrow \mathbb{C}$ ;
- $\left\langle \frac{dz}{z-c}, \varphi(z) \right\rangle_z := \int_{-\infty}^{\infty} \frac{dz}{z-c} \varphi(z)$ , for  $c \in \mathbb{R}$ , the Cauchy integral of  $\frac{\varphi(z)}{z-c}$ , for  $\varphi \in \mathcal{S}(\mathbb{R})$ , and  $\frac{dz}{z-c}$  the corresponding principal value distribution  $\mathcal{S}(\mathbb{R}) \ni \varphi \mapsto \left\langle \frac{dz}{z-c}, \varphi(z) \right\rangle_z$  (see Section A for further notation and details).

The paper is organized as follows. Section 2 poses a weak, distributional formulation of the continuum spanning problem and examines its correspondence with strong formulations. Section 3 shows that a unique solution exists if and only if the payoff function is even and absolutely homogeneous, and provides a Fourier-based formula (3.3) for its calculation. Section 4 derives explicit solutions of function-, measure-, and principal value-type for specific payoff examples. Section 5 showcases how feedforward neural networks, which have a natural interpretation for discrete payoff spanning, may constitute an effective numerical method compared to other restricted spanning strategies. Section 6 gives our conclusions and perspectives for future research. The most technical proof of the paper is built upon Sections A and B and deferred to Section C.

## 2 The Continuum Spanning Problem

In dimension  $d = 1$ , the Carr and Madan (1998) spanning formula states that any twice differentiable European payoff function  $F(x), x \in \mathbb{R}_+$  can be perfectly replicated by a “continuous portfolio” of vanilla calls in quantities  $F''(K) dK$  with strike prices spanning the continuum  $K \in \mathbb{R}_+$ , together with fixed cash and underlying asset positions in ad-hoc quantities, as

$$F(x) = F(0) + F'(0)x + \int_0^\infty (x - K)^+ F''(K) dK, \quad x \in \mathbb{R}_+. \quad (2.1)$$

It is worth observing that the regularity condition on  $F$  formally excludes most financial payoffs, such as a simple straddle payoff  $F(x) = |x - 1|$  which is not differentiable at  $x = 1$ . To reconcile the trivial replication identity “long straddle = long 2 calls, short 1 forward contract” with the Carr-Madan formula, we may write

$$|x - 1| = -(x - 1) + 2(x - 1)^+ = 1 - x + 2 \int_0^\infty (x - K)^+ \delta_1(dK), \quad \mathbf{x} \in \mathbb{R}_+.$$

Here, a theoretical extension is needed to give definite meaning to such statements as “ $F''(K) dK = 2\delta_1(dK)$ ” in order to match (2.1). This is the aim of Sections 2 to 4 of this paper, in general dimension  $d \geq 1$  for which solutions become nontrivial.

Bossu, Carr, and Papanicolaou (2022) and Bossu (2022) explored replication identities for European multi-asset options with absolutely homogeneous payoffs  $F(\lambda \mathbf{x}, \lambda k) = |\lambda| F(\mathbf{x}, k)$  for  $\lambda \in \mathbb{R}^*$ , such as

$$F(\mathbf{x}, k) = \int_{\mathbb{R}^d} (\mathbf{w} \cdot \mathbf{x} - k)^+ \nu(d\mathbf{w}), \quad \mathbf{x} \in \mathbb{R}^d, \quad (2.2)$$

where  $k$  is a real parameter and  $\nu(d\mathbf{w})$  is the quantity of vanilla basket calls with basket weights  $\mathbf{w}$  spanning the continuum  $\mathbf{w} \in \mathbb{R}^d$ . In this multidimensional version of the spanning problem, the cash and asset terms  $\alpha + \boldsymbol{\mu} \cdot \mathbf{x}$  are moved to the left-hand side and inside the arbitrary target payoff function  $F(\mathbf{x}, k)$  for ease of writing. Section D shows how the Carr-Madan formula (2.1) may be rewritten as (2.2) for  $d = 1$  under specific conditions.

## 2.1 Weak Formulation of the Continuum Spanning Problem

Our continuum spanning results are established in general dimension  $d \geq 1$  for a class of distributions which are not necessarily of measure-type. This theoretical extension is necessary to give precise mathematical meaning to spanning solutions found for even simple financial payoffs, such as formula (4.7) for the dispersion call in dimension 2. To this end, we define the function set

$$\mathcal{S}_0 = \{\phi_0 \in \mathcal{C}^\infty(\mathbb{R}^{d+1}, \mathbb{C}); \phi_0(\mathbf{x}, k) = e^{-irk} h(\mathbf{x}) \text{ for some } r \in \mathbb{R}^* \text{ and } h \text{ in } \mathcal{S}(\mathbb{R}^d)\},$$

and vector spaces

$$\mathcal{S}_e = \{\varphi_e \text{ even in } \mathcal{S}(\mathbb{R}^d)\},$$

$$\Sigma_e = \{\psi_e \in \mathcal{C}^\infty(\mathbb{R}^d); \psi_e = \varphi_e - \varphi_e(\mathbf{0}) \text{ for some } \varphi_e \in \mathcal{S}_e\}.$$

Note that  $\mathcal{S}_e$  and  $\Sigma_e$  are isomorphic via the maps

$$\begin{aligned} \mathcal{S}_e &\longleftrightarrow \Sigma_e \\ \varphi_e &\longmapsto \varphi_e - \varphi_e(\mathbf{0}) \\ \psi_e - \lim_{\infty} \psi_e &\longleftarrow \psi_e, \end{aligned} \quad (2.3)$$

wherein  $\lim_{\infty} \psi_e := \lim_{|\mathbf{w}| \rightarrow \infty} \psi_e(\mathbf{w})$ . This one-to-one correspondence is illustrated in the left panel of Figure 1 for  $d = 1$ .

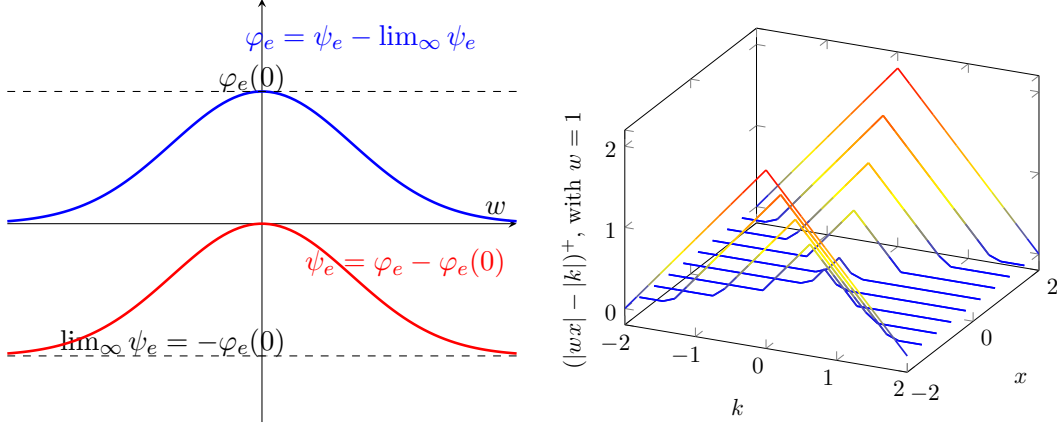


Figure 1:  $d = 1$ . (Left) One-to-one correspondence (2.3) between the example functions  $\varphi_e(w) = e^{-w^2}$  and  $\psi_e(w) = e^{-w^2} - 1$ . (Right) “Basket call” payoff  $F(x, k) = (|wx| - |k|)^+$  in dimension  $d = 1$  used in the spanning formula (2.7) with  $w = 1$ .

**Lemma 2.1.** (i) For any  $\phi_0(\mathbf{x}, k) = e^{-ikr}h(\mathbf{x}) \in \mathcal{S}_0$  (with  $r \neq 0$  and  $h \in \mathcal{S}(\mathbb{R}^d)$ ), the function

$$\psi(\mathbf{w}) := -r^2 \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\mathbf{x}, k) = 2 \int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) (\cos(\mathbf{w} \cdot \mathbf{x}) - 1) \quad (2.4)$$

is in  $\Sigma_e$ .

(ii) The linear operator

$$\begin{aligned} \mathcal{S}_0 &\longrightarrow \Sigma_e \\ \phi_0 &\longmapsto \psi_e : \mathbf{w} \mapsto \int_{\mathbb{R}^{d+1}} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\mathbf{x}, k) dk d\mathbf{x} \end{aligned} \quad (2.5)$$

is well defined.

**Proof.** (i) For any  $\phi_0(\mathbf{x}, k) = e^{-ikr}h(\mathbf{x}) \in \mathcal{S}_0$  with  $r \neq 0$  and  $h \in \mathcal{S}(\mathbb{R}^d)$ , by definition,  $-\frac{1}{r^2}\psi(\mathbf{w}) = \int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}} dk e^{-irk} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ = \int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) \mathcal{F}_k[(|\mathbf{w} \cdot \mathbf{x}| - |k|)^+](r)$ .

Substituting (B.2) for the Fourier transform of the basket call above,

$$\begin{aligned} \psi(\mathbf{w}) &= 2 \int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) (\cos(\mathbf{w} \cdot \mathbf{x}) - 1) \\ &= \int_{\mathbb{R}^d} e^{ir\mathbf{w} \cdot \mathbf{x}} h(\mathbf{x}) d\mathbf{x} + \int_{\mathbb{R}^d} e^{-ir\mathbf{w} \cdot \mathbf{x}} h(\mathbf{x}) d\mathbf{x} - 2 \int_{\mathbb{R}^d} h(\mathbf{x}) d\mathbf{x}, \end{aligned}$$

which establishes (2.4). From  $\lim_{|\mathbf{w}| \rightarrow \infty} \int_{\mathbb{R}^d} e^{\pm ir\mathbf{w} \cdot \mathbf{x}} h(\mathbf{x}) d\mathbf{x} = 0$  (see Remark B.2), we get  $\lim_{\infty} \psi = -2 \int_{\mathbb{R}^d} h(\mathbf{x}) d\mathbf{x}$ . Let  $\varphi = \psi - \lim_{\infty} \psi$ . Then  $\varphi(\mathbf{w}) = \int_{\mathbb{R}^d} e^{ir\mathbf{w} \cdot \mathbf{x}} h(\mathbf{x}) d\mathbf{x} + \int_{\mathbb{R}^d} e^{-ir\mathbf{w} \cdot \mathbf{x}} h(\mathbf{x}) d\mathbf{x}$  is even, and in  $\mathcal{S}(\mathbb{R}^d)$  as a property of Fourier transforms (Kanwal, 2004, Theorem 2 page 143). Therefore,  $\varphi$  belongs to  $\mathcal{S}_e$ , and  $\psi$  belongs to  $\Sigma_e$  in view of the bijection (2.3).

(ii) proceeds immediately from (i) since  $r \neq 0$  and  $\psi_e = -\psi/r^2$ .  $\square$

Throughout Sections 2 to 4, the target payoff function  $F$  satisfies the following

regularity condition:

**Assumption 2.1.** The payoff function  $F(\mathbf{x}, k)$  is continuous in  $\mathbf{x}$ , in  $k$ , and such that

$$\int_{\mathbb{R}^{d+1}} d\mathbf{x} dk |F(\mathbf{x}, k)h(\mathbf{x})| < \infty, \quad h \in \mathcal{S}(\mathbb{R}^d). \quad (2.6)$$

Observe that the above continuity assumption is less restrictive than the Carr-Madan differentiability condition in dimension  $d = 1$ . The following distributional equation is then a weak formulation of the continuum spanning problem.

**Problem 2.1.** Given a target payoff function  $F(\mathbf{x}, k)$  satisfying Assumption 2.1, find a linear form  $N$  on  $\Sigma_e$  such that

$$\int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk F(\mathbf{x}, k) \phi_0(\mathbf{x}, k) = \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\mathbf{x}, k) \right\rangle_{\mathbf{w}}, \quad \phi_0 \in \mathcal{S}_0; \quad (2.7)$$

or, equivalently by correspondence (2.3), find a linear form  $T$  on  $\mathcal{S}_e$  such that

$$\langle N, \psi_e \rangle := \langle T, \varphi_e \rangle, \quad \text{where } \psi_e = \varphi_e - \varphi_e(\mathbf{0}), \quad (2.8)$$

solves (2.7).

*Remark 2.1.* (i) If such  $T$  is continuous for the topology of  $\mathcal{S}(\mathbb{R}^d)$ , it defines a distribution on  $\mathcal{S}_e$  (Kanwal, 2004, Definitions page 22).

(ii) By Lemma 2.1(ii), for any  $\phi_0(\mathbf{x}, k) \in \mathcal{S}_0$  the function  $\mathbf{w} \mapsto \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\mathbf{x}, k)$  is indeed in  $\Sigma_e$ .

(iii) The operator  $N_{d\mathbf{w}}$  represents the quantity of basket call payoffs  $(|\mathbf{w} \cdot \mathbf{x}| - |k|)^+$  for each vector  $\mathbf{w}$  of basket weights across the  $\mathbb{R}^d$  continuum. Note that  $(|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ = (\mathbf{w} \cdot \mathbf{x} - |k|)^+ + (-\mathbf{w} \cdot \mathbf{x} - |k|)^+$ , which actually corresponds to two proper basket calls with opposite basket weights  $\mathbf{w}$  and  $-\mathbf{w}$  and identical nonnegative strike  $|k|$  (see the right panel in Figure 1).

## 2.2 Recovering Strong Solutions from Weak Solutions

Proposition 2.3 below shows how the weak formulation (2.7) coincides with a strong formulation of the continuum spanning problem (2.2) when the solution  $N_{d\mathbf{w}}$  is a measure-type distribution  $\nu(d\mathbf{w})$  (see Remark 2.2). We begin with the following Dirac approximation lemma.

**Lemma 2.2.** Let  $h_n \in \mathcal{D}(\mathbb{R}^d)$  be the sequence of functions with support inside  $\{\|\mathbf{x}\|_\infty < \frac{1}{n}\}$  given as

$$h_n(\mathbf{x}) = c_n \prod_{i=1}^d \mathbf{1}_{|x_i| < \frac{1}{n}} \exp\left(-\frac{1}{1 - n^2 x_i^2}\right), \quad (2.9)$$

where  $c_n$  is a normalizing factor such that  $\int_{\mathbb{R}^d} h_n(\mathbf{x}) d\mathbf{x} = 1$ . Then  $\int_{\mathbb{R}^d} d\mathbf{x} h_n(\mathbf{x}) g(\mathbf{x}) \rightarrow g(\mathbf{0})$  as  $n \rightarrow \infty$ , for any function  $g: \mathbb{R}^d \rightarrow \mathbb{C}$  which is continuous at the origin.



**Proof.**  $h_n(\mathbf{x})$  is in  $\mathcal{D}(\mathbb{R}^d)$  as (normalized) product of functions  $\mathbb{R} \ni x_i \mapsto \mathbf{1}_{|x_i| < \frac{1}{n}} \exp\left(-\frac{1}{1-n^2x_i^2}\right) \in \mathcal{D}(\mathbb{R})$  (Kanwal, 2004, Lemma 2 page 181), and the support of  $h_n$  is inside  $\{\|\mathbf{x}\|_\infty < \frac{1}{n}\}$  by (2.9). By continuity of  $g$  at the origin, for any  $\epsilon > 0$ , there exists  $n_\epsilon$  such that  $|g(\mathbf{x}) - g(\mathbf{0})| \leq \epsilon$  holds on  $\{\|\mathbf{x}\|_\infty < \frac{1}{n_\epsilon}\}$ . Hence, for  $n \geq n_\epsilon$ ,  $|\int_{\mathbb{R}^d} d\mathbf{x} h_n(\mathbf{x})(g(\mathbf{x}) - g(\mathbf{0}))| \leq \int_{\{\|\mathbf{x}\|_\infty < \frac{1}{n_\epsilon}\}} d\mathbf{x} h_n(\mathbf{x})|g(\mathbf{x}) - g(\mathbf{0})| \leq \epsilon$  since  $h_n$  integrates to 1.  $\square$

**Proposition 2.3.** *If a solution  $N$  to the weak spanning problem (2.7) exists in the form of an integration operator on  $\Sigma_\epsilon$  against an even measure  $\nu$  with finite second moment  $\int_{\mathbb{R}^d} |\mathbf{w}|^2 \nu(d\mathbf{w}) < \infty$ , then the following strong, pointwise representation holds:*

$$F(\mathbf{x}, k) = \int_{\mathbb{R}^d} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \nu(d\mathbf{w}), \quad (\mathbf{x}, k) \in \mathbb{R}^d \times \mathbb{R}. \quad (2.10)$$

**Proof.** Let  $\phi_0(\mathbf{x}, k) = e^{-irk} h(\mathbf{x}) \in \mathcal{S}_0$  with  $r \neq 0$  and  $h \in \mathcal{S}(\mathbb{R}^d)$ . Substituting  $N_{d\mathbf{w}} = \nu(d\mathbf{w})$  into (2.7),

$$\int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk F(\mathbf{x}, k) \phi_0(\mathbf{x}, k) = \int_{\mathbb{R}^d} \nu(d\mathbf{w}) \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\mathbf{x}, k). \quad (2.11)$$

The second moment condition on  $\nu$  together with the Cauchy-Schwarz inequality implies

$$\begin{aligned} \int_{\mathbb{R}^d} d\mathbf{x} |h(\mathbf{x})| \int_{\mathbb{R}^{d+1}} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \nu(d\mathbf{w}) dk &\leq \int_{\mathbb{R}^d} d\mathbf{x} |h(\mathbf{x})| \int_{\mathbb{R}^d} (\mathbf{w} \cdot \mathbf{x})^2 \nu(d\mathbf{w}) \\ &\leq \int_{\mathbb{R}^d} d\mathbf{x} |h(\mathbf{x})| |\mathbf{x}|^2 \int_{\mathbb{R}^d} |\mathbf{w}|^2 \nu(d\mathbf{w}), \end{aligned}$$

which is finite for all  $h \in \mathcal{S}(\mathbb{R}^d)$  (Kanwal, 2004, Theorem page 141). By dominated convergence left and right in (2.11) (due respectively to (2.6) and the above inequality), (2.11) also holds for  $r$  formally set to 0 in  $\phi_0$ . Replacing  $\phi_0(\mathbf{x}, k)$  with  $e^{-irk} h(\mathbf{x})$  in (2.11), where  $r \in \mathbb{R}$  may now be 0, and reordering integrals,

$$\int_{\mathbb{R}} dk e^{-irk} \int_{\mathbb{R}^d} d\mathbf{x} F(\mathbf{x}, k) h(\mathbf{x}) = \int_{\mathbb{R}} dk e^{-irk} \int_{\mathbb{R}^d} \nu(d\mathbf{w}) \int_{\mathbb{R}^d} d\mathbf{x} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ h(\mathbf{x}).$$

Recognizing Fourier transforms over  $k$ ,

$$\mathcal{F}_k \left[ \int_{\mathbb{R}^d} F(\mathbf{x}, k) h(\mathbf{x}) d\mathbf{x} \right] = \mathcal{F}_k \left[ \int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}^d} \nu(d\mathbf{w}) (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \right].$$

Since the Fourier transform with respect to  $k$  is injective on  $L^1(\mathbb{R})$  (see Remark B.2), we recover for all  $h \in \mathcal{S}(\mathbb{R}^d)$ :

$$\int_{\mathbb{R}^d} F(\mathbf{x}, k) h(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}^d} \nu(d\mathbf{w}) (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \quad (2.12)$$

$dk$ -almost everywhere. For each  $k$  satisfying (2.12), by continuity in  $\mathbf{x}$  of  $F(\mathbf{x}, k)$  and  $\int_{\mathbb{R}^d} \nu(d\mathbf{w}) (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+$ , taking  $h(\mathbf{x}) = h_n(\mathbf{x} - \mathbf{x}_0)$  and  $n \rightarrow \infty$  by Lemma 2.2, we recover  $F(\mathbf{x}_0, k) = \int_{\mathbb{R}^d} (|\mathbf{w} \cdot \mathbf{x}_0| - |k|)^+ \nu(d\mathbf{w})$  for any  $\mathbf{x}_0 \in \mathbb{R}^d$ . The validity of (2.10) then follows by continuity in  $k$  applied to both sides.  $\square$

*Remark 2.2.* For fixed  $k \geq 0$ , (2.10) can be seen as a variation of (2.2) up to a doubling factor. Indeed, substituting  $(|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ = (\mathbf{w} \cdot \mathbf{x} - k)^+ + (-\mathbf{w} \cdot \mathbf{x} - k)^+$  and splitting

the integrand,

$$F(\mathbf{x}, k) = \int_{\mathbb{R}^d} (\mathbf{w} \cdot \mathbf{x} - k)^+ \nu(d\mathbf{w}) + \int_{\mathbb{R}^d} (-\mathbf{w} \cdot \mathbf{x} - k)^+ \nu(d\mathbf{w}).$$

Substituting  $\mathbf{w} \mapsto -\mathbf{w}$  then  $\nu(-d\mathbf{w}) = \nu(d\mathbf{w})$  into the second integral above,

$$F(\mathbf{x}, k) = 2 \int_{\mathbb{R}^d} (\mathbf{w} \cdot \mathbf{x} - k)^+ \nu(d\mathbf{w}), \quad \mathbf{x} \in \mathbb{R}^d, k \in \mathbb{R}_+.$$

If  $N$  is not induced by a measure, then integration over  $k$  and the action of  $N$  do not necessarily commute and the proof of Proposition 2.3 no longer works. Hence, in general, strong representations of the form “ $F(\mathbf{x}, k) = \langle N_{d\mathbf{w}}, (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \rangle_{\mathbf{w}}, (\mathbf{x}, k) \in \mathbb{R}^{d+1}$ ” cannot be readily obtained from (2.7). Nevertheless, we have the following regularized representation.

