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A Hyperbolic – Parabolic Framework to Manage Traffic Generated Pollution

Rinaldo M. Colombo¹ Paola Goatin² Elena Rossi³

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Abstract

Vehicular traffic flows through a merge regulated by traffic lights and produces pollutant that diffuses in the surrounding region. This situation motivates a general hyperbolic - parabolic system, whose well-posedness and stability are here proved in \mathbf{L}^1 . Roads are allowed to be also 2-dimensional. The effects of stop & go waves are comprised, leading to measure source terms in the parabolic equation. The traffic lights, as well as inflows and outflows, can be regulated to minimize the presence of pollutant in given regions.

2020 Mathematics Subject Classification: 35L65, 76A30, 35Q93, 35K15

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1 Introduction

We study a macroscopic model describing how vehicular traffic along a road network produces pollution and how this pollution propagates. The well-posedness and stability properties obtained allow to tackle the problem of optimal management of traffic lights in order to minimize the presence of pollution in given regions.

The network we consider consists of a merge with 2 incoming roads \mathcal{R}^1 and \mathcal{R}^2 and one outgoing road \mathcal{R}^3 . All roads may also have a 2-dimensional geometry, so that $\mathcal{R}^i \subset \mathbb{R}^2$, and the traffic along them is described by a 2-dimensional Lighthill–Whitham [15] and Richards [18] model

$$\left\{ \begin{array}{lll} \partial_t \rho^i + \nabla \cdot (\vec{q}(x, \rho^i)) = 0 & x \in \mathcal{R}^i & i = 1, 2, 3 \\ \rho^i(0, x) = \rho_o^i(x) & x \in \mathcal{R}^i & i = 1, 2, 3 \\ \vec{q}(\xi, \rho^i(t, \xi)) = f_{in}^i(t, \xi) & \xi \in \text{entry to } \mathcal{R}^i & i = 1, 2 \\ \vec{q}(\xi, \rho^i(t, \xi)) = f_{out}^i(t, \xi) & \xi \in \text{exit to } \mathcal{R}^i & i = 3. \end{array} \right. \quad (1.1)$$

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As usual, $\rho = \rho(t, x)$ is the traffic density and $\vec{q} = \vec{q}(x, \rho(t, x))$ the corresponding traffic flow. A precise statement of the whole model is presented below.

Traffic is controlled through the inflows f_{in}^1, f_{in}^2 , the outflow f_{out}^3 and through traffic lights at the merge. The latter consists in selecting time intervals J_j where the light is green for \mathcal{R}^1 , while in the complement $\mathbb{R}^+ \setminus \bigcup_j J_j$ the light is green for \mathcal{R}^2 , i.e.,

$$\begin{aligned} t \in \bigcup_j J_j & \quad f_{out}^1 = f_{in}^3 \quad \text{and} \quad f_{out}^2 = 0; \\ t \in \mathbb{R}^+ \setminus \bigcup_j J_j & \quad f_{out}^1 = 0 \quad \text{and} \quad f_{out}^2 = f_{in}^3. \end{aligned}$$

Traffic flow produces pollutant whose density $u = u(t, x)$ propagates according to

$$\begin{cases} \partial_t u + \nabla \cdot (u \beta(t, x) - \mu(t, x) \nabla u) + \kappa(t, x) u = G & x \in \mathbb{R}^2 \\ u(0, x) = u_o(x) \end{cases} \quad (1.2)$$

where $\mu = \mu(t, x)$ is the diffusion coefficient, $\beta = \beta(t, x)$ represents the wind velocity field, $\kappa = \kappa(t, x)$ is the pollutant decay rate and $u_o = u_o(x)$ is the initial datum. The source term G models the pollutant production and is a function possibly depending locally or non locally on ρ, \vec{q} . We underline that G may well depend also on the spatial derivative of the flow \vec{q} , so that it accounts also for accelerations, well known to be a major source of pollutant.

We remark that, differently from other results in the literature, we carefully adopt a conservative form for (1.2) so that the production of pollutant is exclusively due to vehicular traffic, while its decay is that explicitly prescribed in the equation through the term κu .

Analytically, we develop an \mathbf{L}^1 well-posedness theory for solutions, understood in the weak sense, to the system consisting of the non linear hyperbolic equation (1.1) on the merge coupled to the parabolic equation (1.2). The coupling, non linear and possibly non local, may also contain in measure terms.

At the hyperbolic level, this is achieved first extending [6, 9] to a 1-dimensional version of (1.1), i.e., to an initial boundary value problems with flux constraints. Then, we introduce a framework for a 2-dimensional road and pass to the 2-dimensional merge. In this part, the techniques developed below rely on the definitions of solution from [17, 19, 22], notably stable with respect to \mathbf{L}^1 convergence. Moreover, precise \mathbf{BV} estimates are obtained to allow that the distributional spatial derivatives of ρ and \vec{q} appear in G , as required by the physical application considered. Note that the roads' geometry is essentially arbitrary and the extension to more general networks is a matter of iterating the techniques below to more roads or junctions.

The parabolic part is settled in \mathbf{L}^1 , as in [14], a norm whose physical meaning in (1.2) is evident. This choice requires our introduction of an *ad hoc* definition of weak solution and of the related uniqueness and continuous dependence theorems, obtained essentially extending the classical results limited to strong solutions in [12]. The case of the source term G in (1.2) being a Radon measure falls within the general well-posedness framework here developed and allows to account for stop & go waves.

Once stability estimates are available, we optimize the timing of the traffic lights, the inflows f_{in}^1, f_{in}^2 and the outflow f_{out}^3 , so that an integral functional measuring the presence of pollutant in a given region is minimized. For completeness, we refer to [21] for a numerical approach to the control of a conservative equation like (1.2).

Thus, we provide a rigorous analytical framework to various results in the literature motivated by pollution production and propagation. We refer for instance to [1, 2], consisting mainly in numerical simulations devoted to general networks. Second order models along a junction connecting several 1-dimensional roads are considered in [4, 5]. The minimization of pollution by means of *ad hoc* speed limits is presented in the recent work [13].

The use of mixed hyperbolic–parabolic systems is frequent in various mathematical models: see for instance [20] motivated by tumor growth and analytically limited to radially symmetric solutions, or [16] devoted to the circulation of blood on a network, analytically settled in \mathbf{L}^2 and considering strong solutions on a bounded domain with Robin boundary conditions. A related mixed ODE – parabolic PDE model is considered in [3].

The necessary 1-dimensional results require minor modifications of known theorems and are collected in Section 2. The analytical framework describing a 2-dimensional road together with the corresponding well-posedness and stability results related to (1.1) are in Section 3. The parabolic model (1.2) is treated in Section 4, while the optimal management problems is deferred to Section 5. All proofs are collected in Section 6.

2 IBVP on a 1D Network with a Flux Constraint

Following the classical Lighthill–Whitham [15] and Richards [18] model, the vehicle speed $v = v(\rho)$ at traffic density ρ is prescribed, for all $\rho \in [0, R]$, $R > 0$ being fixed. Throughout, we require the following assumption on v , stated by means of the corresponding flow $q(\rho) = \rho v(\rho)$:

- (**q**) $q \in \mathbf{C}^2([0, R]; \mathbb{R}_+)$ is such that $q(0) = q(R) = 0$, $q'(\rho) > 0$ for $\rho \in [0, \bar{\rho}]$ and $q'(\rho) < 0$ for $\rho \in]\bar{\rho}, R]$, for a fixed $\bar{\rho} \in]0, R[$.

The restrictions $q|_{[0, \bar{\rho}]}$ of q to $[0, \bar{\rho}]$ and $q|_{[\bar{\rho}, R]}$ of q to $[\bar{\rho}, R]$ are invertible and we denote them by $q|_{[0, \bar{\rho}]}^{-1}$ and $q|_{[\bar{\rho}, R]}^{-1}$ respectively.

For completeness, we recall the basic results concerning the following Initial–Boundary Value Problem (IBVP) on the (non trivial) bounded open interval I :

$$\left\{ \begin{array}{ll} \partial_t \rho + \partial_s q(\rho) = 0 & (t, s) \in [0, T] \times I \\ \rho(0, s) = \rho_o(s) & s \in I \\ q(\rho(t, s_{in})) = f_{in}(t) & t \in [0, T] \\ q(\rho(t, s_{out})) = f_{out}(t) & t \in [0, T], \end{array} \right. \quad (2.1)$$

where we set $s_{in} = \inf I$ and $s_{out} = \sup I$.

Particular care has to be taken to correctly evaluate the effects and the meaning of the boundary conditions. First, as it is usual in the context of hyperbolic conservation laws, a solution to (2.1) needs not to satisfy these conditions at all t , not even at a.e. t , see [17, 19, 22]. Second, due to our considering only fluxes satisfying (**q**), the boundary conditions in (2.1) can be equivalently understood as *constraints*. Indeed, the trace $(q \circ \rho(t))(s_{in}+)$ at s_{in} , respectively $(q \circ \rho(t))(s_{out}-)$ at s_{out} , of the flow $q(\rho)$ of the solution ρ is the maximal possible flow of a solution compatible with $(q \circ \rho(t))(s_{in}+) \leq f_{in}(t)$, respectively $(q \circ \rho(t))(s_{out}-) \leq f_{out}(t)$.

The specific form of the following definition is inspired by [17, Definition 1] and [22, Definition 1].

Definition 2.1 ([9, Definition 3.1]). *A solution to (2.1) is a map $\rho \in \mathbf{L}^\infty([0, T] \times I; \mathbb{R})$ such that for any test function $\varphi \in \mathbf{C}_c^1(\mathbb{R}^2; \mathbb{R}_+)$ and for any $\kappa \in \mathbb{R}$,*

$$\begin{aligned} & \int_0^T \int_I (\rho(t, s) - \kappa)^\pm \partial_t \varphi(t, s) \, ds \, dt \\ & + \int_0^T \int_I \operatorname{sgn}^\pm (\rho(t, s) - \kappa) \left(q(\rho(t, s)) - q(\kappa) \right) \partial_s \varphi(t, s) \, ds \, dt \\ & + \int_I (\rho_o(s) - \kappa)^\pm \varphi(0, s) \, ds + \int_I (\rho(T, s) - \kappa)^\pm \varphi(T, s) \, ds \\ & + \|q'\|_{\mathbf{L}^\infty([0, R]; \mathbb{R})} \int_0^T \left(q_{|[0, \bar{\rho}]}^{-1}(f_{in}(t)) - \kappa \right) \varphi(t, s_{in}) \, dt \\ & + \|q'\|_{\mathbf{L}^\infty([0, R]; \mathbb{R})} \int_0^T \left(q_{|[\bar{\rho}, R]}^{-1}(f_{out}(t)) - \kappa \right) \varphi(t, s_{out}) \, dt \geq 0. \end{aligned}$$

Here and in what follows, we use the standard notation $x^+ = \max\{x, 0\}$, $x^- = \min\{x, 0\}$, for any $x \in \mathbb{R}$, as well as $\operatorname{sgn}^+ x = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$ and $\operatorname{sgn}^- x = \begin{cases} 0 & x \geq 0 \\ -1 & x < 0 \end{cases}$.

The well-posedness of (2.1) and detailed stability estimates are ensured extending [9, Propositions 3.2 and 3.3] to comprise non **BV** initial and boundary data, see also [6, Theorem 2.2]. The following result also provides estimates on the total variation of the nonlinear map $x \mapsto \Psi(\rho(t, x))$, where

$$\Psi(\rho) = \operatorname{sgn}(\rho - \bar{\rho}) (q(\bar{\rho}) - q(\rho)). \quad (2.2)$$

This bound is instrumental in ensuring the existence of traces of ρ and in proving that $q(\rho)$ has bounded total variation, while $x \mapsto \rho(t, x)$ may well have infinite total variation.

Proposition 2.2. *Let q satisfy (q), $\rho_o \in \mathbf{L}^1(I; [0, R])$ and $f_{in}, f_{out} \in \mathbf{L}^1([0, T]; [0, q(\bar{\rho})])$. Then, problem (2.1) admits a unique solution ρ in the sense of Definition 2.1 which satisfies, for all $t \in [0, T]$,*

$$\|\rho(t)\|_{\mathbf{L}^\infty(I; \mathbb{R})} \leq \max \left\{ \|\rho_o\|_{\mathbf{L}^\infty(I; \mathbb{R})}, \|q_{|[0, \bar{\rho}]}^{-1} \circ f_{in}\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})}, \|q_{|[\bar{\rho}, R]}^{-1} \circ f_{out}\|_{\mathbf{L}^\infty([0, t]; \mathbb{R})} \right\}. \quad (2.3)$$

Moreover, if $\hat{\rho}_o, \hat{f}_{in}$ and \hat{f}_{out} satisfy the same assumptions, the corresponding solution $\hat{\rho}$ satisfies, for all $t \in [0, T]$,

$$\|\rho(t) - \hat{\rho}(t)\|_{\mathbf{L}^1(I; \mathbb{R})} \leq \|\rho_o - \hat{\rho}_o\|_{\mathbf{L}^1(I; \mathbb{R})} + \|f_{in} - \hat{f}_{in}\|_{\mathbf{L}^1([0, t]; \mathbb{R})} + \|f_{out} - \hat{f}_{out}\|_{\mathbf{L}^1([0, t]; \mathbb{R})}. \quad (2.4)$$

If in addition, with reference to (2.2), $\Psi(\rho_o) \in \mathbf{BV}(I; \mathbb{R})$ and $f_{in}, f_{out} \in \mathbf{BV}([0, T]; [0, q(\bar{\rho})])$ then, for all $t \in [0, T]$, the map $\Psi(\rho(t))$ has uniformly bounded total variation in the space variable.

The proof of Proposition 2.2 is deferred to § 6.1. We remark that an estimate analogous to (2.4) is also obtained in [6, Theorem 2.2], where, as in the present setting, the incoming flux is assigned as (Dirichlet) boundary datum.

Consider now three 1-dimensional (one-way) roads, where roads 1 and 2 merge into road 3. Denote by I_i the road segment parametrizing the i -th road, and by ρ^i the corresponding traffic density. All densities vary in the same interval $[0, R]$. Moreover, the speed law v is the same on all the three roads, with $q(\rho) = \rho v(\rho)$ satisfying **(q)**. All this leads to consider

$$\begin{cases} \partial_t \rho^i + \partial_s q(\rho^i) = 0 & (t, s) \in [0, T] \times I_i & i = 1, 2, 3 \\ \rho^i(0, s) = \rho_o^i(s) & s \in I_i & i = 1, 2, 3 \\ q(\rho^i(t, s_{in}^i)) = f_{in}^i(t) & t \in [0, T] & i = 1, 2 \\ q(\rho^i(t, s_{out}^i)) = f_{out}^i(t) & t \in [0, T] & i = 3 \end{cases} \quad (2.5)$$

where $s_{in}^i = \inf I_i$ and $s_{out}^i = \sup I_i$, $i = 1, 2, 3$.

