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► **To cite this version:**

Lisa Morhaim, Ayşegül Yıldız Ulus. History-dependent optimization with applications to circular and sustainable economics. 2023. hal-04505111

**HAL Id: hal-04505111**

**<https://hal.science/hal-04505111>**

Preprint submitted on 14 Mar 2024

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# HISTORY-DEPENDENT OPTIMIZATION WITH APPLICATIONS TO CIRCULAR AND SUSTAINABLE ECONOMICS

Lisa Morhaim\* and Ayşegül Yıldız Ulus†

Thursday 14<sup>th</sup> March, 2024

## ABSTRACT

We propose a general history-dependent framework (GHDF), an intertemporal optimization framework in which the instantaneous reward function depends on the memory of (eventually all) the previous decisions. By considering a very general history-formation process, through the introduction of a memory function, we provide a wide embracing framework, keeping it being both tractable and interpretable within many diverse contexts and allowing to enrich the analysis by treating at the same time several kinds of history dependencies. An easily implementable history-dependent framework (EIHDF) is provided, a version of our general history-dependent framework (GHDF) for which the primer (i.e. instantaneous reward function, feasible set and history-formation process) are defined in a recursive way so that it is easily implementable while still as general as needed to be widely applicable.

Taking into account the fact that the environment keeps in memory our activities and decisions, we further provide a general sustainable framework (GSF) which introduces a basis for future analysis in environmental and sustainable issues and encompasses many existing models in the environmental literature (including circular economy models). It is designed in a very amenable and flexible manner so that it can be adapted to many contexts and one can easily remove or add different effects that will be needed to be addressed. The mathematical results (including existence of a solution and dynamic programming tools) are derived as an application of our general history-dependent (GHDF) framework and can be directly used.

As examples, we address the model by Morhaim and Ulus[37] and all its history-dependent and habit formation applications, as well as many environmental models. These include optimal management of natural resources, circular economy (CE) models (with or without recycling habits) and circular and cumulative causation (CCC) models, all of which being particular cases of our framework.

**KEYWORDS:** History-dependent model, Circular Economy, Environment, Pollution, Sustainability, Growth, Habits, Satiation, Optimal management of natural resources, Optimal growth, Intertemporal decisions with instantaneous history-dependencies, Dynamic programming.

**JEL CLASSIFICATION:** C61, D90.

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We are grateful to Rabah Amir for useful discussions and comments during this work. We thank Manjira Datta, Kevin Reffett and Juan-Pablo Rincon Zapatero for having outreached important questions about this work. We also thank Lucasz Wozny, Agnieszka Wiszniewska-Matyszekiel, Jean-Pierre Drugeon, Luciano de Castro, Dominika Machowska and the SAET 2023 participants.

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# 1 Introduction

History-dependencies are currently widely taken into account in economics in various domains, among which importantly representation of preferences issues (Rozen[36], Rustichini and Siconolfi[37, 38]) or optimal growth (Ryder and Heal[17], Morhaim and Ulus[32]) and to address environmental issues. In this paper, we provide history-dependent models and dynamic programming tools to deal with such models. The dynamic programming tools (in particular, existence of a solution and that the value function is the unique fixed point of the Bellman operator) are obtained without concavity assumptions. More precisely, we consider the following three history-dependent frameworks

- a general history-dependent framework (GHDF)
- an easily implementable history-dependent framework (EIHDF), a version of our general history-dependent framework (GHDF) in which the primer of the problem are defined in an adequate recursive way so that is both kept easily implementable and as general as needed to be widely applicable.
- a general sustainable framework (GSF) which introduces a very amenable and flexible basis for future analysis in environmental and sustainable issues and encompasses many existing models in the environmental literature (including linear and circular economy models) and is an application of general history-dependent framework (GHDF)

The first history-dependent model, i.e. the general history-dependent framework (GHDF), is a general tractable framework for intertemporal optimization framework in which the instantaneous reward function depends on the memory of (eventually all) the previous decisions. On one hand, we introduce a general function allowing to model many different memory processes and on the other hand, we deal with general decision variables, objective functions and feasible sets. The formalism we propose allows to discuss and study the memory process formation (through a function  $m$ ) as well as the way the history enters the instantaneous reward function. We provide dynamic programming tools for such models. Without concavity assumptions, we show the existence of a solution and that the value function is the unique fixed point of the Bellman operator.

The framework in Morhaim and Ulus[32], in which *consumption* history is considered becomes a particular case of transforming the decisions into a history sequence. Indeed, the framework in Morhaim and Ulus[32] is a case in which *consumption* is kept in memory, which is a function of previous date and current capital stock decisions. Thus, it means keeping in memory a function of these decisions. Our general history-dependent framework allows, not only to keep in memory *the particular function* defining consumption, but *any function* of the previous date and current decisions. By this way, the model provided in Morhaim and Ulus[32] as well as the examples presented in Morhaim and Ulus[32]<sup>1</sup> become particular cases of our general history-dependent framework.

By generalizing the history formation process, through the introduction of a *memory* function, we provide a very general framework that keeps the framework being tractable as well as the elements of the basic model being interpretable within different contexts. This includes all the history-dependent models that are considered in Morhaim and Ulus[32] (i.e. Ryder and Heal[17]) but also sustainability issues, in particular environmental models (Van Der Ploeg and Withagen[44], Ikefuji[18], Löfgren[27]),

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<sup>1</sup>including Ryder and Heal[17], Rozen[36], Rustichini and Siconolfi[38], Caroll, Overland and Weil[6], He, Dyer and Butler[16], Baucells and Sarin[4], Ikefuji[18], Löfgren[27], Safi and Ben Hassen[39].

optimal management of natural resource (Smulders, Toman and Withagen[40], Ulus[42]), circular economy (CE) models (George, Chi-ang Lin and Chen[14], and circular and causation (CCC) models (Donaghy[10, 11]) Kasioumi[20, 21, 23]).

Further, we provide an easily implementable history-dependent framework (EIHDF), a version of our general history-dependent framework (GHDF) which is both easily implementable and as general as needed to be widely applicable. This is allowed by providing a history-dependent framework in which the primer of the problem, in particular the instantaneous reward function and the feasible set  $\Gamma$ , are defined in an adequate recursive way answering both issues simultaneously. We then show that such a model is a particular case of our general history-dependent framework and the results for this particular case are derived as corollaries of the results for the general history-dependent framework.

Further, taking into account the fact that the environment keeps in memory our activities and decisions, we provide a general sustainable framework (GSF) which introduces a basis for future analysis in environmental and sustainable issues and encompasses many existing models in the environmental literature (including circular economy models). It is designed in a very amenable and flexible manner so that it can be adapted to many contexts and one can easily remove or add different effects that will be needed to be addressed. The mathematical results (including existence of a solution and dynamic programming tools) are derived as an application of our general history-dependent (GHDF) framework and can be directly used.

Environmental and sustainable variables in the economy can be interpreted as being influenced by (the memory of our past) decisions. Through such a point of view, our framework fits many environmental and sustainable model, including circular economy models.

Circular economy (CE) is an important issue on current policy agendas worldwide (Abad-Segura, de la Fuente, González-Zamar and Belmonte-Ureña[1], Fitch-Roy, Benson and Monciardini [12], de Melo, de Oliveira, de Sousa, Vieira and Amaral[9])<sup>2</sup>, CE aiming to increase the efficiency of resources use to achieve a better balance and harmony between economy, environment and society (Ghisellini, Cialani and Ulgiati[15])<sup>3</sup>. Among scholars, discussions and ideas about CE date at least back to the second half of the the twentieth century (Boulding[5], Pearce and Turner [34]). The conceptual and theoretical CE understandings as well as CE strategies and implementations are still currently discussed (Andersen[2], Kirchherr, Reike and Hekkert[24], Kalmykova Sadagopan and Rosado[19])<sup>4</sup> Nevertheless, CE is most frequently depicted as a combination of reduce, reuse and recycle activities (Kirchherr, Reike and Hekkert[24]), but also design, implying a focus on the entire life cycle of the processes the interaction between the process and the environment and the economy in which it is embedded (Ghisellini, Cialani and Ulgiati[15]). Although not systematically, a large strand of research analyze CE interlinked with sustainable development (Ghisellini, Cialani and Ulgiati[15], Ritzén and Sandström[35])<sup>5</sup>. The Ellen Mac Arthur Foundation Report[28] thus aimed

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<sup>2</sup>((a enlever)) cf aussi European Commission[7, 8], China 2008 Circular Economy Promotion Law of the People's Republic, cf comparaison Europe/China dans McDowall et al.[30]

<sup>3</sup>((a enlever)) “By promoting the adoption of closing-the-loop production patterns within an economic system Circular Economy (CE) aims to increase the efficiency of resource use, with special focus on urban and industrial waste, to achieve a better balance and harmony between economy, environment and society” ; “over the entire life cycle of any process as well as at the interaction between the process and the environment and the economy in which it is embedded” (Ghisellini, Cialani and Ulgiati[15])

<sup>4</sup>((a enlever)) Kalmykova cité 1400 fois

<sup>5</sup>((a enlever)) ritzen est cité 700 fois

to demonstrate how circular economy principles and strategies significantly reduce greenhouse gas emissions and calls for integrating efforts to respond to climate change with those to accelerate the transition to a circular economy.

While incorporating the concept of circular economic activities, George Lin and Chen[14] considers a social planner who maximizes an intertemporal utility function, which depends on aggregate consumption and the stock of pollution. This model was later generalized to incorporate the intensity of recycling by Stengos[22], and to further incorporate recycling habits by Kasioumi[20, 21]. The theoretical circular-economy model of economic growth with circular and cumulative causation (CCC)<sup>6</sup> is presented in Donaghy[10, 11]. The George Lin and Chen[14]’s model is modified by Donaghy[10, 11] including physical capital  $K$ , human capital  $HC$ , labor  $L$ , and other materials  $OM$ , as productive factors. A whole section is devoted to such applications and examples.

While we detail extensively some of these models within our framework and discuss how our framework is fitted for future research as it is amenable not only to treat various sustainable and environmental issues but also allows to interlink these with many kind of effects and history-dependencies (consumption, production, saving and investment, human capital, labor, consumption habits, recycling habits, pollution, stock of waste, etc).

The research on circular and sustainable economy is currently vivid. Our framework is fitted to consider and interlink economic, environmental, technological and social issues. The GSF can easily be adapted to already suggested paths for future research (Donaghy[], Ghisellini .... ) and the extensive literature that is developing. The way the GSF may incorporate the history-dependence viewpoint and the memory formation that we introduce open perspectives towards several aspects and interpretations. In particular, the GSF allows to deal with many important features that are coming to be taken into account, such as recycling, reuse, reduction, design, habits, activities of harvesting exhaustible and renewable resources, the assimilative capacity of the natural environment for (non-recyclable) waste, transport activities, management of resources, interaction between the processes and the environment, preventative and regenerative eco-industrial development, etc. We discuss the way existing models are particular cases of our general framework as well as how its flexibility allows to use it in future research. As an example, the sustainable process or design variable  $r$  does not enter directly Donaghy’s utility function in contrast with this possibility which is allowed in our framework. Our general framework allows to study simultaneously many effects and contexts: circular models without production waste (Section 4.4), linear economies with production waste, and furthermore circular economies with production waste, as well as other many effects. These may be interconnected: our general framework allows to enrich the analysis by treating at the same time several kinds of history dependencies. This is crucial for sustainability issues as they involve on one hand habits (consumption habits but also recycling habits), and on another hand pollution stocks and environmental quality (Mazar[29], Moreau[31]). Such a modelling choice also emphasizes the

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<sup>6</sup>The property of circular and cumulative causation (CCC) was given a formulation by Gunnar Myrdal (Fujita[13], Donaghy[10, 11])<sup>7</sup>. “Myrdal characterizes (CCC) as a process in which “a change in one form of an institution will lead to successive changes in other institutions. These changes are circular in that they continue in a cycle, many times in a negative way, in which there is no end, and cumulative in that they persist in each round” (Wikipedia, n.d.). Through CC, initial advantages or disadvantages are reinforced. Examples of CCC include the working of agglomeration economies (Krugman 1995, Baldwin et al. 2001), path dependence (Arthur 1994, durlauf 2005, Donaghy 2009) and poverty traps (Myrdal 1957, Azariadis and Stachurski 2005). Some contributing factors to positive CCC are technologies that display increasing returns to scale (Kaldor 1967, Arthur 1994), learning by doing (Arrow 1962) and human capital deepening (Romer 1986). The principal contribution of the original research reported in this chapter is to introduce CCC through the abovementioned factors contributing to circularity in a model of economic growth to characterize the effects it elicits. (Donaghy[10]).

fact that the environment keeps in memory our activities and decisions.