**Proposition 2.4.** *For any linear form  $N$  on  $\Sigma_e$  solving (2.7), if  $k \mapsto F(\mathbf{x}, k)$  and its Fourier transform  $r \mapsto \mathcal{F}_k[F(\mathbf{x}, k)](r)$  are both in  $L^1(\mathbb{R})$  for each  $\mathbf{x} \in \mathbb{R}^d$ , and  $\mathbf{x} \mapsto \mathcal{F}_k[F(\mathbf{x}, k)](r)$  is continuous on  $\mathbb{R}^d$  for each  $r \in \mathbb{R}^*$ , then the following strong, pointwise representation holds  $dk$ -almost everywhere, for each  $\mathbf{x} \in \mathbb{R}^d$ :*

$$F(\mathbf{x}, k) = \mathcal{F}_r^{-1} \left[ \lim_{n \rightarrow \infty} \mathbf{1}_{r \neq 0} \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{y} h_n(\mathbf{y} - \mathbf{x}) \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right\rangle_{\mathbf{w}} \right] (k),$$

where  $h_n$  is the sequence of Dirac approximation functions in (2.9).

**Proof.** Let  $\phi_0(\mathbf{x}, k) = e^{-irk} h(\mathbf{x}) \in \mathcal{S}_0$  with  $r \neq 0$  and  $h \in \mathcal{S}(\mathbb{R}^d)$ . Substituting into (2.7),

$$\int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}} dk F(\mathbf{x}, k) e^{-irk} = \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{x} h(\mathbf{x}) \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ e^{-irk} \right\rangle_{\mathbf{w}}.$$

Recognizing Fourier transforms, replacing  $\mathbf{x}$  with  $\mathbf{y}$  on both sides and  $k$  with  $\kappa$  on the right-hand side,

$$\int_{\mathbb{R}^d} d\mathbf{y} h(\mathbf{y}) \mathcal{F}_k[F(\mathbf{y}, k)](r) = \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{y} h(\mathbf{y}) \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right\rangle_{\mathbf{w}}.$$

Let  $\mathbf{x} \in \mathbb{R}^d$ . Substituting  $h(\mathbf{y}) \equiv h_n(\mathbf{y} - \mathbf{x})$ ,

$$\int_{\mathbb{R}^d} d\mathbf{y} h_n(\mathbf{y} - \mathbf{x}) \mathcal{F}_k[F(\mathbf{y}, k)](r) = \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{y} h_n(\mathbf{y} - \mathbf{x}) \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right\rangle_{\mathbf{w}}.$$

Taking  $n \rightarrow +\infty$ , by Lemma 2.2 the limit of the left-hand side exists as  $\mathcal{F}_k[F(\mathbf{x}, k)](r)$ , and thus on both sides as

$$\mathcal{F}_k[F(\mathbf{x}, k)](r) = \lim_{n \rightarrow +\infty} \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{y} h_n(\mathbf{y} - \mathbf{x}) \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right\rangle_{\mathbf{w}}, \quad r \neq 0.$$

As such, the functions  $\mathbb{R} \ni r \mapsto \mathcal{F}_k[F(\mathbf{x}, k)](r)$  and

$$r \mapsto \mathbf{1}_{r \neq 0} \lim_{n \rightarrow +\infty} \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{y} h_n(\mathbf{y} - \mathbf{x}) \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right\rangle_{\mathbf{w}}$$

coincide on  $\mathbb{R}^*$ . The inverse Fourier transform of  $r \mapsto \mathcal{F}_k[F(\mathbf{x}, k)](r)$  exists as  $k \mapsto F(\mathbf{x}, k)$  (see Remark B.2), hence both functions admit inverse Fourier transforms, which coincide as functions in  $L^1(\mathbb{R})$  and therefore  $dk$ -almost everywhere.  $\square$

### 3 Solution to the Continuum Spanning Problem

**Lemma 3.1.** *The following identity holds between any two bijectively related test functions  $\varphi_e$  and  $h_e = -\frac{1}{2}\mathcal{F}^{-1}\varphi_e$  that are both in  $\mathcal{S}_e$ :*

$$\psi_e(\mathbf{w}) := \varphi_e(w) - \varphi_e(\mathbf{0}) = \int_{\mathbb{R}^d} d\mathbf{x} h_e(\mathbf{x}) \int_{\mathbb{R}} dk e^{-ik} (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+. \quad (3.1)$$

**Proof.** Let  $\varphi_e \in \mathcal{S}_e$ . By the Fourier inversion theorem (see Remark B.3),  $\varphi_e(\mathbf{w}) = \mathcal{F}[\mathcal{F}^{-1}\varphi_e(\mathbf{x})](\mathbf{w}) = -2 \int_{\mathbb{R}^d} e^{-i\mathbf{x} \cdot \mathbf{w}} h_e(\mathbf{x}) d\mathbf{x}$ . Because  $h_e$  is even,

$$\begin{aligned} \varphi_e(\mathbf{w}) &= - \int_{\mathbb{R}^d} e^{-i\mathbf{x} \cdot \mathbf{w}} h_e(\mathbf{x}) d\mathbf{x} - \int_{\mathbb{R}^d} e^{i\mathbf{x} \cdot \mathbf{w}} h_e(\mathbf{x}) d\mathbf{x} = -2 \int_{\mathbb{R}^d} \cos(\mathbf{x} \cdot \mathbf{w}) h_e(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk e^{ik} h_e(\mathbf{x}) (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ - 2 \int_{\mathbb{R}^d} d\mathbf{x} h_e(\mathbf{x}), \end{aligned} \quad (3.2)$$

where we used (2.4) with  $r = -1$  in the last step. Substituting  $k \mapsto -k$  and

$$-2 \int_{\mathbb{R}^d} d\mathbf{x} h_e(\mathbf{x}) = \int_{\mathbb{R}^d} d\mathbf{x} (\mathcal{F}^{-1}\varphi_e)(\mathbf{x}) = \mathcal{F}\mathcal{F}^{-1}\varphi_e(\mathbf{0}) = \varphi_e(\mathbf{0}),$$

then rearranging terms yields the required result.  $\square$

**Corollary 3.2.** *The function space  $\{\mathbf{w} \mapsto \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\mathbf{x}, k) ; \phi_0 \in \mathcal{S}_0\}$  coincides with  $\Sigma_e$  and Problem 2.1 admits at most one solution.*

**Proof.** This space is a subset of  $\Sigma_e$  by Lemma 2.1(ii), and a superset by Lemma 3.1. Hence any two solutions  $N$  and  $N'$  to Problem 2.1 coincide on their entire domain  $\Sigma_e$ , i.e.  $N = N'$ .  $\square$

**Definition 3.1.** The payoff function  $F(\mathbf{x}, k)$  is said to be of class (AH) if it is absolutely homogeneous in  $(\mathbf{x}, k)$  and even in both  $\mathbf{x}$  and  $k$ , i.e.

$$F(\lambda\mathbf{x}, \lambda k) = |\lambda| F(\mathbf{x}, k), \lambda \in \mathbb{R}^*, \quad \text{and} \quad F(\mathbf{x}, k) = F(\mathbf{x}, -k) = F(-\mathbf{x}, k), (\mathbf{x}, k) \in \mathbb{R}^{d+1}.$$

*Remark 3.1.* Examples of absolutely homogeneous payoffs can be found in the last column in Table 1, page 2. By continuity of  $F$  in  $\mathbf{x}$  or  $k$  postulated in Assumption 2.1, the payoff function must vanish at the origin, since e.g.  $F(\mathbf{0}, 0) = \lim_{n \rightarrow \infty} F(\mathbf{0}, 1/n) = \lim_{n \rightarrow \infty} 1/n F(\mathbf{0}, 1) = 0$ .

#### 3.1 General Weak Solution

**Lemma 3.3** (and definition). *Given a payoff  $F(\mathbf{x}, k)$ , the associated linear forms  $N^F, T^F$  below are well defined:*

$$\begin{aligned} \langle N_{d\mathbf{w}}^F, \psi_e(\mathbf{w}) \rangle_{\mathbf{w}} &= \langle T_{d\mathbf{w}}^F, \varphi_e(\mathbf{w}) \rangle_{\mathbf{w}} := -\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} \mathcal{F}^{-1}\varphi_e(\mathbf{x}) \mathcal{F}_k[F(\mathbf{x}, k)](1) \\ &:= -\frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{w} e^{i\mathbf{w} \cdot \mathbf{x}} \varphi_e(\mathbf{w}) \int_{\mathbb{R}} dk e^{-ik} F(\mathbf{x}, k), \end{aligned} \quad (3.3)$$

for any  $\varphi_e \in \mathcal{S}_e$ ,  $\psi_e = \varphi_e - \varphi_e(\mathbf{0}) \in \Sigma_e$ . If, in addition,  $d\mathbf{x} \mathcal{F}_k[F(\mathbf{x}, k)](1) = d\mathbf{x} (\int_{\mathbb{R}} dk F(\mathbf{x}, k) e^{-ik})$  is a function-type distribution on  $\mathcal{S}(\mathbb{R}^d)$ , then  $T^F$  is a distribution on  $\mathcal{S}_e$ .

**Proof.** Observe that  $\mathcal{F}^{-1}\varphi_e \in \mathcal{S}_e$  for any  $\varphi_e \in \mathcal{S}_e$  (Milton, 1974, Theorem 3.3 and 4.1). By Assumption 2.1,  $\int_{\mathbb{R}^d} d\mathbf{x} \mathcal{F}^{-1}\varphi_e(\mathbf{x}) \mathcal{F}_k[F(\mathbf{x}, k)](1)$  is therefore well defined for any  $\varphi_e \in \mathcal{S}_e$ , and it is linear in  $\varphi_e$ . In addition, if  $(\int dk F(\mathbf{x}, k) e^{-ik}) d\mathbf{x}$  is a distribution on  $\mathcal{S}(\mathbb{R}^d)$  and thus on  $\mathcal{S}_e$  (Milton, 1974, Theorem 4.4), because  $\mathcal{F}^{-1}$  is a continuous linear operator on  $\mathcal{S}(\mathbb{R}^d) \supset \mathcal{S}_e$  (see Remark B.3), the right-hand side of (3.3) is continuous with respect to  $\varphi_e \in \mathcal{S}_e$ , hence  $T^F$  given by (3.3) defines a distribution on  $\mathcal{S}_e$ .  $\square$

**Theorem 3.4.** *Problem 2.1 admits a solution  $N$ , or equivalently  $T$  by correspondence (2.8), if and only if the payoff  $F(\mathbf{x}, k)$  is of class (AH), in which case the solution is unique and given by (3.3).*

**Proof.** The uniqueness of a solution to Problem 2.1 was established in Corollary 3.2. ( $\Rightarrow$ ) Suppose that a solution  $T$  to (2.7)-(2.8) exists, and let  $\phi_0 \in \mathcal{S}_0$  and  $\lambda \in \mathbb{R}^*$ . The function  $(\mathbf{x}, k) \mapsto \phi_0(\lambda^{-1}\mathbf{x}, \lambda^{-1}k)$  is again in  $\mathcal{S}_0$  and may be substituted into (2.7) to obtain

$$\int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk F(\mathbf{x}, k) \phi_0(\lambda^{-1}\mathbf{x}, \lambda^{-1}k) = \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\lambda^{-1}\mathbf{x}, \lambda^{-1}k) \right\rangle_{\mathbf{w}}. \quad (3.4)$$

By linear change of variable  $(\mathbf{x}, k) \mapsto (\lambda\mathbf{x}, \lambda k)$  on both sides, and absolute homogeneity of  $(|\mathbf{w} \cdot \mathbf{x}| - |k|)^+$  in  $(\mathbf{x}, k)$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk |\lambda|^{-d-1} F(\lambda\mathbf{x}, \lambda k) \phi_0(\mathbf{x}, k) \\ &= \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk |\lambda|^{-d-1} (|\mathbf{w} \cdot \lambda\mathbf{x}| - |\lambda k|)^+ \phi_0(\mathbf{x}, k) \right\rangle_{\mathbf{w}} \\ &= |\lambda|^{-d} \left\langle N_{d\mathbf{w}}, \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \phi_0(\mathbf{x}, k) \right\rangle_{\mathbf{w}} \\ &= |\lambda|^{-d} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk F(\mathbf{x}, k) \phi_0(\mathbf{x}, k), \end{aligned}$$

where we substituted (2.7) in the last step. Multiplying both sides of the above by  $|\lambda|^{d+1}$ ,

$$\int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk F(\lambda\mathbf{x} \lambda k) \phi_0(\mathbf{x}, k) = |\lambda| \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk F(\mathbf{x}, k) \phi_0(\mathbf{x}, k), \quad (3.5)$$

which holds for any  $\phi_0 \in \mathcal{S}_0$ , and also for  $r$  formally set to 0 in  $\phi_0$  as a result of dominated convergence left and right in (3.5). By the same reasoning that concluded the proof of Proposition 2.3, we recover  $F(\lambda\mathbf{x}, \lambda k) = |\lambda| F(\mathbf{x}, k)$  for all  $(\mathbf{x}, k) \in \mathbb{R}^{d+1}$ . In addition,  $(\mathbf{x}, k) \mapsto \phi_0(\mathbf{x}, -k)$  and  $(\mathbf{x}, k) \mapsto \phi_0(-\mathbf{x}, k)$  are also in  $\mathcal{S}_0$ , and we similarly obtain  $F(\mathbf{x}, k) = F(\mathbf{x}, -k) = F(-\mathbf{x}, k)$  for  $(\mathbf{x}, k) \in \mathbb{R}^{d+1}$ . Therefore,  $F$  is of class (AH) as required.

( $\Leftarrow$ ) Suppose that  $F$  is of class (AH), and let  $\psi_e \in \Sigma_e$ ,  $\varphi_e = \psi_e - \lim_{\infty} \psi$ , and  $r \in \mathbb{R}^*$ . By absolute homogeneity of  $F(\mathbf{x}, k)$ , we may substitute  $F(\mathbf{x}, k) = |r| F(r^{-1}\mathbf{x}, r^{-1}k)$  in (3.3) to get

$$\langle N_{d\mathbf{w}}^F, \psi_e(\mathbf{w}) \rangle_{\mathbf{w}} = -\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk |r| F(r^{-1}\mathbf{x}, r^{-1}k) e^{-ik} \mathcal{F}^{-1}\varphi_e(\mathbf{x}).$$

By linear change of variables  $(\mathbf{y}, \kappa) = (r^{-1}\mathbf{x}, r^{-1}k)$ ,

$$\langle N_{\mathbf{d}\mathbf{w}}^F, \psi_\epsilon(\mathbf{w}) \rangle_{\mathbf{w}} = -\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}} d\kappa |r|^{d+2} F(\mathbf{y}, \kappa) e^{-ir\kappa} \mathcal{F}^{-1} \varphi_\epsilon(r\mathbf{y}). \quad (3.6)$$

By Lemma 3.1 with  $h_\epsilon = -\frac{1}{2}\mathcal{F}^{-1}\varphi_\epsilon$ , the left-hand side satisfies

$$\begin{aligned} \langle N_{\mathbf{d}\mathbf{w}}^F, \psi_\epsilon(\mathbf{w}) \rangle_{\mathbf{w}} &= \left\langle N_{\mathbf{d}\mathbf{w}}^F, \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk e^{-ik} h_\epsilon(\mathbf{x}) (|\mathbf{w} \cdot \mathbf{x}| - |k|)^+ \right\rangle_{\mathbf{w}} \\ &= \left\langle N_{\mathbf{d}\mathbf{w}}^F, |r|^{d+2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}} d\kappa e^{-ir\kappa} h_\epsilon(r\mathbf{y}) (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right\rangle_{\mathbf{w}}, \end{aligned} \quad (3.7)$$

where we applied the change of variable  $(\mathbf{x}, k) \mapsto (\mathbf{y}, \kappa) = (r^{-1}\mathbf{x}, r^{-1}k)$  in the last step. Connecting with (3.6) and dividing both sides by  $|r|^{d+2}$ ,

$$\int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}} d\kappa F(\mathbf{y}, \kappa) e^{-ir\kappa} h_\epsilon(r\mathbf{y}) = \left\langle N_{\mathbf{d}\mathbf{w}}^F, \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}} d\kappa e^{-ir\kappa} h_\epsilon(r\mathbf{y}) (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right\rangle_{\mathbf{w}} \quad (3.8)$$

which holds for any  $h_\epsilon(r\mathbf{y}) = -\frac{1}{2}\mathcal{F}^{-1}\varphi_\epsilon(r\mathbf{y}) \in \mathcal{S}_e$ . In addition, for any  $h_o$  odd in  $\mathcal{S}(\mathbb{R}^d)$ , the even-odd product  $F(\mathbf{y}, \kappa)h_o(r\mathbf{y})$  is odd in  $\mathbf{y}$ , so that

$$\int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}} d\kappa F(\mathbf{y}, \kappa) e^{-ir\kappa} h_o(r\mathbf{y}) = \int_{\mathbb{R}} d\kappa e^{-ir\kappa} \int_{\mathbb{R}^d} d\mathbf{y} F(\mathbf{y}, \kappa) h_o(r\mathbf{y}) = 0$$

as the  $d\mathbf{y}$  integral over  $\mathbb{R}^d$  of an odd function vanishes. Consequently, replacing  $h_\epsilon$  by any odd function  $h_o$  to the left-hand side of (3.8) yields 0, and the equation thus holds for any  $h \in \mathcal{S}(\mathbb{R}^d)$ , so that  $N^F$  solves (2.7) as required.  $\square$

### 3.2 Connection With Radon Transform

In general, the integrals in (3.3) may not commute, so that the solution  $T$  (or  $N$ ) is not necessarily “of function-type”. The following proposition connects our Theorem 3.4 with the approach of Bossu (2022) based on Radon transforms and the Fourier slice inversion formula (3.10) in the Lebesgue-integrable case:

**Proposition 3.5.** *Let  $F$  be a payoff function of class (AH) which is  $L^1(\mathbb{R})$  in  $k$  and such that  $\mathcal{F}_k[F(\mathbf{x}, k)](1)$  is  $L^1(\mathbb{R}^d)$  in  $\mathbf{x}$ . Then:*

(i) *The corresponding solution (3.3) to Problem 2.1 is of function-type  $T_{\mathbf{d}\mathbf{w}}^F = f(\mathbf{w}) d\mathbf{w}$ , where*

$$f(\mathbf{w}) = -\frac{1}{2} \mathcal{F}_{\mathbf{x}}^{-1} [\mathcal{F}_k[F(\mathbf{x}, k)](1)](\mathbf{w}); \quad (3.9)$$

(ii) *If  $F(\mathbf{x}, k)$  is  $n$ -times continuously differentiable against  $k$  and all the derivatives are  $L^1(\mathbb{R})$  in  $k$ , then we may replace  $F$  by  $(-i)^n \partial_{k^n} F$  in (3.9), e.g., for  $n = 2$ ,*

$$f(\mathbf{w}) = \frac{1}{2} \mathcal{F}_{\mathbf{x}}^{-1} [\mathcal{F}_k[\partial_{k^2}^2 F(\mathbf{x}, k)](1)](\mathbf{w}); \quad (3.10)$$

(iii) *If  $\partial_k F, \partial_{k^2}^2 F$  exist and are  $L^1(\mathbb{R})$  in  $k$ , and  $\frac{1}{2} \partial_{k^2}^2 F(\mathbf{x}, k)$  is the Radon transform  $\mathcal{R}g(\mathbf{x}, k)$  defined below of some function  $g \in L^1(\mathbb{R}^d)$ , then  $f = g$  almost everywhere:*

$$\mathcal{R}g(\mathbf{x}, k) := \frac{\mathbf{1}_{\mathbf{x} \neq \mathbf{0}}}{|x_j|} \int_{\mathbb{R}^{d-1}} g\left(\mathbf{y}_{<j}, \frac{k - \mathbf{x}_{\neq j} \cdot \mathbf{y}_{\neq j}}{x_j}, \mathbf{y}_{>j}\right) d\mathbf{y}_{\neq j}, \quad (3.11)$$

which is independent from the choice of  $j$  such that  $x_j \neq 0$ , for any  $\mathbf{x} \neq \mathbf{0}$  (Rubin, 2015, page 130).

**Proof.** (i) All integrands being now Lebesgue-integrable, for any  $\psi_e \in \Sigma_e$  and  $\varphi_e$  such that  $\psi_e = \varphi_e - \varphi_e(\mathbf{0})$ , we may reorder integrals in (3.3) to get

$$\begin{aligned} \langle T_{d\mathbf{w}}^F, \varphi_e(\mathbf{w}) \rangle_{\mathbf{w}} &= -\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{w} \varphi_e(\mathbf{w}) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{x} e^{i\mathbf{w} \cdot \mathbf{x}} \int_{\mathbb{R}} dk e^{-ik} F(\mathbf{x}, k) \\ &= -\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{w} \varphi_e(\mathbf{w}) \mathcal{F}_{\mathbf{x}}^{-1} [\mathcal{F}_k[F(\mathbf{x}, k)](1)](\mathbf{w}), \quad \text{as required.} \end{aligned}$$

(ii) follows from the derivative rule for Fourier transforms applied to  $\mathcal{F}_k[F(\mathbf{x}, k)](1)$  (Iosevich and Liflyand, 2014, Theorem 1.8 page 12).

