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# LAYER POTENTIAL OPERATORS FOR TRANSMISSION PROBLEMS ON EXTENSION DOMAINS

GABRIEL CLARET, MICHAEL HINZ, ANNA ROZANOVA-PIERRAT,  
AND ALEXANDER TEPLYAEV

**ABSTRACT.** We use the well-posedness of transmission problems on classes of two-sided Sobolev extension domains to give variational definitions for (boundary) layer potential operators and Neumann-Poincaré operators. These classes of domains contain Lipschitz domains, and also domains with fractal boundaries. Although our variational formulation does not involve any measures on the boundary, we recover the classical results in smooth domains by considering the surface measure on the boundary. We discuss properties of these operators and generalize basic results in imaging beyond the Lipschitz case.

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## 1. INTRODUCTION

The aim of our work is to introduce a purely variational approach in order to extend the classical theory of layer potential operators, Neumann-Poincaré operators, boundary layer operators and transmission problems to a class of rough (two-sided) extension domains, with possibly non-Lipschitz or even fractal boundaries. For rectifiable boundaries a deep study of layer potential operators and related topics has been carried out in the fundamental recent books [75–79]. Here we extend several basic results to a much wider class of boundaries with no rectifiability assumption.

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In the classical theory of partial differential equations on smooth domains, the (boundary) layer potential and Neumann-Poincaré operators are defined as convolutions with Green’s functions and their first derivatives, see for instance [41, 63, 72, 91, 96] or [5]. They provide explicit representation formulas for solutions to boundary value problems, and they are fundamental tools in inverse methods, numerical analysis and certain areas of spectral theory.

A systematic study of the classical approach to layer potentials in the context of Lipschitz domains was provided in [95], following seminal studies on boundary value problems [52], and singular integrals [19, 25]. Since then, the Lipschitz case has become a standard level of generality for many applications, see for instance [3, 4, 27, 28, 31, 32, 50, 73, 82, 89]. A different approach was adopted in [11], where the weak well-posedness of transmission problems was used to define the layer potentials. In view of the well established use of Hilbert space methods in potential theory — see [26, 29, 39, 97] for classical references and [16, 24, 43, 67] for later developments — the variational definitions in [11] are very natural.

Boundary layer operators on piecewise smooth, not necessarily Lipschitz domains were already studied in [72, Chapter 5]. Results for layer potentials in the context of Riemannian manifolds were obtained in [81], results for layer potentials on half-spaces with boundary data in Besov spaces in [12]. The research on the boundedness of singular integral operators in [19, 25, 95] culminated in the comprehensive study [75–79] of such operators on uniformly rectifiable sets [34, 35]. Uniformly rectifiable sets in  $\mathbb{R}^n$  are Ahlfors  $(n - 1)$ -regular closed subsets having “big pieces of Lipschitz images”, see [75, Definitions 5.10.1 and 5.10.2], and the class of such sets is basically characterized by the  $L^2$ -boundedness of singular integral operators, [75, Theorem 5.10.2]. A different major stream of research focused on the behaviour of harmonic functions on a domain  $\Omega$  and harmonic measures on its boundary  $\partial\Omega$ , see [30, 51, 52, 55] and the later references [2, 13, 20, 33]. A essential notion is that of NTA domains, [52, Section 3]. The class of NTA domains includes all Lipschitz domains, but also certain domains with fractal boundaries such as quasidisks [8, 52, 53, 84], and among them are examples where different parts of the boundary have different Hausdorff dimensions. A short introduction to harmonic measures and NTA domains is given in [90]. Another question about a domain  $\Omega$  is whether it is a Sobolev extension domain, *i.e.*, whether there is a bounded linear extension operator mapping Sobolev functions on  $\Omega$  to Sobolev functions on  $\mathbb{R}^n$ . In [54] it was shown that uniform domains [70, 93], and, more generally,  $(\varepsilon, \delta)$ -domains [54], are Sobolev extension domains. Any NTA domain is uniform, see for instance [8, Theorem 2.15] and any uniform domain with uniformly rectifiable boundary is NTA [8, Theorem 1.1].

We are interested in boundary value problems involving rough boundaries that may be fractal and may even have different parts of different Hausdorff dimensions. Well-known and more specific references on linear elliptic partial differential equations on fractal domains are [58, 65, 66, 84], and for the applications to the heat equation [68, 94]. Some further articles close to our work based on the elliptic operators are [47, 49], on Helmholtz models [10, 17, 18, 22, 45, 48, 69], on the non-linear wave equation (the Westervelt equation) [37, 38]. The highly active research around domains with rough boundaries motivates an extension of the variational approach in [11] beyond the Lipschitz case, and potential follow-up questions about convergence and optimization make it desirable to have a generalization which is flexible and rather easy to handle.

The purpose of our work is to propose a lightweight and streamlined generalization. The domains we consider are “first order” Sobolev extension domains  $\Omega$  in  $\mathbb{R}^n$  [46, 54], having a boundary  $\partial\Omega$  of positive capacity [71]. Basic examples in the planar case are quasidisks or complements of large enough Cantor sets. Domains having sharp inward or outward cusps or “collapsing” boundaries are not covered; in particular, certain fractal trees [1] do not fall within the scope of our work. We use established results [15] to introduce trace and extension operators between the domain  $\Omega$  and its boundary  $\partial\Omega$  in the context of the Sobolev space  $H^1(\Omega)$  (respectively, the homogeneous Sobolev space  $\dot{H}^1(\Omega)$ ). On the boundary  $\partial\Omega$ , we use the corresponding abstract trace space  $\mathcal{B}(\partial\Omega)$  (respectively,  $\dot{\mathcal{B}}(\partial\Omega)$ ), endowed with natural trace norms. That approach is *measure free* in the sense that we do not specify any “surface

measure” on  $\partial\Omega$ , nor do we consider  $L^2$ -spaces on  $\partial\Omega$ , unlike [95]. Instead, we make a systematic use of the trace spaces  $\mathcal{B}(\partial\Omega)$  and  $\dot{\mathcal{B}}(\partial\Omega)$  and their respective duals  $\mathcal{B}'(\partial\Omega)$  and  $\dot{\mathcal{B}}'(\partial\Omega)$ . As a consequence, we observe a variety of *natural isometries*; the preservation of those isometries might be considered a guiding theme of our work. In the special case of a Lipschitz domain  $\Omega$ , the spaces  $\mathcal{B}(\partial\Omega)$  and  $\mathcal{B}'(\partial\Omega)$  are the usual spaces  $H^{1/2}(\partial\Omega)$  and  $H^{-1/2}(\partial\Omega)$ , up to equivalent norms; the “dot” versions correspond to their homogeneous counterparts. In cases when the boundary  $\partial\Omega$  carries a measure satisfying certain scaling conditions, the trace space  $\mathcal{B}(\partial\Omega)$  admits an explicit descriptions in terms of equivalent Besov space norms, see [56, 57] and the comments in [48, Section 5]. In general, the present measure free formulation does not give any explicit norm representation for the trace space  $\mathcal{B}(\partial\Omega)$  and may therefore not be sufficient to discuss regularity features. But it works under minimal assumptions on the boundary and this seems particularly useful in view of possible studies of convergence and compactness properties as in [47–49].

To set the spaces, operators and notations, we briefly survey Dirichlet and Neumann problems in the weak (variational) sense for one-sided domains in Section 2. They are formulated for the operator  $\Delta - 1$  in the context of the space  $H^1(\Omega)$  and for the operator  $\Delta$  in the context of the space  $\dot{H}^1(\Omega)$ . The use of the space  $\dot{H}^1(\Omega)$  may have the disadvantage that constants are factored out, but it has the advantage that arguments can proceed very much in parallel to the inhomogeneous case. In Section 3, we briefly discuss basic features of the related Poincaré-Steklov operators. Suitable two-sided domains are then introduced in Section 4. Roughly speaking, we require that both the inner domain  $\Omega$  and the outer domain  $\mathbb{R}^n \setminus \bar{\Omega}$  be non-empty and Sobolev extension domains, and we assume that the separating boundary  $\partial\Omega$  has zero Lebesgue measure. In the homogeneous case, we additionally assume that  $\Omega$  is bounded. Similarly to [11], the weak well-posedness of transmission problems is used to give variational definitions of double and single layer potential operators for those two-sided domains, now measure free, see Subsections 4.3 and 4.4. Their basic properties are discussed in those subsections. In Section 5, resolvent representations of double and single layer potential operators are surveyed. Neumann-Poincaré operators are well understood on smooth or Lipschitz domains [5, 59, 63] and their spectral properties are known to reflect the regularity of  $\partial\Omega$ : they are compact in the  $C^1$  case [41] but have a continuous non-real spectrum if  $\partial\Omega$  has a corner [5, 21]. In Section 6, we define Neumann-Poincaré operators  $\mathcal{K}$  and  $\dot{\mathcal{K}}$  on the trace spaces  $\mathcal{B}(\partial\Omega)$  and  $\dot{\mathcal{B}}(\partial\Omega)$  respectively. As in the Lipschitz case [89], they are bounded operators and satisfy the well-known jump relations, see Theorems 6.1 and 6.2. Following [89], we define boundary layer potential operators associated with  $(\Delta - 1)$  and  $\Delta$  in (6.1) and (6.5) respectively and using those operators, we obtain generalizations of the classical Calderón projectors, see Theorems 6.5 and 6.7. As in [89], the boundary layer potentials give rise to equivalent Hilbert space norms on the trace spaces and their duals. Those norms make the single and double layer potentials isometries, see Theorem 6.10 in Subsection 6.3. Again, similarly to [89], we observe that  $(\pm \frac{1}{2}I + \mathcal{K})$  are coercive contractions and, in particular, bijective, see Theorem 6.10 (iii). The contractivity of  $(\pm \frac{1}{2}I + \mathcal{K})$  ensures the convergence of the associated Neumann series [83]: recovering the jump in trace  $f \in \mathcal{B}(\partial\Omega)$  of a transmission solution for  $(\Delta - 1)$  with zero jump in normal derivative from its exterior trace  $\varphi \in \mathcal{B}(\partial\Omega)$  amounts to solving the boundary integral equation of the second kind  $-\frac{1}{2}f - \mathcal{K}f = \varphi$  in  $\mathcal{B}(\partial\Omega)$ , and its unique solution  $f$  is given by the Neumann series  $f = \sum_{\ell=0}^{+\infty} (\frac{1}{2}I + \mathcal{K})^\ell \varphi$  which converges in  $\mathcal{B}(\partial\Omega)$ . Similarly, one can invert the operator  $-\frac{1}{2}I + \mathcal{K}^*$  on  $\mathcal{B}'(\partial\Omega)$  to recover the jump in normal derivative of a solution with no jump in trace from its exterior normal derivative, see [89, Theorem 3.2, p.741]. As in the classical case [40, 41, 95], Theorem 6.10 (iii) can be used to show that the complex spectra of  $\mathcal{K}$  and  $\mathcal{K}^*$  lie in the intersection of the circles of radius 1 and center  $\pm \frac{1}{2}$ , Theorem 6.12. Those results have homogeneous counterparts, see Theorems 6.11 and 6.13.

As an application, we generalize three results from [3]. There they were shown for Lipschitz domains; here we establish them for two-sided Sobolev extension domains. Theorem 7.1 is a boundary representation formula for the unique weak solution to a specific transmission

problem for the Laplacian and generalizes [3, Theorem 2.17], see also [60, 61]. Theorem 7.2 is a uniqueness results for subdomain identification through a single boundary measurement in the monotone case, partly generalizing [3, Theorem A.7, p.220], see also [14]. Finally, we prove a similar result but in the case of disks in  $\mathbb{R}^2$  in Theorem 7.4, generalizing [60, Theorem A] in the case of a surrounding extension domain and boundary data understood in the sense of the dual trace space  $\dot{\mathcal{B}}'(\partial\Omega)$  instead of  $L_0^2(\partial\Omega)$ .

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## 2. BOUNDARY VALUE PROBLEMS ON ADMISSIBLE DOMAINS

**2.1. Admissible domains and traces.** Let  $\Omega$  be a nonempty open subset of  $\mathbb{R}^n$ . As usual, we write  $H^1(\Omega)$  for the Hilbert space of all  $u \in L^2(\Omega)$  such that  $\nabla u \in L^2(\Omega, \mathbb{R}^n)$  and having the scalar product

$$(2.1) \quad \langle u, v \rangle_{H^1(\Omega)} = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\Omega} uv \, dx;$$

here  $\nabla u$  is interpreted in distributional sense. We write  $\dot{H}^1(\Omega)$  for the Hilbert space formed by the vector space of all  $u \in L_{\text{loc}}^2(\Omega)$  with  $\nabla u \in L^2(\Omega, \mathbb{R}^n)$  modulo locally constant functions, endowed with the scalar product

$$(2.2) \quad \langle u, v \rangle_{\dot{H}^1(\Omega)} = \int_{\Omega} \nabla u \nabla v \, dx.$$

As usual, we call  $\Omega$  a domain if it is connected. Details on  $\dot{H}^1(\Omega)$  in the case of a domain  $\Omega$  can be found in [24, Section 2.2.4], [39] or [71, Section 1.1.13]; the generalization to open subsets with multiple connected components is straightforward. We agree to use the notations (2.1) and (2.2) whenever the right-hand side makes sense.

We use the notions of capacity, quasi-continuous representatives and quasi-everywhere (q.e.) valid statements exclusively with respect to the space  $H^1(\mathbb{R}^n)$ ; background may be found in [43, Section 2.1] or [71, Section 7.2]. Each  $u \in H^1(\mathbb{R}^n)$  has a quasi-continuous representative  $\tilde{u}$  – see for instance [24, Theorem 2.3.4] or [71, Section 7.2.4]. For each class  $u \in \dot{H}^1(\mathbb{R}^n)$  we can find a quasi-continuous representative of a representative modulo constants. This can for instance be seen from [24, Theorem 2.3.4] together with [24, Theorems 2.2.12 and 2.2.13], see (A.2) and (A.3) in the Appendix. For brevity we will again call it a quasi-continuous representative of  $u$ .

Recall that  $\Omega$  is called an  $H^1$ -extension domain [46, 54], if there is a bounded linear extension operator  $E_{\Omega} : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ . We say that a domain  $\Omega \subset \mathbb{R}^n$  is  $H^1$ -admissible if it is an  $H^1$ -extension domain and its boundary  $\partial\Omega$  has positive capacity. In this case we write  $\mathcal{B}(\partial\Omega)$  for the vector space of all q.e. equivalence classes of pointwise restrictions  $\tilde{u}|_{\partial\Omega}$  of quasi-continuous representatives  $\tilde{u}$  of classes  $u \in H^1(\mathbb{R}^n)$ . The following result on the existence and uniqueness of natural trace operators from  $H^1(\Omega)$  onto  $\mathcal{B}(\partial\Omega)$  is immediate from [15, Theorem 6.1, Remark 6.2 and Corollary 6.3].

**Proposition 2.1.** *Let  $\Omega$  be  $H^1$ -admissible. Given  $u \in H^1(\Omega)$ , let  $\text{Tr}_i u := (E_{\Omega} u)|_{\partial\Omega}$  be the restriction of any quasi-continuous representative  $(E_{\Omega} u)^{\sim}$  of  $E_{\Omega} u$ . Then  $\text{Tr}_i : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  is a well-defined linear surjection.*

We use the subscript  $i$  in  $\text{Tr}_i$  because we will later discuss two-sided domains and  $\text{Tr}_i$  as defined here will play the role of an *interior* trace operator with respect to  $\Omega$ .

We say that a domain  $\Omega \subset \mathbb{R}^n$  is an  $\dot{H}^1$ -extension domain if there is a bounded linear extension operator  $\dot{E}_{\Omega} : \dot{H}^1(\Omega) \rightarrow \dot{H}^1(\mathbb{R}^n)$ , and we call it  $\dot{H}^1$ -admissible if it is an  $\dot{H}^1$ -extension domain and  $\partial\Omega$  is compact and of positive capacity. If so, we write  $\dot{\mathcal{B}}(\partial\Omega)$  for the vector space modulo constants of all q.e. equivalence classes of pointwise restrictions  $\tilde{w}|_{\partial\Omega}$  of quasi-continuous representatives  $\tilde{w}$  of classes  $u \in \dot{H}^1(\mathbb{R}^n)$ .

A variant of Proposition 2.1 gives natural trace operators from  $\dot{H}^1(\Omega)$  onto  $\dot{\mathcal{B}}(\partial\Omega)$ .

**Proposition 2.2.** *Let  $\Omega$  be  $\dot{H}^1$ -admissible. Given  $u \in \dot{H}^1(\Omega)$ , let  $\tilde{w}$  be a quasi-continuous representative of  $\dot{E}_\Omega u$ . For  $\dot{\text{Tr}}_i u$  defined as the  $\dot{\mathcal{B}}(\partial\Omega)$ -equivalence class of  $\tilde{w}|_{\partial\Omega}$ , the operator  $\dot{\text{Tr}}_i : \dot{H}^1(\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$  is a well-defined linear surjection.*

*Proof.* Suppose that  $u \in \dot{H}^1(\Omega)$  and let  $w \in \dot{E}_\Omega u$  be a representative of  $\dot{E}_\Omega u$  modulo constants. If  $\Omega$  is bounded, let  $B$  be an open ball containing  $\overline{\Omega}$  and let  $\chi \in C_c^\infty(B)$  be such that  $0 \leq \chi \leq 1$  and  $\chi \equiv 1$  on  $\overline{\Omega}$ . Then  $v := \chi w$  is an element of  $H^1(\mathbb{R}^n)$ , and by [15, Theorem 6.1, Remark 6.2 and Corollary 6.3] the restriction  $\tilde{w}|_{\partial\Omega} = \tilde{v}|_{\partial\Omega}$  is uniquely determined in the q.e. sense. Clearly its class modulo constants in  $\dot{\mathcal{B}}(\partial\Omega)$  does not depend on the choice of  $w$ . If instead the set  $\mathbb{R}^n \setminus \Omega$  is bounded, let  $B$  be an open ball containing it. Then  $\Omega \cap B$  is a bounded  $\dot{H}^1$ -extension domain, the elementary arguments are spelled out in Lemma A.1. Since  $u|_{\Omega \cap B} \in \dot{H}^1(\Omega \cap B)$ , we can proceed as before.  $\square$

**Remark 2.3.**

- (i) *Suppose that  $\Omega$  is a bounded  $H^1$ -extension domain. Then the vector space  $\dot{H}^1(\Omega)$  is isomorphic to the space of all  $u \in H^1(\Omega)$  with  $\int_\Omega u(x) \, dx = 0$ , and by Poincaré's inequality, (2.1) and (2.2) are equivalent scalar products on this space. In particular, the vector spaces  $H^1(\Omega)$  and  $\dot{H}^1(\Omega) \oplus \mathbb{R}$  are isomorphic.*
- (ii) *For a bounded domain  $\Omega$  which is both  $H^1$ - and  $\dot{H}^1$ -admissible, the vector spaces  $\mathcal{B}(\partial\Omega)$  and  $\dot{\mathcal{B}}(\partial\Omega) \oplus \mathbb{R}$  are isomorphic.*

**Examples 2.4.**

- (i) *For  $n \geq 2$  any  $(\varepsilon, \delta)$ -domain  $\Omega \subset \mathbb{R}^n$  is an  $H^1$ -extension domain, [54, Theorem 1], and any  $(\varepsilon, \infty)$ -domain  $\Omega \subset \mathbb{R}^n$  is an  $\dot{H}^1$ -extension domain, [54, Theorem 2].*
- (ii) *For  $n \geq 2$  any  $(\varepsilon, \infty)$ -domain  $\Omega \subset \mathbb{R}^n$  with  $\mathbb{R}^n \setminus \overline{\Omega}$  nonempty is  $H^1$ -admissible, and if one of the two open sets is bounded, it is also  $\dot{H}^1$ -admissible.*
- (iii) *For  $n = 1$  any interval  $(a, b)$  with  $a$  or  $b$  finite is  $H^1$ -admissible, and if both are finite, also  $\dot{H}^1$ -admissible. For  $n \geq 2$  the domain  $\Omega = \mathbb{R}^n \setminus \{0\}$  is not  $H^1$ -admissible.*

**Remark 2.5.** *Classical references on the spaces  $\dot{H}^1(\Omega)$  are [39] and [71, Sections 1.1.2 and 1.1.13]; a more recent discussion may be found in [44, Section II.6]. In those references, different symbols are used to denote these spaces. For the full space case  $\Omega = \mathbb{R}^n$ , the notation  $\dot{H}^1(\mathbb{R}^n)$  is established, see for instance [9, Section 1.3] and [92, Chapter 5]. Since the domains we consider are  $H^1$ - respectively  $\dot{H}^1$ -extension domains, we decided to follow this notation for the full space case and to use  $\dot{H}^1(\Omega)$ .*

**2.2. Orthogonality and isometries.** We center our discussion of boundary value problems around restrictions, duals and inverses of trace operators.