The paper is organized as follows. Section 2 introduces the general history-dependent framework (GHDF) an easily implementable history-dependent framework (EIHDF), a version of our general history-dependent framework (GHDF) in which the primer of the problem are defined in an adequate recursive way so that is both kept easily implementable and as general as needed to be widely applicable. We provide the results on the existence of a solution and dynamic programming tools for such models. Section 3 provides a general sustainable framework (GSF) which introduces a very amenable and flexible basis for future analysis in environmental and sustainable issues and encompasses many existing models in the environmental literature (including linear and circular economy models) and is an application of our general history-dependent framework (GHDF). Section 4 details and addresses many examples and applications of our general framework from the related literature. The proofs are in the Appendix.

## 2 A general history-dependent framework (GHDF)

We propose a general intertemporal optimization framework in which the instantaneous reward function depends on memory of all the previous decisions. The formalism we propose allows to discuss and study the memory process formation (through a function  $m$ ) as well as the way the history enters the instantaneous reward function (through a function  $\varphi$ ). We further provide a general history-dependent framework which is both easily implementable and as general as needed to be widely applicable. This is allowed by providing a history-dependent framework in which the primer of the problem, in particular the instantaneous reward function and the feasible set  $\Gamma$ , are defined in an adequate recursive way answering both issues simultaneously. We then show that such a model is a particular case of our general history-dependent framework and the results for this particular case are derived as corollaries of the results for the general history-dependent framework.

### 2.1 The general history-dependent model

We propose a general intertemporal optimization framework in which the instantaneous reward function depends on memory of all the previous decisions. The formalism we propose allows to discuss and study the memory process formation (through a function  $m$ ) as well as the way the history enters the instantaneous reward function (through a function  $\varphi$ ).

We describe a general framework of an infinite horizon intertemporal history dependent decision problem of a social planner which may be a problem of optimal growth related to consumption, saving, conservation, accumulation, pollution or sustainable issues. depending on the objectives and the constraints of the model.

Let  $X$  be a topological space which will be the set of state and control variables of the problem.<sup>8</sup> Let us consider an intertemporal decision process. Let  $Y$  be a topological space. We define a *memory function/process* by  $m : X \times X \rightarrow Y$  such that for time  $t$ ,  $m(x_t, x_{t+1})$  is the *memory* for the decision  $x_{t+1} \in X$  given  $x_t$ . By this *memory function/process*  $m$  we will obtain the history of the memories over time which is modeled in the following way:

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<sup>8</sup>For example, in Morhaim and Ulus[32],  $X = \mathbb{R}^+$ , in George, Lin and Chen[14],  $X = \mathbb{R} \times \mathbb{R}^+$

Let us consider the set  $l_+^\infty$  defined by

$$l_+^\infty = \{\tilde{x} = (x_t)_{t=0}^\infty \in (\mathbb{R}^+)^{\mathbb{N}}, \|\tilde{x}\|_\infty := \sup_{t \in \mathbb{N}} x_t < +\infty\}$$

At time  $t = 0$ , she has a *time-0 history* denoted by  $\tilde{h}^{(0)} := (h_0^{(0)}, h_1^{(0)}, \dots) = (h_j^{(0)})_{j=1}^\infty$  lying in  $l_+^\infty$ .

Having on memory  $m(x_0, x_1)$  at time 0, she has then *time-1 history* equal to  $\tilde{h}^{(1)} := (m(x_0, x_1), \tilde{h}^{(0)})$  at time  $t = 1$ . That is, for any time  $t \geq 1$ , *time-t history* will be

$$\forall t \geq 1, \tilde{h}^{(t)} = (h_j^{(t)})_{j=1}^\infty := (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$$

The  $j$ -th coordinate of the sequence  $\tilde{h}^{(t)}$  is denoted by  $h_j^{(t)}$ . For  $j$  such that  $1 \leq j \leq t$ , the term  $h_j^{(t)}$  is the decision  $j$  periods prior to time  $t$ , i.e.

$$h_j^{(t)} = m(x_{t-j}, x_{t-j+1})$$

For  $j \geq t + 1$ , the terms  $h_j^{(t)}$  of the sequence  $\tilde{h}^{(t)}$  are defined by the terms of  $\tilde{h}^{(0)}$ , i.e.

$$(h_j^{(t)})_{j=t+1}^\infty = \tilde{h}^{(0)}$$

Let us consider an agent facing a time  $t$  objective function  $F$  which is defined on a subset  $\mathcal{D}_F$  of  $l_+^\infty \times X \times X$

$$F : \mathcal{D}_F \subseteq (l_+^\infty \times X \times X) \rightarrow \mathbb{R}$$

The feasible correspondence  $\Gamma : (l_+^\infty \times X) \rightarrow X$  is given such that for any  $(\tilde{h}, x) \in l_+^\infty \times X$

$$\Gamma(\tilde{h}, x) \subseteq \{x' \in X, (\tilde{h}, x, x') \in \mathcal{D}_F\}$$

Assuming a fixed discount factor  $\beta \in (0, 1)$ , for initially given state stock  $x_0 > 0$  and time-0 history  $\tilde{h}^{(0)} \in l_+^\infty$ ; with feasible state and control variables at time  $t$ , that is, satisfying the process  $x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t)$ , for every  $t \geq 0$ , the general framework optimization problem can then be written as follows:

$$P_{F, \Gamma, m}(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Max} & \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & x_0 > 0 \text{ and } \tilde{h}^{(0)} \in l_+^\infty \text{ are given} \end{cases}$$

### 2.1.1 The problem

Let  $X, Y$  be topological spaces. Let us consider  $F : X \times l_+^\infty \times X \rightarrow \mathbb{R}$  the instantaneous reward function,  $m : X \times X \rightarrow Y$  the memory function and the (fixed) discount factor  $\beta \in (0, 1)$ . Let  $x_0 \in X$  and  $\tilde{h}^{(0)} \in l_+^\infty$  be given. Let us consider the problem

$$P_{F, \beta, m}(x_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \end{cases}$$



### 2.1.2 Notations and feasible sets

DEFINITION 2.1.— For any given initial data  $x_0 > 0$ , and initial time-0 history  $\tilde{h}^{(0)} \in l_+^\infty$ , the feasible set  $\Pi(k_0, \tilde{h}^{(0)})$  is defined by the set of sequences feasible from  $x_0$  and  $\tilde{h}^{(0)}$ , i.e. for any  $x_0 > 0$ , for any  $\tilde{h}^{(0)} \in l_+^\infty$ ,

$$\Pi(\tilde{h}^{(0)}, x_0) = \{\tilde{x} = (x_t)_{t=1}^{+\infty} \in X^{\mathbb{N}}, \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t), \tilde{h}^{(t+1)} = (m(x_t, x_{t+1}), \tilde{h}^{(t)})\}$$

The process is analogous as usual. The feasible set is defined from  $x_0$  but here also from the given (infinite) sequence  $\tilde{h}^{(0)}$ . Once  $x_1$  is chosen in the feasible set  $\Gamma(x_0, \tilde{h}^{(0)})$ , the sequence  $\tilde{h}^{(1)}$  is updated from the current chosen variable  $x_1$  and the given variable  $x_0$ . Then,  $x_2$  is chosen given  $x_1$  and  $\tilde{h}^{(1)}$  in the feasible set  $\Gamma(x_1, \tilde{h}^{(1)})$ . And so on, at time  $t + 1$ , given  $x_t$  and  $\tilde{h}^{(t)}$ , the variable  $x_{t+1}$  must be chosen in the feasible set  $\Gamma(x_t, \tilde{h}^{(t)})$ . Once  $x_{t+1}$  is chosen, at time  $t + 2$ ,  $x_{t+2}$  is chosen given  $x_{t+1}$  and  $\tilde{h}^{(t+1)}$  that has been updated from the previous history sequence  $\tilde{h}^{(t)}$  and the variables  $(x_t, x_{t+1})$ . Such a process allows to keep decisions in memory while choosing the variable  $x_t$  at each time  $t$ .

For a sequence  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h}^{(0)})$ , we denote by  $\mathcal{U}(\tilde{x})$  the objective i.e.

$$\mathcal{U}(\tilde{x}) = \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1})$$

### 2.1.3 Assumptions

We set that  $(F, \Gamma)$  satisfies the set  $(\mathcal{A})$  of assumptions if the following assumptions **(F)** on the feasible set, **(A)** on the instantaneous function, and **(M)** on the memory function are satisfied

- (F)**  
(F1)  $\Gamma$  is a continuous nonempty compact-valued correspondence from  $X \times l_+^\infty$  into  $X$ .  
(F2) There exists  $a \geq 0, a \neq 1$  and  $a' \geq 0$  such that for any  $x \in X$  and any history  $\tilde{h} \in l_+^\infty$ ,  $x' \in \Gamma(\tilde{h}, x) \Rightarrow \|x'\| \leq a'\|x\| + a$

- (A)**  
(A1)  $\forall x_0 > 0$  and  $\tilde{h}^{(0)} \neq 0, \exists \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$  such that  $\mathcal{U}(\tilde{x}) > -\infty$ .  
(A2) There exist  $a \in \mathbb{R}^+$  with  $a\beta < 1$ ,  $a_2 \in \mathbb{R}^+$  and a continuous<sup>9</sup> function  $a_1 : X \rightarrow \mathbb{R}$  such that for any  $x_0 > 0$  and  $\tilde{h}^{(0)} \in l_+^\infty$ , for any feasible sequence  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$  and its associated history  $\tilde{h}^{(t)}$ , for any  $t \geq 0$ ,

$$F^+(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

- (M)**  
(m1) The function  $m$  is continuous.

### 2.1.4 The objective is well-defined

The next result shows that, under the set of assumptions  $(\mathcal{A})$ , the objective function of the optimization problem  $\mathcal{P}_{F,\beta,m}(\tilde{h}^{(0)}, x_0)$  is well-defined. The proof is given in the Appendix.

<sup>9</sup>This continuity assumption is needed to prove the upper semi-continuity of the value function (Proposition 4.3).

PROPOSITION 2.1.— Assume  $(\mathcal{A})$  and let  $x_0 \in X$  and  $\tilde{h}^{(0)} \in l_+^\infty$  be given. Then for any feasible sequence  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$ , the limit<sup>10</sup>  $\lim_{T \rightarrow +\infty} \sum_{t=0}^T \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1})$ , with  $\forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$ , is well-defined.

The proof can be found in the Appendix.

### 2.1.5 Existence and uniqueness of the solution

In this section, we give a proposition which shows the existence of a solution to the optimization problem  $\mathcal{P}_{F,\beta,m}(x_0, \tilde{h}^{(0)})$ . The uniqueness is obtained under additional assumptions, requiring strict concavity of the instantaneous function  $F$ . The proof is given in the Appendix.

PROPOSITION 2.2.— Assume  $(\mathcal{A})$  and assume that  $\Pi(x_0, \tilde{h}^{(0)}) \neq \emptyset$ . Then there exists an optimal solution. Moreover, if  $F$  is jointly strictly concave in  $(y, x, x')$  on  $l_+^\infty \times X \times X$ , then the solution is unique.

We give the existence and uniqueness results in a general framework, in particular without any differentiability assumptions. The strict joint concavity of  $F$  allows to guarantee uniqueness.