(iii) Substituting  $\frac{1}{2} \partial_{k^2}^2 F(\mathbf{x}, k) = \mathcal{R}g(\mathbf{x}, k)$  and (3.11) into the right-hand side of (3.10),

$$\begin{aligned} f(\mathbf{w}) &= \frac{1}{2} \mathcal{F}_{\mathbf{x}}^{-1} [\mathcal{F}_k[\partial_{k^2}^2 F(\mathbf{x}, k)](1)](\mathbf{w}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{x} e^{i\mathbf{w} \cdot \mathbf{x}} \int_{\mathbb{R}} dk e^{-ik} \int_{\mathbb{R}^d} \delta_{\mathbf{x} \cdot \mathbf{y} = k} (d\mathbf{y}) g(\mathbf{y}) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{x} e^{i\mathbf{w} \cdot \mathbf{x}} \int_{\mathbb{R}} dk e^{-ik} \frac{\mathbf{1}_{\mathbf{x} \neq \mathbf{0}}}{|x_{j_{\mathbf{x}}}|} \int_{\mathbb{R}^{d-1}} g\left(\mathbf{y}_{< j_{\mathbf{x}}}, \frac{k - \mathbf{x}_{\neq j_{\mathbf{x}}} \cdot \mathbf{y}_{\neq j_{\mathbf{x}}}}{x_{j_{\mathbf{x}}}}, \mathbf{y}_{> j_{\mathbf{x}}}\right) d\mathbf{y}_{\neq j_{\mathbf{x}}}, \end{aligned}$$

where  $j_{\mathbf{x}}$  measurably picks a nonzero coordinate of  $\mathbf{x}$  (i.e.  $x_{j_{\mathbf{x}}} \neq 0$ ), for every  $\mathbf{x} \neq \mathbf{0}$ . By change of variable  $k \mapsto y_{j_{\mathbf{x}}} = \frac{k - \mathbf{x}_{\neq j_{\mathbf{x}}} \cdot \mathbf{y}_{\neq j_{\mathbf{x}}}}{x_{j_{\mathbf{x}}}}$  for each  $\mathbf{x} \neq \mathbf{0}$ ,

$$\begin{aligned} f(\mathbf{w}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{x} e^{i\mathbf{w} \cdot \mathbf{x}} \int_{\mathbb{R}} dy_{j_{\mathbf{x}}} \int_{\mathbb{R}^{d-1}} d\mathbf{y}_{\neq j_{\mathbf{x}}} e^{-i\mathbf{x} \cdot \mathbf{y}} g(\mathbf{y}) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\mathbf{x} e^{i\mathbf{w} \cdot \mathbf{x}} \int_{\mathbb{R}} d\mathbf{y} e^{-i\mathbf{x} \cdot \mathbf{y}} g(\mathbf{y}) = \mathcal{F}^{-1} \mathcal{F}g(\mathbf{w}) = g(\mathbf{w}) \text{ a.e. } \square \end{aligned}$$

*Remark 3.2.* Proposition 3.5(iii) is tantamount to saying that  $f(\mathbf{w})$  is the inverse Radon transform of  $\frac{1}{2} \partial_{k^2}^2 F(\mathbf{x}, k)$ , in agreement with the solution derived in Bossu (2022, Section 3) for the related strong spanning problem (2.2) when  $\nu(d\mathbf{w}) = f(\mathbf{w}) d\mathbf{w}$ .

## 4 Explicit Examples

In this section, we provide explicit solutions to Problem 2.1, encompassing the function, measure and principal value types that may be encountered depending on the regularity of the target payoff  $F$ .

### 4.1 Smooth Payoff Function

Let  $G_d(\mathbf{x}, k) = \mathbf{1}_{\mathbf{x} \neq \mathbf{0}} |\mathbf{x}| e^{-\frac{k^2}{|\mathbf{x}|^2}}$  be a smooth payoff function. This payoff does not trade but provides a valuable example where the solution is of function-type and available in closed form, as shown below.

**Proposition 4.1.** *For  $F = G_d$ , Problem 2.1 admits the unique solution  $T_{d\mathbf{w}}^{G_d} = g_d(\mathbf{w}) d\mathbf{w}$  on  $\mathcal{S}_e$  and  $N_{d\mathbf{w}}^{G_d} = g_d(\mathbf{w}) d\mathbf{w}$  on  $\Sigma_e$ , where  $g_d(\mathbf{w}) = -\frac{\sqrt{\pi}}{2} \mathcal{F}_{\mathbf{x}}^{-1} [|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}](\mathbf{w})$ . In particular,*

$$\begin{aligned} g_1(w) &= e^{-w^2} (2w^2 - 1), \quad g_2(w_1, w_2) = \frac{2}{\sqrt{\pi}} e^{-|\mathbf{w}|^2} (|\mathbf{w}|^2 - 1), \\ g_3(w_1, w_2, w_3) &= \frac{1}{\pi} e^{-|\mathbf{w}|^2} (2|\mathbf{w}|^2 - 3). \end{aligned} \tag{4.1}$$

**Proof.** The function  $G_d$  is of joint class  $\mathcal{C}^\infty(\mathbb{R}^{d+1})$ , (AH), and  $L^1(\mathbb{R})$  in  $k$ . By Kammler (2008, Example page 139),  $\mathcal{F}_k[G_d(\mathbf{x}, k)](1) = \sqrt{\pi}|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}$  is  $L^1(\mathbb{R}^d)$  in  $\mathbf{x}$ . Hence  $F = G_d$  satisfies the assumptions of Proposition 3.5(i), and the corresponding solution  $T^{G_d}$  is of function-type  $T_{d\mathbf{w}}^{G_d} = g_d(\mathbf{w}) d\mathbf{w}$  with

$$g_d(\mathbf{w}) = -\frac{1}{2}\mathcal{F}_{\mathbf{x}}^{-1}[\mathcal{F}_k[G_d(\mathbf{x}, k)](1)](\mathbf{w}) = -\frac{\sqrt{\pi}}{2}\mathcal{F}_{\mathbf{x}}^{-1}\left[|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}\right](\mathbf{w}).$$

The expressions for  $g_1, g_2$  and  $g_3$  given in (4.1) can be obtained by Hankel transform computations (Iosevich and Lifyand, 2014, Theorem 4.1 page 93). In addition,

$$\int_{\mathbb{R}^d} g_d(\mathbf{w}) d\mathbf{w} = \mathcal{F}g_d(\mathbf{0}) = -\frac{\sqrt{\pi}}{2}|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}}\Big|_{\mathbf{x}=\mathbf{0}} = 0. \quad (4.2)$$

Consequently, for any pair  $(\varphi_e, \psi_e)$  in correspondence (2.3), we have  $\int_{\mathbb{R}} \varphi_e(\mathbf{w})g_d(\mathbf{w}) d\mathbf{w} = \int_{\mathbb{R}} \psi_e(\mathbf{w})g_d(\mathbf{w}) d\mathbf{w}$ , so that  $N_{d\mathbf{w}}^{G_d} = g_d(\mathbf{w}) d\mathbf{w}$  holds on  $\Sigma_e$  as required.  $\square$

**Corollary 4.2.** *The following strong spanning formula holds, for the solution  $g_d$  given in Proposition 4.1:*

$$G_d(\mathbf{x}, k) = \int_{\mathbb{R}^d} (|\mathbf{x} \cdot \mathbf{w}| - |k|)^+ g_d(\mathbf{w}) d\mathbf{w}, \quad (\mathbf{x}, k) \in \mathbb{R}^{d+1}. \quad (4.3)$$

**Proof.** Since  $|\mathbf{x}|^2 e^{-\frac{|\mathbf{x}|^2}{4}} \in \mathcal{S}(\mathbb{R}^d)$ ,  $g_d(\mathbf{w})$  is also in  $\mathcal{S}(\mathbb{R}^d)$  and thus  $\int_{\mathbb{R}^d} |\mathbf{w}|^2 |g_d(\mathbf{w})| d\mathbf{w} < \infty$  (Kanwal, 2004, Theorem page 141), yielding that the even measure  $N_{d\mathbf{w}} = g_d(\mathbf{w}) d\mathbf{w}$  has finite second moment. (4.3) then follows by an application of Propositions 4.1 and 2.3.  $\square$

*Remark 4.1* (sanity check). For  $d = 1$ , we may verify by the following direct computation that, for  $(x, k) \in \mathbb{R}^* \times \mathbb{R}$ ,

$$\begin{aligned} \int_{\mathbb{R}} (|xw| - |k|)^+ g_1(w) dw &= \int_{\frac{|k|}{|x|}}^{\infty} (|xw| - |k|) g_1(w) dw + \int_{-\infty}^{-\frac{|k|}{|x|}} (-|xw| - |k|) g_1(w) dw \\ &= 2 \int_{\frac{|k|}{|x|}}^{\infty} (|xw| - |k|) g_1(w) dw = \int_{\frac{|k|}{|x|}}^{\infty} (|xw| - |k|) e^{-w^2} (2w^2 - 1) dw \\ &= -e^{-w^2} (2w^2|x| + |x| - 2kw) \Big|_{\frac{|k|}{|x|}}^{\infty} = |x|e^{-\frac{k^2}{x^2}} = G_1(x, k), \end{aligned}$$

while  $\int_{\mathbb{R}} (|xw| - |k|)^+ g_1(w) dw = G_1(x, k) = 0$  for  $x = 0$ . In dimension  $d = 2$  and  $d = 3$ , (4.3) may be verified by change of variables to polar coordinates.

## 4.2 Dispersion Call

For  $d \geq 1$ , let

$$C_d(\mathbf{x}, k) = \left( \sum_{j=1}^d |x_j| - |k| \right)^+ = (\|\mathbf{x}\|_1 - |k|)^+, \quad (\mathbf{x}, k) \in \mathbb{R}^{d+1} \quad (4.4)$$

denote the absolutely homogeneous variation of the  $\ell^1$  dispersion call payoff listed in Table 1, page 2. This type of payoff trades on exotic derivatives markets (Bossu, Carr, and Papanicolaou, 2022, Figure 1).

**Definition 4.1.** Let  $S_e \subset \mathcal{S}_e$  be the Schwartz subspace generated by functions of the form  $\varphi(\mathbf{w}) + \varphi(-\mathbf{w})$  where  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  is multiplicatively separable, i.e.  $\varphi(\mathbf{w}) = \prod_{i=1}^d \varphi_i(w_i)$  for  $\varphi_i \in \mathcal{S}(\mathbb{R}), i = 1, \dots, d$ . (For  $d = 1, S_e = \mathcal{S}_e$ .)

(i) By Definition A.2(iii) and Lemma A.2, we define the following maps, where  $c$  is a constant and  $\varphi_i \in \mathcal{S}(\mathbb{R})$  for every  $i, j \in 1..d$ :

$$\begin{aligned} \left\langle \delta_c(dw_j) \prod_{i \in 1..d \setminus \{j\}} \frac{dw_i}{w_i \pm w_j}, \prod_{i=1}^d \varphi_i(w_i) \right\rangle_{\mathbf{w}} &:= \varphi_j(c) \prod_{i \in 1..d \setminus \{j\}} \int_{-\infty}^{\infty} \frac{dw_i}{w_i \pm c} \varphi_i(w_i), \\ \left\langle \frac{dw_j}{w_j - c} \prod_{i \in 1..d \setminus \{j\}} \frac{dw_i}{w_i \pm w_j}, \prod_{i=1}^d \varphi_i(w_i) \right\rangle_{\mathbf{w}} &:= \\ &\int_{-\infty}^{\infty} \frac{dw_j}{w_j - c} \left( \varphi_j(w_j) \prod_{i \in 1..d \setminus \{j\}} \int_{-\infty}^{\infty} \frac{dw_i}{w_i \pm w_j} \varphi_i(w_i) \right) \end{aligned} \quad (4.5)$$

and, in line with the distributivity principle,

$$\begin{aligned} \left\langle \left( \delta_1(dw_j) + \delta_{-1}(dw_j) \right) \prod_{i \in 1..d \setminus \{j\}} \left( \frac{dw_i}{w_i + w_j} - \frac{dw_i}{w_i - w_j} \right), \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} &:= \\ \sum_{\substack{c \in \{-1, 1\} \\ (\varepsilon_i)_{i \in 1..d \setminus \{j\}} \in \{-1, 1\}^{d-1}}} \varepsilon_i \left\langle \delta_c(dw_j) \prod_{i \in 1..d \setminus \{j\}} \frac{dw_i}{w_i + \varepsilon_i w_j}, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}}, & \\ \left\langle \left( \frac{dw_j}{w_j - 1} + \frac{dw_j}{w_j + 1} \right) \prod_{i \in 1..d \setminus \{j\}} \left( \frac{dw_i}{w_i + w_j} - \frac{dw_i}{w_i - w_j} \right), \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} &:= \\ \sum_{\substack{c \in \{-1, 1\} \\ (\varepsilon_i)_{i \in 1..d \setminus \{j\}} \in \{-1, 1\}^{d-1}}} \varepsilon_i \left\langle \frac{dw_j}{w_j + c} \prod_{i \in 1..d \setminus \{j\}} \frac{dw_i}{w_i + \varepsilon_i w_j}, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}}, & \phi_e \in S_e, \end{aligned} \quad (4.6)$$

which we both extend by linearity as linear forms on  $S_e$ .

(ii) We denote by  $T^d$  the following linear form on  $S_e$ :

- $T_{dw}^1 = \frac{1}{2} \delta_1(dw) + \frac{1}{2} \delta_{-1}(dw)$ ,

$$\begin{aligned} T_{d\mathbf{w}}^2 &= \frac{1}{4\pi^2} \left( \frac{dw_1}{w_1 + 1} + \frac{dw_1}{w_1 - 1} \right) \left( \frac{dw_2}{w_2 + w_1} - \frac{dw_2}{w_2 - w_1} \right) \\ &+ \frac{1}{4\pi^2} \left( \frac{dw_2}{w_2 + 1} + \frac{dw_2}{w_2 - 1} \right) \left( \frac{dw_1}{w_1 + w_2} - \frac{dw_1}{w_1 - w_2} \right); \end{aligned} \quad (4.7)$$

- more generally, for  $d = 2l - 1$  odd or  $2l$  even (with  $l$  positive integer),

$$\begin{aligned} T_{d\mathbf{w}}^{2l-1} &= \frac{(-1)^{l-1} \pi}{(2\pi)^d} \sum_{j=1}^d \left( \left( \delta_1(dw_j) + \delta_{-1}(dw_j) \right) \prod_{i \in 1..d \setminus \{j\}} \left( \frac{dw_i}{w_i + w_j} - \frac{dw_i}{w_i - w_j} \right) \right) \\ T_{d\mathbf{w}}^{2l} &= \frac{(-1)^{l-1}}{(2\pi)^d} \sum_{j=1}^d \left( \left( \frac{dw_j}{w_j - 1} + \frac{dw_j}{w_j + 1} \right) \prod_{i \in 1..d \setminus \{j\}} \left( \frac{dw_i}{w_i + w_j} - \frac{dw_i}{w_i - w_j} \right) \right). \end{aligned} \quad (4.8)$$

*Remark 4.2.* The underlying combinatorial structure of  $T^d$  is apparent in (4.6). In



particular,  $T^d$  embeds a sum of  $d \times 2^d$  products and/or iterations of Cauchy integrals. For large  $d$  (e.g.  $d > 20$ ), the corresponding quadrature would be numerically infeasible.

**Proposition 4.3.** *For  $F = C_d$ , Problem 2.1 admits a unique solution  $T^{C_d}$ , which is a distribution on  $\mathcal{S}_e$ , namely the continuous extension of  $T^d - \delta_{\mathbf{0}}$  to  $\mathcal{S}_e$ . In particular,*

$$\left\langle T_{\mathbf{d}\mathbf{w}}^{C_d}, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} = \left\langle T_{\mathbf{d}\mathbf{w}}^d, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} - \phi_e(\mathbf{0}), \quad \phi_e \in \mathcal{S}_e. \quad (4.9)$$

**Proof.** The function  $C_d$  is of class (AH) and, by equivalence of finite-dimensional norms, there exist positive constants  $B, M$  such that

$$\left| \int_{\mathbb{R}} dk C_d(\mathbf{x}, k) e^{-ik} \right| \leq \int_{\mathbb{R}} dk (\|\mathbf{x}\|_1 - |k|)^+ = \|\mathbf{x}\|_1^2 \leq B(1 + |\mathbf{x}|)^M, \quad \mathbf{x} \in \mathbb{R}^d. \quad (4.10)$$

Consequently, the function  $\mathbf{x} \mapsto \int dk C_d(\mathbf{x}, k) e^{-ik}$  induces a distribution on  $\mathcal{S}(\mathbb{R}^d)$  (Friedlander, 1998, Eqn. (8.3.2) page 97). Hence by Theorem 3.4 and Lemma 3.3 the corresponding unique solution  $T^{C_d}$  defines a distribution on  $\mathcal{S}_e$ . The subspace of  $\mathcal{D}(\mathbb{R}^d)$  generated by functions of the form  $\prod_{i=1}^d \varphi_i(x_i)$ ,  $\varphi_i \in \mathcal{D}(\mathbb{R})$ ,  $i = 1, \dots, d$ , is dense in  $\mathcal{D}(\mathbb{R}^d)$  (Kanwal, 2004, Lemma 2 page 181), which itself is dense in  $\mathcal{S}(\mathbb{R}^d)$  (Kanwal, 2004, Remark page 140). As a result,  $\mathcal{S}_e$  is dense in  $\mathcal{S}_e$ .

Turning our attention to (4.9), for  $\psi_e \in \Sigma_e$  and  $\varphi_e$  such that  $\psi_e = \varphi_e - \varphi_e(\mathbf{0})$ , (3.3) yields

$$\begin{aligned} \left\langle N_{\mathbf{d}\mathbf{w}}^{C_d}, \psi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} &= -\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} dk C_d(\mathbf{x}, k) e^{-ik} \mathcal{F}^{-1} \varphi_e(\mathbf{x}) \\ &= -\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} \mathcal{F}^{-1} \varphi_e(\mathbf{x}) \mathcal{F}_k [\|\mathbf{x}\|_1 - |k|] (1). \end{aligned} \quad (4.11)$$

Substituting (B.2) with  $c = \|\mathbf{x}\|_1$  and  $r = 1$ ,

$$\begin{aligned} \left\langle N_{\mathbf{d}\mathbf{w}}^{C_d}, \psi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} &= \int_{\mathbb{R}^d} d\mathbf{x} \mathcal{F}^{-1} \varphi_e(\mathbf{x}) (\cos \|\mathbf{x}\|_1 - 1) = \\ &= - \int_{\mathbb{R}^d} d\mathbf{x} \mathcal{F}^{-1} \varphi_e(\mathbf{x}) + \frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} e^{-i\|\mathbf{x}\|_1} \mathcal{F}^{-1} \varphi_e(\mathbf{x}) + \frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} e^{i\|\mathbf{x}\|_1} \mathcal{F}^{-1} \varphi_e(\mathbf{x}). \end{aligned} \quad (4.12)$$

As shown in Section C, for  $\varphi_e = \phi_e \in \mathcal{S}_e$ ,

$$\frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} e^{-i\|\mathbf{x}\|_1} \mathcal{F}^{-1} \phi_e(\mathbf{x}) + \frac{1}{2} \int_{\mathbb{R}^d} d\mathbf{x} e^{i\|\mathbf{x}\|_1} \mathcal{F}^{-1} \phi_e(\mathbf{x}) = \left\langle T_{\mathbf{d}\mathbf{w}}^d, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}}. \quad (4.13)$$

Plugging  $\int_{\mathbb{R}^d} d\mathbf{x} \mathcal{F}^{-1} \varphi_e(\mathbf{x}) = \varphi_e(\mathbf{0})$  and (4.13) into (4.12),

$$\left\langle T_{\mathbf{d}\mathbf{w}}^d, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} - \phi_e(\mathbf{0}) = \left\langle N_{\mathbf{d}\mathbf{w}}^{C_d}, \psi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} = \left\langle T_{\mathbf{d}\mathbf{w}}^{C_d}, \varphi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} = \left\langle T_{\mathbf{d}\mathbf{w}}^{C_d}, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}} \quad (4.14)$$

which proves (4.9).  $\square$

*Remark 4.3* (sanity check). For  $d = 1$  and  $T^1 = \frac{1}{2}(\delta_1 + \delta_{-1})$ , (4.14) yields

$$\begin{aligned} \left\langle N_{\mathbf{d}\mathbf{w}}^{C_1}, \psi_e(w) \right\rangle_w &= \left\langle T_{\mathbf{d}\mathbf{w}}^1, \phi_e(w) \right\rangle_w - \phi_e(0) = \left\langle \frac{1}{2}(\delta_1(dw) + \delta_{-1}(dw)), \phi_e \right\rangle_w - \phi_e(0) \\ &= \left\langle \frac{1}{2}(\delta_1(dw) + \delta_{-1}(dw)), \phi_e - \phi_e(0) \right\rangle_w = \frac{1}{2}(\psi_e(1) + \psi_e(-1)), \end{aligned}$$

hence  $N_{\mathbf{d}\mathbf{w}}^{C_1} = \frac{1}{2}(\delta_1(dw) + \delta_{-1}(dw))$  on  $\Sigma_e$  (recalling that  $\mathcal{S}_e = \mathcal{S}_e$  for  $d = 1$ ). Since  $N^{C_1}$  is a measure-type distribution with finite second moment  $\frac{1}{2} \int_{\mathbb{R}} w^2 (\delta_1(dw) + \delta_{-1}(dw)) = 1$ ,

we recover via Proposition 2.3 the trivial spanning identity, for  $(x, k) \in \mathbb{R}^2$ ,

$$(|x| - |k|)^+ = \left\langle N_{dw}^{C_1}, (|wx| - |k|)^+ \right\rangle_w = \frac{1}{2}((|x| - |k|)^+ + (|-x| - |k|)^+). \quad (4.15)$$

In higher dimension  $d \geq 2$ , the distribution  $T^{C_d}$  is not of measure-type and there is no strong spanning representation readily following from (2.7). However, by (B.2),  $\mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) = \frac{2-2\cos(r\mathbf{w} \cdot \mathbf{y})}{r^2}$ , which is continuous in  $\mathbf{y}$ . The strong representation from Proposition 2.4 is thus applicable and can be turned into the following more explicit representation, derived for ease of writing in dimension  $d = 2$  only.