The regulation of the merge is described by suitable conditions on the outflows f_{out}^1, f_{out}^2 of the incoming roads and on the inflow f_{in}^3 of the outgoing one. For instance, we introduce traffic lights, displaying alternatively red or green lights at the end of each incoming road. This leads to introduce the time intervals J_j where the green light is between roads 1 and 3. Correspondingly, consider the class \mathcal{W} of functions w that can be written as follows, for a suitable $n \in \mathbb{N}$:

$$\begin{aligned} w &: [0, T[\rightarrow \{0, 1\} & J_j \subseteq [0, T] \text{ is an interval} \\ t &\mapsto \sum_{j=1}^n \chi_{J_j}(t) & \text{where } J_j \neq \emptyset \\ & & j \neq k \implies J_j \cap J_k = \emptyset. \end{aligned} \quad (2.6)$$

Definition 2.3. By solution to the IBVP (2.5) regulated by w as in (2.6) we mean a triple of maps ρ^1, ρ^2, ρ^3 such that, for $i = 1, 2, 3$, $\rho^i \in \mathbf{L}^\infty([0, T] \times I_i; [0, R])$ solves (2.1) in the sense of Definition 2.1 where, for $t \in [0, T]$,

$$\begin{aligned} f_{out}^1(t) &= w(t) \, q\left(\max\{\bar{\rho}, \rho^3(t, s_{in}^3)\}\right) \\ f_{out}^2(t) &= (1 - w(t)) \, q\left(\max\{\bar{\rho}, \rho^3(t, s_{in}^3)\}\right) \\ f_{in}^3(t) &= w(t) \, q\left(\min\{\rho^1(t, s_{out}^1), \bar{\rho}\}\right) + (1 - w(t)) \, q\left(\min\{\rho^2(t, s_{out}^2), \bar{\rho}\}\right). \end{aligned} \quad (2.7)$$

Theorem 2.4. Let q satisfy **(q)**. For any $\rho_o^i \in \mathbf{L}^1(I_i; [0, R])$, $i = 1, 2, 3$, for any $f_{in}^1, f_{in}^2, f_{out}^3 \in \mathbf{L}^1([0, T]; [0, q(\bar{\rho})])$ and for any w as in (2.6), the IBVP (2.5) regulated by w admits a unique solution ρ^1, ρ^2, ρ^3 in the sense of Definition 2.3. Moreover:

(1D.1) If $\widehat{\rho}_o^i \in \mathbf{L}^1(I_i; [0, R])$, for $i = 1, 2, 3$, and $\widehat{f_{in}^1}, \widehat{f_{in}^2}, \widehat{f_{out}^3} \in \mathbf{L}^1([0, T]; [0, q(\bar{\rho})])$, the corresponding solution $\widehat{\rho}^1, \widehat{\rho}^2, \widehat{\rho}^3$ satisfies for all $t \in [0, T]$,

$$\begin{aligned} & \sum_{i=1}^3 \left\| \rho^i(t) - \widehat{\rho}^i(t) \right\|_{\mathbf{L}^1(I_i; \mathbb{R})} \\ & \leq \sum_{i=1}^3 \left\| \rho_o^i - \widehat{\rho}_o^i \right\|_{\mathbf{L}^1(I_i; \mathbb{R})} + \sum_{i=1}^2 \left\| f_{in}^i - \widehat{f_{in}^i} \right\|_{\mathbf{L}^1([0, t]; \mathbb{R})} + \left\| f_{out}^3 - \widehat{f_{out}^3} \right\|_{\mathbf{L}^1([0, t]; \mathbb{R})}. \end{aligned}$$

(1D.2) If, for $i = 1, 2, 3$, $\Psi(\rho_o^i) \in \mathbf{BV}(I_i; \mathbb{R})$ and $f_{in}^1, f_{in}^2, f_{out}^3 \in \mathbf{BV}([0, T]; [0, q(\bar{\rho})])$, then, for all $t \in [0, T]$, $\sum_{i=1}^3 \text{TV}(\Psi(\rho^i(t)))$ is uniformly bounded on I .

(1D.3) If the triple $\hat{\rho}^1, \hat{\rho}^2, \hat{\rho}^3$ solves (2.5) regulated by a map \hat{w} as in (2.6), then for all $t \in [0, T]$

$$\sum_{i=1}^3 \left\| \rho^i(t) - \hat{\rho}^i(t) \right\|_{\mathbf{L}^1(I_i; \mathbb{R})} \leq 3q(\bar{\rho}) \|w - \hat{w}\|_{\mathbf{L}^1([0, t]; \mathbb{R})}. \quad (2.8)$$

The proof of Theorem 2.4 is deferred to § 6.1.

3 A Merge in a 2D Traffic Flow Model

We first introduce the basic definitions and notations to deal with a 2D road. Any vector $u \in \mathbb{R}^2$ is understood as a *column* vector, i.e., $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. We also set $u^\perp = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}$ and $u^\top = [u_1 \ u_2]$. If, $u, z \in \mathbb{R}^2$, their scalar product is denoted $u \cdot z = u^\top z = u_1 z_1 + u_2 z_2$. Identities useful in the sequel are $u^\perp \cdot z + u \cdot z^\perp = 0$ and $u^\perp \cdot z^\perp = u \cdot z$.

Definition 3.1. Let $I \subseteq \mathbb{R}$ be an open non trivial interval, $L > 0$ and $\gamma \in \mathbf{C}^2(\bar{I}; \mathbb{R}^2)$ be a simple curve such that $\|\gamma'(s)\| = 1$ and $\|\gamma''(s)\| \leq 1/L$ for all $s \in \bar{I}$. A Road is a map

$$\begin{aligned} \mathcal{R}: \bar{I} \times]-L, L[&\rightarrow \mathbb{R}^2 \\ (s, \ell) &\mapsto x \end{aligned} \quad \text{where} \quad x = \gamma(s) + \ell (\gamma'(s))^\perp$$

such that $\mathcal{R}(\bar{I} \times]-L, L[)$ is simply connected.

If I is bounded below, respectively above, we set

$$\begin{aligned} s_{in} &= \inf I & \mathcal{E}_{in} &= \mathcal{R}(s_{in},]-L, L[); \\ s_{out} &= \sup I & \mathcal{E}_{out} &= \mathcal{R}(s_{out},]-L, L[). \end{aligned} \quad (3.1)$$

Below, we occasionally identify the map \mathcal{R} with its support $\mathcal{R}(\bar{I} \times]-L, L[)$.

Lemma 3.2. Let \mathcal{R} be a road in the sense of Definition 3.1. Then $\mathcal{R}: \bar{I} \times]-L, L[\rightarrow \mathcal{R}(\bar{I} \times]-L, L[)$ is bijective and its inverse is of class $\mathbf{C}^1(\mathcal{R}(I \times]-L, L[); \mathbb{R}^2) \cap \mathbf{C}^0(\mathcal{R}(\bar{I} \times]-L, L[); \mathbb{R}^2)$.

The proof is deferred to § 6.2. The above Lemma 3.2 allows to define on \mathcal{R} the \mathbf{C}^1 vector field r by

$$r(x) = \gamma'(s) \iff \exists \ell \in]-L, L[\quad \mathcal{R}(s, \ell) = x$$

so that

$$r(\mathcal{R}(s, \ell)) = \gamma'(s). \quad (3.2)$$

Consider the IBVP for a conservation law on a bounded road \mathcal{R} , understood in the sense of Definition 3.1, where we prescribe the inflow f_{in} and the outflow f_{out} :

$$\begin{cases} \partial_t \rho + \nabla \cdot (q(\rho) r(x)) = 0 & (t, x) \in [0, T] \times \mathcal{R} \\ \rho(0, x) = \rho_o(x) & x \in \mathcal{R} \\ q(\rho(t, \xi)) = f_{in}(t, \xi) & (t, \xi) \in [0, T] \times \mathcal{E}_{in} \\ q(\rho(t, \xi)) = f_{out}(t, \xi) & (t, \xi) \in [0, T] \times \mathcal{E}_{out}, \end{cases} \quad (3.3)$$

where \mathcal{E}_{in} and \mathcal{E}_{out} are as in (3.1).

Definition 3.3. By solution to the IBVP (3.3) on the time interval $[0, T]$ we mean a map $\rho \in \mathbf{L}^\infty([0, T] \times \mathcal{R}; \mathbb{R})$ such that for any test function $\varphi \in \mathbf{C}_c^1(\mathbb{R} \times \mathcal{R}; \mathbb{R}_+)$ and for any $\kappa \in \mathbb{R}$,

$$\begin{aligned} & \int_0^T \int_{\mathcal{R}} (\rho(t, x) - \kappa)^\pm \partial_t \varphi(t, x) \, dx \, dt \\ & + \int_0^T \int_{\mathcal{R}} \operatorname{sgn}^\pm (\rho(t, x) - \kappa) \left(q(\rho(t, x)) - q(\kappa) \right) \nabla_x \varphi(t, x) \cdot r(x) \, dx \, dt \\ & + \int_{\mathcal{R}} (\rho_o(x) - \kappa)^\pm \varphi(0, x) \, dx + \int_{\mathcal{R}} (\rho(T, x) - \kappa)^\pm \varphi(T, x) \, dx \\ & + \|q'\|_{\mathbf{L}^\infty([0, R]; \mathbb{R})} \int_0^T \int_{\mathcal{E}_{in}} \left(q_{[\bar{\rho}]}^{-1}(f_{in}(t, \xi)) - \kappa \right)^\pm \varphi(t, \xi) \, d\xi \, dt \\ & + \|q'\|_{\mathbf{L}^\infty([0, R]; \mathbb{R})} \int_0^T \int_{\mathcal{E}_{out}} \left(q_{[\bar{\rho}]}^{-1}(f_{out}(t, \xi)) - \kappa \right)^\pm \varphi(t, \xi) \, d\xi \, dt \geq 0. \end{aligned}$$

In the above integral inequality, the usual term with $\nabla_x \cdot r$ is missing since the vector field r , defined in (3.2), is divergence free by construction, as proved in Lemma 6.1.

The next Lemma 3.4 relates the 2D IBVP (3.3) to the 1D IBVP (2.1).

Lemma 3.4. Let q satisfy **(q)**. For $i = 1, 2, 3$, let \mathcal{R}^i be a road as in Definition 3.1. For any $\rho_o \in \mathbf{L}^1(\mathcal{R}; [0, R])$, $f_{in} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}; [0, q(\bar{\rho})])$, $f_{out} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}; [0, q(\bar{\rho})])$, the Cauchy problem (3.3), in the sense of Definition 3.3, is equivalent to the family of 1D Cauchy problems parametrized by $\ell \in]-L, L[$,

$$\begin{cases} \partial_t \rho^\ell + \partial_\sigma q(\rho^\ell) = 0 & (t, \sigma) \in [0, T] \times I^\ell \\ \rho^\ell(0, \sigma) = \rho_o^\ell(\sigma) & \sigma \in I^\ell \\ q(\rho^\ell(t, s_{in}^\ell)) = f_{in}^\ell(t) & t \in [0, T] \\ q(\rho^\ell(t, s_{out}^\ell)) = f_{out}^\ell(t) & t \in [0, T] \end{cases} \quad (3.4)$$

understood in the sense of Definition 2.1, where $I^\ell =]s_{in}^\ell, s_{out}^\ell[$ and for all $\ell \in]-L, L[$, $s \in I$

$$\begin{aligned} \frac{d\sigma}{ds}(s) &= 1 - \ell (\gamma'(s))^\perp \cdot \gamma''(s); & s_{in}^\ell &= s_{in}, \\ s_{out}^\ell &= s_{out} - \ell \int_{s_{in}}^{s_{out}} (\gamma'(s))^\perp \cdot \gamma''(s) \, ds, \end{aligned} \quad (3.5)$$

and moreover for a.e. $t \in [0, T]$, $\ell \in]-L, L[$, $s \in I$, $\sigma \in I^\ell$,

$$\begin{aligned} \rho^\ell(t, \sigma) &= \rho(t, \mathcal{R}(s, \ell)), & f_{in}^\ell(t) &= f_{in}(t, \mathcal{R}(s_{in}, \ell)), \\ \rho_o^\ell(\sigma) &= \rho_o(\mathcal{R}(s, \ell)), & f_{out}^\ell(t) &= f_{out}(t, \mathcal{R}(s_{out}, \ell)). \end{aligned} \quad (3.6)$$

The proof is deferred to § 6.2.

The next theorem extends the well-posedness results of Proposition 2.2 to the present 2D setting by means of Lemma 3.4.

Theorem 3.5. *Let q satisfy **(q)**. For $i = 1, 2, 3$, let \mathcal{R}^i be a road as in Definition 3.1. For any $\rho_o \in \mathbf{L}^1(\mathcal{R}; [0, R])$, $f_{in} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}; [0, q(\bar{\rho})])$, $f_{out} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}; [0, q(\bar{\rho})])$, the IBVP (3.3) admits a unique solution ρ in the sense of Definition 3.3 and it satisfies, for $t \in [0, T]$, the estimate*

$$\|\rho(t)\|_{\mathbf{L}^\infty(\mathcal{R}; \mathbb{R})} \leq \max \left\{ \|\rho_o\|_{\mathbf{L}^\infty(\mathcal{R}; \mathbb{R})}, \left\| q_{|[0, \bar{\rho}]}^{-1} \circ f_{in} \right\|_{\mathbf{L}^\infty([0, t] \times \mathcal{E}_{in}; \mathbb{R})}, \left\| q_{|[\bar{\rho}, R]}^{-1} \circ f_{out} \right\|_{\mathbf{L}^\infty([0, t] \times \mathcal{E}_{out}; \mathbb{R})} \right\},$$

with \mathcal{E}_{in} and \mathcal{E}_{out} as in (3.1). Moreover, if $\hat{\rho}_o, \hat{f}_{in}$ and \hat{f}_{out} satisfy the same assumptions, the corresponding solution $\hat{\rho}$ satisfies, for all $t \in [0, T]$,

$$\|\rho(t) - \hat{\rho}(t)\|_{\mathbf{L}^1(\mathcal{R}; \mathbb{R})} \leq \|\rho_o - \hat{\rho}_o\|_{\mathbf{L}^1(\mathcal{R}; \mathbb{R})} + \|f_{in} - \hat{f}_{in}\|_{\mathbf{L}^1([0, t] \times \mathcal{E}_{in}; \mathbb{R})} + \|f_{out} - \hat{f}_{out}\|_{\mathbf{L}^1([0, t] \times \mathcal{E}_{out}; \mathbb{R})}.$$

If in addition

$$\operatorname{ess\,sup}_{\ell \in]-L, L[} \operatorname{TV} \left(\Psi \left(\rho_o(\mathcal{R}(\cdot, \ell)) \right) \right) < +\infty, \quad \text{and} \quad \begin{aligned} & \operatorname{ess\,sup}_{\xi \in \mathcal{E}_{in}} \operatorname{TV} (f_{in}(\cdot, \xi)) < +\infty, \\ & \operatorname{ess\,sup}_{\xi \in \mathcal{E}_{out}} \operatorname{TV} (f_{out}(\cdot, \xi)) < +\infty, \end{aligned}$$

then the following bound holds

$$\sup_{t \in [0, T]} \operatorname{ess\,sup}_{\ell \in]-L, L[} \operatorname{TV} \left(\Psi \left(\rho(t, \mathcal{R}(\cdot, \ell)) \right) \right) < +\infty. \quad (3.7)$$

The proof is deferred to § 6.2.

Consider 3 roads $\mathcal{R}^1, \mathcal{R}^2$ and \mathcal{R}^3 , understood in the sense of Definition 3.1, with $\mathcal{R}^i: \bar{I}_i \times]-L, L[\rightarrow \mathbb{R}^2$, I_i being a non trivial open and bounded real interval. We say we have a *merge* when, with the notation (3.1),

$$\mathcal{E}_{out}^1 = \mathcal{E}_{out}^2 = \mathcal{E}_{in}^3 \quad \text{and} \quad \gamma_1''(s_{out}^1) = \gamma_2''(s_{out}^2) = \gamma_3''(s_{in}^3),$$

so that, in particular, also $\gamma_1'(s_{out}^1) = \gamma_2'(s_{out}^2) = \gamma_3'(s_{in}^3)$.

Along \mathcal{R}^i , for $i = 1, 2, 3$, the traffic density is $\rho^i = \rho^i(t, x)$, with $\rho^i(t, x) \in [0, R]$, and the traffic flow q , as a function of ρ , satisfies **(q)**. We are thus lead to consider the problem

$$\begin{cases} \partial_t \rho^i + \nabla \cdot (q(\rho^i) r^i(x)) = 0 & (t, x) \in [0, T] \times \mathcal{R}^i & i = 1, 2, 3 \\ \rho^i(0, x) = \rho_o^i(x) & x \in \mathcal{R}^i & i = 1, 2, 3 \\ q(\rho^i(t, \xi)) = f_{in}^i(t, \xi) & (t, \xi) \in [0, T] \times \mathcal{E}_{in}^i & i = 1, 2 \\ q(\rho^i(t, \xi)) = f_{out}^i(t, \xi) & (t, \xi) \in [0, T] \times \mathcal{E}_{out}^i & i = 3. \end{cases} \quad (3.8)$$

We assume that the merge is regulated by traffic lights displaying alternatively red or green lights at the end of \mathcal{R}^1 and \mathcal{R}^2 . As in the 1D case, the traffic light is represented by a function w as in (2.6). Hence, the traffic flow at the junction is described by conditions on f_{out}^1, f_{out}^2 and f_{in}^3 .

