



**HAL**  
open science

# Sticky-threshold diffusions, local time approximation and parameter estimation

Sara Mazzonetto, Alexis Anagnostakis

► **To cite this version:**

Sara Mazzonetto, Alexis Anagnostakis. Sticky-threshold diffusions, local time approximation and parameter estimation. 2024. hal-04503682v2

**HAL Id: hal-04503682**

**<https://hal.science/hal-04503682v2>**

Preprint submitted on 4 Apr 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0  
International License

# Sticky-threshold diffusions, local time approximation and parameter estimation

Alexis Anagnostakis<sup>\*1</sup>, Sara Mazzonetto<sup>†2</sup>

<sup>1</sup>Université Grenoble-Alpes, CNRS, LJK, F-38000, Grenoble, France

<sup>2</sup>Université de Lorraine, CNRS, IECL, Inria, F-54000 Nancy, France

## Abstract

We study a class of high-frequency path functionals of diffusions with a singular threshold, including the case of sticky-reflection. The functionals are built upon a test function and a normalizing sequence. We prove convergence to the local time, advancing existing results on sticky, oscillating (regime-switching), and skew or reflecting diffusions. Notably, we consider any normalizing sequence that diverges slower than the observation frequency. Additionally, we allow for jointly sticky-oscillating-skew thresholds. Combining these findings with an approximation of the occupation time, we propose consistent estimators for skewness and stickiness parameters.

*keywords and phrases:* stickiness, skewness, reflection, oscillating diffusion coefficient, occupation time statistics

*MSC 2020:* Primary 62F12, 60J55; Secondary 60J60

*This version:* April 4, 2024

## 1 Introduction

It is known that the local time at the point  $\zeta$  of an Itô diffusion with smooth coefficients  $X$  can be effectively approximated, up to a multiplicative constant, using high-frequency statistics of the form

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g(u_n(X_{\frac{i-1}{n}} - \zeta)), \quad (1)$$

where  $g$  is an integrable bounded function,  $(u_n)_n$  is a sequence that diverges slower than  $n$ , and  $(X_{i/n})_i$  are observations of the diffusion.

These approximations are used to estimate quantities of interest of a diffusion. For instance, in [5], they are employed for localized estimates of the diffusion coefficient.

---

<sup>\*</sup>alexis.anagnostakis@univ-grenoble-alpes.fr

<sup>†</sup>sara.mazzonetto@univ-lorraine.fr

Statistics like (1) are also used to estimate stickiness, skewness and oscillation jump of a diffusion [1, 13, 14, 15, 16, 18, 20]. It is possible that a diffusion spends positive amount of time at a threshold upon contact. We call such threshold *sticky* and the amount of time the diffusion spends at the threshold is governed by a parameter called *stickiness*. Also, it is possible that the excursions of a diffusion exhibit total or partial reflection at a threshold. We call such threshold *skew* and the reflection rate is governed by a parameter called *skewness*. The *oscillation jump* of a diffusion is the jump-discontinuity of the diffusion coefficient. This defines a regime-switching behavior as the dynamics on each side of the oscillating threshold are different. The simplest cases of diffusion that exhibit these features are the sticky Brownian motion [4], the skew Brownian motion [8], and the oscillating Brownian motion [12]. It is possible to define an elementary process that combines all these features, the *sticky-oscillating-skew Brownian motion* (SOS-BM). This process, defined in Sections 2.2, plays a central role in this work.

In [13, 15, 18], local time approximations are employed to estimate the skewness parameter of a skew Brownian motion. In [16], the diffusion coefficient of the oscillating Brownian motion is estimated using approximations of the occupation time for each half-line. For some Lévy processes, local time approximations are employed to estimate the volatility jump [20]. Convergence rates for the oscillating-skew Brownian motion are obtained in [18] for the case  $u_n = \sqrt{n}$ . In [1], the consistency of an approximation of the local time at a sticky threshold is demonstrated for normalizing sequences  $u_n$  diverging slower than  $\sqrt{n}$ , i.e.,  $(u_n)_n$  diverges and  $(u_n^2/n)_n$  vanishes as  $n$  diverges. This approximation is then leveraged to propose an estimator for the stickiness parameter.

In this paper we aim to achieve the following objectives.

1. Prove that the local time approximation holds at a sticky threshold for any sequence  $(u_n)_n$  that diverge at a rate slower than  $n$ . This question was left open in [1], where the author exclusively considers sequences diverging slower than  $n^{1/2}$  in the case of sticky Brownian motion.
2. Prove that the local time approximation holds at a skew and oscillating threshold for any sequence  $(u_n)_n$  that diverges slower than  $n$ . In existing literature, this has been demonstrated only for  $u_n = \sqrt{n}$  [15, 18].
3. Establish consistency of the approximation at a threshold that exhibits possibly simultaneous oscillation, stickiness, and skew.
4. Devise consistent estimators of stickiness and skewness parameters at an SOS threshold or a sticky-reflecting threshold. When possible, devise also consistent estimators of the diffusion coefficient. The latter extends to SOS-BM the main result of [16].

For simplicity, we consider a reflecting threshold to be an SOS threshold with skewness  $\beta \in \{-1, 1\}$ . Therefore, the term SOS includes the cases of reflection and sticky-reflection.

The main challenge, especially in estimation, is that the limit is intricately dependent on the process behavior in the vicinity of the threshold of interest, thereby encompassing stickiness, skewness, and the volatility coefficient. For simplicity, we suppose there is a unique SOS threshold, located at 0.

Another challenge is that the proofs rely on the time-scaling of the SOS-BM which, at high frequencies, results in an exploding stickiness. This is handled by forcing the test function  $g$  to vanish at the sticky threshold. We then prove that the action of the exploding stickiness on the absolutely continuous part of the transition law is bounded. Last, extending the findings from the SOS-BM to a diffusion with an SOS-threshold necessitates singular versions of stochastic calculus results, specifically Itô-Tanaka and Girsanov. We prove explicit versions of these results in Appendix A.

**Outline:** In Section 2 we state the main results of the paper. In Section 3, we prove the local time approximation holds for the SOS-BM and SOS-diffusions by first assuming a similar result holds for the sticky-skew Brownian motion that is proved in Section 6. In Sections 4 and 5, we provide estimates on the SOS-BM semigroup and a preliminary local time approximation. These are used to prove the main results of the paper. In Section 7, we prove that the occupation times can be approximated by Riemann sums and then use this to prove the estimation results in Section 8.

Regarding auxiliary results, in Appendix A, we prove, in presence of an SOS-threshold, explicit versions of Itô-Tanaka and Girsanov. We also prove the interplay between oscillation and stickiness for the SOS-BM. In Appendix B, we establish the time-scaling of the sticky-skew Brownian motion. In appendix C we provide the proofs for Section 4.

## 2 Main results

We now state the main results of the paper. We first set notations and define notions of convergence in which these results are expressed.

### 2.1 Preliminaries: some notations and notions of convergence

The local time approximation results are expressed in the following type of convergence.

**DEFINITION 2.1.** Let  $(A^n)_{n \geq 0}$  be a sequence of processes defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We say that  $(A^n)_{n \geq 1}$  *converges locally uniformly in time, in probability or u.c.p.* to  $A^0$  if

$$\forall t \geq 0 : \sup_{s \leq t} |A_s^n - A_s^0| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

We denote this convergence with

$$A^n \xrightarrow[n \rightarrow \infty]{\text{P-u.c.p.}} A^0.$$

The following result gives a sufficient condition for u.c.p. convergence to occur.

**LEMMA 2.2** (see [11], §2.2.3). *If  $A^n$  and  $A$  have increasing paths and  $A$  is continuous, then*

$$A_t^n \xrightarrow{\mathbb{P}} A_t, \forall t \in D, \text{ with } D \text{ dense in } \mathbb{R}^+ \implies A^n \xrightarrow[n \rightarrow \infty]{\text{P-u.c.p.}} A.$$

For the estimation results, we consider the notion of *conditional convergence in probability*. The reason is that these estimators are consistent conditionally on the event that the threshold of interest (that we consider to be 0) is reached. For an event  $\mathcal{L}$ , let  $P^{\mathcal{L}}(\cdot) := P(\cdot | \mathcal{L})$  be the conditional probability on  $\mathcal{L}$ .

DEFINITION 2.3. Let  $(A^n)_{n \geq 0}$  be a sequence of processes defined on the probability space  $(\Omega, \mathcal{F}, P)$ . We say that  $(A^n)_{n \geq 1}$  converges to  $A^0$  in probability, conditionally on  $\mathcal{L}$ , if  $A^n \rightarrow A^0$  in  $P^{\mathcal{L}}$ -probability. We denote this convergence with

$$A^n \xrightarrow[n \rightarrow \infty]{P^{\mathcal{L}}} A^0.$$

Let  $X$  be a continuous real semi-martingale. We denote by  $(\ell^y(X))_{y \in \mathbb{R}}$  the right local time field of  $X$  defined in [19, Theorem VI.1.2] via the Tanaka formula. We denote by  $(L^y(X))_{y \in \mathbb{R}}$  the symmetric local time field of the process  $X$ , defined for all  $t \geq 0$  and  $y \in \mathbb{R}$  by

$$L_t^y(X) = \frac{\ell_t^{y+}(X) + \ell_t^{y-}(X)}{2},$$

where

$$\ell_t^{y+}(X) = \lim_{h \rightarrow 0} \ell_t^{y+h}(X), \quad \ell_t^{y-}(X) = \lim_{h \rightarrow 0} \ell_t^{y-h}(X).$$

It is not always the case that the right and symmetric local time fields are equal. An example is the skew Brownian motion, see e.g. [2, Theorem 2.1]. Typically, path-wise characterizations of diffusions are expressed in terms of the symmetric local time, see e.g. [22]. Most results, like the occupation times formula [19, Corollary VI.1.6] and the representation of martingale diffusions [21, Theorem V.47.1] as a time-changed Brownian motion are expressed in terms of the right local time.

## 2.2 Local time approximation for SOS-BM

The SOS-BM is the diffusion with state space  $\mathbb{I} \subseteq \mathbb{R}$  defined through  $(s, m)$ , where  $s, m$  are defined for all  $x \in \mathbb{R}$  by

$$s(x) = \frac{2x}{1 + \operatorname{sgn}(x)\beta} \mathbb{1}_{\mathbb{I}}(x), \quad m(dx) = \frac{1 + \operatorname{sgn}(x)\beta}{\sigma_0^2(x)} dx + \rho \delta_0(dx), \quad (2)$$

with  $\rho > 0$ ,  $\beta \in [-1, 1]$  and

$$\sigma_0(x) := \mathbb{1}_{(-\infty, 0)}(x)\sigma_- + \mathbb{1}_{(0, +\infty)}(x)\sigma_+ > 0.$$

We note that we use the convention  $\operatorname{sgn}(0) = 0$ .

If  $\beta \in (-1, 1)$ , then  $\mathbb{I} = \mathbb{R}$ . The cases  $\beta = -1$  and  $\beta = +1$  correspond to negative and positive reflection. In these case  $\mathbb{I} = (-\infty, 0]$  and  $\mathbb{I} = [0, \infty)$  respectively. For simplicity, we avoid specifying  $\mathbb{I}$ , and we write  $\mathbb{R}$ . The information on the state-space is included in the pair  $(s, m)$ .

If  $X$  is an SOS-BM defined on a probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , then, from [22],  $(X, B)$  solves the system

$$\begin{aligned} X_t &= X_0 + \int_0^t \sigma_0(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \beta L_t^0(X), \\ \int_0^t \mathbb{1}_{\{X_s \neq 0\}} ds &= \frac{\rho}{2} L_t^0(X), \end{aligned} \quad (3)$$

for a standard Brownian motion  $B$ , where  $L_t^0(X)$  is the symmetric local time  $X$  at 0. To make explicit the dependence on the parameters, we refer to the SOS-BM defined in (3) as the  $(\rho, \beta, \sigma_0)$ -SOS-BM. Also, we denote by  $m_{(\rho, \beta, \sigma_0)}$  the measure defined in (2).

We are now ready to provide our first result.

**THEOREM 2.4.** *Let  $X$  be an SOS-BM solution to (3) on the probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$  such that  $X_0 = x$ ,  $\mathbb{P}_x$ -almost surely. Let also  $(u_n)_n$  be a sequence that diverges slower than  $n$ , i.e.*

$$\lim_{n \rightarrow \infty} u_n = \infty, \quad \lim_{n \rightarrow \infty} u_n/n = 0, \quad (4)$$

and let  $g$  be an integrable bounded function with  $g(0) = 0$ . Then,

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g(u_n X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x\text{-u.c.p.}} \left( \int_{\mathbb{R}} g dm \right) L_t^0(X). \quad (5)$$

**REMARK 2.5.** We observe that the limits in Theorem 2.4 do not depend on  $\rho$ . Indeed, since  $g(0) = 0$ ,

$$\int_{\mathbb{R}} g dm = \int_{\mathbb{R} \setminus \{0\}} g dm$$

which does not depend on  $\rho$ .

Theorem 2.4 is a straightforward consequence of a more general result, Theorem 2.6 (taking  $T$  the identity function). Passing through an intermediate function  $T$  is the key for extending Theorem 2.4 to more general diffusions with SOS-threshold. This is done in the next section.

**THEOREM 2.6.** *We consider the assumptions of Theorem 2.4 and let  $T$  be a twice differentiable function such that for some  $\varepsilon > 0$  and all  $x \in \mathbb{R}$ ,*

$$T(0) = 0, \quad T'(0) = 1, \quad \varepsilon \leq T'(x) \leq 1/\varepsilon, \quad |T''(x)| \leq 1/\varepsilon. \quad (6)$$

and  $g_n[T]$  be the sequence of functions defined for all  $n, x$  by

$$g_n[T](x) = g(u_n T(x/u_n)).$$

Then,

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x\text{-u.c.p.}} \left( \int_{\mathbb{R}} g dm \right) L_t^0(X).$$

### 2.3 For an SOS-diffusion

We consider the following system

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s) \mathbb{1}_{\{X_s \neq 0\}} ds + \int_0^t \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \beta L_t^0(X), \\ \int_0^t \mathbb{1}_{\{X_s = 0\}} ds = \frac{\rho}{2} L_t^0(X), \end{cases} \quad (7)$$

where  $B$  is a standard Brownian motion,  $\rho \geq 0$ ,  $\beta \in [-1, 1]$  and  $b, \sigma : \mathbb{R} \mapsto \mathbb{R}$ .

We suppose the following condition on  $(b, \sigma)$  and the state space of the diffusion  $\mathbb{I} \subseteq \mathbb{R}$  that guarantees existence and uniqueness in law (see [22]) and allows the usage of Girsanov theorem in Section 3.

CONDITION 2.7.  $\mathbb{I} = \mathbb{R}$ , or one of the half lines:  $[0, \infty)$  if  $\beta = 1$  or  $(-\infty, 0]$  if  $\beta = -1$ . The coefficients  $(b, \sigma)$  are taken such that

1. weak existence and uniqueness in law holds for (7). This is equivalent to the process that solves the SDE  $dY_t = b(Y_t) dt + \sigma(Y_t) dB_t$  having weak existence and uniqueness in law,
2.  $\sigma \in C^1(\mathbb{I} \setminus \{0\})$ ,
3. if  $X$  solves (7),  $P_x(X_0 = x) = 1$  and  $\theta = (\sigma'(X_t) - \frac{b(X_t)}{\sigma(X_t)})_{t \geq 0}$ , where  $\sigma'$  is the right-derivative of  $\sigma$ . Then, for all  $x \in \mathbb{I}$  and  $t \geq 0$ :  $E_x(\mathcal{E}(\theta)_t) = 1$ , where  $\mathcal{E}(\theta)$  is the process defined for all  $t \geq 0$  by

$$\mathcal{E}(\theta)_t = \exp \left( \int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right).$$

Note that Condition 2.7, Item 3 is equivalent to:  $\mathcal{E}(\theta)$  is a  $P_x$ -martingale. Also, when  $\mathbb{I} = \mathbb{R}$ , the function  $\sigma : \mathbb{R} \rightarrow (0, +\infty)$  may have a finite jump at 0. We define  $\sigma_0$  to be the function defined, for all  $x \in \mathbb{R}$  by

$$\sigma_0(x) := \mathbb{1}_{(-\infty, 0]}(x) \lim_{\varepsilon \rightarrow 0^-} \sigma(\varepsilon) + \mathbb{1}_{(0, +\infty)}(x) \lim_{\varepsilon \rightarrow 0^+} \sigma(\varepsilon). \quad (8)$$

The solution to the system above has an SOS threshold at 0.

THEOREM 2.8. Let  $X$  be a process defined on the probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P_x)$  such that  $X_0 = x$ ,  $P_x$ -almost surely. We assume that  $X$  is the SOS-diffusion such that  $(X, B)$  solves (7) for some Brownian  $B$  and  $(b, \sigma)$  that satisfy Condition 2.7. Then, Theorem 2.4 holds for  $X$  on  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P_x)$ , where  $m$  is the speed measure, defined in (2), of the  $(\rho, \beta, \sigma_0)$ -SOS-BM and  $\sigma_0$  is given by (8).

Clearly, Theorem 2.4 is a special case of Theorem 2.8. Regarding Theorem 2.8, we observe the following. First, the approximation (5) is extremely localized. Indeed, for large  $n$ , only the values of the process in the vicinity of the threshold, 0, matter. Second, the approximated quantity is an additive functional of the process. Hence, it may be possible to obtain a localized version of the results. This is the object of further research.

## 2.4 Occupation times approximation

For any borelian set  $U$  we denote by  $\mathcal{O}^U(X)$  the occupation time of  $U$  by  $X$ , defined for all  $t \geq 0$  by

$$\mathcal{O}_t^U(X) = \int_0^t \mathbb{1}_{X_s \in U} ds.$$

We have already noticed that constants in the limit of the previous results do not depend on the stickiness  $\rho$ . But, if one considers the occupation time of the threshold of the SOS-diffusions we considered, then we have  $\int_0^t \mathbb{1}_{X_s=0} ds = \frac{\rho}{2} L_t^X(0)$ . Hence, to estimate  $\rho$ , we consider statistics which converge to the occupation times.

The following result states that the occupation time can be effectively approximated by Riemann sums. It generalizes, and corrects [1, Lemma 5.1] which concerned sticky-diffusions and  $U = \{0\}$ .

LEMMA 2.9 (Occupation times approximation). *Let one of the following assumptions,*

- (i)  $X$  is an SOS-BM motion and  $U$  be an interval of  $\mathbb{R}$ .
- (ii)  $X$  is an SOS-diffusion solving (7) under Condition 2.7 and  $U \in \{(0, \infty), (-\infty, 0), \{0\}\}$ .