**Lemma 4.4.** *For any  $r \in \mathbb{R}^*$  and multiplicatively separable  $h \in \mathcal{S}(\mathbb{R}^2)$ , letting  $\psi_e(\mathbf{w}) = \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}) \mathcal{F}_\kappa [(|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+] (r)$  and  $\phi_e := \psi_e - \lim_{\infty} \psi_e$ , then  $\phi_e \in S_e$  and*

$$\begin{aligned} \langle T^2, \phi_e \rangle - \phi_e(\mathbf{0}) &= \frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}) \\ &- \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left( \frac{dw_1}{w_1+1} + \frac{dw_1}{w_1-1} \right) \int_{-\infty}^{\infty} \left( \frac{dw_2}{w_2+w_1} - \frac{dw_2}{w_2-w_1} \right) \frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}) \cos(r\mathbf{w} \cdot \mathbf{y}) \\ &- \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left( \frac{dw_2}{w_2+1} + \frac{dw_2}{w_2-1} \right) \int_{-\infty}^{\infty} \left( \frac{dw_1}{w_1+w_2} - \frac{dw_1}{w_1-w_2} \right) \frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}) \cos(r\mathbf{w} \cdot \mathbf{y}). \end{aligned} \quad (4.16)$$

**Proof.** By Lemma 2.1,  $\psi_e(\mathbf{w})$  is in  $\Sigma_e$ . Substituting (B.2) into the definition of  $\psi_e$  yields  $\psi_e(\mathbf{w}) = \frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}) (1 - \cos(r\mathbf{w} \cdot \mathbf{y}))$ . By Riemann-Lebesgue's lemma (see Remark B.2),  $\lim_{\infty} \psi_e = \frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y})$ , so that

$$\phi_e(\mathbf{w}) = -\frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}) \cos(r\mathbf{w} \cdot \mathbf{y}), \quad (4.17)$$

which is in  $\mathcal{S}(\mathbb{R}^2)$  as Fourier transform of an  $\mathcal{S}(\mathbb{R}^2)$  function. As  $h(\mathbf{y})$  is multiplicatively separable, expanding  $\cos(r\mathbf{w} \cdot \mathbf{y}) = \cos(rw_1y_1) \cos(rw_2y_2) - \sin(rw_1y_1) \sin(rw_2y_2)$  and separating integrals proves that  $\phi_e \in S_e$ . Substituting (4.17) into the left-hand side of (4.16),

$$\langle T^2, \phi_e \rangle - \phi_e(\mathbf{0}) = - \left\langle T_{dw}^2, \frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}) \cos(r\mathbf{w} \cdot \mathbf{y}) \right\rangle_{\mathbf{w}} + \frac{2}{r^2} \int_{\mathbb{R}^2} d\mathbf{y} h(\mathbf{y}).$$

Substituting (4.7) into the right-hand side yields the required result.  $\square$

**Proposition 4.5.** *In dimension  $d = 2$  the dispersion call payoff admits the strong*

representation

$$\begin{aligned}
& (|x_1| + |x_2| - |k|)^+ \\
&= \mathcal{F}_r^{-1} \left[ \lim_{n \rightarrow \infty} \mathbf{1}_{r \neq 0} \left( \frac{2}{r^2} + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left( \frac{dw_1}{w_1 + 1} + \frac{dw_1}{w_1 - 1} \right) \int_{-\infty}^{\infty} \left( \frac{dw_2}{w_2 + w_1} - \frac{dw_2}{w_2 - w_1} \right) \times \right. \right. \\
&\quad \left. \left( \mathcal{F}_\kappa \left[ \int_{\mathbb{R}^2} dy h_n(\mathbf{y} - \mathbf{x}) (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) - \frac{2}{r^2} \right) \right. \\
&\quad \left. + \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left( \frac{dw_2}{w_2 + 1} + \frac{dw_2}{w_2 - 1} \right) \int_{-\infty}^{\infty} \left( \frac{dw_1}{w_1 + w_2} - \frac{dw_1}{w_1 - w_2} \right) \times \right. \\
&\quad \left. \left. \left( \mathcal{F}_\kappa \left[ \int_{\mathbb{R}^2} dy h_n(\mathbf{y} - \mathbf{x}) (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) - \frac{2}{r^2} \right) \right) \right] (k),
\end{aligned}$$

$dk$ -almost everywhere for each  $\mathbf{x} \in \mathbb{R}^2$ , where  $h_n$  is the sequence of Dirac approximation functions given in (2.9).

**Proof.** For each  $\mathbf{y} \in \mathbb{R}^2$ , the payoff function  $k \mapsto C_2(\mathbf{y}, k)$  has compact support and is thus in  $L^1(\mathbb{R})$ , while  $r \mapsto \mathcal{F}_k[C_2(\mathbf{y}, k)](r)$  is in  $L^1(\mathbb{R})$  by (B.2) and is continuous in  $\mathbf{y} \in \mathbb{R}^2$  for each  $r \in \mathbb{R}^*$ . Therefore, by Proposition 2.4, we have for each  $\mathbf{x} \in \mathbb{R}^2$ ,

$$C_2(\mathbf{x}, k) = \mathcal{F}_r^{-1} \left[ \lim_{n \rightarrow \infty} \mathbf{1}_{r \neq 0} \left\langle N_{d\mathbf{w}}^{C_2}, \int_{\mathbb{R}^d} dy h_n(\mathbf{y} - \mathbf{x}) \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right\rangle_{\mathbf{w}} \right] (k), \quad (4.18)$$

$dk$ -almost everywhere. Let  $\psi_e^n(\mathbf{w}) = \int_{\mathbb{R}^d} dy h_n(\mathbf{y} - \mathbf{x}) \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r)$  and  $\phi_e^n = \psi_e^n - \lim_{\infty} \psi_e^n$ . By Proposition 4.3 and Lemma 4.4 with  $h(\mathbf{y}) \equiv h_n(\mathbf{y} - \mathbf{x})$  that is multiplicatively separable,

$$\langle N^{C_2}, \psi_e^n \rangle = \langle T^{C_2}, \phi_e^n \rangle = \langle T^2, \phi_e^n \rangle - \phi_e^n(\mathbf{0}).$$

Substituting (4.16) with  $h(\mathbf{y}) = h_n(\mathbf{y} - \mathbf{x})$  and  $\frac{2}{r^2} \cos(r\mathbf{w} \cdot \mathbf{y}) = \frac{2}{r^2} (1 - (1 - \cos(r\mathbf{w} \cdot \mathbf{y}))) = \frac{2}{r^2} - \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r)$  into the right-hand side, then plugging the resulting expression for  $\langle N^{C_2}, \psi_e^n \rangle$  into (4.18), yields

$$\begin{aligned}
C_2(\mathbf{x}, k) &= \mathcal{F}_r^{-1} \left[ \lim_{n \rightarrow \infty} \mathbf{1}_{r \neq 0} \left( \frac{2}{r^2} \int_{\mathbb{R}^2} dy h_n(\mathbf{y} - \mathbf{x}) \right. \right. \\
&\quad - \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left( \frac{dw_1}{w_1 + 1} + \frac{dw_1}{w_1 - 1} \right) \int_{-\infty}^{\infty} \left( \frac{dw_2}{w_2 + w_1} - \frac{dw_2}{w_2 - w_1} \right) \times \\
&\quad \int_{\mathbb{R}^2} dy h_n(\mathbf{y} - \mathbf{x}) \left( \frac{2}{r^2} - \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right) \\
&\quad - \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left( \frac{dw_2}{w_2 + 1} + \frac{dw_2}{w_2 - 1} \right) \int_{-\infty}^{\infty} \left( \frac{dw_1}{w_1 + w_2} - \frac{dw_1}{w_1 - w_2} \right) \times \\
&\quad \left. \left. \int_{\mathbb{R}^2} dy h_n(\mathbf{y} - \mathbf{x}) \left( \frac{2}{r^2} - \mathcal{F}_\kappa \left[ (|\mathbf{w} \cdot \mathbf{y}| - |\kappa|)^+ \right] (r) \right) \right) \right] (k).
\end{aligned}$$

Substituting  $\int_{\mathbb{R}^2} d\mathbf{y} h_n(\mathbf{y} - \mathbf{x}) = 1$ , and taking Fourier transforms with respect to  $\kappa$  out of the  $d\mathbf{y}$  integrals yields the required result.  $\square$

## 5 Benefits of Unrestricted Neural Network Spanning Versus Other Discrete Spanning Strategies

The delicate nature of the explicit distribution derived in the previous section for the dispersion call shows that the solution formula (3.3) for  $N^F$  can be nontrivial and requires case-by-case analysis. When the target payoff  $F$  is not smooth, even if an analytical representation can be derived, it is likely to be numerically intractable in high dimension (see Remark 4.2), and other numerical methods must be developed. Since the hedging error (1.2) corresponds to the prediction error of a one-hidden-layer feedforward neural network, training such a network may constitute an efficient alternative for discrete spanning.

In this empirical section the strike variable  $k$  of the target payoff  $F(\mathbf{x}, k)$  is viewed as a fixed parameter, and we simply write  $F(\mathbf{x})$  for ease of notation. For two positive integers  $d$  and  $n$ , let  $\mathcal{NN}_{d,n}$  denote the family of functions that take a vector  $\mathbf{x} \in \mathbb{R}^d$  as input and return a value in  $\mathbb{R}$  through the sequential mapping

$$\mathbb{R}^d \ni \mathbf{x} \xrightarrow{\tilde{F}} \alpha + \boldsymbol{\mu} \cdot \mathbf{x} + \sum_{i=1}^n \nu_i \left( \eta_i (\mathbf{w}^{(i)} \cdot \mathbf{x} - k_i) \right)^+ \in \mathbb{R}, \quad (5.1)$$

where  $\boldsymbol{\mu}, \mathbf{w}^{(i)} = [w_1^{(i)}, \dots, w_d^{(i)}]^\top$  are vector versions of the quantities introduced in (1.2), while the sign of each parameter  $\eta_i$  determines whether a basket call or put is used. The sign of each parameter  $\nu_i$  determines whether a long or short position is taken in basket option  $i$  with strike  $k_i > 0$ , in quantity  $|\nu_i \eta_i|$ .

*Remark 5.1.* When spanning with basket calls only, i.e.  $\eta_i > 0$ , approximating  $F(\mathbf{x})$  by  $\tilde{F}(\mathbf{x}) \in \mathcal{NN}_{d,n}$  given by (5.1) is equivalent to discretizing (2.2) for fixed  $k > 0$  (in other words,  $\nu$  in (2.2) would be allowed to depend on  $k$ ). This can be seen by rewriting (5.1) as

$$\tilde{F}(\mathbf{x}) = \alpha + \boldsymbol{\mu} \cdot \mathbf{x} + \sum_{i=1}^n \underbrace{\nu_i \eta_i \frac{k_i}{k}}_{=: \nu'_i} \left( \underbrace{\frac{k}{k_i} \mathbf{w}^{(i)} \cdot \mathbf{x} - k}_{=: \mathbf{w}'^{(i)}} \right)^+, \quad (5.2)$$

which is a discretization of (2.2) with explicit affine terms to the right-hand side.

The family of functions (5.1) corresponds to an unrestricted one-hidden-layer residual neural network with ReLU activation function architecture characterized by the set of trainable parameters

$$\theta = (\mathbf{W}, \boldsymbol{\mu}, \mathbf{k}, \alpha, \boldsymbol{\nu}, \boldsymbol{\eta}),$$

where  $\mathbf{W} = [\mathbf{w}^{(1)}, \dots, \mathbf{w}^{(n)}]^\top \in \mathbb{R}^{n \times d}$  stores the  $w_j^{(i)}$  in matrix form, while  $\mathbf{k} \in (\mathbb{R}_+^*)^n$ ,  $\boldsymbol{\nu} \in \mathbb{R}^n$ , and  $\boldsymbol{\eta} \in \mathbb{R}^n$  store the  $k_i, \nu_i$  and  $\eta_i$  in vector form. We seek an ap-

proximation

$$\hat{F} \in \arg \min_{\tilde{F} \in \mathcal{NN}_{d,n}} \hat{\mathbb{E}} \left[ (F(\mathbf{x}) - \tilde{F}(\mathbf{x}))^2 \right], \quad (5.3)$$

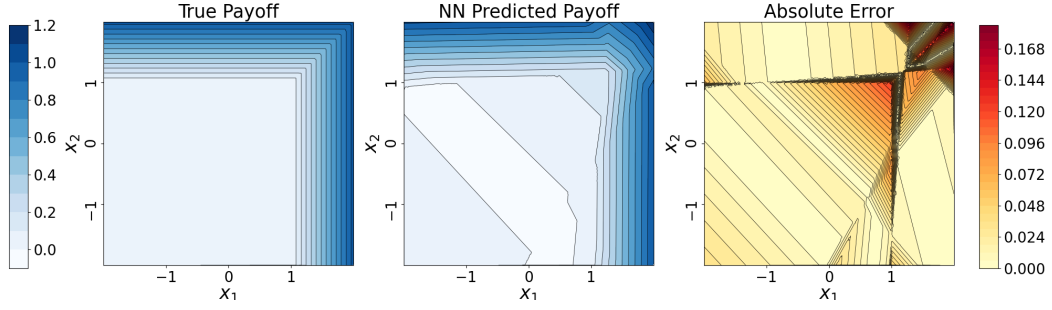
where  $\hat{\mathbb{E}}[\cdot] = \frac{[\cdot]_1 + \dots + [\cdot]_m}{m}$  denotes the sample mean over  $m$  observations drawn from values of  $\mathbf{x}$  that may be deterministically or randomly sampled. Additional methodological details can be found in the paper’s github. Below is a summary of our main empirical results, which we compared to three other spanning strategies:

- Spanning with single-asset payoffs: This “marginal” approach is attractive to practitioners because single-asset vanilla options are more liquid and can often be traded on exchanges.
- Spanning with predetermined basket payoffs, i.e. with fixed basket weights  $w_1^{(i)}, \dots, w_d^{(i)}$  and strikes  $k_i$ : This level of control may be beneficial to practitioners in order to define a tractable universe of spanning instruments. Another benefit of this approach is that it can be solved using classical linear regression techniques.
- Spanning with long-only basket payoffs, i.e. the components of the basket weights  $\mathbf{w}^{(i)}$  of each basket payoff  $1 \leq i \leq n$  are positive.

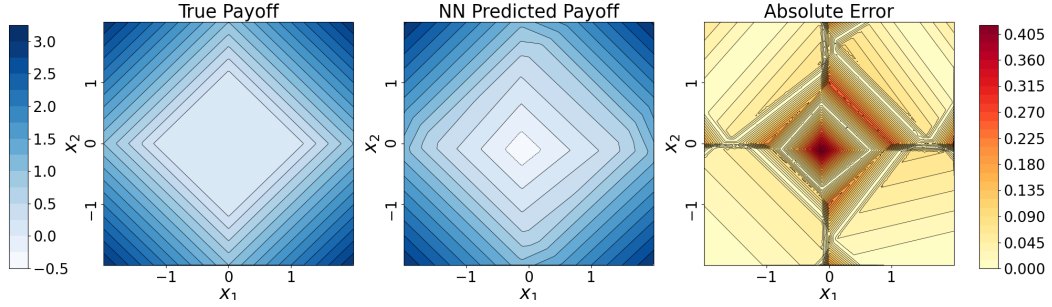
## 5.1 Spanning Metrics

We use two metrics to assess the distance between the target payoff  $F$  and a predictor  $\tilde{F} \in \mathcal{NN}_{d,n}$ : mean squared error  $\text{MSE} = \hat{\mathbb{E}}[(F(\mathbf{x}) - \tilde{F}(\mathbf{x}))^2]$  for the loss function, and mean absolute error  $\text{MAE} = \hat{\mathbb{E}}|F(\mathbf{x}) - \tilde{F}(\mathbf{x})|$  for reporting. The choice of MAE for reporting is motivated by its ease of financial interpretation as average absolute dollar mismatch between the target payoff and the spanning portfolio. However, MAE is not differentiable at certain points and as such it is not a good choice of loss function for gradient calculation, which is why we opted for MSE loss.

Figure 2 shows contour plots reporting the target payoff  $F$ , its approximation  $\hat{F} \in \mathcal{NN}_{2,40}$  and the pointwise absolute spanning error  $|F(x_1, x_2) - \hat{F}(x_1, x_2)|$  for best-of call and dispersion call payoffs. From visual inspection the predicted surface provided by the unrestricted neural network spanning strategy (NN) fits the target payoff reasonably well. However, higher spanning errors are observed around areas where the target payoff is nondifferentiable.



(a) Best-of call  $F(x_1, x_2) = (\max(x_1, x_2) - 1)^+$ . MAE = 0.02 (2 cents per dollar of notional).



(b) Dispersion call  $F(x_1, x_2) = (|x_1| + |x_2| - 1)^+$ . MAE = 0.051 (5.1 cents per dollar of notional).

Figure 2: Contour plots of the target payoff (*left*), NN prediction (*center*) and absolute spanning error (*right*) for the (a) best-of call, (b) dispersion call payoffs.

## 5.2 Limitations of Spanning with Single-Asset Payoffs

When spanning the target payoff  $F(\mathbf{x})$  with single-asset vanilla payoffs only, equations (5.1)-(5.3) become

$$\min_{\alpha, \boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\eta}, \mathbf{k}, \mathbf{E}} \widehat{\mathbb{E}} \left[ \left( F(\mathbf{x}) - \alpha - \boldsymbol{\mu} \cdot \mathbf{x} - \sum_{i=1}^n \nu_i (\eta_i (\mathbf{e}_i \cdot \mathbf{x} - k_i))^+ \right)^2 \right],$$

where  $\mathbf{e}_i \in \{0, 1\}^d$  is a “one-hot” vector with all coefficients equal to 0, except the coefficient corresponding to the selected asset which is equal to 1 (the index of which remains free for optimization), and  $\mathbf{E} = [\mathbf{e}_1, \dots, \mathbf{e}_n]^\top \in \{0, 1\}^{n \times d}$  is the corresponding matrix. We solve this optimization problem numerically by Adam stochastic gradient descent (Kingma and Ba, 2015) with a restricted neural network architecture, for the best-of call and worst-of put payoffs on 2 to 5 underlying assets (see Table 1, page 2). Single-asset spanning does provide some risk reduction compared to the unhedged case for which all parameters are zero and MAE is the payoff average absolute value. However, Figure 3 shows that the spanning error of this strategy is substantially higher than that of our core unrestricted NN approach (5.3). Mathematically, this is hardly surprising given that single-asset option spanning attempts to reproduce a “joint distribution” (the best-of or worst-of payoff) with “marginal distributions” only (the single-asset payoffs).

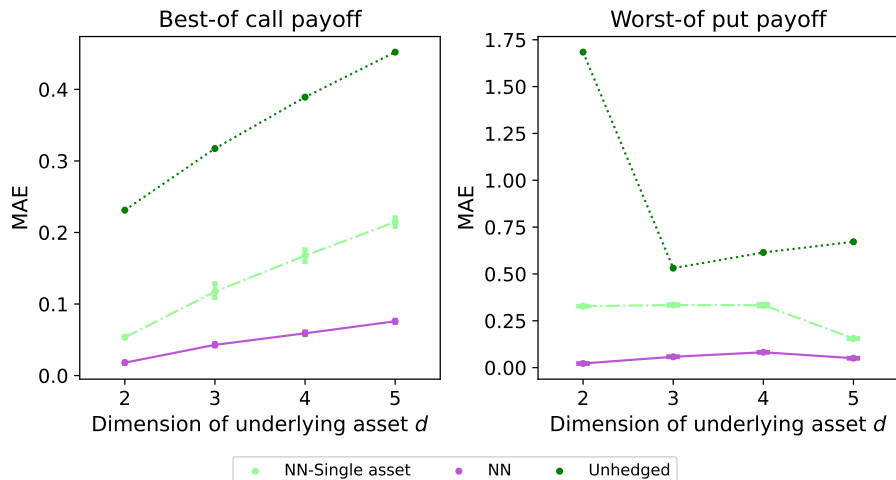


Figure 3: MAE comparison of the single-asset and unrestricted NN spanning strategies for  $d = 2, \dots, 5$  assets: best-of call (*left*) and worst-of put (*right*). Error bars are 95% confidence intervals.