The proof can be found in the Appendix.

### 2.1.6 The value function and Bellman equation

We now define the value function of the optimization problem. We show that under the set of assumptions  $(\mathcal{A})$ , dynamic programming tools can be used to study  $\mathcal{P}_{F,\beta,m}(\tilde{h}^{(0)}, x_0)$ . We first study the properties of the value function. We then provide an appropriate set of functions on which the Bellman operator has a unique fixed point which is the value function.

DEFINITION 2.2.— The value function  $V$  is defined on  $l_+^\infty \times X$  by for any  $(\tilde{h}^{(0)}, x_0) \in l_+^\infty \times X$

$$V(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Max} & \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & \tilde{h}^{(0)} \in l_+^\infty \text{ and } x_0 \in X \text{ are given} \end{cases}$$

PROPOSITION 2.3.— Assume  $(\mathcal{A})$ . Then the value function  $V$  is upper semi-continuous.

The proof of this proposition can be found in Appendix.

### 2.1.7 Properties of the value function

The next proposition states some further properties on the value function and the proof is given in Appendix. These properties, together with the upper-semi continuity, will provide an appropriate set of functions to consider for dynamic programming tools.

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<sup>10</sup>i.e. the objective function

PROPOSITION 2.4.— Assume  $(\mathcal{A})$ . Then the value function  $V$  satisfies

- (i)  $\forall x_0, \tilde{h}^{(0)}, \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0), \overline{\lim}_{t \rightarrow +\infty} \beta^t V(\tilde{h}^{(t)}, x_t) \leq 0$ .
- (ii)  $\forall x_0, \tilde{h}^{(0)}$ , and  $\forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$  such that  $\mathcal{U}(\tilde{x}) > -\infty$ ,  $\lim_{t \rightarrow +\infty} \beta^t V(\tilde{h}^{(t)}, x_t) = 0$ .

The proof can be found in the Appendix.

A standard proof (see Theorem 4.4 p.75 Stokey, Lucas and Prescott[41]) allows to show the following result.

PROPOSITION 2.5.— Assume  $(\mathcal{A})$ . Then  $\tilde{x}^*$  is an optimal solution if and only if

$$\forall t \geq 0, V(\tilde{h}^{*(t)}, x_t^*) = F(\tilde{h}^{*(t)}, x_t^*, x_{t+1}^*) + \beta V(\tilde{h}^{*(t+1)}, x_{t+1}^*)$$

where  $\tilde{h}^{*(t)} = (m(x_{t-1}^*, x_t^*), m(x_{t-2}^*, x_{t-1}^*), \dots, m(x_1^*, x_2^*), m(x_0^*, x_1^*), \tilde{h}^{(0)})$

Let  $B$  be the Bellman operator, i.e.  $B : \mathcal{F}(l_+^\infty \times X, \mathbb{R}) \rightarrow \mathcal{F}(l_+^\infty \times X, \mathbb{R})$  be defined by

$$\forall w \in \mathcal{F}(l_+^\infty \times X, \mathbb{R}), Bw(\tilde{h}, x) = \max_{x' \in \Gamma(\tilde{h}, x)} \{F(\tilde{h}, x, x') + \beta w((m(x, x'), \tilde{h}), x')\}$$

DEFINITION 2.3.— Let  $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$  be the set of upper semi-continuous functions  $w \in \mathcal{F}(l_+^\infty \times X, \mathbb{R})$  such that

- (i)  $\forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0), \overline{\lim}_{t \rightarrow +\infty} \beta^t w(\tilde{h}^{(t)}, x_t) \leq 0$ ,
- with  $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$
- (ii)  $\forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$  such that  $\mathcal{U}(\tilde{x}) > -\infty$ , one has  $\lim_{t \rightarrow +\infty} \beta^t w(\tilde{h}^{(t)}, x_t) = 0$

### 2.1.8 The value function is the unique fixed-point of the Bellman operator

We now finally state that the value function is the unique fixed-point of the Bellman operator on this set of functions. The proof is given in Appendix.

PROPOSITION 2.6.— Assume  $(\mathcal{A})$ . Then the value function  $V$  is the unique fixed-point of the Bellman operator on the set of functions  $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$ .

Finally, this shows that dynamic programming tools can be used to deal with general history-dependent optimal growth models.

## 2.2 An easily implementable history-dependent model (EIHDF)

In this section, we provide a general history-dependent framework which is both easily implementable and as general as needed to be widely applicable. This is allowed by providing a history-dependent framework in which the primer of the problem, in particular the instantaneous reward function and the feasible set  $\Gamma$ , are defined in an adequate recursive way answering both issues simultaneously. We then show that such a model is a particular case of our general history-dependent framework and the results for this particular case are derived as corollaries of the results for the general history-dependent framework.

### 2.2.1 The problem

Let  $X, Y$  be topological spaces. Let us consider  $F : \mathbb{R}^N \times X \times X \rightarrow \mathbb{R}$  the instantaneous reward function,  $m : X \times X \rightarrow Y$  the memory function and the (fixed) discount factor  $\beta \in (0, 1)$ . Let us consider an adjustment level function  $\varphi : l_+^\infty \rightarrow \mathbb{R}^N$ , with  $N \in \mathbb{N}$ . Let  $x_0 \in X$  and  $\tilde{h}^{(0)} \in l_+^\infty$  be given. Let us consider the problem

$$\mathcal{P}_{F, \Gamma, m}^\varphi(x_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t F(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \end{cases}$$

est-ce que c'est  $\mathbb{R}^N$  ou  $X^N$  ?

Recall that for any given initial data  $x_0 > 0$ , and initial time-0 history  $\tilde{h}^{(0)} \in l_+^\infty$ , the feasible set  $\Pi(k_0, \tilde{h}^{(0)})$  is defined by the set of sequences feasible from  $x_0$  and  $\tilde{h}^{(0)}$ , i.e. for any  $x_0 > 0$ , for any  $\tilde{h}^{(0)} \in l_+^\infty$ ,

$$\Pi(\tilde{h}^{(0)}, x_0) = \{\tilde{x} = (x_t)_{t=1}^{+\infty} \in X^{\mathbb{N}}, \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t), \tilde{h}^{(t+1)} = (m(x_t, x_{t+1}), \tilde{h}^{(t)})\}$$

### 2.2.2 Assumptions

Let us give a set  $(\mathcal{A}')$  of assumptions: **(F)** on the feasible set, **(A')** on the instantaneous function, and **(M)** on the memory function.

**(A')**

(A1')  $\forall x_0 > 0$  and  $\tilde{h}^{(0)} \neq 0$ ,  $\exists \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$  such that  $\sum_{t=0}^{+\infty} \beta^t F(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}) > -\infty$ .

(A2') The function  $\varphi$  is continuous and there exist  $a \in \mathbb{R}^+$  with  $a\beta < 1$ ,  $a_2 \in \mathbb{R}^+$  and a continuous<sup>11</sup> function  $a_1 : X \rightarrow \mathbb{R}$  such that for any  $x_0 > 0$  and  $\tilde{h}^{(0)} \in l_+^\infty$ , for any feasible sequence  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$  and its associated history  $\tilde{h}^{(t)}$ , for any  $t \geq 0$ ,

$$F^+(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

### 2.2.3 An easily implementable case

An easily implementable case is when the function  $\varphi : l_+^\infty \rightarrow \mathbb{R}^N$ , with  $N \in \mathbb{N}$  can be defined recursively through a function  $G : Y \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ , i.e. for any  $v \in Y$  and any  $\tilde{h} \in l_+^\infty$ , as<sup>12</sup>

$$\varphi(v, \tilde{h}) = G(v, \varphi(\tilde{h}))$$

and when the correspondence  $\Gamma$  can be defined (and easily calculated) by  $\varphi$

$$\Gamma(\tilde{h}, x) = g(\varphi(\tilde{h}), x)$$

with  $g : \mathbb{R}^N \times X \rightarrow X$  is given<sup>13</sup>.

<sup>11</sup>This continuity assumption is needed to prove the upper semi-continuity of the value function (Proposition 4.3).

<sup>12</sup>In the problem, we then will have for any  $t \geq 1$ ,  $\varphi(\tilde{h}^{(t)}) = G(m(x_{t-1}, x_t), \varphi(\tilde{h}^{(t-1)}))$

<sup>13</sup>In the problem, we then will have for any  $t \geq 0$ ,  $\Gamma(\tilde{h}^{(t)}, x_t) = g(\varphi(\tilde{h}^{(t)}), x_t)$

The problem is then written

$$\mathcal{P}_{F,\Gamma,m}^\varphi(x_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t F(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & \forall t \geq 1, \varphi(\tilde{h}^{(t)}) = G(m(x_{t-1}, x_t), \varphi(\tilde{h}^{(t-1)})) \\ & x_0 > 0 \text{ and } \tilde{h}^{(0)} \in l_+^\infty \text{ are given} \end{cases}$$

with  $\forall t \geq 1, \varphi(\tilde{h}^{(t)}) = G(m(x_{t-1}, x_t), \varphi(\tilde{h}^{(t-1)}))$  and  $\Gamma(\tilde{h}^{(t)}, x_t) = g(\varphi(\tilde{h}^{(t)}), x_t)$ .

## 2.2.4 A particular case of our general history-dependent framework

It is a particular case of our general history-dependent framework. Let us define the following instantaneous reward function  $\hat{F} : l_+^\infty \times X \times X \rightarrow \mathbb{R}$  by

$$\hat{F}(\tilde{h}^{(t)}, x_t, x_{t+1}) = F(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1})$$

It is straightforward to check that the problem  $\mathcal{P}_{F,\Gamma,m}^\varphi(x_0, \tilde{h}^{(0)})$  is equivalent to the general history-dependent problem  $\mathcal{P}_{\hat{F},\Gamma,m}(x_0, \tilde{h}^{(0)})$  and that  $(\hat{F}, \Gamma)$  satisfies the set  $(\mathcal{A})$  of assumptions (by  $(\mathcal{A}')$ ),

## 2.2.5 Theorem/proposition (existence d'une solution, programmation dynamique etc)

The following results are then derived as corollaries of the ones previously shown for the general history-dependent model. Assume  $(\mathcal{A}')$  in this subsection hereafter.

**COROLLARY 2.1.**— *Assume  $(\mathcal{A}')$ . The problem  $\mathcal{P}_{F,\Gamma,m}^\varphi(x_0, \tilde{h}^{(0)})$  is well-defined. Assume moreover that  $\Pi(x_0, \tilde{h}^{(0)}) \neq \emptyset$ . Then there exists an optimal solution. Moreover, if  $F$  is jointly strictly concave in  $(y, x, x')$  on  $l_+^\infty \times X \times X$  and  $\varphi$  is linear, then the solution is unique.*

**DEFINITION 2.4.**— *The value function  $V$  is defined on  $l_+^\infty \times X$  by for any  $(\tilde{h}^{(0)}, x_0) \in l_+^\infty \times X$*

$$V(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Max} & \sum_{t=0}^{+\infty} \beta^t F(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & \tilde{h}^{(0)} \in l_+^\infty \text{ and } x_0 \in X \text{ are given} \end{cases}$$

**COROLLARY 2.2.**— *Assume  $(\mathcal{A}')$ . Then the value function  $V$  is upper semi-continuous.*

**COROLLARY 2.3.**— *Assume  $(\mathcal{A}')$ . Then the value function  $V$  satisfies*

- (i)  $\forall x_0, \tilde{h}^{(0)}, \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0), \overline{\lim}_{t \rightarrow +\infty} \beta^t V(\tilde{h}^{(t)}, x_t) \leq 0$ .
- (ii)  $\forall x_0, \tilde{h}^{(0)}$ , and  $\forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$  such that  $\mathcal{U}(\tilde{x}) > -\infty$ ,  $\lim_{t \rightarrow +\infty} \beta^t V(\tilde{h}^{(t)}, x_t) = 0$ .