Definition 3.6. By solution to the IBVP (3.8) regulated by w as in (2.6) we mean a triple of maps ρ^1, ρ^2, ρ^3 such that for $i = 1, 2, 3$, $\rho^i \in \mathbf{L}^\infty([0, T] \times \mathcal{R}^i; [0, R])$ solves (3.3) in the sense of Definition 3.3, where, for $(t, \ell) \in [0, T] \times]-L, L[$,

$$\begin{aligned} f_{out}^1(t, \mathcal{R}^1(s_{out}^1, \ell)) &= w(t) q \left(\max \left\{ \bar{\rho}, \rho^3(t, \mathcal{R}^3(s_{in}^3, \ell)) \right\} \right) \\ f_{out}^2(t, \mathcal{R}^2(s_{out}^2, \ell)) &= (1 - w(t)) q \left(\max \left\{ \bar{\rho}, \rho^3(t, \mathcal{R}^3(s_{in}^3, \ell)) \right\} \right) \\ f_{in}^3(t, \mathcal{R}^3(s_{in}^3, \ell)) &= w(t) q \left(\min \left\{ \rho^1(t, \mathcal{R}^1(s_{out}^1, \ell)), \bar{\rho} \right\} \right) \\ &\quad + (1 - w(t)) q \left(\min \left\{ \rho^2(t, \mathcal{R}^2(s_{out}^2, \ell)), \bar{\rho} \right\} \right). \end{aligned} \quad (3.9)$$

We now state the main result about the hyperbolic part (3.8) of our model (3.8)–(4.1).

Theorem 3.7. Let q satisfy **(q)**. For $i = 1, 2, 3$, let \mathcal{R}^i be a road as in Definition 3.1. F

$$\begin{aligned} &f_{in}^1 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^1; [0, q(\bar{\rho})]), \\ \text{For any } \rho_o^i \in \mathbf{L}^1(\mathcal{R}^i; [0, R]) \quad \text{and} \quad &f_{in}^2 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^2; [0, q(\bar{\rho})]), \\ &f_{out}^3 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}^3; [0, q(\bar{\rho})]), \end{aligned} \quad (3.10)$$

and for any w as in (2.6), the IBVP (3.8) regulated by w admits a unique solution ρ^1, ρ^2, ρ^3 in the sense of Definition 3.6. Moreover,

(2D.1) If $\hat{\rho}_o^i \in \mathbf{L}^1(\mathcal{R}^i; [0, R])$, for $i = 1, 2, 3$, and $\hat{f}_{in}^1 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^1; [0, q(\bar{\rho})])$, $\hat{f}_{in}^2 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^2; [0, q(\bar{\rho})])$, $\hat{f}_{out}^3 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}^3; [0, q(\bar{\rho})])$, the corresponding solution $\hat{\rho}^1, \hat{\rho}^2, \hat{\rho}^3$ satisfies for all $t \in [0, T]$,

$$\begin{aligned} &\sum_{i=1}^3 \left\| \rho^i(t) - \hat{\rho}^i(t) \right\|_{\mathbf{L}^1(\mathcal{R}^i; \mathbb{R})} \\ &\leq \sum_{i=1}^3 \left\| \rho_o^i - \hat{\rho}_o^i \right\|_{\mathbf{L}^1(\mathcal{R}^i; \mathbb{R})} + \sum_{i=1}^2 \left\| f_{in}^i - \hat{f}_{in}^i \right\|_{\mathbf{L}^1([0, t] \times \mathcal{E}_{in}^i; \mathbb{R})} + \left\| f_{out}^3 - \hat{f}_{out}^3 \right\|_{\mathbf{L}^1([0, t] \times \mathcal{E}_{out}^3; \mathbb{R})}. \end{aligned}$$

(2D.2) If moreover for $i = 1, 2, 3$,

$$\begin{aligned} &\text{ess sup}_{\xi \in \mathcal{E}_{in}^1} \text{TV} \left(f_{in}^1(\cdot, \xi) \right) < +\infty, \\ \text{ess sup}_{\ell \in]-L, L[} \text{TV} \left(\Psi \left(\rho_o^i \left(\mathcal{R}^i(\cdot, \ell) \right) \right) \right) < +\infty, & \text{ess sup}_{\xi \in \mathcal{E}_{in}^2} \text{TV} \left(f_{in}^2(\cdot, \xi) \right) < +\infty, \\ &\text{ess sup}_{\xi \in \mathcal{E}_{out}^3} \text{TV} \left(f_{out}^3(\cdot, \xi) \right) < +\infty, \end{aligned} \quad (3.11)$$

then the following bound holds

$$\sum_{i=1}^3 \sup_{t \in [0, T]} \text{ess sup}_{\ell \in]-L, L[} \text{TV} \left(\Psi \left(\rho^i(t, \mathcal{R}^i(\cdot, \ell)) \right) \right) < +\infty. \quad (3.12)$$

(2D.3) If the triple $\hat{\rho}^1, \hat{\rho}^2, \hat{\rho}^3$ solves (2.5) regulated by a map \hat{w} as in (2.6), then for all $t \in [0, T]$

$$\sum_{i=1}^3 \left\| \rho^i(t) - \hat{\rho}^i(t) \right\|_{\mathbf{L}^1(\mathcal{R}^i; \mathbb{R})} \leq 6 L q(\bar{\rho}) \|w - \hat{w}\|_{\mathbf{L}^1([0, t]; \mathbb{R})}. \quad (3.13)$$

The proof of Theorem 3.7 follows from Theorem 3.5, exactly as the proof of Theorem 2.4 follows from that of Proposition 2.2.

4 The Conservative Parabolic Pollution Model

We assume the concentration of air pollutant $u = u(t, x)$ is described by the Cauchy problem

$$\begin{cases} \partial_t u + \nabla \cdot (u \beta(t, x) - \mu(t, x) \nabla u) + \kappa(t, x) u = g(t, x) & (t, x) \in]0, T] \times \mathbb{R}^2 \\ u(0, x) = u_o(x) & x \in \mathbb{R}^2. \end{cases} \quad (4.1)$$

The role of the various symbols is as described in the Introduction.

(HP) On the functions in (4.1) we assume the following conditions:

(HP.1) $\mu \in (\mathbf{C}^1 \cap \mathbf{W}^{1, \infty})([0, T] \times \mathbb{R}^2; \mathbb{R}_+)$ is such that $\nabla \mu$ is Hölder continuous in x uniformly in t and there exist $\hat{\mu}, \check{\mu}$ with $\hat{\mu} > \check{\mu} > 0$ such that $\mu(t, x) \in [\check{\mu}, \hat{\mu}]$ for all $(t, x) \in [0, T] \times \mathbb{R}^2$;

(HP.2) $\beta \in (\mathbf{C}^1 \cap \mathbf{W}^{1, \infty})([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$ is such that $\nabla \beta$ is Hölder continuous in x uniformly in t ;

(HP.3) $\kappa \in (\mathbf{C}^0 \cap \mathbf{L}^\infty)([0, T] \times \mathbb{R}^2; \mathbb{R})$ is Hölder continuous in x uniformly in t .

See Definition 6.5 for details on Hölder continuity.

Definition 4.1. A function $u \in \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}))$ is a weak solution to (4.1) if for any

$$\begin{aligned} & \varphi \in \mathbf{C}^0([0, T] \times \mathbb{R}^2; \mathbb{R}) \cap \mathbf{C}^1(]0, T[\times \mathbb{R}^2; \mathbb{R}) \quad \text{such that} \\ & \forall t \in]0, T[\quad x \mapsto \varphi(t, x) \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R}) \\ & \forall t \in]0, T[\quad \lim_{r \rightarrow +\infty} \sup_{\|x\| \geq r} |\varphi(t, x)| = 0 \\ & \forall t \in]0, T[\quad \lim_{r \rightarrow +\infty} \sup_{\|x\| \geq r} |\partial_t \varphi(t, x)| = 0 \\ & \forall t \in]0, T[\quad \lim_{r \rightarrow +\infty} \sup_{\|x\| \geq r} \|\nabla \varphi(t, x)\| = 0 \\ & \forall t \in]0, T[\quad \lim_{r \rightarrow +\infty} \sup_{\|x\| \geq r} \|\nabla^2 \varphi(t, x)\| = 0 \end{aligned} \quad (4.2)$$

the following equality holds

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} u (\partial_t \varphi + \nabla \cdot (\mu \nabla \varphi) + \beta \cdot \nabla \varphi - \kappa \varphi) dx dt \\ & + \int_{\mathbb{R}^2} u_o(x) \varphi(0, x) dx - \int_{\mathbb{R}^2} u(T, x) \varphi(T, x) dx + \int_0^T \int_{\mathbb{R}^2} g \varphi dx dt = 0. \end{aligned} \quad (4.3)$$

Below, we extensively use the Green function Γ associated to the homogeneous version of (4.1), refer to Lemma 6.6 for further details.

Theorem 4.2. *Assume that (HP) holds. Let $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ and $g \in \mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$. Then, Problem (4.1) admits a unique weak solution in the sense of Definition 4.1. Moreover:*

(P.1) *The following representation formula holds:*

$$u(t, x) = \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) u_o(\xi) d\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) g(\tau, \xi) d\xi d\tau \quad (4.4)$$

where Γ is defined in Lemma 6.6.

(P.2) *If also $\widehat{u}_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ and $\widehat{g} \in \mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$, calling \widehat{u} the corresponding solution, the following estimate holds:*

$$\|u(t) - \widehat{u}(t)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \leq 4\pi C \widehat{\mu} \left(\|u_o - \widehat{u}_o\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} + \|g - \widehat{g}\|_{\mathbf{L}^1([0, t] \times \mathbb{R}^2; \mathbb{R})} \right) \quad (4.5)$$

where C is defined in (F.3) of Lemma 6.6.

(P.3) *If $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}_+)$ and $g, \widehat{g} \in \mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$ satisfy $g \geq \widehat{g}$, then the corresponding solutions u, \widehat{u} satisfy $u \geq \widehat{u}$.*

The proof is deferred to § 6.3.

Consider now the case of a measure source. Call \mathcal{M} the set of Radon measures on \mathbb{R}^2 and denote by $\|\mu\|_{\mathcal{M}}$ the total variation norm, see [10, Chapter 7, § 3, p.216].

Definition 4.3. *A function $u \in \mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$ and $u \in \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}))$ is a weak solution to (4.1) with $g: [0, T] \mapsto \mathcal{M}$ being measurable, if for any test function φ as in (4.2) the following equality holds*

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} u (\partial_t \varphi + \nabla \cdot (\mu \nabla \varphi) + \beta \cdot \nabla \varphi - \kappa \varphi) dx dt \\ & + \int_{\mathbb{R}^2} u_o(x) \varphi(0, x) dx - \int_{\mathbb{R}^2} u(T, x) \varphi(T, x) dx + \int_0^T \int_{\mathbb{R}^2} \varphi(t, x) dg_t(x) dt = 0 \end{aligned}$$

where $g_t(x) = g(t, x)$.

Theorem 4.4. *Assume that (HP) holds. Let $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ and $g: [0, T] \rightarrow \mathcal{M}$ be measurable with*

$$\sup_{\tau \in [0, T]} \|g_\tau\|_{\mathcal{M}} < +\infty. \quad (4.6)$$

Then, Problem (4.1) admits a unique weak solution in the sense of Definition 4.3. Moreover:

(M.1) *The following representation formula holds:*

$$u(t, x) = \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) u_o(\xi) d\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) dg_\tau(\xi) d\tau \quad (4.7)$$

where Γ is defined in Lemma 6.6.

(M.2) If also $\hat{u}_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ and $\hat{g}: [0, T] \rightarrow \mathcal{M}$ is measurable and satisfies (4.6), calling \hat{u} the corresponding solution, the following estimate holds:

$$\|u(t) - \hat{u}(t)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \leq 4\pi C \hat{\mu} \left(\|u_o - \hat{u}_o\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} + t \sup_{[0, t]} \|g_\tau - \hat{g}_\tau\|_{\mathcal{M}} \right) \quad (4.8)$$

where C is defined in (F.3) of Lemma 6.6.

(M.3) If $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}_+)$ and $g, \hat{g}: [0, T] \rightarrow \mathcal{M}$ satisfy (4.6) and $g_\tau \geq \hat{g}_\tau$ for all $\tau \in [0, T]$, then the corresponding solutions u, \hat{u} satisfy $u \geq \hat{u}$.

5 The Coupled Problem

5.1 Well-Posedness

We now couple the traffic model (3.8) to the pollution diffusion model (4.1), choosing a source term g in (4.1) that depends on the traffic evolution described by (3.8). To this aim, for $i = 1, 2, 3$, extend each ρ^i to \mathbb{R}^2 assigning the value 0 for all $x \in \mathbb{R}^2 \setminus \mathcal{R}_i$.

A first reasonable choice is

$$g(t, x) = G(t, \rho(t, x), q(t, x)) \quad \text{where} \quad \begin{aligned} \rho &= (\rho^1, \rho^2, \rho^3) \\ q &= (q(\rho^1), q(\rho^2), q(\rho^3)) \end{aligned} \quad (5.1)$$

meaning that the production of pollutant at (t, x) depends on the traffic density $\rho(t, x)$ and on the traffic flow $q(t, x)$ at (t, x) .

Theorem 5.1. *Let q satisfy (q). For $i = 1, 2, 3$, let \mathcal{R}^i be a road as in Definition 3.1. Assume that, for $i = 1, 2, 3$, $\rho_o^i \in \mathbf{L}^1(\mathcal{R}^i; [0, R])$, $f_{in}^1 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^1; [0, q(\bar{\rho})])$, $f_{in}^2 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^2; [0, q(\bar{\rho})])$, $f_{out}^3 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}^3; [0, q(\bar{\rho})])$, and $w \in \mathcal{W}$ is as in (2.6). Assume that (HP) holds, $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}_+)$ and moreover*

(G) $G: [0, T] \times [0, R]^3 \times [0, q(\bar{\rho})]^3 \rightarrow \mathbb{R}$ is such that

$$\begin{aligned} \forall t \in [0, T] \quad & (\rho, q) \mapsto G(t, \rho, q) \in \mathbf{C}^{0,1}([0, R]^3 \times [0, q(\bar{\rho})]^3; \mathbb{R}) \\ \forall (\rho, q) \in [0, R]^3 \times [0, q(\bar{\rho})]^3 \quad & t \mapsto G(t, \rho, q) \in \mathbf{L}^1([0, T]; \mathbb{R}) \end{aligned}$$

Then, there exists a unique solution $(\rho^1, \rho^2, \rho^3, u)$ to the coupled problem (3.8)–(4.1)–(5.1).

The proof follows from Theorem 3.7 and Theorem 4.2, which can be applied since the source term g provided by (G) is in $\mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$.

Remark that physical reasons induce to consider only positive functions G in (G). However, this assumption is not necessary in Theorem 5.1. The proof is deferred to § 6.4.

We now take into consideration the pollution produced by variations in the traffic flow, such as those due to accelerations or breakings. To this aim, we consider source terms in (4.1) that depend on the total variation of the traffic flow. The analytical basis for this relies on the information provided by Theorem 3.7.

Proposition 5.2. *Under the same assumptions of Theorem 3.7, including also the requirement in (2D.2), each component ρ^i of the solution to (3.8)–(2.6) defines for $i = 1, 2, 3$, the map $g^i: [0, T] \rightarrow \mathcal{M}$ setting, for any test function $\varphi \in \mathbf{C}_c^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$,*

$$g_t^i(\varphi) = \int_{-L}^L \int_I \varphi(t, \mathcal{R}^i(s, \ell)) \left(1 - \ell (\gamma'(s))^\perp \cdot \gamma''(s)\right) d\mu_t^{i, \ell}(s) d\ell \quad (5.2)$$

where $\mu_t^{i, \ell}$ is the total variation measure of the distributional derivative of the map $s \mapsto q\left(\rho^i(t, \mathcal{R}^i(s, \ell))\right)$. Moreover, for $i = 1, 2, 3$, the measure g_t^i is supported in \mathcal{R}^i and the map $g = g^1 + g^2 + g^3$ satisfies (4.6).