Then,

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U} \xrightarrow[n \rightarrow \infty]{P_x \text{ - u.c.p.}} \mathcal{O}_t^U(X).$$

An important consequence of this result is the following:

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} \xrightarrow[n \rightarrow \infty]{P_x} \frac{\rho}{2} L_t^X(0). \quad (9)$$

## 2.5 Estimation

Let us first assume  $\sigma_0$ , defined in (8), is known. The result of the previous sections imply that the parameters  $(\rho, \beta)$  can be estimated as follows.

For any non-trivial function  $g$ , let  $g_{>0} := \mathbb{1}_{(0, \infty)}g$  and  $g_{<0} := \mathbb{1}_{(-\infty, 0)}g$ . Also, for a process  $X$ , let  $S_n^{g^+}(X)$  and  $S_n^{g^-}(X)$  be statistics defined, for all  $n$ , by

$$S_n^{g^+}(X) := \frac{u_n}{n} \frac{\sigma_0^2(1)}{\int_{\mathbb{I}} g_{>0}(x) dx} \sum_{i=1}^{[nt]} g_{>0}(u_n X_{\frac{i-1}{n}}), \quad S_n^{g^-}(X) := \frac{u_n}{n} \frac{\sigma_0^2(-1)}{\int_{\mathbb{I}} g_{<0}(x) dx} \sum_{i=1}^{[nt]} g_{<0}(u_n X_{\frac{i-1}{n}}).$$

PROPOSITION 2.10. *We consider  $X, g$  and  $(u_n)_n$  as in Theorem 2.8. If  $\tau_0^X$  is the hitting time of 0 by  $X$  and  $\mathcal{L} = \{\tau_0^X < t\}$ , then, the following convergences hold*

$$\widehat{\rho}_n(X) := \frac{2}{n} \frac{\sum_{i=1}^{[nt]} \mathbb{1}_{\{X_{\frac{i-1}{n}}=0\}}}{S_n^{g^+}(X) + S_n^{g^-}(X)} \xrightarrow[n \rightarrow \infty]{P_x^{\mathcal{L}}} \rho, \quad \widehat{\beta}_n(X) := \frac{S_n^{g^+}(X) - S_n^{g^-}(X)}{S_n^{g^+}(X) + S_n^{g^-}(X)} \xrightarrow[n \rightarrow \infty]{P_x^{\mathcal{L}}} \beta.$$

This yields consistent estimators of stickiness and skewness parameters.



In the proof of Proposition 2.10, in Section 8.1, we see that the estimators can only be consistent on  $\mathcal{L}$ . Indeed, on  $\mathcal{L}^c$ ,  $\mathcal{O}_t^0(X) = 0$ ,  $L_t^0(X) = 0$ , hence  $\widehat{\rho}_n(X)$ ,  $\widehat{\beta}_n(X)$  are asymptotically 0/0 divisions. This confirms the following intuition: On  $\mathcal{L}^c$  we observe nothing in the vicinity of the threshold of interest. Thus, it is not possible to assess properties on the behavior of the process on the threshold with no relevant observations.

Let us now consider the case that  $\sigma_0$  is unknown. Note that in Theorem 2.8 there are 5 unknown factors in the right-hand-side of (5), namely  $(\rho, \beta, \sigma(0+), \sigma(0-), L_t^0(X))$  and 3 asymptotic results (9) and

$$S_n^{g+}(X) \xrightarrow[n \rightarrow \infty]{P_x} (1 + \beta)L_t^X(0), \quad S_n^{g-}(X) \xrightarrow[n \rightarrow \infty]{P_x} (1 - \beta)L_t^X(0). \quad (10)$$

In order to perform algebraic operations with these equations, one needs  $L_t^0(X) > 0$ . Thus, on the event  $\mathcal{L} = \{L_t^0(X) > 0\}$  we need to know at least 2 of the unknowns to infer all of them, of whom  $L_t^0(X)$  is always unknown.

In the particular case of the SOS-BM, it is possible to bypass this *uncertainty principle* and estimate all parameters  $(\rho, \beta, \sigma_-, \sigma_+)$  by combining local time statistics with occupation time statistics. Indeed,  $(\sigma_-, \sigma_+)$  can be estimated as follows.

**PROPOSITION 2.11.** *Let  $X$  be an SOS-BM defined on the probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P_x)$  such that  $P_x(X_0 = x) = 1$ . Also, let*

$$\tau^+ = \inf\{t \geq 0 : X_t > 0\}, \quad \tau^- = \inf\{t \geq 0 : X_t < 0\},$$

$\mathcal{L}_t^+ = \{\tau_+ < t\}$  and  $\mathcal{L}_t^- = \{\tau_- < t\}$ . Then, for all  $t > 0$ , in  $P_0$ -probability,

$$\sqrt{\frac{\sum_{i=1}^{\lfloor nt \rfloor} (X_{\frac{i}{n}}^+ - X_{\frac{i-1}{n}}^+)^2}{\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{1}_{X_{\frac{i-1}{n}} > 0}}} \xrightarrow[n \rightarrow \infty]{P_x^{\mathcal{L}_t^+}} \sigma_+, \quad \sqrt{\frac{\sum_{i=1}^{\lfloor nt \rfloor} (X_{\frac{i}{n}}^- - X_{\frac{i-1}{n}}^-)^2}{\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{1}_{X_{\frac{i-1}{n}} < 0}}} \xrightarrow[n \rightarrow \infty]{P_x^{\mathcal{L}_t^-}} \sigma_-$$

where  $X^+ = \max\{0, X\}$  and  $X^- = \max\{0, -X\}$ .

We note that the case  $(\rho, \beta) = (0, 0)$  is shown in [16] and the case  $\rho = 0$  in [18].

### 3 Proofs: Local time approximation for SOS-diffusions

In this section, we prove Theorem 2.6 and Theorem 2.8. The necessary theoretical foundations, the sticky versions of the Itô-Tanaka formula and Girsanov theorem, are proved in Appendix A.

We demonstrate that the problem in Theorem 2.8, stated for SOS-diffusions, can be reduced to the “simplest” similar process: SOS-BM. Indeed, the SOS-diffusion defined by (7), like the SOS-BM, exhibits an SOS threshold at 0. Similarly, in the seminal work of Jacod [10], the approximation of the local time of a smooth diffusion using the functional (1) is initially established by reduction to the standard Brownian motion. Using the Itô formula and Girsanov theorem, the result is then extended to stochastic differential equations (SDEs) with smooth coefficients. The tools used to reduce the problem (Itô formula, Girsanov theorem) cannot remove the SOS feature of the threshold (see Section A).

### 3.1 Proof of Theorem 2.6

We further reduce the problem: using the Itô-Tanaka formula, we can turn the oscillation to skewness and vice versa (see Appendix A.1). Hence, we can prove Theorem 2.6, assuming that we proved it in a special case: *sticky-skew Brownian motion*, the SOS-BM with  $\sigma_0 \equiv 1$ . We denote this process by  $(\rho, \beta, \cdot)$ -SOS-BM or  $(\rho, \beta)$ -SOS-BM. We formulate Theorem 2.6 for the sticky-skew Brownian motion in Theorem 6.1 and prove it in Section 6.

**PROOF OF THEOREM 2.6.** We consider the process  $X^{(\rho', \beta', \cdot)} := X^{(\rho, \beta, \sigma_0)} / \sigma(X^{(\rho, \beta, \sigma_0)})$ . By Proposition A.3, it is a  $(\rho', \beta', \cdot)$ -SOS-BM, where

$$(\rho', \beta') = \left( \rho \frac{2\sigma_- \sigma_+}{\sigma_-(1+\beta) + \sigma_+(1-\beta)}, \frac{\sigma_-(1+\beta) - \sigma_+(1-\beta)}{\sigma_-(1+\beta) + \sigma_+(1-\beta)} \right).$$

Also, let  $S$  be the function defined for all  $x \in \mathbb{R}$  by

$$S(x) := \frac{T(\sigma_0(x)x)}{\sigma_0(x)}$$

which, like  $T$ , preserves the sign, is  $C^1(\mathbb{R})$ ,  $C^2$  on each half-plane and satisfies (6).

We observe that

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \sigma_0)}) = \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T] \left( u_n \sigma_0 \left( X_{\frac{i-1}{n}}^{(\rho, \beta, \sigma_0)} \right) \frac{X_{\frac{i-1}{n}}^{(\rho, \beta, \sigma_0)}}{\sigma_0 \left( X_{\frac{i-1}{n}}^{(\rho, \beta, \sigma_0)} \right)} \right).$$

Moreover, by the definition of  $X'$ , we have that  $\sigma_0(X_t) = \sigma_0(X'_t)$ , hence the latter quantity is equal to

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} (g_{<0} \circ \phi_{\sigma_-})_n[S](u_n X_{\frac{i-1}{n}}^{(\rho', \beta', \cdot)}) + \frac{u_n}{n} \sum_{i=1}^{[nt]} (g_{>0} \circ \phi_{\sigma_+})_n[S](u_n X_{\frac{i-1}{n}}^{(\rho', \beta', \cdot)}),$$

where for all  $c > 0$ ,  $\phi_c$  is the function defined for all  $x \in \mathbb{R}$  by  $\phi_c(x) = cx$ . Thus, from Theorem 6.1,

$$\begin{aligned} \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \sigma_0)}) &\xrightarrow[n \rightarrow \infty]{\text{P}_x\text{-u.c.p.}} \left( \frac{1-\beta'}{2} \int g_{<0} \circ \phi_{\sigma_-}(x) dx \right. \\ &\quad \left. + \frac{1+\beta'}{2} \int g_{>0} \circ \phi_{\sigma_+}(x) dx \right) L_t^0(X^{(\rho, \beta', \cdot)}). \end{aligned}$$

By Lemma A.1 (or Proposition A.3), we have

$$\begin{aligned} \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \sigma_0)}) &\xrightarrow[n \rightarrow \infty]{\text{P}_x\text{-u.c.p.}} \left( \frac{\sigma_+(1-\beta)}{\sigma_- 2\sigma_+\sigma_-} \int_{\mathbb{R}} g_{<0}(x) dx \right. \\ &\quad \left. + \frac{\sigma_-(1+\beta)}{\sigma_+ 2\sigma_+\sigma_-} \int_{\mathbb{R}} g_{>0}(x) dx \right) L_t^0(X^{(\rho, \beta, \sigma_0)}). \end{aligned}$$

This completes the proof. □

### 3.2 Proof of Theorem 2.8

Using Theorem 2.6, the Itô-Tanaka formula and Girsanov theorem in Appendix A, we prove the result for the process of interest with non-trivial drift and diffusion coefficients.

**PROOF OF THEOREM 2.8.** Let us first assume that the diffusion coefficient and its derivative are bounded. More precisely, we suppose there exists a real constant  $\delta > 0$  such that

$$\delta \leq \sigma(x) \leq 1/\delta, \quad |\sigma'(x)| \leq 1/\delta, \quad (11)$$

for every  $x \in \mathbb{I}$ . There exists a Brownian motion  $B$  such that  $(X, B)$  jointly solve (7) [22].

*We then remove the drift.* Let  $\mathbb{Q}_x$  be the probability measure such that  $d\mathbb{Q}_x = \mathcal{E}_t(\theta) d\mathbb{P}_x$  where  $\mathcal{E}_t(\theta), \theta$  are processes defined for all  $t \geq 0$  by

$$\mathcal{E}_t(\theta) = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right), \quad \theta_t = \sigma'(X_t) - \frac{b(X_t)}{\sigma(X_t)}.$$

From Lemma A.2, under the new measure the process  $(X, \tilde{B})$  jointly solves

$$\begin{cases} dX_t = \frac{1}{2} \sigma(X_t) \sigma'(X_t) \mathbb{1}_{\{X_t \neq 0\}} dt + \sigma(X_t) \mathbb{1}_{\{X_t \neq 0\}} d\tilde{B}_t + \beta dL_t^0(X), \\ \mathbb{1}_{X_t=0} dt = \frac{\rho}{2} dL_t^0(X), \end{cases}$$

where  $\tilde{B} = (B_t - \int_0^t \theta_s ds)_{t \geq 0}$  is a  $\mathbb{Q}_x$ -Brownian motion. Let  $S$  be the function defined for every  $x \in \mathbb{I}$  by

$$S(x) = \int_0^x \frac{\sigma_0(y)}{\sigma(y)} dy, \quad (12)$$

where  $\sigma_0$  is given by (8). The function  $S$  is strictly increasing,  $S(0) = 0$ ,  $S \in C^2(\mathbb{I} \setminus \{0\})$ , and for all  $x \in \mathbb{I} \setminus \{0\}$  it holds

$$S'(x) = \frac{\sigma_0(x)}{\sigma(x)}, \quad S''(x) = -\sigma_0(x) \frac{\sigma'(x)}{\sigma^2(x)}.$$

Also, from (11), it satisfies (6). Thus, from Lemma A.1, the process  $(X' = (S(X_t))_{t \geq 0}, \tilde{B}_t)$  solves under  $\mathbb{Q}_x$

$$\begin{cases} dX'_t = \frac{\sigma_0(X_t)}{\sigma(X_t)} dX_t - \frac{1}{2} \sigma_0(X_t) \frac{\sigma'(X_t)}{\sigma^2(X_t)} d\langle X \rangle_t + \beta dL_t^0(X) \\ \quad = \mathbb{1}_{\{X'_t \neq 0\}} \sigma_0(X'_t) d\tilde{B}_t + \beta dL_t^0(X'), \\ \mathbb{1}_{X'_t=0} dt = \frac{\rho}{2} dL_t^0(X'), \end{cases}$$

and hence is an  $(\rho, \beta, \sigma_0)$ -SOS-BM.

From Theorem 2.6,

$$\frac{u_n}{n} \sum_{i=1}^{\lfloor nt \rfloor} g_n[S](u_n X'_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\mathbb{Q}_x\text{-u.c.p.}} m_{(\rho, \beta, \sigma_0)}(g) L_t^0(X').$$

From the definition of  $g_n[S]$  and  $X'$ , the fact that  $L_t^0(X)$  is a version of  $L_t^0(X')$  (see Lemma A.1), and as  $P_x \sim Q_x$ ,

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g(u_n X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{P_x\text{-u.c.p.}} m_{(\rho, \beta, \sigma_0)}(g) L_t^0(X').$$

This proves Theorem 2.8 in the case of bounded  $\sigma$ ,  $1/\sigma$ ,  $\sigma'$ . Using the localization argument of [10, Section 2.5], the proof is extended to any locally bounded  $\sigma \in C^1(\mathbb{I} \setminus \{0\})$ .  $\square$

## 4 Bounds on the law of the SOS-BM

Throughout the paper, let

1.  $(P_t^{(\rho, \beta, \sigma_0)})_{t \geq 0}$  be the semigroup,
2.  $m_{(\rho, \beta, \sigma_0)}$  the speed measure,
3.  $p_{(\rho, \beta, \sigma_0)}$  the probability transition kernel with respect to  $m_{(\rho, \beta, \sigma_0)}$

of the  $(\rho, \beta, \sigma_0)$ -SOS-BM. Also, let  $X^{(\rho, \beta, \sigma_0)}$  be the  $(\rho, \beta, \sigma_0)$ -SOS-BM defined on a probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P_x)$  such that  $X_0 = x \in \mathbb{I}$ ,  $P_x$ -almost surely. For all  $t \geq 0$ ,  $x \in \mathbb{I}$  and any measurable functions  $h$  such that  $h(X_t^{(\rho, \beta, \sigma_0)}) \in L^1(\Omega, \mathcal{F}, P_x)$ , we have

$$P_t^{(\rho, \beta, \sigma_0)} h(x) := E_x(h(X_t^{(\rho, \beta, \sigma_0)})) = \int_{\mathbb{R}} h(y) p_{(\rho, \beta, \sigma_0)}(t, x, y) m_{(\rho, \beta, \sigma_0)}(dy). \quad (13)$$

Let  $\gamma_n$  be the aggregate action of the semigroup, defined for all measurable  $h$  as above,  $t \geq 0$ ,  $x \in \mathbb{I}$  and  $n \in \mathbb{N}$  by

$$\gamma_n^{(\rho, \beta, \sigma_0)}[h](t, x) = \sum_{i=2}^{[nt]} E_x(h(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \sigma_0)})), \quad (14)$$

Also, for a measure  $m$  and  $\gamma > 0$  and a measurable function  $h$  such that  $\int |h| dm < \infty$ , let

$$m(h) := \int_{\mathbb{R}} h(x) m(dx), \quad m^{(\gamma)}(h) := \int_{\mathbb{R}} |x|^\gamma |h(x)| m(dx).$$

The Lebesgue measure is denoted by  $\lambda$ .

In this section, we consider only the sticky-skew Brownian motion or  $(\rho, \beta, \cdot)$ -SOS-BM. The results of this section are easily extended to the SOS-BM. We can remove or reintroduce the oscillation effect via a space transformation and the Itô-Tanaka formula, see Appendix A.1: if  $X$  is the  $(\rho, \beta, \sigma_0)$ -SOS-BM, then  $X/\sigma_0(X)$  is a sticky-skew Brownian motion. We denote with  $(P_t^{(\rho, \beta, \cdot)})_{t \geq 0}$ ,  $m_{(\rho, \beta, \cdot)}$ ,  $p_{(\rho, \beta, \cdot)}$  the semi-group, speed measure and probability transition kernel of the  $(\rho, \beta, \cdot)$ -SOS-BM. Similarly, we denote  $(P_t^{(\cdot, \beta, \cdot)})_{t \geq 0}$ ,  $m_{(\cdot, \beta, \cdot)}$ ,  $p_{(\cdot, \beta, \cdot)}$  the same quantities for the skew Brownian motion or  $(\cdot, \beta, \cdot)$ -SOS-BM and with  $(P_t^{(\cdot, \cdot, \cdot)})_{t \geq 0}$ ,  $m_{(\cdot, \cdot, \cdot)}$ ,  $p_{(\cdot, \cdot, \cdot)}$  for the standard Brownian motion or  $(\cdot, \cdot, \cdot)$ -SOS-BM.

In this section we only state the results. The proofs are given in Appendix C.

## 4.1 Transition density bound

The probability transition kernel  $p_{(\rho, \beta, \cdot)}$  of the  $(\rho, \beta, \cdot)$ -SOS-BM with respect to  $m_{(\rho, \beta, \cdot)}$ , computed in [23, Theorem 2.4], is given for all  $t > 0$  and  $x, y \in \mathbb{R}$  by

$$p_{(\rho, \beta, \cdot)}(t, x, y) = \frac{1}{a(y)} (u_1(t, x, y) - u_2(t, x, y)) + v_\rho(t, x, y), \quad (15)$$

where  $a$  is the function defined for all  $x \in \mathbb{R}$  by

$$a(x) = \frac{1 + \operatorname{sgn}(x)\beta}{2}$$

and  $u_1, u_2, v_\rho$  are defined for all  $t > 0, x, y \in \mathbb{R}$  by

$$\begin{cases} u_1(t, x, y) = \frac{1}{2\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \\ u_2(t, x, y) = \frac{1}{2\sqrt{2\pi t}} e^{-(|x|+|y|)^2/2t}, \\ v_\rho(t, x, y) = \frac{1}{\rho} e^{4(|x|+|y|)/\rho+8t/\rho^2} \operatorname{erfc}\left(\frac{|x|+|y|}{\sqrt{2t}} + \frac{2\sqrt{2t}}{\rho}\right), \quad \forall \rho > 0, \\ v_0(t, x, y) = u_2(t, x, y). \end{cases} \quad (16)$$

REMARK 4.1. If  $\rho > 0$  and  $xy \leq 0$ , then  $p_{(\rho, \beta, \cdot)}(t, x, y) = v_\rho(t, x, y)$ .