**COROLLARY 2.4.**— *Assume  $(\mathcal{A}')$ . Then  $\tilde{x}^*$  is an optimal solution if and only if*

$$\forall t \geq 0, V(\tilde{h}^{*(t)}, x_t^*) = F(\varphi(\tilde{h}^{*(t)}), x_t^*, x_{t+1}^*) + \beta V(\tilde{h}^{*(t+1)}, x_{t+1}^*)$$

where  $\tilde{h}^{*(t)} = (m(x_{t-1}^*, x_t^*), m(x_{t-2}^*, x_{t-1}^*), \dots, m(x_1^*, x_2^*), m(x_0^*, x_1^*), \tilde{h}^{(0)})$

Let  $B$  be the Bellman operator, i.e.  $B : \mathcal{F}(l_+^\infty \times X, \mathbb{R}) \rightarrow \mathcal{F}(l_+^\infty \times X, \mathbb{R})$  be defined by

$$\begin{aligned} \forall w \in \mathcal{F}(l_+^\infty \times X, \mathbb{R}), Bw(\tilde{h}, x) &= \max_{x' \in \Gamma(\tilde{h}, x)} \{F(\varphi(\tilde{h}), x, x') + \beta w((m(x, x'), \tilde{h}), x')\} \\ &= \max_{x' \in g(\varphi(\tilde{h}), x)} \{F(\varphi(\tilde{h}), x, x') + \beta w((m(x, x'), \tilde{h}), x')\} \end{aligned}$$

DEFINITION 2.5.— Let  $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$  be the set of upper semi-continuous functions  $w \in \mathcal{F}(l_+^\infty \times X, \mathbb{R})$  such that

$$(i) \forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0), \overline{\lim}_{t \rightarrow +\infty} \beta^t w(\tilde{h}^{(t)}, x_t) \leq 0,$$

with  $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$

$$(ii) \forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0) \text{ such that } \mathcal{U}(\tilde{x}) > -\infty, \text{ one has } \lim_{t \rightarrow +\infty} \beta^t w(\tilde{h}^{(t)}, x_t) = 0$$

COROLLARY 2.5.— Assume  $(\mathcal{A}')$ . Then the value function  $V$  is the unique fixed-point of the Bellman operator on the set of functions  $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$ .

We next propose a general sustainable framework that provides a basis for future research, including environmental and sustainable analysis. It is a particular case of our general history-dependent framework. This illustrates how our general history-dependent framework can be used to deal with many economic issues.

### 3 Application: a general sustainable framework (GSF)

In this section, we present a general sustainable framework. It provides a basis for future analysis in environmental and sustainable issues. It is designed in a very flexible manner. It can be adapted to many contexts and one can easily remove or add different effects that are needed to be addressed. The mathematical results (including existence of a solution and dynamic programming tools) can be directly used. They are derived as an application of the general history-dependent framework presented in the previous section.

It encompasses many existing models in the literature. The next section is devoted to applications ((CHANGER “applications” en “examples” ?)): we detail extensively some of these models within our framework and discuss how our framework is fitted for future research as it is amenable not only to treat various sustainable and environmental issues but also allows to interlink these with many kind of effects and history-dependencies (consumption, production, saving and investment, human capital, labor, consumption habits, recycling habits, pollution, stock of waste, etc).

#### 3.1 The model

We consider an economy in which a unique final good is produced using two factors of production. The *sustainable inputs* obtained from the sustainable process (from waste, recycling actions, reuse, etc) are denoted by  $\xi$  and the *other inputs* (as polluting resources, capital, human capital, labor, other materials, etc) are denoted by  $\kappa$ . At each date  $t$ , the production level  $y_t$  is thus given by the production function  $\hat{f}$  depending on  $\xi_t \in \mathbb{R}$  and  $\kappa_t \in \mathbb{R}^{N_\kappa}$  (with  $N_\kappa \in \mathbb{N}$ ) as

$$y_t = \hat{f}(\xi_t, \kappa_t)$$

In this economy, the representative consumer cares for the state of environment and sustainability. She derives utility from consumption  $c_t$ , from the sustainable process or design  $r_t$  (as for example the recycling level), and from environmental and sustainability variables  $E_t$  (which can be the stock of pollution, the recycling habits, etc). These environmental and sustainability variables depend on all previous decisions-history, thus inducing instantaneous history-dependent preferences. At each date  $t$ , the instantaneous utility of the representative agent thus depends on  $c_t, r_t \in \mathbb{R}^+$  and  $E_t \in \mathbb{R}^{N_E}$  (with  $N_E \in \mathbb{N}$ ) as

$$u(c_t, r_t, E_t)$$

The economy accumulates waste. Indeed, production and consumption generate discards. Discards generated from production come both from the production process itself and the use of the polluting input for producing, so the level  $\hat{D}_p$  of discard from production is a function of the production level  $y_t$  and the other inputs level  $\kappa_t$ . The level  $\hat{D}_c$  of discards from consumption depend on the consumption level  $c_t$ . All what is produced ( $y_t$ ) and all discards that are neither consumed ( $c_t$ ), invested ( $i_t$ ) nor used in the sustainable process ( $r_t$ ), accumulate as waste:

$$s_{t+1} - s_t = \hat{f}(\xi_t, \kappa_t) + \hat{D}_p(y_t, \kappa_t) + \hat{D}_c(c_t) - c_t - i_t - r_t$$

with  $r_t$  depending on the instantaneous waste stock  $s_t$  and on the environmental and sustainability variables  $E_t$ . This is modelled through a function  $\mathcal{R}$  so that we have  $r_t = \mathcal{R}(s_t, E_t)$  such that for all  $s, E$ , one has  $0 \leq \mathcal{R}(s, E) \leq s$ . The investment  $i_t$  depends on the instantaneous and the next period input levels, i.e.  $i_t = \hat{I}(\xi_t, \kappa_t, \xi_{t+1}, \kappa_{t+1})$ . The waste accumulation dynamics is thus given by

$$s_{t+1} - s_t = \hat{f}(\xi_t, \kappa_t) + \hat{D}_p(y_t, \kappa_t) + \hat{D}_c(c_t) - c_t - \hat{I}(\xi_t, \kappa_t, \xi_{t+1}, \kappa_{t+1}) - \mathcal{R}(s_t, E_t)$$

The environmental and sustainability variables  $E_t$  evolve in function of their previous state and the current decisions. The law of motion of the environmental and sustainability variables is thus defined through a given function  $G : \mathbb{R} \times \mathbb{R}^{N_\kappa} \times \mathbb{R} \times \mathbb{R}^{N_E} \rightarrow \mathbb{R}^{N_E}$  by, at each date  $t$ ,

$$E_{t+1} = G((s_t, \kappa_t, c_t), E_t)$$

Given the initial stock of waste  $s_0$ , available inputs  $\kappa_0$ , and initial environmental and sustainability variables  $E_0$ , the representative agent solves the following optimization problem

$$\mathcal{P} = \left\{ \begin{array}{l} \text{Maximize } \sum_{t=0}^{+\infty} \beta^t u(c_t, r_t, E_t) \\ \text{s.t. } \forall t \geq 0, \\ \quad s_{t+1} - s_t = \hat{f}(\xi_t, \kappa_t) + \hat{D}_p(y_t, \kappa_t) + \hat{D}_c(c_t) - c_t - \hat{I}(\xi_t, \kappa_t, \xi_{t+1}, \kappa_{t+1}) - \mathcal{R}(s_t, E_t) \\ \quad E_{t+1} = G(s_t, \kappa_t, c_t), E_t) \\ \quad r_t = \mathcal{R}(s_t, E_t) \in [0, s_t] \\ \quad s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{array} \right.$$

In a sustainable economy, a key assumption is that production involves an input which is obtained through sustainable actions. Here, it is assumed that

$$\xi_t = \mathcal{R}(s_t, E_t)$$

Then, by defining  $f(s_t, \kappa_t, E_t) := \hat{f}(\mathcal{R}(s_t, E_t), \kappa_t)$ ,  $\mathcal{D}_p(s_t, \kappa_t) := \hat{\mathcal{D}}_p(f(s_t, \kappa_t), \kappa_t)$  and  $\mathcal{I}(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}) = \hat{\mathcal{I}}(\mathcal{R}(s_t, E_t), \kappa_t, \mathcal{R}(s_{t+1}, E_{t+1}), \kappa_{t+1})$ , the problem is written

$$\mathcal{P} = \begin{cases} \text{Maximize } \sum_{t=0}^{+\infty} \beta^t u(c_t, r_t, E_t) \\ \text{s.t. } \forall t \geq 0, \\ s_{t+1} - s_t = f(s_t, \kappa_t, E_t) + \mathcal{D}_p(s_t, \kappa_t) + \mathcal{D}_c(c_t) - c_t - \mathcal{I}(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}) - \mathcal{R}(s_t, E_t) \\ E_{t+1} = G((s_t, \kappa_t, c_t), E_t) \\ r_t = \mathcal{R}(s_t, E_t) \in [0, s_t] \\ s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{cases}$$

This framework can be easily adapted to remove or add any environmental and sustainability variables  $E$  related effects, as for example the ones played by recycling habits, or pollution, etc, adding or removing them in the preferences and corresponding accumulation laws of motion. For simplicity of exposition, we assumed that  $\xi$  and  $s$  belong to  $\mathbb{R}$  but the framework can easily be adapted to deal with several kinds of sustainable inputs and several kinds of waste that may play different roles in the economy. Linear economy models are encompassed by assuming there is no input coming from any circular process (i.e. all the functions involved are constant with respect to  $\xi$  and there is no  $\mathcal{R}$  involved). This will be more extensively discussed afterwards.

### 3.2 Assumptions

Let us assume that the function  $c \rightarrow c - \mathcal{D}_c(c)$  is bijective and let us define the function  $C : \mathbb{R}^{N_v} \rightarrow \mathbb{R}$  by for all  $(s, \kappa, s', \kappa', E) \in \mathbb{R}^{N_v}$  (with  $N_v := 2 + 2N_\kappa + N_E$ ),

$$C(s, \kappa, s', \kappa', E) = (Id - \mathcal{D}_c)^{-1}(f(s, \kappa, E) + \mathcal{D}_p(s, \kappa) - s' - \mathcal{I}(s, \kappa, s', \kappa') + s - \mathcal{R}(s, E))$$

This gives the consumption level as a function of the stocks of waste, the inputs, and the environmental and sustainability variables.

The general sustainable problem is equivalent to

$$\mathcal{P} = \begin{cases} \text{Maximize } \sum_{t=0}^{+\infty} \beta^t u(C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t), \mathcal{R}(s_t, E_t), E_t) \\ \text{s.t. } \forall t \geq 0, \\ C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t) \geq 0 \\ \forall t \geq 0, E_{t+1} = G((s_t, \kappa_t, C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t)), E_t) \\ \mathcal{R}(s_t, E_t) \in [0, s_t] \\ s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{cases}$$

Let us consider the following set  $\mathcal{S}$  of assumptions<sup>14</sup>.

(S1) The functions  $u, f, \mathcal{D}_c, \mathcal{D}_p, \mathcal{I}, \mathcal{R}, G$  are continuous, discards are bounded (both below and from above), and the function  $c \rightarrow c - \mathcal{D}_c(c)$  is bijective.