### 5.3 Limitations of Spanning with Predetermined Basket Payoffs

When spanning the target payoff  $F(\mathbf{x})$  with predetermined basket option payoffs, the strikes  $k_i$ , call/put selectors  $\eta_i$ , and basket weights  $\mathbf{w}^{(i)}$  of each basket payoff  $1 \leq i \leq n$  are fixed and the spanning problem takes the simpler form

$$\min_{\alpha, \boldsymbol{\mu}, \boldsymbol{\nu}} \widehat{\mathbb{E}} \left[ \left( F(\mathbf{x}) - \alpha - \boldsymbol{\mu} \cdot \mathbf{x} - \sum_{i=1}^n \nu_i \left( \eta_i (\mathbf{w}^{(i)} \cdot \mathbf{x} - k_i) \right)^+ \right)^2 \right], \quad (5.4)$$

which is a classic linear least-squares regression problem. Due to parameter redundancy we choose unit strikes  $k_i = 1$ , and we also set selectors  $\eta_i = 1$ . The explicit solution to (5.4) is therefore given by the regression coefficients  $\widehat{\boldsymbol{\beta}} = \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\nu} \end{bmatrix} \in \mathbb{R}^{d+n}$  and constant  $\widehat{\alpha} \in \mathbb{R}$  with

$$\widehat{\boldsymbol{\beta}} = \widehat{\text{Var}}(\mathbf{z})^{-1} \widehat{\text{Cov}}(\mathbf{z}, F(\mathbf{x})), \quad \widehat{\alpha} = \widehat{\mathbb{E}}[F(\mathbf{x})] - \widehat{\boldsymbol{\beta}}^\top \widehat{\mathbb{E}}[\mathbf{z}],$$

where  $\mathbf{z} = [x_1, \dots, x_d, (\mathbf{w}^{(1)} \cdot \mathbf{x} - 1)^+, \dots, (\mathbf{w}^{(n)} \cdot \mathbf{x} - 1)^+]^\top \in \mathbb{R}^{d+n}$  is the vector of explanatory variables (underlying assets and basket payoffs),  $\widehat{\text{Cov}}$  is the sample covariance operator, and  $\widehat{\text{Var}}(\mathbf{z})$  is the sample covariance matrix which must be nonsingular. In practice, we found this calculation to be numerically unstable due to conditioning issues: basket payoffs tend to overlap, making the columns of  $\mathbf{z}$  loosely dependent, particularly for large  $n$ . This issue is further compounded when the sampling of  $\mathbf{x}$  (and thus  $\mathbf{z}$ ) is sparse. To circumvent this practical difficulty, we used singular value decomposition (SVD) (Golub and Van Loan, 2013, Theorems 2.5.2 and 5.5.1).

Figure 4 reports the spanning error for the best-of call on 2 to 5 underlying assets, together with 95% confidence intervals over 30 different runs that were obtained using the following methods:

- (a) SVD with regular grid sampling of basket weights  $\mathbf{w}$  (regular+SVD);

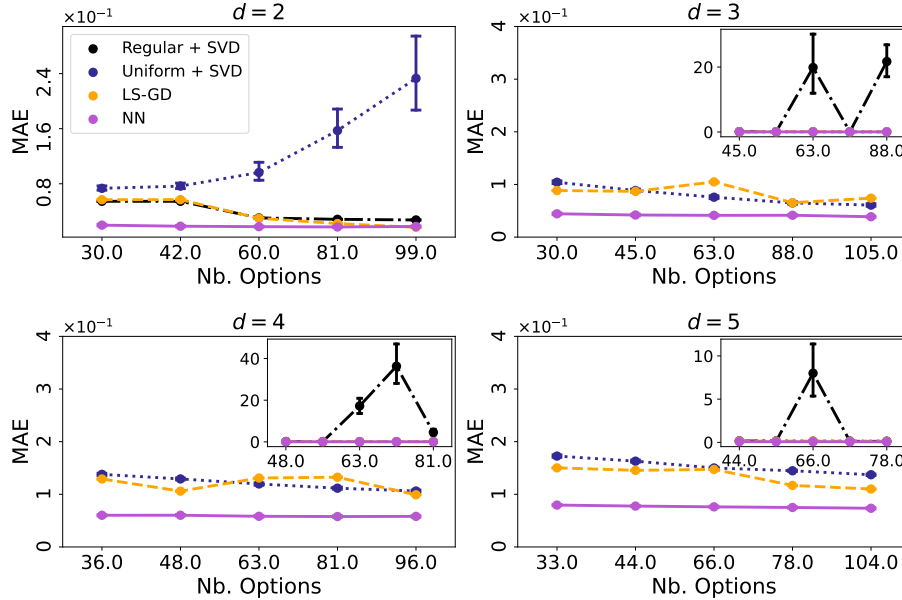


Figure 4: MAE with 95% confidence intervals for the best-of call on  $d = 2$  to 5 underlying assets for methods (a) to (d), as a function of the number of basket options  $n$ . Large errors obtained with (a) are reported in inside panels for  $d = 3, 4, 5$  for readability.

- (b) SVD with i.i.d. uniform random sampling of  $\mathbf{w}$  (uniform+SVD);
- (c) stochastic gradient descent with regular grid sampling of  $\mathbf{w}$  (LS-GD);
- (d) unrestricted neural network approach with free  $\mathbf{w}$  initialized with i.i.d. uniform random sampling (NN).

Each run is a new training routine with the same fixed weights  $\mathbf{w}$  for methods (a) and (c), and with new random weights  $\mathbf{w}$  for methods (b) and (d). We can see that all fixed-weights methods (a) to (c) resulted in substantial spanning errors compared to (d) unrestricted NN, except perhaps in dimension  $d = 2$  where the error magnitude is smaller. Remarkably enough, increasing the number  $n$  of basket payoffs does not materially reduce the spanning error for any method (a) to (d). This suggests that only a limited number of basket payoffs may be needed to obtain a satisfactory hedge of the best-of call, in contrast to the combinatorial issue identified in Remark 4.2. Finally, SVD methods (a) and (b) often perform poorly: when sampling  $\mathbf{w}$  along a regular grid, the spanning error (a) even “exploded” in dimensions  $d = 3, 4, 5$  due to ill-conditioning of the design matrix generated by observations of  $\mathbf{z}$ , as shown in inner panels.

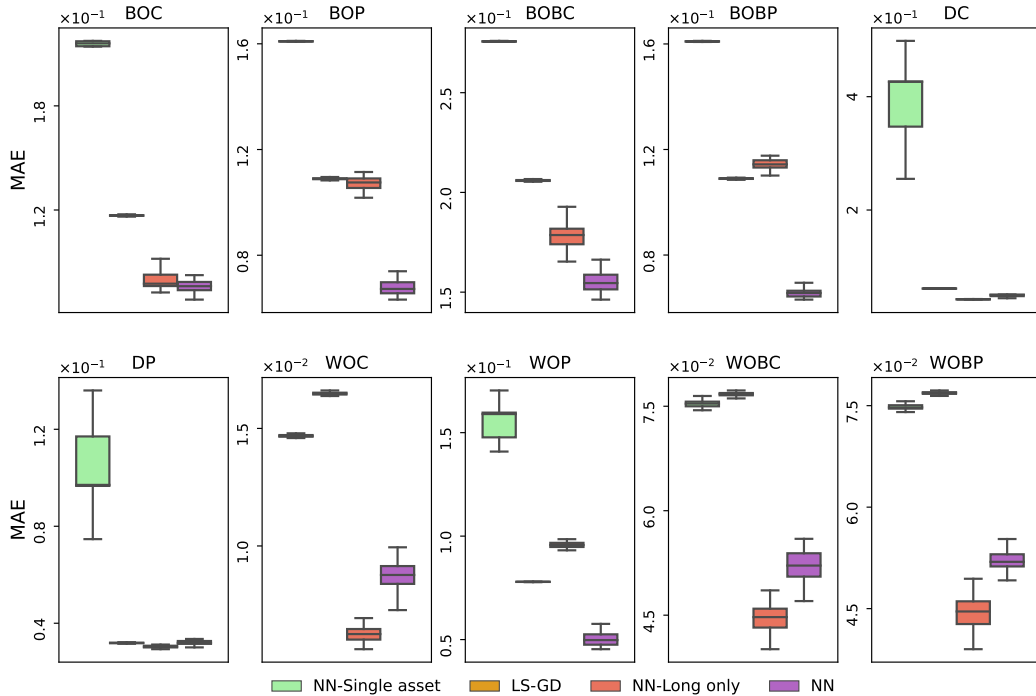
It is worth noting that, from a neural network architecture perspective, the spanning problem (5.4) corresponds to a particular category of extreme learning machines or ELMs (Huang, Zhu, and Siew, 2004, 2006). ELMs are known to be universal approximators, but they typically require a very large number  $n$  of hidden units to achieve satisfactory performance. In the context of payoff spanning, this suggests that only a large number  $n$  of predetermined basket payoffs in methods (a) to (c) would be able to approach the performance of the unrestricted NN method (d), which would be



impractical for hedge execution.

### 5.4 Summary of Restricted and Unrestricted Spanning Results

Figure 5 reports the performance of four spanning strategies in dimension  $d = 5, 20$  and  $50$  using 50 different network initializations: (i) Adam training of single asset portfolio (NN-Single asset) as explained in Section 5.2, (ii) least-squares approach with gradient descent (LS-GD) as explained in Section 5.3, (iii) Adam training of long only basket portfolio (NN-Long only), and (iv) unrestricted neural network with Adam training (NN). We can see that the spanning portfolios suggested by the neural network give satisfying MAE results in most cases. Unrestricted NN outperforms all other strategies in terms of MAE and standard deviation: MAE increases with the underlying asset dimension  $d$  but remains fairly low. The low standard deviation figures also signals that unrestricted NN is the most stable among the four strategies.



(a)  $d = 5$  and  $n = 78$ .

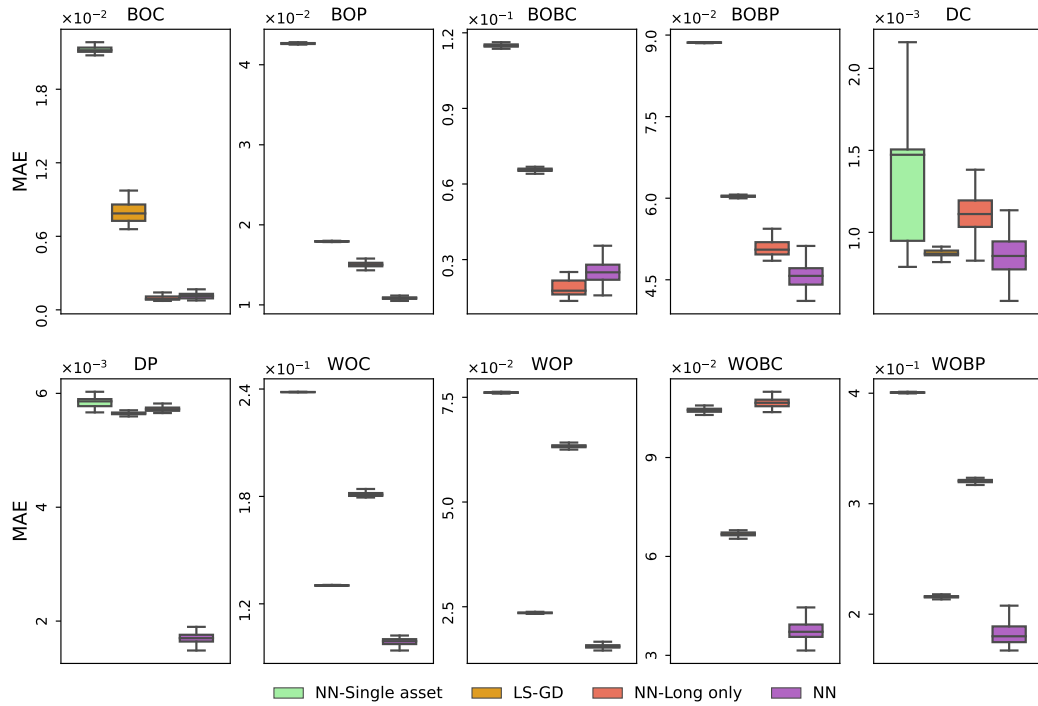
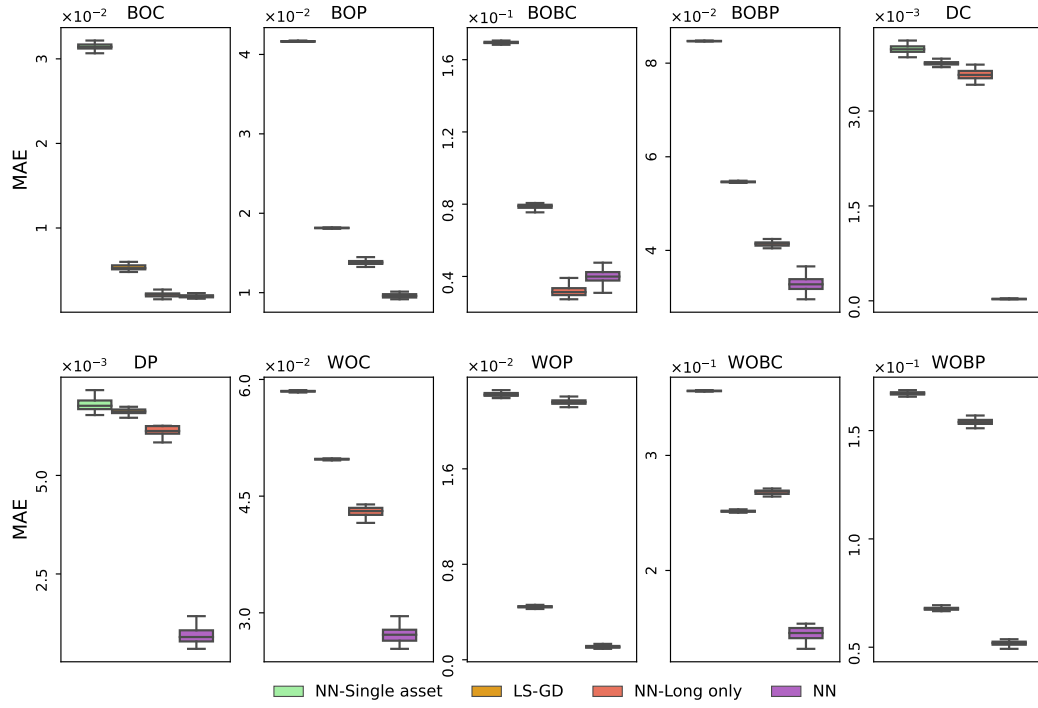


Figure 5: Average MAE and 95% error bars over 50 runs by spanning strategy and target payoff in dimension  $d = 5, 20$  and  $50$ .

## 5.5 Stability Issues

In linear regression models, the loss function is convex and thus easier to optimize. In contrast, it is nonconvex in neural network parameterizations such as (5.1)-(5.3): optimization is more difficult, algorithms such as Adam method typically converge to different local minima for different initializations, yet loss values usually remain small (Choromanska, Henaff, Mathieu, Arous, and LeCun, 2015). We observed this phenomenon in our study: the optimal neural network parameters vary with each training, but we obtained persistently small and stable errors. Financially, this means that spanning performance remains strong but the particular static hedge identified based on a particular training set does not have a stable interpretative meaning with respect to the target option.

Figures 6 and 7 show how the optimizer solution for the dispersion call approaches the theoretical solution that we derived in Proposition 4.3 in dimensions  $d = 1, 2$ . Our results are indeed remarkably consistent with the accumulation of basket call quantities  $\nu_i$  predicted by theory. In dimension  $d = 1$  our numerical results perfectly match the exact solution (4.15) (see also Remark 4.3) after we process the learning with an  $\ell^2$  regularization (Figure 6, right panel.) In dimension  $d = 2$  our numerical results after regularization are also consistent with the accumulation of basket calls around discontinuity points  $w_1 = \pm 1$  and  $w_2 = \pm 1$  predicted by the theoretical formulas (4.7) and (4.18). (Figure 7, right panel).

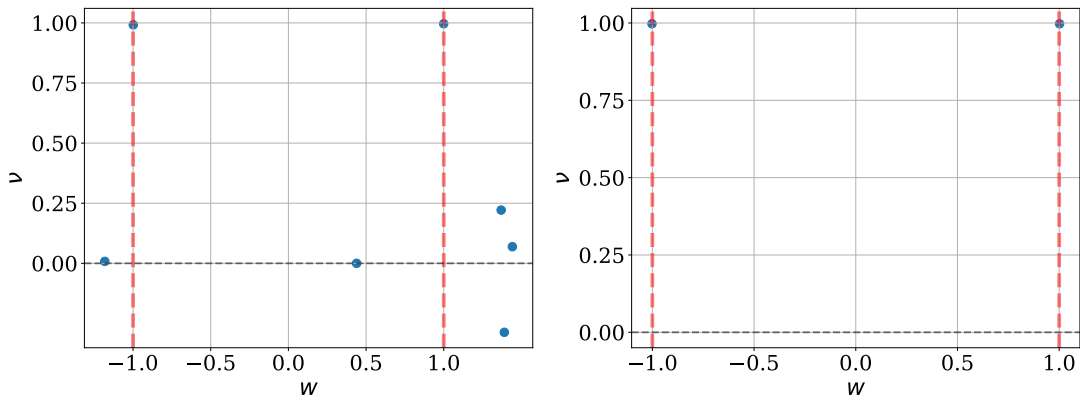


Figure 6: Scatter plots of the optimal basket call quantity  $\nu_i$  against the optimal basket weight  $w^{(i)}$  (see (5.2)) predicted by (5.1)-(5.3) for the single-asset dispersion call payoff  $F(x) = (|x| - 1)^+$  when  $n = 10$  and the training asset price  $x$  is sampled in  $[-2, 2]$ . (*Left*) without regularization; (*Right*) with regularization.

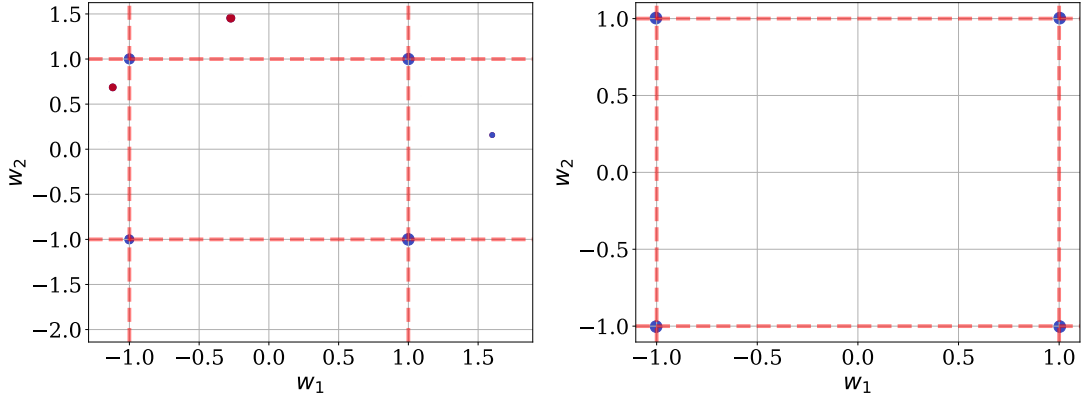


Figure 7: Scatter plots of the optimal basket call weights  $(w_1^{(i)}, w_2^{(i)})$  (see. (5.2)) predicted by (5.1)-(5.3) for the two-asset dispersion call with payoff  $F(x_1, x_2) = (|x_1| + |x_2| - 1)^+$  when  $n = 50$ . The training asset prices  $(x_1, x_2)$  are sampled in  $[-1, 1]^2$ . The blue and red points represent respectively long ( $\nu_i > 0$ ) and short ( $\nu_i < 0$ ) positions while point sizes reflect absolute quantities  $|\nu_i|$ . (Left) without regularization; (Right) with regularization.

## 6 Conclusion and Perspectives

Identifying static hedges for exotic option payoffs is important for both theory and practice. Theorem 3.4 expands on existing continuum replication theory of European option payoffs with vanilla options to formulate a general and rigorous solution of the continuum spanning problem. As an application, Proposition 4.3 derives the continuum solution replicating the industry  $\ell^1$  dispersion call. In addition to the derivation of explicit solutions for other absolutely homogeneous payoffs such as best-of and worst-of options, other formulations of the continuum spanning problem are open for future research: for example, letting  $\nu$  in (1.1) depend on  $k$ , or integrating the spanning portfolio over  $k$ , may relax the absolute homogeneity restriction to yield solutions for non-homogeneous payoffs.

Leveraging the parallel between vanilla basket calls and ReLU functions, we examined how neural networks can be used to numerically solve the corresponding discrete spanning problem and identify finite static hedges, in comparison to other restricted spanning strategies and optimization schemes such as least-squares SVD. Our empirical study suggests that our unrestricted NN approach yields superior results in terms of static hedging error for any dimension 2 to 50. This approach may be of practical interest for the exotic derivatives industry, particularly if combined with delta-hedging of the residual payoff mismatch, which could be investigated in future research.