REMARK 4.2. The term  $v_\rho(t, x, y)$  does not appear in the probability transition kernel of the standard BM and skew BM. The probability transition kernel of the skew Brownian motion for  $\beta \neq 0$  and of the standard Brownian motion with respect to their speed measure are respectively

$$p_{(\cdot, \beta, \cdot)}(t, x, y) = \frac{1}{a(y)} (u_1(t, x, y) - u_2(t, x, y)) + u_2(t, x, y), \quad p_{(\cdot, \cdot, \cdot)} = u_1(t, x, y).$$

LEMMA 4.3. *There exists a constant  $K > 0$  such that for all  $t > 0$  and  $x, y \in \mathbb{R}$ ,*

$$p_{(\rho, \beta, \cdot)}(t, x, y) \leq K u_1(t, x, y)$$

and  $K$  does not depend on  $\rho$ . In particular there exists a constant  $K > 0$  (not depending on  $\rho$ ) such that for all  $t > 0, \rho \geq 0$  and  $x, y \in \mathbb{R}$ ,

$$v_\rho(t, x, y) \leq K u_1(t, x, y).$$

## 4.2 Semigroup bounds

LEMMA 4.4. *There exists a constant  $K > 0$  such that, for all  $h \in L^1(\mathbb{R}), t > 0$  and  $x \in \mathbb{R}$ ,*

$$|P_t^{(\rho, \beta, \cdot)} h(x)| \leq K \frac{\rho\sqrt{2t}}{\rho|x|/2 + 2t} |h(0)| + \frac{K}{\sqrt{t}} m_{(\cdot, \beta, \cdot)}(|h|) \leq \frac{K}{\sqrt{t}} m_{(\rho, \beta, \cdot)}(|h|).$$

LEMMA 4.5. For all  $\gamma \geq 0$ , there exists a constant  $K_\gamma > 0$ , that does not depend on  $(\rho, \beta)$ , such that, for all  $h \in L^1(\mathbb{R})$ ,  $t > 0$  and  $x \in \mathbb{R}$ ,

$$|P_t^{(\rho, \beta, \cdot)} h(x) - m_{(\rho, \beta, \cdot)}(h) p_{(\rho, \beta, \cdot)}(t, x, 0)| \leq K_\gamma \frac{1}{t} \left( m_{(\cdot, \beta, \cdot)}^{(1)}(h) + \frac{m_{(\cdot, \beta, \cdot)}^{(1)}(h)}{1 + |x/\sqrt{t}|^\gamma} + \frac{m_{(\cdot, \beta, \cdot)}^{(\gamma)}(h)}{1 + |x|^\gamma} \right),$$

for all  $x \in \mathbb{R}$  and  $t \geq 0$ .

LEMMA 4.6. Let  $\gamma_n^{(\rho, \beta, \cdot)}[h](x, t)$  be the functional defined in (14).

1. There exists a positive constant  $K$  such that

$$|\gamma_n^{(\rho, \beta, \cdot)}[h](x, t)| \leq K m_{(\rho\sqrt{n}, \beta, \cdot)}(|h|) \sqrt{nt}.$$

2. If  $m_{(\sqrt{n}\rho, \beta, \cdot)}(h) = 0$  then there exists a positive constant  $K$

$$\gamma_n^{(\rho, \beta, \cdot)}[h](x, t) \leq K m_{(\cdot, \beta, \cdot)}(h) (1 + \max(0, \log(nt))).$$

## 5 A preliminary approximation of the local time

In this section we establish a preliminary local time approximation result based on the Tanaka formula. We prove also a property of this approximation.

The statistic used for our first approximation of the local time is the rescaled mean absolute displacement. It is defined as follows.

Let  $\rho \geq 0$  and  $\beta \in [-1, 1]$ . Let us consider the sequence of functions  $(\widehat{g}_n)_n$  defined, for all  $n \in \mathbb{N}$  and  $x \in \mathbb{I}$ , by

$$\widehat{g}_n(x) = \mathbb{E}_x (|X_1^{(\rho\sqrt{n}, \beta, \cdot)}| - |x|). \quad (17)$$

PROPOSITION 5.1 (local time approximation). *The following convergence holds:*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \widehat{g}_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{\text{P-u.c.p.}} L_t^0(X^{(\rho, \beta, \cdot)}).$$

PROOF. Since the process  $(X_t^{(\rho, \beta, \cdot)})_{t \geq 0}$  is a semi-martingale, the Tanaka formula (see e.g. Itô-Tanaka formula in Lemma A.1) ensures that

$$|X_t^{(\rho, \beta, \cdot)}| - |x| = \int_0^t \text{sgn}(X_s^{(\rho, \beta, \cdot)}) \mathbb{1}_{\{X_s \neq 0\}} dB_s + L_t^0(X^{(\rho, \beta, \cdot)}).$$

Since  $\int_0^t \text{sgn}(X_s^{(\rho, \beta, \cdot)}) \mathbb{1}_{\{X_s \neq 0\}} dB_s$  is a martingale,

$$\mathbb{E}_x (|X_t^{(\rho, \beta, \cdot)}| - |X_s^{(\rho, \beta, \cdot)}|) = \mathbb{E}_x [L_t^0(X^{(\rho, \beta, \cdot)}) - L_s^0(X^{(\rho, \beta, \cdot)})].$$

Also, from the scaling property (Corollary B.3),

$$\frac{1}{\sqrt{n}} \widehat{g}_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) = \mathbb{E}_x [ |X_{\frac{i}{n}}^{(\rho, \beta, \cdot)}| - |X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}| | \mathcal{F}_{\frac{i-1}{n}} ].$$

Thus, from [9, Lemma 2.14],

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \widehat{g}_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) = \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}_x \left[ L_{\frac{i}{n}}^0(X^{(\rho, \beta, \cdot)}) - L_{\frac{i-1}{n}}^0(X^{(\rho, \beta, \cdot)}) \mid \mathcal{F}_{\frac{i-1}{n}} \right] \xrightarrow[n \rightarrow \infty]{P_x} L_t^0(X^{(\rho, \beta, \cdot)})$$

and, from Lemma 2.2, the convergence is locally uniform in time, in probability (u.c.p.).  $\square$

We now provide a property of  $(\widehat{g}_n)_n$ , that is useful in the next section.

**PROPOSITION 5.2.** *Let  $(\widehat{g}_n)_n$  be the sequence of functions defined in (17). Then,*

$$m_{(\rho\sqrt{n}, \beta, \cdot)}(\widehat{g}_n) \xrightarrow[n \rightarrow \infty]{} 1.$$

**PROOF.** For simplicity, let  $m_n := m_{(\rho\sqrt{n}, \beta, \cdot)}$ . We observe that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (y-x) \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}} dy dx = \frac{1}{2}, \quad \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} x \frac{e^{-(y+x)^2/2}}{\sqrt{2\pi}} dy dx = \frac{1}{4}. \quad (18)$$

We first consider the case  $\rho = 0$ . Then,

$$m_{(\rho\sqrt{n}, \beta, \cdot)} = m_{(\cdot, \beta, \cdot)}, \quad \widehat{g}_n = \widehat{g} : x \mapsto \mathbb{E}_x (|X_1^{(\cdot, \beta, \cdot)}| - |x|).$$

Thus, from (15)-(16),

$$\begin{aligned} m_n(\widehat{g}_n) &= \int_{\mathbb{R}} \widehat{g}(x) m_{(\cdot, \beta, \cdot)}(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(\cdot, \beta, \cdot)}(1, x, y) m_{(\cdot, \beta, \cdot)}(dy) m_{(\cdot, \beta, \cdot)}(dx) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_1(1, x, y) 4a(x) dy dx - \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) 4a(x) dy dx \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) 4a(x) a(y) dy dx. \end{aligned}$$

From (18) and since for all  $t > 0$  and  $x, y \in \mathbb{R}$ ,  $u_2(t, x, y) = u_2(t, y, x)$ ,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_1(1, x, y) 4a(x) dy dx &= 1, \\ \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) 4a(x) dy dx &= 0, \\ \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) 4a(x) a(y) dy dx &= 0. \end{aligned} \quad (19)$$

Thus,  $m_n(\widehat{g}_n) = 1$ .

We now suppose that  $\rho > 0$ . Then,

$$m_n(\widehat{g}_n) = \int_{\mathbb{R}} \widehat{g}_n(x) m_n(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(\sqrt{n}\rho, \beta, \cdot)}(1, x, y) m_n(dy) m_n(dx).$$

Since  $m_n$  is defined for all  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$  by  $m_n(dx) = 2a(x)dx + \rho\sqrt{n}\delta_0(dx)$ ,

$$m_n(\widehat{g}_n) = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(\sqrt{n}\rho, \beta, \cdot)}(1, x, y) 4a(x)a(y) dy dx \\ + \sqrt{n}\rho \left( \int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta, \cdot)}(1, 0, y) 2a(y) dy - \int_{\mathbb{R}} |x| p_{(\sqrt{n}\rho, \beta, \cdot)}(1, x, 0) 2a(x) dx \right). \quad (20)$$

We now show that the last additive term of the right-hand-side of (20) vanishes. Indeed, we observe that  $\int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta, \cdot)}(t, 0, y) a(y) dy < \infty$  and that for all  $t > 0$  and  $x, y \in \mathbb{R}$ ,  $p_{(\sqrt{n}\rho, \beta, \cdot)}(t, x, y) = p_{(\sqrt{n}\rho, \beta, \cdot)}(t, y, x)$ . Thus,

$$\int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta, \cdot)}(t, 0, y) a(y) dy = \int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta, \cdot)}(t, y, 0) a(y) dy$$

and

$$\sqrt{n}\rho \left( \int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta, \cdot)}(1, 0, y) 2a(y) dy - \int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta, \cdot)}(1, y, 0) 2a(y) dy \right) = 0.$$

For the first additive term of the right-hand-side of (20), from (15),

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(\sqrt{n}\rho, \beta, \cdot)}(1, x, y) 4a(x)a(y) dy dx \\ = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) (u_1(1, x, y) - u_2(1, x, y)) 4a(x) dy dx \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) v_{\sqrt{n}\rho}(1, x, y) 4a(x)a(y) dy dx.$$

We observe that  $v_{\sqrt{n}\rho}(t, x, y)$  vanishes as  $n \rightarrow \infty$ . Also, from Lemma 4.3,

$$(|y| - |x|) v_{\sqrt{n}\rho}(t, x, y) a(x)a(y) \leq K(|y| + |x|) u_1(t, x, y)$$

which is an integrable function with respect to  $(x, y) \in \mathbb{R}^2$ . Thus, from Lebesgue's convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) v_{\sqrt{n}\rho}(1, x, y) a(x)a(y) dy dx = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} m_n(\widehat{g}_n) = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) (u_1(1, x, y) - u_2(1, x, y)) 4a(x) dy dx$$

which is 1 by (19). This proves the result.  $\square$

## 6 Proof of Theorem 2.6 for $(\rho, \beta, \cdot)$ -SOS-BM

In this section we prove Theorem 2.6 in the special case of sticky-skew Brownian motion, stated in Theorem 6.1. Throughout this section,  $X^{(\rho, \beta, \cdot)}$  is a  $(\rho, \beta)$ -SOS-BM with  $\rho \geq 0$  and  $\beta \in [-1, 1]$  defined on the probability space  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P_x)$  such that  $X_0^{(\rho, \beta, \cdot)} = x \in \mathbb{I}$  (recall  $\mathbb{I} = \mathbb{R}$  if  $|\beta| \neq 1$ ),  $P_x$ -almost surely.



THEOREM 6.1 (Local time approximation for sticky-skew Brownian motion).

$$\frac{u_n}{n} \sum_{i=1}^{\lfloor nt \rfloor} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-u.c.p.}} \left( \int_{\mathbb{R}} g \, dm_{(\rho, \beta, \cdot)} \right) L_t^0(X^{(\rho, \beta, \cdot)}),$$

where  $m_{(\rho, \beta, \cdot)}$  is the speed measure of  $X^{(\rho, \beta, \cdot)}$  given by (2) with  $\sigma_0 \equiv 1$ .

We prove Theorem 6.1 considering separately normalizing sequences with different modes of convergence.

1. The mode  $(u_n)_n : n^{1/2}$ . In this case we prove a more general result, Proposition 6.2. The proof relies on linearization of the statistic with respect to the approximation in Lemma 5.1.
2. The mode  $(u_n)_n : \log n / u_n \rightarrow 0, u_n \rightarrow \infty$ . We prove Proposition 6.7 via rescaling to the mode  $n^{1/2}$ . This case includes the case  $u_n = n^\alpha$  with  $\alpha \in (0, 1)$  (and so, also  $\alpha = 1/2$ ) but excludes logarithmic cases.
3. The mode  $(u_n)_n : u_n^2 / n \rightarrow 0, u_n \rightarrow \infty$ . We prove Proposition 6.9 using properties of the local time of the process, namely its continuity on each half-plane and its time-scaling (Lemma B.2).

## 6.1 The case: $u_n = n^{1/2}$

PROPOSITION 6.2. Let  $(g_n)_n$  be a sequence of function such that for all  $x \in \mathbb{I}$

$$\lim_{n \rightarrow \infty} \left( \frac{g_n^2(\sqrt{n}x)}{n} + \frac{m_{(\rho\sqrt{n}, \beta, \cdot)}(g_n^2)}{\sqrt{n}} + \frac{m_{(\cdot, \beta, \cdot)}^{(1)}(g_n)(1 + \log(n))g_n(\sqrt{n}x)}{n} + \frac{(1 + \log(n))m_{(\rho\sqrt{n}, \beta, \cdot)}(|g_n|)}{\sqrt{n}} \right) = 0 \quad (21)$$

and  $\lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta, \cdot)}(g_n) = M$ . Then, for all  $t \geq 0$ ,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \rightarrow M L_t^0(X^{(\rho, \beta, \cdot)}) \quad (22)$$

in  $P_x$ -probability. Also, if  $\sup_n (m_{(\sqrt{n}\rho, \beta, \cdot)}(|g_n|)) < \infty$  then the above convergence is locally uniform in time, in  $P_x$ -probability.

REMARK 6.3. We observe that if two sequences of functions  $(f_n)_n, (g_n)_n$  satisfy (21), then the sequence  $(f_n + g_n)_n$  also satisfies (21).

Before proving Proposition 6.2, we need to provide some other results which are relevant on their own.

LEMMA 6.4. Let  $(g_n)_n$  be a sequence of function such that for all  $x \in \mathbb{R}$ , as  $n \rightarrow \infty$ ,  $m_{(\sqrt{n}\rho, \beta, \cdot)}(|g_n|) + g_n(\sqrt{n}x)/\sqrt{n} \rightarrow 0$ . Then,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \rightarrow 0$$

locally uniformly in time, in  $L^1(\mathbb{P}_x)$ .

PROOF. We observe that

$$\mathbb{E}_x \left( \sup_{s \in [0, t]} \left| \sum_{i=1}^{\lfloor ns \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right| \right) \leq |g_n(x\sqrt{n})| + \gamma_n[|g_n|](x, t).$$

Thus, from Item (1) in Lemma 4.6,

$$\mathbb{E}_x \left( \sup_{s \in [0, t]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right| \right) \leq \frac{|g_n(x\sqrt{n})|}{\sqrt{n}} + K m_{(\sqrt{n}\rho, \beta, \cdot)}(|g_n|) \sqrt{t}$$

which completes the proof.  $\square$

The next result is an adaptation to this context of [10, Lemma 4.2].

LEMMA 6.5. *Let  $(g_n)_n$  be a sequence of functions that such that (21) holds and for all  $n \in \mathbb{N}, x \in \mathbb{R}$ :  $m_{(\sqrt{n}\rho, \beta, \cdot)}(g_n) = 0$ . Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_x \left( \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right|^2 \right) = 0.$$

PROOF. It holds that

$$\begin{aligned} \mathbb{E}_x \left( \left| \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right|^2 \right) &\leq g_n^2(\sqrt{n}x) + \gamma_n[g_n^2](t, x) \\ &\quad + 2 \sup_{y \in \mathbb{R}; s \leq t} \left\{ |\gamma_n[g_n](s, y)| \right\} \left( g_n(\sqrt{n}x) + \gamma_n[g_n](t - 1/n, x) \right). \end{aligned}$$

From Lemma 4.6, for some  $K_t > 0$ ,

$$\begin{aligned} \sup_{y \in \mathbb{R}; s \leq t} \left\{ |\gamma_n[g_n](s, y)| \right\} &\leq K_t m_{(\cdot, \beta, \cdot)}^{(1)}(g_n) (1 + \log(n)), \\ \gamma_n[g_n](t, x) &\leq K_t m_{(\sqrt{n}\rho, \beta, \cdot)}(|g_n|) \sqrt{n}, \\ \gamma_n[g_n^2](t, x) &\leq K_t m_{(\sqrt{n}\rho, \beta, \cdot)}(g_n^2) \sqrt{n}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}_x \left( \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right|^2 \right) &\leq K_t \left( \frac{g_n^2(\sqrt{n}x)}{n} + \frac{m_{(\rho\sqrt{n}, \beta, \cdot)}(g_n^2)}{\sqrt{n}} \right. \\ &\quad \left. + \frac{m_{(\cdot, \beta, \cdot)}^{(1)}(g_n) (1 + \log(n)) g_n(\sqrt{n}x)}{n} + \frac{(1 + \log(n)) m_{(\rho\sqrt{n}, \beta, \cdot)}(|g_n|)}{\sqrt{n}} \right), \end{aligned}$$

which converges to 0 as  $n \rightarrow \infty$ .  $\square$

LEMMA 6.6. *The sequence of functions  $(\widehat{g}_n)_n$  in (17) satisfies (21). Moreover,*

$$\lim_{n \rightarrow \infty} \widehat{g}_n(0) = 0.$$

PROOF. We observe that  $s(X^{(\rho, \beta, \cdot)})$  is a martingale and  $x \mapsto |s^{-1}(x)|$  is convex. Thus,  $|X^{(\rho, \beta, \cdot)}| = |s^{-1} \circ s(X^{(\rho, \beta, \cdot)})|$  is a supermartingale. From [21, p.277] or [23], for all  $n$ , there exists a skew Brownian motion  $X^{(\cdot, \beta, \cdot)}$  and a time change  $\gamma_n$  defined via its right-inverse by

$$\gamma_n^{-1}(t) = t + \frac{\rho\sqrt{n}}{2} L_t^0(X^{(\cdot, \beta, \cdot)}).$$