(S2) (i) There exists a bounded function  $E : \mathbb{R}^{N_\kappa + N_Z} \rightarrow \mathbb{R}$  such that for all  $(s, \kappa, s', \kappa', E) \in \mathbb{R}^{N_v}$ ,

$$C(s, \kappa, s', \kappa', E) \geq 0 \Rightarrow \|(s', \kappa')\| \leq E(s, \kappa, E)$$

<sup>14</sup>((enlever cette footnote)) on a besoin de supposer que  $x, z \geq 0$  pour compacite de  $\Gamma$

La partie recyclee  $\mathcal{R}(\pi_1(x), \varphi(\hat{h}))$  du waste  $s_t$  doit etre plus petite que  $s_t$ , autrement dit la partie non recycled  $\mathcal{N}\mathcal{R}(\pi_1(x), \varphi(\hat{h})) = s_t - \mathcal{R}(\pi_1(x), \varphi(\hat{h})) \geq 0$  doit etre positive (ou nulle)



(ii) There exists  $a \geq 0, a \neq 1$  and  $a' \geq 0$  such that for all  $(s, \kappa, E) \in \mathbb{R}^{N_\kappa + N_E}$ ,

$$E(s, \kappa, E) \leq a' \|(s, \kappa)\| + a$$

(S3) There exist  $a \in \mathbb{R}^+$  with  $a\beta < 1$ ,  $a_2 \in \mathbb{R}^+$  and a continuous<sup>15</sup> function  $a_1 : X \rightarrow \mathbb{R}$  such that for any  $x_0 = (s_0, \kappa_0) > 0$  and  $E_0$ , for any feasible sequence  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(E_0, x_0)$  and its associated history  $\tilde{h}^{(t)}$ , for any  $t \geq 0$ ,

$$u^+(C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t), \mathcal{R}(s_t, E_t), E_t) \leq a_1(x_0)a^t + a_2$$

These assumptions are usual (see Le Van Dana[43]). They are not quite restrictive (see Le Van and Morhaim[25]). They allow to cover situations with unboundedness in the objective function combined with various types of the feasible set as used in the literature (i.e. with various returns to scale technology).

They ensure that the assumptions in Section 2.2.2 are satisfied. The assumption that  $(Id - \mathcal{D}_c)$  is bijective allows to uniquely express the instantaneous consumption in terms of the waste stocks, the other inputs stocks, and environmental and sustainable variables, and greatly simplifies the notations. The continuity in (S1) and (S2) (i) ensure (F1) so that  $\Gamma$  is compact-valued. The existence of a function  $E$  in Assumption (S2) (ii) ensures Assumption (F2) and Assumptions (S3) and (S4) ensure Assumption (A2').

### 3.3 The general sustainable framework is a particular case of the general history-dependent framework

In this section, we show that the general sustainable framework is a particular case of the general history-dependent framework.

The general sustainable framework is

$$\mathcal{P} = \begin{cases} \text{Maximize } \sum_{t=0}^{+\infty} \beta^t u(C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t), \mathcal{R}(s_t, E_t), E_t) \\ \text{s.t. } \forall t \geq 0, \\ C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t) \geq 0 \\ E_{t+1} = G((s_t, \kappa_t, C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t)), E_t) \\ \mathcal{R}(s_t, E_t) \in [0, s_t] \\ s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{cases}$$

We explain hereafter that the problem is indeed a particular case of our general history-dependent framework.

Let us define  $X := \mathbb{R}^{1+N_\kappa}$  and  $x := (s, \kappa)$ . Let the memory function  $m : X \times X \rightarrow Y$  be defined with  $Y := \mathbb{R}^{2+2N_\kappa}$  by for any  $(x, x') \in X \times X$ ,<sup>16</sup>

$$m(x, x') = (x, x')$$

<sup>15</sup>This continuity assumption is needed to prove the upper semi-continuity of the value function (Proposition 4.3).

<sup>16</sup>((a enlever))

$$m(x_t, x_{t+1}) = (x_t, x_{t+1})$$

(so  $x_t := (s_t, z_t)$  et  $x_{t+1} := (s_{t+1}, z_{t+1})$ ). This means that  $m(x_t, x_{t+1}) = (x_t, x_{t+1}) = (s_t, z_t, s_{t+1}, z_{t+1})$ . For  $(x, x') = (s, z, s', z')$ , we denote  $\pi_{11}(x, x') = s, \pi_1(x, x') = x, \pi_2(x, x') = x'$ . Note that  $m$  is continuous and satisfies (m1).

Let us define the function  $\hat{G}$  by for any  $(x, x') \in X \times X, E \in \mathbb{R}^{N_E}$ ,

$$\hat{G}((x, x'), E) = G((x, C(x, x', E)), E)$$

The adjustment level function  $\varphi : l_+^\infty \rightarrow Z$  with<sup>17</sup>  $Z = \mathbb{R}^{N_M}$  is defined recursively in the following way:<sup>18</sup>

$$\forall \tilde{h} \in l_+^\infty, \forall (x, x') \in X \times X, \varphi(m(x, x'), \tilde{h}) = \hat{G}(m(x, x'), \varphi(\tilde{h}))$$

By defining<sup>19</sup>

$$F(\tilde{h}, x, x') = u(C(x, x', \varphi(\tilde{h})), \mathcal{R}(\pi_1(x), \varphi(\tilde{h})), \varphi(\tilde{h}))$$

and  $\Gamma$  is defined by for all  $\tilde{h} \in l_+^\infty$  and  $x \in \mathbb{R}^2$ ,

$$\Gamma(\tilde{h}, x) = \{x' \in (\mathbb{R}^+)^2, C(x, x', \varphi(\tilde{h})) \geq 0\}$$

The feasible correspondence  $\Gamma$  can also be written<sup>20</sup>

$$\Gamma(\tilde{h}, x) = g(\varphi(\tilde{h}), x)$$

with the correspondence  $g : Z \times X \rightarrow X$  defined for all  $(k, x) \in Z \times X$  by

$$g(k, x) = \{x' \in \mathbb{R}^2, C(x, x', k) \geq 0\}$$

Then the general sustainable model is written

$$P(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Max} & \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & x_0 > 0 \text{ and } \tilde{h}^{(0)} \in l_+^\infty \text{ are given} \end{cases}$$

which shows that our general sustainable framework fits general history-dependent model. Hence, the results shown in Section 2 can be applied to the general sustainable problem. In particular, we

<sup>17</sup>pour  $\Gamma$  on a One has for all  $x, \Gamma(\tilde{h}, x)$  is bounded from below ( $x \in (\mathbb{R}^+)^2$ ) and by the assumption on the functions  $\mathcal{D}, f$  etc  $\Gamma(\tilde{h}, x)$  is bounded above.

$Z = \mathbb{R}^2$  parce qu'on a  $H_{t+1} \in \mathbb{R}$  et  $P_{t+1} \in \mathbb{R}$

<sup>18</sup>((a enlever))  $\forall \tilde{h}^{(t+1)} \in l_+^\infty, \varphi(\tilde{h}^{(t+1)}) = (\varphi_1(\tilde{h}^{(t+1)}), \varphi_2(\tilde{h}^{(t+1)}))$

$$\forall \tilde{h}^{(t+1)} \in l_+^\infty, \varphi(\tilde{h}^{(t+1)}) = \hat{G}(m(x_t, x_{t+1}), \varphi(\tilde{h}^{(t)}))$$

<sup>19</sup>Indeed, by the definition of  $F$ , then

$$F(\tilde{h}^{(t)}, x_t, x_{t+1}) = u(C(x_t, x_{t+1}, \varphi(\tilde{h}^{(t)})), \mathcal{R}(\pi_1(x_t), \varphi(\tilde{h}^{(t)})), \varphi(\tilde{h}^{(t)}))$$

i.e.

$$F(\tilde{h}^{(t)}, x_t, x_{t+1}) = u(C(x_t, x_{t+1}, \varphi_2(\tilde{h}^{(t)}), \varphi_2(\tilde{h}^{(t)})), \mathcal{R}(\pi_1(x_t), \varphi_1(\tilde{h}^{(t)}), \varphi_2(\tilde{h}^{(t)})), \varphi_1(\tilde{h}^{(t)}), \varphi_2(\tilde{h}^{(t)}))$$

i.e.

$$F(\tilde{h}^{(t)}, x_t, x_{t+1}) = u(C(s_t, z_t, s_{t+1}, z_{t+1}, P_t, H_t), \mathcal{R}(s_t, H_t, P_t), H_t, P_t)$$

<sup>20</sup>((a enlever)) au lieu de considerer la correspondance  $\Gamma$  sur les suites infinies  $\tilde{h}$  on peut ecrire le probleme implementable en definissant la correspondance  $g$  sur  $\varphi(\tilde{h})$

obtain the existence of a solution and the dynamic programming tools described in Section 2 can be implemented.

This general sustainable framework can be easily adapted to various contexts and models. In the next section, we give existing models in the related literature as applications of our framework.

## 4 Some models and examples

In this section, we discuss how our general framework allows to study issues and models from various literature strands. As we introduced a general function allowing to model many different memory processes, general decision variables, objective functions and feasible sets, we are able to encompass many existing models. It generalizes the history-dependent intertemporal optimization models provided in Morhaim and Ulus[32]. Thus, the applications presented in Morhaim and Ulus[32] are encompassed, including seminal models dealing with habit formation (Ryder and Heal[17], Rozen[36], Rustichini and Siconolfi[38], Carroll, Overland and Weil[6]) and satiation (He, Dyer and Butler[16], Baucells and Sarin[4]), as well as environmental models (Ikefuji[18], Löfgren[27]) and Safi and Ben Hassen[39]) and optimal management of natural resources (Smulders, Toman and Withagen[40], Ulus[42]). Moreover, it encompasses circular economy models (George, Chi-ang Lin and Chen[14], Kasioumi[20], Kasioumi and Stengos[23]) and circular and causation models (Donaghy[10, 11]).

We next discuss the way these existing models are particular cases of our general framework as well as how its flexibility allows to use it in future research. Our general framework allows to study simultaneously many effects and contexts: circular models without production waste (Section 4.4), linear economies with production waste, and furthermore circular economies with production waste, as well as other many effects.

### 4.1 History-dependent optimal growth models with (consumption) habit formation or satiation

In Morhaim and Ulus[32], a representative agent consumes a single good on periods  $t = 0, 1, 2, \dots$  and maximizes her intertemporal utility over the consumption stream  $\tilde{c} = (c_0, c_1, \dots)$  in  $l^\infty$ . At date  $t$ , the consumer's instantaneous utility depends on her current consumption  $c_t$ . But it also depends on her *time- $t$  (consumption) history*  $\tilde{h}^{(t)}$  which is defined from an initial history  $\tilde{h}^{(0)} \in l_+^\infty$  and keeps in memory the consumption decisions as follows:

$$\forall t \geq 1, \tilde{h}^{(t)} = (h_j^{(t)})_{j=1}^\infty := (c_{t-1}, \tilde{h}^{(t-1)}) = (c_{t-1}, \dots, c_0, \tilde{h}^{(0)})$$

Her time  $t$ -utility  $u : \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \rightarrow \mathbb{R} \cup \{-\infty\}$  changes endogenously from her time- $t$  consumption history  $\tilde{h}^{(t)}$  through the adjustment level function  $\varphi : l_+^\infty \rightarrow R$  with  $R = (\mathbb{R}^+)^n$  (where  $n \geq 1$ ). For initially given capital stock  $k_0 > 0$  and time-0 history  $\tilde{h}^{(0)} \in l_+^\infty$ , the general framework and optimization problem, with  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  the production function,  $k_t$  the capital stock at time  $t$ , and  $\beta \in (0, 1)$  the fixed discount factor, is given as follows:

$$\mathcal{P}_{u, \varphi, \beta}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} = f(k_t) - c_t, k_t \geq 0 \text{ and } c_t \geq 0 \\ & \forall t \geq 1, \tilde{h}^{(t)} = (c_{t-1}, \dots, c_1, c_0, \tilde{h}^{(0)}) \\ & \forall t \geq 0, (c_t, \varphi(\tilde{h}^{(t)})) \in \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \end{cases}$$

The problem can be rewritten as follows:

$$\mathcal{P}_{u,\varphi,\beta}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} \in [0, f(k_t)] \\ & \forall t \geq 1, \tilde{h}^{(t)} = (f(k_{t-1}) - k_t, f(k_{t-2}) - k_{t-1}, \dots, f(k_1) - k_2, f(k_0) - k_1, \tilde{h}^{(0)}) \\ & \forall t \geq 0, (f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \in \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \end{cases}$$

This is a case in which *consumption* is kept in memory, which is a function of previous date and current capital stock decisions. Thus, it means keeping in memory a function of these decisions. Our general history-dependent framework allows, not only to keep in memory *the particular function* defining consumption, but *any function* of the previous date and current decisions. By this way, the model provided in Morhaim and Ulus[32] as well as the models presented in Morhaim and Ulus[32]<sup>21</sup> become particular cases of our general history-dependent framework. Indeed, let us define  $X = Y = \mathbb{R}^+$ ,  $x_t = k_t \in X$  for all  $t = 0, \dots, +\infty$ , and  $F : l_+^\infty \times X \times X \rightarrow \mathbb{R}$  by for any  $\tilde{h} \in l_+^\infty$  and  $x, x' \in X$ ,

$$F(\tilde{h}, x, x') := u(f(x) - x', \varphi(\tilde{h}))$$

We define for any decision  $x'$  given  $x$ , the *memory function/process*  $m : X \times X \rightarrow Y$  by

$$m(x, x') = f(x) - x'$$

and for history  $\tilde{h} \in l_+^\infty$  and for any  $x \in \mathbb{R}^+$ , the feasible correspondence  $\Gamma$  is given by

$$\Gamma(\tilde{h}, x) = \{x' \in [0, f(x)], (f(x) - x', \varphi(\tilde{h})) \in \mathcal{D}_u\}$$

The introduction of the memory function  $m$  to the modelling allows to explicitly study many memory processes. Here, it is done by defining  $m$  which associates the consumption to the decisions.