Let  $\Omega \subset \mathbb{R}^n$  be an  $H^1$ -admissible domain. We write  $H_0^1(\Omega)$  for the closure in  $H^1(\Omega)$  of the set  $C_c^\infty(\Omega)$  of infinitely differentiable functions with compact support in  $\Omega$ . Let  $V_1(\Omega)$  denote the orthogonal complement of  $H_0^1(\Omega)$  in  $H^1(\Omega)$ ,

$$(2.3) \quad H^1(\Omega) = H_0^1(\Omega) \oplus V_1(\Omega).$$

The functions in  $V_1(\Omega)$  are called 1-harmonic on  $\Omega$ . We write  $\text{tr}_i := \text{Tr}_i|_{V_1(\Omega)}$  for the restriction of  $\text{Tr}_i$  defined in Proposition 2.1 to  $V_1(\Omega)$ . The following theorem generalizes well-known results to the framework of  $H^1$ -admissible domains.

**Theorem 2.6.** *Let  $\Omega \subset \mathbb{R}^n$  be an  $H^1$ -admissible domain. Then,*

- (i) *the space  $H_0^1(\Omega)$  is the kernel of  $\text{Tr}_i$ , that is,  $H_0^1(\Omega) = \ker \text{Tr}_i$ .*
- (ii) *endowed with the norm*

$$(2.4) \quad \|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega)} \mid v \in H^1(\Omega) \text{ and } \text{Tr}_i v = f\},$$

*the space  $\mathcal{B}(\partial\Omega)$  is a Hilbert space.*

- (iii) *with respect to  $\|\cdot\|_{\mathcal{B}(\partial\Omega)}$ , the trace operator  $\text{tr}_i$  is bounded with operator norm one. Its restriction  $\text{tr}_i : V_1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  to  $V_1(\Omega)$  is an isometry and onto.*

Details on statement (i) can be found in Appendix A, statements (ii) and (iii) are direct consequences.

We write  $\mathcal{B}'(\partial\Omega)$  and  $(H^1(\Omega))'$  for the dual spaces of  $\mathcal{B}(\partial\Omega)$  and  $H^1(\Omega)$ , and we use the notation

$$\langle \cdot, \cdot \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{(H^1(\Omega))', H^1(\Omega)}$$

for the corresponding dual pairings.

**Remark 2.7.** *Since  $\mathcal{B}(\partial\Omega)$  is a Hilbert space, for any  $f \in \mathcal{B}(\partial\Omega)$ , the assignment  $\iota(f)(h) := \langle f, h \rangle_{\mathcal{B}(\partial\Omega)}$ ,  $h \in \mathcal{B}(\partial\Omega)$ , defines an isometric isomorphism  $\iota$  from  $\mathcal{B}(\partial\Omega)$  onto  $\mathcal{B}'(\partial\Omega)$ . The dual pairing is defined by  $\langle g, f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} := \langle \iota^{-1}(g), f \rangle_{\mathcal{B}(\partial\Omega)} = \langle g, \iota(f) \rangle_{\mathcal{B}'(\partial\Omega)}$ ,  $f \in \mathcal{B}(\partial\Omega)$ ,  $g \in \mathcal{B}'(\partial\Omega)$ . We may identify  $\mathcal{B}(\partial\Omega)$  with its image  $\iota(\mathcal{B}(\partial\Omega)) \subset \mathcal{B}'(\partial\Omega)$  under  $\iota$ .*

**Remark 2.8.** *If  $V_1'(\Omega)$  denotes the dual of the closed subspace  $V_1(\Omega)$  of  $H^1(\Omega)$ , then  $(H^1(\Omega))' \subset V_1'(\Omega)$  by restriction. However, by the Riesz representation theorem, any  $w \in V_1'(\Omega)$  is represented as  $w = \langle v, \cdot \rangle_{H^1(\Omega)}$  with some suitable  $v \in V_1(\Omega)$ . The orthogonal decomposition (2.3) then implies that  $w$  automatically extends to a unique bounded linear functional  $w' \in (H^1(\Omega))'$  on all of  $H^1(\Omega)$  and zero on  $H_0^1(\Omega)$ , and that extension is an isometry:  $\|w'\|_{(H^1(\Omega))'} = \|w\|_{V_1'(\Omega)}$ . We agree to make silent use of this extension: we write  $w$  to denote  $w'$  and use  $\|\cdot\|_{(H^1(\Omega))'}$  in place of  $\|\cdot\|_{V_1'(\Omega)}$  on  $V_1'(\Omega)$ .*

By Remark 2.8, the dual  $\text{tr}_i^* : \mathcal{B}'(\partial\Omega) \rightarrow V_1'(\Omega)$  of  $\text{tr}_i$  is seen to be characterized by

$$(2.5) \quad \langle g, \text{Tr}_i v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \langle \text{tr}_i^* g, v \rangle_{(H^1(\Omega))', H^1(\Omega)}, \quad v \in H^1(\Omega), \quad g \in \mathcal{B}'(\partial\Omega).$$

**Corollary 2.9.** *Let  $\Omega \subset \mathbb{R}^n$  be  $H^1$ -admissible. Then the operator  $\text{tr}_i^* : \mathcal{B}'(\partial\Omega) \rightarrow V_1'(\Omega)$  is an isometry,  $\|\text{tr}_i^* g\|_{(H^1(\Omega))'} = \|g\|_{\mathcal{B}'(\partial\Omega)}$ ,  $g \in \mathcal{B}'(\partial\Omega)$ , and onto.*

Now let  $\Omega \subset \mathbb{R}^n$  be an  $\dot{H}^1$ -admissible domain. The space

$$(2.6) \quad \dot{V}_0(\Omega) := \left\{ u \in \dot{H}^1(\Omega) \mid \int_{\Omega} \nabla u \nabla v \, dx = 0 \text{ for all } v \in C_c^\infty(\Omega) \right\}$$

is a closed subspace of  $\dot{H}^1(\Omega)$ . We write  $\dot{\text{tr}}_i := \dot{\text{Tr}}_i|_{\dot{V}_0(\Omega)}$  for the restriction of  $\dot{\text{Tr}}_i$  to  $\dot{V}_0(\Omega)$ . The following counterpart of Theorem 2.6 holds; a proof of (i) is given in Appendix A.

**Theorem 2.10.** *Let  $\Omega \subset \mathbb{R}^n$  be an  $\dot{H}^1$ -admissible domain. Then the following assertions hold:*

- (i) *The kernel  $\ker \dot{\text{Tr}}_i|_{\dot{H}^1(\Omega)}$  is the orthogonal complement  $(\dot{V}_0(\Omega))^\perp$  of  $\dot{V}_0(\Omega)$  in  $\dot{H}^1(\Omega)$ .*
- (ii) *Endowed with the norm  $\|f\|_{\dot{\mathcal{B}}(\partial\Omega)} := \min\{\|v\|_{\dot{H}^1(\Omega)} \mid v \in \dot{H}^1(\Omega) \text{ and } \dot{\text{Tr}}_i v = f\}$ , the space  $\dot{\mathcal{B}}(\partial\Omega)$  is a Hilbert space.*
- (iii) *With respect to  $\|\cdot\|_{\dot{\mathcal{B}}(\partial\Omega)}$ , the operator  $\dot{\text{Tr}}_i$  is bounded with operator norm one. Its restriction  $\dot{\text{tr}}_i : \dot{V}_0(\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$  to  $\dot{V}_0(\Omega)$  is an isometry and onto.*

We write  $(\dot{H}^1(\Omega))'$ ,  $\dot{V}_0'(\Omega)$  and  $\dot{\mathcal{B}}'(\partial\Omega)$  for the dual spaces of  $\dot{H}^1(\Omega)$ ,  $\dot{V}_0(\Omega)$  and  $\dot{\mathcal{B}}(\partial\Omega)$  respectively. With a similar agreement as in Remark 2.8, the dual  $\dot{\text{tr}}_i^* : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{V}_0'(\Omega)$  of  $\dot{\text{tr}}_i$  is now seen to be characterized by

$$\langle g, \dot{\text{Tr}}_i v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} = \langle \dot{\text{tr}}_i^* g, v \rangle_{(\dot{H}^1(\Omega))', \dot{H}^1(\Omega)}, \quad v \in \dot{H}^1(\Omega), \quad g \in \dot{\mathcal{B}}'(\partial\Omega).$$

**Corollary 2.11.** *Let  $\Omega \subset \mathbb{R}^n$  be  $\dot{H}^1$ -admissible. Then the operator  $\dot{\text{tr}}_i^* : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{V}_0'(\Omega)$  is an isometry,  $\|\dot{\text{tr}}_i^* g\|_{(\dot{H}^1(\Omega))'} = \|g\|_{\dot{\mathcal{B}}'(\partial\Omega)}$ ,  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , and onto.*

**2.3. Harmonic extensions.** Suppose that  $\Omega \subset \mathbb{R}^n$  is an  $H^1$ -admissible domain. Given  $f \in \mathcal{B}(\partial\Omega)$ , an element  $u$  of  $H^1(\Omega)$  is called a *weak solution* of the Dirichlet problem

$$(2.7) \quad \begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f \end{cases}$$

if  $\text{Tr}_i u = f$  and  $\langle u, v \rangle_{H^1(\Omega)} = 0$  for all  $v \in C_c^\infty(\Omega)$ . The symbol  $u|_{\partial\Omega}$  in the formal problem (2.7) stands for the restriction of  $u$  to the boundary; in the context of weak solutions, it is made rigorous through the trace condition. For any  $f \in \mathcal{B}(\partial\Omega)$ , the Dirichlet problem (2.7) has a unique weak solution  $u^f$ , namely  $u^f = \text{tr}_i^{-1} f \in V_1(\Omega)$ ; this is well-known and follows by using (2.3). By Theorem 2.6, the operator  $\text{tr}_i^{-1} : \mathcal{B}(\partial\Omega) \rightarrow V_1(\Omega)$  is an isometry.

Suppose that  $\Omega$  is an  $\dot{H}^1$ -admissible domain. Given  $f \in \dot{\mathcal{B}}(\partial\Omega)$ , we call an element  $u$  of  $\dot{H}^1(\Omega)$  a *weak solution in the  $\dot{H}^1$ -sense* of the Dirichlet problem

$$(2.8) \quad \begin{cases} -\Delta u &= 0 & \text{in } \Omega \\ u|_{\partial\Omega} &= f \end{cases}$$

if  $\text{Tr}_i u = f$  and  $\langle u, v \rangle_{\dot{H}^1(\Omega)} = 0$  for all  $v \in C_c^\infty(\Omega)$ . For any  $f \in \dot{\mathcal{B}}(\partial\Omega)$  the Dirichlet problem (2.8) has the unique weak solution  $u^f = \text{tr}_i^{-1} f \in \dot{V}_0(\Omega)$  in the  $\dot{H}^1$ -sense. By Theorem 2.10 the operator  $\text{tr}_i^{-1}$  is an isometry.

**Remark 2.12.**

- (i) If  $\Omega$  is bounded and  $f \in \mathcal{B}(\partial\Omega)$ , then  $w \in H^1(\Omega)$  is called a *weak solution to (2.8)* if  $\text{Tr}_i w = f$  and  $\langle w, v \rangle_{\dot{H}^1(\Omega)} = 0$  for all  $v \in C_c^\infty(\Omega)$ . It is well-known that for any  $f \in \mathcal{B}(\partial\Omega)$ , there is a unique weak solution  $w^f$  in this sense of (2.8) and that  $w^f$  is an element of the space  $V_0(\Omega)$  of all  $u \in H^1(\Omega)$  with  $\langle u, v \rangle_{\dot{H}^1(\Omega)} = 0$  for all  $v \in H_0^1(\Omega)$ .
- (ii) If  $\Omega$  is bounded and  $f \in \mathcal{B}(\partial\Omega)$ , then the equivalence class  $[f]$  of  $f$  modulo constants is in  $\dot{\mathcal{B}}(\partial\Omega)$ , cf. Remark 2.3 (ii), and the unique weak solution  $u^{[f]} \in \dot{H}^1(\Omega)$  in the  $\dot{H}^1$ -sense of (2.8) with this class  $[f]$  in place of  $f$  contains exactly one representative  $w \in u^{[f]}$  modulo constants such that  $\tilde{w}|_{\partial\Omega} = f$  q.e. This particular representative  $w$  is exactly the  $w^f$  from (i).
- (iii) In [88] a different type of function spaces based on [39] was used in order to handle Dirichlet boundary conditions for unbounded domains without losing constants. However, to avoid technicalities in later sections, we decided to accept a loss of constants.

**2.4. Neumann solutions.** Suppose  $\Omega \subset \mathbb{R}^n$  is an  $H^1$ -admissible domain. Given  $g \in \mathcal{B}'(\partial\Omega)$ , we call  $u \in H^1(\Omega)$  a *weak solution* of the Neumann problem

$$(2.9) \quad \begin{cases} -\Delta u + u &= 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} &= g \end{cases}$$

if for all  $v \in H^1(\Omega)$ , we have

$$(2.10) \quad \langle u, v \rangle_{H^1(\Omega)} = \langle g, \text{Tr}_i v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}.$$

The symbol  $\frac{\partial u}{\partial \nu}$  in the formal problem (2.9) stands for the (interior) normal derivative; in the context of weak solutions it is implicitly made rigorous by (2.10). It is well-known and easily seen from the Riesz representation theorem that for any  $g \in \mathcal{B}'(\partial\Omega)$ , the Neumann problem (2.9) has a unique weak solution  $u_g$ , it is an element of  $V_1(\Omega)$ . We write  $\mathcal{N}_1 : \mathcal{B}'(\partial\Omega) \rightarrow V_1(\Omega)$ ,  $\mathcal{N}_1 g := u_g$  for the linear operator mapping a given element  $g$  of  $\mathcal{B}'(\partial\Omega)$  to the unique weak solution of (2.9). By (2.5), we have

$$\langle \mathcal{N}_1 g, v \rangle_{H^1(\Omega)} = \langle \text{tr}_i^* g, v \rangle_{(H^1(\Omega))', H^1(\Omega)}, \quad v \in H^1(\Omega),$$

hence  $\|\mathcal{N}_1 g\|_{H^1(\Omega)} = \|\text{tr}_i^* g\|_{(H^1(\Omega))'} = \|g\|_{\mathcal{B}'(\partial\Omega)}$ ,  $g \in \mathcal{B}'(\partial\Omega)$ , that is,  $\mathcal{N}_1$  is an isometry.

**Corollary 2.13.** *Let  $\Omega$  be  $H^1$ -admissible. The linear operator  $\text{tr}_i \circ \mathcal{N}_1 : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  is an isometry and onto. It satisfies*

$$\langle g, \text{tr}_i \circ \mathcal{N}_1 h \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \langle g, h \rangle_{\mathcal{B}'(\partial\Omega)} = \langle h, \text{tr}_i \circ \mathcal{N}_1 g \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}, \quad g, h \in \mathcal{B}'(\partial\Omega).$$



*Proof.* The first statement is clear from (2.10), (2.5) and the preceding. The second follows because for all  $g, h \in \mathcal{B}'(\partial\Omega)$ , we have

$$\begin{aligned} \langle g, \text{tr}_i \circ \mathcal{N}_1 h \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} &= \langle g, \text{tr}_i u_h \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} \\ &= \langle \text{tr}_i^* g, u_h \rangle_{(H^1(\Omega))', H^1(\Omega)} = \langle u_g, u_h \rangle_{H^1(\Omega)}. \end{aligned}$$

□

Now suppose that  $\Omega$  is  $\dot{H}^1$ -admissible. Given  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , we call  $u \in \dot{H}^1(\Omega)$  a *weak solution in the  $\dot{H}^1$ -sense* of the Neumann problem

$$(2.11) \quad \begin{cases} -\Delta u &= 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} |_{\partial\Omega} &= g \end{cases}$$

if for all  $v \in \dot{H}^1(\Omega)$  we have

$$(2.12) \quad \langle u, v \rangle_{\dot{H}^1(\Omega)} = \langle g, \dot{\text{Tr}}_i v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}.$$

For any  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , the Neumann problem (2.11) has a unique weak solution  $u_g$  in the  $\dot{H}^1$ -sense, and  $u_g$  is an element of  $\dot{V}_0(\Omega)$  defined in (2.6). We write  $\dot{\mathcal{N}}_0 : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{V}_0(\Omega)$ ,  $\dot{\mathcal{N}}_0 g := u_g$ , for the linear operator mapping a given element of  $\dot{\mathcal{B}}'(\partial\Omega)$  to  $u_g$ . As before, we see that  $\dot{\mathcal{N}}_0$  is an isometry:  $\|\dot{\mathcal{N}}_0 g\|_{\dot{H}^1(\Omega)} = \|\dot{\text{tr}}_i^* g\|_{(\dot{H}^1(\Omega))'} = \|g\|_{\dot{\mathcal{B}}'(\partial\Omega)}$ ,  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ .

**Corollary 2.14.** *Let  $\Omega$  be  $\dot{H}^1$ -admissible. Then the linear operator  $\text{tr}_i \circ \dot{\mathcal{N}}_0 : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$  is an isometry and onto. It satisfies*

$$\langle g, \dot{\text{tr}}_i \circ \dot{\mathcal{N}}_0 h \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} = \langle g, h \rangle_{\dot{\mathcal{B}}'(\partial\Omega)} = \langle h, \dot{\text{tr}}_i \circ \dot{\mathcal{N}}_0 g \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}, \quad g, h \in \dot{\mathcal{B}}'(\partial\Omega).$$

**2.5. Normal derivatives.** Abstract normal derivatives have been defined and studied by various authors in different contexts, see for instance [26, p. 218], [67, Section 3.2] and [64]. We formulate a variant of those definitions which suits our purposes.

