A singular perturbation result for infinite-dimensional coupled systems

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Abstract

This paper deals with an infinite dimensional version of the singular perturbation method. It is, more precisely, an exponential stability result for a system composed by a fast and a slow dynamics that may involve infinite-dimensional dynamics where, roughly speaking, the system can be approximated into two decoupled systems for which the stability conditions are easier to check. The classical singular perturbation result states that, as soon as the fast system is sufficiently fast, then the exponential stability conditions for the full-system can be reduced to the exponential stability conditions for the two decoupled systems. In this article, it is proved that the classical singular perturbation method can be applied when facing with a slow infinite-dimensional system coupled with a fast finite-dimensional system. The proof relies on a Lyapunov approach, more precisely with the construction a functional inspired by the forwarding method. The well-posedness proof is obtained thanks to the regular system theory. In addition to these result, a Tikhnov theorem is proved. Namely, supposing that the initial condition depends on ε, the parameter modeling the difference between the two time-scales, the trajectory of the full-system is proved to depend linearly on ε.

Keywords: infinite-dimensional systems, singular perturbation, Input-to-State Stability, Regular systems

1. Introduction

This paper states a singular perturbation result for a class of infinite-dimensional involving a fast infinite-dimensional system and a slow finite-dimensional system. The singular perturbation method applies to coupled systems admitting two different time-scales. In other words, when facing with only two different time-scales - which is the case of this article -, one has to study a fast subsystem coupled with a slow subsystem. The singular perturbation method consists in decoupling the system into two subsystems and states that, if the fast system is sufficiently fast and if the two decoupled subsystems are asymptotically, then the asymptotic stability of the full-system holds true. This makes really easier the analysis of coupled system since the terms of coupling are more or less ignored. Moreover, this result may have practical impacts. Indeed, the existence of different time-scales appears in many applications, since electrical and mechanical behaviors might be coupled in situations where an actuator is designed with an electronic device.

Many results have been obtained in the finite-dimensional setting, as illustrated in (Kokotović et al., 1999; Lorenzetti and Weiss, 2022, 2023), and are used nowadays as a classical tool for the analysis of nonlinear finite-dimensional systems. To the best of our knowledge, at least in control theory, there exist very few results dealing with the infinite-dimensional case. Most of them, moreover, focus on the one-dimensional hyperbolic case, as illustrated in (Tang et al., 2015; Tang and Mazanti, 2017; Cerpa and Prieur, 2020; Arias et al., 2023), even if, recently, the Korteweg-de Vries equation coupled with an ordinary differential equation (for short, ODE) has been treated in (Marx and Cerpa, 2024). Most of these results rely on a Lyapunov approach (Tang and Mazanti, 2017; Marx and Cerpa, 2024), even if very recently the frequency domain approach has been used (Arias et al., 2023), allowing for more precise results in this context. The lack of spectral results is due mostly to the fact that having a precise description of the spectrum is, in general, very difficult. Furthermore, even if one has in hand the spectrum, manipulating the latter requires sophisticated tools that are only known in some specific cases.

Having singular perturbation results for coupled systems may make easier the asymptotic stability of the latter, even if, nowadays, some powerful tools are known for the asymptotic stability analysis of systems composed by PDEs coupled with ODEs. The backstepping method (Krstic and Smyshlyaev, 2008), for instance, has been proved to be applicable in many situations, especially in the case of PDEs coupled with ODEs (Auriol et al., 2018; Deutscher, 2017). When facing with coupled PDE/ODE, one may also use Lyapunov methods and, more precisely, the Legendre polynomials that can be used in order to construct Lyapunov functionals (Barreau et al., 2018). Finally, one may use Sylvester equations as in (Natarajan, 2021; Paunonen and Pohjolainen, 2010) to study and to stabilize coupled systems.

It is worth noticing that counterexamples for the singular perturbation have been provided for the infinite-dimensional case. In other words, and to be more precise, there exist unstable infinite-dimensional systems for which there exist different time-scales. However, the related subsystems are asymptotically stables. Such counterexamples have been provided in (Cerpa and Prieur, 2017) and (Tang and Mazanti, 2017) and are based on the link between transport equation and difference equations. Indeed, the spectrum of difference equations is nowadays well-known. However, it is important to

note that these counterexamples are composed by a fast finite-dimensional system coupled with a slow infinite-dimensional system.

This fact leads to the present article, which proposes some sufficient conditions so that the singular perturbation method can be applied for a specific infinite-dimensional system composed by a slow ODE coupled with a fast infinite-dimensional system. To be more precise, three results will be stated and proved: first, a well-posedness theorem whose proof is based on the regular system theory (Weiss, 1994); second, an exponential stability theorem whose proof is based on the existence of an input-to-state stable Lyapunov functional (for short ISS) (Mironchenko and Prieur, 2020) and on the forwarding method, known in finite-dimension (Mazenc and Praly, 1996) and extended to the infinite-dimensional setting recently in (Marx et al., 2021; Balogoun et al., 2023; Terrand-Jeanne et al., 2019); third, and finally, an infinite-dimensional Tikhonov theorem is provided in the case of a fast infinite-dimensional system coupled with a slow ODE.