## 4.2 History-dependent optimal growth models with environmental effects

History-dependence is important in environmental economics models. Morhaim and Ulus[32] already underlined that their general framework allows to deal with environmental effects. We show in this section how Löfgren[27]'s model writes in our new general history-dependent and further describe how the discrete time version of Ikefuji[18]'s model is also encompassed.

Löfgren[27] proposes a model with environmental quality habit formation and in which a consumption good moreover causes a negative external effect on the environment. The social planner maximizes the utility given the negative effect of the consumption good on the environment and taking into account that there is habit formation in environmental quality. The instantaneous utility  $u(n_t, x_t, z_t, s_t)$  depends on  $n_t$  which is the environment that displays habit formation,  $x_t$  the “dirty” consumption good (the environmental bad),  $z_t$  the “clean” consumption good and  $s_t$  the habit level related to the environment. The following relations are satisfied with  $\gamma \in (0, 1), \beta \in (0, 1), \delta \in (0, 1)$ ,  $y$  being an exogenously given income and  $n$  is a given initial environment

$$\begin{cases} n_t = n - \gamma x_t \\ z_t = y - x_t \\ s_{t+1} = \beta n_t + (1 - \delta) s_t \end{cases}$$

<sup>21</sup>including Ryder and Heal[17], Rozen[36], Rustichini and Siconolfi[38], Carroll, Overland and Weil[6], He, Dyer and Butler[16], Baucells and Sarin[4], Ikefuji[18], Löfgren[27], Safi and Ben Hassen[39].

Löfgren[27]'s model is encompassed in our general framework. Let us define  $X = Y = \mathbb{R}^+$   $x_t = k_t \in X$  for all  $t = 0 \dots \infty$ , and  $\varphi$  is defined through the following recurrence relation

$$\forall c \in \mathbb{R}^+, \forall \tilde{h} \in l_+^\infty, \varphi((c, \tilde{h})) = \beta n + (1 - \delta)\varphi(\tilde{h}) - \beta\gamma c$$

Let us define  $F : l_+^\infty \times X \times X \rightarrow \mathbb{R}$  by, for any  $\tilde{h} \in l_+^\infty$  and  $x, x' \in X$ ,

$$F(\tilde{h}, x, x') := u(n - \gamma(f(x) - x'), f(x) - x', y - (f(x) - x'), \varphi(\tilde{h}))$$

We define for any decision  $x'$  given  $x$ , the *memory function/process*  $m : X \times X \rightarrow Y$  by  $m(x, x') = f(x) - x'$ , and for any  $\tilde{h} \in l_+^\infty$  and  $x \in \mathbb{R}^+$ , the feasible correspondence  $\Gamma$  is given by

$$\Gamma(\tilde{h}, x) = \{x' \in [0, f(x)], (n - \gamma(f(x) - x'), f(x) - x', y - (f(x) - x'), \varphi(\tilde{h})) \in \mathcal{D}_u\}$$

We further describe how the discrete time version of Ikefuji[18]'s model is encompassed in our framework. Ikefuji[18] studies habit formation in consumption and pollution abatement activities when agents derive disutility both from the habit stock and pollution. The pollution  $P_t$  in period  $t$  is generated by the capital stock  $k_t$  used in production and reduced by abatement activities  $a_t$  in the same period. The problem is written as follows.

$$P(h_0, k_0, m_0) = \begin{cases} \text{Max} & \sum_{t=0}^{+\infty} \beta^t u(c_t, H_t, P_t) \\ \text{s.t.} & \forall t \geq 1, H_{t+1} = \rho c_t + (1 - \rho)H_t \text{ and } P_t = \left(\frac{k_t}{a_t}\right)^\phi \\ & \forall t \geq 0, k_{t+1} = Ak_t - c_t - a_t - k_t \\ & k_0 > 0, m_0 > 0 \text{ and } h_0 \text{ are given} \end{cases}$$

where  $c_t$  denotes the consumption in period  $t$ ,  $H_t$  denotes the consumption habit, and  $P_t$  is the level of aggregated pollution in the economy.

In order to cover this problem in our general framework, let us define  $X = (\mathbb{R}^+)^2, Y = \mathbb{R}^+$ , and  $x_t = (k_t, a_t) \in X$  for all  $t = 0 \dots \infty$ . The function  $\varphi : l_+^\infty \rightarrow \mathbb{R}$  is defined by  $\varphi(\tilde{h}^{(0)}) = h_0$  and

$$\forall \tilde{h} \in l_+^\infty, \forall c \in \mathbb{R}_+, \varphi(c, \tilde{h}) = G(c, \varphi(\tilde{h})).$$

with the function  $G$  defined by  $G(c, y) = \rho c + (1 - \rho)y$ .

Also, let us define  $F : l_+^\infty \times X \times X \rightarrow \mathbb{R}$  by, for any  $\tilde{h} \in l_+^\infty$  and  $x, x' \in X$ ,

$$F(\tilde{h}, x, x') := u(f(\pi_1(x)) - \pi_1(x'), \varphi(\tilde{h}), \left(\frac{\pi_1(x)}{\pi_2(x)}\right)^\phi)$$

We define for any decision  $x'$  given  $x$ , the *memory function/process*  $m : X \times X \rightarrow Y$  by

$$m(x, x') = f(\pi_1(x)) - \pi_1(x')$$

and for history  $\tilde{h} \in l_+^\infty$  and for any  $x \in \mathbb{R}^+$ , the feasible correspondence  $\Gamma$  is given by

$$\Gamma(\tilde{h}, x) = \{x' \in [0, f(\pi_1(x))], \pi_2(x') > 0, (f(\pi_1(x)) - \pi_1(x'), \varphi(\tilde{h}), \left(\frac{\pi_1(x)}{\pi_2(x)}\right)^\phi) \in \mathcal{D}_u\}$$

These two examples illustrate how our framework is on one hand as general and easy to use as needed and on the other hand allows to keep the effects to be studied visible and interpretable, specially to deal with environmental economic issues. In the next section, we show how it is also fitted to optimal management of natural resources.

### 4.3 Optimal management of natural resources

Smulders, Toman and Withagen[40] and Ulus[42] present models referring to the optimal management of an exhaustible natural resource like oil, coal, gas and etc. In these models, there exists a single planner who manages the decision of extraction and consumption of a single exhaustible natural resource on periods  $t = 0, 1, 2, \dots$  in order to maximize her intertemporal utility. At each period  $t$ , the instantaneous utility depends on its consumption  $c_t$  and also on the stock of the natural resource, which is denoted by  $s_t$ . The output  $y_t \in \mathbb{R}_+$  is produced from capital ( $k_t$ ) and extracted quantity of resource ( $r_t$ ) (or the extraction flow of natural resource per unit of time) by a production function  $f : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  where  $y_t = f(k_t, r_t)$ . The output is either consumed as  $c_t \geq 0$  or saved as capital to the next period as  $k_{t+1}$  satisfying:

$$c_t + k_{t+1} \leq f(k_t, r_t) + (1 - \delta)k_t \text{ with } k_t \geq 0$$

where  $\delta$  stands for the depreciation rate of capital.

For initially given stocks of  $k_0, s_0, r_0 > 0$  and  $\beta \in (0, 1)$  the fixed discount factor, the problem of the planner is given as follows:

$$\mathcal{P}((k_0, r_0), s_0) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, s_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} = s_t - r_t \\ & \forall t \geq 0, 0 \leq c_t = f(k_t, r_t) + (1 - \delta)k_t - k_{t+1} \\ & s_0, k_0, r_0 > 0 \text{ are given} \end{cases}$$

which can be rewritten as

$$\mathcal{P}((k_0, r_0), s_0) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(f(k_t, r_t) + (1 - \delta)k_t - k_{t+1}, s_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} = s_t - r_t \\ & \forall t \geq 0, 0 \leq k_{t+1} \leq f(k_t, r_t) + (1 - \delta)k_t \\ & s_0, k_0, r_0 > 0 \text{ are given} \end{cases}$$

Let us define  $G$  by  $G(s, r) := s - r$ . Note that our framework allows to consider more general forms of  $G$ , for example if it comes to taking into account the resources regeneration (whether naturally or otherwise). This model can be written similarly to Morhaim and Ulus[32] framework, defining  $\varphi$  by the law of motion

$$\forall t \geq 0, \varphi(\tilde{h}^{(t+1)}) = G(r_t, \varphi(\tilde{h}^{(t)}))$$

so that the instantaneous utility  $u(c_t, s_t)$  is  $u(c_t, \varphi(\tilde{h}^{(t)}))$ . By defining  $\tilde{h}^{(0)}$  such that  $\varphi(\tilde{h}^{(0)}) = s_0$  and for any  $t \geq 0$ ,  $\tilde{h}^{(t+1)}$  such that  $\tilde{h}^{(t+1)} = (r_t, \tilde{h}^{(t)})$ , the model can be written

$$\mathcal{P}_{u, \varphi, \beta}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(f(k_t, r_t) + (1 - \delta)k_t - k_{t+1}, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} \in [0, f(k_t, r_t) + (1 - \delta)k_t] \\ & \forall t \geq 0, \tilde{h}^{(t+1)} = (r_t, \dots, \tilde{h}^{(0)}) \end{cases}$$

The model can be written in our general framework by defining  $x := (k, r)$  and the memory function by  $m(x, x') = \pi_2(x)$ . Let us define the objective function  $F$  by, for any  $(x, x', \tilde{h})$ ,

$$F(x, x', \tilde{h}) := u(f(\pi_1(x), \pi_2(x)) + (1 - \delta)\pi_1(x) - \pi_1(x'), \varphi(\tilde{h}))$$

and the feasible set by, for any  $(\tilde{h}, x)$ ,

$$\Gamma(\tilde{h}, x) := \{x', 0 \leq \pi_1(x') \leq f(\pi_1(x), \pi_2(x)) + (1 - \delta)\pi_1(x), 0 \leq \pi_2(x') \leq \varphi(\tilde{h})\}$$

In the next section, we show that our results are also fitted for circular economy issues.

In the next section, we show that our results are also fitted for circular economy issues.

## 4.4 Circular economy (CE) and circular and cumulative causation (CCC) models

Circular economy is most frequently depicted as a combination of reduce, reuse and recycle activities (Kirchherr, Reike and Hekkert[24]), but also design, implying a focus on the entire life cycle of the processes as well as the interaction between the process and the environment and the economy in which it is embedded (Ghisellini, Cialani and Ulgiati[15]).