Suppose  $\Omega \subset \mathbb{R}^n$  is  $H^1$ -admissible. Let

$$(2.13) \quad H_\Delta^1(\Omega) := \{u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega)\},$$

with  $\Delta u$  understood in distributional sense. Clearly,  $V_1(\Omega) \subset H_\Delta^1(\Omega)$ . Given  $u \in H_\Delta^1(\Omega)$ , there is a unique element  $g$  of  $\mathcal{B}'(\partial\Omega)$  such that

$$(2.14) \quad \langle g, \text{Tr}_i v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \int_\Omega (\Delta u)v \, dx + \int_\Omega \nabla u \nabla v \, dx, \quad v \in H^1(\Omega).$$

We call this element  $g$  the *weak interior normal derivative* of  $u$  (with respect to  $\Omega$ ) and denote it by  $\frac{\partial_i u}{\partial \nu} := g$ . The operator  $\frac{\partial_i}{\partial \nu} : H_\Delta^1(\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$  is linear and bounded in the sense that

$$(2.15) \quad \left\| \frac{\partial_i u}{\partial \nu} \right\|_{\mathcal{B}'(\partial\Omega)} \leq \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}.$$

**Corollary 2.15.** *Let  $\Omega$  be  $H^1$ -admissible. Then the following assertions hold:*

- (i) *Both  $\mathcal{N}_1$  and the operator  $\frac{\partial_i}{\partial \nu} : V_1(\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$  are isometries and onto, and  $\frac{\partial_i}{\partial \nu} = \mathcal{N}_1^{-1}$  on  $V_1(\Omega)$ .*
- (ii) *For  $u, v \in V_1(\Omega)$ , it holds*

$$(2.16) \quad \left\langle \frac{\partial_i u}{\partial \nu}, \text{tr}_i v \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \langle u, v \rangle_{H^1(\Omega)} = \left\langle \frac{\partial_i v}{\partial \nu}, \text{tr}_i u \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}.$$

- (iii) *The dual  $(\frac{\partial_i}{\partial \nu})^* : \mathcal{B}(\partial\Omega) \rightarrow V_1'(\Omega)$  of  $\frac{\partial_i}{\partial \nu}$  on  $V_1(\Omega)$  is an isometry and onto.*

*Proof.* Statement (i) follows using the surjectivity of  $\text{tr}_i$  observed in Theorem 2.6 (iii) and a comparison of (2.10) and (2.14). Statement (ii) is a special case of (2.14), and (iii) is a consequence of (i). □

Now suppose  $\Omega \subset \mathbb{R}^n$  is  $\dot{H}^1$ -admissible. We then consider the space

$$\dot{H}_\Delta^1(\Omega) := \{u \in \dot{H}^1(\Omega) \mid \Delta u \in L^2(\Omega)\},$$

containing  $\dot{V}_0(\Omega)$ . Given  $u \in \dot{H}_\Delta^1(\Omega)$ , there is a unique element  $g$  of  $\dot{\mathcal{B}}'(\partial\Omega)$  such that

$$\langle g, \dot{\text{Tr}}_i v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} = \int_\Omega (\Delta u)v \, dx + \int_\Omega \nabla u \nabla v \, dx, \quad v \in \dot{H}^1(\Omega),$$

and we call also  $\frac{\partial_i u}{\partial \nu} := g$  the *weak interior normal derivative of  $u$  in the  $\dot{H}^1$ -sense* (with respect to  $\Omega$ ). The operator  $\frac{\partial_i}{\partial \nu} : \dot{H}_\Delta^1(\Omega) \rightarrow \dot{\mathcal{B}}'(\partial\Omega)$  is linear and

$$(2.17) \quad \left\| \frac{\partial_i u}{\partial \nu} \right\|_{\dot{\mathcal{B}}'(\partial\Omega)} \leq \|u\|_{\dot{H}^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}.$$

**Remark 2.16.** *Suppose that  $\Omega$  is  $\dot{H}^1$ -admissible and bounded. Then by Remark 2.3 (ii), the spaces  $\mathcal{B}(\partial\Omega)$  and  $\dot{\mathcal{B}}(\partial\Omega) \oplus \mathbb{R}$  can be identified. Each bounded linear functional  $g \in \dot{\mathcal{B}}'(\partial\Omega)$  on  $\dot{\mathcal{B}}(\partial\Omega)$  induces a linear functional on  $\mathcal{B}(\partial\Omega)$  through extension by zero on  $\mathbb{R}$ . In this sense, an equality of two elements of  $\dot{\mathcal{B}}'(\partial\Omega)$ , such as the Neumann boundary condition  $\frac{\partial_i}{\partial \nu}(\mathcal{N}_0 g) = g$  for  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , may be seen as an equality of two linear functionals on  $\mathcal{B}(\partial\Omega)$ .*

**Corollary 2.17.** *Let  $\Omega$  be  $\dot{H}^1$ -admissible. Then:*

- (i) *Both  $\mathcal{N}_0$  and the operator  $\frac{\partial_i}{\partial \nu} : \dot{V}_0(\Omega) \rightarrow \dot{\mathcal{B}}'(\partial\Omega)$  are isometries and onto, and  $\frac{\partial_i}{\partial \nu} = \mathcal{N}_0^{-1}$  on  $\dot{V}_0(\Omega)$ .*
- (ii) *For  $u, v \in \dot{V}_0(\Omega)$ , it holds*

$$(2.18) \quad \left\langle \frac{\partial_i u}{\partial \nu}, \dot{\text{tr}}_i v \right\rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} = \langle u, v \rangle_{\dot{H}^1(\Omega)} = \left\langle \frac{\partial_i v}{\partial \nu}, \dot{\text{tr}}_i u \right\rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}.$$

- (iii) *The dual  $(\frac{\partial_i}{\partial \nu})^* : \dot{\mathcal{B}}(\partial\Omega) \rightarrow \dot{V}_0'(\Omega)$  of  $\frac{\partial_i}{\partial \nu}$  on  $\dot{V}_0(\Omega)$  is an isometry and onto.*

**Remark 2.18.** *Weak normal derivatives can also be used to characterize weak solutions of (2.7): an element  $u$  of  $H^1(\Omega)$  is a weak solution to (2.7) if and only if  $\langle u, v \rangle_{H^1(\Omega)} = 0$  for all  $v \in C_c^\infty(\Omega)$  and  $\langle v, u \rangle_{H^1(\Omega)} = \langle \frac{\partial_i v}{\partial \nu}, f \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}$  for all  $v \in V_1(\Omega)$ . Similarly for (2.8).*

### 3. POINCARÉ-STEKLOV OPERATORS FOR ADMISSIBLE DOMAINS

We provide some comments on Poincaré-Steklov operators for bounded admissible domains.

Let  $\Omega \subset \mathbb{R}^n$  be a bounded  $H^1$ -admissible domain. We write  $\Delta_D$  for the self-adjoint Dirichlet Laplacian on  $L^2(\Omega)$  and denote its spectrum by  $\sigma(\Delta_D)$ . Recall that  $\sigma(\Delta_D) \subset (-\infty, 0)$  is pure point with eigenvalues accumulating at minus infinity.

Given  $k \in \mathbb{R}$  and  $f \in \mathcal{B}(\partial\Omega)$ , we call  $u \in H^1(\Omega)$  a weak solution to the Dirichlet problem

$$(3.1) \quad \begin{cases} -\Delta u + ku &= 0 & \text{in } \Omega \\ u|_{\partial\Omega} &= f \end{cases}$$

if  $\text{Tr}_i u = f$  and  $\langle u, v \rangle_{\dot{H}^1(\Omega)} + k \langle u, v \rangle_{L^2(\Omega)} = 0$  for all  $v \in C_c^\infty(\Omega)$ . This corresponds to (2.7) with  $-\Delta u + ku$  in place of  $-\Delta u + u$ ; problem (2.7) is the special case for  $k = 1$ .

If  $k \in \mathbb{R} \setminus \sigma(\Delta_D)$ , then for any  $f \in \mathcal{B}(\partial\Omega)$  there is a unique weak solution  $u_f \in H_\Delta(\Omega)$  of (3.1). For a such  $k$ , one can define a linear operator  $\mathcal{A}_k : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$  by

$$\mathcal{A}_k f := \frac{\partial_i u_f}{\partial \nu}.$$

That operator is called the *Poincaré-Steklov* (or *Dirichlet-to-Neumann*) *operator* associated with  $(\Delta - k)$  on  $\Omega$ . See for instance [6, 7, 42, 86] for studies of Poincaré-Steklov operators under more restrictive assumptions on  $\Omega$ .

**Lemma 3.1.** *Let  $\Omega$  be a bounded  $H^1$ -admissible domain. Then the following assertions hold:*

- (i) For any  $k \in \mathbb{R} \setminus \sigma(\Delta_D)$ , the Poincaré-Steklov operator  $\mathcal{A}_k : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$  is a bounded linear operator and coincides with its adjoint. It is injective if and only if  $k$  is not an eigenvalue of the self-adjoint Neumann Laplacian for  $\Omega$ .
- (ii) The Poincaré-Steklov operator  $\mathcal{A}_1 : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$  is an isometry and

$$\mathcal{A}_1 = \frac{\partial_i}{\partial\nu} \circ (\text{tr}_i)^{-1}.$$

*Proof.* By (2.15) and (3.1) we have

$$\|\mathcal{A}_k f\|_{\mathcal{B}'(\partial\Omega)} \leq (1+k) \|u_f\|_{H^1(\Omega)} = (1+k) \|f\|_{\mathcal{B}(\partial\Omega)}, \quad f \in \mathcal{B}(\partial\Omega).$$

From (2.14) it is easily seen that

$$\langle \mathcal{A}_k f_1, f_2 \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \langle \mathcal{A}_k f_2, f_1 \rangle_{\mathcal{B}(\partial\Omega), \mathcal{B}'(\partial\Omega)},$$

the special case for  $k = 1$  was stated in (2.16). The statement on injectivity is clear. Item (ii) follows using Theorem 2.6 (iii) and Corollary 2.15 (i).  $\square$

Now suppose that  $\Omega \subset \mathbb{R}^n$  is bounded and  $\dot{H}^1$ -admissible. Then the linear operator  $\dot{\mathcal{A}}_0 : \dot{\mathcal{B}}(\partial\Omega) \rightarrow \dot{\mathcal{B}}'(\partial\Omega)$ , defined as

$$\dot{\mathcal{A}}_0 := \frac{\dot{\partial}_i}{\dot{\partial}\nu} \circ (\text{tr}_i)^{-1},$$

is called the *Poincaré-Steklov* (or *Dirichlet-to-Neumann*) operator in the  $\dot{H}^1$ -sense associated with  $\Delta$  on  $\Omega$ . Theorem 2.10 (iii) and Corollary 2.17 (i) give the following analog of Lemma 3.1.

**Lemma 3.2.** *Let  $\Omega$  be a bounded  $\dot{H}^1$ -admissible domain. Then  $\dot{\mathcal{A}}_0 : \dot{\mathcal{B}}(\partial\Omega) \rightarrow \dot{\mathcal{B}}'(\partial\Omega)$  is a linear isometry, and it coincides with its adjoint.*

Writing  $\iota$  to denote the Riesz isomorphism from a Hilbert space to its dual (regardless of that space), we obtain the following commutative diagram (3.2):

$$(3.2) \quad \begin{array}{ccc} V_1(\Omega) = H_0^1(\Omega)^\perp & \begin{array}{c} \xrightarrow{\text{tr}_i} \\ \xleftarrow{\text{tr}_i^{-1}} \end{array} & \text{Tr}_i(H^1(\Omega)) = \mathcal{B}(\partial\Omega) \\ \begin{array}{c} \uparrow \iota^{-1} \\ \downarrow \iota \end{array} & \begin{array}{c} \searrow \frac{\partial_i}{\partial\nu} \\ \swarrow (\frac{\partial_i}{\partial\nu})^{-1} \end{array} & \begin{array}{c} \downarrow \mathcal{A}_1^{-1} = \iota^{-1} \\ \uparrow \mathcal{A}_1 = \iota \end{array} \\ V_1'(\Omega) = (H_0^1(\Omega)^\perp)' & \begin{array}{c} \xleftarrow{(\text{tr}_i^*)^{-1}} \\ \xrightarrow{\text{tr}_i^*} \end{array} & (\text{Tr}_i(H^1(\Omega)))' = \mathcal{B}'(\partial\Omega) \end{array}$$

Note that for any  $f_1, f_2 \in \mathcal{B}(\partial\Omega)$ , we have indeed

$$\begin{aligned} \langle \iota(f_1), f_2 \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} &= \langle f_1, f_2 \rangle_{\mathcal{B}(\partial\Omega)} \\ &= \langle \text{tr}_i^{-1} f_1, \text{tr}_i^{-1} f_2 \rangle_{H^1(\Omega)} = \left\langle \frac{\partial_i}{\partial\nu} \circ (\text{tr}_i)^{-1} f_1, f_2 \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}. \end{aligned}$$

Obvious adjustments give an analogous commutative diagram involving the Poincaré-Steklov operator in the  $\dot{H}^1$ -sense.

#### 4. TRANSMISSION PROBLEMS FOR ADMISSIBLE DOMAINS

We introduce layer potential operators for transmission problems on two-sided rough domains through well-posedness results for transmission problems in the form

$$(4.1) \quad \begin{cases} (-\Delta + k)u & = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = f \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g; \end{cases}$$

here  $\Omega \subset \mathbb{R}^n$  is a bounded domain,  $u_i$  and  $u_e$  are the restrictions of the prospective solution to  $\Omega$  and  $\mathbb{R}^n \setminus \overline{\Omega}$  respectively,  $f$  and  $g$  are given data on  $\partial\Omega$  and  $k = 0, 1$ . First, we describe our setup for (4.1) and then proceed to well-posedness results, the definitions of layer potentials and statements on some of their properties.

**4.1. Two-sided admissible domains and jumps.** We call  $\Omega \subset \mathbb{R}^n$  a *two-sided  $H^1$ -admissible domain* if both  $\Omega$  and  $\mathbb{R}^n \setminus \overline{\Omega}$  are nonempty and  $H^1$ -extension domains and the Lebesgue measure of  $\partial\Omega$  is zero. This implies in particular that the Hausdorff dimension of  $\partial\Omega$  is at least  $n - 1$ , hence its capacity is positive.

Suppose that  $\Omega \subset \mathbb{R}^n$  is a two-sided  $H^1$ -admissible domain. We discuss (4.1) with  $k = 1$  in terms of the space  $H^1(\mathbb{R}^n \setminus \partial\Omega)$ . It admits the orthogonal decomposition  $H^1(\mathbb{R}^n \setminus \partial\Omega) = H^1(\Omega) \oplus H^1(\mathbb{R}^n \setminus \overline{\Omega})$ , so every  $u \in H^1(\mathbb{R}^n \setminus \partial\Omega)$  can be written  $u = u_i + u_e$  with uniquely determined  $u_i \in H^1(\Omega)$  and  $u_e \in H^1(\mathbb{R}^n \setminus \overline{\Omega})$ . We denote the closure of  $C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$  in  $H^1(\mathbb{R}^n \setminus \partial\Omega)$  by  $H_0^1(\mathbb{R}^n \setminus \partial\Omega)$  and write  $V_1(\mathbb{R}^n \setminus \partial\Omega)$  for its orthogonal complement, which is the space of functions that are 1-harmonic in  $\mathbb{R}^n \setminus \partial\Omega$ .

To distinguish between the two operators, we now write  $\text{Tr}_i : H^1(\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  to denote the boundary trace operator for  $\Omega$  and  $\text{Tr}_e : H^1(\mathbb{R}^n \setminus \overline{\Omega}) \rightarrow \mathcal{B}(\partial\Omega)$  to denote the boundary trace operator for  $\mathbb{R}^n \setminus \overline{\Omega}$ ; by construction both map onto  $\mathcal{B}(\partial\Omega)$ . We refer to  $\text{Tr}_i$  as the *interior* trace operator with respect to  $\Omega$  and to  $\text{Tr}_e$  as the *exterior* one. In the context of transmission problems, we endow  $\mathcal{B}(\partial\Omega)$  with the Hilbert space norm

$$(4.2) \quad \|f\|_{\mathcal{B}(\partial\Omega),t} := (\|f\|_{\mathcal{B}(\partial\Omega),i}^2 + \|f\|_{\mathcal{B}(\partial\Omega),e}^2)^{1/2},$$

where  $\|\cdot\|_{\mathcal{B}(\partial\Omega),i}$  denotes the norm defined in (2.4) with respect to  $\Omega$  and  $\|\cdot\|_{\mathcal{B}(\partial\Omega),e}$  denotes the norm defined similarly but with  $\mathbb{R}^n \setminus \overline{\Omega}$  in place of  $\Omega$ . Since both  $\Omega$  and  $\mathbb{R}^n \setminus \overline{\Omega}$  are  $H^1$ -extension domains, the norms  $\|\cdot\|_{\mathcal{B}(\partial\Omega),i}$ ,  $\|\cdot\|_{\mathcal{B}(\partial\Omega),e}$  and  $\|\cdot\|_{\mathcal{B}(\partial\Omega),t}$  are all equivalent. As a consequence, we may view  $\text{Tr}_i$  and  $\text{Tr}_e$  as bounded linear operators from  $H^1(\mathbb{R}^n \setminus \partial\Omega)$  onto  $\mathcal{B}(\partial\Omega)$ , no matter which norm is used. Only  $\|\cdot\|_{\mathcal{B}(\partial\Omega),t}$  will be used in the sequel; we agree to denote it by  $\|\cdot\|_{\mathcal{B}(\partial\Omega)}$  again.

Given  $u \in H^1(\mathbb{R}^n \setminus \partial\Omega)$ , we write

$$[\text{Tr } u] := \text{Tr}_i u - \text{Tr}_e u$$

for its *jump in trace across  $\partial\Omega$* . The map  $u \mapsto [\text{Tr } u]$  defines a bounded linear operator  $[\text{Tr}] : H^1(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  and is onto. In the same spirit, we write  $[\text{tr}] := \text{tr}_i - \text{tr}_e : V_1(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ , also bounded, linear and onto.

If  $\Delta u \in L^2(\Omega)$ , then the *interior* normal derivative  $\frac{\partial_i u}{\partial\nu}$  of  $u$  with respect to  $\Omega$  is as defined in (2.14). If  $\Delta u \in L^2(\mathbb{R}^n \setminus \overline{\Omega})$ , then we define the *exterior* normal derivative  $\frac{\partial_e u}{\partial\nu}$  of  $u$  with respect to  $\Omega$  as minus the interior normal derivative of  $u$  with respect to  $\mathbb{R}^n \setminus \overline{\Omega}$ , that is, the unique element  $g \in \mathcal{B}'(\partial\Omega)$  such that

$$(4.3) \quad \langle g, \text{Tr}_e v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = - \int_{\mathbb{R}^n \setminus \overline{\Omega}} (\Delta u) v \, dx - \int_{\mathbb{R}^n \setminus \overline{\Omega}} \nabla u \cdot \nabla v \, dx, \quad v \in H^1(\mathbb{R}^n \setminus \overline{\Omega}).$$

Now let

$$H_\Delta^1(\mathbb{R}^n \setminus \partial\Omega) := \{u \in H^1(\mathbb{R}^n \setminus \partial\Omega) \mid \Delta u \in L^2(\mathbb{R}^n \setminus \partial\Omega)\}.$$

Clearly this space contains  $V_1(\mathbb{R}^n \setminus \partial\Omega)$ . For  $u \in H_\Delta^1(\mathbb{R}^n \setminus \partial\Omega)$  we write

$$\left[ \frac{\partial u}{\partial\nu} \right] := \frac{\partial_i u}{\partial\nu} - \frac{\partial_e u}{\partial\nu}$$

for the *jump of its normal derivative across  $\partial\Omega$* . This defines a linear operator  $u \mapsto \left[ \frac{\partial u}{\partial\nu} \right]$  from  $H_\Delta^1(\mathbb{R}^n \setminus \partial\Omega)$  onto  $\mathcal{B}'(\partial\Omega)$ , bounded in the sense that

$$\left\| \left[ \frac{\partial u}{\partial\nu} \right] \right\|_{\mathcal{B}'(\partial\Omega)} \leq \|u\|_{H^1(\mathbb{R}^n \setminus \partial\Omega)} + \|\Delta u\|_{L^2(\mathbb{R}^n \setminus \partial\Omega)}.$$

We call  $\Omega \subset \mathbb{R}^n$  a *two-sided  $\dot{H}^1$ -admissible domain* if both  $\Omega$  and  $\mathbb{R}^n \setminus \overline{\Omega}$  are nonempty and  $\dot{H}^1$ -extension domains,  $\Omega$  is bounded and the Lebesgue measure of  $\partial\Omega$  is zero.