This article is organized as follows. In Section 2, some regularity properties of the full system are given. Section 3 deals with the approximation of the full-system into two subsystems and provides some regularity properties of the latter. Section 4 states and proves the first main result of the paper, i.e. the singular perturbation theorem for the system under consideration. Section 5 introduces and proves an infinite-dimensional version of a Tikhonov theorem. Finally, Section 6 recalls the main results obtained in this paper and collects some future research lines to be followed.

2. Regularity properties of the full system

This section is devoted to the regularity properties of the full system. We use in particular results of system theory given in (Weiss, 1994), also available, in a more general setting in (Salamon, 1987).

Define Z, W, U_1 and U_2 Hilbert spaces and denote by I_Z (resp.. I_W) the identity operator in Z (resp. W). Denote $\|\cdot\|Z$ (resp. $\|\cdot\|_W$, $\|\cdot\|_W$, $\|\cdot\|_{U_1}$ and $\|\cdot\|_{U_2}$) the norm of Z (resp. W, U_1 and U_2). In this paper, the focus is on the following coupled linear infinite-dimensional system

$$\begin{cases} \varepsilon\dfrac{d}{dt}z = A_1 z + B_1 C_1 w \\ \dfrac{d}{dt}w = A_2 w + B_2 C_2 z \\ z(0) = z_0,\ w(0) = w_0. \end{cases} \tag{1}$$

where $A_1 : D(A_1) \subset Z \to Z$, whose resolvent set is denoted by $\sigma(A_1)$, (resp. $A_2 : D(A_2) \subset W \to W$ with a resolvent set denoted by $\sigma(A_2)$) generates a strongly continuous semigroup $(\mathbb{T}_1(t))_{t\geq 0}$ on Z (resp. $(\mathbb{T}_2(t))_{t\geq 0}$) on W). We suppose that $B_1 \in \mathcal{L}(U_1, Z_{-1})$ (resp. $B_2 \in \mathcal{L}(U_2, W_{-1})$ is infinite-time p-admissible for $\mathbb{T}_1$ (resp. for $\mathbb{T}_2$), where Z_{-1} (resp. W_{-1}) is defined as the completion of Z (resp. W) with respect to the norm $\|z\|_{Z_{-1}} := \|(A_1 - \lambda \mathrm{I}_Z)^{-1} z\|_Z$ for every $\lambda \in \sigma(A_1)$ (resp. the norm $\|w\|_{W_{-1}} := \|(A_2 - \lambda \mathrm{I}_W)^{-1} w\|_W$ for every $\lambda \in \sigma(A_2)$). We furthermore assume that $C_1 \in \mathcal{L}(D(A_2), U_1)$ (resp. $C_2 \in \mathcal{L}(D(A_1), U_2)$) is infinite-time p-admissible for $(\mathbb{T}_2(t))_{t\geq 0}$ (resp. for $(\mathbb{T}_1(t))_{t\geq 0}$)). As an admissible output, C_1 (resp. C_2) admits a continuous extension, denoted by C_1^L (resp. C_2^L), and defined as follows, for all $w_0 \in W$

$$C_1^L w_0 = \lim_{\tau\to 0} C_1 \frac{1}{\tau}\int_0^\tau \mathbb{T}_2(t) w_0 dt, \tag{2}$$

with the domain given by

$$D(C_1^L) = \{w_0 \in W \mid \text{ the limit of (2) exists}\}, \tag{3}$$

satisfying $W_1 \subset D(C_1^L) \subset W$. We can define similarly C_2^L, i.e., for all $z_0 \in Z$

$$C_2^L z_0 = \lim_{\tau\to 0} C_2 \frac{1}{\tau}\int_0^\tau \mathbb{T}_1(t) z_0 dt, \tag{4}$$

with the domain given by

$$D(C_2^L) = \{z_0 \in Z \mid \text{ the limit of (4) exists}\}, \tag{5}$$

This extension is necessary in order to ensure the well-posedness (in a sense that will be given later on) of (1). It is a quite common tool in system theory, as illustrated in (Weiss, 1994; Staffans, 2005). For another properties dealing with the space $D(C_1^L)$, we refer the interested reader to (Weiss, 1994).

From a well-posedness point of view, one can consider $\varepsilon = 1$ without loss of generality. Let us rewrite (1) as follows:

$$\begin{cases} \dfrac{d}{dt}\eta = \mathcal{A}\eta + \mathcal{B}y = (\mathcal{A} + \mathcal{B}C)\eta \\ y = C\eta, \\ \eta(0) = \begin{pmatrix} z_0 \\ w_0 \end{pmatrix}, \end{cases} \tag{6}$$

where $\eta := \begin{pmatrix} z \\ w \end{pmatrix}$, $\mathcal{A} := \begin{pmatrix} \frac{A_1}{\varepsilon} & 0 \\ 0 & A_2 \end{pmatrix}$, $C := \begin{pmatrix} 0 & \frac{C_1}{\varepsilon} \\ C_2 & 0 \end{pmatrix}$, $\mathcal{B} := \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$. We define furthermore $\mathcal{X} = Z \times W$ and $\mathcal{U} := U_1 \times U_2$. The operator $\mathcal{A}$ generates a strongly continuous semigroup, called $(\mathbb{S}(t))_{t\geq 0}$, and defined, for all $t \geq 0$, by $\mathbb{S}(t) = \begin{pmatrix} \mathbb{T}_1(t) \\ \mathbb{T}_2(t) \end{pmatrix}$. Since C_1 and C_2 are admissible, there exists a continuous extension C^L defined as

$$C^L \eta_0 = \lim_{\tau\to 0} C \frac{1}{\tau}\int_0^\tau \mathbb{S}(t) dt, \tag{7}$$

with the domain given by

$$D\left(C^L\right) = \{\eta_0 \in \mathcal{X} \mid \text{the limit of (7) exists}\}.$$