We observe that  $\gamma_n^{-1}(0) = 0$ ,  $\gamma_n^{-1}$  is strictly increasing and that for all  $t > 0$ ,  $t \leq \gamma_n^{-1}(t)$ . Hence, for all  $t > 0$ ,  $0 < \gamma_n(t) \leq t$  and

$$\begin{aligned} 0 \leq \widehat{g}_n(x) &= \mathbb{E}_x (|X_1^{(\rho\sqrt{n}, \beta, \cdot)}| - |x|) = \mathbb{E}_x (|X_{\gamma_n^{-1}(1)}^{(\cdot, \beta, \cdot)}| - |x|) \\ &\leq \mathbb{E}_x (|X_1^{(\cdot, \beta, \cdot)}| - |x|) =: G_\beta(x) \leq K \mathbb{E}_x (|B_1| - |x|) =: K\widehat{g}(x). \end{aligned}$$

Since, from [10]  $\widehat{g}$  satisfies (21), to show that  $\widehat{g}_n$  does too it remains to show that if  $n \rightarrow \infty$

$$\widehat{g}_n^2(0) \rightarrow 0, \quad (1 + \log n)\widehat{g}_n(0) \rightarrow 0. \quad (23)$$

Indeed, since  $0 < a(y) \leq 1$ ,

$$\begin{aligned} \widehat{g}_n(0) &= \mathbb{E}_0 (|X_1^{(\rho\sqrt{n}, \beta, \cdot)}|) = \int_{\mathbb{I}} |y| p_{(\rho\sqrt{n}, \beta, \cdot)}(1, 0, y) m_{(\rho\sqrt{n}, \beta, \cdot)}(dy) \\ &\leq 2 \int_{\mathbb{R}} |y| v_{\rho\sqrt{n}}(1, 0, y) dy = \frac{2}{\rho\sqrt{n}} \int_{\mathbb{R}} |y| e^{4|y|/\rho\sqrt{n} + 8/\rho^2 n} \operatorname{erfc}\left(\frac{|y|}{\sqrt{2}} + \frac{2\sqrt{2}}{\rho\sqrt{n}}\right) dy \end{aligned}$$

hence  $((1 + \log n)\widehat{g}_n(0)) \rightarrow 0$  as  $n \rightarrow \infty$ . Similarly,

$$\begin{aligned} (\widehat{g}_n(0))^2 &\leq \mathbb{E}_0 (|X_1^{(\rho\sqrt{n}, \beta, \cdot)}|^2) \leq 2 \int_{\mathbb{R}} |y|^2 v_{\rho\sqrt{n}}(1, 0, y) dy \\ &= \frac{2}{\rho\sqrt{n}} \int_{\mathbb{R}} |y|^2 e^{4|y|/\rho\sqrt{n} + 8/\rho^2 n} \operatorname{erfc}\left(\frac{|y|}{\sqrt{2}} + \frac{2\sqrt{2}}{\rho\sqrt{n}}\right) dy, \end{aligned}$$

hence  $\widehat{g}_n^2(0) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus, (23) is satisfied. This finishes the proof.  $\square$

PROOF OF PROPOSITION 6.2. Let  $(\widehat{g}_n)_{n \in \mathbb{N}}, (h_n)_{n \in \mathbb{N}}$  be the sequences of functions defined for all  $x \in \mathbb{R}$ ,  $n \in \mathbb{N}$  by

$$h_n(x) = g_n(x) - \frac{m_{(\sqrt{n}\rho, \beta, \cdot)}(g_n)}{m_{(\sqrt{n}\rho, \beta, \cdot)}(\widehat{g}_n)} \widehat{g}_n(x), \quad \widehat{g}_n(x) = \mathbb{E}_x (|X_1^{(\rho\sqrt{n}, \beta, \cdot)}| - |x|).$$

Note that  $\widehat{g}_n$  was already defined in (17).

From Proposition 5.2, Remark 6.3 and Lemma 6.6,  $(h_n)_n$  satisfies (21). Also, for all  $n \in \mathbb{N}$ ,  $m_{(\sqrt{n}\rho, \beta, \cdot)}(h_n) = 0$ . Thus, from Lemma 6.5,

$$\mathbb{E}_x \left( \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} h_n(\sqrt{n}X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right|^2 \right) \xrightarrow[n \rightarrow \infty]{} 0$$

which proves (22).

It remains to prove that the convergence is locally uniform in time, in  $P_x$ -probability. If  $(g_n)_n$  are all positive, the processes  $\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} g_n(\sqrt{n}X_{\frac{i-1}{n}}^{(\rho, \cdot, \sigma_0)})$  are non-decreasing with  $P_x$ -almost surely, a continuous limit. Thus, from Lemma 2.2, the convergence is locally uniform in time, in probability (u.c.p.). More precisely, for an arbitrary  $(g_n)_n$  satisfying the conditions of Proposition 6.2, let  $g_n = g_n^+ - g_n^-$ , where  $g_n^+(x) = \max\{g_n(x), 0\}$  and  $g_n^-(x) = \max\{-g_n(x), 0\}$ . Since  $(g_n^+)_n$  and  $(g_n^-)_n$  are both sequences of positive functions, if  $\lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta, \cdot)}(g_n^+) =: M_+$  and  $\lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta, \cdot)}(g_n^-) =: M_-$ , then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} g_n^+(n^\alpha X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-u.c.p.}} M_+ L_t^0(X), \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} g_n^-(n^\alpha X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-u.c.p.}} M_- L_t^0(X).$$

From the triangle inequality for the absolute value and the  $L^\infty(0, t)$ -norm, the local uniform convergence in time, in  $P_x$ -probability of (22) is proven.  $\square$

## 6.2 A case including $u_n = n^\alpha$ , $\alpha \in (0, 1)$

PROPOSITION 6.7. *Theorem 6.1 holds under the additional assumption that*

$$\log n/u_n \xrightarrow[n \rightarrow \infty]{} 0.$$

The proof relies on the following result.

PROPOSITION 6.8. *Let  $(g_n)_n, (u_n)_n$  be respectively a sequence of functions and a sequence of real numbers such that*

1. *if as  $n \rightarrow \infty$ :  $\log n/u_n \rightarrow 0$ ,  $u_n \rightarrow \infty$  and  $u_n/n \rightarrow 0$ ,*
2.  *$(g_n)_n$  a sequence of functions such that*

$$\begin{aligned} \sup \lambda(|g_n|) &< \infty, & \sup_{n,x} |g_n(x)| &< \infty, \\ \lim_{n \rightarrow \infty} m_{(\cdot, \beta, \cdot)}(g_n) &=: M, & \lim_q \limsup_n \int_{|x|>q} |g_n(x)| dx &= 0. \end{aligned}$$

Then,

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-u.c.p.}} M L_t^0(X^{(\rho, \beta, \cdot)}).$$

PROOF. Let  $(k_n)_n$  be the sequence of functions defined for all  $x \in \mathbb{R}$  by  $k_n(x) = \frac{u_n}{\sqrt{n}} g_n\left(\frac{u_n}{\sqrt{n}}x\right)$ . Let  $\sup_n \lambda^{(1)}(g_n) < \infty$ . Then, for some  $K > 0$ ,

$$\begin{aligned} |k_n(x)| &\leq K \frac{u_n}{\sqrt{n}}, & m_{(\cdot, \beta, \cdot)}(|k_n|) &= m_{(\cdot, \beta, \cdot)}(|g_n|) \leq K, \\ m_{(\cdot, \beta, \cdot)}^{(1)}(k_n) &= \frac{\sqrt{n}}{u_n} m_{(\cdot, \beta, \cdot)}^{(1)}(g_n) \leq \frac{\sqrt{n}}{u_n} K, & \lambda(k_n^2) &= \frac{u_n}{\sqrt{n}} \lambda(g_n^2) \leq \frac{u_n}{\sqrt{n}} K. \end{aligned}$$

We observe that from the hypothesis we made on  $(u_n)_n$ ,  $(k_n)_n$  satisfies (21). Thus, from Proposition 6.2,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} k_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{\text{P}_x\text{-u.c.p.}} ML_t^0(X^{(\rho, \beta, \cdot)}). \quad (24)$$

In the case  $\sup_n \lambda^{(1)}(g_n) = \infty$ , for all  $r \geq 1$  consider  $h_{n,r} := g_n \mathbb{1}_{(-r, r)}$ . Then,  $m_{(\cdot, \beta, \cdot)}^{(1)}(h_{n,r}) \leq r m_{(\cdot, \beta, \cdot)}(|g_n|) < \infty$ . Moreover, for all  $r$ ,  $m_{(\cdot, \beta, \cdot)}(h_{n,r}) \rightarrow \alpha_r$  as  $n \rightarrow \infty$  (up to considering a subsequence) for  $\alpha_r$  such that  $\alpha_r \rightarrow M$  as  $r \rightarrow \infty$ . Hence, from (24),

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} h_{n,r}(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{\text{P}_x\text{-u.c.p.}} \alpha_r L_t^0(X^{(\rho, \beta, \cdot)}).$$

Since  $\alpha_r \rightarrow M$  as  $r \rightarrow \infty$ , it remains to show that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_x \left( \sup_{s \in [0, t]} \left| \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) - \frac{u_n}{n} \sum_{i=1}^{[nt]} h_{n,r}(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right| \right) = 0. \quad (25)$$

If  $\bar{h}_{n,r}(x) = g_n(x) - h_{n,r}(x) = g_n(x) \mathbb{1}_{|x| > r}$ , then the expectation in (25) is bounded by

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} \mathbb{E}_x \left( |\bar{h}_{n,r}(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)})| \right) \leq \frac{u_n}{n} \bar{h}_{n,r}(u_n x) + \frac{1}{\sqrt{n}} \gamma_n[\tilde{h}_{n,r}](x, t),$$

where  $\tilde{h}_{n,r} = \frac{u_n}{\sqrt{n}} \bar{h}_{n,r}\left(\frac{u_n}{\sqrt{n}}x\right) = |k_n| \mathbb{1}_{|x| > r\sqrt{n}/u_n}$  and  $\gamma_n[\tilde{h}_{n,r}](x, t)$  defined in (14). Since  $|k_n|$  is bounded by  $K u_n / \sqrt{n}$ ,

$$\lim_{n \rightarrow \infty} \frac{u_n}{n} \bar{h}_{n,r}(u_n x) = 0.$$

It remains to check that for all  $x \in \mathbb{R}$

$$\limsup_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \gamma_n[\tilde{h}_{n,r}](x, t) = 0.$$

Indeed, since  $\tilde{h}_{n,r}(0) = 0$ , Lemma 4.6 yields

$$\frac{1}{\sqrt{n}} \gamma_n[\tilde{h}_{n,r}](x, t) \leq K m_{(\cdot, \beta, \cdot)}(|\tilde{h}_{n,r}|) \sqrt{t},$$

which converges to 0 as  $r \rightarrow \infty$ . □

PROOF OF PROPOSITION 6.7. We observe

$$\sup_{n,x} g_n[T](x) \leq \|g\|_\infty, \quad \lambda(g_n[T]) \leq \lambda(g)\|1/T\|_\infty \leq \lambda(g)/\varepsilon,$$

$$\lim_{n \rightarrow \infty} m_{(\cdot, \beta, \cdot)}(g_n[T]) = m_{(\cdot, \beta, \cdot)}(g).$$

Also,

$$\begin{aligned} \int_{|x|>q} |g_n[T](x)| dx &= \int_{|x|>\frac{q}{u_n}} g(u_n T(x)) u_n dx \\ &\leq \frac{1}{\varepsilon} \int_{u_n|x|>q} g(u_n x) u_n dx \leq \frac{1}{\varepsilon} \int_{|x|>q} g(x) dx, \end{aligned}$$

which converges to 0 as  $q \rightarrow \infty$ . Since  $g(0) = 0$  and  $m_{(\cdot, \beta, \cdot)}$  is equal to  $\lambda$  up to a multiplicative bounded weight, the conditions of Proposition 6.8 are satisfied for  $(g_n)_n = (g_n[T])_n$ . The proof is thus completed.  $\square$

### 6.3 A case including $u_n = \log n$

Sections 6.1-6.2 are an improvement with respect to [1, Theorem 1.2] because they allow to consider  $n^\alpha$  for  $\alpha \in [1/2, 1)$  for sticky-BM. They are also an improvement with respect to the results for skew BM in [15] and oscillating-skew BM in [18] where only  $u_n = \sqrt{n}$  was considered. But, the results of Sections 6.1-6.2 do not consider  $(u_n)_n$  with log  $n$ -like asymptotic behaviors.

In this section we prove that the result holds for  $(u_n^2)_n$  that satisfy (4), *i.e.*

$$u_n^2/n \xrightarrow{n \rightarrow \infty} 0, \quad u_n^2 \xrightarrow{n \rightarrow \infty} 0. \quad (26)$$

PROPOSITION 6.9. *Theorem 6.1 holds under the additional assumption that  $(u_n)_n$  satisfies (26).*

For this, we use the following preliminary results.

LEMMA 6.10. *If  $(u_n)_n$  is a diverging sequence and  $T$  a function that satisfies (6), then the following convergence holds*

$$\int_{\mathbb{R}} g_n[T](x) \ell_{[nt]/n}^{(x/u_n)}(X^{(\rho, \beta, \cdot)}) dx \xrightarrow[n \rightarrow \infty]{P_x} m_{(\rho, \beta, \cdot)}(g) L_t^0(X^{(\rho, \beta, \cdot)}).$$

PROOF. Note that there exists a skew Brownian motion  $X^{(\cdot, \beta, \cdot)}$  defined on an extension of the probability space such that  $X^{(\rho, \beta, \cdot)} = (X_{\gamma(t)}^{(\cdot, \beta, \cdot)})_{t \geq 0}$ , where  $\gamma$  is the time-change defined by its right-inverse  $\gamma^{-1}(t) = t + \frac{\rho}{2} \ell_t^{0+}(X^{(\cdot, \beta, \cdot)})$  (cf. [23]).

The skew Brownian motion  $X^{(\cdot, \beta, \cdot)}$  is the unique strong solution to the following equation

$$X^{(\cdot, \beta, \cdot)} = B'_t + \beta L_t^0(X^{(\cdot, \beta, \cdot)}),$$

where  $B'$  is a standard Brownian motion (see [8]) and

$$\ell_t^{0+}(X^{(\cdot,\beta,\cdot)}) = (1 + \beta)L_t^0(X^{(\cdot,\beta,\cdot)}), \quad \ell_t^{0-}(X^{(\cdot,\beta,\cdot)}) = (1 - \beta)L_t^0(X^{(\cdot,\beta,\cdot)}).$$

We observe that  $d\langle X^{(\cdot,\beta,\cdot)} \rangle_t = dt$ . From [19, Theorem VI.1.6], the right local time field the symmetric local time  $[(t, y) \rightarrow \ell_t^y(X^{(\cdot,\beta,\cdot)})]$  is  $(t, y)$ -jointly continuous on each half-plane  $(\mathbb{R}_+ \times (-\infty, 0))$ ,  $(\mathbb{R}_+ \times [0, \infty))$ . From [19, Exercice VI.1.27],  $\ell_t^0(X^{(\rho,\beta,\cdot)}) = \ell_{\gamma(t)}^0(X^{(\cdot,\beta,\cdot)})$ . Thus, since  $\gamma^{-1}$  is almost surely continuous,  $t \mapsto \ell_t^y(X^{(\rho,\beta,\cdot)})$  is also  $(t, y)$ -jointly continuous on each half-plane and

$$\ell_t^{0+}(X^{(\rho,\beta,\cdot)}) = (1 + \beta)L_t^0(X^{(\rho,\beta,\cdot)}), \quad \ell_t^{0-}(X^{(\rho,\beta,\cdot)}) = (1 - \beta)L_t^0(X^{(\rho,\beta,\cdot)}). \quad (27)$$

From the *a.s.* piece-wise continuity of  $(t, y) \rightarrow \ell_t^y(X^{(\rho,\beta,\cdot)})$ , for all  $t \geq 0$  and  $x > 0$ ,

$$|\ell_{[nt]/n}^{(x/u_n)}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0+}(X^{(\rho,\beta,\cdot)})| \xrightarrow[n \rightarrow \infty]{a.s.} 0, \quad |\ell_{[nt]/n}^{-(x/u_n)}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0-}(X^{(\rho,\beta,\cdot)})| \xrightarrow[n \rightarrow \infty]{a.s.} 0. \quad (28)$$

We observe that almost surely

$$-\infty < \inf_{s \leq t} X_s < \sup_{s \leq t} X_s < \infty. \quad (29)$$

Thus, from the Hölder continuity of  $[t \rightarrow \ell_t^{x-}(X^{(\rho,\beta,\cdot)})]$ ,  $[x \rightarrow \ell_t^{x+}(X^{(\rho,\beta,\cdot)})]$  on  $[0, \infty) \times (-\infty, 0)$  and  $[0, \infty) \times [0, \infty)$ , there exists positive finite random variables  $U_-, U_+$  such that, almost surely, for all  $x \neq 0$  and  $n \in \mathbb{N}$ ,

$$|\ell_{[nt]/n}^{-(x/u_n)}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0-}(X^{(\rho,\beta,\cdot)})| \leq U_-, \quad |\ell_{[nt]/n}^{(x/u_n)}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0+}(X^{(\rho,\beta,\cdot)})| \leq U_+.$$

Let us fix an event  $\Omega_0$  of full probability such that the above convergences (28) hold and such that the process  $X$  is finite, i.e. (29) holds for all  $\omega \in \Omega_0$ .

We now prove that, on  $\Omega_0$ ,

$$\int_0^\infty g_n[T](x) \ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) dx \xrightarrow[n \rightarrow \infty]{} \ell_t^{0+}(X^{(\rho,\beta,\cdot)}) \int_0^\infty g(x) dx$$

and

$$\int_{-\infty}^0 g_n[T](x) \ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) dx \xrightarrow[n \rightarrow \infty]{} \ell_t^{0-}(X^{(\rho,\beta,\cdot)}) \int_{-\infty}^0 g(x) dx.$$

We focus on the first convergence. The second convergence can be proven with similar arguments.