The measure g^i essentially replaces the total variation of the distributional derivative of the map $x \mapsto q\left(\rho^i(t, x)\right)$.

Theorem 5.3. *Let the assumptions of Proposition 5.2 hold. Assume that (HP) holds and let $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}_+)$. Let g be defined as in Proposition 5.2. Then, there exists a unique solution $(\rho^1, \rho^2, \rho^3, u)$ to the coupled problem (3.8)–(4.1)–(5.1).*

The proof follows applying subsequently Theorem 3.7, Proposition 5.2 and Theorem 4.4.

5.2 Optimal Management

To limit the effects of pollutant, the controller acts on the evolution of traffic through the function w in (2.6) that regulates the merge, or also on the inflows f_{in}^1, f_{in}^2 and on the outflow f_{out}^3 . The target is the minimization of a functional of the type

$$\mathcal{F}(w, f_{in}^1, f_{in}^2, f_{out}^3) = \int_0^T \int_{\mathbb{R}^2} p(t, x) u(t, x) dx dt \quad (5.3)$$

where u is the pollutant's density and p is a suitable weight assigning to different regions different relevance according, for instance, to population density.

Theorem 5.4. *Let q satisfy (q). For $i = 1, 2, 3$, let \mathcal{R}^i be a road as in Definition 3.1. Assume that, for $i = 1, 2, 3$, $\rho_o^i \in \mathbf{L}^1(\mathcal{R}^i; [0, R])$. Assume that (HP) and (G) hold, $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}_+)$ and moreover \mathcal{F} is as in (5.3) with $p \in \mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})$.*

Let $\rho^1, \rho^2, \rho^3, u$ be the unique solution to the coupled problem (3.8)–(4.1)–(5.1) regulated by w in \mathcal{W} as in (2.6) with inflows $f_{in}^1 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^1; [0, q(\bar{\rho})])$, $f_{in}^2 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^2; [0, q(\bar{\rho})])$, and outflow $f_{out}^3 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}^3; [0, q(\bar{\rho})])$. Then, the map

$$\mathcal{F}: \mathcal{W} \times \prod_{i=1}^2 \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^i; [0, q(\bar{\rho})]) \times \mathbf{L}^1([0, T] \times \mathcal{E}_{out}^3; [0, q(\bar{\rho})]) \rightarrow \mathbb{R} \quad (5.4)$$

in (5.3) is \mathbf{L}^1 -Lipschitz continuous.

The proof is deferred to § 6.4.

As for G in (G), physical reasons induce to consider only positive functions p in the functional \mathcal{F} (5.3). We remark that these positivity assumptions on both G and p are not necessary in the proof of Theorem 5.4.

Theorem 5.4 allows to prove the existence of optimal management strategies as soon as w , f_{in}^1 , f_{in}^2 and f_{out}^3 are suitably selected in a compact set, thanks to Weierstrass Theorem.

An example tailored to the present physical setting is as follows. For a $N \in \mathbb{N}$, introduce the sets $\mathbf{PC}_N([0, T]; \{0, 1\})$, respectively $\mathbf{PC}_N([0, T]; \{0, q(\bar{\rho})\})$, of piecewise constant functions defined on $[0, T]$ with values in $\{0, 1\}$, respectively $\{0, q(\bar{\rho})\}$, that change value at most N times, i.e., with at most N jumps. This choice corresponds to traffic lights at the merge and at $\mathcal{E}_{in}^1, \mathcal{E}_{in}^2, \mathcal{E}_{out}^3$ that alternate between green and red lights at most N times in $[0, T]$.

Corollary 5.5. *Under the same assumptions of Theorem 5.4, the map \mathcal{F} in (5.4) restricted to $\mathbf{PC}_N([0, T]; \{0, 1\}) \times \mathbf{PC}_N([0, T]; \{0, q(\bar{\rho})\})^3$ admits a point of global minimum.*

Alternatively, Theorem 5.4 also comprises the case of assigned – possibly non maximal – inflows and outflows.

Corollary 5.6. *Under the same assumptions of Theorem 5.4, fix $f_{in}^1 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^1; [0, q(\bar{\rho})])$, $f_{in}^2 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^2; [0, q(\bar{\rho})])$, and $f_{out}^3 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}^3; [0, q(\bar{\rho})])$. Then, the map $w \mapsto \mathcal{F}(w, f_{in}^1, f_{in}^2, f_{out}^3)$ with \mathcal{F} as in (5.4) admits a point of global minimum in $\mathbf{PC}_N([0, T]; \{0, 1\})$.*

6 Technical Details

6.1 Proofs Related to the 1D Hyperbolic Traffic Model

Proof of Proposition 2.2. We distinguish different steps.

Claim 1: The BV case. We assume in addition that $\rho_o \in \mathbf{BV}(I; [0, R])$ and $(q_{|[0, \bar{\rho}]})^{-1}(f_{in})$, $(q_{|[\bar{\rho}, R]})^{-1}(f_{out}) \in \mathbf{BV}([0, T]; \mathbb{R})$. Then, the existence of solutions and the estimate (2.3) follow from [9, Proposition 3.3].

Assume that ρ and $\hat{\rho}$ are solutions to (2.1) in the sense of Definition 2.1. To prove the bound (2.4), we refer to the detailed proof in [8, Theorem 4.3], devoted to the multi-dimensional case. Exploit the fact that in problem (2.1) the fluxes at the boundaries are assigned. The key point is the estimate of the term

$$-\operatorname{sgn}(\operatorname{tr} \rho - \operatorname{tr} \hat{\rho}) [q(\operatorname{tr} \rho) - q(\operatorname{tr} \hat{\rho})] \cdot \nu, \quad (6.1)$$

ν being the exterior normal vector, here $\nu = \pm 1$, and in the present setting the traces correspond to the two boundary terms, one at s_{in} and one at s_{out} . Let us focus on the left boundary at s_{in} . Recall that for a.e. $t \in [0, T]$

$$\begin{aligned} (\operatorname{tr} \rho)(t, s_{in}) &\in \left\{ (q_{|[0, \bar{\rho}]})^{-1}(f_{in}(t)) \right\} \cup \left[(q_{|[\bar{\rho}, R]})^{-1}(f_{in}(t)), R \right], \\ (\operatorname{tr} \hat{\rho})(t, s_{in}) &\in \left\{ (q_{|[0, \bar{\rho}]})^{-1}(\hat{f}_{in}(t)) \right\} \cup \left[(q_{|[\bar{\rho}, R]})^{-1}(\hat{f}_{in}(t)), R \right]. \end{aligned}$$

A case by case analysis shows that

$$-\operatorname{sgn}((\operatorname{tr} \rho)(t, s_{in}) - (\operatorname{tr} \hat{\rho})(t, s_{in})) \left[q((\operatorname{tr} \rho)(t, s_{in})) - q((\operatorname{tr} \hat{\rho})(t, s_{in})) \right] \cdot \nu \leq |f_{in}(t) - \hat{f}_{in}(t)|.$$

This term is then integrated over the time interval $[0, t]$ leading to the second term in the right hand side of (2.4). The term related to the right boundary s_{out} is treated in an entirely similar way.

Claim 2: The general case. Introduce a sequence of mollifiers $\eta_n(\xi) = n\widehat{\eta}(n\xi)$ with $\widehat{\eta} \in \mathbf{C}_c^\infty(\mathbb{R}; \mathbb{R})$, $\widehat{\eta} \geq 0$, $\text{spt } \widehat{\eta} = [-1, 1]$ and $\int_{\mathbb{R}} \widehat{\eta} = 1$. Define

$$\begin{aligned} \widehat{\rho}_o^n(x) &= \begin{cases} \rho_o(x) & x \in I \\ 0 & x \in \mathbb{R} \setminus I \end{cases} & \rho_o^n(x) &= (\widehat{\rho}_o^n * \eta_n)(x) & x \in I, \\ \widehat{f}_{in}^n(t) &= \begin{cases} \min \left\{ f_{in}(t), q \left(\bar{\rho} - \frac{1}{n} \right) \right\} & t \in [0, T] \\ 0 & t \in \mathbb{R} \setminus [0, T] \end{cases} & f_{in}^n(t) &= (\widehat{f}_{in}^n * \eta_n)(t) & t \in [0, T], \\ \widehat{f}_{out}^n(t) &= \begin{cases} \min \left\{ f_{out}(t), q \left(\bar{\rho} + \frac{1}{n} \right) \right\} & t \in [0, T] \\ 0 & t \in \mathbb{R} \setminus [0, T] \end{cases} & f_{out}^n(t) &= (\widehat{f}_{out}^n * \eta_n)(t) & t \in [0, T]. \end{aligned}$$

Clearly, $\rho_o^n \in \mathbf{W}^{1,\infty}(I; [0, R])$ and $\lim_{n \rightarrow +\infty} \rho_o^n = \rho_o$ in $\mathbf{L}^1(I; [0, R])$. At the same time, also $f_{in}^n \in \mathbf{W}^{1,\infty}([0, T]; [0, q(\bar{\rho} - \frac{1}{n})])$, so that $(q_{[0, \bar{\rho}]})^{-1} \circ f_{in}^n \in \mathbf{W}^{1,\infty}([0, T]; [0, \bar{q}])$ since $(q_{[0, \bar{\rho}]})^{-1} \in \mathbf{W}^{1,\infty}([0, q(\bar{\rho} - \frac{1}{n})]; [0, \bar{\rho} - \frac{1}{n}])$. Moreover, the map $(q_{[0, \bar{\rho}]})^{-1}(f_{in}^n)$ converges to $(q_{[0, \bar{\rho}]})^{-1}(f_{in})$ in $\mathbf{L}^1([0, T]; \mathbb{R})$. The outflow \widehat{f}_{out}^n is treated similarly.

Note also, for later use, that by construction

$$\begin{aligned} \|\rho_o^n\|_{\mathbf{L}^\infty(I; \mathbb{R})} &\leq \|\rho_o\|_{\mathbf{L}^\infty(I; \mathbb{R})} \quad \text{and} \quad \|f_{in}^n\|_{\mathbf{L}^\infty([0, T]; \mathbb{R})} \leq \|f_{in}\|_{\mathbf{L}^\infty([0, T]; \mathbb{R})}, \\ \|f_{out}^n\|_{\mathbf{L}^\infty([0, T]; \mathbb{R})} &\leq \|f_{out}\|_{\mathbf{L}^\infty([0, T]; \mathbb{R})}. \end{aligned} \tag{6.2}$$

Moreover, $\rho_o^n \in \mathbf{BV}(I; [0, R])$, $(q_{[0, \bar{\rho}]})^{-1}(f_{in}^n(t)) \in \mathbf{BV}([0, T]; \mathbb{R})$ and $(q_{[\bar{\rho}, R]})^{-1}(f_{out}^n(t)) \in \mathbf{BV}([0, T]; \mathbb{R})$ so that Claim 1 applies.

Call ρ_n the corresponding solution to (2.1). By (2.4), the sequence ρ_n is a Cauchy sequence in $\mathbf{L}^1([0, T] \times I; \mathbb{R})$. Call ρ_∞ the corresponding \mathbf{L}^1 limit. Since Definition 2.1 is stable with respect to \mathbf{L}^1 convergence and thanks to (\mathbf{q}) , we obtain that ρ solves (2.1) in the sense of Definition 2.1. Also (2.4) immediately follows. Since ρ_n satisfies (2.3), (6.2) ensures that ρ_∞ satisfies (2.3).

Finally, the **BV** estimate follows from [6, Theorem 2.2]. \square

Proof of Theorem 2.4. We assume that, with a suitable reparametrization, $\sup I_1 = \sup I_2 = \inf I_3$.

Existence and uniqueness of solution to (2.5)–(2.6). A solution to (2.5) regulated by (2.6) can be iteratively constructed as follows. Assume for simplicity that $J_1 = [0, \tau_1[$.

Then, for all $t \in J_1$ consider separately the problems

$$\begin{array}{cc} s \in]s_{in}^1, s_{out}^3[& s \in I_2 \\ \left\{ \begin{array}{l} \partial_t \rho + \partial_s q(\rho) = 0 \\ \rho(0, s) = \begin{cases} \rho_o^1(s) & s \in I_1 \\ \rho_o^3(s) & s \in I_3 \end{cases} \\ q(\rho(t, s_{in}^1)) = f_{in}^1(t) \\ q(\rho(t, s_{out}^3)) = f_{out}^3(t) \end{array} \right. & \left\{ \begin{array}{l} \partial_t \rho^2 + \partial_s q(\rho^2) = 0 \\ \rho(0, s) = \rho_o^2(s) \\ q(\rho^2(t, s_{in})) = f_{in}^2(t) \\ q(\rho^2(t, s_{out})) = 0. \end{array} \right. \end{array} \quad (6.3)$$

Both of them fit into Proposition 2.2 and thus admit unique solutions ρ and ρ^2 defined for $t \in J_1$. Thanks to (2.7), we immediately have that, setting $\rho^1 = \rho|_{I_1}$ and $\rho^3 = \rho|_{I_3}$, the triple ρ^1, ρ^2, ρ^3 is the unique solution to (2.5) regulated by w as in (2.6) for $t \in J_1$.

This procedure can be iterated on the time interval $[\sup J_1, \inf J_2[$, gluing road 2 with road 3, taking as initial datum at time τ_1 , $\rho^1(\tau_1)$ and $\rho(\tau_1, s) = \begin{cases} \rho^2(\tau_1, s) & s \in I_2 \\ \rho^3(\tau_1, s) & s \in I_3 \end{cases}$. Hence, (2.5)–(2.6) admits a unique solution in the sense of Definition 2.3.

Proof of (1D.1) Use the construction above to define the solutions ρ^1, ρ^2, ρ^3 and $\widehat{\rho}^1, \widehat{\rho}^2, \widehat{\rho}^3$. On each time interval, the estimate (2.4) applies. Note that this estimate is additive both in time and in space over non overlapping time or space intervals.

Proof of (1D.2) The construction above of ρ^1, ρ^2, ρ^3 allows to iteratively use the bound on the total variation of $\Psi(\rho^1), \Psi(\rho^2), \Psi(\rho^3)$ in Proposition 2.2.

Proof of (1D.3) Introduce a partition of $[0, T]$ in intervals $[\tau_j, \tau_{j+1}[$ for $i = 1, \dots, N$ and values $\vartheta_1, \dots, \vartheta_N, \widehat{\vartheta}_1, \dots, \widehat{\vartheta}_N$ in $\{0, 1\}$ such that

$$\begin{aligned} w &= \sum_{j=1}^N \vartheta_j \chi_{[\tau_j, \tau_{j+1}[}, & \text{and} & \forall j \in \{1, \dots, N-1\} \quad \vartheta_j \neq \vartheta_{j+1} \text{ or } \widehat{\vartheta}_j \neq \widehat{\vartheta}_{j+1} \\ \widehat{w} &= \sum_{j=1}^N \widehat{\vartheta}_j \chi_{[\tau_j, \tau_{j+1}[}, & & 0 = \tau_1 < \tau_2 < \dots < \tau_N < \tau_{N+1} = T. \end{aligned}$$

Assume $t \in [\tau_1, \tau_2[$. Then, these cases are possible:

1. $\vartheta_1 = 0, \widehat{\vartheta}_1 = 0$: then, problem (2.5) can be solved considering roads 2 and 3 as a unique road in the time interval $[\tau_1, \tau_2[$, the outflow of road 1 being $f_{out}^1 = 0$. Hence, we have $\sum_{i=1}^3 \|\rho^i(t) - \widehat{\rho}^i(t)\|_{\mathbf{L}^1(I_i; \mathbb{R})} = 0$.
2. $\vartheta_1 \neq \widehat{\vartheta}_1$: by the stability estimate (2.4) we have for $i = 1, 2$

$$\begin{aligned} \|\rho^i(t) - \widehat{\rho}^i(t)\|_{\mathbf{L}^1(I_i; \mathbb{R})} &\leq \|\rho^i(\tau_1) - \widehat{\rho}^i(\tau_1)\|_{\mathbf{L}^1(I_i; \mathbb{R})} + q(\bar{\rho}) \left| \vartheta_1 - \widehat{\vartheta}_1 \right| (t - \tau_1) \\ &= q(\bar{\rho}) \|w - \widehat{w}\|_{\mathbf{L}^1([0, t]; \mathbb{R})} \end{aligned}$$

$$\begin{aligned}\left\|\rho^3(t) - \hat{\rho}^3(t)\right\|_{\mathbf{L}^1(I_3; \mathbb{R})} &\leq \left\|\rho^3(\tau_1) - \hat{\rho}^3(\tau_1)\right\|_{\mathbf{L}^1(I_3; \mathbb{R})} + q(\bar{\rho}) \left|\vartheta_1 - \hat{\vartheta}_1\right|(t - \tau_1) \\ &= q(\bar{\rho}) \|w - \hat{w}\|_{\mathbf{L}^1([0, t]; \mathbb{R})}.\end{aligned}$$

- 3.** $\vartheta_1 = 1, \hat{\vartheta}_1 = 1$: problem (2.5) can be solved considering roads 1 and 3 as a unique road in the time interval $[\tau_1, \tau_2[$, the outflow of road 2 being $f_{out}^2 = 0$. Hence we have $\sum_{i=1}^3 \left\|\rho^i(t) - \hat{\rho}^i(t)\right\|_{\mathbf{L}^1(I_i; \mathbb{R})} = 0$.