Suppose that  $\Omega \subset \mathbb{R}^n$  is a two-sided  $\dot{H}^1$ -admissible domain. We discuss (4.1) with  $k = 0$  using the space  $\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$ , which decomposes as  $\dot{H}^1(\Omega) \oplus \dot{H}^1(\mathbb{R}^n \setminus \bar{\Omega})$ . By  $\dot{H}_0^1(\mathbb{R}^n \setminus \partial\Omega)$ , we denote the closure of  $C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$  in  $\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  and by  $\dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$ , its orthogonal complement.

By  $\dot{\text{Tr}}_i : \dot{H}^1(\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$  and  $\dot{\text{Tr}}_e : \dot{H}^1(\mathbb{R}^n \setminus \bar{\Omega}) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$ , we denote the interior and exterior trace with respect to  $\Omega$ , defined in an analogous manner. We endow  $\dot{\mathcal{B}}(\partial\Omega)$  with the Hilbert space norm

$$\|f\|_{\dot{\mathcal{B}}(\partial\Omega),t} := (\|f\|_{\dot{\mathcal{B}}(\partial\Omega),i}^2 + \|f\|_{\dot{\mathcal{B}}(\partial\Omega),e}^2)^{1/2}$$

with the summands defined similarly as before. Only  $\|\cdot\|_{\dot{\mathcal{B}}(\partial\Omega),t}$  will be used in the sequel; we agree to denote it by  $\|\cdot\|_{\dot{\mathcal{B}}(\partial\Omega)}$  again.

Given  $u \in \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$ , we set

$$\llbracket \dot{\text{Tr}} u \rrbracket := \dot{\text{Tr}}_i u - \dot{\text{Tr}}_e u$$

for its jump in trace across  $\Omega$  in the  $\dot{H}^1$ -sense. The map  $\llbracket \dot{\text{Tr}} \rrbracket : \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$ ,  $u \mapsto \llbracket \dot{\text{Tr}} u \rrbracket$ , defines a bounded linear operator which is onto. In the same spirit, we write  $\llbracket \dot{\text{tr}} \rrbracket := \dot{\text{tr}}_i - \dot{\text{tr}}_e : \dot{V}_0(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$ , also bounded, linear and onto.

Let  $\dot{H}_\Delta^1(\mathbb{R}^n \setminus \partial\Omega)$  denote the space of all  $u \in \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  with  $\Delta u \in L^2(\mathbb{R}^n \setminus \partial\Omega)$ ; it contains  $\dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$ . Given an element  $u \in \dot{H}_\Delta^1(\mathbb{R}^n \setminus \partial\Omega)$ , we write  $\frac{\partial_i u}{\partial \nu}$  for its interior normal derivative in the  $\dot{H}^1$ -sense with respect to  $\Omega$  and  $\frac{\partial_e u}{\partial \nu}$  for its exterior, again defined as minus the interior with respect to  $\mathbb{R}^n \setminus \bar{\Omega}$ . We then write

$$\llbracket \frac{\partial u}{\partial \nu} \rrbracket := \frac{\partial_i u}{\partial \nu} - \frac{\partial_e u}{\partial \nu}$$

for the jump of its normal derivative across  $\partial\Omega$  in the  $\dot{H}^1$ -sense. This defines a linear operator  $u \mapsto \llbracket \frac{\partial u}{\partial \nu} \rrbracket$  from  $\dot{H}_\Delta^1(\mathbb{R}^n \setminus \partial\Omega)$  onto  $\dot{\mathcal{B}}'(\partial\Omega)$ , bounded in the sense that

$$\left\| \llbracket \frac{\partial u}{\partial \nu} \rrbracket \right\|_{\dot{\mathcal{B}}'(\partial\Omega)} \leq \|u\|_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} + \|\Delta u\|_{L^2(\mathbb{R}^n \setminus \partial\Omega)}.$$

**Remark 4.1.** *By elliptic regularity, any element of  $V_1(\mathbb{R}^n \setminus \partial\Omega)$  and any representative modulo (two) constants of any element  $\dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$  are in  $C^\infty(\mathbb{R}^n \setminus \partial\Omega)$ .*

**4.2. Subspaces and orthogonality.** Suppose that  $\Omega \subset \mathbb{R}^n$  is a two-sided  $H^1$ -admissible domain. We write

$$V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega) := \{u \in V_1(\mathbb{R}^n \setminus \partial\Omega) \mid \llbracket \text{tr} u \rrbracket = 0\}$$

and

$$V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega) := \{u \in V_1(\mathbb{R}^n \setminus \partial\Omega) \mid \llbracket \frac{\partial u}{\partial \nu} \rrbracket = 0\}.$$

**Lemma 4.2.** *Let  $\Omega \subset \mathbb{R}^n$  be two-sided  $H^1$ -admissible. Then the following statements hold:*

- (i) *We have  $\ker \llbracket \text{Tr} \rrbracket = H^1(\mathbb{R}^n)$ . The linear operator  $\text{Tr} : H^1(\mathbb{R}^n) \rightarrow \mathcal{B}(\partial\Omega)$ , defined as  $\text{Tr} := \text{Tr}_i|_{H^1(\mathbb{R}^n)} = \text{Tr}_e|_{H^1(\mathbb{R}^n)}$ , is bounded with operator norm one. Its kernel agrees with  $H_0^1(\mathbb{R}^n \setminus \partial\Omega)$ , which is the orthogonal complement of  $V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  in  $H^1(\mathbb{R}^n)$ .*
- (ii) *The space  $V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  is a closed subspace of  $H^1(\mathbb{R}^n \setminus \partial\Omega)$ . The operator*

$$(4.4) \quad \text{tr} : V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \mathcal{B}(\partial\Omega),$$

*defined as the restriction  $\text{tr} := \text{Tr}|_{V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)}$ , is a linear isometry and onto.*

- (iii) *The space  $V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  is a closed subspace of  $H^1(\mathbb{R}^n \setminus \partial\Omega)$ . The operator*

$$(4.5) \quad \frac{\partial}{\partial \nu} : V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega),$$

*defined as  $\frac{\partial}{\partial \nu} := \frac{\partial_i}{\partial \nu} = \frac{\partial_e}{\partial \nu}$ , is a linear isometry and onto.*

- (iv) *The space  $V_1(\mathbb{R}^n \setminus \partial\Omega)$  admits the orthogonal decomposition*

$$V_1(\mathbb{R}^n \setminus \partial\Omega) = V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega) \oplus V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega).$$

*Proof.* For (i), note that  $u \in V_1(\mathbb{R}^n \setminus \Omega) \cap \ker \llbracket \text{Tr} \rrbracket$  agrees with the unique element  $w$  of  $H^1(\mathbb{R}^n)$  of minimal norm such that  $w|_{\partial\Omega} = \text{Tr}_i u = \text{Tr}_i u$ . The remaining part of (i) now follows by using (4.2) and (A.1). Statements (ii) and (iii) follow from Theorem 2.6 (iii) and Corollary 2.15 (i). To prove (iv), note that given  $u \in V_1(\mathbb{R}^n \setminus \partial\Omega)$ , we find that for any  $v \in V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$ , we have

$$(4.6) \quad \left\langle \left[ \frac{\partial u}{\partial \nu} \right], \text{tr } v \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \int_{\mathbb{R}^n \setminus \partial\Omega} (\Delta u) v \, dx + \int_{\mathbb{R}^n \setminus \partial\Omega} \nabla u \nabla v \, dx = \langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)}.$$

Since  $\text{tr}$  in (4.4) is surjective, it follows that  $\left[ \frac{\partial u}{\partial \nu} \right] = 0$  in  $\mathcal{B}'(\partial\Omega)$  if and only if (4.6) is zero for all  $v \in V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$ , and this is the case if and only if  $u$  belongs to the orthogonal complement of  $V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  in  $V_1(\mathbb{R}^n \setminus \partial\Omega)$ .  $\square$

**Remark 4.3.** *Alternatively, we could observe that given  $v \in V_1(\mathbb{R}^n \setminus \partial\Omega)$ , we have*

$$(4.7) \quad \left\langle \frac{\partial u}{\partial \nu}, \llbracket \text{tr } v \rrbracket \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)}.$$

for all  $u \in V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$ . Since  $\frac{\partial}{\partial \nu}$  in (4.5) is surjective and  $\mathcal{B}(\partial\Omega)$  is a Hilbert space, it follows that  $\llbracket \text{tr } v \rrbracket = 0$  in  $\mathcal{B}(\partial\Omega)$  if and only if  $v$  belongs to the orthogonal complement of  $V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  in  $V_1(\mathbb{R}^n \setminus \partial\Omega)$ .

By  $P_{1,\mathcal{S}}$  and  $P_{1,\mathcal{D}}$ , we denote the orthogonal projections from  $H^1(\mathbb{R}^n \setminus \partial\Omega)$  onto the closed subspaces  $V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  and  $V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  respectively. The following is a straightforward consequence of Lemma 4.2 (iv).

**Corollary 4.4.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. If  $u, v \in V_1(\mathbb{R}^n \setminus \partial\Omega)$ , then*

$$\langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = \left\langle \left[ \frac{\partial u}{\partial \nu} \right], \text{tr} \circ P_{1,\mathcal{S}} v \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} + \left\langle \frac{\partial}{\partial \nu} \circ P_{1,\mathcal{D}} v, \llbracket \text{tr } u \rrbracket \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}.$$

Now suppose that  $\Omega \subset \mathbb{R}^n$  is a two-sided  $H^1$ -admissible domain. We write

$$\dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega) := \{u \in \dot{V}_0(\mathbb{R}^n \setminus \partial\Omega) \mid \llbracket \text{tr } u \rrbracket = 0\}$$

and

$$\dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega) := \{u \in \dot{V}_0(\mathbb{R}^n \setminus \partial\Omega) \mid \left[ \frac{\partial u}{\partial \nu} \right] = 0\}.$$

The following is seen similarly as before.

**Lemma 4.5.** *Let  $\Omega \subset \mathbb{R}^n$  be two-sided  $\dot{H}^1$ -admissible. Then the following statements hold:*

- (i) *The linear operator  $\dot{\text{Tr}} : \ker \llbracket \dot{\text{Tr}} \rrbracket \rightarrow \dot{\mathcal{B}}(\partial\Omega)$  defined as  $\dot{\text{Tr}} := \dot{\text{Tr}}_i = \dot{\text{Tr}}_e$  is bounded with operator norm one. Its kernel is the orthogonal complement of  $\dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  in  $\ker \llbracket \dot{\text{Tr}} \rrbracket$ , and this complement is the closure of  $C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$ .*
- (ii) *The space  $\dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  is a closed subspace of  $\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$ . The operator*

$$\dot{\text{tr}} : \dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega),$$

*defined as the restriction  $\dot{\text{tr}} := \dot{\text{Tr}}|_{\dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)}$ , is a linear isometry and onto.*

- (iii) *The space  $\dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  is a closed subspace of  $\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$ . The operator*

$$\frac{\partial}{\partial \nu} : \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega),$$

*defined as  $\frac{\partial}{\partial \nu} := \frac{\partial_i}{\partial \nu} = \frac{\partial_e}{\partial \nu}$ , is a linear isometry and onto.*

- (iv) *The space  $\dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$  admits the orthogonal decomposition*

$$\dot{V}_0(\mathbb{R}^n \setminus \partial\Omega) = \dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega) \oplus \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega).$$

By  $P_{0,\mathcal{S}}$  and  $P_{0,\mathcal{D}}$ , we denote the orthogonal projections from  $\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  onto the closed subspaces  $\dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  and  $\dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  respectively. Lemma 4.2 (iv) now gives the following.

**Corollary 4.6.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. If  $u, v \in \dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$ , then*

$$\langle u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = \left\langle \left[ \frac{\partial u}{\partial \nu} \right], \dot{\text{tr}} \circ \text{P}_{0, \dot{\mathcal{S}}} v \right\rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} + \left\langle \frac{\partial}{\partial \nu} \circ \text{P}_{0, \dot{\mathcal{D}}} v, \llbracket \dot{\text{tr}} u \rrbracket \right\rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}.$$

**Remark 4.7.** *Suppose that  $\Omega$  is two-sided  $\dot{H}^1$ -admissible and  $u \in \ker \llbracket \dot{\text{Tr}} \rrbracket$ . Let  $w_i \in u_i$  and  $w_e \in u_e$  be representatives modulo constants of  $u_i$  and  $u_e$ . By the arguments used to show Proposition 2.2, both  $\text{Tr}_i w_i$  and  $\text{Tr}_e w_e$  are well-defined elements of  $\mathcal{B}(\partial\Omega)$ , and by construction, there is a constant  $c \in \mathbb{R}$  such that  $\text{Tr}_i w_i - \text{Tr}_e w_e = c$ . Setting  $\bar{w} := w_i + w_e + c\mathbf{1}_{\mathbb{R}^n \setminus \bar{\Omega}}$  we obtain a representative  $\bar{w}$  of  $u$  which satisfies  $\llbracket \text{Tr} \bar{w} \rrbracket = 0$  in  $\mathcal{B}(\partial\Omega)$ . The equivalence class  $\bar{u}$  modulo single constants is a uniquely determined element of  $\dot{H}^1(\mathbb{R}^n)$ , which can be called the zero trace jump readjusted representative of  $u$ .*

*The zero trace jump readjusted representative  $\bar{u}$  of  $u \in \dot{V}_{0, \dot{\mathcal{S}}}(\mathbb{R}^n \setminus \partial\Omega)$  is an element of the orthogonal complement of  $C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$  in  $\dot{H}^1(\mathbb{R}^n)$ ; we denote it by  $\bar{V}_{0, \dot{\mathcal{S}}}(\mathbb{R}^n \setminus \partial\Omega)$ . Setting  $\bar{\text{tr}} \bar{u} := \dot{\text{tr}} u$ , we obtain a linear bijection*

$$(4.8) \quad \bar{\text{tr}} : \bar{V}_{0, \dot{\mathcal{S}}}(\mathbb{R}^n \setminus \partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega).$$

**4.3. Double layer potentials.** Let  $\Omega \subset \mathbb{R}^n$  be two-sided  $H^1$ -admissible. Given  $f \in \mathcal{B}(\partial\Omega)$ , we call  $u \in H^1(\mathbb{R}^n \setminus \partial\Omega)$  a *weak solution* of (4.1) with  $k = 1$  and  $g = 0$ , that is, a weak solution to

$$(4.9) \quad \begin{cases} -\Delta u + u & = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \\ \frac{\partial_i u_i}{\partial \nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial \nu}|_{\partial\Omega} & = 0, \end{cases}$$

if we have  $\langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = 0$  for all  $v \in C_c^\infty(\mathbb{R}^n \setminus \partial\Omega) \cup V_{1, \mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  and  $\langle v, u \rangle_{H^1(\mathbb{R}^n \setminus \Omega)} = -\langle \frac{\partial v}{\partial \nu}, f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$  for all  $v \in V_{1, \mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$ .

**Lemma 4.8.** *Let  $\Omega \subset \mathbb{R}^n$  be two-sided  $H^1$ -admissible. For any  $f \in \mathcal{B}(\partial\Omega)$  there is a unique weak solution  $u^f$  of (4.9). It is an element of  $V_{1, \mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  and satisfies  $\|u^f\|_{H^1(\mathbb{R}^n \setminus \partial\Omega)} \leq \|f\|_{\mathcal{B}(\partial\Omega)}$ .*

*Proof.* Since, by Lemma 4.2 (iii), the linear functional  $v \mapsto \langle \frac{\partial v}{\partial \nu}, f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$  is bounded on the closed subspace  $V_{1, \mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  of  $H^1(\mathbb{R}^n \setminus \partial\Omega)$ , the result follows from the Riesz representation theorem.  $\square$

We refer to the bounded linear operator  $\mathcal{D} : \mathcal{B}(\partial\Omega) \rightarrow V_{1, \mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  defined by

$$\mathcal{D}f := u^f$$

as the *double layer potential operator associated with the transmission problem for  $1 - \Delta$  and  $\Omega$* .

**Corollary 4.9.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. The operator  $\mathcal{D}$  is bijective, and its inverse is  $\mathcal{D}^{-1} = -\llbracket \text{tr} \rrbracket$ .*

Now let  $\Omega \subset \mathbb{R}^n$  be two-sided  $\dot{H}^1$ -admissible. Given  $f \in \dot{\mathcal{B}}(\partial\Omega)$ , we call  $u \in \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  a *weak solution in the  $\dot{H}^1$ -sense* of (4.1) with  $k = 0$  and  $g = 0$ , that is,

$$(4.10) \quad \begin{cases} -\Delta u & = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \\ \frac{\partial_i u_i}{\partial \nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial \nu}|_{\partial\Omega} & = 0, \end{cases}$$

if it satisfies  $\langle u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = 0$  for all  $v \in C_c^\infty(\mathbb{R}^n \setminus \partial\Omega) \cup \dot{V}_{0, \dot{\mathcal{S}}}(\mathbb{R}^n \setminus \partial\Omega)$  and  $\langle v, u \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \Omega)} = -\langle \frac{\partial v}{\partial \nu}, f \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}$  for all  $v \in \dot{V}_{0, \dot{\mathcal{D}}}(\mathbb{R}^n \setminus \partial\Omega)$ .

**Lemma 4.10.** *Let  $\Omega \subset \mathbb{R}^n$  be two-sided  $\dot{H}^1$ -admissible. For any  $f \in \dot{\mathcal{B}}(\partial\Omega)$ , there is a unique weak solution  $u^f$  of (4.10) in the  $\dot{H}^1$ -sense. It is an element of  $\dot{V}_{0, \dot{\mathcal{D}}}(\mathbb{R}^n \setminus \partial\Omega)$  and satisfies  $\|u^f\|_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} \leq \|f\|_{\dot{\mathcal{B}}(\partial\Omega)}$ .*

We refer to the bounded linear operator  $\dot{\mathcal{D}} : \dot{\mathcal{B}}(\partial\Omega) \rightarrow \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  defined by  $\dot{\mathcal{D}}f := u^f$  as the *double layer potential operator associated with the transmission problem for  $-\Delta$  and  $\Omega$  in the  $\dot{H}^1$ -sense*.

**Corollary 4.11.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. The operator  $\dot{\mathcal{D}}$  is bijective, and its inverse is  $\dot{\mathcal{D}}^{-1} = -[[\text{tr}]]$ .*

**Remark 4.12.** *A priori, the trace jump condition in (4.10) is only an equality modulo additive constants. However, by arguments similar to Remarks 2.12 and 4.7, the situation is auto-improving: if  $\Omega$  is two-sided  $\dot{H}^1$ -admissible and  $f \in \mathcal{B}(\partial\Omega)$  is given, let  $u^{[f]}$  be the unique weak solution in the  $\dot{H}^1$ -sense of (4.10) with the equivalence class  $[f] \in \dot{\mathcal{B}}'(\partial\Omega)$  of  $f$  modulo constants in place of  $f$ . We can find representatives  $w_i \in u_i^{[f]}$  and  $w_e \in u_e^{[f]}$  modulo constants and some  $c \in \mathbb{R}$  such that  $\text{Tr}_i w_i - \text{Tr}_e w_e = -f + c$  in  $\mathcal{B}(\partial\Omega)$ . We set  $\bar{w} := w_i + w_e + c\mathbb{1}_{\mathbb{R}^n \setminus \bar{\Omega}}$ . The equivalence class  $\bar{w}^{[f]}$  of  $\bar{w}$  modulo a single constant is uniquely determined. It satisfies  $\Delta \bar{w}^{[f]} = 0$  on  $\mathbb{R}^n \setminus \partial\Omega$ ,  $[[\text{tr} \bar{w}^{[f]}]] = -f$  as an equality in  $\mathcal{B}(\partial\Omega)$  and  $[[\frac{\partial \bar{w}^{[f]}}{\partial \nu}]] = 0$ , which by Remark 2.16 may be interpreted as an equality of linear functionals on  $\mathcal{B}(\partial\Omega)$ . One could call  $\bar{w}^{[f]}$  the trace jump readjusted solution to (4.10).*

**4.4. Single layer potentials.** Let  $\Omega$  be two-sided  $H^1$ -admissible. Given  $g \in \mathcal{B}'(\partial\Omega)$ , we call  $u \in H^1(\mathbb{R}^n \setminus \partial\Omega)$  a *weak solution* of (4.1) with  $k = 1$  and  $f = 0$ , that is,

$$(4.11) \quad \begin{cases} -\Delta u + u & = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = 0 \\ \frac{\partial_i u_i}{\partial \nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial \nu}|_{\partial\Omega} & = g, \end{cases}$$

if  $\langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = 0$  for all  $v \in V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  and  $\langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle g, \text{tr } v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$  for all  $v \in H^1(\mathbb{R}^n)$ .