From dominated convergence, since  $|T'|$  is bounded and  $g$  integrable, we can easily show that  $\int_0^\infty g_n[T](x) dx$  converges towards  $\int_0^\infty g(x) dx$ . Hence, it remains to prove that on  $\Omega_0$

$$\int_0^\infty (g_n[T](x) - g(x)) (\ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0+}(X^{(\rho,\beta,\cdot)})) dx \xrightarrow[n \rightarrow \infty]{} 0$$

and

$$\int_0^\infty g(x) (\ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0+}(X^{(\rho,\beta,\cdot)})) dx \xrightarrow[n \rightarrow \infty]{} 0.$$

This holds by dominated convergence. Indeed, as  $g$  is bounded for all  $q > 0$ , it holds on  $\Omega_0$

$$\begin{aligned} & \left| \int_0^\infty (g_n[T](x) - g(x)) (\ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0+}(X^{(\rho,\beta,\cdot)})) dx \right| \\ & \leq 2\|g\|_\infty \int_{0 < x \leq q} |\ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) - \ell_t^{0+}(X^{(\rho,\beta,\cdot)})| dx + U_+ \int_{x > q} |g_n[T](x) - g(x)| dx \\ & \leq 2\|g\|_\infty U_+ q + 2U_+ \sup_{x \in \mathbb{R}} |T'(x)|^{-1} \int_q^\infty |g(x)| dx < \infty. \end{aligned}$$

Thus, from dominated convergence yields

$$\int_0^\infty g_n[T](x) \ell_{[nt]/n}^{(x/u_n)}(X^{(\rho,\beta,\cdot)}) dx \xrightarrow[n \rightarrow \infty]{a.s.} \left( \int_0^\infty g(x) dx \right) \ell_t^{0+}(X^{(\rho,\beta,\cdot)}).$$

With similar arguments on  $(-\infty, 0)$ , we have

$$\begin{aligned} & \int_{\mathbb{R}} g_n[T](x) \ell_{[nt]/n}^{(x/u_n)}(X^{(\rho,\beta,\cdot)}) dx \\ & \xrightarrow[n \rightarrow \infty]{a.s.} \left( \int_{-\infty}^0 g(x) dx \right) \ell_t^{0-}(X^{(\rho,\beta,\cdot)}) + \left( \int_0^\infty g(x) dx \right) \ell_t^{0+}(X^{(\rho,\beta,\cdot)}). \end{aligned}$$

This and (27) completes the proof.  $\square$

LEMMA 6.11. *Let  $k$  be a bounded Lipschitz function with compact support that vanishes on an open interval around 0,  $(u_n)_n$  be a sequence that satisfies (26),  $T$  a function that satisfies (6) and  $(k_n[T])_n, (F_n[k])_n$  be the sequences of functions defined for all  $x \in \mathbb{R}$ ,  $n \in \mathbb{N}$  by*

$$k_n[T](x) = k(u_n T(x/u_n)), \quad F_n[k](x) = \int_0^1 (P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](x) - k_n[T](x)) ds.$$

Then,

$$m_{(u_n \rho, \beta, \cdot)}(F_n[k]) \xrightarrow[n \rightarrow \infty]{} 0.$$

PROOF. For all  $\delta > 0$ , let  $U_\delta = \{x \in \mathbb{R} : |x| \in (\delta, 1/\delta)\}$ . We choose  $\delta > 0$  such that for all  $x \notin U_\delta$ ,  $k(x) = 0$ . We observe that there exists some  $n_0$  such that, for all  $n \geq n_0$  and  $|x| \notin U_{\delta/2}$ ,  $k_n[T](x) = 0$ .

In what follows, we study separately the absolutely continuous and singular parts of  $m_{(u_n \rho, \beta, \cdot)}$ . By Fubini's theorem,

$$\begin{aligned} |m_{(\cdot, \beta, \cdot)}(F_n[k])| &= \left| \int_{\mathbb{R}} \left( \int_0^1 (P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](x) - k_n[T](x)) (ds) \right) m_{(\cdot, \beta, \cdot)}(dx) \right| \\ &\leq \int_0^1 \left( \int_{\mathbb{R}} |P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](x) - k_n[T](x)| m_{(\cdot, \beta, \cdot)}(dx) \right) ds. \end{aligned}$$

Also,

$$|F_n[k](0)| \leq \int_0^1 |P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](0) - k_n[T](0)| ds.$$



Lemma 4.3 yields

$$\begin{aligned}
& |P_{su_n^2/n}^{(u_n\rho,\beta)} k_n[T](x) - k_n[T](x)| \\
& \leq \int_{\mathbb{R}} |k_n[T](y) - k_n[T](x)| p_{(u_n\rho,\beta,\cdot)}(su_n^2/n, x, y) m_{(u_n\rho,\beta,\cdot)} dy \\
& \leq K \int_{\mathbb{R}} |k_n[T](y) - k_n[T](x)| u_1(su_n^2/n, x, y) (dy) + K |k_n[T](x)| u_1(su_n^2/n, x, 0). \quad (30)
\end{aligned}$$

Regarding the second additive term at the right-hand-side of (30),

- for all  $x \in \mathbb{R}$ ,

$$\lim_{n \rightarrow \infty} K |k_n[T](x)| u_1(su_n^2/n, x, 0) = 0,$$

- from the convergence on  $u_n^2/n$  to 0, there exists some  $n_1 \geq n_0$  such that for all  $n \geq n_1$ ,

$$\begin{aligned}
& K |k_n[T](x)| u_1(su_n^2/n, x, 0) \leq K \|k\|_{\infty} u_1(su_n^2/n, 0, x) \mathbb{1}_{U_{\delta/2}} \\
& \leq K \|k\|_{\infty} \sup_{|x| > \delta/2; n \geq n_1} \{u_1(su_n^2/n, 0, x)\} \mathbb{1}_{U_{\delta/2}} \leq K \|k\|_{\infty} u_1(u_{n_1}^2/n_1, 0, \delta/2) \mathbb{1}_{U_{\delta/2}}
\end{aligned}$$

which is  $L^1(\mathbb{R})$ .

Thus, from dominated convergence,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} K |k_n[T](x)| u_1(su_n^2/n, 0, x) m_{(\rho,\beta,\cdot)}(dx) = 0.$$

Regarding the second additive term, if  $Z$  is a Gaussian  $\mathcal{N}(0, 1)$  random variable, then for every  $n \geq n_0$  and  $s \in [0, 1]$ ,

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{R}} |k_n[T](y) - k_n[T](x)| u_1(su_n^2/n, x, y) (dy) m_{(\cdot,\beta,\cdot)}(dx) \\
& = \mathbb{E} \left( \int_{\mathbb{R}} \left| k_n[T]\left(x + \frac{u_n\sqrt{s}}{\sqrt{n}}Z\right) - k_n[T](x) \right| m_{(\cdot,\beta,\cdot)}(dx) \right) \\
& \leq \mathbb{E} \left( \int_{\mathbb{R}} \mathbb{1}_{\{|x + \frac{u_n\sqrt{s}}{\sqrt{n}}Z| < 2/\delta\} \cup \{|x| < 2/\delta\}} \left| k_n[T]\left(x + \frac{u_n\sqrt{s}}{\sqrt{n}}Z\right) - k_n[T](x) \right| m_{(\cdot,\beta,\cdot)}(dx) \right) \\
& \leq \|k\|_{\text{Lip}} \|T\|_{\text{Lip}} \frac{u_n}{\sqrt{n}} \mathbb{E} \left( |Z| \int_{\mathbb{R}} \mathbb{1}_{\{|x| < 2/\delta + \frac{u_n}{\sqrt{n}}|Z|\}} m_{(\cdot,\beta,\cdot)}(dx) \right) \\
& \leq 4 \|k\|_{\text{Lip}} \|T\|_{\text{Lip}} \frac{u_n}{\sqrt{n}} \mathbb{E} \left( |Z| \left( \frac{2}{\delta} + \frac{u_n}{\sqrt{n}}|Z| \right) \right)
\end{aligned}$$

which, since  $E(|Z|), E(Z^2) < \infty$ , converges to 0 as  $n \rightarrow \infty$ .

Since the bounds do not depend on  $s$ , from Lebesgue's dominated convergence theorem on  $L^1([0, 1])$ ,

$$m_{(\cdot,\beta,\cdot)}(F_n[k]) \xrightarrow[n \rightarrow \infty]{} 0.$$

Regarding the singular part of the measures, from (15),

$$\begin{aligned} \rho u_n |P_{su_n^2/n}^{(u_n\rho, \beta)} k_n[T](0) - k_n[T](0)| &\leq \rho u_n K \int_{\mathbb{R}} |k_n[T](y) - k_n[T](0)| v_{u_n\rho}(su_n^2/n, 0, y) (dy) \\ &= \rho u_n K \int_{U_{\delta/2}} |k_n[T](y)| v_{u_n\rho}(su_n^2/n, 0, y) (dy). \end{aligned}$$

Thus,

$$\begin{aligned} \rho u_n \int_{U_{\delta/2}} |k_n[T](y)| v_{u_n\rho}(su_n^2/n, 0, y) dy \\ \leq \|k\|_{\infty} \int_{\delta/2}^{2/\delta} e^{4(|y|)/u_n\rho + 8u_n^2 s/nu_n^2\rho^2} \operatorname{erfc}\left(\frac{|y|}{\sqrt{2su_n^2/n}} + \frac{2\sqrt{2su_n^2/n}}{u_n\rho}\right) dy \\ \leq \|k\|_{\infty} K_{\text{Mills}} \int_{\delta/2}^{2/\delta} \frac{u_n\rho}{u_n\rho|y| + 4t} e^{-y^2n/2su_n^2} dy \leq \|k\|_{\infty} K_{\text{Mills}} \int_{\delta/2}^{2/\delta} \frac{2}{\delta} e^{-y^2n/2su_n^2} dy \end{aligned}$$

which converges to 0 as  $n \rightarrow \infty$ .

Again, since the bounds do not depend on  $s$ , from Lebesgue's dominated convergence theorem on  $L^1([0, 1])$ ,

$$\lim_{n \rightarrow \infty} (\rho u_n) F_n[k](0) = 0.$$

This finishes the proof.  $\square$

LEMMA 6.12. *Let  $X^{(\rho, \beta, \cdot)}$  be the  $(\rho, \beta)$ -sticky-skew Brownian motion and  $g$  an integrable function. Then,*

$$\mathbb{E}_x \left( \sup_{s \leq t} \sum_{i=1}^{\lfloor ns \rfloor} g(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right) \leq |g(u_n x)| + \left( \rho |g(0)| + \frac{1}{u_n} m_{(\cdot, \beta, \cdot)}(|g|) \right) n\sqrt{2t}.$$

PROOF. We observe that

$$\int_{\mathbb{R}} |g(u_n x)| m_{(\rho, \beta, \cdot)}(dx) = \rho |g(0)| + \frac{1}{u_n} \int_{\mathbb{R}} |g(x)| m_{(\cdot, \beta, \cdot)}(dx).$$

From Lemma 4.4 and the fact that  $\sum_{i=1}^{\lfloor nt \rfloor} i^{-1/2} \leq 2\sqrt{nt}$ , we have

$$\begin{aligned} \mathbb{E}_x \left( \sup_{s \leq t} \sum_{i=1}^{\lfloor ns \rfloor} g(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \right) &\leq \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}_x \left( |g(u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)})| \right) \\ &\leq |g(u_n x)| + \left( \rho |g(0)| + \frac{1}{u_n} m_{(\cdot, \beta, \cdot)}(|g|) \right) \sum_{i=2}^{\lfloor nt \rfloor} \frac{\sqrt{n}}{\sqrt{i}} \\ &\leq |g(u_n x)| + \left( \rho |g(0)| + \frac{1}{u_n} m_{(\cdot, \beta, \cdot)}(|g|) \right) n\sqrt{2t}. \end{aligned}$$

This finishes the proof.  $\square$

PROOF OF PROPOSITION 6.9. We observe that  $X^{(\rho,\beta,\cdot)}$  is a semi-martingale with quadratic variation  $d\langle X^{(\rho,\beta,\cdot)} \rangle_t = \mathbb{1}_{X_t^{(\rho,\beta,\cdot)} \neq 0} dt$ . Thus, from the diffusion version of the occupation times formula [19, p.224],

$$\int_0^t f(X_s^{(\rho,\beta,\cdot)}) \mathbb{1}_{X_s^{(\rho,\beta,\cdot)} \neq 0} ds = \int_{\mathbb{R}} f(y) \mathbb{1}_{y \neq 0} \ell_t^y(X^{(\rho,\beta,\cdot)}) dy$$

and if  $f(0) = 0$ ,

$$\int_0^t f(X_s^{(\rho,\beta,\cdot)}) ds = \int_{\mathbb{R}} f(y) \ell_t^y(X^{(\rho,\beta,\cdot)}) dy. \quad (31)$$

Applying consecutive change of variables and (31) yields

$$\begin{aligned} \frac{u_n}{n} \int_0^{[nt]} g_n[T](u_n X_{s/n}^{(\rho,\beta,\cdot)}) ds &= u_n \int_0^{[nt]/n} g_n[T](u_n X_s^{(\rho,\beta,\cdot)}) ds \\ &= u_n \int_{\mathbb{R}} g_n[T](u_n x) \ell_{[nt]/n}^x(X^{(\rho,\beta,\cdot)}) dx = \int_{\mathbb{R}} g_n[T](x) \ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) dx. \end{aligned}$$

And, from Lemma 6.10, we have

$$\int_{\mathbb{R}} g_n[T](x) \ell_{[nt]/n}^{x/u_n}(X^{(\rho,\beta,\cdot)}) dx \xrightarrow[n \rightarrow \infty]{P_x\text{-u.c.p.}} m_{(\rho,\beta,\cdot)}(g) L_t^0(X^{(\rho,\beta,\cdot)}).$$

Therefore, by the two latter relations, to complete the proof, it suffices to prove that

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho,\beta,\cdot)}) - \frac{u_n}{n} \int_0^{[nt]} g_n[T](u_n X_{s/n}^{(\rho,\beta,\cdot)}) ds \xrightarrow[n \rightarrow \infty]{P_x\text{-u.c.p.}} 0.$$

First, let us note that

$$\begin{aligned} \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho,\beta,\cdot)}) - \frac{u_n}{n} \int_0^{[nt]} g_n[T](u_n X_{s/n}^{(\rho,\beta,\cdot)}) ds \\ = \frac{u_n}{n} \sum_{i=1}^{[nt]} \int_0^1 \left( g_n[T](X_{\frac{i-1}{n}}^{(\rho,\beta,\cdot)}) - g_n[T](X_{\frac{i-1}{n} + \frac{s}{n}}^{(\rho,\beta,\cdot)}) \right) ds \\ + \frac{u_n}{n} \int_0^{nt-[nt]} \left( g_n[T](X_{\frac{[nt]}{n}}^{(\rho,\beta,\cdot)}) - g_n[T](X_{\frac{[nt]}{n} + \frac{s}{n}}^{(\rho,\beta,\cdot)}) \right) ds. \quad (32) \end{aligned}$$

Next, observe that the second additive term of the right-hand-side of (32) is bounded by  $2\|g\|_{\infty} u_n/n$ , hence

$$\frac{u_n}{n} \int_0^{nt-[nt]} \left( g_n[T](X_{\frac{[nt]}{n}}^{(\rho,\beta,\cdot)}) - g_n[T](X_{\frac{[nt]}{n} + \frac{s}{n}}^{(\rho,\beta,\cdot)}) \right) ds \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

To show that the first additive term of the right-hand-side of (32) converges to 0 in probability, since  $g$  is bounded, from [6, Lemma 9] it suffices to prove that

$$B_t^{(n)}(g) := \frac{u_n}{n} \sum_{i=1}^{[nt]} \int_0^1 \mathbb{E}_x \left( g_n[T](X_{\frac{i-1}{n}}^{(\rho,\beta,\cdot)}) - g_n[T](X_{\frac{i-1}{n} + \frac{s}{n}}^{(\rho,\beta,\cdot)}) \middle| \mathcal{F}_{\frac{i-1}{n}} \right) ds \xrightarrow[n \rightarrow \infty]{P_x} 0.$$

Using the notation of Lemma 6.11,  $B_t^n$  rewrites

$$B_t^{(n)}(g) = \frac{u_n}{n} \sum_{i=1}^{[nt]} F_n[g](X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}).$$

From Lemma 6.12,

$$\mathbb{E}_x (|B_t^{(n)}|) \leq \frac{u_n}{n} |F_n[g](u_n x)| + \frac{u_n}{n} \left( \rho |F_n[g](0)| + \frac{1}{u_n} m_{(\cdot, \beta, \cdot)}(|F_n[g]|) \right) n \sqrt{2t}.$$

We observe that  $|F_n[g](u_n x)| \leq 2\|g\|_\infty$ . We observe that, if  $g$  is Lipschitz continuous, with compact support and vanishes on an open interval around 0, then, from Lemma 6.11, the upper bound of the above inequality vanishes as  $n \rightarrow \infty$ . Thus, in this case,

$$B_t^{(n)}(g) \xrightarrow[n \rightarrow \infty]{L^1(\mathbb{P}_x)} 0.$$

Since the limit is finite, convergence in probability also holds.