## A Cauchy Integral

For a function  $\varphi \in L^1(\mathbb{R})$  and a constant  $c \in \mathbb{R}$ , the singular integral

$$\int_{-\infty}^{+\infty} \frac{\varphi(w)}{w-c} dw = \lim_{\alpha \rightarrow 0^+} \int_{-\infty}^{c-\alpha} \frac{\varphi(w)}{w-c} dw + \lim_{\beta \rightarrow 0^+} \int_{c+\beta}^{+\infty} \frac{\varphi(w)}{w-c} dw$$

may not exist when  $\alpha$  and  $\beta$  tend to  $0^+$  independently. However, the integral exists restricted to the diagonal  $\alpha = \beta$  (Estrada and Kanwal, 2000, Section 1.5).

**Definition A.1.** The diagonal limit

$$\lim_{\epsilon \rightarrow 0^+} \left( \int_{-\infty}^{c-\epsilon} \frac{\varphi(w)}{w-c} dw + \int_{c+\epsilon}^{+\infty} \frac{\varphi(w)}{w-c} dw \right) = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w-c|} \frac{\varphi(w)}{w-c} dw \in \mathbb{R}, \quad (\text{A.1})$$

denoted by  $\int_{-\infty}^{+\infty} \frac{dw}{w-c} \varphi(w)$ , is called the Cauchy principal value integral of  $\frac{\varphi(w)}{w-c}$  against  $dw$ .

**Definition A.2.** (i) The Hilbert transform (up to a multiplicative factor  $\pi$ , see King (2009, Section 3.1 page 83))

$$\int_{-\infty}^{\infty} \frac{dw}{w-c} \varphi(w), \quad \varphi \in \mathcal{S}(\mathbb{R}), \quad (\text{A.2})$$

is a distribution on  $\mathcal{S}(\mathbb{R})$  (Reed and Simon, 1980, Proposition 6 page 136), which we denote by  $\frac{dw}{w-c}$  in abstract distribution notation.

(ii) The linear form

$$\int_{\mathbb{R}^2} \left( \frac{dw_1}{w_1-c_1} \frac{dw_2}{w_2-c_2} \right) \varphi(w_1, w_2) := \int_{-\infty}^{\infty} \frac{dw_1}{w_1-c_1} \left( \int_{-\infty}^{\infty} \frac{dw_2}{w_2-c_2} \varphi(w_1, w_2) \right), \quad \varphi \in \mathcal{S}(\mathbb{R}^2)$$

is a well defined distribution on  $\mathcal{S}(\mathbb{R}^2)$ , known as the tensor (or direct) product of the distributions  $\frac{dw_1}{w_1-c_1}$  and  $\frac{dw_2}{w_2-c_2}$  on  $\mathcal{S}(\mathbb{R}^2)$ , which is commutative (Kanwal, 2004, page 185).

(iii) The previous construction can be iterated to define the distribution  $\prod_{j \in 1..q} \frac{dw_j}{w_j-c_j}$  for any  $q \geq 1$  and  $(c_j) \in \mathbb{R}^q$  as

$$\int_{\mathbb{R}^q} \left( \prod_{j \in 1..q} \frac{dw_j}{w_j-c_j} \right) \varphi(\mathbf{w}), \quad \varphi \in \mathcal{S}(\mathbb{R}^q). \quad (\text{A.3})$$

We may also define, in line with the distributivity principle of commutative products,

$$\int_{\mathbb{R}^q} \left( \prod_{j \in 1..q} \left( \frac{dw_j}{w_j-c_j} + \frac{dw_j}{w_j+c_j} \right) \right) \varphi(\mathbf{w}) := \sum_{(\epsilon_j) \in \{-1,1\}^q} \int_{\mathbb{R}^q} \left( \prod_{j \in 1..q} \frac{dw_j}{w_j+\epsilon_j c_j} \right) \varphi(\mathbf{w}), \quad (\text{A.4})$$

as a distribution on  $\mathcal{S}(\mathbb{R}^q)$ , abstractly denoted  $\prod_{j \in 1..q} \left( \frac{dw_j}{w_j-c_j} + \frac{dw_j}{w_j+c_j} \right)$ .

*Remark A.1.* In view of the above definitions and properties, we may commute Cauchy integrals to write

$$\begin{aligned} \int_{\mathbb{R}^q} \left( \prod_{j \in 1..q} \left( \frac{dw_j}{w_j-c_j} + \frac{dw_j}{w_j+c_j} \right) \right) \varphi(\mathbf{w}) = \\ \int_{\mathbb{R}^{q-1}} \left( \prod_{j \in 2..q} \left( \frac{dw_j}{w_j-c_j} + \frac{dw_j}{w_j+c_j} \right) \right) \int_{\mathbb{R}} \left( \frac{dw_1}{w_1-c_1} + \frac{dw_1}{w_1+c_1} \right) \varphi(\mathbf{w}). \end{aligned} \quad (\text{A.5})$$

This is in line with the definition of the tensor product of the distribution  $(\frac{dw_1}{w_1-c_1} + \frac{dw_1}{w_1+c_1})$  with  $\prod_{j \in 2..q} (\frac{dw_j}{w_j-c_j} + \frac{dw_j}{w_j+c_j})$ , equal to  $\prod_{j \in 1..q} (\frac{dw_j}{w_j-c_j} + \frac{dw_j}{w_j+c_j})$  (Kanwal, 2004, page 185); in bracket notation,

$$\left\langle \prod_{j \in 1..q} \left( \frac{dw_j}{w_j - c_j} + \frac{dw_j}{w_j + c_j} \right), \varphi(\mathbf{w}) \right\rangle_{w_j; j \in 1..q} = \left\langle \prod_{j \in 2..q} \left( \frac{dw_j}{w_j - c_j} + \frac{dw_j}{w_j + c_j} \right), \left\langle \frac{dw_1}{w_1 - c_1} + \frac{dw_1}{w_1 + c_1}, \varphi(\mathbf{w}) \right\rangle_{w_1} \right\rangle_{w_j; j \in 2..q}.$$

The following technical lemmas establish properties related to Cauchy integrals that will be used in Section C.

**Lemma A.1.** *For any functions  $\varphi_j \in \mathcal{S}(\mathbb{R})$ ,  $j = 1, \dots, q$ , there exists a constant  $C > 0$  such that*

$$\left| \int_{\epsilon < |w_2 - w_1|} \frac{dw_2}{w_2 - w_1} \left( \dots \int_{\epsilon < |w_q - w_1|} \frac{dw_q}{w_q - w_1} \prod_{j=1}^q \varphi_j(w_j) \right) \right| \leq C |\varphi_1(w_1)|, \quad w_1 \in \mathbb{R}, 0 < \epsilon < 1,$$

where the constant  $C$  may depend on  $\varphi$  but not on  $w_1$  or  $\epsilon$ .

**Proof.** Factoring  $\varphi_1(w_1)$  out and separating integrals,

$$\begin{aligned} & \int_{\epsilon < |w_2 - w_1|} \frac{dw_2}{w_2 - w_1} \left( \dots \int_{\epsilon < |w_q - w_1|} \frac{dw_q}{w_q - w_1} \prod_{j=1}^q \varphi_j(w_j) \right) \\ &= \varphi_1(w_1) \prod_{j=2}^q \int_{\epsilon < |w_j - w_1|} \frac{dw_j}{w_j - w_1} \varphi_j(w_j). \end{aligned} \tag{A.6}$$

For  $j = 2, \dots, q$ ,

$$\begin{aligned} \int_{\epsilon < |w_j - w_1|} \frac{dw_j}{w_j - w_1} \varphi_j(w_j) &= \int_{\epsilon < |w_j - w_1| < 1} \frac{dw_j}{w_j - w_1} \varphi_j(w_j) + \int_{|w_j - w_1| \geq 1} \frac{dw_j}{w_j - w_1} \varphi_j(w_j) \\ &= \int_{\epsilon < |w_j - w_1| < 1} \frac{dw_j}{w_j - w_1} (\varphi_j(w_j) - \varphi_j(w_1)) + \int_{|w_j - w_1| \geq 1} \frac{dw_j}{w_j - w_1} \varphi_j(w_j), \end{aligned}$$

where we used  $\varphi_j(w_1) \int_{\epsilon < |w_j - w_1| < 1} \frac{dw_j}{w_j - w_1} = 0$  in the last step (as integral of an odd function over a symmetric domain). Since  $\varphi_j \in \mathcal{S}(\mathbb{R})$ ,  $\varphi_j'$  is bounded and by the mean value inequality,  $|\varphi_j(w_j) - \varphi_j(w_1)| \leq |w_j - w_1| \sup |\varphi_j'|$ , whence

$$\left| \int_{\epsilon < |w_j - w_1| < 1} \frac{dw_j}{w_j - w_1} (\varphi_j(w_j) - \varphi_j(w_1)) \right| \leq 2 \sup |\varphi_j'|. \tag{A.7}$$

Turning our attention to the second integral,

$$\left| \int_{|w_j - w_1| \geq 1} \frac{dw_j}{w_j - w_1} \varphi_j(w_j) \right| \leq \int_{\mathbb{R}} dw_j |\varphi_j(w_j)| = \|\varphi_j\|_{L^1(\mathbb{R})} < +\infty, \tag{A.8}$$

since  $\varphi_j \in \mathcal{S}(\mathbb{R})$ . Combining (A.7) and (A.8) yields  $\left| \int_{\epsilon < |w_j - w_1|} \frac{dw_j}{w_j - w_1} \varphi_j(w_j) \right| < +\infty$ ,  $j = 2, \dots, q$ . In view of (A.6) the lemma is thus proven.  $\square$

**Lemma A.2.** For any  $c \in \mathbb{R}$  and functions  $\varphi_j \in \mathcal{S}(\mathbb{R}), j = 1, \dots, q$ ,

$$\int_{-\infty}^{\infty} \frac{dw_1}{w_1 - c} \left( \varphi_1(w_1) \prod_{j=2}^q \int_{-\infty}^{\infty} \frac{dw_j}{w_j - w_1} \varphi_j(w_j) \right) \quad (\text{A.9})$$

is well defined.

**Proof.** By Definitions A.1 and A.2,

$$\varphi_1(w_1) \prod_{j=2}^q \int_{-\infty}^{\infty} \frac{dw_j}{w_j - w_1} \varphi_j(w_j) = \varphi_1(w_1) \prod_{j=2}^q \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_j - w_1|} \frac{dw_j}{w_j - w_1} \varphi_j(w_j),$$

which in view of Lemma A.1 is Lebesgue-integrable with respect to  $w_1$ . Hence (A.9) is well defined in accordance with (A.1).  $\square$

**Lemma A.3.** For any  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R})$ ,

$$\int_{\mathbb{R}} dw_1 \int_{\mathbb{R}} \frac{dw_2}{w_1 - w_2} \varphi_1(w_1) \varphi_2(w_2) = \int_{\mathbb{R}} dw_2 \int_{\mathbb{R}} \frac{dw_1}{w_1 - w_2} \varphi_1(w_1) \varphi_2(w_2). \quad (\text{A.10})$$

**Proof.** By Lemma A.1, both sides of (A.10) are well defined. By Definition A.1, dominated convergence, Fubini's theorem, and dominated convergence again,

$$\begin{aligned} & \int_{\mathbb{R}} dw_1 \int_{\mathbb{R}} \frac{dw_2}{w_1 - w_2} \varphi_1(w_1) \varphi_2(w_2) = \int_{\mathbb{R}} dw_1 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} dw_2 \frac{\varphi_1(w_1) \varphi_2(w_2) \mathbf{1}_{\epsilon < |w_1 - w_2|}}{w_1 - w_2} \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} dw_1 \int_{\mathbb{R}} \frac{dw_2 \varphi_1(w_1) \varphi_2(w_2) \mathbf{1}_{\epsilon < |w_1 - w_2|}}{w_1 - w_2} = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} dw_2 \int_{\mathbb{R}} \frac{dw_1 \varphi_1(w_1) \varphi_2(w_2) \mathbf{1}_{\epsilon < |w_1 - w_2|}}{w_1 - w_2} \\ &= \int_{\mathbb{R}} dw_2 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} \frac{dw_1}{w_1 - w_2} \varphi_1(w_1) \varphi_2(w_2) \mathbf{1}_{\epsilon < |w_1 - w_2|} = \int_{\mathbb{R}} dw_2 \int_{\mathbb{R}} \frac{dw_1}{w_1 - w_2} \varphi_1(w_1) \varphi_2(w_2). \quad \square \end{aligned}$$

## B Fourier Transform

**Definition B.1.** For  $f \in L^1(\mathbb{R}^q)$ , the Fourier and inverse Fourier transforms  $\mathcal{F}f$  and  $\mathcal{F}^{-1}f$  of  $f$  are the following functions on  $\mathbb{R}^q$  (King, 2009, Section 2.6 and 15.6):

$$\mathcal{F}f(\mathbf{z}) = \int_{\mathbb{R}^q} f(\mathbf{s}) e^{-i\mathbf{z} \cdot \mathbf{s}} d\mathbf{s}, \quad \mathcal{F}^{-1}f(\mathbf{s}) = \frac{1}{(2\pi)^q} \int_{\mathbb{R}^q} f(\mathbf{z}) e^{i\mathbf{s} \cdot \mathbf{z}} d\mathbf{z}. \quad (\text{B.1})$$

When  $f$  depends on several variables, we write e.g.  $\mathcal{F}_x[f(x, y)](t)$  to indicate that the transform is taken with respect to specific variables only.

*Example B.1.* For  $c \in \mathbb{R}$ , we compute

$$\begin{aligned} \mathcal{F}_k \left[ (|c| - |k|)^+ \right] (r) &= \int_{\mathbb{R}} dk e^{-irk} (|c| - |k|)^+ = \int_{-|c|}^{|c|} dk e^{-irk} (|c| - |k|) = \\ &= \int_0^{|c|} dk \left( e^{-irk} + e^{irk} \right) (|c| - k) = \frac{1}{r^2} (2 - 2 \cos(rc)) = \frac{1}{r^2} (2 - e^{irc} - e^{-irc}). \end{aligned}$$

Since  $\int_{\mathbb{R}} \frac{dr}{r^2} (2 - 2 \cos(rc)) = 2\pi|c|$ , we have

$$\mathcal{F}_k \left[ (|c| - |k|)^+ \right] (r) = \frac{2 - 2 \cos(rc)}{r^2} \in L^1(\mathbb{R}). \quad (\text{B.2})$$

*Remark B.2.* The Fourier transform is injective on  $L^1(\mathbb{R}^q)$  (Knapp, 2007, Corollary 8.5). If  $f \in L^1(\mathbb{R}^q)$ , then  $g = \mathcal{F}f$  is bounded, uniformly continuous and vanishes at infinity (Knapp, 2007, Proposition 8.1 and Theorem 8.3). If, in addition,  $g \in L^1(\mathbb{R}^q)$ , then  $\mathcal{F}^{-1}g = f$  (Knapp, 2007, Theorem 8.4).

*Remark B.3.*  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  are continuous automorphisms of  $\mathcal{S}(\mathbb{R}^q)$  (Kanwal, 2004, Theorem 2 page 143), hence  $\varphi = \mathcal{F}\mathcal{F}^{-1}\varphi = \mathcal{F}^{-1}\mathcal{F}\varphi$ , for  $\varphi \in \mathcal{S}(\mathbb{R}^q)$ .

**Definition B.2.** For a distribution  $T$  on  $\mathcal{S}(\mathbb{R}^q)$ , its Fourier and inverse Fourier transforms  $\mathcal{F}T$  and  $\mathcal{F}^{-1}T$  are the distributions on  $\mathcal{S}(\mathbb{R}^q)$  given as (Kanwal, 2004, Theorem 3 page 147), (King, 2009, Section 10.4 and 10.10)

$$\langle (\mathcal{F}T)_{\mathbf{ds}}, \varphi(\mathbf{s}) \rangle_{\mathbf{s}} = \langle T_{\mathbf{dz}}, \mathcal{F}\varphi(\mathbf{z}) \rangle_{\mathbf{z}}, \quad \langle (\mathcal{F}^{-1}T)_{\mathbf{dz}}, \varphi(\mathbf{z}) \rangle_{\mathbf{z}} = \langle T_{\mathbf{ds}}, \mathcal{F}^{-1}\varphi(\mathbf{s}) \rangle_{\mathbf{s}}, \quad \varphi \in \mathcal{S}(\mathbb{R}^q). \quad (\text{B.3})$$

*Example B.4.* Below are two well-known examples of distributional Fourier transforms (King, 2009, page 489), (Kammler, 2008, page 415):

$$\mathcal{F}_x[\text{sgn } x \, dx]_{d\lambda} = \frac{2}{i} \frac{d\lambda}{\lambda}, \quad \mathcal{F}_x[\cos x \, dx](d\lambda) = \delta_1(d\lambda) + \delta_{-1}(d\lambda),$$

i.e., for  $\varphi \in \mathcal{S}(\mathbb{R})$ ,

$$\langle \mathcal{F}_x[\text{sgn } x \, dx]_{d\lambda}, \varphi(\lambda) \rangle_{\lambda} = \int_{\mathbb{R}} dx \, \text{sgn } x \int_{\mathbb{R}} d\lambda e^{-i\lambda x} \varphi(\lambda) = \frac{2}{i} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \varphi(\lambda), \quad (\text{B.4})$$

$$\langle \mathcal{F}_x[\cos x \, dx](d\lambda), \varphi(\lambda) \rangle_{\lambda} = \int_{\mathbb{R}} dx \, \cos x \int_{\mathbb{R}} d\lambda e^{-i\lambda x} \varphi(\lambda) = \pi\varphi(1) + \pi\varphi(-1). \quad (\text{B.5})$$

## C Proof of (4.13)

**Lemma C.1.** *The measure  $dx\delta_{\|\mathbf{x}\|_1}(dk) + dx\delta_{-\|\mathbf{x}\|_1}(dk)$  is equal to*

$$dk(\mathbf{1}_{|x_d| < |k|} dx_d)(\mathbf{1}_{|x_{d-1}| < |k| - |x_d|} dx_{d-1}) \cdots (\mathbf{1}_{|x_2| < |k| - \|\mathbf{x}_{>2}\|_1} dx_2) \quad (\text{C.1}) \\ (\delta_{|k| - \|\mathbf{x}_{>1}\|_1}(dx_1) + \delta_{-|k| + \|\mathbf{x}_{>1}\|_1}(dx_1)) =: \delta_D(d(\mathbf{x}, k)).$$

**Proof.** Let  $f : \mathbb{R}^{d+1} \rightarrow \mathbb{C}$  be any  $\delta_D$ -integrable function. By sifting property of the Dirac mass distributions  $\delta_{\|\mathbf{x}\|_1}(dk)$  and  $\delta_{-\|\mathbf{x}\|_1}(dk)$ ,

$$\int_{\mathbb{R}^{d+1}} (dx\delta_{\|\mathbf{x}\|_1}(dk) + dx\delta_{-\|\mathbf{x}\|_1}(dk)) f(\mathbf{x}, k) = \int_{\mathbb{R}^d} dx f(\mathbf{x}, \|\mathbf{x}\|_1) + \int_{\mathbb{R}^d} dx f(\mathbf{x}, -\|\mathbf{x}\|_1) \quad (\text{C.2})$$

Splitting the first right-hand side integral above,

$$\int_{\mathbb{R}^d} dx f(\mathbf{x}, \|\mathbf{x}\|_1) = \int_{\mathbb{R}^{d-1}} dx_{>1} \int_0^{\infty} dx_1 f(\mathbf{x}, \|\mathbf{x}_{>1}\|_1 + x_1) \quad (\text{C.3}) \\ + \int_{\mathbb{R}^{d-1}} dx_{>1} \int_{-\infty}^0 dx_1 f(\mathbf{x}, \|\mathbf{x}_{>1}\|_1 - x_1).$$

By change of variables  $(x_1, \dots, x_d) \mapsto (k = \|\mathbf{x}_{>1}\|_1 + x_1, x_2, \dots, x_d)$  in the first right-hand side integral above, and change of variables  $(x_1, \dots, x_d) \mapsto (k = \|\mathbf{x}_{>1}\|_1 - x_1, x_2, \dots, x_d)$  in the second integral, where both Jacobians are 1, and adapting in-



tegral regions,

$$\begin{aligned} \int_{\mathbb{R}^d} d\mathbf{x} f(\mathbf{x}, \|\mathbf{x}\|_1) &= \int_0^\infty dk \int_{-k}^k dx_d \cdots \int_{-k+\|\mathbf{x}_{>2}\|_1}^{k-\|\mathbf{x}_{>2}\|_1} dx_2 f(k - \|\mathbf{x}_{>1}\|_1, \mathbf{x}_{>1}, k) \\ &+ \int_0^\infty dk \int_{-k}^k dx_d \cdots \int_{-k+\|\mathbf{x}_{>2}\|_1}^{k-\|\mathbf{x}_{>2}\|_1} dx_2 f(\|\mathbf{x}_{>1}\|_1 - k, \mathbf{x}_{>1}, k). \end{aligned} \quad (\text{C.4})$$