Proceed now iteratively and assume that $t \in [\tau_j, \tau_{j+1}[$. Then,

- 1.** $\vartheta_j = 0, \hat{\vartheta}_j = 0$: problem (2.5) can be solved considering roads 2 and 3 as a unique road in the time interval $[\tau_j, \tau_{j+1}[$, the outflow of road 1 being $f_{out}^1 = 0$. Hence we have $\sum_{i=1}^3 \left\|\rho^i(t) - \hat{\rho}^i(t)\right\|_{\mathbf{L}^1(I_i; \mathbb{R})} \leq \sum_{i=1}^3 \left\|\rho^i(\tau_j) - \hat{\rho}^i(\tau_j)\right\|_{\mathbf{L}^1(I_i; \mathbb{R})}$ and observe that, clearly, $\|w - \hat{w}\|_{\mathbf{L}^1([\tau_j, t]; \mathbb{R})} = 0$.

- 2.** $\vartheta_j \neq \hat{\vartheta}_j$: by the stability estimate (2.4) we have for $i = 1, 2$

$$\begin{aligned}\left\|\rho^i(t) - \hat{\rho}^i(t)\right\|_{\mathbf{L}^1(I_i; \mathbb{R})} &\leq \left\|\rho^i(\tau_j) - \hat{\rho}^i(\tau_j)\right\|_{\mathbf{L}^1(I_i; \mathbb{R})} + q(\bar{\rho}) \left|\vartheta_j - \hat{\vartheta}_j\right|(t - \tau_j) \\ \left\|\rho^3(t) - \hat{\rho}^3(t)\right\|_{\mathbf{L}^1(I_3; \mathbb{R})} &\leq \left\|\rho^3(\tau_j) - \hat{\rho}^3(\tau_j)\right\|_{\mathbf{L}^1(I_3; \mathbb{R})} + q(\bar{\rho}) \left|\vartheta_j - \hat{\vartheta}_j\right|(t - \tau_j)\end{aligned}$$

and observe that $\left|\vartheta_j - \hat{\vartheta}_j\right|(t - \tau_j) = \|w - \hat{w}\|_{\mathbf{L}^1([\tau_j, t]; \mathbb{R})}$.

- 3.** $\vartheta_j = 1, \hat{\vartheta}_j = 1$: problem (2.5) can be solved considering roads 1 and 3 as a unique road in the time interval $[\tau_j, \tau_{j+1}[$, the outflow of road 2 being $f_{out}^2 = 0$. Hence, we have $\sum_{i=1}^3 \left\|\rho^i(t) - \hat{\rho}^i(t)\right\|_{\mathbf{L}^1(I_i; \mathbb{R})} \leq \sum_{i=1}^3 \left\|\rho^i(\tau_j) - \hat{\rho}^i(\tau_j)\right\|_{\mathbf{L}^1(I_i; \mathbb{R})}$ and observe that, clearly, $\|w - \hat{w}\|_{\mathbf{L}^1([\tau_j, t]; \mathbb{R})} = 0$.

The bound (2.8) follows. \square

6.2 Proofs Related to the 2D Hyperbolic Traffic Model

Lemma 6.1. *Let \mathcal{R} be a road as in Definition 3.1. Then, the following identities hold:*

$$D\mathcal{R}(s, \ell) = \begin{bmatrix} \gamma'(s) + \ell (\gamma''(s))^\perp & (\gamma'(s))^\perp \end{bmatrix} \quad (6.4)$$

$$\det D\mathcal{R}(s, \ell) = 1 - \ell (\gamma'(s))^\perp \cdot \gamma''(s) \quad (6.5)$$

$$[D\mathcal{R}(s, \ell)]^{-1} = \frac{1}{1 - \ell (\gamma'(s))^\perp \cdot \gamma''(s)} \begin{bmatrix} (\gamma'(s))^\top \\ ((\gamma'(s))^\perp - \ell \gamma''(s))^\top \end{bmatrix} \quad (6.6)$$

$$\nabla_x s = \frac{1}{1 - \ell (\gamma'(s))^\perp \cdot \gamma''(s)} (\gamma'(s))^\top. \quad (6.7)$$

Moreover, if r is as defined in (3.2), then

$$\nabla_x \cdot r = 0. \quad (6.8)$$

In connection with (6.5), observe that, since $\gamma'(s) \cdot \gamma''(s) = 0$, we have

$$\left| (\gamma'(s))^\perp \cdot \gamma''(s) \right| = \left| \gamma'(s) \cdot \left(\|\gamma''(s)\| \gamma'(s) \right) \right| = \|\gamma''(s)\|,$$

which is the *curvature* of γ at $\gamma(s)$.

Proof of Lemma 6.1. The proofs of (6.4), (6.5) and (6.6) are immediate. Relation (6.7) is the first line in (6.6). To verify (6.8), use (6.7) and $\partial_\ell r = 0$, for $x = \mathcal{R}(s, \ell)$, so that

$$\begin{aligned} \nabla_x \cdot r &= \partial_{x_1} r_1 + \partial_{x_2} r_2 \\ &= \partial_s \gamma'_1 \partial_{x_1} s + \partial_s \gamma'_2 \partial_{x_2} s \\ &= \begin{bmatrix} \partial_{x_1} s & \partial_{x_2} s \end{bmatrix} \begin{bmatrix} \partial_s \gamma'_1 \\ \partial_s \gamma'_2 \end{bmatrix} \\ &= \nabla_x s \gamma'' \\ &= \frac{1}{1 - \ell (\gamma')^\perp \cdot \gamma''} (\gamma')^\top \gamma'' \\ &= 0, \end{aligned}$$

completing the proof. \square

For later use, we provide the following equalities, where $\varphi: \mathbb{R} \times \mathcal{R} \rightarrow \mathbb{R}_+$ is smooth.

$$\nabla_{(s, \ell)} \varphi(\mathcal{R}(s, \ell)) = \nabla_x \varphi(x)_{|x=\mathcal{R}(s, \ell)} [D\mathcal{R}(s, \ell)]. \quad (6.9)$$

$$\nabla_x \varphi(\mathcal{R}^{-1}(x)) = \nabla_{(s, \ell)} \varphi(s, \ell)_{|(s, \ell)=\mathcal{R}^{-1}(x)} [D\mathcal{R}(s, \ell)]. \quad (6.10)$$

Proof of Lemma 3.2. By construction, $\mathcal{R} \in \mathbf{C}^1(\bar{I} \times]-L, L[, \mathbb{R}^2)$. \mathcal{R} is locally invertible in each point of $I \times]-L, L[$, by (6.5)

$$\det D\mathcal{R}(s, \ell) > 1 - L \|\gamma'(s)\| \|\gamma''(s)\| = 1 - L \|\gamma''(s)\| > 0$$

which shows that $\det D\mathcal{R}(s, \ell) \neq 0$. The Implicit Function Theorem ensures local invertibility at any $(s, \ell) \in I \times]-L, L[$.

Since $\mathcal{R}(I \times]-L, L[)$ is simply connected, the map $\mathcal{R}: I \times]-L, L[\rightarrow \mathcal{R}(I \times]-L, L[)$ is globally invertible and its inverse is of class \mathbf{C}^1 .

The conditions $\|\gamma'\| = 1$, $\|\gamma''\| \leq 1/L$, $|\ell| < L$ and (6.4) ensure that \mathcal{R} is uniformly continuous and can be uniquely extended to $\bar{I} \times]-L, L[$. \mathcal{R} is injective also on $\bar{I} \times]-L, L[$, otherwise $\mathcal{R}(\bar{I} \times]-L, L[)$ can not be simply connected. \square

Lemma 6.2. *Let $f \in \mathbf{BV}(]-L, L[; \mathbb{R})$. Then,*

$$\forall \eta \in \mathbf{C}^1([-L, L]; \mathbb{R}_+) \quad \int_{-L}^L f(\ell) \eta(\ell) d\ell \geq 0 \implies f(\ell) \geq 0 \text{ for a.e. } \ell \in [-L, L].$$

Proof of Lemma 6.2. If f is continuous, then the conclusion follows by a standard elementary procedure. Assume $f \in \mathbf{BV}(]-L, L[; \mathbb{R})$ and, by contradiction, that $f(\ell) < 0$ for $\ell \in E$,

E being a subset of $] -L, L[$ of positive measure. Call η_n a sequence in $\mathbf{C}^1([-L, L]; \mathbb{R}_+)$ such that $\left\| \eta_n - \chi_E \right\|_{\mathbf{L}^1([-L, L]; \mathbb{R})} \rightarrow 0$ as $n \rightarrow +\infty$, χ_E being the characteristic function of E . Then,

$$0 > \int_E f(\ell) d\ell = \lim_{n \rightarrow +\infty} \int_{-L}^L f(\ell) \eta_n(\ell) d\ell \geq 0,$$

where the latter inequality follows from the assumption. \square

Lemma 6.3. *Let $K \subset \mathbb{R}$ be a compact interval and $\varphi \in \mathbf{C}^1(K \times [0, 1]^2; \mathbb{R}_+)$. Then, for every $\varepsilon > 0$, there exist functions $f_1, \dots, f_n, g_1, \dots, g_n \in \mathbf{C}^1([0, 1]; [0, 1])$ and $c_{ij} \in \mathbf{C}^1(K; \mathbb{R}_+)$ for $i, j = 1, \dots, n$ such that*

$$\left| \varphi(t, x, y) - \sum_{i=1}^n \sum_{j=1}^n c_{ij}(t) f_i(x) g_j(y) \right| < \varepsilon \quad \text{for all } (t, x, y) \in K \times [0, 1]^2.$$

Proof. Define

$$\begin{aligned} \Upsilon &: \mathbb{R} \rightarrow \mathbb{R} \\ x &\mapsto (2|x| + 1) (|x| - 1)^2 \chi_{[-1, 1]}(x). \end{aligned} \quad (6.11)$$

Clearly, Υ is even, non negative and in $\mathbf{C}^1(\mathbb{R}; \mathbb{R})$. Moreover, for all $x \in [-1, 1]$, Υ is such that $\Upsilon(x - 1) + \Upsilon(x) + \Upsilon(x + 1) = 1$, since at any $x \in [-1, 1]$ only two summands in the latter sum are non 0.

Choose n so that $1/n$ is a modulus of uniform continuity of φ on $[0, 1]^2$ and set for $t \in K$

$$c_{ij}(t) = \varphi\left(t, \frac{i}{n}, \frac{j}{n}\right), \quad f_i(x) = \Upsilon(nx - i), \quad g_j(y) = \Upsilon(ny - j). \quad (6.12)$$

The proof follows by direct computations, since $\sum_{i=1}^n f_i(x) = \sum_{j=1}^n g_j(y) = 1$ for $(x, y) \in [0, 1]^2$. \square

Proof of Lemma 3.4. In this proof, we extensively use the change of variable $x = \mathcal{R}(s, \ell)$, see Lemma 3.2 for its properties, and, with the notation (3.5), the change of variables $(\sigma, \ell) = \mathcal{S}(s, \ell)$ where

$$\begin{aligned} \mathcal{S}: \bar{I} \times]-L, L[&\rightarrow \left\{ (\sigma, \ell) \in \mathbb{R}^2: \sigma \in \bar{I}^\ell \text{ and } \ell \in]-L, L[\right\} \\ (s, \ell) &\mapsto (\sigma, \ell) \quad \text{where } \frac{d\sigma}{ds}(s) = 1 - \ell (\gamma'(s))^\perp \cdot \gamma''(s). \end{aligned} \quad (6.13)$$

Note that \mathcal{S} is a diffeomorphism of class $\mathbf{C}^1(I \times]-L, L[; \mathbb{R}^2) \cap \mathbf{C}^0(\bar{I} \times]-L, L[; \mathbb{R}^2)$ and the same computations as in the proof of Lemma 3.2 ensure that $s_{out}^\ell > s_{in}^\ell$.

Let ρ solve (3.3) in the sense of Definition 3.3. Let $\varphi \in \mathbf{C}_c^1(\mathbb{R} \times \mathcal{R}; \mathbb{R}_+)$ be a test function and call

$$\tilde{\varphi}(t, s, \ell) = \varphi(t, \mathcal{R}(s, \ell)) \quad \text{and} \quad \varphi^\ell(t, \sigma) = \tilde{\varphi}\left(t, \mathcal{S}^{-1}(\sigma, \ell)\right). \quad (6.14)$$

Clearly, $\varphi^\ell \in \mathbf{C}_c^1(\mathbb{R} \times I^\ell; \mathbb{R}_+)$. Moreover, setting for instance $\Phi^\ell(t, \sigma) = \eta(\ell) \hat{\varphi}(t, \sigma)$ with $\eta \in \mathbf{C}_c^1([-L, L]; \mathbb{R}_+)$ and $\hat{\varphi} \in \mathbf{C}_c^1(\mathbb{R} \times I^\ell; \mathbb{R}_+)$, reversing the changes of variables \mathcal{R} and (6.13), through (6.14) one gets that $\Phi^\ell = \varphi^\ell$ for a $\varphi \in \mathbf{C}_c^1(\mathbb{R} \times \mathcal{R}; \mathbb{R}_+)$.

Similarly, also recalling (3.6), denote for a.e. $t \in [0, T]$, $\ell \in]-L, L[$, $s \in I$, $\sigma \in I^\ell$,

$$\tilde{\rho}(t, s, \ell) = \rho(t, \mathcal{R}(s, \ell)) \quad \text{and} \quad \rho^\ell(t, \sigma) = \tilde{\rho}\left(t, \mathcal{S}^{-1}(\sigma, \ell)\right). \quad (6.15)$$

Compute each line in the integral inequality in Definition 3.3, using first the change of variables $x = \mathcal{R}(s, \ell)$ and then the change of variables (6.13).

$$\begin{aligned} & \int_0^T \int_{\mathcal{R}} (\rho(t, x) - \kappa)^\pm \partial_t \varphi(t, x) \, dx \, dt \\ &= \int_0^T \int_{-L}^L \int_I \left(\rho(t, \mathcal{R}(s, \ell)) - \kappa \right)^\pm \partial_t \varphi(t, \mathcal{R}(s, \ell)) |\det D\mathcal{R}(s, \ell)| \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_I \left(\tilde{\rho}(t, s, \ell) - \kappa \right)^\pm \partial_t \tilde{\varphi}(t, s, \ell) \left(1 - \ell \left(\gamma'(s) \right)^\perp \cdot \gamma''(s) \right) \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_{I^\ell} \left(\rho^\ell(t, \sigma) - \kappa \right)^\pm \partial_t \varphi^\ell(t, \sigma) \, d\sigma \, d\ell \, dt \\ &= \int_{-L}^L \left(\int_0^T \int_{I^\ell} \left(\rho^\ell(t, \sigma) - \kappa \right)^\pm \partial_t \varphi^\ell(t, \sigma) \, d\sigma \, dt \right) \, d\ell, \end{aligned}$$

where we used the notations (6.14) and (6.15).