We now approximate  $g$  by functions which satisfy the assumptions of Lemma 6.11. Since  $g$  is bounded and integrable, for every  $p > 0$ , there exists a Lipschitz-continuous function with compact support  $k^{(p)}$  that vanishes in the vicinity of 0 such that  $m_{(\rho, \beta, \cdot)}(|g - k^{(p)}|) < 1/p$  (see the proof of [10, Lemma 4.5]). By combining this, Fubini's theorem and the fact that

$$\int_{-\infty}^{+\infty} p_{(\rho\sqrt{n}, \beta, \sigma_0)}(t, x, y) m_{(\rho, \beta, \sigma_0)}(dx) \leq \int_{-\infty}^{+\infty} p_{(\rho\sqrt{n}, \beta, \sigma_0)}(t, x, y) m_{(\rho\sqrt{n}, \beta, \sigma_0)}(dx) = 1,$$

we show that  $m_{(\rho, \beta, \cdot)}(|P_t^{(\sqrt{n}\rho, \beta, \cdot)} g - P_t^{(\sqrt{n}\rho, \beta, \cdot)} k^{(p)}|) < 1/p$ . Similarly, from (6) and a suitable change of variables we obtain for all  $t > 0$

$$\begin{aligned} m_{(\rho, \beta, \cdot)}(|g_n[T] - k_n^{(p)}[T]|) &< 1/p\varepsilon, \\ m_{(\rho, \beta, \cdot)}(|P_t^{(\sqrt{n}\rho, \beta, \cdot)} g_n[T] - P_t^{(\sqrt{n}\rho, \beta, \cdot)} k_n^{(p)}[T]|) &< 1/p\varepsilon. \end{aligned}$$

Thus, Fubini's theorem ensures

$$\begin{aligned} m_{(\rho, \beta, \cdot)}(|F_n[g] - F_n[k^{(p)}]|) &\leq \int_0^1 m_{(\rho, \beta, \cdot)}(|P_{su_n^2/n}^{(u_n\rho, \beta, \cdot)} g_n[T] - P_{su_n^2/n}^{(u_n\rho, \beta, \cdot)} k_n^{(p)}[T]|) ds \\ &\quad + \int_0^1 m_{(\rho, \beta, \cdot)}(|g_n[T] - k_n^{(p)}[T]|) ds \leq 2/p\varepsilon. \end{aligned}$$

Also, from (15)-(16),

$$\begin{aligned} (\rho u_n) p_{(u_n\rho, \beta, \cdot)}(tu_n^2/n, 0, y) &= (\rho u_n) v_{u_n\rho}(tu_n^2/n, 0, y) \\ &= 2e^{4|y|/u_n\rho + 8t/n\rho^2} \operatorname{erfc}\left(\frac{|y|\sqrt{n}}{\sqrt{2}tu_n} + \frac{2\sqrt{2}t}{\sqrt{n}\rho}\right) \xrightarrow[n \rightarrow \infty]{} C > 0. \end{aligned}$$

Hence, by dominated convergence

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} u_n |F_n[g](0) - F_n[k^{(p)}](0)| \tag{33} \\
& \leq \limsup_{n \rightarrow \infty} \int_0^1 u_n |P_{su_n^2/n}^{(u_n \rho, \beta, \cdot)} g_n[T](0) - P_{su_n^2/n}^{(u_n \rho, \beta, \cdot)} k_n^{(p)}[T](0)| ds \\
& \leq \limsup_{n \rightarrow \infty} \int_0^1 \int_{\mathbb{R}} |g_n[T](y) - k_n^{(p)}[T](y)| u_n p_{u_n \rho}(su_n^2/n, 0, y) m_{(u_n \rho, \beta, \cdot)}(dy) ds \\
& \leq (C/\rho) m_{(\rho, \beta, \cdot)}(|g[T] - k^{(p)}[T]|).
\end{aligned}$$

From Lemma 6.12 and (33),

$$\mathbb{E}_x(|B_t^{(n)}|) \leq \frac{u_n}{n} |F_n[g](u_n x)| + \sqrt{2t} \left( u_n \rho |F_n[k^{(p)}](0)| + m_{(\cdot, \beta, \cdot)}(|F_n[k^{(p)}]|) \right) + \frac{C}{p\varepsilon} + \frac{2}{p\varepsilon}.$$

Thus, from Lemma 6.11,

$$\forall p > 0 : \limsup_{n \rightarrow \infty} \mathbb{E}_x(|B_t^{(n)}|) \leq 2/p \implies \lim_{n \rightarrow \infty} \mathbb{E}_x(|B_t^{(n)}|) = 0$$

which is what we wanted to prove to conclude that

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)}) \xrightarrow[n \rightarrow \infty]{P_x} m_{(\rho, \beta, \cdot)}(g) L_t^0(X^{(\rho, \beta, \cdot)}).$$

With the same arguments, based on Lemma 2.2, as in the closure of the proof of Proposition 6.2, the locally uniform in time convergence, in  $P_x$ -probability ( $P_x$ -u.c.p.) is proven.  $\square$

## 7 Proof of Lemma 2.9 on occupation time approximation

**PROOF OF ITEM (i).** *Step 1:* Let us first consider a closed interval  $U$ . Let us fix  $t > 0$ . For all  $\varepsilon > 0$ , let  $U_\varepsilon = \{y \in \mathbb{R} : \text{dist}(y, U) < \varepsilon\}$ . For all  $\varepsilon > 0$ ,

$$\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U} - \mathcal{O}_t^U(X) \right| \leq \left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_\varepsilon} - \mathcal{O}_t^U(X) \right| + \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_\varepsilon \setminus U}. \tag{34}$$

Proposition 6.2 with  $g_n(x) := \mathbb{1}_{U_\varepsilon \setminus U}(x/\sqrt{n})/\sqrt{n}$  ensures that

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_\varepsilon \setminus U} \xrightarrow[n \rightarrow \infty]{P_x} C_\varepsilon L_t^0(X^{(\rho, \beta, \sigma_0)})$$

where  $C_\varepsilon = \lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta, \cdot)}(g_n) \leq K\varepsilon + \rho \mathbb{1}_{U_\varepsilon \setminus U}(0)$  for some constant  $K \in (0, \infty)$  depending on  $\beta$  and  $\sigma_0$ . Note that  $\varepsilon$  can be chosen small enough so that 0 does not belong to  $U_\varepsilon \setminus U$ , hence  $C_\varepsilon$  vanishes as  $\varepsilon$  goes to 0.

We now deal with the other term on the right hand side of (34). For all  $\varepsilon > 0$ , let  $\phi_\varepsilon, \psi_\varepsilon$  be the continuous functions defined by

$$\begin{aligned}\phi_\varepsilon(x) &= \mathbb{1}_{U_\varepsilon}(x) + \max\left\{1 - \frac{1}{\varepsilon} \text{dist}(x, U_\varepsilon), 0\right\} \mathbb{1}_{\mathbb{R} \setminus U_\varepsilon}(x), \\ \psi_\varepsilon(x) &= \mathbb{1}_{U_\varepsilon}(x) - \max\left\{1 - \frac{1}{\varepsilon} \text{dist}(x, U_\varepsilon), 0\right\} \mathbb{1}_{U_\varepsilon \setminus U}(x).\end{aligned}$$

For all  $\varepsilon > 0$ , we observe that

$$\mathbb{1}_U \leq \psi_\varepsilon(x) \leq \mathbb{1}_{U_\varepsilon} \leq \phi_\varepsilon(x) \leq \mathbb{1}_{U_{2\varepsilon}} \leq 1.$$

We observe that

$$\frac{1}{n} \sum_{i=1}^{[nt]} \psi_\varepsilon(X_{\frac{i-1}{n}}) - \mathcal{O}_t^U(X) \leq \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_\varepsilon} - \mathcal{O}_t^U(X) \leq \frac{1}{n} \sum_{i=1}^{[nt]} \phi_\varepsilon(X_{\frac{i-1}{n}}) - \mathcal{O}_t^U(X).$$

Hence

$$\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_\varepsilon} - \mathcal{O}_t^U(X) \right| \leq \left| \frac{1}{n} \sum_{i=1}^{[nt]} \phi_\varepsilon(X_{\frac{i-1}{n}}) - \mathcal{O}_t^U(X) \right| + \left| \frac{1}{n} \sum_{i=1}^{[nt]} \psi_\varepsilon(X_{\frac{i-1}{n}}) - \mathcal{O}_t^U(X) \right|$$

Also, the composed function  $\phi_\varepsilon(X_s)$  and  $\psi_\varepsilon(X_s)$  are both a.s. continuous functions of  $s$ , hence a.s. Riemann integrable. Thus, almost surely

$$\frac{1}{n} \sum_{i=1}^{[nt]} \phi_\varepsilon(X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{} \int_0^t \phi_\varepsilon(X_s) ds, \quad \frac{1}{n} \sum_{i=1}^{[nt]} \psi_\varepsilon(X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{} \int_0^t \psi_\varepsilon(X_s) ds. \quad (35)$$

From (35),

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^{[nt]} \psi_\varepsilon(X_{\frac{i-1}{n}}) - \mathcal{O}_t^U(X) \right| = \left| \int_0^t \psi_\varepsilon(X_s) ds - \mathcal{O}_t^U(X) \right| \leq \int_0^t \mathbb{1}_{U_\varepsilon \setminus U}(X_s) ds$$

Similarly for  $\phi_\varepsilon$ ,

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^{[nt]} \phi_\varepsilon(X_{\frac{i-1}{n}}) - \mathcal{O}_t^U(X) \right| = \left| \int_0^t \phi_\varepsilon(X_s) ds - \mathcal{O}_t^U(X) \right| \leq \int_0^t \mathbb{1}_{U_{2\varepsilon} \setminus U_\varepsilon}(X_s) ds.$$

Hence

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_\varepsilon} - \mathcal{O}_t^U(X) \right| \leq \int_0^t \mathbb{1}_{U_\varepsilon \setminus U}(X_s) ds + \int_0^t \mathbb{1}_{U_{2\varepsilon} \setminus U_\varepsilon}(X_s) ds.$$

Lemma 4.4 and the fact that we can suppose that  $0 \notin U_{2\varepsilon} \setminus U$ , ensure that

$$\begin{aligned}\mathbb{E}_x \left( \int_0^t \mathbb{1}_{U_\varepsilon \setminus U}(X_s) + \mathbb{1}_{U_{2\varepsilon} \setminus U_\varepsilon}(X_s) ds \right) &\leq \int_0^t \mathbb{E}_x \left( \mathbb{1}_{U_{2\varepsilon} \setminus U}(X_s) \right) ds \\ &\leq K \lambda(U_{2\varepsilon} \setminus U) \int_0^t s^{-1/2} ds \leq 4K\varepsilon\sqrt{t}\end{aligned}$$

for some constant  $K \in (0, \infty)$  depending on  $\rho, \beta, \sigma_0$ . The right hand side of the latter inequality, vanishes as  $\varepsilon \rightarrow 0$  which implies convergence probability. Let us consider a subsequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  converging  $\mathbb{P}_x$ -almost surely.

Combining this with (34), we obtain that for all  $\varepsilon_k > 0$

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U} - \mathcal{O}_t^U(X) \right| &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_{\varepsilon_k} \setminus U} \\ &+ \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U_{\varepsilon_k}} - \mathcal{O}_t^U(X) \right| \leq C_{\varepsilon_k} L_t^0(X^{(\rho, \beta, \sigma_0)}) + \int_0^t \mathbb{1}_{U_{2\varepsilon_k} \setminus U}(X_s) ds. \end{aligned}$$

Taking the limit as  $k \rightarrow \infty$  we obtain that  $\mathbb{P}_x$ -a.s.

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U} - \mathcal{O}_t^U(X) \right| \leq \lim_{k \rightarrow \infty} C_{\varepsilon_k} L_t^0(X^{(\rho, \beta, \sigma_0)}) + \int_0^t \lim_{k \rightarrow \infty} \mathbb{1}_{U_{2\varepsilon_k} \setminus U}(X_s) ds = 0.$$

This ensures that  $\left| \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U} - \mathcal{O}_t^U(X) \right|$  converges a.s. to 0 as  $n$  goes to infinity.

*Step 2: Extension to non-closed intervals.* Let  $U$  half-open and  $a \in \bar{U} \setminus U$ . From the previous case, since  $\bar{U}$  is closed, we get

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U} = \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in \bar{U}} - \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} = a} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \mathcal{O}_t^{\bar{U}}(X) - \mathcal{O}_t^{\{a\}}(X) = \mathcal{O}_t^U(X)$$

With a similar argument the convergence also holds for an open interval  $U$ .

*Step 3: u.c.p. convergence.* We observe that  $(\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U})_{t \geq 0, n \in \mathbb{N}}$  is a sequence of increasing processes with a limit in probability  $(\mathcal{O}_t^U(X))_{t \geq 0}$  that is also increasing. Thus, from Lemma 2.2, the convergence is locally uniform in time, in  $\mathbb{P}_x$ -probability. This completes the proof in the case of SOS-BM.  $\square$

**PROOF OF ITEM (ii).** Proceeding as in the proof of Theorem 2.8 (Section 3.2), letting  $S$  be the function defined in (12), then under an equivalent measure  $\mathbb{Q}_x \sim \mathbb{P}_x$ , the process  $X' = S(X)$  is an  $(\rho, \beta, \sigma_0)$ -SOS-BM.

From Item (i),

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{S(X_{\frac{i-1}{n}}) \in U} \xrightarrow[n \rightarrow \infty]{\mathbb{Q}_x} \int_0^t \mathbb{1}_{S(X_s) \in U} ds,$$

or equivalently,

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in S^{-1}(U)} \xrightarrow[n \rightarrow \infty]{\mathbb{Q}_x} \int_0^t \mathbb{1}_{X_s \in S^{-1}(U)} ds.$$

Since  $\mathbb{Q}_x \sim \mathbb{P}_x$ ,

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in S^{-1}(U)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \int_0^t \mathbb{1}_{X_s \in S^{-1}(U)} ds.$$

Since  $S$  is strictly increasing and preserves the sign,  $S^{-1}(U) = U$ . The proof is thus completed.  $\square$

## 8 Parameter estimation

In this section we prove the results regarding parameter estimation. We combine the results on occupation time and local time approximations to find consistent estimators of stickiness and skewness parameters.

In [18, 20], it is proven that one can infer the values  $\sigma(0+) - \sigma(0-)$  of an oscillating diffusion using local time approximations if  $\beta$  is known. Indeed  $\sigma(0+) - \sigma(0-)$  and  $\beta$  aggregate. We first assume that  $\sigma$  is known. Similarly to [18] (where  $\rho = 0$ ), we estimate separately  $\sigma$  for SOS-BM in Section 8.2.

### 8.1 Proof of Propositions 2.10

We consider the following statistics

$$V_n^{(0)} := \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0}, \quad V_n^{(+)} := \frac{u_n}{n} \sum_{i=1}^{[nt]} g_{>0}(u_n X_{\frac{i-1}{n}}), \quad V_n^{(-)} := \frac{u_n}{n} \sum_{i=1}^{[nt]} g_{<0}(u_n X_{\frac{i-1}{n}}),$$

and

$$W_n^{(+)} := \frac{S_n^{g^+}(X) + S_n^{g^-}(X)}{2}, \quad W_n^{(-)} := \frac{S_n^{g^+}(X) - S_n^{g^-}(X)}{2}.$$

Let us recall (9), which follows from (7) and Lemma 2.9:

$$V_n^{(0)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} (\rho/2) L_t^X(0).$$

From Theorem 2.8,

$$\begin{aligned} \frac{u_n}{n} \sum_{i=1}^{[nt]} g_{>0}(u_n X_{\frac{i-1}{n}}) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \frac{1 + \beta}{\sigma_0^2(0+)} \left( \int_0^\infty g(x) dx \right) L_t^0(X), \\ \frac{u_n}{n} \sum_{i=1}^{[nt]} g_{<0}(u_n X_{\frac{i-1}{n}}) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \frac{1 - \beta}{\sigma_0^2(0-)} \left( \int_{-\infty}^0 g(x) dx \right) L_t^0(X). \end{aligned}$$

Thus, (10) holds, and so

$$W_n^{(+)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} L_t^0(X), \quad W_n^{(-)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \beta L_t^0(X). \quad (36)$$

We consider the event  $\mathcal{L} = \{\tau_0 < t\}$ . We observe that on  $(\mathcal{L})^c$  all the aforementioned statistics converge to 0. On  $\mathcal{L} = \{\tau_0 < t\}$  they converge to non-zero random quantities. Thus, from (36),

$$\beta_n(X) = W_n^{(-)} / W_n^{(+)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \beta.$$

Also, from (9), (36),

$$\rho_n(X) = 2V_n^{(0)} / W_n^{(+)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \rho.$$

This finishes the proof.



## 8.2 Proof of Proposition 2.11

Let  $X$  be the SOS-BM such that  $(X, B)$  solves the system (3) for a standard Brownian motion  $B$ . We denote here  $\sigma_0$  by simply  $\sigma$ . From the Itô-Tanaka formula,  $X^+ = \max\{X, 0\}$  solves

$$\begin{cases} dX_t^+ = \sigma(X_t) \mathbb{1}_{X_t > 0} dB_t + \frac{\beta+1}{2} dL_t^0(X), \\ \mathbb{1}_{X_t=0} dt = (\rho/2) dL_t^0(X). \end{cases}$$

It is thus a semi-martingale with explicit Doob-Meyer decomposition and quadratic variation given by

$$d\langle X^+ \rangle_t = \sigma^2(X_t) \mathbb{1}_{X_t > 0} dt, \quad \langle X^+ \rangle_t = \int_0^t \sigma^2(X_s) \mathbb{1}_{X_s > 0} ds = \sigma_+^2 \int_0^t \mathbb{1}_{X_s > 0} ds.$$

From [19, Definition I.2.3] and Lemma 2.2,

$$\sum_{i=1}^{[nt]} (X_{\frac{i}{n}}^+ - X_{\frac{i-1}{n}}^+)^2 \xrightarrow[n \rightarrow \infty]{\text{P-u.c.p.}} \langle X^+ \rangle_t = \sigma_+^2 \int_0^t \mathbb{1}_{X_s > 0} ds.$$

With similar arguments,

$$\sum_{i=1}^{[nt]} (X_{\frac{i}{n}}^- - X_{\frac{i-1}{n}}^-)^2 \xrightarrow[n \rightarrow \infty]{\text{P-u.c.p.}} \langle X^- \rangle_t = \sigma_-^2 \int_0^t \mathbb{1}_{X_s > 0} ds.$$

This, combined with Lemma 2.9 implies the results.

## A Singular Itô calculus

In this section we prove stochastic calculus results regarding a process  $X$  that solves the system

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s) \mathbb{1}_{\{X_s \neq 0\}} ds + \int_0^t \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \beta L_t^0(X) \\ \int_0^t \mathbb{1}_{\{X_s=0\}} ds = \frac{\rho}{2} L_t^0(X), \end{cases} \quad (37)$$

where  $B$  is a standard Brownian motion and  $L_t^0(X)$  is the symmetric local time of  $X$  at 0.

**LEMMA A.1** (Singular Itô-Tanaka formula). *Let  $(X, B)$  be a weak solution to (37). If  $f$  be a difference of convex functions, such that  $f' \in C^1(\mathbb{R} \setminus \{0\})$ , then  $f(X) = (f(X)_t)_{t \geq 0}$  satisfies*

$$\begin{cases} f(X_t) = f(X_0) + \int_0^t \left( f'(X_s) b(X_s) + \frac{1}{2} f''(X_s) \sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds \\ + \int_0^t f'(X_s) \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \frac{\Sigma f'(0) \beta + \Delta f'(0)}{2} L_t^0(X), \\ \int_0^t \mathbb{1}_{\{X_s=0\}} ds = \frac{\rho}{2} L_t^0(X), \end{cases}$$

where  $\Delta f'(0) = f'(0+) - f'(0-)$  and  $\Sigma f'(0) = f'(0+) + f'(0-)$ .

If in addition,  $f$  is invertible and such that  $f(0) = 0$ ,  $\Delta f'(0)\beta + \Sigma f'(0) \neq 0$ , the process  $Y = (f(X_t))_{t \geq 0}$  is an Itô diffusion, singular at 0, that is solution to

$$\begin{cases} Y_t = Y_0 + \int_0^t \left( f'(f^{-1}(Y_s))b(f^{-1}(Y_s)) + \frac{1}{2}f''(f^{-1}(Y_s))\sigma(f^{-1}(Y_s))^2 \right) \mathbb{1}_{\{Y_s \neq 0\}} ds \\ + \int_0^t f'(f^{-1}(Y_s))\sigma(f^{-1}(Y_s)) \mathbb{1}_{\{Y_s \neq 0\}} dB_s + \frac{\Sigma f'(0)\beta + \Delta f'(0)}{\Delta f'(0)\beta + \Sigma f'(0)} L_t^0(Y), \\ \int_0^t \mathbb{1}_{\{Y_s = 0\}} ds = \frac{\rho}{\Delta f'(0)\beta + \Sigma f'(0)} L_t^0(Y); \end{cases}$$

in particular it holds that

$$L_t^0(Y) = \frac{\Delta f'(0)\beta + \Sigma f'(0)}{2} L_t^0(X).$$

PROOF. The process  $X$  is a semi-martingale as,

$$X_t = X_0 + \int_0^t b(X_s) \mathbb{1}_{X_s \neq 0} ds + \int_0^t \sigma(X_s) \mathbb{1}_{X_s \neq 0} dB_s + \beta L_t^0(X),$$

where

1.  $\left( \int_0^t b(X_s) \mathbb{1}_{X_s \neq 0} ds + \beta L_t^0(X) \right)_{t \geq 0}$  is a process of bounded variation,
2.  $\left( \int_0^t \sigma(X_s) \mathbb{1}_{X_s \neq 0} dB_s \right)_{t \geq 0}$  is a local martingale.