Following similar steps, the second right-hand side integral in (C.2) may be rewritten as

$$\begin{aligned} \int_{\mathbb{R}^d} d\mathbf{x} f(\mathbf{x}, -\|\mathbf{x}\|_1) &= \int_{-\infty}^0 dk \int_k^{-k} dx_d \cdots \int_{k+\|\mathbf{x}_{>2}\|_1}^{-k-\|\mathbf{x}_{>2}\|_1} dx_2 f(-\|\mathbf{x}_{>1}\|_1 - k, \mathbf{x}_{>1}, k) \\ &+ \int_{-\infty}^0 dk \int_k^{-k} dx_d \cdots \int_{k+\|\mathbf{x}_{>2}\|_1}^{-k-\|\mathbf{x}_{>2}\|_1} dx_2 f(k + \|\mathbf{x}_{>1}\|_1, \mathbf{x}_{>1}, k). \end{aligned} \quad (\text{C.5})$$

Combining (C.4) and (C.5) and piecing  $dk$  integrals together,

$$\begin{aligned} &\int_{\mathbb{R}^{d+1}} (d\mathbf{x} \delta_{\|\mathbf{x}\|_1}(dk) + d\mathbf{x} \delta_{-\|\mathbf{x}\|_1}) f(\mathbf{x}, k) \\ &= \int_{-\infty}^\infty dk \int_{-|k|}^{|k|} dx_d \cdots \int_{-|k|+\|\mathbf{x}_{>2}\|_1}^{|k|-\|\mathbf{x}_{>2}\|_1} dx_2 f(|k| - \|\mathbf{x}_{>1}\|_1, \mathbf{x}_{>1}, k) \\ &\quad + \int_{-\infty}^\infty dk \int_{-|k|}^{|k|} dx_d \cdots \int_{-|k|+\|\mathbf{x}_{>2}\|_1}^{|k|-\|\mathbf{x}_{>2}\|_1} dx_2 f(\|\mathbf{x}_{>1}\|_1 - |k|, \mathbf{x}_{>1}, k) \\ &= \int_{\mathbb{R}} dk \int_{-|k|}^{|k|} dx_d \cdots \int_{-|k|+\|\mathbf{x}_{>2}\|_1}^{|k|-\|\mathbf{x}_{>2}\|_1} dx_2 \int_{\mathbb{R}} dx_1 \delta_{|k|-\|\mathbf{x}_{>1}\|_1} (dx_1) f(x_1, \mathbf{x}_{>1}, k) \\ &\quad + \int_{\mathbb{R}} dk \int_{-|k|}^{|k|} dx_d \cdots \int_{-|k|+\|\mathbf{x}_{>2}\|_1}^{|k|-\|\mathbf{x}_{>2}\|_1} dx_2 \int_{\mathbb{R}} dx_1 \delta_{-|k|+\|\mathbf{x}_{>1}\|_1} (dx_1) f(x_1, \mathbf{x}_{>1}, k) \\ &= \int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) f(\mathbf{x}, k), \end{aligned}$$

where we used the sifting property again in the second equality.  $\square$

By sifting property of the Dirac mass distribution, the left-hand side in (4.13) satisfies for any  $\phi_e \in S_e$

$$\begin{aligned} &\int_{\mathbb{R}^d} d\mathbf{x} e^{-i\|\mathbf{x}\|_1} \mathcal{F}^{-1} \phi_e(\mathbf{x}) + \int_{\mathbb{R}^d} d\mathbf{x} e^{i\|\mathbf{x}\|_1} \mathcal{F}^{-1} \phi_e(\mathbf{x}) \\ &= \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} \delta_{\|\mathbf{x}\|_1}(dk) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) + \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}} \delta_{-\|\mathbf{x}\|_1}(dk) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) \\ &= \frac{1}{2} \int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}), \end{aligned}$$

where we used Lemma C.1 in the last step. Therefore, showing (4.13) reduces to proving that

$$\frac{1}{2} \int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) = \left\langle T_{d\mathbf{w}}^d, \phi_e(\mathbf{w}) \right\rangle_{\mathbf{w}}, \quad (\text{C.6})$$

where  $T^d$  is defined by (4.8). This is done in the remaining subsections by an induction procedure in line with the combinatorial structure of  $T^d$  highlighted in Remark 4.2.

### C.1 Dimension $d = 1$

By definition of  $\delta_D$  in (C.1),

$$\begin{aligned} \int_{\mathbb{R}^2} \delta_D(d(x, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(x) &= \int_{\mathbb{R}} dk e^{-ik} \int_{\mathbb{R}} (\delta_k(dx) + \delta_{-k}(dx)) \mathcal{F}^{-1} \phi_e(x) \\ &= \mathcal{F}_k[\mathcal{F}^{-1} \phi_e(k) + \mathcal{F}^{-1} \phi_e(-k)](1) = \phi_e(1) + \phi_e(-1) \\ &= \langle \delta_1(dw) + \delta_{-1}(dw), \phi_e(w) \rangle_w = 2 \langle T_{dw}^1, \phi_e(w) \rangle_w, \end{aligned}$$

which is (C.6) for  $d = 1$ .

### C.2 Dimension $d = 2$

Lemma C.2 below is necessary to initialize the proof by induction of Lemma C.3 in general dimension  $d \geq 2$ . We also include the proof of (C.6) for  $d = 2$  in this section for ease of reading.

**Lemma C.2.** For  $d = 2$  and  $\varphi \in \mathcal{S}(\mathbb{R}^2)$ , let

$$\Psi_{\pm}(k) := \langle (\mathbf{1}_{|x_2| < |k|} dx_2) \delta_{\pm|k| \mp |x_2|}(dx_1), \mathcal{F}^{-1} \varphi(\mathbf{x}) \rangle_{x_1, x_2}.$$

If  $\varphi = \phi_e \in S_e$ , then

$$\begin{aligned} \Psi_+(k) + \Psi_-(k) &= \frac{1}{2\pi^2} \int_{\mathbb{R}} dw_1 \sin(w_1|k|) \int_{\mathbb{R}} \left( \frac{dw_2}{w_2 + w_1} - \frac{dw_2}{w_2 - w_1} \right) \phi_e(w_1, w_2) \\ &\quad + \frac{1}{2\pi^2} \int_{\mathbb{R}} dw_2 \sin(w_2|k|) \int_{\mathbb{R}} \left( \frac{dw_1}{w_1 + w_2} - \frac{dw_1}{w_1 - w_2} \right) \phi_e(w_1, w_2). \end{aligned} \quad (\text{C.7})$$

**Proof.** For  $\varphi \in \mathcal{S}(\mathbb{R}^2)$ , we have

$$\begin{aligned} \Psi_+(k) &= \left\langle (\mathbf{1}_{|x_2| < |k|} dx_2), \left\langle \delta_{|k| - |x_2|}(dx_1), \mathcal{F}^{-1} \varphi(\mathbf{x}) \right\rangle_{x_1} \right\rangle_{x_2} \\ &= \int_{\mathbb{R}} dx_2 \mathbf{1}_{|x_2| < |k|} \mathcal{F}^{-1} \varphi(|k| - |x_2|, x_2) \\ &= \frac{1}{(2\pi)^2} \int_{|x_2| < |k|} dx_2 \int_{\mathbb{R}^2} dw_2 dw_1 e^{iw_1(|k| - |x_2|) + iw_2 x_2} \varphi(w_1, w_2). \end{aligned}$$

Splitting the  $dx_2$  integral at the origin, then by dominated convergence

$$\begin{aligned} \Psi_+(k) &= \frac{1}{(2\pi)^2} \int_0^{|k|} dx_2 \int_{\mathbb{R}^2} dw_2 dw_1 e^{ix_2(w_2 - w_1)} e^{iw_1|k|} \varphi(w_1, w_2) \\ &\quad + \frac{1}{(2\pi)^2} \int_{-|k|}^0 dx_2 \int_{\mathbb{R}^2} dw_2 dw_1 e^{ix_2(w_2 + w_1)} e^{iw_1|k|} \varphi(w_1, w_2) \\ &= \frac{1}{(2\pi)^2} \int_0^{|k|} dx_2 \left( \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 - w_1| \leq \epsilon\}} dw_2 dw_1 e^{ix_2(w_2 - w_1)} e^{iw_1|k|} \varphi(w_1, w_2) \right) \\ &\quad + \frac{1}{(2\pi)^2} \int_{-|k|}^0 dx_2 \left( \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 + w_1| \leq \epsilon\}} dw_2 dw_1 e^{ix_2(w_2 + w_1)} e^{iw_1|k|} \varphi(w_1, w_2) \right). \end{aligned} \quad (\text{C.8})$$

Since  $\varphi \in \mathcal{S}(\mathbb{R}^2)$ , for any  $\epsilon \geq 0$ , there exists  $B > 0$  such that

$$\begin{aligned} & \left| \int_{\mathbb{R}^2 \setminus \{|w_2 - w_1| \leq \epsilon\}} dw_2 dw_1 \varphi(w_1, w_2) e^{ix_2(w_2 \pm w_1)} e^{iw_1|k|} \right| \\ & \leq \int_{\mathbb{R}^2 \setminus \{|w_2 - w_1| \leq \epsilon\}} dw_2 dw_1 \left| \varphi(w_1, w_2) e^{ix_2(w_2 \pm w_1)} e^{iw_1|k|} \right| \\ & \leq \int_{\mathbb{R}^2 \setminus \{|w_2 - w_1| \leq \epsilon\}} dw_2 dw_1 |\varphi(w_1, w_2)| < B < \infty, \end{aligned}$$

and the integral  $\int_0^{\pm|k|} B dx_2$  is finite for any  $|k| < \infty$ . Hence, by dominated convergence and Fubini's theorem,

$$\begin{aligned} \Psi_+(k) &= \frac{1}{(2\pi)^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 - w_1| \leq \epsilon\}} dw_2 dw_1 \varphi(w_1, w_2) \int_0^{|k|} dx_2 e^{ix_2(w_2 - w_1)} e^{iw_1|k|} \\ & \quad + \frac{1}{(2\pi)^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 + w_1| \leq \epsilon\}} dw_2 dw_1 \varphi(w_1, w_2) \int_{-|k|}^0 dx_2 e^{ix_2(w_2 + w_1)} e^{iw_1|k|}. \end{aligned}$$

Solving integrals with respect to  $x_2$ ,

$$\begin{aligned} \Psi_+(k) &= \frac{1}{(2\pi)^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 - w_1| \leq \epsilon\}} dw_2 dw_1 \varphi(w_1, w_2) \frac{-ie^{i|k|w_2} + ie^{iw_1|k|}}{w_2 - w_1} \\ & \quad + \frac{1}{(2\pi)^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 + w_1| \leq \epsilon\}} dw_2 dw_1 \varphi(w_1, w_2) \frac{ie^{-i|k|w_2} - ie^{iw_1|k|}}{w_2 + w_1}. \end{aligned}$$

Similar steps would show that

$$\begin{aligned} \Psi_-(k) &= \frac{1}{(2\pi)^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 - w_1| \leq \epsilon\}} dw_2 dw_1 \varphi(w_1, w_2) \frac{ie^{-i|k|w_2} - ie^{-iw_1|k|}}{w_2 - w_1} \\ & \quad + \frac{1}{(2\pi)^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^2 \setminus \{|w_2 + w_1| \leq \epsilon\}} dw_2 dw_1 \varphi(w_1, w_2) \frac{-ie^{i|k|w_2} + ie^{-iw_1|k|}}{w_2 + w_1}. \end{aligned}$$

Plugging Euler's sine formula into the above expressions and combining,

$$\begin{aligned} & \Psi_+(k) + \Psi_-(k) \\ &= \frac{1}{2\pi^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} dw_2 \sin(w_2) \left( \int_{\epsilon < |w_1 + w_2|} \frac{dw_1}{w_1 + w_2} \varphi(w_1, w_2) - \int_{\epsilon < |w_1 - w_2|} \frac{dw_1}{w_1 - w_2} \varphi(w_1, w_2) \right) \\ & \quad + \frac{1}{2\pi^2} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} dw_1 \sin(w_1) \left( \int_{\epsilon < |w_2 + w_1|} \frac{dw_2}{w_2 + w_1} \varphi(w_1, w_2) - \int_{\epsilon < |w_2 - w_1|} \frac{dw_2}{w_2 - w_1} \varphi(w_1, w_2) \right). \end{aligned} \tag{C.9}$$

For  $\varphi(w_1, w_2) = \varphi_1(w_2)\varphi_2(w_1)$  with  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R})$ , in view of Lemma A.1, we have by dominated convergence

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}} dw_2 \sin(w_2) \int_{\epsilon < |w_1 + w_2|} \frac{dw_1}{w_1 + w_2} \varphi(w_1, w_2) \\ &= \int_{\mathbb{R}} dw_2 \sin(w_2) \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_1 + w_2|} \frac{dw_1}{w_1 + w_2} \varphi(w_1, w_2) = \int_{\mathbb{R}} dw_2 \sin(w_2) \int_{\mathbb{R}} \frac{dw_1}{w_1 + w_2} \varphi(w_1, w_2). \end{aligned}$$

Following similar steps for the remaining terms in (C.9), we recover (C.7) for  $\varphi(w_1, w_2) = \varphi_1(w_2)\varphi_2(w_1)$  with  $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R})$ , from which the result for  $\varphi = \phi_e \in S_e$  follows by

linearity.  $\square$

We are now in a position to prove (C.6) for  $d = 2$ . By definition of  $\delta_D$  in (C.1),

$$\delta_D(d(\mathbf{x}, k)) = dk(\mathbf{1}_{|x_2| < |k|} dx_2)(\delta_{|k| - |x_2|} + \delta_{-|k| + |x_2|})(dx_1).$$

Therefore, by Lemma C.2,

$$\begin{aligned} \int_{\mathbb{R}^3} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) &= \int_{\mathbb{R}} dk e^{-ik} (\Psi_+(k) + \Psi_-(k)) = \\ &= \frac{1}{2\pi^2} \int_{\mathbb{R}} dk e^{-ik} \int_{\mathbb{R}} dw_1 \sin(w_1 |k|) \int_{\mathbb{R}} \left( \frac{dw_2}{w_2 + w_1} - \frac{dw_2}{w_2 - w_1} \right) \phi_e(w_1, w_2) \\ &+ \frac{1}{2\pi^2} \int_{\mathbb{R}} dk e^{-ik} \int_{\mathbb{R}} dw_2 \sin(w_2 |k|) \int_{\mathbb{R}} \left( \frac{dw_1}{w_1 + w_2} - \frac{dw_1}{w_1 - w_2} \right) \phi_e(w_1, w_2). \end{aligned} \quad (\text{C.10})$$

Let  $f(w_1) := \int_{\mathbb{R}} \left( \frac{dw_2}{w_2 + w_1} - \frac{dw_2}{w_2 - w_1} \right) \phi_e(w_1, w_2)$ . Substituting Euler's sine formula, then splitting the integrand while applying the change of variable  $k \mapsto -k$  to the second resulting integral, the first term in (C.10) may be rewritten as

$$\begin{aligned} & \frac{i}{4\pi^2} \int_{\mathbb{R}} dk e^{-ik} \operatorname{sgn}(k) \int_{\mathbb{R}} dw_1 (e^{-iw_1 k} - e^{iw_1 k}) f(w_1) \\ &= \frac{i}{4\pi^2} \left( \int_{-\infty}^{\infty} dk \operatorname{sgn}(k) \int_{\mathbb{R}} dw_1 e^{-ik(w_1+1)} f(w_1) + \int_{-\infty}^{\infty} dk \operatorname{sgn}(k) \int_{\mathbb{R}} dw_1 e^{-ik(w_1-1)} f(w_1) \right) \\ &= \frac{1}{2\pi^2} \int_{\mathbb{R}} \left( \frac{dw_1}{w_1 + 1} + \frac{dw_1}{w_1 - 1} \right) f(w_1), \end{aligned}$$

where we substituted (B.4) in the last step. In addition,

$$\begin{aligned} & \frac{1}{2\pi^2} \int_{\mathbb{R}} \left( \frac{dw_1}{w_1 + 1} + \frac{dw_1}{w_1 - 1} \right) f(w_1) = \\ & \frac{1}{2\pi^2} \int_{\mathbb{R}} \left( \frac{dw_1}{w_1 + 1} + \frac{dw_1}{w_1 - 1} \right) \int_{\mathbb{R}} \left( \frac{dw_2}{w_2 + w_1} - \frac{dw_2}{w_2 - w_1} \right) \phi_e(w_1, w_2). \end{aligned} \quad (\text{C.11})$$

Following similar steps for the remaining terms to the right-hand side of (C.10), we may conclude in view of the definition (4.7) of  $T_2$  that

$$\frac{1}{2} \int_{\mathbb{R}^3} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) = \langle T_{d\mathbf{w}}^2, \phi_e(\mathbf{w}) \rangle_{\mathbf{w}},$$

which is (C.6) for  $d = 2$ .

### C.3 General Dimension $d \geq 2$

For  $\phi_e \in S_e, k \in \mathbb{R}$ , let  $\Psi_d(k) \equiv \Psi_d(\mathbf{x}_{>d}, k)$  result from the following finite recurrence:

$$\begin{aligned} \Psi_1(\mathbf{x}_{>1}, k) &= \left\langle (\delta_{|k| - \|\mathbf{x}_{>1}\|_1} + \delta_{-|k| + \|\mathbf{x}_{>1}\|_1})(dx_1), \mathcal{F}^{-1} \phi_e(x_1, \mathbf{x}_{>1}) \right\rangle_{x_1}, \mathbf{x}_{>1} \in \mathbb{R}^{d-1}, \\ \Psi_l(\mathbf{x}_{>l}, k) &= \left\langle \mathbf{1}_{|x_l| < |k| - \|\mathbf{x}_{>l}\|_1} dx_l, \Psi_{l-1}(x_l, \mathbf{x}_{>l}, k) \right\rangle_{x_l}, \mathbf{x}_{>l} \in \mathbb{R}^{d-l}, l = 2, \dots, d. \end{aligned} \quad (\text{C.12})$$

**Lemma C.3.** For any  $\phi_e \in S_e, l = 2 \dots, d$ , and  $(\mathbf{x}, k) \in \mathbb{R}^{d+1}$  such that  $|\mathbf{x}_{>l}| \leq |k|$ ,

$$\begin{aligned} \Psi_l(\mathbf{x}_{>l}, k) &= \frac{2}{(2\pi)^l} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \operatorname{sc}_l(w_j(|k| - \|\mathbf{x}_{>l}\|_1)) \times \\ &\int_{\mathbb{R}^{l-1}} \left( \prod_{i \in 1..l \setminus \{j\}} \left( \frac{dw_i}{w_i + w_j} - \frac{dw_i}{w_i - w_j} \right) \right) \Phi_l(\mathbf{w}_{\leq l}, \mathbf{x}_{>l}), \end{aligned} \quad (\text{C.13})$$

where  $\Phi_l(\mathbf{w}_{\leq l}, \mathbf{x}_{>l}) := \mathcal{F}_{\mathbf{w}_{>l}}^{-1}[\phi_e(\mathbf{w})](w_1, \dots, w_l, \mathbf{x}_{>l}) \in S_e$ , and  $\operatorname{sc}_l(\cdot) = (-1)^{\frac{l-1}{2}} \cos(\cdot)$  if  $l$  is odd,  $(-1)^{\frac{l-2}{2}} \sin(\cdot)$  if  $l$  is even.