**Lemma 4.13.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. For any  $g \in \mathcal{B}'(\partial\Omega)$ , there is a unique weak solution  $u_g$  of (4.11). It is in  $V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  and satisfies  $\|u_g\|_{H^1(\mathbb{R}^n \setminus \partial\Omega)} \leq \|g\|_{\mathcal{B}'(\partial\Omega)}$ .*

*Proof.* By Lemma 4.2 (ii) the linear functional  $v \mapsto \langle g, \text{tr } v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$  is bounded on the closed subspace  $V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  of  $H^1(\mathbb{R}^n \setminus \partial\Omega)$ , so the result follows from the Riesz representation theorem once again.  $\square$

We refer to the bounded linear operator  $\mathcal{S} : \mathcal{B}'(\partial\Omega) \rightarrow V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$ , defined by

$$\mathcal{S}g := u_g$$

as the *single layer potential operator associated with the transmission problem for  $1 - \Delta$  and  $\Omega$* .

**Corollary 4.14.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. The operator  $\mathcal{S}$  is bijective, and its inverse is  $\mathcal{S}^{-1} = [[\frac{\partial}{\partial \nu}]]$ .*

Suppose that  $\Omega$  is two-sided  $\dot{H}^1$ -admissible. Given  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , we call  $u \in \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  a *weak solution in the  $\dot{H}^1$ -sense* of (4.1) with  $k = 0$  and  $f = 0$ , that is,

$$(4.12) \quad \begin{cases} -\Delta u & = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = 0 \\ \frac{\partial_i u_i}{\partial \nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial \nu}|_{\partial\Omega} & = g, \end{cases}$$

if  $\langle u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = 0$  for all  $v \in \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  and  $\langle u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle g, \dot{\text{Tr}} v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}$  for all  $v \in \dot{H}^1(\mathbb{R}^n)$ .

**Lemma 4.15.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. For any  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , there is a unique weak solution  $u_g$  of (4.12) in the  $\dot{H}^1$ -sense. It is an element of  $\dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  and satisfies  $\|u_g\|_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} \leq \|g\|_{\dot{\mathcal{B}}'(\partial\Omega)}$ .*



We refer to the bounded linear operator  $\dot{\mathcal{S}} : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  defined by  $\dot{\mathcal{S}}g := u_g$  as the *single layer potential operator associated with the transmission problem for  $-\Delta$  and  $\Omega$  in the  $\dot{H}^1$ -sense*.

**Corollary 4.16.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. The operator  $\dot{\mathcal{S}}$  is bijective, and its inverse is  $\dot{\mathcal{S}}^{-1} = \llbracket \frac{\partial}{\partial\nu} \rrbracket$ .*

**Remark 4.17.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible,  $g \in \dot{\mathcal{B}}'(\partial\Omega)$  and let  $u_g$  be as above. Let  $\bar{u}_g \in \bar{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  be the zero trace jump readjusted representative of  $u_g$  as in Remark 4.7. Since it is uniquely determined,  $\bar{\mathcal{S}}g := \bar{u}_g$  defines a bounded linear map*

$$(4.13) \quad \bar{\mathcal{S}} : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \bar{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega);$$

*it is a zero trace jump readjusted variant of  $\dot{\mathcal{S}}$ .*

**4.5. Superposition.** Superposition gives well-posedness for (4.1). Let  $\Omega$  be  $H^1$ -admissible. Given  $f \in \mathcal{B}(\partial\Omega)$  and  $g \in \mathcal{B}'(\partial\Omega)$ , we call  $u \in H^1(\mathbb{R}^n \setminus \partial\Omega)$  a *weak solution* for (4.1) with  $k = 1$  if  $\langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = 0$  for all  $v \in C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$ ,  $\langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle g, \text{tr } v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$  for all  $v \in V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  and  $\langle u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle \llbracket \frac{\partial v}{\partial\nu} \rrbracket, f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$  for all  $v \in V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$ .

**Corollary 4.18.** *Let  $\Omega$  be  $H^1$ -admissible. For any  $f \in \mathcal{B}(\partial\Omega)$  and  $g \in \mathcal{B}'(\partial\Omega)$ , the unique weak solution  $u$  of (4.1) with  $k = 1$  is  $u = \mathcal{S}g - \mathcal{D}f$ . It is an element of  $V_1(\mathbb{R}^n \setminus \partial\Omega)$  and satisfies  $\|u\|_{H^1(\mathbb{R}^n \setminus \partial\Omega)} \leq \|f\|_{\mathcal{B}(\partial\Omega)} + \|g\|_{\mathcal{B}'(\partial\Omega)}$ .*

If  $\Omega$  is  $\dot{H}^1$ -admissible,  $f \in \dot{\mathcal{B}}(\partial\Omega)$  and  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , then we call  $u \in \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  a *weak solution* for (4.1) with  $k = 0$  in the  $\dot{H}^1$ -sense if  $\langle u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = 0$  for all  $v \in C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$ ,  $\langle u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle g, \text{tr } v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}$  for all  $v \in \dot{V}_{0,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$  and we have  $\langle u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle \llbracket \frac{\partial v}{\partial\nu} \rrbracket, f \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}$  for all  $v \in \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$ .

**Corollary 4.19.** *Let  $\Omega$  be  $\dot{H}^1$ -admissible. For any  $f \in \dot{\mathcal{B}}(\partial\Omega)$  and  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , the unique weak solution  $u$  in the  $\dot{H}^1$ -sense of (4.1) with  $k = 0$  is  $u = \dot{\mathcal{S}}g - \dot{\mathcal{D}}f$ . It is an element of  $\dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$  and satisfies  $\|u\|_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} \leq \|f\|_{\dot{\mathcal{B}}(\partial\Omega)} + \|g\|_{\dot{\mathcal{B}}'(\partial\Omega)}$ .*

## 5. RESOLVENT REPRESENTATIONS

We give resolvent representations of the layer potential operators defined in the variational sense in Section 4. This allows to recover the classical integral formulas when the domain  $\Omega$  is a Lipschitz domain and the boundary is endowed with the surface measure.

**5.1. Representations of single layer potentials.** Let  $\mathcal{G} : u \mapsto \mathcal{G}u = ((1 + |\xi|^2)^{-1}\hat{u})^\vee$  be the Bessel potential operator of order 2, where  $u \mapsto \hat{u}$  denotes the Fourier transform and  $u \mapsto \tilde{u}$  its inverse, a priori defined on Schwartz functions. It is well-known that  $\mathcal{G} = (I - \Delta)^{-1}$  on Schwartz functions and that  $\mathcal{G}$  extends to a bounded linear operator of  $L^2(\mathbb{R}^2)$ .

If  $\Omega \subset \mathbb{R}^n$  is two-sided  $H^1$ -admissible, then this operator  $\mathcal{G}$  is the resolvent operator uniquely associated with the symmetric bilinear form (2.1) when endowed with the domain  $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus V_{1,\mathcal{S}}(\mathbb{R}^n \setminus \partial\Omega)$ .

The operator  $\mathcal{G}$  extends further to an isometric isomorphism  $H^{-1}(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}^n)$ . Using a similar agreement as in Remark 2.8 together with Lemma 4.2 (ii) the adjoint  $\text{tr}^* : \mathcal{B}'(\partial\Omega) \rightarrow V_{1,\mathcal{S}}'(\mathbb{R}^n \setminus \partial\Omega)$  of the restricted trace operator  $\text{tr}$  as in Lemma 4.2 (iii) can be viewed as a bounded linear operator  $\text{tr}^* : \mathcal{B}'(\partial\Omega) \rightarrow H^{-1}(\mathbb{R}^n)$ . We obtain the following representation for the single layer potential operator.

**Lemma 5.1.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. Then  $\mathcal{S} = \mathcal{G} \circ \text{tr}^*$ .*

*Proof.* Let  $g \in \mathcal{B}'(\partial\Omega)$ . Then  $\text{tr}^*g \in H^{-1}(\mathbb{R}^n)$  and consequently  $\mathcal{G} \circ \text{tr}^*g \in H^1(\mathbb{R}^n)$ . Using the fact that  $\mathcal{G} : H^{-1}(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}^n)$  is a Riesz isometry, together with (2.5) and the definition of  $\mathcal{S}$  we find that

$$(5.1) \quad \langle \mathcal{G} \circ \text{tr}^*g, v \rangle_{H^1(\mathbb{R}^n)} = \langle \text{tr}^*g, v \rangle_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} = \langle g, \text{Tr } v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \langle \mathcal{S}g, v \rangle_{H^1(\mathbb{R}^n)}$$

for all  $v \in H^1(\mathbb{R}^n)$ .  $\square$

Let  $G$  be the Bessel kernel of order two, that is, the fundamental solution for  $\Delta - 1$  on  $\mathbb{R}^n$ . It is well-known that for any  $w \in L^2(\mathbb{R}^n)$ , we have  $\mathcal{G}w = G * w$ . For a nonnegative Radon measure  $\nu$  on  $\partial\Omega$ , one defines

$$G * \nu(x) := \int_{\partial\Omega} G(x-y)\nu(dy), \quad x \in \mathbb{R}^n;$$

this is a lower semi-continuous function taking values in  $[0, +\infty]$ , finite at all points  $x \in \mathbb{R}^n \setminus \partial\Omega$ . For a finite signed Borel measure  $\nu$  on  $\partial\Omega$ , we set

$$G * \nu(x) := G * \nu^+(x) - G * \nu^-(x), \quad x \in \mathbb{R}^n \setminus \partial\Omega;$$

this gives a well-defined integrable function  $G * \nu$  on  $\mathbb{R}^n$ .

We say that a nonnegative Radon measure  $\nu$  on  $\partial\Omega$  has *finite energy* if there is a constant  $c > 0$  such that

$$(5.2) \quad \int_{\partial\Omega} |v| d\nu \leq c \|v\|_{H^1(\mathbb{R}^n)}, \quad v \in H^1(\mathbb{R}^n) \cap C_c(\mathbb{R}^n).$$

Given a finite signed Borel measure  $\nu$  on  $\partial\Omega$ , we say that it has finite energy if its total variation measure  $\nu^+ + \nu^-$  has finite energy.

**Remark 5.2.** *It is well-known that, by the Riesz representation theorem for measures, the cone of nonnegative elements of  $H^{-1}(\mathbb{R}^n)$  is in one-to-one correspondence with the cone of nonnegative Radon measures of finite energy on  $\mathbb{R}^n$ . This can be seen using [87, Chapter 6, Exercise 4]; a variant of the argument is provided in [16, Proposition 9.2.1].*

**Proposition 5.3.** *Let  $\Omega$  be two-sided  $H^1$ -admissible and let  $\nu$  be a finite signed Borel measure on  $\partial\Omega$  of finite energy. Then sets of zero capacity have zero  $\nu$ -measure, and  $\nu$  defines an element of  $\mathcal{B}'(\partial\Omega)$  by*

$$(5.3) \quad \langle \nu, f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} := \int_{\partial\Omega} f d\nu, \quad f \in \mathcal{B}(\partial\Omega).$$

Moreover,  $\mathcal{S}\nu(x) = G * \nu(x)$ ,  $x \in \mathbb{R}^n \setminus \partial\Omega$ .

*Proof.* It suffices to prove the result for a nonnegative Radon measure  $\nu$  of finite energy. The first claim is proved in [43, Lemma 2.2.3]. One can follow [43, Theorem 2.2.2] to see that

$$(5.4) \quad \int_{\partial\Omega} \tilde{v} d\nu = \langle G * \nu, v \rangle_{H^1(\mathbb{R}^n)}, \quad v \in H^1(\mathbb{R}^n).$$

Estimate (5.2) extends to all  $v \in H^1(\mathbb{R}^n)$  and gives  $|\int_{\partial\Omega} \tilde{v} d\nu| \leq c\|v\|_{H^1(\mathbb{R}^n)}$ ; optimizing over  $v$  gives  $|\langle \nu, f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}| \leq c\|f\|_{\mathcal{B}(\partial\Omega), t}$ ,  $f \in \mathcal{B}(\partial\Omega)$ , with a (different) constant  $c > 0$ . Consequently,  $\nu \in \mathcal{B}'(\partial\Omega)$ . The rest now follows from (5.1), (5.3) and (5.4).  $\square$

**Corollary 5.4.** *Let  $\Omega$  be two-sided  $H^1$ -admissible and let  $\mu$  be a nonnegative Radon measure on  $\partial\Omega$  such that  $\mathcal{B}(\partial\Omega) \subset L^2(\partial\Omega, \mu)$  is a continuous embedding, see [15]. Then  $L^2(\partial\Omega, \mu) \subset \mathcal{B}'(\partial\Omega)$ , for any  $g \in L^2(\partial\Omega, \mu)$ , the finite signed measure  $g \cdot \mu$  is of finite energy and*

$$(5.5) \quad \mathcal{S}g(x) = \int_{\partial\Omega} G(x-y)g(y)\mu(dy), \quad x \in \mathbb{R}^n \setminus \partial\Omega.$$

**Remark 5.5.**

- (i) *Suppose that  $\partial\Omega$  is compact,  $\text{supp } \nu = \partial\Omega$ ,  $n-2 < d < n$ ,  $c > 0$  and  $\nu^\pm(B(x, r)) \leq cr^d$  for all  $x \in \partial\Omega$ ,  $0 < r < 1$ . Then  $x \mapsto G * \nu(x)$  is bounded and continuous on  $\mathbb{R}^n$ .*
- (ii) *If  $\Omega$  is a bounded Lipschitz domain and  $\mu$  is the surface measure  $\sigma$  on  $\partial\Omega$ , then  $\mathcal{B}'(\partial\Omega)$  equals  $H^{-1/2}(\partial\Omega)$ , and for any  $g \in L^2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$ , the right-hand side of (5.5) is a weak solution to (4.11). In that case (5.5) can alternatively be derived from the uniqueness part of Lemma 4.13.*

(iii) Corollary 5.4 implies that, under suitable conditions on measure  $\mu$ , see [15], we obtain a so-called Gelfand triple  $\mathcal{B}(\partial\Omega) \hookrightarrow L^2(\partial\Omega, \mu) \hookrightarrow \mathcal{B}'(\partial\Omega)$ , also referred to as a rigged Hilbert space  $(\mathcal{B}(\partial\Omega), L^2(\partial\Omega, \mu), \mathcal{B}'(\partial\Omega))$ . Thus, the commutative diagram (3.2) can be extended to a larger diagram (5.6) which is not commutative. Thus, a natural infinite collection of Gelfand triples are indexed by measures  $\mu$ . This is especially interesting when embedding  $\mathcal{B}(\partial\Omega) \hookrightarrow L^2(\partial\Omega, \mu)$  is compact, see [48, 49].

$$(5.6) \quad \begin{array}{ccc} V_1(\Omega) = H_0^1(\Omega)^\perp & \begin{array}{c} \xrightarrow{\text{tr}_i} \\ \xleftarrow{\text{tr}_i^{-1}} \end{array} & \text{Tr}_i(H^1(\Omega)) = \mathcal{B}(\partial\Omega) \\ \begin{array}{c} \curvearrowright \\ \iota^{-1} \\ \curvearrowleft \end{array} & \begin{array}{c} \xrightarrow{\frac{\partial_i}{\partial v}} \\ \xleftarrow{(\frac{\partial_i}{\partial v})^{-1}} \end{array} & \begin{array}{c} \curvearrowright \\ \mathcal{A}_1^{-1} = \iota^{-1} \\ \curvearrowleft \end{array} & L^2(\partial\Omega, \mu) \\ \begin{array}{c} \xrightarrow{(\text{tr}_i^*)^{-1}} \\ \xleftarrow{\text{tr}_i^*} \end{array} & & \begin{array}{c} \xrightarrow{\text{tr}_i^*} \\ \xleftarrow{(\text{tr}_i^*)^{-1}} \end{array} & \end{array}$$

Now assume that  $n \geq 2$ . Then one can, with some care, introduce the Newton potential operator  $u \mapsto \mathcal{I}u = (|\xi|^{-2}\hat{u})^\vee$ . It is well-known that  $\mathcal{I}$  extends to an isometric isomorphism  $\dot{H}^{-1}(\mathbb{R}^n) \rightarrow \dot{H}^1(\mathbb{R}^n)$ .

Suppose that  $\Omega$  is two-sided  $\dot{H}^1$ -admissible and recall the operator (4.8). Similarly as before, we may view its dual  $\overline{\text{tr}}^*$  as a bounded linear operator  $\overline{\text{tr}}^* : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{H}^{-1}(\mathbb{R}^n)$ .

**Lemma 5.6.** *Let  $n \geq 2$  and let  $\Omega$  be a two-sided  $\dot{H}^1$ -admissible domain in  $\mathbb{R}^n$ . Then  $\overline{\mathcal{S}} = \mathcal{I} \circ \overline{\text{tr}}^*$ .*

*Proof.* Given  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , we have  $\mathcal{I} \circ \overline{\text{tr}}^* g \in \dot{H}^1(\mathbb{R}^n)$ , and for all  $v \in \dot{H}^1(\mathbb{R}^n)$  then

$$(5.7) \quad \begin{aligned} \langle \mathcal{I} \circ \overline{\text{tr}}^* g, v \rangle_{\dot{H}^1(\mathbb{R}^n)} &= \langle \overline{\text{tr}}^* g, v \rangle_{\dot{H}^{-1}(\mathbb{R}^n), \dot{H}^1(\mathbb{R}^n)} = \langle g, \text{Tr} v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} \\ &= \langle \dot{\mathcal{S}}g, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle \overline{\mathcal{S}}g, v \rangle_{\dot{H}^1(\mathbb{R}^n)}. \end{aligned}$$

□

Let  $K$  denote the Newton kernel, that is, the fundamental solution (Green's function) to  $\Delta$  on  $\mathbb{R}^n$ . It is well-known that  $\mathcal{I}w = K * w$  for all  $w \in L^2(\mathbb{R}^n)$ . For a nonnegative Radon measure  $\nu$  on  $\partial\Omega$ , the function

$$K * \nu(x) := \int_{\mathbb{R}^n} K(x-y)\nu(dy), \quad x \in \mathbb{R}^n$$

is lower semi-continuous. In the case  $n \geq 3$ , it takes values in  $[0, +\infty]$ , in the case  $n = 2$  in  $(-\infty, +\infty]$ , and in either case it is finite on  $\mathbb{R}^n \setminus \partial\Omega$ . Given a finite signed Borel measure  $\nu$ , we consider  $K * \nu := K * \nu^+ - K * \nu^-$  on  $\mathbb{R}^n \setminus \partial\Omega$ . We call  $\nu$  centered if  $\nu(\mathbb{R}^n) = 0$ .