Set temporarily $F(t, x) = \text{sgn}^\pm(\rho(t, x) - \kappa) \left(q(\rho(t, x)) - q(\kappa) \right)$ and $\tilde{F}(t, s, \ell) = F(t, \mathcal{R}(s, \ell))$. Then, the second line in the integral inequality in Definition 3.3, thanks to (6.9), (6.10) and the notations (6.14)–(6.15), becomes

$$\begin{aligned} & \int_0^T \int_{\mathcal{R}} \text{sgn}^\pm(\rho(t, x) - \kappa) \left(q(\rho(t, x)) - q(\kappa) \right) \nabla_x \varphi(t, x) \cdot r(x) \, dx \, dt \\ &= \int_0^T \int_{\mathcal{R}} F(t, x) \nabla_x \varphi(t, x) \cdot r(x) \, dx \, dt \\ &= \int_0^T \int_{-L}^L \int_I F(t, \mathcal{R}(s, \ell)) \nabla_{(s, \ell)} \varphi(t, \mathcal{R}(s, \ell)) [D\mathcal{R}(s, \ell)]^{-1} \cdot r(\mathcal{R}(s, \ell)) \det D\mathcal{R}(s, \ell) \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_I \tilde{F}(t, s, \ell) \nabla_{(s, \ell)} \tilde{\varphi}(t, s, \ell) \left[\begin{array}{c} (\gamma'(s))^\top \\ \left((\gamma'(s))^\perp - \ell \gamma''(s) \right)^\top \end{array} \right] \gamma'(s) \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_I \tilde{F}(t, s, \ell) \nabla_{(s, \ell)} \tilde{\varphi}(t, s, \ell) \left[\begin{array}{c} \gamma'(s) \cdot \gamma'(s) \\ -\ell \gamma'(s) \cdot \gamma''(s) \end{array} \right] \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_I \tilde{F}(t, s, \ell) \nabla_{(s, \ell)} \tilde{\varphi}(t, s, \ell) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_I \tilde{F}(t, s, \ell) \partial_s \tilde{\varphi}(t, s, \ell) \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_I \text{sgn}^\pm \left(\rho(t, \mathcal{R}(s, \ell)) - \kappa \right) \left(q(\rho(t, \mathcal{R}(s, \ell))) - q(\kappa) \right) \partial_s \tilde{\varphi}(t, s, \ell) \, ds \, d\ell \, dt \\ &= \int_0^T \int_{-L}^L \int_I \text{sgn}^\pm \left(\tilde{\rho}(t, s, \ell) - \kappa \right) \left(q(\tilde{\rho}(t, s, \ell)) - q(\kappa) \right) \partial_s \tilde{\varphi}(t, s, \ell) \, ds \, d\ell \, dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^T \int_{-L}^L \int_{I^\ell} \operatorname{sgn}^\pm \left(\rho^\ell(t, \sigma) - \kappa \right) \left(q \left(\rho^\ell(t, \sigma) \right) - q(\kappa) \right) \partial_\sigma \varphi^\ell(t, \sigma) \, d\sigma \, d\ell \, dt \\
&= \int_{-L}^L \left(\int_0^T \int_{I^\ell} \operatorname{sgn}^\pm \left(\rho^\ell(t, \sigma) - \kappa \right) \left(q \left(\rho^\ell(t, \sigma) \right) - q(\kappa) \right) \partial_\sigma \varphi^\ell(t, \sigma) \, d\sigma \, dt \right) d\ell .
\end{aligned}$$

Now, the first summand in the third line in the integral inequality in Definition 3.3 is

$$\begin{aligned}
&\int_{\mathcal{R}} (\rho_o(x) - \kappa)^\pm \varphi(0, x) \, dx \\
&= \int_{-L}^L \int_I (\rho_o(\mathcal{R}(s, \ell)) - \kappa)^\pm \varphi(0, \mathcal{R}(s, \ell)) \, |\det D\mathcal{R}(s, \ell)| \, ds \, d\ell \\
&= \int_{-L}^L \int_{I^\ell} (\rho_o^\ell(\sigma) - \kappa)^\pm \varphi^\ell(0, \sigma) \, d\sigma \, d\ell \\
&= \int_{-L}^L \left(\int_{I^\ell} (\rho_o^\ell(\sigma) - \kappa)^\pm \varphi^\ell(0, \sigma) \, d\sigma \right) d\ell ,
\end{aligned}$$

where we used (3.6). The second summand in the same line is entirely analogous. The fourth line in the integral inequality in Definition 3.3, by (3.6) and (6.14), is

$$\begin{aligned}
&\int_0^T \int_{\mathcal{E}_{in}} \left(q_{|[0, \bar{\rho}]}^{-1} (f_{in}(t, \xi)) - \kappa \right)^\pm \varphi(t, \xi) \, d\xi \, dt \\
&= \int_0^T \int_{-L}^L \left(q_{|[0, \bar{\rho}]}^{-1} (f_{in}(t, \mathcal{R}(s_{in}, \ell))) - \kappa \right)^\pm \varphi(t, \mathcal{R}(s_{in}, \ell)) \, d\ell \, dt \\
&= \int_0^T \int_{-L}^L \left(q_{|[0, \bar{\rho}]}^{-1} (f_{in}^\ell(t)) - \kappa \right)^\pm \varphi^\ell(t, s_{in}^\ell) \, d\ell \, dt \\
&= \int_{-L}^L \left(\int_0^T \left(q_{|[0, \bar{\rho}]}^{-1} (f_{in}^\ell(t)) - \kappa \right)^\pm \varphi^\ell(t, s_{in}^\ell) \, dt \right) d\ell .
\end{aligned}$$

Finally, the fifth line is entirely analogous.

By the above computations, for any $\eta \in \mathbf{C}_c^1(]-L, L[; \mathbb{R}_+)$, for any $\ell \in]-L, L[$ and any $\widehat{\varphi} \in \mathbf{C}_c^1(\mathbb{R} \times I^\ell; \mathbb{R}_+)$, choosing a function φ so that $\varphi^\ell(t, \sigma) = \eta(\ell) \widehat{\varphi}(t, \sigma)$, we have

$$\begin{aligned}
&\int_{-L}^L \left[\int_0^T \int_{I^\ell} (\rho^\ell(t, \sigma) - \kappa)^\pm \partial_t \widehat{\varphi}(t, \sigma) \, d\sigma \, dt \right. \\
&+ \int_0^T \int_{I^\ell} \operatorname{sgn}^\pm (\rho^\ell(t, \sigma) - \kappa) \left(q(\rho^\ell(t, \sigma)) - q(\kappa) \right) \partial_\sigma \widehat{\varphi}(t, \sigma) \, d\sigma \, dt \\
&+ \int_{I^\ell} (\rho_o^\ell(\sigma) - \kappa)^\pm \widehat{\varphi}(0, \sigma) \, d\sigma + \int_{I^\ell} (\rho^\ell(T, \sigma) - \kappa)^\pm \widehat{\varphi}(T, \sigma) \, d\sigma \\
&\quad + \int_0^T \left(q_{|[0, \bar{\rho}]}^{-1} (f_{in}^\ell(t)) - \kappa \right)^\pm \widehat{\varphi}(t, s_{in}) \, dt \\
&\quad \left. + \int_0^T \left(q_{|[\bar{\rho}, R]}^{-1} (f_{out}^\ell(t)) - \kappa \right)^\pm \widehat{\varphi}(t, s_{out}) \, dt \right] \eta(\ell) \, d\ell \geq 0 .
\end{aligned}$$

An application of Lemma 6.2, thanks to the arbitrariness of η , shows that ρ^ℓ solves (3.4), for a.e. $\ell \in]-L, L[$.

Conversely, assume ρ^ℓ solves (3.4), for any $\ell \in]-L, L[$. Then, any $\varphi \in \mathbf{C}_c^1(\mathbb{R} \times \mathcal{R}; \mathbb{R}_+)$ yields a function φ^ℓ as defined in (6.14) which is of class \mathbf{C}^1 in all three variables (t, ℓ, σ) and compactly supported. Call φ^ℓ the map defined in (6.14) with s_{out}^ℓ as in (3.5). Then, a direct consequence of Lemma 6.3 ensures that for any $\varepsilon > 0$ there exist functions $F_1, \dots, F_n \in \mathbf{C}_c^1(]-L, L[; \mathbb{R}_+)$, $G_1^\ell, \dots, G_n^\ell \in \mathbf{C}^1(\overline{I}^\ell; \mathbb{R}_+)$ and $c_{ij} \in \mathbf{C}_c^1(]-\infty, T[; \mathbb{R}_+)$ such that for all $t \in]-\infty, T[$, $\ell \in]-L, L[$ and $\sigma \in \overline{I}^\ell$

$$\left| \varphi^\ell(t, \sigma) - \sum_{i=1}^n \sum_{j=1}^n c_{ij}(t) F_i(\ell) G_j^\ell(\sigma) \right| < \varepsilon.$$

By the above computations, the left hand side of the inequality in Definition 3.3 satisfies

$$\begin{aligned} & \int_0^T \int_{\mathcal{R}} (\rho(t, x) - \kappa)^\pm \partial_t \varphi(t, x) \, dx \, dt \\ & + \int_0^T \int_{\mathcal{R}} \operatorname{sgn}^\pm (\rho(t, x) - \kappa) \left(q(\rho(t, x)) - q(\kappa) \right) \nabla_x \varphi(t, x) \cdot r(x) \, dx \, dt \\ & + \int_{\mathcal{R}} (\rho_o(x) - \kappa)^\pm \varphi(0, x) \, dx + \int_{\mathcal{R}} (\rho(T, x) - \kappa)^\pm \varphi(T, x) \, dx \\ & + \|q'\|_{\mathbf{L}^\infty([0, R]; \mathbb{R})} \int_0^T \int_{\mathcal{E}_{in}} \left(q_{|[0, \bar{\rho}]}^{-1}(f_{in}(t, \xi)) - \kappa \right)^\pm \varphi(t, \xi) \, d\xi \, dt \\ & + \|q'\|_{\mathbf{L}^\infty([0, R]; \mathbb{R})} \int_0^T \int_{\mathcal{E}_{out}} \left(q_{|[\bar{\rho}, R]}^{-1}(f_{out}(t, \xi)) - \kappa \right)^\pm \varphi(t, \xi) \, d\xi \, dt \geq \\ & -C\varepsilon + \sum_{i,j=1}^n \int_{-L}^L \left[\int_0^T \int_{I^\ell} (\rho^\ell(t, \sigma) - \kappa)^\pm \partial_t c_{ij}(t) G_j^\ell(\sigma) \, d\sigma \, dt \right. \\ & + \int_0^T \int_{I^\ell} \operatorname{sgn}^\pm (\rho^\ell(t, \sigma) - \kappa) \left(q(\rho^\ell(t, \sigma)) - q(\kappa) \right) c_{ij}(t) \partial_\sigma G_j^\ell(\sigma) \, d\sigma \, dt \\ & + \int_{I^\ell} (\rho_o^\ell(\sigma) - \kappa)^\pm c_{ij}(0) G_j^\ell(\sigma) \, d\sigma + \int_{I^\ell} (\rho^\ell(T, \sigma) - \kappa)^\pm c_{ij}(T) G_j^\ell(\sigma) \, d\sigma \\ & + \int_0^T \left(q_{|[0, \bar{\rho}]}^{-1}(f_{in}^\ell(t)) - \kappa \right)^\pm c_{ij}(t) G_j^\ell(s_{in}) \, dt \\ & \left. + \int_0^T \left(q_{|[\bar{\rho}, R]}^{-1}(f_{out}^\ell(t)) - \kappa \right)^\pm c_{ij}(t) G_j^\ell(s_{out}^\ell) \, dt \right] F_i(\ell) \, d\ell \geq -C\varepsilon, \end{aligned}$$

for a constant C dependent only on $R, \bar{\rho}, s_{in}, s_{out}, L$ and norms of q . To get to the latter inequality above, use the fact that, for any ℓ , ρ^ℓ solves (3.4) in the sense of Definition 2.1. \square

Proof of Theorem 3.5. The existence and uniqueness of a solution to (3.3) is a consequence of Lemma 3.4 and of Proposition 2.2. The \mathbf{L}^∞ estimate directly follows from (2.3).

To prove the \mathbf{L}^1 Lipschitz estimate, compute:

$$\|\rho(t) - \hat{\rho}(t)\|_{\mathbf{L}^1(\mathcal{R}; \mathbb{R})}$$

$$\begin{aligned}
&= \int_{\mathcal{R}} |\rho(t, x) - \widehat{\rho}(t, x)| \, dx \\
&= \int_{-L}^L \int_I |\rho(t, \mathcal{R}(s, \ell)) - \widehat{\rho}(t, \mathcal{R}(s, \ell))| \left(1 - \ell(\gamma'(s))^\perp \cdot \gamma''(s)\right) \, ds \, d\ell \quad [\text{By (6.5)}] \\
&= \int_{-L}^L \int_{I^\ell} |\rho^\ell(t, \sigma) - \widehat{\rho}^\ell(t, \sigma)| \, d\sigma \, d\ell \quad [\text{By (3.6)}] \\
&= \int_{-L}^L \left\| \rho^\ell(t) - \widehat{\rho}^\ell(t) \right\|_{\mathbf{L}^1(I^\ell; \mathbb{R})} \, d\ell \\
&\leq \int_{-L}^L \left(\left\| \rho_o^\ell - \widehat{\rho}_o^\ell \right\|_{\mathbf{L}^1(I^\ell; \mathbb{R})} + \left\| f_{in}^\ell - \widehat{f}_{in}^\ell \right\|_{\mathbf{L}^1([0, t]; \mathbb{R})} + \left\| f_{out}^\ell - \widehat{f}_{out}^\ell \right\|_{\mathbf{L}^1([0, t]; \mathbb{R})} \right) \, d\ell \quad [\text{By (2.4)}] \\
&= \left\| \rho_o - \widehat{\rho}_o \right\|_{\mathbf{L}^1(\mathcal{R}; \mathbb{R})} + \left\| f_{in} - \widehat{f}_{in} \right\|_{\mathbf{L}^1([0, t] \times \mathcal{E}_{in}; \mathbb{R})} + \left\| f_{out} - \widehat{f}_{out} \right\|_{\mathbf{L}^1([0, t] \times \mathcal{E}_{out}; \mathbb{R})} \quad [\text{By (3.6) and (6.5)}]
\end{aligned}$$

where the latter equality follows from the changes of coordinates (3.5) and $x = \mathcal{R}(s, \ell)$.

The proof of the bound (3.7) follows immediately from Proposition 2.2. \square

6.3 Proofs Related to the Parabolic Pollution Model

Definition 6.4. A strong solution to (4.1) is a map $u \in \mathbf{C}^0([0, T] \times \mathbb{R}^2; \mathbb{R})$ such that

$$\forall t \in]0, T[\quad x \mapsto u(t, x) \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R}) \quad \text{and} \quad \forall x \in \mathbb{R}^2 \quad t \mapsto u(t, x) \in \mathbf{C}^1([0, T]; \mathbb{R})$$

which satisfies the equation in (4.1) at every $(t, x) \in]0, T[\times \mathbb{R}^2$ and the initial condition in (4.1) at every $x \in \mathbb{R}^2$.

Recall the following definition.

Definition 6.5. Let $k \in \mathbb{N} \setminus \{0\}$. A function $f: [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^k$ is Hölder continuous if there exist $M > 0$ and $\alpha \in]0, 1]$ such that $\|f(t_1, x_1) - f(t_2, x_2)\| \leq M \left(|t_1 - t_2|^{\alpha/2} + \|x_1 - x_2\|^\alpha \right)$ for all $(t_1, x_1), (t_2, x_2) \in [0, T] \times \mathbb{R}^2$.