Thus, from [19, Theorem VI.1.5],

$$f(X_t) = f(X_0) + \int_0^t \frac{f'(X_{s+}) + f'(X_{s-})}{2} dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^y(X) f''(dy) \quad (38)$$

where  $f''(dx)$  is the measure defined such that for every  $g \in C_c^1$ ,

$$\int_{\mathbb{R}} f'g dx = - \int_{\mathbb{R}} gf''(dx).$$

Since  $f \in C^1(\mathbb{R} \setminus \{0\})$ ,  $f''(dx) = f''(x) \mathbb{1}_{\mathbb{R} \setminus \{0\}}(x) dx + \Delta f'(0) \delta_0(dx)$  and from the occupation times formula

$$\frac{1}{2} \int_{\mathbb{R}} L_t^y(X) f''(dy) = \frac{\Delta f'(0)}{2} L_t^0(X) + \frac{1}{2} \int_0^t f''(X_s) \mathbb{1}_{\{X_s \neq 0\}} \sigma(X_s)^2 ds. \quad (39)$$

From (38) and (39),

$$\begin{aligned} f(X_t) = f(X_0) + \int_0^t \left( f'(X_s)b(X_s) + \frac{1}{2}f''(X_s)\sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds \\ + \int_0^t f'(X_s)\sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \frac{\Sigma f'(0)\beta + \Delta f'(0)}{2} L_t^0(X). \end{aligned} \quad (40)$$

This proves the first statement.

Assume now  $f$  is invertible and  $f(0) = 0$ . To complete the proof, it suffices to verify the relationship between the local times. Applying (40) to the function  $|f|$  yields

$$|f(X_t)| = |f(X_0)| + \int_0^t \operatorname{sgn}(f(X_s)) \left( f'(X_s)b(X_s) + \frac{1}{2}f''(X_s)\sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds + \int_0^t \operatorname{sgn}(f(X_s))f'(X_s)\sigma(X_s)\mathbb{1}_{\{X_s \neq 0\}} dB_s + \frac{\Delta f'(0)\beta + \Sigma f'(0)}{2} L_t^0(X). \quad (41)$$

The Itô-Tanaka formula also yields

$$|f(X_t)| = |f(X_0)| + \int_0^t \operatorname{sgn}(f(X_s)) df(X_s) + L_t^0(f(X)).$$

Thus, since  $\operatorname{sgn}(f(0)) = \operatorname{sgn}(0) = 0$ ,

$$|f(X_t)| = |f(X_0)| + \int_0^t \operatorname{sgn}(f(X_s)) \left( f'(X_s)b(X_s) + \frac{1}{2}f''(X_s)\sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds + \int_0^t \operatorname{sgn}(f(X_s))f'(X_s)\sigma(X_s)\mathbb{1}_{\{X_s \neq 0\}} dB_s + L_t^0(f(X)). \quad (42)$$

From (41) and (42),

$$L_t^0(f(X)) = \frac{\Delta f'(0)\beta + \Sigma f'(0)}{2} L_t^0(X).$$

The proof is thus completed.  $\square$

LEMMA A.2 (Singular Girsanov theorem). *Let  $(\Omega, (\mathcal{F})_{t \geq 0}, \mathbb{P})$  be a probability space and  $X$  the process that solves (37) with  $B$  a  $\mathbb{P}$ -Brownian motion. Let  $\theta$  be a processes such that  $\mathbb{P}(\int_0^T \theta_s ds < \infty) = 1$ ,  $\mathcal{E}(\theta)$  the process such that*

$$\mathcal{E}_t(\theta) = \exp \left( \int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right),$$

*for every  $t \geq 0$  and  $\mathbb{Q}$  the probability measure such that  $d\mathbb{Q} = \mathcal{E}_t(\theta) d\mathbb{P}$ . Then, if  $\mathbb{E}_{\mathbb{P}}$  is the expectation under  $\mathbb{P}$  and  $\mathbb{E}_{\mathbb{P}}(\mathcal{E}_t(\theta)) = 1$ , the process  $X$  solves*

$$\begin{cases} d\tilde{X}_t = (b(\tilde{X}_t) + \theta_t \sigma(\tilde{X}_t)) \mathbb{1}_{\tilde{X}_t \neq 0} dt + \sigma(\tilde{X}_t) \mathbb{1}_{\tilde{X}_t \neq 0} d\tilde{B}_t + \beta dL_t^0(\tilde{X}), \\ \mathbb{1}_{\tilde{X}_t=0} dt = \frac{\rho}{2} dL_t^0(\tilde{X}), \end{cases} \quad (43)$$

*where  $\tilde{B} = (B_t - \int_0^t \theta_s ds)_{t \geq 0}$  is a standard Brownian motion under  $\mathbb{Q}$ .*

PROOF. Let  $\gamma$  be the time-change  $\gamma = [t \rightarrow \int_0^t \mathbb{1}_{X_s \neq 0} ds]$ ,  $A$  the right-inverse of  $\gamma$  and  $Y = (X_{A(t)})_{t \geq 0}$ . Let  $\tilde{B}$  be the process defined by  $\tilde{B}_t = B_t - \int_0^t \theta_s ds$  for every  $t \geq 0$ . Then, from [17, Theorem 6.3],  $\tilde{B}$  is a standard Brownian motion under  $\mathbb{Q}$  and the probability measures  $\mathbb{P}$  and  $\mathbb{Q}$  are equivalent. By substitution,

$$dX_t = (b(X_t) + \theta_t \sigma(X_t)) \mathbb{1}_{X_t \neq 0} dt + \sigma(X_t) \mathbb{1}_{X_t \neq 0} d\tilde{B}_t + \beta dL_t^0(X).$$

Moreover, since  $P \sim Q$  and  $L_t^0(X), \langle X \rangle_t$  are defined as limits in probability,

$$\langle X \rangle_t = \int_0^t \mathbb{1}_{X_s \neq 0} ds = t - \int_0^t \mathbb{1}_{X_s = 0} ds.$$

Thus, under  $Q$ ,  $X$  solves (43). □

## A.1 Interplay between skew and oscillation

PROPOSITION A.3. *If  $X \sim (\rho, \beta, \sigma_0)$ -SOS-BM, then*

1.  $X/\sigma_0(X) \sim \left( \rho \frac{2\sigma_- \sigma_+}{\sigma_- (1+\beta) + \sigma_+ (1-\beta)}, \frac{\sigma_- (1+\beta) - \sigma_+ (1-\beta)}{\sigma_- (1+\beta) + \sigma_+ (1-\beta)}, \cdot \right)$ -SOS-BM, *that is a sticky-skew BM,*
2.  $X/2a(X) \sim \left( \rho, \cdot, \left( \frac{\sigma_-}{1-\beta}, \frac{\sigma_+}{1+\beta} \right) \right)$ -SOS-BM, *that is an oscillating sticky BM.*

PROOF. The function  $f = [x \rightarrow x/\sigma_0(x)]$  is difference of two convex function, invertible, and  $xf(x) > 0$  for all  $x \neq 0$ . Thus, from Lemma A.1, the process  $Y = (X_t/\sigma_0(X_t))_{t \geq 0}$  solves

$$\begin{cases} Y_t = Y_0 + \int_0^t \mathbb{1}_{\{Y_s \neq 0\}} dW_s + \frac{(\sigma_- + \sigma_+)\beta + (\sigma_- - \sigma_+)}{(\sigma_- - \sigma_+)\beta + (\sigma_- + \sigma_+)} L_t^0(Y) \\ \int_0^t \mathbb{1}_{\{Y_s = 0\}} ds = \frac{\rho \sigma_- \sigma_+}{(\sigma_- - \sigma_+)\beta + (\sigma_- + \sigma_+)} L_t^0(Y), \end{cases}$$

which proves Item (1).

The function  $f = [x \rightarrow x/2a(x)] = [x \rightarrow x/(1 + \text{sgn}(x)\beta)]$  is difference of two convex function, invertible, and  $xf(x) > 0$  for all  $x \neq 0$ . Thus, from Lemma A.1, the process  $Y = (X_t/2a(X_t))_{t \geq 0}$  solves

$$\begin{cases} Y_t = Y_0 + \int_0^t \frac{\sigma_0(Y_s)}{2a(Y_s)} \mathbb{1}_{\{Y_s \neq 0\}} dW_s \\ \int_0^t \mathbb{1}_{\{Y_s = 0\}} ds = \frac{\rho}{2} L_t^0(Y), \end{cases}$$

which proves Item (2). □

REMARK A.4. For  $\rho = 0$ , we recover the interplay between skew and oscillating BM (without stickiness), see [8, 18].

## B Time-scaling of the sticky-skew Brownian motion

Let  $\mathcal{O}_t^+(X)$  be the occupation time of  $[0, \infty)$  by the process  $X$ , defined for all  $t \geq 0$  by

$$\mathcal{O}_t^+(X) = \int_0^t \mathbb{1}_{X_s \geq 0} ds.$$

Then, the following results hold.

LEMMA B.1. *The joint law of the  $(\rho, \beta, \cdot)$ -SOS-BM is*

$$\begin{aligned} P_x(X_t \in dy, L_t^0(X) \in dl, \mathcal{O}_t^+(X) \in do) \\ = g(t, x - y)\delta_0(dl)(\delta_t(do)\mathbb{1}_{x,y>0} + \delta_0(do)\mathbb{1}_{x,y<0}) dy \\ + h(o - \rho\ell, a\ell + x_+ + y_+)h(t - o, (1 - a)\ell - x_- - y_-)\mathbb{1}_{\ell\rho/2 < o < t}m(dy) dl do, \end{aligned}$$

where

$$g(t, x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}, \quad h(t, x) = \frac{|x|}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}}.$$

PROOF. See [23] and the corrections [3].  $\square$

LEMMA B.2 (Scaling property). *For any  $\rho > 0$ ,  $\beta \in (0, 1)$  and  $x \in \mathbb{R}$  let*

$$\mathcal{P}_x^{(\rho, \beta, \cdot)} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_x^{(\rho, \beta, \cdot)})$$

be a family of filtered probability spaces and  $X^{(\rho, \beta, \cdot)}$  a process defined on  $\mathcal{P}_x^{(\rho, \beta, \cdot)}$  such that under  $P_x^{(\rho, \beta, \cdot)}$ ,  $X^{(\rho, \beta, \cdot)}$  is the  $(\rho, \beta, \cdot)$ -SOS-BM and that  $P_x^{(\rho, \beta, \cdot)}(X_0^{(\rho, \beta, \cdot)} = x) = 1$ . Then,

$$\begin{aligned} \text{Law}_{P_x^{(\rho, \beta, \cdot)}}(X_{ct}^{(\rho, \beta, \cdot)}, L_{ct}^0(X^{(\rho, \beta, \cdot)}), \mathcal{O}_{ct}^+(X^{(\rho, \beta, \cdot)}); t \geq 0) \\ = \text{Law}_{P_x^{(\rho/\sqrt{c}, \beta, \cdot)}}(\sqrt{c}X_t^{(\rho/\sqrt{c}, \beta, \cdot)}, \sqrt{c}L_t^0(X^{(\rho/\sqrt{c}, \beta, \cdot)}), c\mathcal{O}_t^+(X^{(\rho/\sqrt{c}, \beta, \cdot)}); t \geq 0), \end{aligned}$$

where  $P_x^{(\rho, \beta, \cdot)}(X_0^{(\rho/\sqrt{c}, \beta, \cdot)} = \sqrt{c}x) = 1$  and

$$(L^0(X^{(\rho, \beta, \cdot)}), \mathcal{O}^+(X^{(\rho, \beta, \cdot)})), (L^0(X^{(\rho/\sqrt{c}, \beta, \cdot)}), \mathcal{O}^+(X^{(\rho/\sqrt{c}, \beta, \cdot)}))$$

are the local times at 0, occupation time of  $\mathbb{R}_+$  pairs of  $X^{(\rho, \beta, \cdot)}$  and  $X^{(\rho/\sqrt{c}, \beta, \cdot)}$  respectively.

PROOF. We observe that

$$\begin{aligned} P_x(X_{ct}^{(\rho, \beta, \cdot)} \in dy, L_{ct}^0(X^{(\rho, \beta, \cdot)}) \in dl, \mathcal{O}_{ct}^+(X^{(\rho, \beta, \cdot)}) \in do) \\ = g(ct, x - y)\delta_0(dl)(\delta_{ct}(do)\mathbb{1}_{x,y>0} + \delta_0(do)\mathbb{1}_{x,y<0}) dy \\ + h(o - \rho\ell, a\ell + x_+ + y_+)h(ct - o, (1 - a)\ell - x_- - y_-)\mathbb{1}_{\ell\rho/2 < o < ct}m(dy) dl do. \end{aligned}$$

In particular,

$$\begin{aligned} g(ct, x - y)\delta_0(dl)(\delta_{ct}(do)\mathbb{1}_{x,y>0} + \delta_0(do)\mathbb{1}_{x,y<0}) dy \\ = g\left(t, \frac{x}{\sqrt{c}} - \frac{y}{\sqrt{c}}\right)\delta_0\left(\frac{dl}{\sqrt{c}}\right)\left(\delta_t\left(\frac{do}{c}\right)\mathbb{1}_{x,y>0} + \delta_0\left(\frac{do}{c}\right)\mathbb{1}_{x,y<0}\right)\frac{dy}{\sqrt{c}} \end{aligned}$$

and since for all  $c > 0$ ,  $t > 0$  and  $x \in \mathbb{R}$ ,  $h(ct, x) = c^{-1}h(t, x/\sqrt{c})$ ,  $a(cx) = a(x)$ ,  $(cx)_+ = cx_+$ ,  $(cx)_- = cx_-$

$$\begin{aligned} h(o - \rho\ell, a\ell + x_+ + y_+)h(ct - o, (1 - a)\ell - x_- - y_-)\mathbb{1}_{\frac{\ell\rho}{2} < o < ct}2a(y) dy dl do \\ = h\left(c\left(\frac{o}{c} - \frac{\rho}{\sqrt{c}}\frac{\ell}{\sqrt{c}}\right), a\ell + x_+ + y_+\right)h\left(c\left(t - \frac{o}{c}\right), (1 - a)\ell - x_- - y_-\right)\mathbb{1}_{\frac{\ell\rho}{2c} < \frac{o}{c} < t}2a(y) dy dl do \\ = h\left(\frac{o}{c} - \frac{\rho}{\sqrt{c}}\frac{\ell}{\sqrt{c}}, a\frac{\ell}{\sqrt{c}} + \left(\frac{x}{\sqrt{c}}\right)_+ + \left(\frac{y}{\sqrt{c}}\right)_+\right) \\ h\left(t - \frac{o}{c}, (1 - a)\frac{\ell}{\sqrt{c}} - \left(\frac{x}{\sqrt{c}}\right)_- - \left(\frac{y}{\sqrt{c}}\right)_-\right)\mathbb{1}_{\frac{\ell\rho}{2c} < \frac{o}{c} < t}2a(y/\sqrt{c})\left(\frac{dy}{\sqrt{c}}\right)\left(\frac{dl}{\sqrt{c}}\right)\left(\frac{do}{c}\right). \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{P}_x(X_{ct}^{(\rho,\beta,\cdot)} \in dy, L_{ct}^0(X^{(\rho,\beta,\cdot)}) \in d\ell, \mathcal{O}_{ct}^+(X^{(\rho,\beta,\cdot)}) \in do) \\ = \mathbb{P}_{x/\sqrt{c}}\left(X_t^{(\rho/\sqrt{c},\beta,\cdot)} \in \frac{dy}{\sqrt{c}}, L_t^0(X^{(\rho/\sqrt{c},\beta,\cdot)}) \in \frac{d\ell}{\sqrt{c}}, \mathcal{O}_t^+(X^{(\rho/\sqrt{c},\beta,\cdot)}) \in \frac{do}{c}\right) \end{aligned}$$

which is the desired result.  $\square$

**COROLLARY B.3 (Density scaling).** *Let  $p_{(\rho,\beta,\cdot)}$  be the probability transition kernel of the  $(\rho, \beta, \cdot)$ -SOS-BM with respect to its speed measure  $m_{(\rho,\beta,\cdot)}$ . Then, for any measurable  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,*

$$\int_{\mathbb{R}} h(x, y) p_{(\rho,\beta,\cdot)}(ct, x, y) m_{(\rho,\beta,\cdot)}(dy) = \int_{\mathbb{R}} h(x, \sqrt{c}y) p_{(\rho/\sqrt{c},\beta,\cdot)}\left(t, \frac{x}{\sqrt{c}}, y\right) m_{(\rho/\sqrt{c},\beta,\cdot)}(dy).$$

**PROOF.** From the proof of Proposition B.2,

$$\begin{aligned} & \int_{\mathbb{R}} h(x, y) p_{(\rho,\beta,\cdot)}(ct, x, y) m_{(\rho,\beta,\cdot)}(dy) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} h(x, y) \mathbb{P}_x(X_{ct}^{(\rho,\beta,\cdot)} \in dy, L_{ct}^0(X^{(\rho,\beta,\cdot)}) \in d\ell, \mathcal{O}_{ct}^+(X^{(\rho,\beta,\cdot)}) \in do) \\ &= \int_{\mathbb{R}} \iint_{\mathbb{R}_+^2} h(x, y) \mathbb{P}_{\frac{x}{\sqrt{c}}}\left(X_t^{(\frac{\rho}{\sqrt{c}},\beta,\cdot)} \in \frac{dy}{\sqrt{c}}, L_t^0(X^{(\frac{\rho}{\sqrt{c}},\beta,\cdot)}) \in \frac{d\ell}{\sqrt{c}}, \mathcal{O}_t^+(X^{(\frac{\rho}{\sqrt{c}},\beta,\cdot)}) \in \frac{do}{c}\right) \\ &= \int_{\mathbb{R}} h(x, y) p_{(\frac{\rho}{\sqrt{c}},\beta,\cdot)}\left(t, \frac{x}{\sqrt{c}}, \frac{y}{\sqrt{c}}\right) d\left(\frac{y}{\sqrt{c}}\right) = \int_{\mathbb{R}} h(x, \sqrt{c}y) p_{(\frac{\rho}{\sqrt{c}},\beta,\cdot)}\left(t, \frac{x}{\sqrt{c}}, y\right) dy, \end{aligned}$$

which completes the proof.  $\square$

**COROLLARY B.4 (Semigroup scaling).** *Let  $X$  be the  $(\rho, \beta, \cdot)$ -SOS-BM and  $(P_t^{(\rho,\beta,\cdot)})_{t \geq 0}$  the associated semigroup. Then,*

$$P_t^{(\rho\sqrt{n},\beta,\cdot)} h(x\sqrt{n}) = \mathbb{E}_x \left( h(\sqrt{n} X_{\frac{t}{n}}^{(\rho,\beta,\cdot)}) \right).$$