**Proposition 5.7.** *Let  $n \geq 2$ , let  $\Omega$  be  $\dot{H}^1$ -admissible and let  $\nu$  be a centered finite signed Borel measure on  $\partial\Omega$  of finite energy. Then sets of zero capacity have zero  $\nu$ -measure, and  $\nu$  defines an element of  $\dot{\mathcal{B}}'(\partial\Omega)$  by*

$$(5.8) \quad \langle \nu, f \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} := \int_{\partial\Omega} f d\nu, \quad f \in \dot{\mathcal{B}}(\partial\Omega).$$

Moreover,  $K * \nu$  is a representative modulo constants of  $\overline{\mathcal{S}}\nu = \mathcal{I} \circ \overline{\text{tr}}^* \nu$ .

*Proof.* By Proposition 5.3, the measure  $\nu$  charges no set of zero capacity, and since  $\nu$  is centered, the right hand side of (5.8) is linear in  $f \in \dot{\mathcal{B}}(\partial\Omega)$ . Let  $B$  be a ball containing  $\partial\Omega$ , let  $\chi \in C_c^\infty(B)$  be such that  $0 \leq \chi \leq 1$  and  $\chi \equiv 1$  on  $\overline{\Omega}$ . Given  $f \in \dot{\mathcal{B}}'(\partial\Omega)$ , let  $v \in \dot{H}^1(\mathbb{R}^n)$  be such that  $\text{Tr} v = f$  and let  $w \in v$  be such that  $\int_B \chi w dx = 0$ . By Poincaré's inequality for  $B$ , we find that  $\|\chi w\|_{H^1(\mathbb{R}^n)} \leq c \|\chi\|_{C^1(B)} \|v\|_{\dot{H}^1(\mathbb{R}^n)}$  with  $c > 0$  independent of  $v$ . By (5.2)

the function  $\tilde{w}|_{\partial\Omega} = (\chi w)^\sim|_{\partial\Omega}$  is integrable with respect to  $\nu^+$  and  $\nu^-$ . As before, we find that

$$(5.9) \quad \int_{\partial\Omega} \tilde{w} \, d\nu = \langle K * \nu, v \rangle_{\dot{H}^1(\mathbb{R}^n)}, \quad v \in \dot{H}^1(\mathbb{R}^n),$$

and this implies that  $|\langle \nu, f \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}| \leq c \|f\|_{\dot{\mathcal{B}}(\partial\Omega)}$ ,  $f \in \dot{\mathcal{B}}(\partial\Omega)$ , where  $c > 0$  is another constant. The rest now follows from (5.7), (5.8) and (5.9).  $\square$

**Corollary 5.8.** *Let  $n \geq 2$ , let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible and let  $\mu$  be a nonnegative Radon measure on  $\partial\Omega$  such that  $\mathcal{B}(\partial\Omega) \subset L^2(\partial\Omega, \mu)$  is a continuous embedding, see [15]. Then  $L_0^2(\partial\Omega, \mu) = \{f \in L^2(\partial\Omega, \mu) \mid \int_{\partial\Omega} f \, d\mu = 0\}$  is a subspace of  $\dot{\mathcal{B}}'(\partial\Omega)$ . Indeed, by Remark 2.3,  $\dot{\mathcal{B}}'(\partial\Omega)$  is isomorphic to the elements of  $\mathcal{B}'(\partial\Omega)$  null on constants. For any  $g \in L^2(\partial\Omega, \mu)$ , such that the finite signed measure  $g \cdot \mu$  is centered and of finite energy, and  $K * (g \cdot \mu)(x) = \int_{\partial\Omega} K(x-y)g(y)\mu(dy)$ ,  $x \in \mathbb{R}^n \setminus \partial\Omega$ , is a representative modulo constants of  $\overline{\mathcal{S}}g$ .*

**Remark 5.9.** *One can formulate a corresponding analog of Remark 5.5 and, by doing so, recover the classical formula for the single layer potential operator [74, 95] in the case of a bounded Lipschitz domain endowed with the surface measure on the boundary.*

**5.2. Representations of double layer potentials.** Given a two-sided  $H^1$ -admissible domain  $\Omega \subset \mathbb{R}^n$ , let  $\mathcal{R}$  denote the unique bounded linear operator from  $L^2(\mathbb{R}^n \setminus \partial\Omega)$  into  $H_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  such that

$$\langle \mathcal{R}u, v \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle u, v \rangle_{L^2(\mathbb{R}^n \setminus \partial\Omega)}$$

for all  $u \in L^2(\mathbb{R}^n \setminus \partial\Omega)$  and  $v \in H_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$ . This operator is symmetric on  $L^2(\mathbb{R}^n \setminus \partial\Omega)$  and symmetric with respect  $\langle \cdot, \cdot \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)}$ . It extends to an isometric isomorphism from  $(H_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega))'$  onto  $H_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$ . The adjoint  $(\frac{\partial}{\partial\nu})^*$  of  $\frac{\partial}{\partial\nu}$  as in Lemma 4.2 (iii) maps from  $\mathcal{B}(\partial\Omega)$  into  $(V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega))'$ . In the spirit of Remark 2.8, it can be viewed as a bounded linear operator from  $\mathcal{B}(\partial\Omega)$  into  $(H^1(\mathbb{R}^n \setminus \partial\Omega))'$  if for each  $f \in \mathcal{B}(\partial\Omega)$ , the functional  $(\frac{\partial}{\partial\nu})^* f$  is silently extended by zero to a bounded linear functional on all of  $H^1(\mathbb{R}^n \setminus \partial\Omega)$ . The operator  $\mathcal{D}$  can be represented in terms of those operators.

**Lemma 5.10.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. Then  $\mathcal{D} = -\mathcal{R} \circ (\frac{\partial}{\partial\nu})^*$ .*

*Proof.* Given  $f \in \mathcal{B}(\partial\Omega)$  and  $\varphi \in H_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus V_{1,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  Corollary 4.4 gives

$$\begin{aligned} -\langle \mathcal{D}f, \varphi \rangle_{(H^1(\mathbb{R}^n \setminus \partial\Omega))', H^1(\mathbb{R}^n \setminus \partial\Omega)} &= -\langle \mathcal{R} \circ \mathcal{D}f, \varphi \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = -\langle \mathcal{D}f, \mathcal{R}\varphi \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} \\ &= \left\langle \frac{\partial}{\partial\nu} \circ P_{1,\mathcal{D}} \circ \mathcal{R}\varphi, f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \left\langle \left(\frac{\partial}{\partial\nu}\right)^* f, P_{1,\mathcal{D}} \circ \mathcal{R}\varphi \right\rangle_{(H^1(\mathbb{R}^n \setminus \partial\Omega))', H^1(\mathbb{R}^n \setminus \partial\Omega)} \\ &= \left\langle \left(\frac{\partial}{\partial\nu}\right)^* f, \mathcal{R}\varphi \right\rangle_{(H^1(\mathbb{R}^n \setminus \partial\Omega))', H^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle \mathcal{R} \circ \left(\frac{\partial}{\partial\nu}\right)^* f, \varphi \rangle_{(H^1(\mathbb{R}^n \setminus \partial\Omega))', H^1(\mathbb{R}^n \setminus \partial\Omega)}. \end{aligned}$$

$\square$

**Proposition 5.11.** *Let  $\Omega$  be a bounded Lipschitz domain. Then  $\mathcal{B}(\partial\Omega)$  equals  $H^{1/2}(\partial\Omega)$ , and for any  $f \in H^{1/2}(\partial\Omega)$ , we have*

$$(5.10) \quad \mathcal{D}f(x) = - \int_{\partial\Omega} \frac{\partial}{\partial\nu_y} G(x-y) f(y) \sigma(dy), \quad x \in \mathbb{R}^n \setminus \partial\Omega;$$

here  $\frac{\partial}{\partial\nu_y}$  denotes the classical interior normal derivative and  $\sigma$ , the surface measure on  $\partial\Omega$ .

*Proof.* The classical double layer potential on the right hand side of (5.10) is known to solve (4.9) in the weak sense, so the uniqueness part of Lemma 4.8 implies (5.10).  $\square$

Given a two-sided  $\dot{H}^1$ -admissible domain  $\Omega \subset \mathbb{R}^n$ . Let  $\dot{H}_0^1(\mathbb{R}^n \setminus \partial\Omega)$  denote the closure of  $C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$  in  $\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$ . Let  $\dot{\mathcal{R}}$  denote the unique bounded linear operator from  $(\dot{H}_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega))'$  into  $\dot{H}_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$  such that

$$\langle \dot{\mathcal{R}}u, v \rangle_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} = \langle u, v \rangle_{(\dot{H}_0^1(\mathbb{R}^n \setminus \partial\Omega))', \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)}$$

for all  $u \in (\dot{H}_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega))'$  and  $v \in \dot{H}_0^1(\mathbb{R}^n \setminus \partial\Omega) \oplus \dot{V}_{0,\mathcal{D}}(\mathbb{R}^n \setminus \partial\Omega)$ . With a similar interpretation as before, we obtain the following.

**Lemma 5.12.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. Then  $\dot{\mathcal{D}} = -\dot{\mathcal{R}} \circ \left(\frac{\partial}{\partial\nu}\right)^*$ .*

**Proposition 5.13.** *Let  $n \geq 2$ , let  $\Omega$  be a bounded Lipschitz domain. Then  $\dot{\mathcal{B}}(\partial\Omega)$  equals  $\dot{H}^{1/2}(\partial\Omega)$ , and for any  $f \in \dot{H}^{1/2}(\partial\Omega)$ , the trace jump readjusted version of  $\dot{\mathcal{D}}f$  differs only by a constant from  $-\int_{\partial\Omega} \frac{\partial}{\partial\nu_y} K(\cdot - y) f(y) \mu(dy)$  on  $\mathbb{R}^n \setminus \partial\Omega$ .*

**Remark 5.14.** *Proposition 5.13 states that, in the case of a bounded Lipschitz domain endowed with the surface measure on the boundary, the double layer potential  $\dot{\mathcal{D}}$  defined in Subsection 4.3 can be represented using the classical formula [74, 95], up to a constant.*

## 6. NEUMANN-POINCARÉ OPERATORS FOR ADMISSIBLE DOMAINS

**6.1. Neumann-Poincaré operators.** Suppose that  $\Omega \subset \mathbb{R}^n$  is a two-sided  $H^1$ -admissible domain. We refer to the bounded linear operator  $\mathcal{K} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$ , defined by

$$\mathcal{K} := \frac{1}{2}(\text{tr}_i + \text{tr}_e) \circ \mathcal{D},$$

as the *Neumann-Poincaré operator* for (4.9), where  $\text{tr}_i := \text{Tr}_i|_{V_1(\Omega)}$  and  $\text{tr}_e := \text{Tr}_e|_{V_1(\mathbb{R}^n \setminus \bar{\Omega})}$ . By  $\mathcal{K}^* : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ , we denote its dual.

In the sequel, we use the symbol  $I$  to denote the identity operator; the space on which it acts will be clear from the context.

The following identities generalize results well-known in the Lipschitz case, see [3, Theorem 2.4] or [95].

**Theorem 6.1.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. Then*

- (i)  $\text{tr}_i \circ \mathcal{D} = -\frac{1}{2}I + \mathcal{K}$  and  $\text{tr}_e \circ \mathcal{D} = \frac{1}{2}I + \mathcal{K}$ .
- (ii)  $\frac{\partial_i}{\partial\nu} \circ \mathcal{S} = \frac{1}{2}I + \mathcal{K}^*$  and  $\frac{\partial_e}{\partial\nu} \circ \mathcal{S} = -\frac{1}{2}I + \mathcal{K}^*$ . In particular,

$$\mathcal{K}^* = \frac{1}{2} \left( \frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu} \right) \circ \mathcal{S}.$$

*Proof.* Statement (i) follows from the definitions of  $\mathcal{D}$ . To see (ii), note that for any  $f \in \mathcal{B}(\partial\Omega)$  and  $g \in \mathcal{B}'(\partial\Omega)$  we have

$$0 = \langle \mathcal{D}f, \mathcal{S}g \rangle_{H^1(\mathbb{R}^n \setminus \partial\Omega)} = \left\langle \frac{\partial_i \mathcal{S}g}{\partial\nu}, \text{tr}_i \mathcal{D}f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} - \left\langle \frac{\partial_e \mathcal{S}g}{\partial\nu}, \text{tr}_e \mathcal{D}f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$$

by Lemma 4.2 (iv), (2.14) and (4.3), and that adding this zero to

$$\left\langle \frac{\partial_i \mathcal{S}g}{\partial\nu}, \text{tr}_e \mathcal{D}f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} - \left\langle \frac{\partial_e \mathcal{S}g}{\partial\nu}, \text{tr}_i \mathcal{D}f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$$

respectively subtracting it gives

$$\begin{aligned} 2 \langle g, \mathcal{K}f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} &= \left\langle \left[ \frac{\partial \mathcal{S}g}{\partial\nu} \right], \text{tr}_i \mathcal{D}f + \text{tr}_e \mathcal{D}f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} \\ &= \left\langle \frac{\partial_i \mathcal{S}g}{\partial\nu} + \frac{\partial_e \mathcal{S}g}{\partial\nu}, -\llbracket \text{tr} \mathcal{D}f \rrbracket \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \left\langle \left( \frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu} \right) \circ \mathcal{S}g, f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}. \end{aligned}$$

□

For a two-sided  $\dot{H}^1$ -admissible domain  $\Omega \subset \mathbb{R}^n$ , similar observations can be made. We refer to the bounded linear operator  $\dot{\mathcal{K}} : \dot{\mathcal{B}}(\partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$ , defined by

$$\dot{\mathcal{K}} := \frac{1}{2}(\text{tr}_i + \text{tr}_e) \circ \dot{\mathcal{D}},$$

as the *Neumann-Poincaré operator* for (4.10). By  $\dot{\mathcal{K}}^* : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{\mathcal{B}}'(\partial\Omega)$ , we denote its dual.

**Theorem 6.2.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. Then*

- (i)  $\text{tr}_i \circ \dot{\mathcal{D}} = -\frac{1}{2}I + \dot{\mathcal{K}}$  and  $\text{tr}_e \circ \dot{\mathcal{D}} = \frac{1}{2}I + \dot{\mathcal{K}}$ .
- (ii)  $\frac{\dot{\partial}_i}{\partial\nu} \circ \dot{\mathcal{S}} = \frac{1}{2}I + \dot{\mathcal{K}}^*$  and  $\frac{\dot{\partial}_e}{\partial\nu} \circ \dot{\mathcal{S}} = -\frac{1}{2}I + \dot{\mathcal{K}}^*$ . In particular,

$$\dot{\mathcal{K}}^* = \frac{1}{2}\left(\frac{\dot{\partial}_i}{\partial\nu} + \frac{\dot{\partial}_e}{\partial\nu}\right) \circ \dot{\mathcal{S}} = \frac{1}{2}\left(\frac{\dot{\partial}_i}{\partial\nu} + \frac{\dot{\partial}_e}{\partial\nu}\right) \circ \bar{\mathcal{S}},$$

where  $\bar{\mathcal{S}}$  is as in Remark 4.13.

**Remark 6.3.** *If  $\Omega$  is a bounded Lipschitz domain and  $\partial\Omega$  is endowed with the surface measure  $\sigma$ , then it follows from Remark 5.9 that  $\bar{\mathcal{S}}$  admits the usual kernel representation for a single layer potential. Since, in that case, the normal derivatives can be defined in the strong sense  $\sigma$ -a.e. and coincide with the weak normal derivatives defined in Subsections 2.5 and 4.1, this definition of  $\dot{\mathcal{K}}^*$  coincides with the usual one for Lipschitz boundaries [74, 95]. See Corollary 5.8 for an interpretation of  $\dot{\mathcal{B}}'(\partial\Omega)$ , and also Corollary 5.4 and Propositions 5.11 and 5.13.*

**6.2. Boundary layer potentials and Calderón projectors.** Let  $\Omega$  be two-sided  $H^1$ -admissible. By Lemma 4.2, Corollary 4.9 and Corollary 4.14 the boundary single layer potential operator  $\mathcal{V} : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  and the boundary double layer potential operator  $\mathcal{W} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$ , defined by

$$(6.1) \quad \mathcal{V} := \text{tr} \circ \mathcal{S} \quad \text{respectively} \quad \mathcal{W} := -\frac{\partial}{\partial\nu} \circ \mathcal{D},$$

are bounded linear bijections with bounded inverses.

**Remark 6.4.** *In a classical context, the trace  $\mathcal{V}g$  on  $\partial\Omega$  of the single layer potential  $\mathcal{S}g$  can be defined as a pointwise limit, while the (co-)normal derivative  $\mathcal{W}f$  of the double layer potential  $\mathcal{D}f$  is a hypersingular integral, which, for smooth enough  $\partial\Omega$  and  $f$ , can be expressed as a principal value. See for instance [89, Section 2] for details; note that our notation differs slightly from the one used there.*

Now let  $\mathcal{M} : \mathcal{B}(\partial\Omega) \times \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega) \times \mathcal{B}'(\partial\Omega)$  be the linear operator defined by

$$\mathcal{M} := \begin{pmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}^* \end{pmatrix}$$

and set

$$\mathcal{C}_i := \frac{1}{2}I + \mathcal{M} \quad \text{and} \quad \mathcal{C}_e := \frac{1}{2}I - \mathcal{M}.$$

One refers to the linear operators  $\mathcal{C}_i$  and  $\mathcal{C}_e$  as the *interior* respectively *exterior Calderón projector* for  $\Omega$ . We obtain generalizations of known symmetrization formulas referred to as *Calderón relations*, see [50, Lemma 1.2.4], [82, Theorem 3.1.3] or [89, Proposition 5.1].