A function $f: [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^k$ is Hölder continuous in x uniformly in t if there exist $M > 0$ and $\alpha \in]0, 1]$ such that $\|f(t, x_1) - f(t, x_2)\| \leq M \|x_1 - x_2\|^\alpha$ for all $t \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^2$.

Lemma 6.6. [12, Chapter 1, Section 6, Theorem 10 and Section 7, Theorem 12] Let **(HP)** hold. Then, there exists a function

$$\Gamma \in \mathbf{C}^0 \left(\left\{ (t, x, \tau, \xi) \in [0, T] \times \mathbb{R}^2 \times [0, T] \times \mathbb{R}^2 : t > \tau \right\}; \mathbb{R}_+ \right) \quad (6.16)$$

such that

(Γ.1) For all $(\tau, \xi) \in [0, T] \times \mathbb{R}^2$, $(t, x) \mapsto \Gamma(t, x, \tau, \xi)$ is a strong solution, in the sense of Definition 6.4, to the parabolic equation in (4.1) with $g = 0$.

(Γ.2) For all $x \in \mathbb{R}^2$ and for all $u_o \in \mathbf{C}^0(\mathbb{R}^2; \mathbb{R})$, $\lim_{t \rightarrow \tau+} \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) u_o(\xi) \, d\xi = u_o(x)$.

(**Γ.3**) *There exists a constant $C > 0$ such that for all choices of the variables (t, x, τ, ξ) in $\{(t, x, \tau, \xi) \in [0, T] \times \mathbb{R}^2 \times [0, T] \times \mathbb{R}^2 : t > \tau\}$*

$$\Gamma(t, x, \tau, \xi) < \frac{C}{(t - \tau)} \exp \left(-\frac{\|x - \xi\|^2}{4\hat{\mu}(t - \tau)} \right), \quad (6.17)$$

$$|\partial_{x_i} \Gamma(t, x, \tau, \xi)| < \frac{C}{(t - \tau)^{3/2}} \exp \left(-\frac{\|x - \xi\|^2}{4\hat{\mu}(t - \tau)} \right) \quad i = 1, 2, \quad (6.18)$$

$$|\partial_{x_i x_j}^2 \Gamma(t, x, \tau, \xi)| < \frac{C}{(t - \tau)^2} \exp \left(-\frac{\|x - \xi\|^2}{4\hat{\mu}(t - \tau)} \right) \quad i, j = 1, 2. \quad (6.19)$$

Moreover, if

(**R.1**) $g \in \mathbf{C}^0([0, T] \times \mathbb{R}^2; \mathbb{R})$; for every bounded set $B \subset \mathbb{R}^2$ there exists $M_B > 0$ and $\alpha_B \in]0, 1]$ such that $|g(t, x_1) - g(t, x_2)| \leq M_B \|x_1 - x_2\|^{\alpha_B}$ for all $t \in [0, T]$ and $x_1, x_2 \in B$ and, for suitable $C > 0$ and $h \in]0, 1/(4T\hat{\mu})]$, satisfies $|g(t, x)| \leq C e^{h\|x\|^2}$ for all $(t, x) \in [0, T] \times \mathbb{R}^2$;

(**R.2**) $u_o \in \mathbf{C}^0(\mathbb{R}^2; \mathbb{R})$ and, for suitable $C > 0$ and $h \in]0, 1/(4T\hat{\mu})]$, satisfies $|u_o(x)| \leq C e^{h\|x\|^2}$ for all $x \in \mathbb{R}^2$;

then, the Cauchy problem (4.1) admits the strong solution

$$u(t, x) = \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) u_o(\xi) d\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) g(\tau, \xi) d\xi d\tau. \quad (6.20)$$

Proof of Lemma 6.6. Observe that (4.1) takes the form $\partial_t u - \sum_{ij} a_{ij} \partial_{ij}^2 u - b \cdot \nabla u - c u = -f$ as soon as $a_{ii} = \mu$, $a_{ij} = 0$, $b = \nabla \mu - \beta$, $c = -\kappa + \nabla \cdot \beta$, $f = -g$.

The existence of Γ , together with (**Γ.1**) and (**Γ.2**), follow from [12, Chapter 1, Section 6, Theorem 10]. The continuity of Γ follows from [12, Chapter 1, Section 6], where the results for bounded domains in [12, Section 4] are extended to unbounded domains. The positivity of Γ follows from [12, Chapter 2, Section 4, Theorem 11]. The estimates (6.17) and (6.18) on Γ are explained after [12, Chapter 1, Section 6, Theorem 11]. The estimate (6.19) is obtained in a similar manner, exploiting [12, Chapter 1, (4.11)], with, in his notation, $\mu = (n + 2)/2$.

The existence of the solution to (4.1) and its expression follow from [12, Chapter 1, Section 7, Theorem 12]. \square

Lemma 6.7 ([12, Chapter 1, Section 9, Theorem 16]). *Let (**HP**), (**R.1**) and (**R.2**) hold. Assume moreover that $D_x^2 \mu$ is in $(\mathbf{C}^0 \cap \mathbf{L}^\infty)([0, T] \times \mathbb{R}^2)$ and Hölder continuous in x uniformly in t . Then, (6.20) is the unique strong solution to (4.1).*

Proof of Lemma 6.7. To apply [12, Chapter 1, Section 9, Theorem 16] we need to verify the boundedness condition [12, Chapter 1, Section 9, Formula (9.1)], namely that there exists a $k > 0$ such that $\int_0^T \int_{\mathbb{R}^2} |u(t, x)| e^{-k\|x\|^2} dx dt < +\infty$. Indeed,

$$\int_0^T \int_{\mathbb{R}^2} |u(t, x)| e^{-k\|x\|^2} dx dt \quad [\text{Use (6.20)}]$$

$$\begin{aligned}
&\leq \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) |u_o(\xi)| e^{-k\|x\|^2} d\xi dx dt && [\text{Use (6.17) and (R.2)}] \\
&\quad + \int_0^T \int_{\mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) |g(\tau, \xi)| e^{-k\|x\|^2} d\xi d\tau dx dt && [\text{Use (6.17) and (R.1)}] \\
&\leq \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{C}{t} e^{-\frac{\|x-\xi\|^2}{4\tilde{\mu}t}} e^{h\|\xi\|^2} e^{-k\|x\|^2} d\xi dx dt && (6.21)
\end{aligned}$$

$$\begin{aligned}
&\quad + \int_0^T \int_{\mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} \frac{C}{(t-\tau)} e^{-\frac{\|x-\xi\|^2}{4\tilde{\mu}(t-\tau)}} e^{h\|\xi\|^2} e^{-k\|x\|^2} d\xi d\tau dx dt . && (6.22)
\end{aligned}$$

The latter terms (6.21) and (6.22) can be treated similarly. By (R.2) for an $\varepsilon > 0$ we have $h \leq (1 - \varepsilon)/(4\tilde{\mu}T)$ and set $\xi - x = y$, so that

$$\begin{aligned}
[(6.21)] &\leq \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{t} e^{-\frac{\|y\|^2}{4\tilde{\mu}t}} e^{h\|x+y\|^2} e^{-k\|x\|^2} dy dx dt \\
&\leq \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{t} e^{-\frac{\|y\|^2}{4\tilde{\mu}t} + h\|y\|^2} e^{-k\|x\|^2 + h\|x\|^2} dy dx dt \\
&\leq \int_0^T \int_{\mathbb{R}^2} \frac{1}{t} e^{-\frac{\varepsilon\|y\|^2}{4\tilde{\mu}t}} dy dt \int_{\mathbb{R}^2} e^{-(k-h)\|x\|^2} dx , \\
[(6.22)] &\leq \int_0^T \int_{\mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} \frac{C}{(t-\tau)} e^{-\frac{\|y\|^2}{4\tilde{\mu}(t-\tau)}} e^{h\|x+y\|^2} e^{-k\|x\|^2} dy d\tau dx dt \\
&\leq \int_0^T \int_{\mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} \frac{C}{(t-\tau)} e^{-\frac{\|y\|^2}{4\tilde{\mu}(t-\tau)} + h\|y\|^2} e^{-k\|x\|^2 + h\|x\|^2} dy d\tau dx dt \\
&\leq \int_0^T \int_{\mathbb{R}^2} \int_0^t \frac{C}{(t-\tau)} e^{-\frac{\varepsilon\|y\|^2}{4\tilde{\mu}(t-\tau)}} d\tau dy dt \int_{\mathbb{R}^2} e^{-(k-h)\|x\|^2} dx ,
\end{aligned}$$

and both are finite for $k > h$. \square

Lemma 6.8. *Under the same assumptions of Lemma 6.7, suppose in addition that $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ and $g \in \mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$. Then, every strong solution to (4.1) in the sense of Definition 6.4 is also a weak solution in the sense of Definition 4.1.*

Proof. By Lemma 6.6 and Lemma 6.7, the unique strong solution u to (4.1) can be written as in (6.20). The \mathbf{L}^1 in space regularity and the \mathbf{L}^1 continuity required in Definition 4.1 follow from (6.17), since $u_o \in \mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ and $g \in \mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$.

We now have to show that u written as in (6.20) satisfies the integral equality (4.3). First, observe that the requirements in (4.2) imply that φ , $\partial_t \varphi$, $\nabla \varphi$, $\nabla^2 \varphi$, $\varphi(0, \cdot)$ and $\varphi(T, \cdot)$ are all in \mathbf{L}^∞ on \mathbb{R}^2 . Hence the integrals appearing in (4.3) are all finite.

A standard procedure allows to obtain (4.3) by subsequent integrations by parts, exploiting the hypotheses (4.2) on the test function φ and (F.1) in Lemma 6.6. \square

Proof of Theorem 4.2. We deal first with existence and, separately, with uniqueness. The other claims are proved subsequently.

Existence: For $n \in \mathbb{N} \setminus \{0\}$, call η_n a mollifier in $\mathbf{C}_c^\infty(\mathbb{R}^3; \mathbb{R}_+)$ with the properties $\text{spt } \eta_n \subseteq B_{\mathbb{R}^3}(0, 1/n)$, $\int_{\mathbb{R}^3} \eta_n(t, x) dt dx = 1$ so that (extending g to \mathbb{R}^3 with value 0 outside $[0, T] \times \mathbb{R}^2$)

$$(g * \eta_n) \in \mathbf{C}^1([0, T] \times \mathbb{R}^2; \mathbb{R}) \quad \text{and} \quad \lim_{n \rightarrow +\infty} g * \eta_n = g \text{ in } \mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R}). \quad (6.23)$$

Similarly, for $n \in \mathbb{N} \setminus \{0\}$, call ϑ_n a mollifier in $\mathbf{C}_c^\infty(\mathbb{R}^2; \mathbb{R}_+)$ with the properties $\text{spt } \vartheta_n \subseteq B_{\mathbb{R}^2}(0, 1/n)$, $\int_{\mathbb{R}^2} \vartheta_n(t, x) dt dx = 1$ so that

$$(u_o * \vartheta_n) \in \mathbf{C}^1(\mathbb{R}^2; \mathbb{R}) \quad \text{and} \quad \lim_{n \rightarrow +\infty} u_o * \vartheta_n = u_o \text{ in } \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}). \quad (6.24)$$

Note that $g * \eta_n$ satisfies **(R.1)** and $u_o * \vartheta_n$ satisfies **(R.2)**. Hence, the problem

$$\begin{cases} \partial_t u + \nabla \cdot (u \beta(t, x) - \mu(t, x) \nabla u) + \kappa(t, x) u = (g * \eta_n)(t, x) & (t, x) \in]0, T] \times \mathbb{R}^2 \\ u(0, x) = (u_o * \vartheta_n)(x) & x \in \mathbb{R}^2. \end{cases} \quad (6.25)$$

admits a strong solution u_n which belongs to $\mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$, due to the construction of $g * \eta_n$ and $u_o * \vartheta_n$. The function u_n can be written as in (6.20):

$$u_n(t, x) = \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) (u_o * \vartheta_n)(\xi) d\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) (g * \eta_n)(\tau, \xi) d\xi d\tau. \quad (6.26)$$

The sequence $t \mapsto u_n(t)$ is a Cauchy sequence for the uniform convergence in $\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$, indeed using (6.17)

$$\begin{aligned} & \|u_n(t) - u_m(t)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \\ & \leq \frac{C}{t} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \exp\left(-\frac{\|x - \xi\|^2}{4\hat{\mu}t}\right) ((u_o * \vartheta_n)(\xi) - (u_o * \vartheta_m)(\xi)) d\xi dx \\ & + \int_0^t \frac{C}{t - \tau} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \exp\left(-\frac{\|x - \xi\|^2}{4\hat{\mu}(t - \tau)}\right) ((g * \eta_n)(\tau, \xi) - (g * \eta_m)(\tau, \xi)) d\xi dx d\tau \\ & \leq \frac{C}{t} \int_{\mathbb{R}^2} \exp\left(-\frac{\|\xi\|^2}{4\hat{\mu}t}\right) d\xi \int_{\mathbb{R}^2} ((u_o * \vartheta_n)(\xi) - (u_o * \vartheta_m)(\xi)) d\xi \\ & + \int_0^t \int_{\mathbb{R}^2} \frac{C}{t - \tau} \exp\left(-\frac{\|\xi\|^2}{4\hat{\mu}(t - \tau)}\right) d\xi \int_{\mathbb{R}^2} ((g * \eta_n)(\tau, \xi) - (g * \eta_m)(\tau, \xi)) d\xi d\tau \\ & \leq 4\pi C \hat{\mu} \left(\|u_o * \vartheta_n - u_o * \vartheta_m\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} + \|g * \eta_n - g * \eta_m\|_{\mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})} \right). \end{aligned} \quad (6.28)$$

Recall that $(u_o * \vartheta_n)$ is a Cauchy sequence in $\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})$ by (6.24) and $(g * \eta_n)$ is a Cauchy sequence in $\mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$ by (6.23).

Call $u \in \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^2; \mathbb{R}))$ the \mathbf{L}^1 uniform limit of the u_n . Clearly, $u \in \mathbf{L}^1([0, T] \times \mathbb{R}^2; \mathbb{R})$. To prove that u is a weak solution to (4.1) in the sense of Definition 4.1, since u_n is a strong – and hence also weak – solution to (4.1), compute

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} u (\partial_t \varphi + \nabla \cdot (\mu \nabla \varphi) + \beta \cdot \nabla \varphi - \kappa \varphi) dx dt \\ & + \int_{\mathbb{R}^2} u_o(x) \varphi(0, x) dx - \int_{\mathbb{R}^2} u(T, x) \varphi(T, x) dx + \int_0^T \int_{\mathbb{R}^2} g \varphi dx dt \\ & = \int_0^T \int_{\mathbb{R}^2} (u - u_n) (\partial_t \varphi + \nabla \cdot (\mu \nabla \varphi) + \beta \cdot \nabla \varphi - \kappa \varphi) dx dt \\ & + \int_{\mathbb{R}^2} (u_o - u_o * \vartheta_n)(x) \varphi(0, x) dx - \int_{\mathbb{R}^2} (u - u_n)(T, x) \varphi(T, x) dx \end{aligned}$$

$$\begin{aligned}
& + \int_0^T \int_{\mathbb{R}^2} (g - g * \eta_n) \varphi \, dx \, dt \\
& \rightarrow 0 \quad \text{as } n \rightarrow +\infty
\end{aligned}$$

by the Dominated Convergence Theorem.