**PROOF.** It is a special case of Corollary B.3.  $\square$

## C Proofs of the results in Section 4

**PROOF OF LEMMA 4.3.** Let  $u_1, u_2, v_\rho$  be the functions defined in (15)-(16). We first observe that

$$\frac{1}{a(y)} (u_1(t, x, y) - u_2(t, x, y)) \leq \frac{1}{a(y)} u_1(t, x, y) \mathbb{1}_{\mathbb{I}}(y) \leq K u_1(t, x, y) \quad (44)$$

for some positive constant  $K$ , since  $a(y) \geq (1 - |\beta|)/2$  if  $\beta \in (-1, 1)$ , otherwise  $a(y) \equiv 1$ . From the Mill's ratio (see [7, p.98]),  $\operatorname{erfc}(x) \sim e^{-x^2}/x$  and thus there is a constant

$K_{\text{Mills}} > 0$  such that  $\operatorname{erfc}(x) \leq K_{\text{Mills}}e^{-x^2}/x$ . Thus, if  $\rho > 0$ ,

$$\begin{aligned} v_\rho(t, x, y) &= \frac{2}{\rho} e^{4(|x|+|y|)/\rho+8t/\rho^2} \operatorname{erfc}\left(\frac{|x|+|y|}{\sqrt{2t}} + \frac{2\sqrt{2t}}{\rho}\right) \\ &\leq K_{\text{Mills}} \frac{2}{\rho} \frac{\rho\sqrt{2t}}{\rho(|x|+|y|)+4t} e^{-(|x|+|y|)^2/2t} \leq K_{\text{Mills}} \frac{\sqrt{2t}}{\rho|x|/2+2t} e^{-(|x|+|y|)^2/2t} \\ &\leq \frac{K_{\text{Mills}}}{\sqrt{2t}} e^{-(|x|+|y|)^2/2t} \leq 2\sqrt{\pi}K_{\text{Mills}}u_1(t, x, y). \end{aligned} \quad (45)$$

If  $\rho = 0$ , from (16),

$$v_0(t, x, y) = u_2(t, x, y) \leq u_1(t, x, y). \quad (46)$$

Combining (44),(45) and (46), completes the proof.  $\square$

PROOF OF LEMMA 4.4. From (13),

$$P_t^{(\rho, \beta, \cdot)} h(x) = \int_{\mathbb{R}} h(y) p_{(\rho, \beta, \cdot)}(t, x, y) m_{(\rho, \beta, \cdot)}(dy).$$

This, with Lemma 4.3 and since  $p_{(\rho, \beta, \cdot)}(t, x, 0) = v_\rho(t, x, 0)$ , yields

$$\begin{aligned} |P_t^{(\rho, \beta, \cdot)} h(x)| &\leq \int_{\mathbb{R}} |h(y)| p_{(\rho, \beta, \cdot)}(t, x, y) m_{(\rho, \beta, \cdot)}(dy) \\ &= \rho|h(0)|p_{(\rho, \beta, \cdot)}(t, x, 0) + \int_{\mathbb{R}} |h(y)| p_{(\rho, \beta, \cdot)}(t, x, y) m_{(\cdot, \beta, \cdot)}(dy) \\ &\leq K_{\text{Mills}} \frac{\rho\sqrt{2t}}{\rho|x|/2+2t} |h(0)| + \frac{K}{\sqrt{t}} m_{(\cdot, \beta, \cdot)}(|h|) \leq \frac{K}{\sqrt{t}} (|h(0)| + m_{(\cdot, \beta, \cdot)}(|h|)), \end{aligned}$$

which proves the results.  $\square$

PROOF OF LEMMA 4.5. We observe that

$$\begin{aligned} &|P^{(\rho, \beta, \cdot)} h(x) - m_{(\rho, \beta, \cdot)}(h)p_{(\rho, \beta, \cdot)}(t, x, 0)| \\ &= \left| \int_{\mathbb{R}} h(y) (p_{(\rho, \beta, \cdot)}(t, x, y) - p_{(\rho, \beta, \cdot)}(t, x, 0)) m_{(\cdot, \beta, \cdot)}(dy) \right| \\ &\leq \left| \int_{\mathbb{R}} h(y) (u_1(t, x, y) - u_1(t, x, 0)) (2 dy) \right| + \left| \int_{\mathbb{R}} h(y) (u_2(t, x, y) - u_2(t, x, 0)) (2 dy) \right| \\ &\quad + \left| \int_{\mathbb{R}} h(y) (v_\rho(t, x, y) - v_\rho(t, x, 0)) m_{(\cdot, \beta, \cdot)}(dy) \right|. \end{aligned} \quad (47)$$

For every  $\gamma \geq 0$ , there exists a constant  $K''_\gamma > 0$  such that for every  $x \in \mathbb{R}$ ,

$$e^{-x^2/t} \leq e^{-x^2/2t} \leq K''_\gamma \frac{1}{1+|x/t|^\gamma}. \quad (48)$$

From [10, Lemma 3.1], for every  $\gamma \geq 0$ , there exists a  $K'_\gamma > 0$  such that

$$\left| \int_{\mathbb{R}} h(y) [u_1(t, x, y) - u_1(t, x, 0)] dy \right| \leq \frac{K'_\gamma}{t} \left( \lambda^{(1)}(h) + \frac{\lambda^{(1)}(h)}{1+|x/\sqrt{t}|^\gamma} + \frac{\lambda^{(\gamma)}(h)}{1+|x|^\gamma} \right). \quad (49)$$

For the second additive term at the right-hand-side of (47),

$$\int_{\mathbb{R}} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy = \int_0^1 \int_{\mathbb{R}} h(\zeta) \zeta \frac{\partial}{\partial \zeta} u_2(t, x, \theta \zeta) d\zeta d\theta.$$

There exists a constant  $c > 0$  such that for every  $x \in \mathbb{R}$ :  $x e^{-x^2} \leq c$ . Thus,

$$\left| \frac{\partial}{\partial \zeta} u_2(t, x, y) \right| = \frac{\operatorname{sgn}(y)}{2\sqrt{2\pi}} \frac{1}{t\sqrt{t}} e^{-(|x|+|y|)^2/2t} \leq \frac{c/2\sqrt{2\pi}}{t}$$

and for  $c' = c/2\sqrt{2\pi}$ ,

$$\left| \int_{\mathbb{R}} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy \right| \leq \int_0^1 \int_{\mathbb{R}} \left| h(\zeta) \zeta \frac{\partial}{\partial \zeta} u_2(t, x, \theta \zeta) \right| d\zeta d\theta \leq c' \frac{\lambda^{(1)}(h)}{t}. \quad (50)$$

We now bound the third additive term at the right-hand-side of (47) and first consider the case  $\rho = 0$ , where for all  $(t, x, y) \in (0, \infty) \times \mathbb{R}^2$ ,  $v_0(t, x, y) = u_2(t, x, y)$ . We observe that

$$\begin{aligned} \int_{\mathbb{R}_+} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy &= \int_0^1 \int_{\mathbb{R}_+} h(\zeta) \zeta \frac{\partial}{\partial \zeta} u_2(t, x, \theta \zeta) d\zeta d\theta, \\ \int_{\mathbb{R}_-} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy &= \int_0^1 \int_{\mathbb{R}_-} h(\zeta) \zeta \frac{\partial}{\partial \zeta} u_2(t, x, \theta \zeta) d\zeta d\theta. \end{aligned}$$

Thus, with the same arguments as for the second additive term,

$$\left| \int_{\mathbb{R}} h(y) (v_0(t, x, y) - v_0(t, x, 0)) m_{(\cdot, \beta, \cdot)}(dy) \right| \leq c' \frac{m_{(\cdot, \beta, \cdot)}^{(1)}(h)}{t}. \quad (51)$$

For the case  $\rho > 0$ , let  $M$  be the function defined for all  $t \geq 0$  and  $x, y \in \mathbb{R}$  by

$$M(t, x, y) := \frac{v_\rho(t, x, y)}{u_2(t, x, y)} = \frac{\sqrt{\pi}}{\sqrt{2t}} \frac{4t}{\rho} f\left(\frac{|x| + |y| + 4t/\rho}{\sqrt{2t}}\right)$$

where  $f(z) := e^{z^2} \operatorname{erfc}(z)$ . Then,

$$v_\rho(t, x, y) = u_2(t, x, y) M(t, x, y) = u_2(t, x, y) (M(t, x, y) - M(t, x, 0) + M(t, x, 0))$$

and

$$\begin{aligned} v_\rho(t, x, y) - v_\rho(t, x, 0) &= u_2(t, x, y) (M(t, x, y) - M(t, x, 0)) + (u_2(t, x, y) - u_2(t, x, 0)) M(t, x, 0). \end{aligned} \quad (52)$$

We observe that for all  $t, z \geq 0$  and  $x, y \in \mathbb{R}$ ,

$$0 \leq M(t, x, y) \leq M(t, x, 0) \leq \frac{\sqrt{\pi}}{\sqrt{2t}} \frac{4t}{\rho} f\left(\frac{4t/\rho}{\sqrt{2t}}\right).$$

Since  $0 \leq z f(z) < 1/\sqrt{\pi}$ , then and  $0 < M(t, x, y) \leq M(t, x, 0) \leq 1$ . Thus, from (51),

$$\left| \int_{\mathbb{R}} h(y) [u_2(t, x, y) - u_2(t, x, 0)] |M(t, x, y)| m_{(\cdot, \beta, \cdot)}(dy) \right| \leq c' \frac{m_{(\cdot, \beta, \cdot)}^{(1)}(h)}{t}. \quad (53)$$



Moreover

$$M(t, x, 0) - M(t, x, y) = \frac{\sqrt{\pi}}{\sqrt{2t}} \frac{4t}{\rho} \int_0^{|y|/\sqrt{2t}} \left| f' \left( \frac{|x| + 4t/\rho}{\sqrt{2t}} + \zeta \right) \right| d\zeta.$$

Since  $|f'(\zeta)| = \frac{2}{\sqrt{\pi}}(1 - \sqrt{\pi}\zeta f(\zeta))$  is decreasing,

$$M(t, x, 0) - M(t, x, y) \leq |y| \frac{2\sqrt{\pi}}{\rho} \left| f' \left( \frac{4t}{\rho\sqrt{2t}} \right) \right| = |y| \frac{4}{\rho} \left( 1 - \sqrt{\pi} \frac{4t}{\rho\sqrt{2t}} f \left( \frac{4t}{\rho\sqrt{2t}} \right) \right).$$

Note that  $[0, \infty) \ni x \mapsto x(1 - \sqrt{\pi}xe^{x^2} \operatorname{erfc}(x)) \in \mathbb{R}$  takes values in of  $[0, 1/4)$ . Hence, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}} h(y) [u_2(t, x, y)] [M(t, x, y) - M(t, x, 0)] m_{(\cdot, \beta, \cdot)}(dy) \right| \\ & \leq \frac{1}{2\sqrt{2t}} \int_{\mathbb{R}} |y| |h(y)| |u_2(t, x, y)| m_{(\cdot, \beta, \cdot)}(dy) \\ & \leq \frac{1}{4t\sqrt{\pi}} m_{(\cdot, \beta, \cdot)}^{(1)}(h) e^{-\frac{x^2}{2t}} \leq \frac{1}{t} \frac{K_\gamma''}{4\sqrt{\pi}} m_{(\cdot, \beta, \cdot)}^{(1)}(h) \frac{1}{1 + |x/t|^\gamma}, \end{aligned} \quad (54)$$

where the last inequality comes from (48). From (52),(53),(54),

$$\left| \int_{\mathbb{R}} h(y) (v_\rho(t, x, y) - v_\rho(t, x, 0)) m_{(\cdot, \beta, \cdot)}(dy) \right| \leq \frac{1}{t} \left( c' + \frac{K_\gamma''}{4\sqrt{\pi}} \right) \left( m_{(\cdot, \beta, \cdot)}^{(1)}(h) + \frac{m_{(\cdot, \beta, \cdot)}^{(1)}(h)}{1 + |x/t|^\gamma} \right). \quad (55)$$

From (47),(49),(50),(51),(55) and since that for all  $\gamma \geq 0$ ,

$$(1 - |\beta|)\lambda^{(\gamma)}(h) \leq m_{(\cdot, \beta, \cdot)}^{(\gamma)}(h) \leq (1 + |\beta|)\lambda^{(\gamma)}(h),$$

if  $\mathbb{I} = \mathbb{R}$  (equivalently  $\beta \in (-1, 1)$ ), otherwise  $m_{(\cdot, \beta, \cdot)}^{(\gamma)}(h) = 2\lambda^{(\gamma)}(h)$ , the desired bound holds for

$$K_\gamma = \frac{2}{1 - |\beta|} (K_\gamma' + c') + \left( c' + \frac{K_\gamma''}{4\sqrt{\pi}} \right)$$

if  $|\beta| \neq 1$ , otherwise take  $\beta = 0$  in the formula above. This completes the proof of Lemma 4.5.  $\square$

**PROOF OF LEMMA 4.6.** From Corollary B.4, which is a consequence of the scaling property,

$$\gamma_n^{(\rho, \beta, \cdot)}[h](x, t) = \sum_{i=2}^{[nt]} \mathbb{E}_x [h(\sqrt{n}X_{\frac{i-1}{n}}^{(\rho, \beta, \cdot)})] = \sum_{i=2}^{[nt]} P_{i-1}^{(\sqrt{n}\rho, \beta, \cdot)} h(\sqrt{n}x).$$

Thus, from Lemma 4.4 and since  $\sum_{i=1}^{[nt]-1} i^{-\frac{1}{2}} \leq 2\sqrt{nt}$ ,

$$\begin{aligned} |\gamma_n^{(\rho, \beta, \cdot)}[h](x, t)| & \leq \sum_{i=2}^{[nt]} |P_{i-1}^{(\sqrt{n}\rho, \beta, \cdot)} h(\sqrt{n}x)| \\ & \leq K \sum_{i=1}^{[nt]-1} \frac{1}{\sqrt{i}} m_{(\rho\sqrt{n}, \beta, \cdot)}(|h|) \leq 2K m_{(\rho\sqrt{n}, \beta, \cdot)}(|h|) \sqrt{nt}, \end{aligned}$$

which completes the proof of the first statement.

When  $m_{(\sqrt{n}\rho, \beta, \cdot)}(h) = 0$ , from Lemma (4.5) (with  $\gamma = 1$ ) and the fact that  $\sum_{i=1}^{[nt]-1} i^{-1} \leq 1 + \log(nt)$  we obtain

$$\begin{aligned} |\gamma_n^{(\rho, \beta, \cdot)}[h](x, t)| &\leq \sum_{i=2}^{[nt]} |P_{i-1}^{(\rho, \beta, \cdot)} h(\sqrt{nx})| \\ &\leq Km_{(\cdot, \beta, \cdot)}^{(1)}(h) \sum_{i=2}^{[nt]} \frac{1}{i} \leq Km_{(\cdot, \beta, \cdot)}^{(1)}(h)(1 + \max(0, \log(nt))) \end{aligned}$$

which proves the second and last statement. The proof is thus completed.  $\square$

## References

- [1] A. Anagnostakis. Functional convergence to the local time of a sticky diffusion. *Electronic Journal of Probability*, 28(88):1–26, 2023.
- [2] A. Borodin and P. Salminen. On the local time process of a skew Brownian motion. *Trans. Am. Math. Soc.*, 372(5):3597–3618, 2019.
- [3] J.-B. Casteras and L. Monsaingeon. Trivariate distribution of sticky brownian motion, 2023.
- [4] H.-J. Engelbert and G. Peskir. Stochastic differential equations for sticky Brownian motion. *Stochastics*, 86(6):993–1021, 2014.
- [5] D. Florens-Zmirou. On estimating the diffusion coefficient from discrete observations. *J. Appl. Probab.*, 30(4):790–804, 1993.
- [6] V. Genon-Catalot and J. Jacod. On the estimation of the diffusion coefficient for multi-dimensional diffusion processes. *Annales de l’I.H.P. Probabilités et statistiques*, 29(1):119–151, 1993.
- [7] G. R. Grimmett and D. R. Stirzaker. *Probability and random processes*. Oxford University Press, New York, third edition, 2001.
- [8] J. M. Harrison and L. A. Shepp. On skew Brownian motion. *Ann. Probab.*, 9:309–313, 1981.
- [9] J. Jacod. Une généralisation des semimartingales: les processus admettant un processus à accroissements indépendants tangent. In *Seminar on probability, XVIII*, volume 1059 of *Lecture Notes in Math.*, pages 91–118. Springer, Berlin, 1984.
- [10] J. Jacod. Rates of convergence to the local time of a diffusion. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(4):505–544, 1998.
- [11] J. Jacod and P. Protter. *Discretization of processes*, volume 67 of *Stochastic Modelling and Applied Probability*. Springer, Heidelberg, 2012.
- [12] J. Keilson and J. A. Wellner. Oscillating Brownian motion. *J. Appl. Probab.*, 15:300–310, 1978.

- [13] A. Lejay and S. Mazzonetto. Maximum likelihood estimator for skew brownian motion: the convergence rate. *To appear in Scandinavian Journal of Statistics*, 2023.
- [14] A. Lejay, E. Mordecki, and S. Torres. Is a Brownian motion skew? *Scand. J. Stat.*, 41(2):346–364, 2014.
- [15] A. Lejay, E. Mordecki, and S. Torres. Two consistent estimators for the skew brownian motion. *ESAIM: Probability and Statistics*, 23:567–583, 2019.
- [16] A. Lejay and P. Pigato. Statistical estimation of the oscillating Brownian motion. *Bernoulli*, 24(4B):3568–3602, 2018.
- [17] R. S. Liptser and A. N. Shiryaev. *Statistics of random processes. I*, volume 5 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, expanded edition, 2001. General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability.
- [18] S. Mazzonetto. Rates of convergence to the local time of oscillating and skew brownian motions (preprint arxiv: 1912.04858), 2019.
- [19] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [20] C. Y. Robert. How large is the jump discontinuity in the diffusion coefficient of a time-homogeneous diffusion? *Econom. Theory*, 39(4):848–880, 2023.
- [21] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.
- [22] M. Salins and K. Spiliopoulos. Markov processes with spatial delay: path space characterization, occupation time and properties. *Stoch. Dyn.*, 17(6):21, 2017. Id/No 1750042.
- [23] W. Touhami. On skew sticky Brownian motion. *Stat. Probab. Lett.*, 173:9, 2021. Id/No 109086.