From the assumptions given at the beginning of this section, and supposing moreover that

$$C^L(s\mathrm{I}_{Z\times W} - \mathcal{A})^{-1}\mathcal{B}$$

makes sense for some (hence for every) $s \in \rho(\mathcal{A})$ and that $s \mapsto \|C^L(s\mathrm{I}_{Z\times W} - \mathcal{A})^{-1}\mathcal{B}\|_{\mathcal{L}(\mathcal{U})}$ is bounded in some right half-plane,

one can check that the triple $(\mathcal{A}, \mathcal{B}, C)$ is regular, in the sense given by (Weiss and Curtain, 1997, Proposition 2.1). Invoking moreover (Weiss, 1994, Theorem 7.2) (or similarly (Salamon, 1987, Theorem 4.2)), one proves that the operator $\mathcal{F} := \mathcal{A} + \mathcal{B}\mathcal{C}^L$ generates a strongly continuous semigroup. The domain of $\mathcal{F}$ is defined as follows:

$$D(\mathcal{F}) := \left\{\eta_0 \in D\left(C^L\right) \middle| C^L\eta_0 \in \mathcal{U} \text{ and } \mathcal{F}\eta_0 \in \mathcal{X}\right\} \tag{8}$$

As a consequence, one can state the following proposition (which does need any proof, since $\mathcal{F}$ generates strongly continuous semigroup):

Proposition 1. *Given $T > 0$, for any $(z, w) \in Z \times W$ (resp. $D(\mathcal{F})$), there exists a unique mild solution $(z, w) \in C(0, T; Z \times W)$ (resp. a unique strong solution $(z, w) \in C(0, T; D(\mathcal{F})) \cap C^1(0, T; Z \times W)$).*

As already mentioned in the introduction, the singular perturbation method consists in decoupling a coupled system into a fast system (modeled with the state z) called the *boundary layer system* and a slow system (modeled with the state w) called the *reduced order system*. The computation of these systems will be explained in the sequel.

3. Approximation of (1) with two systems

As already explained in the introduction, one of the goals of the singular perturbation method is to find a way to approximate (1) into two decoupled systems modeling the slow and the fast dynamics. One says that the singular perturbation method works as soon as, for a sufficiently small ε, studying the stability of (1) reduces to show the stability of the two approximated systems, called the reduced order system and the boundary layer system. This section explains how one computes the reduced order system and the boundary layer system. It is worth noticing that these computations are formal.

Reduced order system.. Assume that $\varepsilon = 0$. Therefore, supposing that A_1 is invertible (that may happen if A_1 generates a strongly continuous semigroup that is exponentially stable or if A_1 is skew adjoint), one can compute the following "quasi-steady state":

$$\bar{z} = -A_1^{-1}B_1C_1\bar{w}. \tag{9}$$

The invertibility of A_1 implies in particular that:

$$0 \in \rho(A_1). \tag{10}$$

Injecting this steady state in the first line of (1), one obtains the reduced order system:

$$\begin{cases} \dfrac{d}{dt}\bar{w} = (A_2 - B_2C_2A_1^{-1}B_1C_1)\bar{w}, \\ \bar{w}(0) = \bar{w}_0. \end{cases} \tag{11}$$

Looking at the domain and the range of the operators involved in (11), one may think that the operator $A_2 - B_2C_2A_1^{-1}B_1C_1$ does not make sense. It is because, instead of considering (11), one should look rather at the following system:

$$\begin{cases} \dfrac{d}{dt}\bar{w} = (A_2 - B_2C_2^LA_1^{-1}B_1C_1^L)\bar{w}, \\ \bar{w}(0) = \bar{w}_0, \end{cases} \tag{12}$$

where we recall that C_1^L and C_2^L exist because of the admissibility of the output $y = C\eta$. Since the system (1) is well-posed in the sense given in (Weiss and Curtain, 1997, Proposition 2.1), and noticing that $0 \in \rho(A_1)$, one can invoke (Weiss, 1994, Proposition 5.4) to prove that the operator $C_2^LA_1^{-1}B_1$ is bounded from U_1 to U_2, i.e. $C_2^LA_1^{-1}B_1 \in \mathcal{L}(U_1, U_2)$. Setting $\mathcal{K} := C_2^LA_1^{-1}B_1 \in \mathcal{L}(U_1, U_2)$, (12) can be seen as a regular (or equivalently well-posed) system with (static output) feedback. Hence, invoking (Weiss, 1994, Theorem 7.2), one can prove that $\tilde{A}_2 := A_2 - B_2C_2^LA_1^{-1}B_1C_1^L$, defined with the domain

$$D(\tilde{A}_2) := \{w_0 \in D(C_1^L) \mid C_1^Lw_0 \in U_1 \text{ and } \tilde{A}_2w_0 \in W\}.$$

generates a strongly continuous semigroup, leading therefore to the following proposition:

Proposition 2. *Given any $T > 0$, for any initial condition $w_0 \in W$ (resp. $w_0 \in D(\tilde{A}_2)$), there exists a unique mild solution $w \in C(0, T; W)$ to (12) (resp. there exists a unique strong solution $w \in C(0, T; D(\tilde{A}_2)) \cap C^1(0, T; W)$.*

Throughout the paper, one will suppose that 0 of (12) is exponentially stable, i.e., the following assumption holds true:

Assumption 1. *The semigroup generated by $\tilde{A}_2 : D(\tilde{A}_2) \subset W \to W$ is exponentially stable, i.e., there exists $K_2 > 0$ and $\omega_2 > 0$ such that, for all $w_0 \in W$ and all $t \geq 0$*

$$\|\tilde{\mathbb{T}}_2(t)w_0\|_W \leq K_2e^{-\omega_2 t}\|w_0\|_W. \tag{13}$$

Boundary layer system. Let us introduce $\tau = \frac{t}{\varepsilon}$. Consider $\bar{z}$ defined as $\bar{z} = z + A_1^{-1}B_1C_1w$. We therefore have:

$$\frac{d}{d\tau}\bar{z} = A_1z + B_1C_1w + \varepsilon\frac{d}{dt}A_1^{-1}B_1C_1w.$$

Assuming that $\varepsilon = 0$ and knowing that A_1 is invertible, one therefore obtains the following reduced order system

$$\begin{cases} \dfrac{d}{d\tau}\bar{z} = A_1\bar{z}, \\ \bar{z}(0) = z_0 + A_1^{-1}B_1C_1w_0, \end{cases} \tag{14}$$

for which one knows that, for a given pair $(z_0, w_0) \in Z \times D(C_1^L)$ and a given $T > 0$, there exists a unique mild solution (i.e., $z \in C(0, T; Z)$) since A_1 generates a strongly continuous semigroup (by assumption) and also knowing that C_1 can be extended into C_1^L (since C_1 is admissible).

As in the case of the reduced order system, the following hypothesis is assumed to be true.

Assumption 2. *The semigroup generated by $A_1 : D(A_1) \subset Z \to Z$ is exponentially stable, i.e., there exists $K_1 > 0$ and $\omega_1 > 0$ such that, for all $z_0 \in Z$ and all $t \geq 0$*

$$\|\mathbb{T}_1(t)z_0\|_Z \leq K_1e^{-\omega_1 t}\|z_0\|_Z. \tag{15}$$

Note that supposing that Assumption 2 holds true implies that A_1 is invertible. This justifies even more the computation made until now.

4. A singular perturbation result

The goal of this section is to provide sufficient conditions so that the singular perturbation method applies. However, one cannot hope having a general result on the singular perturbation. Indeed, in (Tang and Mazanti, 2017; Cerpa and Prieur, 2017) some counter-examples have been provided in the case of a fast ODE coupled with a slow transport equation (or a system of transport equations). In those papers, it is proved that, even if the related reduced order system and boundary layer system are asymptotically stable, it may happen that the full-system is not asymptotically stable for every $\varepsilon > 0$. This result is proved using the link between transport equations and difference equations, and especially the precise characterization (in terms of necessary and sufficient conditions) of the spectrum of the latter. Therefore, either one focuses on special structures of systems (parabolic systems, for instance) or one tries to obtain a general result on the case of coupled PDE-ODE where the PDE is fast and the ODE slow.

Our goal is to treat the second situation. Therefore, throughout the sequel of the paper, the functional spaces W and U_2 will be supposed of finite-dimension, i.e.:

Assumption 3. *The functional spaces W and U_2 are of finite-dimension, i.e., $W = \mathbb{R}^n$ and $U_2 = \mathbb{R}^m$, with $n, m \in \mathbb{N}$.*

In addition to Assumptions 1, 2 and 3, the results of this article rely on some ISS property that comes from the admissibility of the operator B_1. Indeed, it is known from (Mironchenko and Prieur, 2020, Theorem 3.18) (or equivalently (Mironchenko and Schwenninger, 2023, Proposition 1)) that a control system is ISS if it is uniformly globally asymptotically stable and if the control operator is p-admissible. However, the ISS property does not guarantee, in general, the existence of an ISS Lyapunov functional, except in some cases given in (Mironchenko and Schwenninger, 2023). In this paper, the existence of such a functional will be supposed, together with a coercivity property (for non-coercive Lyapunov functional, the interested reader may refer to (Mironchenko and Prieur, 2020)).