In this section, we show that the circular economy models provided by George, Chi-ang Lin and Chen[14], Kasioumi and Stengos[22], Kasioumi[21] and Donaghy[10, 11] are particular cases of our general sustainable framework. We first consider circular economy models without recycling habits (George, Chi-ang Lin and Chen[14] and Kasioumi and Stengos[22]), then circular economy models with recycling habits (Kasioumi[20]). We also show that our framework is adapted to the circular and cumulative causation models as developed by Donaghy[10, 11] .

### 4.4.1 CE models without recycling habits

In this section, we consider the circular models without recycling habits proposed by George, Chi-ang Lin and Chen[14], and Kasioumi and Stengos[22].

George, Chi-ang Lin and Chen[14] and Kasioumi and Stengos[22] consider a closed economy with zero population growth. They abstract from capital accumulation and technical progress<sup>22</sup>. The social planner maximizes an intertemporal utility where the instantaneous utility  $u(c, P)$  depends on consumption  $c$  and the stock of pollution  $P$ . The output is produced via a (concave) production function  $\phi$  using two factors of production, one which corresponds to the rate of use of the recyclable resource and another ( $z$ ) corresponding to the rate of use of the environmentally polluting resource. Output produced in any given period but not consumed or used for the employment of the polluting resource, accumulates as (potentially recyclable) waste. Recycling turns waste into a useful factor of production: a proportion  $b$  of the waste stock  $s$  with intensity of recycling  $\tau$  is supposed to be recycled each period. George, Chi-ang Lin and Chen[14]'s model is a particular case of Kasioumi and Stengos[22] in which the intensity of recycling  $\tau$  is equal to one. The social planner solves the following optimization problem

$$\mathcal{P}((s_0, z_0), p_0) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, p_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} - s_t = \phi(\tau b s_t, z_t) - c_t - \alpha z_{t+1} - \tau b s_t \\ & \forall t \geq 0, p_{t+1} - p_t = \theta z_t - \delta p_t + (1 - b)s_t \\ & s_0, z_0 > 0, p_0 \geq 0 \text{ are given} \end{cases}$$

By defining  $\kappa := z$  (thus  $N_\kappa = 1$ ),  $E := p$  (thus  $N_E = 1$ ),  $f(s, \kappa, E) := \phi(\tau b s, \kappa)$ ,  $\mathcal{D}_p(s, \kappa) := 0$ ,  $\mathcal{D}_c(c) := 0$ ,  $\mathcal{I}(s, \kappa, s', \kappa') := \alpha \kappa'$ ,  $\mathcal{R}(s, E) := \tau b s$  and  $G((s, \kappa, c), E) := (1 - \delta)E + \theta \kappa + (1 - b)s$ ,

<sup>22</sup>However, our results apply to such models with capital accumulation and technical progress (see Section 4.4.3).

both models (George, Chi-ang Lin and Chen[14] and Kasioumi and Stengos[22])) fit our general sustainable framework.

#### 4.4.2 CE models with recycling habits

In this section, we consider the circular models with recycling habits proposed by Kasioumi[20, 21]. It is an extension of the theoretical work of Kasioumi and Stengos[22], combining elements of the circular economy model of George, Chi-ang Lin and Chen[14] with the habit formation framework of Ikefuji[18]. Kasioumi[20, 21] deals with a fixed intensity of recycling while Kasioumi[21] deals further with an intensity  $\tau(H_t)$  depending (through an affine function  $\tau(H_t) := \eta + rH_t$ ) on the level of recycling habits. Note that our framework fits also recycling intensity functions  $\tau$  that are not necessarily affine. The social planner solves the following optimization problem

$$\mathcal{P}((s_0, z_0), p_0) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, r_t, H_t, p_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} - s_t = \phi(\tau(H_t)bs_t, z_t) - c_t - \alpha z_{t+1} - \tau(H_t)bs_t \\ & \forall t \geq 0, p_{t+1} - p_t = \theta z_t - \delta p_t + (1 - b)s_t + \rho c_t \\ & H_{t+1} - H_t = \mu(\tau(H_t)bs_t - H_t) \\ & s_0, z_0 > 0, p_0 \geq 0 \text{ are given} \end{cases}$$

By defining  $\kappa := z$  (thus  $N_\kappa = 1$ ),  $E := (p, H)$  (thus  $N_E = 2$ ),  $f(s, \kappa, E) := \phi(\tau(\pi_2(E))bs, e)$ ,  $\mathcal{D}_p(s, \kappa) := 0$ ,  $\mathcal{D}_c(c) := 0$ ,  $\mathcal{I}(s, \kappa, s', \kappa') := \alpha\kappa'$ ,  $\mathcal{R}(s, E) := \tau(\pi_2(E))bs$  and

$$G((s, \kappa, c), E) := ((1 - \delta)\pi_1(E) + \theta\kappa + (1 - b)s + \rho c, \mu(\tau(\pi_2(E))bs - \pi_2(E))$$

these models (Kasioumi[20, 21]) fit our general sustainable framework.

#### 4.4.3 CCC models

The theoretical circular-economy model of economic growth with circular and cumulative causation (CCC) is presented in Donaghy[10, 11] as follows. The George Lin and Chen[14]'s model is modified by modelling capital formation and technical change and including physical capital  $K$ , human capital  $HC$ , labor  $L$ , and other materials  $OM$ , as productive factors. The optimization problem for the social planner is to choose control variables, consumption  $c$  and environmentally polluting resource  $z$  but also investments in physical and social capital to maximize the intertemporal utility function.

Given the functions  $b : \mathbb{R} \rightarrow \mathbb{R}$  accounting for increased efficiencies in recycling (thus depends on the accumulated recycling experience  $R \in \mathbb{R}$ ), the function  $\vartheta : \mathbb{R}^2 \rightarrow \mathbb{R}$  characterizing technical progress in reducing pollution from the polluting resource as a function of human capital deepening (thus depends on  $(K, HC) \in \mathbb{R}^2$ ), the production function  $\phi$ , and the instantaneous utility function



$u$  depending on consumption and pollution  $(c, P) \in \mathbb{R}^2$ , the model in Donaghy[10, 11] is as follows<sup>23</sup>

$$\mathcal{P} = \left\{ \begin{array}{l} \text{Maximize } \sum_{t=0}^{+\infty} \beta^t u(c_t, P_t) \\ \text{subject to, for all } t \geq 0, \\ s_{t+1} - s_t = \phi(s_t, z_t, K_t, HC_t, L_t, OM_t, R_t) - c_t - \alpha z_{t+1} \\ \quad \quad \quad - (K_{t+1} - K_t)(1 + \frac{a_1}{2K_t}) - (HC_{t+1} - HC_t)(1 + \frac{a_2}{2HC_t}) - b(R_t)s_t \\ P_{t+1} - P_t = \vartheta(K_t, HC_t)z_t - \delta P_t + (1 - b(R_t))s_t \\ R_{t+1} - R_t = b(R_t)s_t \\ s_0, z_0 > 0, P_0 \geq 0 \text{ are given} \\ K_0, R_0, HC_0, L_0 \geq 0 \text{ are given} \end{array} \right.$$

Let us define  $\kappa := (z, K, L, HC, OM)$  (thus  $N_\kappa = 5$ ),  $E := (P, R)$ ,  $\mathcal{I}(s, \kappa, s', \kappa') := \alpha\pi_1(\kappa') + (\pi_2(\kappa') - \pi_2(\kappa))(1 + \frac{a_1}{2\pi_2(\kappa)}) + (\pi_4(\kappa') - \pi_4(\kappa))(1 + \frac{a_2}{2\pi_4(\kappa)})$  with for any  $i = 1, \dots, 5$ ,  $\pi_i$  is the  $i$ -th projection<sup>24</sup>,  $\mathcal{D}_p(s, \kappa) = 0$ ,  $\mathcal{D}_c(c) = 0$  and  $\mathcal{R}(s, E) = b(\pi_2(E))s$  and  $G$  by

$$G((s, \kappa, c), E) = (\vartheta(\pi_2(\kappa), \pi_4(\kappa))\pi_1(\kappa) + (1 - \delta)\pi_1(E) + (1 - b(\pi_2(E)))s, \pi_2(E) + b(\pi_2(E))s).$$

The sustainable process or design variable  $r$  does not enter directly Donaghy's utility function in contrast with this possibility which is allowed in our framework. When the process does not enter directly in the utility, it suffices to define the utility function  $\hat{u}(c, r, M) := u(c, M)$ . This shows that circular and cumulative causation models fit our general sustainable framework.

#### 4.4.4 On the circular and sustainable economy research agenda

The research on circular and sustainable economy is currently vivid. Our framework is fitted to consider and interlink economic, environmental, technological and social issues. The GSF can easily be adapted to already suggested paths for future research (Donaghy[], Ghisellini .... ) and the extensive literature that is developing. The way the GSF may incorporate the history-dependence viewpoint and the memory formation that we introduce open perspectives towards several aspects and interpretations. In particular, as a by-product, the GSF allows to deal with many important features that are coming to be taken into account, such as recycling, reuse, reduction, design, habits, activities of harvesting exhaustible and renewable resources, the assimilative capacity of the natural environment for (non-recyclable) waste, transport activities, management of resources, interaction between the processes and the environment, preventative and regenerative eco-industrial development, etc. These may be interconnected.

<sup>23</sup>Donaghy[10] considers the particular functions  $b(R) = (\frac{\phi}{R} + \zeta)^{-1}$ ,  $\vartheta(K, HC) = [g - \chi(\frac{HC}{K})^\epsilon]$ ,  $u(c, P) = [\frac{1}{\gamma}(\omega c P^{-\eta})^\gamma]$  and  $\phi(s, z, K, HC, L, OM, R) = \psi[\theta_1(b(R)s)^{-\sigma} + \theta_2 z^{-\sigma} + \theta_3 K^{-\sigma} + \theta_4 HC^{-\sigma} + \theta_5 L^{-\sigma} + \theta_6 OM^{-\sigma}]^{-\frac{\kappa}{\sigma}}$ .

<sup>24</sup>i.e.  $\pi_i : \mathbb{R}^5 \rightarrow \mathbb{R}$  is defined by, for any  $a = (a_j)_{j=1}^5 \in \mathbb{R}^5$ ,  $\pi_i(a) = a_i$

## 5 Appendix

### 5.1 Proof of Proposition 2.1

By the assumption (A), along a feasible path  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(x_0, \tilde{h}^{(0)})$ , with associated history  $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$ ,

$$\forall t, F(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

Then for all  $T \in \mathbb{N}$ ,

$$\begin{aligned} \sum_{t=0}^T \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) &\leq \sum_{t=0}^T \beta^t (a_1(x_0)a^t + a_2) \\ &= a_1(x_0) \sum_{t=0}^T (a\beta)^t + a_2 \sum_{t=0}^T \beta^t \end{aligned}$$

since  $0 < a\beta < 1$  and  $0 < \beta < 1$  the conclusion follows.

### 5.2 Proof of Proposition 2.2

Let us show that the objective is upper semi-continuous and the feasible sequence set is a compact set of the product topology. By (F), one can check by induction that for any given  $k_0 \in \mathbb{R}^+$  and  $\tilde{h}^{(0)} \in l_+^\infty$ , for all feasible sequence  $\tilde{x} = (x_t)_t \in \Pi(\tilde{h}^{(0)}, x_0)$ , for all  $t, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t)$ , we have

$$\forall t, x_t \leq a^t \|x_0\| + \frac{1-a^t}{1-a} a' = (\|x_0\| - \frac{a'}{1-a}) a^t + \frac{a'}{1-a}$$

The feasible set  $\Pi(\tilde{h}^{(0)}, x_0)$  is included in a compact set for the product topology. Moreover, it is closed. So the feasible set  $\Pi(\tilde{h}^{(0)}, x_0)$  is also compact.

We next show that the objective function  $\mathcal{U}$  is upper semi-continuous.