**Proof.** We proceed by induction on  $l \geq 2$ . Note that  $\Phi_2 \in S_e$ . For any fixed  $\mathbf{x} \in \mathbb{R}^d$ , an application of Lemma C.2 to  $(w_1, w_2) \xrightarrow{\phi_e} \Phi_2(w_1, w_2, \mathbf{x}_{>2})$  and to the strike  $\kappa := |k| - \|\mathbf{x}_{>2}\|_1 \geq 0$  yields

$$\begin{aligned} \Psi_2(\mathbf{x}_{>2}, k) &= \frac{1}{2\pi^2} \sum_{j=1}^2 \int_{\mathbb{R}} dw_j \sin(w_j \kappa) \times \\ &\int_{\mathbb{R}} \left( \prod_{i \in 1..2 \setminus \{j\}} \left( \frac{dw_i}{w_i + w_j} - \frac{dw_i}{w_i - w_j} \right) \right) \Phi_2(w_1, w_2, \mathbf{x}_{>2}), \end{aligned} \quad (\text{C.14})$$

which is (C.13) for  $l = 2$ . It remains to prove that if (C.13) is satisfied for some index  $l = 2, \dots, d-1$ , then it also holds at index  $l+1$ , i.e.

$$\begin{aligned} \Psi_{l+1}(\mathbf{x}_{>l+1}, k) &= \frac{2}{(2\pi)^{l+1}} \sum_{j=1}^{l+1} \int_{\mathbb{R}} dw_j \operatorname{sc}_{l+1}(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1)) \times \\ &\int_{\mathbb{R}^l} \left( \prod_{j \in 1..(l+1) \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}). \end{aligned} \quad (\text{C.15})$$

Assuming (C.13) for some  $l \geq 2$ , let us thus prove (C.15), first in the case where  $l$  is even. Then  $\operatorname{sc}_l(\cdot) = (-1)^{\frac{l-2}{2}} \sin(\cdot)$  and

$$\begin{aligned} \Psi_{l+1}(\mathbf{x}_{>l+1}, k) &= \left\langle \mathbf{1}_{|x_{l+1}| < |k| - \|\mathbf{x}_{>l+1}\|_1} dx_{l+1}, \Psi_l(\mathbf{x}_{>l}, k) \right\rangle_{x_{l+1}} = \frac{(-1)^{\frac{l-2}{2}} 2}{(2\pi)^l} \times \\ &\left[ \int_0^{|k| - \|\mathbf{x}_{>l+1}\|_1} dx_{l+1} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \sin(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1 - x_{l+1})) \times \right. \\ &\int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \Phi_l(\mathbf{w}_{\leq l}, \mathbf{x}_{>l}) \\ &+ \int_{-|k| + \|\mathbf{x}_{>l+1}\|_1}^0 dx_{l+1} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \sin(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1 + x_{l+1})) \times \\ &\left. \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \Phi_l(\mathbf{w}_{\leq l}, \mathbf{x}_{>l}) \right]. \end{aligned} \quad (\text{C.16})$$

Substituting Euler's sine formula into the above together with

$$\begin{aligned}\Phi_l(\mathbf{w}_{\leq l}, \mathbf{x}_{> l}) &= \frac{1}{2\pi} \int_{\mathbb{R}} dw_{l+1} e^{-ix_{l+1}w_{l+1}} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{> l+1}) \\ &= \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} \pm w_j|} dw_{l+1} e^{-ix_{l+1}w_{l+1}} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{> l+1}),\end{aligned}$$

which stems from slicing the Fourier transform  $\Phi_l(\mathbf{w}_{\leq l}, \mathbf{x}_{> l})$  along  $x_{l+1}$ , then using dominated convergence, the first term inside the square bracket in (C.16) may be rewritten as

$$\begin{aligned}& \frac{i}{4\pi} \int_0^{|k| - \|\mathbf{x}_{> l+1}\|_1} dx_{l+1} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \times \\ & \quad \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} - w_j|} dw_{l+1} e^{-ix_{l+1}w_{l+1}} e^{-iw_j(|k| - \|\mathbf{x}_{> l+1}\|_1 - x_{l+1})} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{> l+1}) \\ & - \frac{i}{4\pi} \int_0^{|k| - \|\mathbf{x}_{> l+1}\|_1} dx_{l+1} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \times \\ & \quad \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} + w_j|} dw_{l+1} e^{-ix_{l+1}w_{l+1}} e^{iw_j(|k| - \|\mathbf{x}_{> l+1}\|_1 - x_{l+1})} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{> l+1}).\end{aligned}\tag{C.17}$$

By a combination of Definition A.1, dominated convergence and Fubini's theorem similarly to the proof of Lemma C.2, we may bring the outer  $dx_{l+1}$ -integrals above inside all others. As such, the first term in (C.17) is equal to

$$\begin{aligned}& \frac{i}{4\pi} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \left( \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} + w_j|} dw_{l+1} \times \right. \\ & \quad \left. \int_0^{|k| - \|\mathbf{x}_{> l+1}\|_1} dx_{l+1} e^{-ix_{l+1}(w_{l+1} - w_j) - iw_j(|k| - \|\mathbf{x}_{> l+1}\|_1)} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{> l+1}) \right) \\ & = \frac{1}{4\pi} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \left( \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} - w_j|} dw_{l+1} \times \right. \\ & \quad \left. \frac{e^{iw_{l+1}(|k| - \|\mathbf{x}_{> l+1}\|_1)} - e^{iw_j(|k| - \|\mathbf{x}_{> l+1}\|_1)}}{w_j - w_{l+1}} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{> l+1}) \right),\end{aligned}$$

where we solved the  $dx_{l+1}$  integral in the last step. Following the same steps for the second term in (C.17), and then the second term within square brackets in (C.16), we

obtain

$$\begin{aligned}
\Psi_{l+1}(\mathbf{x}_{>l+1}, k) &= \frac{(-1)^{\frac{l-2}{2}}}{(2\pi)^{l+1}} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \times \\
&\left( - \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} - w_j|} dw_{l+1} \frac{e^{iw_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)} - e^{iw_j(|k| - \|\mathbf{x}_{>l+1}\|_1)}}{w_{l+1} - w_j} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \right. \\
&+ \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} + w_j|} dw_{l+1} \frac{e^{iw_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)} - e^{-iw_j(|k| - \|\mathbf{x}_{>l+1}\|_1)}}{w_{l+1} + w_j} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \\
&- \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} - w_j|} dw_{l+1} \frac{e^{-iw_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)} - e^{-iw_j(|k| - \|\mathbf{x}_{>l+1}\|_1)}}{w_{l+1} - w_j} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \\
&\left. + \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon < |w_{l+1} + w_j|} dw_{l+1} \frac{e^{-iw_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)} - e^{iw_j(|k| - \|\mathbf{x}_{>l+1}\|_1)}}{w_{l+1} + w_j} \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \right).
\end{aligned}$$

Splitting and rearranging integrands, substituting Euler's cosine formula, and recognizing Cauchy principal integrals,

$$\begin{aligned}
\Psi_{l+1}(\mathbf{x}_{>l+1}, k) &= \frac{(-1)^{\frac{l-2}{2}}}{(2\pi)^{l+1}} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \times \\
&\left( \int_{\mathbb{R}} \frac{dw_{l+1}}{w_{l+1} + w_j} \cos(w_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \right. \\
&- \int_{\mathbb{R}} \frac{dw_{l+1}}{w_{l+1} - w_j} \cos(w_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \\
&- \int_{\mathbb{R}} \frac{dw_{l+1}}{w_{l+1} + w_j} \cos(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1)) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \\
&\left. + \int_{\mathbb{R}} \frac{dw_{l+1}}{w_{l+1} - w_j} \cos(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1)) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \right).
\end{aligned}$$

By (A.4) for  $q = 2$ , this is equal to

$$\begin{aligned}
\Psi_{l+1}(\mathbf{x}_{>l+1}, k) &= \frac{(-1)^{\frac{l-2}{2}}}{(2\pi)^{l+1}} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \times \\
&\left( \int_{\mathbb{R}} \left( \frac{dw_{l+1}}{w_{l+1} + w_j} - \frac{dw_{l+1}}{w_{l+1} - w_j} \right) \cos(w_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \right. \\
&\left. - \int_{\mathbb{R}} \left( \frac{dw_{l+1}}{w_{l+1} + w_j} - \frac{dw_{l+1}}{w_{l+1} - w_j} \right) \cos(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1)) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \right). \tag{C.18}
\end{aligned}$$

By (A.5), the first term above corresponding to  $\cos(w_{l+1} \dots)$  may be rewritten as

$$\begin{aligned}
& \frac{(-1)^{\frac{l-2}{2}}}{(2\pi)^{l+1}} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \int_{\mathbb{R}} \left( \frac{dw_{l+1}}{w_{l+1} + w_j} - \frac{dw_{l+1}}{w_{l+1} - w_j} \right) \times \\
& \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \cos(w_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}) \\
& = \frac{(-1)^{\frac{l-2}{2}}}{(2\pi)^{l+1}} \sum_{j=1}^l \int_{\mathbb{R}} dw_{l+1} \cos(w_{l+1}(|k| - \|\mathbf{x}_{>l+1}\|_1)) \times \\
& \int_{\mathbb{R}} \left( \frac{dw_j}{w_j + w_{l+1}} + \frac{dw_j}{w_j - w_{l+1}} \right) \int_{\mathbb{R}^{l-1}} \left( \prod_{j \in 1..l \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}),
\end{aligned} \tag{C.19}$$

where the equality follows from Lemma A.3 applied to the variables  $w_{l+1}$  and  $w_j$ . By reflective change of variables  $(w_j, w_{l+1}) \mapsto (-w_j, -w_{l+1})$  to change minus signs into plus signs within Cauchy integrals, the second term in (C.18) corresponding to  $\cos(w_j \dots)$  may be rewritten as

$$\begin{aligned}
& \frac{(-1)^{\frac{l-2}{2}}}{(2\pi)^{l+1}} \sum_{j=1}^l \int_{\mathbb{R}} dw_j \cos(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1)) \times \\
& \int_{\mathbb{R}^l} \left( \prod_{j \in 1..(l+1) \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}).
\end{aligned} \tag{C.20}$$

Substituting (C.19) and (C.20) into (C.18) yields

$$\begin{aligned}
\Psi_{l+1}(\mathbf{x}_{>l+1}, k) &= \frac{(-1)^{\frac{l-2}{2}}}{(2\pi)^{l+1}} \sum_{j=1}^{l+1} \int_{\mathbb{R}} dw_j \cos(w_j(|k| - \|\mathbf{x}_{>l+1}\|_1)) \times \\
& \int_{\mathbb{R}^l} \left( \prod_{j \in 1..(l+1) \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \Phi_{l+1}(\mathbf{w}_{\leq l+1}, \mathbf{x}_{>l+1}),
\end{aligned}$$

thereby proving that (C.15) holds for  $l$  even. If  $l$  is odd, then  $\text{sc}_l(\cdot) = (-1)^{\frac{l-1}{2}} \cos(\cdot)$  and following similar steps would prove (C.15), thereby establishing the identity (C.13) for any  $l \in 2..d$ .  $\square$

We are now in position to prove (C.6) for any  $d \geq 2$ . By definition of  $\delta_D$  in (C.1),

$$\int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) = \int_{\mathbb{R}} dk e^{-ik} \Psi_d(k), \tag{C.21}$$

where  $\Psi_d(k) \equiv \Psi_d(\mathbf{x}_{>d}, k)$  as defined by the recurrence (C.12). By Lemma C.3,

$$\Psi_d(k) = \frac{2}{(2\pi)^d} \sum_{j=1}^d \int_{\mathbb{R}} dw_j \text{sc}_d(w_j |k|) \int_{\mathbb{R}^{d-1}} \left( \prod_{j \in 1..d \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \phi_e(\mathbf{w}). \tag{C.22}$$



Substituting (C.22) into (C.21),

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) &= \frac{2}{(2\pi)^d} \sum_{j=1}^d \int_{\mathbb{R}} dk e^{-ik} \times \\ &\int_{\mathbb{R}} dw_j \operatorname{sc}_d(w_j |k|) \int_{\mathbb{R}^{d-1}} \left( \prod_{j \in 1..d \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \phi_e(\mathbf{w}). \end{aligned} \quad (\text{C.23})$$

In odd dimension  $d \geq 3$ , substituting (B.5) yields (C.6) as required. In even dimension, replacing  $\operatorname{sc}_d(\cdot) = (-1)^{\frac{d-2}{2}} \sin(\cdot)$  into (C.23) yields

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) &= \frac{(-1)^{\frac{d-2}{2}} 2}{(2\pi)^d} \sum_{j=1}^d \int_{\mathbb{R}} dk e^{-ik} \int_{\mathbb{R}} dw_j \sin(w_j |k|) \times \\ &\underbrace{\int_{\mathbb{R}^{d-1}} \left( \prod_{j \in 1..d \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \phi_e(\mathbf{w})}_{=: f_j(w_j)}, \end{aligned}$$

by which we recognize the same integrals as in (C.10). Substituting Euler's sine formula, then splitting the integrand while applying the change of variable  $k \mapsto -k$  to the second resulting integral,

$$\begin{aligned} &\int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) \\ &= \frac{i(-1)^{\frac{d-2}{2}}}{(2\pi)^d} \sum_{j=1}^d \int_{\mathbb{R}} dk e^{-ik} \operatorname{sgn}(k) \int_{\mathbb{R}} dw_j (e^{-iw_j k} - e^{iw_j k}) f_j(w_j) \\ &= \frac{i(-1)^{\frac{d-2}{2}}}{(2\pi)^d} \sum_{j=1}^d \left( \int_{-\infty}^{\infty} dk \operatorname{sgn}(k) \int_{\mathbb{R}} dw_j e^{-ik(w_j+1)} f_j(w_j) + \right. \\ &\quad \left. \int_{-\infty}^{\infty} dk \operatorname{sgn}(k) \int_{\mathbb{R}} dw_j e^{-ik(w_j-1)} f_j(w_j) \right) \\ &= \frac{(-1)^{\frac{d-2}{2}} 2}{(2\pi)^d} \sum_{j=1}^d \int_{\mathbb{R}} \left( \frac{dw_j}{w_j + 1} + \frac{dw_j}{w_j - 1} \right) f_j(w_j), \end{aligned}$$

where we substituted (B.4) in the last step. Substituting  $f_j(w_j)$  into the above equation, we recover

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} \delta_D(d(\mathbf{x}, k)) e^{-ik} \mathcal{F}^{-1} \phi_e(\mathbf{x}) &= \frac{(-1)^{\frac{d-2}{2}} 2}{(2\pi)^d} \times \\ &\sum_{j=1}^d \int_{\mathbb{R}} \left( \frac{dw_j}{w_j - 1} + \frac{dw_j}{w_j + 1} \right) \int_{\mathbb{R}^{d-1}} \left( \prod_{j \in 1..d \setminus \{j\}} \left( \frac{dw_j}{w_j + w_j} - \frac{dw_j}{w_j - w_j} \right) \right) \phi_e(\mathbf{w}), \end{aligned}$$

which is (C.6).

## D Correspondence Between Carr-Madan Spanning and Basket Call Spanning in Dimension $d = 1$

**Proposition D.1.** *In dimension  $d = 1$  and for  $k > 0$ , the Carr-Madan spanning formula (2.1) of a twice differentiable payoff  $F(x) \equiv F(x, k)$  such that  $F(\lambda x, \lambda k) = \lambda F(x, k)$  for all  $\lambda > 0$ , and  $\partial_x F(0, k) = \partial_{x^2}^2 F(0, k) = 0$ , can be rewritten in the form (2.2) for  $x \geq 0$ , with*

$$\nu(dw) = \frac{1}{w^3} (\partial_{x^2}^2 F) \left( \frac{1}{w}, 1 \right) dw. \quad (\text{D.1})$$

**Proof.** By change of variable  $K \mapsto w = k/K$  in (2.1),

$$\begin{aligned} F(x) &= \int_0^\infty \frac{k}{w^2} \left( x - \frac{k}{w} \right)^+ (\partial_{x^2}^2 F) \left( \frac{k}{w}, k \right) dw = \int_0^\infty \frac{k}{w^3} (wx - k)^+ (\partial_{x^2}^2 F) \left( \frac{k}{w}, k \right) dw \\ &= \int_{-\infty}^\infty \frac{k}{w^3} (wx - k)^+ (\partial_{x^2}^2 F) \left( \frac{k}{w}, k \right) dw, \quad x \in \mathbb{R}_+, \end{aligned} \quad (\text{D.2})$$

where we used in the last step that  $(wx - k)^+ = 0$  for  $w < 0 < \frac{k}{x}$ . Differentiating  $F(\lambda x, \lambda k) = \lambda F(x, k)$  twice with respect to  $x$  yields  $\partial_{x^2}^2 F(\lambda x, \lambda k) = \lambda^{-1} \partial_{x^2}^2 F(x, k)$ , whence

$$\frac{k}{w^3} (\partial_{x^2}^2 F) \left( \frac{k}{w}, k \right) = \frac{1}{w^3} (\partial_{x^2}^2 F) \left( \frac{1}{w}, 1 \right), \quad k > 0. \quad (\text{D.3})$$

Substituting the above into (D.2), we recover (2.2) for  $x \geq 0$  and  $\nu(dw)$  given by (D.1).  $\square$

*Example D.1.* Consider the one-dimensional smooth payoff  $F(x, k) = G_1(x, k) = \mathbf{1}_{x \neq 0} \sqrt{x^2} e^{-\frac{k^2}{x^2}}$ ,  $k > 0, x \in \mathbb{R}_+$  (see Propositions 4.1-4.2). We have

$$\partial_x G_1(x, k) = \mathbf{1}_{x \neq 0} \frac{e^{-\frac{k^2}{x^2}} (2k^2 + x^2)}{x\sqrt{x^2}}, \quad \partial_{x^2}^2 G_1(x, k) = \mathbf{1}_{x \neq 0} \frac{2k^2 e^{-\frac{k^2}{x^2}} (2k^2 - x^2)}{x^4 \sqrt{x^2}},$$

hence  $G_1(0, k) = \partial_x G_1(0, k) = 0$ . The Carr-Madan spanning formula for  $G_1$  then reads

$$G_1(x, k) = \int_0^\infty (x - K)^+ \frac{2k^2 e^{-\frac{k^2}{K^2}} (2k^2 - K^2)}{K^4 \sqrt{K^2}} dK.$$

Equivalently, Proposition D.1 yields

$$G_1(x, k) = \int_{-\infty}^\infty (wx - k)^+ 2e^{-w^2} (2w^2 - 1) dw, \quad k > 0, x \in \mathbb{R}_+,$$

which is a representation of the form (2.2) (with  $(wx - k)^+ = 0$  for  $w < 0$ ).

## References

Alexander, C. and A. Venkatramanan (2012). Analytic approximations for multi-asset option pricing. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics* 22(4), 667–689.

- Antonov, A. and V. Piterbarg (2022). Alternatives to deep neural networks in finance. *Risk Magazine October*.
- Bossu, S. (2014). *Advanced Equity Derivatives*. John Wiley & Sons.
- Bossu, S. (2022). Static replication of european multi-asset options with homogeneous payoff. *Applied Mathematical Finance* 8(25), 381–394.
- Bossu, S., P. Carr, and A. Papanicolaou (2022). Static replication of European standard dispersion options. *Quantitative Finance* 22(5), 799–811.
- Carr, P. and P. Laurence (2011). Multi-asset stochastic local variance contracts. *Mathematical Finance* 21(1), 21–52.
- Carr, P. and D. Madan (1998). Towards a theory of volatility trading. In R. A. Jarrow (Ed.), *Volatility: New Estimation Techniques for Pricing Derivatives*, Volume 29, pp. 417–427. Risk books.
- Chiu, H. and R. Cont (2023). A model-free approach to continuous-time finance. *Mathematical Finance* 33(2), 257–273.
- Choromanska, A., M. Henaff, M. Mathieu, G. B. Arous, and Y. LeCun (2015). The loss surfaces of multilayer networks. In *Artificial intelligence and statistics*, pp. 192–204. PMLR.
- Cui, Z. and Y. Xu (2022). A new representation of the risk-neutral distribution and its applications. *Quantitative Finance* 22(5), 817–834.
- Estrada, R. and R. P. Kanwal (2000). *Singular Integral Equations*. Springer Science & Business Media.
- Friedlander, F. G. (1998). *Introduction to the Theory of Distributions*. Cambridge University Press.
- Golub, G. H. and C. F. Van Loan (2013). *Matrix Computations*. JHU press.
- Huang, G.-B., Q.-Y. Zhu, and C.-K. Siew (2004). Extreme learning machine: a new learning scheme of feedforward neural networks. In *2004 IEEE International Joint Conference on Neural Networks (IEEE Cat. No. 04CH37541)*, Volume 2, pp. 985–990.
- Huang, G.-B., Q.-Y. Zhu, and C.-K. Siew (2006). Extreme learning machine: theory and applications. *Neurocomputing* 70(1-3), 489–501.
- Iosevich, A. and E. Liflyand (2014). *Decay of the Fourier Transform*. Birkhäuser.
- Kammler, D. W. (2008). *A First Course in Fourier Analysis* (2nd ed.). Cambridge University Press.
- Kanwal, R. P. (2004). *Generalized Functions: Theory and Applications* (3rd ed.). Birkhäuser.

- King, F. W. (2009). *Hilbert Transforms*. Encyclopedia of Mathematics and its Applications. Cambridge University Press.
- Kingma, D. P. and J. Ba (2015). Adam: A method for stochastic optimization. In Y. Bengio and Y. LeCun (Eds.), *3rd International Conference on Learning Representations, San Diego, CA, USA, May 7-9, 2015, Conference Track Proceedings*.
- Knapp, A. W. (2007). *Basic Real Analysis*. Springer Science & Business Media.
- Lokeshwar, V., V. Bhardawaj, and S. Jain (2022). Explainable neural network for pricing and universal static hedging of contingent claims. *Applied Mathematics and Computation* 417, Article No. 126775.
- Lyons, T., S. Nejad, and I. P. Arribas (2020). Non-parametric pricing and hedging of exotic derivatives. *Applied Mathematical Finance* 27(6), 457–494.
- Milton, E. (1974). Fourier transforms of odd and even tempered distributions. *Pacific Journal of Mathematics* 50(2), 563–572.
- Molchanov, I. and M. Schmutz (2014). Multiasset derivatives and joint distributions of asset prices. In *Inspired by Finance: The Musiela Festschrift*, pp. 439–459. Springer.
- Ramm, A. G. and A. I. Katsevich (1996). *The Radon Transform and Local Tomography*. CRC press.
- Reed, M. and B. Simon (1980). *Functional Analysis*, Volume 1. Gulf Professional Publishing.
- Rubin, B. (2015). *Introduction to Radon Transforms*, Volume 160. Cambridge University Press.
- Savarese, P., I. Evron, D. Soudry, and N. Srebro (2019). How do infinite width bounded norm networks look in function space? In *Conference on Learning Theory*, pp. 2667–2690. PMLR.
- Schofield, N. C. (2017). *Equity Derivatives Corporate and Institutional Applications*. Palgrave Macmillan UK.
- İlhan, A., M. Jonsson, and R. Sircar (2009). Optimal static-dynamic hedges for exotic options under convex risk measures. *Stochastic Processes and their Applications* 119(10), 3608–3632.