Assumption 4. *There exist positive constants a_1, a_2, a_3 and a_4 and a quadratic and coercive Lyapunov functional $V(z) = \langle P_1 z, z\rangle$, with $P_1 \in \mathcal{L}(Z)$ such that*

$$a_1\|z\|_Z^2 \leq V(z) \leq a_2\|z\|_Z^2. \tag{16}$$

Moreover, one has:

$$\begin{aligned}&\langle P_1(A_1 z + B_1C_1 w), z\rangle_Z + \langle P_1 z, A_1 z + B_1C_1 w\rangle_Z \leq \\ &- a_3\|z\|_Z^2 + a_4\|C_1 w\|_{U_1}^2.\end{aligned} \tag{17}$$

Before proving the main result of this paper, which is based on Assumptions 1, 2, 3 and 4, let us introduce the Lyapunov functional that will be used to this end. Since (12) is supposed to be exponentially stable and W is supposed to be finite-dimensional, then, for any positive and symmetric matrix $Q_2 \in \mathcal{L}(W)$, there exists a positive and symmetric matrix $P_2 \in \mathcal{L}(W)$ such that

$$\langle P_2\tilde{A}_2 w, w\rangle_W + \langle P_2 w, \tilde{A}_2 w\rangle_W = -\langle Q_2 w, w\rangle_W.$$

where we recall that $\tilde{A}_2 := A_2 - B_2C_2A_1^{-1}B_1C_1$.

Let us focus on the following Lyapunov functional:

$$W(z, w) = \varepsilon V(z) + \langle P_2(w - \varepsilon Mz), w - \varepsilon Mz\rangle_W, \tag{18}$$

with M defined as

$$M := B_2C_2A_1^{-1} \in \mathcal{L}(Z, W). \tag{19}$$

It is worth noticing that this Lyapunov functional is inspired by the forwarding method, well-known in the finite-dimensional setting (Mazenc and Praly, 1996), and extended recently in the infinite-dimensional setting (Marx et al., 2021; Vanspranghe and Brivadis, 2023; Terrand-Jeanne et al., 2019). It is also the one used in the recent article (Marx and Cerpa, 2024).

Lemma 1. *Suppose that Assumptions 3 and 4 hold true. Then, there exists $\bar{\nu}, \underline{\nu} > 0$ such that*

$$\underline{\nu}\left(\|z\|_Z^2 + \|w\|_W^2\right) \leq W(z, w) \leq \bar{\nu}\left(\|z\|_Z^2 + \|w\|_W^2\right), \tag{20}$$

where

$$\bar{\nu} := \max(\varepsilon a_2 + \varepsilon^2\|M\|_{\mathcal{L}(Z,W)}^2\|P_2\|_{\mathcal{L}(W)}, \|P_2\|_{\mathcal{L}(W)}),$$

and

$$\underline{\nu} := \min\left(\frac{a_1\varepsilon}{2}, \frac{1}{2}\frac{a_1\varepsilon}{\varepsilon^2\lambda\|M\|_{\mathcal{L}(Z,W)}^2 + a_1\varepsilon}\right),$$

with λ the eigenvalue of lowest value of P_2.

Proof. Using Young's Lemma and the fact that Assumption 4 is satisfied, it is clear that

$$\begin{aligned}W(z, w) \leq &\|P_2\|_{\mathcal{L}(W)}\|w\|_W^2 + \varepsilon a_2\|z\|_Z^2 \\ &+ \varepsilon^2\|M\|_{\mathcal{L}(Z,W)}^2\|P_2\|_{\mathcal{L}(W)}\|z\|_Z^2.\end{aligned} \tag{21}$$

Moreover, using again Young's Lemma and the fact that Assumptions 4 and 4 are satisfied, one obtains:

$$\begin{aligned}W(z, w) \geq &a_1\varepsilon\|z\|_Z^2 \\ &+ \lambda(\varepsilon^2\|Mz\|_W^2 + \|w\|_W^2 - 2\langle\varepsilon Mz, w\rangle_W) \\ \geq &a_1\varepsilon\|z\|_Z^2 + \frac{\lambda\varepsilon^2}{2}\left(1 - \frac{1}{\alpha}\right)\|Mz\|_W^2 \\ &+ \frac{\lambda}{2}(1 - \alpha)\|w\|_W^2\end{aligned}$$

Select $\alpha := \frac{\lambda\varepsilon^2\|M\|_{\mathcal{L}(Z,W)}^2}{\lambda\varepsilon^2\|M\|_{\mathcal{L}(Z,W)}^2 + a_1\varepsilon}$. Then, one has $1 - \frac{1}{\alpha} < 0$. Moreover,

$$\begin{aligned}W(z, w) \geq &a_1\varepsilon\|z\|_Z^2 \\ &- \frac{1}{2}\left(\frac{a_1\varepsilon}{\lambda\varepsilon^2\|M\|_{\mathcal{L}(Z,W)}^2}\right)\varepsilon^2\|M\|_{\mathcal{L}(Z,W)}^2\|z\|_Z^2 \\ &+ \frac{1}{2}\frac{a_1\varepsilon}{\varepsilon^2\lambda\|M\|_{\mathcal{L}(Z,W)}^2 + a_1\varepsilon}\|w\|_W^2,\end{aligned}$$

meaning in particular that the statement of Lemma 1 holds true. This concludes the proof. □

We are now in position to state and prove the main result of this paper.