**Theorem 6.5.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. The operators  $\mathcal{C}_i$  and  $\mathcal{C}_e$  are continuous projectors and satisfy  $\mathcal{C}_i + \mathcal{C}_e = I$ . Moreover, we have  $\mathcal{M}^2 = \frac{1}{4}I$ , that is,*

$$(6.2) \quad \begin{cases} \mathcal{K}\mathcal{V} = \mathcal{V}\mathcal{K}^*, \\ \mathcal{W}\mathcal{K} = \mathcal{K}^*\mathcal{W}, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{K}^2 + \mathcal{V}\mathcal{W} = \frac{1}{4}I, \\ (\mathcal{K}^*)^2 + \mathcal{W}\mathcal{V} = \frac{1}{4}I. \end{cases}$$

*Proof.* Continuity and the first identity are clear. To see that  $\mathcal{C}_i^2 = \mathcal{C}_i$ , suppose that  $f \in \mathcal{B}(\partial\Omega)$  and  $g \in \mathcal{B}'(\partial\Omega)$  are given, and that  $u = \mathcal{S}g - \mathcal{D}f$  is the unique weak solution to (4.1) with  $k = 1$ , with  $\llbracket \frac{\partial u}{\partial \nu} \rrbracket = g$  and with  $\llbracket \text{tr } u \rrbracket = f$  in place of  $\llbracket \text{tr } u \rrbracket = -f$ . Then (6.1) and Theorem 6.1 give

$$(6.3) \quad \mathcal{C}_i \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \mathcal{V}g - \left(-\frac{1}{2}I + \mathcal{K}\right) f \\ \left(\frac{1}{2}I + \mathcal{K}^*\right) g + \mathcal{W}f \end{pmatrix} = \begin{pmatrix} \text{tr } \mathcal{S}g - \text{tr}_i \mathcal{D}f \\ \frac{\partial_i}{\partial \nu} \mathcal{S}g - \frac{\partial}{\partial \nu} \mathcal{D}f \end{pmatrix} = \begin{pmatrix} \text{tr}_i u \\ \frac{\partial_i u}{\partial \nu} \end{pmatrix}.$$

Now let  $v$  be the unique weak solution to problem (4.1) with  $k = 1$ , with  $\llbracket \text{tr } u \rrbracket = \text{tr}_i u$  and  $\llbracket \frac{\partial u}{\partial \nu} \rrbracket = \frac{\partial_i u}{\partial \nu}$ . Since  $u\mathbf{1}_\Omega$  is also a weak solution to this problem, it follows that  $v = u\mathbf{1}_\Omega$  and consequently

$$\mathcal{C}_i^2 \begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{C}_i \begin{pmatrix} \text{tr}_i u \\ \frac{\partial_i u}{\partial \nu} \end{pmatrix} = \begin{pmatrix} \text{tr}_i v \\ \frac{\partial_i v}{\partial \nu} \end{pmatrix} = \begin{pmatrix} \text{tr}_i u \\ \frac{\partial_i u}{\partial \nu} \end{pmatrix} = \mathcal{C}_i \begin{pmatrix} f \\ g \end{pmatrix}.$$

In the same way, we prove  $\mathcal{C}_e^2 = \mathcal{C}_e$ . The remaining identities then follow.  $\square$

**Remark 6.6.** *The unique weak solution  $u = \mathcal{S}g - \mathcal{D}f$  to (4.1) with  $k = 1$  and boundary data  $f \in \mathcal{B}(\partial\Omega)$  and  $g \in \mathcal{B}'(\partial\Omega)$  is zero on  $\mathbb{R}^n \setminus \bar{\Omega}$  if and only if  $\text{tr}_e u = 0$  and  $\frac{\partial_e u}{\partial \nu} = 0$ . By (6.3) this is the case if and only if  $\mathcal{C}_i \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$ , which happens if and only if  $(f, g)$  is an element of the graph of the Poincaré-Steklov operator  $\mathcal{A}_1$  for  $\Omega$ . As in [89, Section 5], an evaluation of (6.3) gives*

$$(6.4) \quad \mathcal{A}_1 = \mathcal{V}^{-1} \left( \frac{1}{2}I + \mathcal{K} \right) = \mathcal{W} + \left( \frac{1}{2}I + \mathcal{K}^* \right) \mathcal{V}^{-1} \left( \frac{1}{2}I + \mathcal{K} \right).$$

Now suppose that  $\Omega$  is two-sided  $\dot{H}^1$ -admissible. By Lemma 4.5, Corollary 4.11 and Corollary 4.16, the operators  $\dot{\mathcal{V}} : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega)$  and  $\dot{\mathcal{W}} : \dot{\mathcal{B}}(\partial\Omega) \rightarrow \dot{\mathcal{B}}'(\partial\Omega)$ , defined by

$$(6.5) \quad \dot{\mathcal{V}} := \text{tr} \circ \dot{\mathcal{S}} \quad \text{and} \quad \dot{\mathcal{W}} := -\frac{\partial}{\partial \nu} \circ \dot{\mathcal{D}},$$

are bounded linear bijections with bounded inverses. Let  $\dot{\mathcal{M}} : \dot{\mathcal{B}}(\partial\Omega) \times \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{\mathcal{B}}(\partial\Omega) \times \dot{\mathcal{B}}'(\partial\Omega)$  be defined by

$$\dot{\mathcal{M}} := \begin{pmatrix} -\dot{\mathcal{K}} & \dot{\mathcal{V}} \\ \dot{\mathcal{W}} & \dot{\mathcal{K}}^* \end{pmatrix}$$

and set

$$\dot{\mathcal{C}}_i := \frac{1}{2}I + \dot{\mathcal{M}} \quad \text{and} \quad \dot{\mathcal{C}}_e := \frac{1}{2}I - \dot{\mathcal{M}}.$$

The same arguments as before give the following counterpart of Theorem 6.5.

**Theorem 6.7.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. The operators  $\dot{\mathcal{C}}_i$  and  $\dot{\mathcal{C}}_e$  are continuous projectors and satisfy  $\dot{\mathcal{C}}_i + \dot{\mathcal{C}}_e = I$ . In addition,  $\dot{\mathcal{M}}^2 = \frac{1}{4}I$ , that is,*

$$(6.6) \quad \begin{cases} \dot{\mathcal{K}}\dot{\mathcal{V}} = \dot{\mathcal{V}}\dot{\mathcal{K}}^*, \\ \dot{\mathcal{W}}\dot{\mathcal{K}} = \dot{\mathcal{K}}^*\dot{\mathcal{W}}, \end{cases} \quad \text{and} \quad \begin{cases} \dot{\mathcal{K}}^2 + \dot{\mathcal{V}}\dot{\mathcal{W}} = \frac{1}{4}I, \\ (\dot{\mathcal{K}}^*)^2 + \dot{\mathcal{W}}\dot{\mathcal{V}} = \frac{1}{4}I. \end{cases}$$

**6.3. Invertibility and isometries.** As in the classical case, one can observe invertibility properties depending on a spectral parameter  $\lambda \in \mathbb{C}$ . The following observation is similar to [3, Lemma 2.4].

**Lemma 6.8.** *Let  $\Omega$  be two-sided  $H^1$ -admissible and  $\lambda \in \mathbb{R}$  with  $|\lambda| \geq \frac{1}{2}$ . Then  $\lambda I + \mathcal{K}^* : \mathcal{B}'(\partial\Omega) \rightarrow \mathcal{B}'(\partial\Omega)$  is injective.*

*Proof.* Assume that there is some nonzero  $g \in \ker(\lambda I + \mathcal{K}^*)$ . Then  $\mathcal{S}g \neq 0$  by Corollary 4.14, and since  $\mathcal{S}g$  is 1-harmonic in  $\mathbb{R}^n \setminus \partial\Omega$ , both  $\mathcal{S}g|_\Omega$  and  $\mathcal{S}g|_{\mathbb{R}^n \setminus \bar{\Omega}}$  must be nonzero. By Theorem 6.1, together with (2.14) and (4.3), this implies that

$$A := \left(\frac{1}{2} - \lambda\right) \langle g, \operatorname{tr} \mathcal{S}g \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \left\langle \left(\frac{1}{2}I + \mathcal{K}^*\right)g, \operatorname{tr} \mathcal{S}g \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} > 0$$

and

$$B := \left(\frac{1}{2} + \lambda\right) \langle g, \operatorname{tr} \mathcal{S}g \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = -\left\langle \left(-\frac{1}{2}I + \mathcal{K}^*\right)g, \operatorname{tr} \mathcal{S}g \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} > 0.$$

For  $|\lambda| = \frac{1}{2}$  this is impossible. For  $|\lambda| > \frac{1}{2}$  we can use that  $\langle g, \operatorname{tr} \mathcal{S}g \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} > 0$  by Corollary 4.4. This gives  $\lambda = \frac{1}{2} \frac{B-A}{B+A} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ , which again is impossible.  $\square$

The corresponding result for two-sided  $\dot{H}^1$ -admissible domains is seen analogously.

**Lemma 6.9.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible and  $\lambda \in \mathbb{R}$  with  $|\lambda| \geq \frac{1}{2}$ . Then  $\lambda I + \dot{\mathcal{K}}^* : \dot{\mathcal{B}}'(\partial\Omega) \rightarrow \dot{\mathcal{B}}'(\partial\Omega)$  is injective.*

Using the operators in (6.1) as metrics, we can introduce alternative Hilbert space norms on the trace spaces and their duals by

$$\|\cdot\|_{\mathcal{B}'(\partial\Omega), \mathcal{W}}^2 := \langle \mathcal{W}\cdot, \cdot \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} \quad \text{and} \quad \|\cdot\|_{\mathcal{B}'(\partial\Omega), \mathcal{V}}^2 := \langle \cdot, \mathcal{V}\cdot \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}$$

in the case of a two-sided  $H^1$ -admissible domain  $\Omega$ , and by

$$\|\cdot\|_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{W}}}^2 := \langle \dot{\mathcal{W}}\cdot, \cdot \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} \quad \text{and} \quad \|\cdot\|_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{V}}}^2 := \langle \cdot, \dot{\mathcal{V}}\cdot \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}$$

in the case of a two-sided  $\dot{H}^1$ -admissible domain  $\Omega$ . The positive definiteness follows from the boundedness of inverses or, alternatively, from Corollaries 4.4 and 4.6 together with (6.1) and (6.5) respectively. On  $\mathcal{B}(\partial\Omega)$  and  $\dot{\mathcal{B}}(\partial\Omega)$ , we also use the Hilbert space norms

$$\|\cdot\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}}^2 := \langle \mathcal{V}^{-1}\cdot, \cdot \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} \quad \text{and} \quad \|\cdot\|_{\dot{\mathcal{B}}(\partial\Omega), \dot{\mathcal{V}}^{-1}}^2 := \langle \dot{\mathcal{V}}^{-1}\cdot, \cdot \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)},$$

where  $\mathcal{V}^{-1}$  and  $\dot{\mathcal{V}}^{-1}$  denote the inverses of  $\mathcal{V}$  and  $\dot{\mathcal{V}}$  respectively.

**Theorem 6.10.** *Let  $\Omega$  be two-sided  $H^1$ -admissible.*

(i) *There are constants  $\alpha, \beta > 0$  such that*

$$\frac{1}{\alpha} \|g\|_{\mathcal{B}'(\partial\Omega)}^2 \leq \|g\|_{\mathcal{B}'(\partial\Omega), \mathcal{V}}^2 \leq \alpha \|g\|_{\mathcal{B}'(\partial\Omega)}^2, \quad g \in \mathcal{B}'(\partial\Omega),$$

and

$$\frac{1}{\beta} \|f\|_{\mathcal{B}(\partial\Omega)}^2 \leq \|f\|_{\mathcal{B}(\partial\Omega), \mathcal{W}}^2 \leq \beta \|f\|_{\mathcal{B}(\partial\Omega)}^2, \quad f \in \mathcal{B}(\partial\Omega).$$

(ii) *The operator  $\mathcal{S}$  is an isometry with respect to  $\|\cdot\|_{\mathcal{B}'(\partial\Omega), \mathcal{V}}$ , and  $\mathcal{D}$  is an isometry with respect to  $\|\cdot\|_{\mathcal{B}(\partial\Omega), \mathcal{W}}$ .*

(iii) *For any  $f \in \mathcal{B}(\partial\Omega)$  it holds*

$$(1-c) \|f\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}} \leq \left\| \left( \pm \frac{1}{2}I + \mathcal{K} \right) f \right\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}} \leq c \|f\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}},$$

where

$$c = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\alpha\beta}} < 1$$

with constants  $\alpha$  and  $\beta$  as in (i) chosen large enough so that  $\alpha\beta > 4$ . In particular, the operators  $\pm \frac{1}{2}I + \mathcal{K} : \mathcal{B}(\partial\Omega) \rightarrow \mathcal{B}(\partial\Omega)$  are isomorphisms.

*Proof.* Item (i) follows from the boundedness of the operators  $\mathcal{V}$  and  $\mathcal{W}$  and their inverses. Using Corollaries 4.4, 4.9 and Corollary 4.14 also, (ii) follows. Item (iii) can be shown similarly as in [89, p. 744]; note that by (6.4), we have

$$\begin{aligned} \left\| \left( \frac{1}{2} + \mathcal{K} \right) f \right\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}}^2 &= \left\langle \mathcal{V}^{-1} \left( \frac{1}{2} + \mathcal{K} \right) f, \left( \frac{1}{2} + \mathcal{K} \right) f \right\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} \\ &= \langle \mathcal{A}_1 f, f \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} - \langle \mathcal{W}f, f \rangle_{\mathcal{B}(\partial\Omega), \mathcal{B}'(\partial\Omega)} \end{aligned}$$



for any  $f \in \mathcal{B}(\partial\Omega)$ ; the first summand on the right-hand side is

$$\langle \mathcal{V}A_1 f, f \rangle_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}} \leq \left\| \left( \frac{1}{2} + \mathcal{K} \right) f \right\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}} \|f\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}},$$

and by (i), the second is bounded below by  $\frac{1}{\alpha\beta} \|f\|_{\mathcal{B}(\partial\Omega), \mathcal{V}^{-1}}^2$ .  $\square$

**Theorem 6.11.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. Then counterparts of Theorem 6.10 (i), (ii) and (iii) hold for  $\dot{\mathcal{V}}$ ,  $\dot{\mathcal{W}}$ ,  $\dot{\mathcal{S}}$ ,  $\dot{\mathcal{D}}$  and  $\dot{\mathcal{K}}$ .*

We use a result from perturbation theory to establish spectral properties of the Neumann-Poincaré operators, generalizing or in the spirit of well-known results in the Lipschitz case [23, 80, 85].

**Theorem 6.12.** *Let  $\Omega$  be two-sided  $H^1$ -admissible. For  $\lambda \in \mathbb{C}$ , if  $|\lambda - \frac{1}{2}| \geq 1$  or  $|\lambda + \frac{1}{2}| \geq 1$ , then the operators  $\lambda I + \mathcal{K}$  and  $\lambda I + \mathcal{K}^*$  are invertible on  $\mathcal{B}(\partial\Omega)$  and  $\mathcal{B}'(\partial\Omega)$  respectively. In particular, their real spectra are included in  $(-\frac{1}{2}, \frac{1}{2})$ .*

*Proof.* For any  $\lambda \neq -\frac{1}{2}$  we have  $\lambda I + \mathcal{K} = (\lambda + \frac{1}{2})I + (-\frac{1}{2}I + \mathcal{K})$ . By Theorem 6.10,  $\|-\frac{1}{2}I + \mathcal{K}\| < 1$ . Hence, by [36, Theorem 1.2.9],  $\lambda I + \mathcal{K}$  is invertible anytime  $|\lambda + \frac{1}{2}| \geq 1$ . Proceeding in the same way with  $\frac{1}{2}I + \mathcal{K}$  instead, we find that  $\lambda I + \mathcal{K}$  is invertible anytime  $|\lambda - \frac{1}{2}| \geq 1$ . By Banach's closed range theorem and Lemma 6.9, the same holds for  $\mathcal{K}^*$ , see [36, Theorem 8.1.5].  $\square$

Analogous arguments give a parallel result in the homogeneous case.

**Theorem 6.13.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible. For  $\lambda \in \mathbb{C}$ , if  $|\lambda - \frac{1}{2}| \geq 1$  or  $|\lambda + \frac{1}{2}| \geq 1$ , then the operators  $\lambda I + \dot{\mathcal{K}}$  and  $\lambda I + \dot{\mathcal{K}}^*$  are invertible on  $\dot{\mathcal{B}}(\partial\Omega)$  and  $\dot{\mathcal{B}}'(\partial\Omega)$  respectively. In particular, their real spectra are included in  $(-\frac{1}{2}, \frac{1}{2})$ .*

The values of  $\lambda \in \mathbb{C}$  for which Theorems 6.12 and 6.13 are represented in Figure 1.

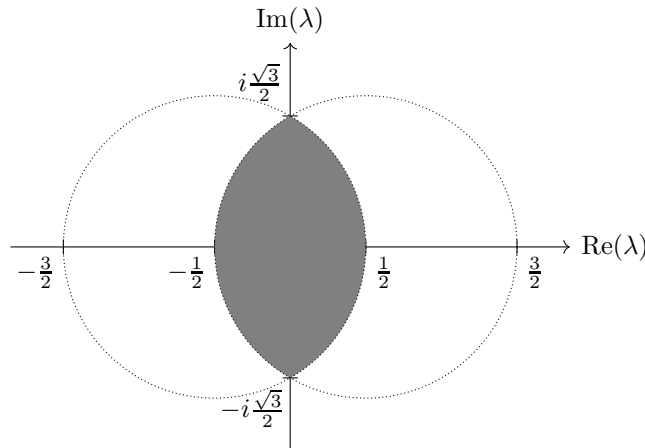


FIGURE 1. Graphic representation of the values of  $\lambda$  for which Theorems 6.12 and 6.13 apply, which are all  $\lambda \in \mathbb{C}$  outside the gray area (or on its boundary). The dotted lines correspond to the circles of center  $\pm\frac{1}{2}$  and radius 1.

## 7. APPLICATIONS TO IMAGING

We generalize three basic results on imaging that are well-known in the case of bounded Lipschitz domains [3, 62].

**7.1. Representation formula.** Let  $\Omega$  and  $D$  be two two-sided  $\dot{H}^1$ -admissible domains in  $\mathbb{R}^n$  such that  $D \subset\subset \Omega$ . We write  $\dot{\text{Tr}}_{\partial\Omega,i}$  and  $\dot{\text{Tr}}_{\partial\Omega,e}$  for the interior and exterior trace and  $\llbracket \dot{\text{Tr}} \rrbracket_{\partial\Omega}$  for the jump in trace with respect to  $\Omega$ . Abusing notation slightly, we write  $\frac{\dot{\partial}}{\partial\nu}|_{\partial\Omega}$  and  $\frac{\dot{\partial}_e}{\partial\nu}|_{\partial\Omega}$  to denote the interior and exterior normal derivative in the  $\dot{H}^1$ -sense and  $\llbracket \frac{\dot{\partial}}{\partial\nu} \rrbracket_{\partial\Omega}$  for the corresponding jump in normal derivatives with respect to  $\partial\Omega$ . Similarly for  $\partial D$  in place of  $\partial\Omega$ .

Given  $k \in (0, 1) \cup (1, +\infty)$  and  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , we call  $u \in \dot{H}^1(\Omega)$  a *weak solution* of the Neumann problem

$$(7.1) \quad \begin{cases} \nabla \cdot \left( (1 + (k-1)\mathbf{1}_D) \nabla u \right) = 0 & \text{on } \Omega, \\ \frac{\dot{\partial}_i u}{\partial\nu} \Big|_{\partial\Omega} = g \end{cases}$$

in the  $\dot{H}^1$ -sense if for any  $v \in \dot{H}^1(\Omega)$  we have

$$\int_{\Omega} (1 + (k-1)\mathbf{1}_D) \nabla u \cdot \nabla v \, dx = \langle g, \dot{\text{Tr}}_{\partial\Omega,i} v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}.$$

Since  $\min(1, k) \leq 1 + (k-1)\mathbf{1}_D \leq \max(1, k)$ , the existence of a unique weak solution  $u$  is clear from the Riesz representation theorem.

We write  $\dot{\mathcal{S}}_{\partial\Omega}$  and  $\dot{\mathcal{S}}_{\partial D}$  for the single layer potential operators with respect to  $\Omega$  and  $D$ ,  $\dot{\mathcal{D}}_{\partial\Omega}$  for the double layer operator with respect to  $\partial\Omega$  and  $\dot{\mathcal{K}}_{\partial D}$  for the Neumann-Poincaré operator with respect to  $\partial D$ , all in the  $\dot{H}^1$ -sense. The following representation formula for the weak solution  $u$  to (7.1) is a generalization of [3, Theorem 2.17].

**Theorem 7.1.** *Let  $\Omega$  and  $D$  be two-sided  $\dot{H}^1$ -admissible and such that  $D \subset\subset \Omega$ , and let  $k \in (0, 1) \cup (1, +\infty)$ . Given  $g \in \dot{\mathcal{B}}'(\partial\Omega)$ , let  $u \in \dot{H}^1(\Omega)$  be the unique weak solution to (7.1) in the  $\dot{H}^1$ -sense, let  $f := \dot{\text{tr}}_{\partial\Omega,i} u$  and  $H = \dot{\mathcal{S}}_{\partial\Omega} g - \dot{\mathcal{D}}_{\partial\Omega} f$ . Then it holds*

$$(7.2) \quad u = H - \dot{\mathcal{S}}_{\partial D} \varphi,$$

seen as an equality in  $\dot{H}^1(\Omega)$ , where  $\varphi$  is the unique element of  $\dot{\mathcal{B}}'(\partial D)$  such that

$$(7.3) \quad \left( \frac{k+1}{2(k-1)} I + \dot{\mathcal{K}}_{\partial D}^* \right) \varphi = \frac{\dot{\partial} H}{\partial\nu} \Big|_{\partial D}.$$

The function  $H - \dot{\mathcal{S}}_{\partial D} \varphi$  is constant on  $\mathbb{R}^n \setminus \overline{\Omega}$ .