Uniqueness: This proof follows the lines of [7, Lemma 5.1]. Let u_1, u_2 be two weak solutions to (4.1) in the sense of Definition 4.1. Then, by a standard approximation argument, the difference $u = u_2 - u_1$ satisfies for all φ as in (4.2) and for all $\tau \in]0, T[$

$$\int_0^\tau \int_{\mathbb{R}^2} u \left(\partial_t \varphi + \nabla \cdot (\mu \nabla \varphi) + \beta \cdot \nabla \varphi - \kappa \varphi \right) dx \, dt - \int_{\mathbb{R}^2} u(\tau, x) \varphi(\tau, x) \, dx = 0. \quad (6.29)$$

For an arbitrary $\omega \in \mathbf{C}_c^0(\mathbb{R}; \mathbb{R})$, consider the (backward) solution to the parabolic problem

$$\begin{cases} \partial_t \varphi + \nabla \cdot (\mu \nabla \varphi + \beta \varphi) - (\kappa + \nabla \cdot \beta) \varphi = 0 & (t, x) \in]0, \tau[\times \mathbb{R}^2 \\ \varphi(\tau, x) = \omega(x) & x \in \mathbb{R}^2. \end{cases} \quad (6.30)$$

This equation fits into (4.1) (with time reversed) and Lemma 6.6 applies. Therefore, (6.30) admits a strong solution which can be written as in (6.20).

We now show that φ satisfies (4.2). Indeed, the $\mathbf{C}^1 - \mathbf{C}^2$ regularity follows from the definition of strong solution. To verify the conditions at infinity, use (6.20) with $g = 0$, bearing in mind that time is reversed, starting from τ and going backward to 0. Assume that $\text{spt } \omega \subseteq B(0, \bar{r})$ for a suitable $\bar{r} > 0$. Compute for $r \gg \bar{r}$, using (6.17):

$$\begin{aligned}
\sup_{\|x\| \geq r} |\varphi(t, x)| &= \sup_{\|x\| \geq r} \left| \int_{\mathbb{R}^2} \Gamma(\tau - t, x, 0, \xi) \omega(\xi) \, d\xi \right| \\
&\leq \sup_{\|x\| \geq r} \int_{\text{spt } \omega} \frac{C}{(\tau - t)} \exp \left(-\frac{\|x - \xi\|^2}{4 \hat{\mu}(\tau - t)} \right) |\omega(\xi)| \, d\xi \\
&\leq \sup_{\|x\| \geq r} \int_{B(0, \bar{r})} \frac{C}{(\tau - t)} \exp \left(-\frac{\|x - \xi\|^2}{4 \hat{\mu}(\tau - t)} \right) d\xi \, \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \\
&\leq \frac{C}{(\tau - t)} \int_{B(0, \bar{r})} \exp \left(-\frac{(r - \bar{r})^2}{4 \hat{\mu}(\tau - t)} \right) d\xi \, \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})} \\
&\leq \frac{C}{(\tau - t)} \mathcal{L}(B(0, \bar{r})) \exp \left(-\frac{(r - \bar{r})^2}{4 \hat{\mu}(\tau - t)} \right) \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})},
\end{aligned}$$

\mathcal{L} being the Lebesgue measure on \mathbb{R}^2 . The latter quantity vanishes as $r \rightarrow +\infty$. We need to repeat similar computations to bound $\|\nabla \varphi\|$, using as above (6.20) with $g = 0$ and (6.18):

$$\begin{aligned}
\sup_{\|x\| \geq r} \|\nabla \varphi(t, x)\| &= \sup_{\|x\| \geq r} \left\| \int_{\mathbb{R}^2} \nabla \Gamma(\tau - t, x, 0, \xi) \omega(\xi) \, d\xi \right\| \\
&\leq \sup_{\|x\| \geq r} \int_{\text{spt } \omega} \frac{C \sqrt{n}}{(\tau - t)^{3/2}} \exp \left(-\frac{\|x - \xi\|^2}{4 \hat{\mu}(\tau - t)} \right) |\omega(\xi)| \, d\xi
\end{aligned}$$

$$\leq \frac{C \sqrt{n}}{(\tau - t)^{3/2}} \mathcal{L}(B(0, \bar{r})) \exp\left(-\frac{(r - \bar{r})^2}{4 \hat{\mu}(\tau - t)}\right) \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})}.$$

Similarly, to bound $\|\nabla^2 \varphi\|$, we use as above (6.20) with $g = 0$ and (6.19):

$$\begin{aligned} \sup_{\|x\| \geq r} \|\nabla^2 \varphi(t, x)\| &= \sup_{\|x\| \geq r} \left\| \int_{\mathbb{R}^2} \nabla^2 \Gamma(\tau - t, x, 0, \xi) \omega(\xi) d\xi \right\| \\ &\leq \sup_{\|x\| \geq r} \int_{\text{spt } \omega} \frac{C \sqrt{n}}{(\tau - t)^2} \exp\left(-\frac{\|x - \xi\|^2}{4 \hat{\mu}(\tau - t)}\right) |\omega(\xi)| d\xi \\ &\leq \frac{C \sqrt{n}}{(\tau - t)^2} \mathcal{L}(B(0, \bar{r})) \exp\left(-\frac{(r - \bar{r})^2}{4 \hat{\mu}(\tau - t)}\right) \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2; \mathbb{R})}. \end{aligned}$$

The bound on $\|\partial_t \varphi\|$ follows from the previous ones, since, by (6.30), $\partial_t \varphi = -\nabla \cdot (\mu \nabla \varphi + \beta \varphi) + (\kappa + \nabla \cdot \beta) \varphi$ and **(HP)** holds. We thus obtain from (6.29) that $\int_{\mathbb{R}^2} u(\tau, x) \omega(x) dx = 0$ for any $\omega \in \mathbf{C}_c^0(\mathbb{R}^2; \mathbb{R})$, which implies that $u \equiv 0$, completing the proof of uniqueness.

Representation: Formula (4.4) follows from (6.26) and the \mathbf{L}^1 convergence of the u_n to u .

Stability: Thanks to (4.4), the estimate (4.5) is obtained repeating the same computations as in (6.27)–(6.28), substituting u_n with u .

Monotonicity: It is an immediate consequence of (6.16) and (4.4). \square

Proof of Theorem 4.4. This result extends Theorem 4.2, we detail the key differences.

Existence: Introduce the mollifiers η_n and the solutions

$$u_n(t, x) = \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) u_o(\xi) d\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) (g_\tau * \eta_n)(\xi) d\xi d\tau \quad (6.31)$$

to (4.1) with the map $(t, x) \mapsto (g_t * \eta_n)(x)$ as source term. Since $g_t * \eta_n$ converges to g_t in \mathcal{D}' , we can pass to the limit $n \rightarrow +\infty$ in (6.31) and define the function u as in (4.7), so that for all $(t, x) \in [0, T] \times \mathbb{R}^2$, $\lim_{n \rightarrow +\infty} u_n(t, x) = u(t, x)$. Observe that

$$\begin{aligned} \|u_n(t)\|_{\mathbf{L}^1} &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) |u_o(\xi)| d\xi dx \\ &\quad + \int_{\mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) |(g_\tau * \eta_n)(\xi)| d\xi d\tau dx \\ &\leq \frac{C}{t} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\|x - \xi\|^2 / (4 \hat{\mu} t)} |u_o(\xi)| d\xi dx \\ &\quad + \int_{\mathbb{R}^2} \int_0^t \int_{\mathbb{R}^2} \frac{C}{t - \tau} e^{-\|x - \xi\|^2 / (4 \hat{\mu} (t - \tau))} |(g_\tau * \eta_n)(\xi)| d\xi d\tau dx \\ &\leq \frac{C}{t} \int_{\mathbb{R}^2} e^{-\|z\|^2 / (4 \hat{\mu} t)} \int_{\mathbb{R}^2} |u_o(z + x)| dx dz \\ &\quad + \int_0^t \int_{\mathbb{R}^2} \frac{C}{t - \tau} e^{-\|z\|^2 / (4 \hat{\mu} (t - \tau))} \int_{\mathbb{R}^2} |(g_\tau * \eta_n)(z + x)| dx dz d\tau \end{aligned}$$

$$\begin{aligned}
&\leq C \|u_o\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} + C \int_0^t \int_{\mathbb{R}^2} |(g_\tau * \eta_n)(\xi)| \, d\xi \, d\tau \\
&\leq C \|u_o\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} + C \int_0^t \|g_\tau\|_{\mathcal{M}} \, d\tau \, \|\eta_n\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})},
\end{aligned}$$

where we used [10, Proposition 8.68, p.282]. Thanks to (4.6), apply Fatou Lemma and obtain that $u(t) \in \mathbf{L}^1(\mathbb{R}^2, \mathbb{R})$ for all $t \in [0, T]$.

The continuity in time of $t \mapsto u(t)$ is proved exactly as in Theorem 4.2.

Proving that u satisfies (4.8) and solves (4.1) is done exactly as in Theorem 4.2, as also the uniqueness of u in the class of weak solutions.

Stability and Monotonicity. Using the representation (4.7), we have

$$\begin{aligned}
\|u(t) - \widehat{u}(t)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} &\leq \int_{\mathbb{R}^2} \Gamma(t, x, 0, \xi) |u_o(\xi) - \widehat{u}_o(\xi)| \, d\xi \\
&\quad + \int_0^t \int_{\mathbb{R}^2} \Gamma(t, x, \tau, \xi) |dg_\tau(\xi) - d\widehat{g}_\tau(\xi)| \, d\tau
\end{aligned}$$

and (4.8) follows by [10, Proposition 8.68, p.282]. Monotonicity is immediate. \square

6.4 Proofs Related to the Coupled Problem

Proof of Proposition 5.2. Each g_t^i in (5.2) is a finite measure by **(2D.2)** in Theorem 3.7, by Lemma 3.2 and by Definition 3.1. Clearly, the distribution defined by (5.2) has order 0, hence it is a Radon measure, see [11, § 1.3, Remark on p.10].

The definition (5.2) also implies that g_t^i is supported in \mathcal{R}^i . Moreover,

$$\begin{aligned}
\sup_{t \in [0, T]} \|g_t^i\|_{\mathcal{M}} &= \sup \left\{ |g_t^i(\varphi)| : \varphi \in \mathbf{C}_c^\infty([0, T] \times \mathbb{R}^2; \mathbb{R}), |\varphi| \leq 1 \right\} \quad [\text{By [10, Ex. 1.4, Chap. 3]}] \\
&\leq \int_{-L}^L \int_I |1 - \ell(\gamma'(s))^\perp \cdot \gamma''(s)| \, d|\mu_t^{i, \ell}(s)| \, d\ell \quad [\text{By (5.2)}] \\
&\leq 4L \, \text{TV} \left(q \left(\rho^i(t, \mathcal{R}^i(\cdot, \ell)) \right) \right) \quad [\text{By Definition 3.1}] \\
&< +\infty \quad [\text{By (2.2) and (3.12)}]
\end{aligned}$$

completing the proof. \square

Proof of Theorem 5.4. We first fix the inflows $f_{in}^1 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^1; [0, q(\bar{\rho})])$, $f_{in}^2 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}^2; [0, q(\bar{\rho})])$, and outflow $f_{out}^3 \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}^3; [0, q(\bar{\rho})])$ and choose w, \widehat{w} in \mathcal{W} . With the notation in (5.1), call (ρ, u) , respectively $(\widehat{\rho}, \widehat{u})$, the solution to the coupled problem (3.8)–(4.1)–(5.1) regulated by w , respectively \widehat{w} , and $f_{in}^1, f_{in}^2, f_{out}^3$. Then,

$$\begin{aligned}
&\left| \mathcal{F}(\widehat{w}, f_{in}^1, f_{in}^2, f_{out}^3) - \mathcal{F}(w, f_{in}^1, f_{in}^2, f_{out}^3) \right| \\
&\leq \|p\|_{\mathbf{L}^\infty([0, T] \times \mathbb{R}^2; \mathbb{R})} \int_0^T \|\widehat{u}(t) - u(t)\|_{\mathbf{L}^1(\mathbb{R}^2; \mathbb{R})} \, dt
\end{aligned}$$

$$\begin{aligned}
&\leq \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \int_0^T 4\pi C \hat{\mu} \|\hat{g} - g\|_{\mathbf{L}^1(0,t]\times\mathbb{R}^2;\mathbb{R})} dt && [\text{By (4.5)}] \\
&= 4\pi C \hat{\mu} \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \\
&\quad \times \int_0^T \int_0^t \int_{\mathbb{R}^2} \left| G\left(\tau, \hat{\rho}(\tau, x), q\left(\hat{\rho}(\tau, x)\right)\right) - G\left(\tau, \rho(\tau, x), q\left(\rho(\tau, x)\right)\right) \right| dx d\tau dt && [\text{By (5.1)}] \\
&\leq 4\pi C \hat{\mu} \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \mathbf{Lip}(G) (1 + \mathbf{Lip}(q)) \\
&\quad \times \int_0^T \int_0^t \sum_{i=1}^3 \left\| \hat{\rho}^i(\tau) - \rho^i(\tau) \right\|_{\mathbf{L}^1(\mathcal{R}^i;\mathbb{R})} d\tau dt && [\text{By (G), (q)}] \\
&\leq 24\pi C \hat{\mu} L \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \mathbf{Lip}(G) (1 + \mathbf{Lip}(q)) \\
&\quad \times q(\bar{\rho}) \int_0^T \int_0^t \|\hat{w} - w\|_{\mathbf{L}^1([0,\tau];\mathbb{R})} d\tau dt && [\text{By (3.13)}] \\
&\leq \pi C \hat{\mu} L \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \mathbf{Lip}(G) (1 + \mathbf{Lip}(q)) q(\bar{\rho}) T^2 \|\hat{w} - w\|_{\mathbf{L}^1([0,T];\mathbb{R})},
\end{aligned}$$

which proves that the map \mathcal{F} , as in (5.4), is \mathbf{L}^1 -Lipschitz continuous with respect to $w \in \mathcal{W}$.

Let now $w \in \mathcal{W}$ be fixed. Choose the inflows and outflows $f_{in}^1, f_{in}^2, f_{out}^3, \hat{f}_{in}^1, \hat{f}_{in}^2, \hat{f}_{out}^3$. With the notation in (5.1), call (ρ, u) , respectively $(\hat{\rho}, \hat{u})$, the solution to the coupled problem (3.8)–(4.1)–(5.1) regulated by w and $f_{in}^1, f_{in}^2, f_{out}^3$, respectively $\hat{f}_{in}^1, \hat{f}_{in}^2, \hat{f}_{out}^3$. Then, proceeding exactly as above,

$$\begin{aligned}
&\left| \mathcal{F}\left(w, \hat{f}_{in}^1, \hat{f}_{in}^2, \hat{f}_{out}^3\right) - \mathcal{F}\left(w, f_{in}^1, f_{in}^2, f_{out}^3\right) \right| \\
&\leq 4\pi C \hat{\mu} \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \mathbf{Lip}(G) (1 + \mathbf{Lip}(q)) \\
&\quad \times \int_0^T \int_0^t \sum_{i=1}^3 \left\| \hat{\rho}^i(\tau) - \rho^i(\tau) \right\|_{\mathbf{L}^1(\mathcal{R}^i;\mathbb{R})} d\tau dt \\
&\leq 4\pi C \hat{\mu} \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \mathbf{Lip}(G) (1 + \mathbf{Lip}(q)) \\
&\quad \times \int_0^T \int_0^t \left(\sum_{i=1}^2 \left\| \hat{f}_{in}^i - f_{in}^i \right\|_{\mathbf{L}^1([0,\tau]\times\mathcal{E}_{in}^i;\mathbb{R})} + \left\| \hat{f}_{out}^3 - f_{out}^3 \right\|_{\mathbf{L}^1([0,\tau]\times\mathcal{E}_{out}^3;\mathbb{R})} \right) d\tau dt && [\text{By (2D.1)}] \\
&\leq \pi C \hat{\mu} \|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \mathbf{Lip}(G) (1 + \mathbf{Lip}(q)) T^2 \\
&\quad \times \left(\sum_{i=1}^2 \left\| \hat{f}_{in}^i - f_{in}^i \right\|_{\mathbf{L}^1([0,T]\times\mathcal{E}_{in}^i;\mathbb{R})} + \left\| \hat{f}_{out}^3 - f_{out}^3 \right\|_{\mathbf{L}^1([0,T]\times\mathcal{E}_{out}^3;\mathbb{R})} \right),
\end{aligned}$$

which proves that the map \mathcal{F} , as in (5.4), is \mathbf{L}^1 -Lipschitz continuous with respect to the inflows and outflow.

Combining together the above results concludes the proof. \square

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