Theorem 1. *Suppose that* (1) *is well-posed. Suppose morever that Assumptions 1, 2, 3 and 4 holds true. Then, there exists* ε^* *such that, for every* $\varepsilon \in (0, \varepsilon^*)$*, the mild solution* $(z, w) \in C(\mathbb{R}_+; Z \times W)$ *to* (1) *is exponentially stable.*

Proof. Throughout this proof, strong solutions will be considered so that time-derivatives of the Lyapunov functional can be considered. Theorem 1 can be then deduced using a standard density argument (see e.g., (Marx et al., 2017, Lemma 1)).

Time derivative of W along trajectories of (1) yields, for every $(z, w) \in D(\mathcal{F})$:

$$\begin{aligned}\frac{d}{dt}W(z,w) \leq & -a_3\|z\|_Z^2 + a_4\|C_1\|^2_{\mathcal{L}(W,U_1)}\|w\|_W^2 \\ & + \langle P_2\tilde{A}_2 w, w - \varepsilon M z\rangle_W \\ & + \langle P_2(w - \varepsilon M z), \tilde{A}_2 w\rangle_W \\ \leq & -a_3\|z\|_Z^2 + a_4\|C_1\|^2_{\mathcal{L}(W,U_1)}\|w\|_W^2 \\ & - \langle Q_2 w, w\rangle_W - 2\varepsilon\langle P_2\tilde{A}_2 w, Mz\rangle_W \\ \leq & -a_3\|z\|_Z^2 + a_4\|C_1\|^2_{\mathcal{L}(W,U_1)}\|w\|_W^2 \\ & - \beta\|w\|_W^2 + \frac{\varepsilon^2}{\mu}\|P_2\tilde{A}_2\|^2_{\mathcal{L}(W)}\|w\|_W^2 \\ & + \mu\|M\|^2_{\mathcal{L}(Z,W)}\|z\|_Z^2,\end{aligned} \tag{22}$$

where β is defined as the eigenvalue of lowest value of Q_2 and, in the third line, the Young's inequality has been applied.

Now, selecting μ and ε such that

$$\mu = \frac{a_3}{2\|M\|^2_{\mathcal{L}(Z,W)}},\ \varepsilon^2 = \frac{\mu\beta}{2\|P_2\tilde{A}_2\|^2_{\mathcal{L}(W)}},$$

one obtains

$$\frac{d}{dt}W(z,w) \leq -\frac{a_3}{2}\|z\|_2^2 - \left(\frac{\beta}{2} - \kappa\|C_1\|^2_{\mathcal{L}(W,U_1)}\right)\|w\|_W^2. \tag{23}$$

Selecting Q_2 such that $\beta > 2a_4\|C_1\|^2_{\mathcal{L}(W,U_1)}$, one obtains the desired result by setting $\varepsilon^* = \sqrt{\frac{\mu\beta}{2\|P_2\tilde{A}_2\|^2_{\mathcal{L}(W)}}}$. □

In addition to this asymptotic stability result, one can show, using Lemma 1, that the following corollary is true.

Corollary 1. *Suppose that Assumptions 1, 2, 3 and 4 hold true. Then, for any initial conditions* $(z_0, w_0) \in Z \times W$ *satisfying*

$$\|z_0\| = O(\varepsilon),\ \|w_0\| = O(\varepsilon),$$

we have the mild solution to (1) $(z, w) \in C(\mathbb{R}_+; Z \times W)$ *that is such that*

$$\|z\|_Z + \|w\|_W = O(\varepsilon) \tag{24}$$

5. An infinite-dimensional Tikhonov result

In the seminal article authored by Tikhonov (Tikonov, 1952), the author proposed to analyze the behavior of the trajectory with respect to ε. Since then, this theorem has been studied in many situations as described in the survey (Kokotovic et al., 1976) and the book (Kokotović et al., 1999). To cite a sentence extracted from (Kokotovic et al., 1976): "When does a reduced solution [...] approximate the original solution [...] and in what sense ?". The Tikhonov result analyzes the influence of the trajectories with respect to the initial condition.

Therefore, to follow the same path, we will emphasize first on the fact that the initial conditions of (1) depend on ε, i.e. one will look rather at

$$\begin{cases}\varepsilon\dfrac{d}{dt}z = A_1 z + B_1 C_1 w, \\ \dfrac{d}{dt}w = A_2 w + B_2 C_2 z, \\ z(0;\varepsilon) = z_0(\varepsilon),\ w(0;\varepsilon) = w_0(\varepsilon),\end{cases} \tag{25}$$

meaning in particular that z and w depend also on ε, i.e $z = z(t;\varepsilon)$ and $w = w(t;\varepsilon)$. It is important to note that $\bar{w}$ and $\bar{z}$ do not depend on ε. Indeed, when computing these states, the variable ε has been supposed to be equal to 0.