The function  $H|_{\Omega}$  is the harmonic part of  $u$  in  $\Omega$ , while  $(-\dot{\mathcal{S}}_{\partial D} \varphi)|_{\Omega}$  is its refraction part. The existence and uniqueness of  $\varphi \in \dot{\mathcal{B}}'(\partial D)$  solving (7.3) follow from Theorem 6.13.

*Proof.* Let  $w \in \dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  be the unique weak solution in the  $\dot{H}^1$ -sense of the transmission problem

$$(7.4) \quad \begin{cases} \nabla \cdot \left( (1 + (k-1)\mathbf{1}_D) \nabla w \right) = 0 & \text{on } \mathbb{R}^n \setminus \partial\Omega \\ w_i|_{\partial\Omega} - w_e|_{\partial\Omega} = f \\ \frac{\dot{\partial}_i w_i}{\partial\nu} \Big|_{\partial\Omega} - \frac{\dot{\partial}_e w_e}{\partial\nu} \Big|_{\partial\Omega} = g, \end{cases}$$

the definition is completely analogous to that in Subsection 4.5. Similarly as for (7.1), a unique weak solution  $w$  in the  $\dot{H}^1$ -sense exists by comparability of quadratic forms. The function  $w$ , defined by  $w := u$  on  $\Omega$  and  $w := 0$  on  $\mathbb{R}^n \setminus \overline{\Omega}$  is a weak solution to (7.4) in the  $\dot{H}^1$ -sense. We now show that also  $\tilde{w} := H - \dot{\mathcal{S}}_{\partial D} \varphi$  is a weak solution to (7.4) in the  $\dot{H}^1$ -sense. Using Corollary 4.16 and Theorem 6.2 (ii), the identity (7.3) can be rewritten as

$$k \frac{\dot{\partial}_i (\dot{\mathcal{S}}_{\partial D} \varphi)}{\partial\nu} \Big|_{\partial D} - \frac{(\dot{\partial}_e \dot{\mathcal{S}}_{\partial D} \varphi)}{\partial\nu} \Big|_{\partial D} = (k-1) \frac{\dot{\partial} H}{\partial\nu} \Big|_{\partial D},$$

and since  $H \in \dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$ , it follows that  $k \frac{\partial_i \tilde{w}}{\partial \nu} \Big|_{\partial D} - \frac{\partial_\varepsilon \tilde{w}}{\partial \nu} \Big|_{\partial D} = 0$ . Given arbitrary  $v \in C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)$ , we find that

$$k \int_D \nabla \tilde{w} \nabla v \, dx + \int_{\Omega \setminus \overline{D}} \nabla \tilde{w} \nabla v \, dx = \left\langle k \frac{\partial_i \tilde{w}}{\partial \nu} \Big|_{\partial D} - \frac{\partial_\varepsilon \tilde{w}}{\partial \nu} \Big|_{\partial D}, \dot{\text{Tr}}_{\partial D} v \right\rangle_{\mathcal{B}'(\partial D), \mathcal{B}(\partial D)} = 0.$$

By definition,  $H$  satisfies the boundary conditions in (7.4) and since  $\dot{\mathcal{S}}_{\partial D} \varphi$  is harmonic on  $\mathbb{R}^n \setminus \overline{D} \supset \partial\Omega$ , it does not alter those boundary conditions. Consequently,  $\tilde{w}$  is a weak solution to (7.4) in the  $\dot{H}^1$ -sense, and  $u = \tilde{w}$  in  $\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)$  by uniqueness.

If (7.2) holds with  $\tilde{H} \in \dot{V}_0(\Omega)$  and  $\tilde{\varphi} \in \dot{\mathcal{B}}'(\partial\Omega)$  in place of  $H$  and  $\varphi$ , then  $H - \dot{\mathcal{S}}_{\partial D} \varphi = \tilde{H} - \dot{\mathcal{S}}_{\partial D} \tilde{\varphi}$ , consequently  $\dot{\mathcal{S}}_{\partial D}(\varphi - \tilde{\varphi}) \in \dot{V}_0(\Omega)$  and

$$\left\| \frac{\partial}{\partial \nu} \dot{\mathcal{S}}_{\partial D}(\varphi - \tilde{\varphi}) \right\|_{\partial D} = 0.$$

But this implies that  $\tilde{\varphi} = \varphi$  and therefore  $\tilde{H} = H$ .  $\square$

**7.2. Subdomain identification.** We give a generalization of a theorem on the identification of “monotone” inclusions through a single boundary measurement. For Lipschitz domains, it can be found as a part of [3, Theorem A.7]; the formulation below works for two-sided  $\dot{H}^1$ -admissible domains.

**Theorem 7.2.** *Let  $\Omega$ ,  $D_1$  and  $D_2$  be two-sided  $\dot{H}^1$ -admissible and such that  $D_1 \subset D_2 \subset\subset \Omega$  and let  $k \in (0, 1) \cup (1, +\infty)$ . Suppose that  $g \in \dot{\mathcal{B}}'(\partial\Omega)$  is nonzero and that  $u_1$  and  $u_2$  are the unique weak solutions in the  $\dot{H}^1$ -sense of (7.1) with  $D_1$  and  $D_2$  in place of  $D$  respectively. Then  $\text{tr}_{\partial\Omega, i} u_1 = \text{tr}_{\partial\Omega, i} u_2$  implies  $D_1 = D_2$ .*

*Proof.* Since

$$\int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) \nabla u_1 \nabla v \, dx = \int_{\Omega} (1 + (k-1)\mathbf{1}_{D_2}) \nabla u_2 \nabla v \, dx$$

for all  $v \in \dot{H}^1(\Omega)$ , it follows that

$$\int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) \nabla(u_1 - u_2) \nabla v \, dx = (k-1) \int_{D_2 \setminus D_1} \nabla u_2 \nabla v \, dx.$$

Testing with  $v = u_1 - u_2$  gives

$$\begin{aligned} \int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) |\nabla(u_1 - u_2)|^2 \, dx + (k-1) \int_{D_2 \setminus D_1} |\nabla u_2|^2 \, dx \\ (7.5) \qquad \qquad \qquad = (k-1) \int_{D_2 \setminus D_1} \nabla u_2 \nabla u_1 \, dx, \end{aligned}$$

and testing with  $v = u_1$  shows that the right-hand side equals

$$(7.6) \qquad \int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) \nabla(u_1 - u_2) \nabla u_1 \, dx.$$

Since  $u_1$  is harmonic with respect to the equivalent scalar product

$$(u, w) \mapsto \int_{\Omega} (1 + (k-1)\mathbf{1}_{D_1}) \nabla u \nabla w \, dx$$

on  $\dot{H}^1(\Omega) \times \dot{H}^1(\Omega)$  and  $u_1 - u_2 \in \ker \text{tr}_{\partial\Omega, i}$ , an analog of Theorem 2.10 (i) shows that (7.6) is zero and therefore also (7.5).

In the case  $k > 1$ , then necessarily  $u_1 = u_2$  in  $\dot{H}^1(\Omega)$ . For  $0 < k < 1$  the result globally follows by the same arguments but with  $D_1$  and  $D_2$  interchanged.  $\square$

The representation formula in Theorem 7.1 can be simplified if  $n = 2$  in the case of a disk. Note that, although the geometry of the inclusion is assumed to be smooth here, the geometry of the larger domain can still be irregular. For that matter, the boundary data is still understood in the sense of  $\dot{\mathcal{B}}$  and  $\dot{\mathcal{B}}'$ , and the operators are defined in the variational sense.

**Corollary 7.3.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible,  $D$  be a disk such that  $D \subset\subset \Omega$ , and let  $k \in (0, 1) \cup (1, +\infty)$ . Given  $g \in \dot{B}'(\partial\Omega)$ , let  $u \in \dot{H}^1(\Omega)$  be the unique weak solution to (7.1) in the  $\dot{H}^1$ -sense, let  $f := \text{tr}_{\partial\Omega, i} u$  and  $H = \dot{S}_{\partial\Omega} g - \dot{D}_{\partial\Omega} f$ . Then we have*

$$(7.7) \quad u = H - \frac{2(k-1)}{k+1} \dot{S}_{\partial D} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D} \right),$$

seen as an equality in  $\dot{H}^1(\Omega)$ .

*Proof.* As it was pointed out in Remark 6.3, if  $D$  is a disk, then  $\dot{K}_{\partial D}^*$  can be represented using the usual kernel formula [74, 95]. By [60, Section 4],  $\dot{K}_{\partial D}^* \psi = 0$  for all  $\psi \in L_0^2(\partial D)$ . By continuity of  $\dot{K}_{\partial D}^*$  on  $\dot{B}'(\partial D)$  and density of  $L_0^2(\partial D)$  in that space, the identity remains valid on  $\dot{B}'(\partial D)$ , hence  $\dot{K}_{\partial D}^* \varphi = 0$ . The representation formula follows by (7.3).  $\square$

Using Theorem 7.2 and Corollary 7.3, the identification of a disk-shaped inclusion when  $\Omega$  is a two-sided extension domain of  $\mathbb{R}^2$  follows using similar methods as in [60, Theorem A].

**Theorem 7.4.** *Let  $\Omega$  be two-sided  $\dot{H}^1$ -admissible,  $D_1$  and  $D_2$  be disks such that  $D_1, D_2 \subset\subset \Omega$  and let  $k \in (0, 1) \cup (1, +\infty)$ . Suppose that  $g \in \dot{B}'(\partial\Omega)$  is nonzero and that  $u_1$  and  $u_2$  are the unique weak solutions in the  $\dot{H}^1$ -sense of (7.1) with  $D_1$  and  $D_2$  in place of  $D$  respectively. Then  $\text{tr}_{\partial\Omega, i} u_1 = \text{tr}_{\partial\Omega, i} u_2$  implies  $D_1 = D_2$ .*

*Proof.* Assume  $D_1 \cap D_2 \not\subset \{D_1, D_2\}$ , otherwise Theorem 7.2 yields the result. Let us denote  $f := \text{Tr}_{\partial\Omega, i} u_1 = \text{Tr}_{\partial\Omega, i} u_2$ . The representation formula (7.7) states that, for  $p \in \{1, 2\}$ :

$$u_p = H - \frac{2(k-1)}{k+1} \dot{S}_{\partial D_p} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D_p} \right)$$

with  $H = \dot{S}_{\partial\Omega} g - \dot{D}_{\partial\Omega} f$ , thought of as elements of  $\dot{H}^1(\mathbb{R}^n)$ . Both  $u_1$  and  $u_2$  are solutions in the  $\dot{H}^1$ -sense to

$$\begin{cases} \Delta u_p = 0 & \text{on } \mathbb{R}^2 \setminus (D_1 \cup D_2) \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} = f \\ \frac{\partial_i u_i}{\partial \nu} \Big|_{\partial\Omega} - \frac{\partial_e u_e}{\partial \nu} \Big|_{\partial\Omega} = g \end{cases}$$

which is well-posed on  $\dot{H}^1(\mathbb{R}^2 \setminus \overline{D_1 \cup D_2})$ , hence  $u_1 = u_2$  on  $\mathbb{R}^2 \setminus D_1 \cup D_2$  by equality of the traces, and it holds:

$$(7.8) \quad \dot{S}_{\partial D_1} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D_1} \right) = \dot{S}_{\partial D_2} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D_2} \right) \quad \text{on } \mathbb{R}^2 \setminus D_1 \cup D_2.$$

If  $D_1 \cap D_2 = \emptyset$ , then, by (7.8), then  $\dot{S}_{\partial D_1}(\frac{\partial H}{\partial \nu}|_{\partial D_1})$  is harmonic on  $\mathbb{R}^2$  since  $D_1$  and  $D_2$  have at most one boundary point in common. Therefore,  $\frac{\partial H}{\partial \nu}|_{\partial D_1} = 0$  and  $H$  is constant, which contradicts  $g \neq 0$ .

Otherwise, since  $\frac{\partial H}{\partial \nu}|_{\partial D_p} \in \dot{B}'(\partial D_p)$  for  $p \in \{1, 2\}$ , it holds  $\dot{K}_{D_p}^*(\frac{\partial H}{\partial \nu}|_{\partial D_p}) = 0$ , therefore:

$$\frac{\partial_i}{\partial \nu} \Big|_{\partial D_p} \circ \dot{S}_{\partial D_p} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D_p} \right) = \frac{1}{2} \frac{\partial H}{\partial \nu} \Big|_{\partial D_p}.$$

Both  $\dot{S}_{\partial D_p}(\frac{\partial H}{\partial \nu}|_{\partial D_p})$  and  $H$  are harmonic on  $D_p$ , hence

$$\dot{S}_{\partial D_p} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D_p} \right) = \frac{1}{2} H \quad \text{on } D_p,$$

which, alongside (7.8), yields

$$\dot{S}_{\partial D_1} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D_1} \right) = \dot{S}_{\partial D_2} \left( \frac{\partial H}{\partial \nu} \Big|_{\partial D_2} \right)$$

on the complement of  $(D_1 \cup D_2) \setminus (D_1 \cap D_2)$ . Similar arguments as in the disjoint case yield that  $H$  is constant, which contradicts  $g \neq 0$  once again.  $\square$

## APPENDIX A. COMMENTS AND PROOFS FOR SECTION 2

**Lemma A.1.** *If  $\Omega \subset \mathbb{R}^n$  is an  $H^1$ -extension domain (resp.  $\dot{H}^1$ -extension domain) and  $\mathbb{R}^n \setminus \Omega$  is contained in an open ball  $B$ , then  $\Omega \cap B$  is an  $H^1$ -extension domain (resp.  $\dot{H}^1$ -extension domain).*

*Proof.* Let  $V$  be an open set containing  $\mathbb{R}^n \setminus \Omega$  and such that  $\overline{V} \subset B$ . Let  $\chi \in C_c^\infty(B)$  be such that  $0 \leq \chi \leq 1$  and  $\chi \equiv 1$  on  $\overline{V}$ . Given  $u \in H^1(\Omega \cap B)$ , the continuation by zero of  $\chi u$  to all of  $\Omega$  is in  $H^1(\Omega)$  and  $\|\chi u\|_{H^1(\Omega)} \leq \|\chi\|_{C^1(B)} \|u\|_{H^1(\Omega \cap B)}$ . Similarly, the continuation by zero of  $(1 - \chi)u$  to all of  $B$  is in  $H^1(B)$  and  $\|\chi u\|_{H^1(B)} \leq \|\chi\|_{C^1(B)} \|u\|_{H^1(\Omega \cap B)}$ . The sum  $E_{\Omega \cap B} u := E_\Omega(\chi u) + E_B((1 - \chi)u)$  is in  $H^1(\mathbb{R}^n)$  and is linear in  $u$ , satisfies  $E_{\Omega \cap B} u|_{\Omega \cap B} = u$  a.e. for all  $u \in H^1(\Omega \cap B)$  and

$$\|E_{\Omega \cap B} u\|_{H^1(\mathbb{R}^n)} \leq \|E_\Omega(\chi u)\|_{H^1(\mathbb{R}^n)} + \|E_B((1 - \chi)u)\|_{H^1(\mathbb{R}^n)} \leq c \|\chi\|_{C^1(B)} \|u\|_{H^1(\Omega \cap B)}.$$

The arguments for an  $\dot{H}^1$ -extension domain are similar.  $\square$

We provide arguments and references for Theorem 2.6 (i).

*Proof of Theorem 2.6 (i).* We consider the quadratic form defined by (2.2) with  $\Omega = \mathbb{R}^n$  and the domain  $H^1(\mathbb{R}^n)$ . For (i) it suffices to note that by [43, Corollary 2.3.1 and Example 2.3.1] we have

$$(A.1) \quad H_0^1(\mathbb{R}^n \setminus \partial\Omega) = \{w \in H^1(\mathbb{R}^n) \mid \tilde{w} = 0 \text{ q.e. on } \partial\Omega\},$$

where  $\tilde{w}$  is a quasi-continuous version of  $w$ . Now  $u \in H^1(\Omega)$  has an extension to an element of  $H_0^1(\mathbb{R}^n \setminus \partial\Omega)$  if and only if  $u \in H_0^1(\Omega)$ , and it has an extension to an element of the right-hand side of (A.1) if and only if  $\text{Tr } u = 0$ .  $\square$

A proof of Theorem 2.10 (i) is given in [43, Example 2.3.2], just in a slightly different language. We sketch how to adapt it to our formulation. For any open set  $\Omega \subset \mathbb{R}^n$ , the bilinear form (2.2), endowed with the domain  $H^1(\Omega)$ , is a Dirichlet form. By  $H_e^1(\Omega)$  we denote its extended Dirichlet space. For  $\Omega = \mathbb{R}^n$  it is well-known that

$$(A.2) \quad \text{if } n \geq 3, \text{ then } (H_e^1(\mathbb{R}^n), \langle \cdot, \cdot \rangle_{\dot{H}^1(\mathbb{R}^n)}) \cong (\dot{H}^1(\mathbb{R}^n), \langle \cdot, \cdot \rangle_{\dot{H}^1(\mathbb{R}^n)})$$

in the sense that the vector spaces are isomorphic and the isomorphism is a Hilbert space isometry, and

$$(A.3) \quad \text{if } n \leq 2, \text{ then } (H_e^1(\mathbb{R}^n), \langle \cdot, \cdot \rangle_{\dot{H}^1(\mathbb{R}^n)}) = (\dot{H}^1(\mathbb{R}^n) \oplus \mathbb{R}, \langle \cdot, \cdot \rangle_{\dot{H}^1(\mathbb{R}^n)}),$$

seen as an equality of vector spaces endowed with bilinear forms. Proofs can for instance be found in [24, Theorems 2.2.12 and 2.2.13].

*Proof of Theorem 2.10 (i).* An element  $u \in \dot{H}^1(\Omega)$  is in  $\ker \dot{\text{Tr}}$  if and only if it is the restriction to  $\Omega$  of an element of

$$(A.4) \quad \{w \in \dot{H}^1(\mathbb{R}^n) \mid \tilde{w} = 0 \text{ q.e. on } \partial\Omega\},$$

where  $\tilde{w} = 0$  is understood representative wise and modulo constants. For  $n \geq 3$  the space (A.4) may be identified with

$$(A.5) \quad \{v \in H_e^1(\mathbb{R}^n) \mid \tilde{v} = 0 \text{ q.e. on } \partial\Omega\}$$

under the isometry in (A.2). By [43, Theorem 2.3.3 and Example 2.3.2] the space (A.5) coincides with  $H_{0,e}^1(\mathbb{R}^n \setminus \partial\Omega)$ , the extended Dirichlet space of (2.2), endowed with the smaller domain  $H_0^1(\mathbb{R}^n \setminus \partial\Omega)$ . The complement of  $H_{0,e}^1(\mathbb{R}^n \setminus \partial\Omega)$  in  $H_e^1(\mathbb{R}^n)$  is isometric to

$$(A.6) \quad \{w \in \dot{H}^1(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \nabla w \nabla v \, dx = 0 \text{ for all } v \in C_c^\infty(\mathbb{R}^n \setminus \partial\Omega)\},$$

and since  $\mathbb{R}^n \setminus \partial\Omega$  is the disjoint union of  $\Omega$  and  $\mathbb{R}^n \setminus \overline{\Omega}$ , the restriction of this space to  $\Omega$  is  $\dot{V}_0(\Omega)$ . For  $n = 2$  it is shown in [43, Example 2.3.2] that the space (A.5) still coincides with  $H_{0,e}^1(\mathbb{R}^n \setminus \partial\Omega)$ , and (simplifications of) the same arguments give this coincidence also

for  $n = 1$ . By (A.3), the space (A.5) may then be regarded as a closed subspace of  $\dot{H}^1(\mathbb{R}^n)$ , and the complement of this closed subspace is (A.6).  $\square$

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