Let us consider a sequence  $\tilde{x}^n = \{(x_t^n)_{t=1}^{+\infty}\}_n \subset \Pi(\tilde{h}^{(0)}, x_0)$  that converges to  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$ . Note that when  $n$  converges to  $+\infty$ , by (m1), the sequence of associated histories  $\forall t \geq 1, \tilde{h}^{n(t)} = (m(x_{t-1}^n, x_t^n), m(x_{t-2}^n, x_{t-1}^n), \dots, m(x_1^n, x_2^n), m(x_0^n, x_1^n), \tilde{h}^{(0)})$  converges to the associated history  $\forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$ .

Let us show that  $\overline{\lim}_{n \rightarrow +\infty} \mathcal{U}(\tilde{x}^n) \leq \mathcal{U}(\tilde{x})$ . The notation  $\overline{\lim}$  means lim sup.

For any  $t \geq 0$ , by (A2), for any  $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$  and with history  $\tilde{h}^{(t)}$  associated to  $\tilde{x}$

$$F^+(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

and by  $0 < a\beta < 1$ , for any  $\varepsilon > 0$ , there exists  $T_\varepsilon$  such that for any  $(x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$ , and for any  $T \geq T_\varepsilon$ ,

$$\sum_{t=T}^{+\infty} \beta^t F^+(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq \varepsilon$$

So for any  $\varepsilon > 0$ , there exists  $T_\varepsilon$  such that for any  $n \in \mathbb{N}$  and for any  $T \geq T_\varepsilon$ ,

$$\sum_{t=T}^{+\infty} \beta^t F^+(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) \leq \varepsilon$$

and for any  $n \in \mathbb{N}$  and for any  $T \geq T_\varepsilon$ ,

$$\begin{aligned} \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) &= \sum_{t=0}^T \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) + \sum_{t=T}^{+\infty} \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) \\ &\leq \sum_{t=0}^T \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) + \sum_{t=T}^{+\infty} \beta^t F^+(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) \\ &\leq \sum_{t=0}^T \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) + \varepsilon \end{aligned}$$

By taking  $n \rightarrow +\infty$  (and using the continuity of  $F$  in the right-hand side of the above inequality),

$$\overline{\lim}_{n \rightarrow +\infty} \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) \leq \sum_{t=0}^T \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \varepsilon$$

Since this is true for any  $T \geq T_\varepsilon$ , by taking  $T \rightarrow +\infty$ ,

$$\overline{\lim}_{n \rightarrow +\infty} \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \varepsilon$$

Since this is true for any  $\varepsilon > 0$ , by taking  $\varepsilon \rightarrow 0$ ,

$$\overline{\lim}_{n \rightarrow +\infty} \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h}^{n(t)}, x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1})$$

So  $\mathcal{U}$  is upper semi-continuous on  $\Pi(\tilde{h}^{(0)}, x_0)$ .

By Weierstrass Theorem (Aubin[3], Theorem 5.3.1), since  $\mathcal{U}$  is upper semi-continuous and  $\Pi(\tilde{h}^{(0)}, x_0)$  is a compact set for the product topology, there exists an optimal solution.

The assumptions that  $F$  is jointly strictly concave ensures the uniqueness of the solution.

### 5.3 Proof of Proposition 2.3

A direct proof using (A2) can be done. Indeed, let us consider a sequence  $(\widetilde{h}^{n(0)}, x_0^n)_n \subset l_+^\infty \times X$  that converges to  $(\tilde{h}^{(0)}, x_0) \in l_+^\infty \times X$ , use the fact that  $x_0^n$  converges to  $x_0$  and let us consider a subsequence  $(\widetilde{h}^{n_i(0)}, x_0^{n_i})_i$  such that

$$\overline{\lim}_{n \rightarrow +\infty} V(\widetilde{h}^{n(0)}, x_0^n) = \lim_{i \rightarrow +\infty} V(\widetilde{h}^{n_i(0)}, x_0^{n_i})$$

Let  $\varepsilon > 0$ . By (A2), there exist  $i_0$  and  $T_0$  such that for any  $i \geq i_0$  and for any  $T \geq T_0$ , and for optimal path  $(\tilde{x}^{n_i})_i \in \Pi(\widetilde{h}^{n_i(0)}, x_0^{n_i})$  and its associated history  $\widetilde{h}^{n_i(t)}$ ,

$$V(\widetilde{h}^{n_i(0)}, x_0^{n_i}) = \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h}^{n_i(t)}, x_t^{n_i}, x_{t+1}^{n_i}) \leq \sum_{t=0}^T \beta^t F(\widetilde{h}^{n_i(t)}, x_t^{n_i}, x_{t+1}^{n_i}) + \varepsilon$$

Fix  $T \geq T_0$ . The subsequence  $(\tilde{x}^{n_i})_i$  that belongs<sup>25</sup> to  $\Pi(\tilde{h}^{n_i(0)}, x_0^{n_i})$  can be assumed to converge to some  $\tilde{x}$  in  $\Pi(\tilde{h}^{(0)}, x_0)$ . By the definition of the associated history and the continuity of  $m$ , this implies that  $(\tilde{h}^{n_i(t)})_i$  converges to  $\tilde{h}^{(t)}$  the history associated to  $\tilde{x}$ .

Let  $i \rightarrow +\infty$ , by the continuity of  $F$ ,

$$\overline{\lim}_{n \rightarrow +\infty} V(\tilde{h}^{n(0)}, x_0^n) \leq \sum_{t=0}^T \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \varepsilon$$

Let  $T \rightarrow +\infty$ ,

$$\overline{\lim}_{n \rightarrow +\infty} V(\tilde{h}^{n(0)}, x_0^n) \leq \mathcal{U}(\tilde{x}) + \varepsilon \leq V(\tilde{h}^{(0)}, x_0)$$

by the arbitrariness of  $\varepsilon$ .

## 5.4 Proof of Proposition 2.4

One can check (see Le Van and Morhaim[26]) that (F2) and (A) imply that

(H)  $\forall x_0 \in X, \exists \mathcal{V}(x_0)$  a compact neighborhood of  $x_0$  in  $X$ ,  $\forall \varepsilon > 0, \exists T_0$  such that  $\forall T \geq T_0, \forall x'_0 \in \mathcal{V}(x_0), \forall \tilde{x}' \in \Pi(\tilde{h}^{(0)}, x'_0)$ , one has with  $\tilde{h}'^{(t)} = (m(x'_{t-1}, x'_t), m(x'_{t-2}, x'_{t-1}), \dots, m(x'_1, x'_2), m(x'_0, x'_1), \tilde{h}^{(0)})$ ,

$$\sum_{t=T}^{+\infty} \beta^t F^+(\tilde{h}'^{(t)}, x'_t, x'_{t+1}) \leq \varepsilon$$

where  $F^+(h, r, r') = \max\{0, F(h, r, r')\}$ .

(i) By (H),  $\exists T_0, \forall T > T_0, \forall x'_0 \in \mathcal{V}(x_0), \varepsilon > 0, \forall \tilde{x}' \in \Pi(\tilde{h}^{(0)}, x_0)$ ,

$$\sum_{t=T}^{+\infty} \beta^t F(\tilde{h}'^{(t)}, x'_t, x'_{t+1}) \leq \sum_{t=T}^{+\infty} \beta^t F^+(\tilde{h}'^{(t)}, x'_t, x'_{t+1}) \leq \varepsilon$$

Let  $\tilde{x}' \in \Pi(\tilde{h}^{(0)}, x'_0), T \geq T_0$ . For any  $\tilde{x}'' = (x''_{T+1}, \dots) \in \Pi(\tilde{h}'^{(T)}, x'_T)$ , one has  $(x'_1, \dots, x'_T, x''_{T+1}, \dots) \in \Pi(\tilde{h}^{(0)}, x'_0)$ , and

$$\beta^T F(\tilde{h}'^{(T)}, x'_T, x''_{T+1}) + \beta^{T+1} F(\tilde{h}'^{(T+1)}, x''_{T+1}, x''_{T+2}) + \dots \leq \varepsilon$$

so  $\beta^T V(\tilde{h}^{(T)}, x'_T) \leq \varepsilon$  which implies (i).

(ii)  $\forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ ,

$$-\infty < \mathcal{U}(\tilde{x}) \leq \sum_{t=0}^T \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \beta^{T+1} V(\tilde{h}^{(T+1)}, x_{T+1})$$

and

$$0 = \lim_{T \rightarrow +\infty} [\mathcal{U}(\tilde{x}) - \sum_{t=0}^T \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1})] \leq \underline{\lim}_{T \rightarrow +\infty} \beta^{T+1} V(\tilde{h}^{(T+1)}, x_{T+1})$$

The notation  $\underline{\lim}$  means  $\liminf$ . From (i) then  $\lim_{T \rightarrow +\infty} \beta^{T+1} V(\tilde{h}^{(T+1)}, x_{T+1}) = 0$

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<sup>25</sup>by the compactness of  $\Pi(\tilde{h}^{n_i(0)}, x_0^{n_i})$

## 5.5 Proof of Proposition 2.6

The proof that  $V$  is a fixed-point of the Bellman operator is standard (see Stokey Lucas and Prescott[41]). Uniqueness of the fixed point is shown by contradiction. Indeed, suppose there exists  $W$  another fixed-point of  $\mathcal{B}$  in  $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$ . Let us first check that  $W \leq V$ . Let  $(\tilde{h}^{(0)}, x_0)$  be given. There exists  $x_1 \in \Gamma(\tilde{h}^{(0)}, x_0)$  such that  $W(\tilde{h}^{(0)}, x_0) = F(\tilde{h}^{(0)}, x_0, k_1) + \beta W(\tilde{h}^{(1)}, x_1)$  and by induction, there exists a sequence  $(x_t)_{t \geq 1}$  with associated history sequence  $(\tilde{h}^{(t)})_t$  such that for any  $T$ ,

$$W(\tilde{h}^{(0)}, x_0) = \sum_{t=0}^{T-1} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \beta^T W(\tilde{h}^{(T)}, x_T)$$

Since  $W$  belongs to  $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$ , one has taking the limit when  $T \rightarrow +\infty$ , and then by  $V$  being the sup of the sum

$$W(\tilde{h}^{(0)}, x_0) \leq \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq V(x_0, \tilde{h}^{(0)})$$

Let us now show that  $V \leq W$ . Let  $x_0 \in X, \tilde{h}^{(0)} \in l_+^\infty$ . For any  $\tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$  such that  $\mathcal{U}(\tilde{x}) > -\infty$ , one has, with  $\tilde{h}^{(1)} = (m(x_0, x_1), \tilde{h}^{(0)})$ ,

$$\begin{aligned} W(\tilde{h}^{(0)}, x_0) &= BW(\tilde{h}^{(0)}, x_0) \\ &\geq F(\tilde{h}^{(0)}, x_0, x_1) + \beta W((m(x_0, x_1), \tilde{h}^{(0)}), x_1) \\ &= F(\tilde{h}^{(0)}, x_0, x_1) + \beta W(\tilde{h}^{(1)}, x_1) \end{aligned}$$

and so by induction, with  $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \dots, m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$ ,

$$\begin{aligned} W(k_0, \tilde{h}^{(0)}) &\geq \sum_{t=0}^T F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \beta^{T+1} W(\tilde{h}^{(T+1)}, x_{T+1}) \\ &\geq \lim_{T \rightarrow +\infty} \sum_{t=0}^T F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \lim_{T \rightarrow +\infty} \beta^{T+1} W(\tilde{h}^{(T+1)}, x_{T+1}) \\ &= \mathcal{U}(\tilde{x}) \end{aligned}$$

which implies that  $W(\tilde{h}^{(0)}, x_0) \geq V(\tilde{h}^{(0)}, x_0)$  (since for any  $\tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ , one has  $W(\tilde{h}^{(0)}, x_0) \geq \mathcal{U}(\tilde{x})$  and  $V(\tilde{h}^{(0)}, x_0)$  is the sup of  $\mathcal{U}(\tilde{x})$  for  $\tilde{x}$  in  $\Pi(\tilde{h}^{(0)}, x_0)$ ).

Finally, this shows that dynamic programming tools can be used to deal with general history-dependent optimal growth models.

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