The goal is to look at the error between the reduced solutions and the solution itself, thus at the following function:

$$\begin{aligned}&\tilde{z} := z + A_1^{-1}B_1C_1 w - \bar{z}\left(\frac{t}{\varepsilon}\right), \\ &\tilde{w} = w - \bar{w},\end{aligned} \tag{26}$$

whose dynamics can be obtained formally as follows

$$\begin{aligned}\varepsilon\frac{d}{dt}\tilde{z} =& A_1 z(t) + B_1C_1 w(t) \\ & + \varepsilon A_1^{-1}B_1C_1(A_2 w(t) + B_2C_2 z(t)) - A_1\bar{z}\left(\frac{t}{\varepsilon}\right) \\ =& (A_1 + \varepsilon A_1^{-1}B_1C_1B_2C_2)\tilde{z} \\ & + \varepsilon A_1^{-1}B_1C_1\left(\tilde{A}_2 w(t) + B_2C_2\bar{z}\left(\frac{t}{\varepsilon}\right)\right),\end{aligned}$$

where, in the second line, the identity (26) has been used, and where we recall that $\tilde{A}_2 = A_2 - B_2C_2^L A^{-1}B_1C_1^L$. The following result holds:

Theorem 2. *Suppose that Assumptions 1, 2 and 3 hold true. There exists* $\varepsilon^* > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon^*)$ *and for any initial condition* $(\tilde{z}_0(\varepsilon), \tilde{w}_0(\varepsilon)) \in Z \times W$ *satisfying*

$$\|\tilde{z}_0\|_Z = O(\varepsilon),\ \|\tilde{w}_0\|_W = O(\varepsilon),$$

we have $\|\tilde{z}\|_Z = O(\varepsilon)$ *and* $\|\tilde{w}\|_W = O(\varepsilon)$*.*

Proof. Since Assumptions 1, 2 and 3 hold true, then Corally 1 applies, i.e. there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_1)$, one has, for all $t \geq 0$

$$\|w(t)\|_W = O(\varepsilon). \tag{27}$$

Moreover, notice that $\bar{w}$ satisfies $\frac{d}{dt}\bar{w} = \tilde{A}_2\bar{w}$, where $\tilde{A}_2$ is supposed to be Hurwitz (see Assumption 2). Then, there exist positive constants K_3 and ω_3, independent on ε, such that

$$\|\bar{w}(t)\|_W \leq K_3 e^{-\omega_3 t}\|\bar{w}_0\|_W. \tag{28}$$

Since K_3 and ω_3 are independent on ε, one can show that

$$\|\tilde{w}\|_W = O(\varepsilon). \tag{29}$$

Using the Duhamel's formula and setting $\tau = \frac{t}{\varepsilon}$, one obtains that

$$\begin{aligned}\tilde{z}(\tau) =& \mathbb{T}_1^p(\tau)\tilde{z}_0 \\ &+ \varepsilon\int_0^\tau \mathbb{T}_1^p(\tau-s)A_1^{-1}B_1C_1\tilde{A}_2w(s)ds \\ &+ \varepsilon\int_0^\tau \mathbb{T}_1^p(\tau-s)A_1^{-1}B_1C_1B_2C_2\bar{z}\left(\frac{s}{\varepsilon}\right)ds,\end{aligned} \tag{30}$$

where $(\mathbb{T}_1^p(\tau))_{\tau\geq 0}$ is the exponential strongly continuous semigroup generated by the operator $A_1 + \varepsilon A_1^{-1}B_1C_1B_2C_2$, as proved in Lemma 2 as soon as ε is sufficiently small. First, note that

$$\begin{aligned}&\left\|\int_0^\tau \mathbb{T}_1^p(\tau-s)A_1^{-1}B_1C_1\tilde{A}_2w(s)ds\right\|_Z \leq \\ &O(\varepsilon)\|A^{-1}B_1C_1\tilde{A}_2\|_{\mathcal{L}(Z,W)}\int_0^\tau \|\mathbb{T}_1^p(\tau-s)\|_{\mathcal{L}(W)}ds,\end{aligned} \tag{31}$$

where one has used the fact that $\|w(t)\|_W = O(\varepsilon)$. Note that

$$\int_0^\tau \|\mathbb{T}_1^p(\tau-s)\|_{\mathcal{L}(W)}ds \leq \int_0^\tau K_1 e^{\alpha(\tau-s)}ds,$$

where $\alpha := -\omega_1 + \varepsilon K_1\|A_1^{-1}B_1C_1^LB_2C_2^L\|_{\mathcal{L}(Z)}$, with K_1 and ω_1 the bounds of the strongly continuous semigroup $(\mathbb{T}_1(t))_{t\geq 0}$, i.e. $\|\mathbb{T}_1(t)\|_{\mathcal{L}(Z)} \leq K_1 e^{-\omega_1 t}$, for all $t \geq 0$. By Assumption 1, and using Lemma 2, there exists $\varepsilon_2 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_2)$, one has $\alpha < 0$. Therefore, for all $\tau \geq 0$

$$\int_0^\tau \|\mathbb{T}_1^p(\tau-s)\|_{\mathcal{L}(W)}ds \leq \frac{C}{\alpha}\exp(\alpha\tau).$$

Therefore, using the fact that $\alpha = O(\varepsilon)$ and that $\tau = \frac{t}{\varepsilon}$, one has, for all $\tau \geq 0$

$$\int_0^\tau \|\mathbb{T}_1^p(\tau-s)\|_{\mathcal{L}(W)}ds \leq O(\varepsilon^{-1}). \tag{32}$$

Gathering (31) and (32), one obtains that, for all $\tau \geq 0$

$$\left\|\int_0^\tau \mathbb{T}_1^p(\tau-s)A_1^{-1}B_1C_1\tilde{A}_2w(s)ds\right\|_Z = O(1). \tag{33}$$

Second, note that, for all $\tau \geq 0$

$$\begin{aligned}&\left\|\int_0^\tau \mathbb{T}_1^p(\tau-s)A_1^{-1}B_1C_1B_2C_2\bar{z}\left(\frac{s}{\varepsilon}\right)ds\right\|_Z \leq \\ &\leq C\|A_1^{-1}B_1C_1B_2\|_{\mathcal{L}(U_2,Z)}\int_0^\tau \left\|C_2\bar{z}\left(\frac{s}{\varepsilon}\right)\right\|_{U_2}ds,\end{aligned} \tag{34}$$

where one has used the fact that $\|T_1^p(\tau-s)\|_{\mathcal{L}(Z)} \leq C$, due to Lemma 2.

Notice that, due to the infinite-time admissibility of the operator C_2, one has, after a suitable change of coordinates:

$$\int_0^\tau \left\|C_2\bar{z}\left(\frac{s}{\varepsilon}\right)\right\|_{U_2}ds \leq K\|\bar{z}_0\|, \tag{35}$$

with K independant of ε. Notice that $\bar{z}_0$ is also independant of ε, since it is related to the approximated system, for which ε has been assumed to be equal to 0. Therefore, gathering (34) and (35) leads to

$$\left\|\int_0^\tau \mathbb{T}_1^p(\tau-s)A_1^{-1}B_1C_1B_2C_2\bar{z}\left(\frac{s}{\varepsilon}\right)ds\right\|_Z = O(1). \tag{36}$$

Since one assumed that $\|\tilde{z}_0\|_Z = O(\varepsilon)$, and by gathering (33) and (36), from (30) one can deduce that, for all $\tau \geq 0$:

$$\|\tilde{z}(\tau)\|_Z = O(\varepsilon).$$

This latter inequality together with (29) and the fact ε has been assumed sufficiently small allow to conclude the proof of Theorem 2. □

6. Conclusion

In this paper, an exponential stability has been obtained for systems admitting two time-scales based on the stability of two other approximated systems and under the condition that the fast system is sufficiently fast. This result can only be applied to fast infinite-dimensional system coupled with a slow ODE, since counter-examples have been already found for a fast ODE coupled with a slow PDE (Tang and Mazanti, 2017; Cerpa and Prieur, 2017). Moreover, a Tikhonov result has been also proved using tools from perturbed operators and admissibility theory. It is worth noticing that well-posedness of the systems under consideration has been proved using regular system theory (Weiss, 1994).

In the future, it could be interesting to provide a sufficient condition for the general case, i.e. without assuming that the slow system is an ODE. One way could consist in focusing on applying the small-gain theory (Mironchenko, 2021).

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Appendix A. Technical results

This appendix is devoted to the statement and the proof of a technical results needed for the theoretical results developed all along the article. This result states that the perturbed version of the generator A_1 is still a generator, that its related semigroup remains exponentially stable for sufficiently small ε and that admissibility properties are conserved under the perturbation.

Lemma 2. *There exists $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*)$, the operator $A_1 + \varepsilon A_1^{-1} B_1 C_1^L B_2 C_2^L : D(A_1) \subset Z \to Z$ generates an exponential stable continuous semigroup $(\mathbb{T}_1^p(t))_{t\geq 0}$.*

Proof. First, it is worth noting that the operator $A_1^{-1} B_1 C_1^L B_2 C_2^L \in \mathcal{L}(Z)$, i.e., it is bounded. Moreover, the operator $A_1 + \varepsilon A_1^{-1} B_1 C_1^L B_2 C_2^L$ is a perturbed version of the generator A_1. In particular, (Tucsnak and Weiss, 2009, Theorem 2.11.2) applies, and one can deduce that the operator $A_1 + \varepsilon A_1^{-1} B_1 C_1^L B_2 C_2^L$ generates a strongly continuous semigroup $(\mathbb{T}_1^p(t))_{t\geq 0}$ with a growth bound given by

$$\alpha := -\omega_1 + K_1 \varepsilon \|A_1^{-1} B_1 C_1^L B_2 C_2^L\|_{\mathcal{L}(Z)}, \tag{A.1}$$

where ω_1 is the growth bound of $(\mathbb{T}_1)_{t\geq 0}$ the semigroup generated by A_1 and $K_1 \geq 1$ is the constant given by the inequality

$$\|\mathbb{T}_1(t)\|_{\mathcal{L}(Z)} \leq K_1 e^{-\omega_1 t}.$$

By Assumption 2, one has $\omega_1 > 0$. Selecting ε such that

$$\varepsilon < \frac{\omega_1}{K_1 \|A_1^{-1} B_1 C_1^L B_2 C_2^L\|_{\mathcal{L}(Z)}},$$

and using (Tucsnak and Weiss, 2009, Corollary 6.1.14), one can show that the semigroup $(\mathbb{T}_1^p(t))_{t\geq 0}$ is exponentially stable. □