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Sticky-Threshold Diffusions, Local Time Approximation and Parameter Estimation

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Abstract

We study a class of high-frequency path functionals for diffusions with singular thresholds or boundaries, where the process exhibits either (i) skewness, oscillating coefficients, and stickiness, or (ii) sticky reflection. The functionals are constructed from a test function and a diverging normalizing sequence. We establish convergence to local time, generalizing existing results for these processes.

Notably, our framework allows for any normalizing sequence diverging slower than the observation frequency and for thresholds that are jointly skew-oscillating-sticky (thresholds where stickiness, oscillations, and skewness occur). Combining our results with occupation time approximations, we develop consistent estimators for stickiness and skewness parameters at thresholds that exhibit any combination of these features (stickiness, oscillation, skewness, and reflection).

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Keywords and phrases: Consistent parameter estimation; high frequency statistics; occupation time approximation; stickiness; skewness; reflection; oscillating diffusion coefficient.

1 Introduction

Consider a stochastic process X that satisfies a time-homogeneous stochastic differential equation (SDE) with smooth coefficients driven by standard Brownian motion. It is known (see [19, Theorem 1.1]) that the local time of X at some threshold ζ can be consistently approximated as $n \rightarrow \infty$ by the empirical process

$$t \mapsto \frac{u_n}{n} \sum_{i=1}^{\lfloor nt \rfloor} g(u_n(X_{\frac{i-1}{n}} - \zeta)), \quad (1.1)$$

where g is a bounded integrable function normalized to $\int g(x)dx = 1$, $(u_n)_n$ is a sequence diverging slower than n , and $(X_{i/n})_{i \in \mathbb{N}}$ are discrete observations of the process. Such approximations

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have applications in the estimation of specific characteristics of a diffusion. (By diffusion we mean a continuous strong Markov process. Such process is not necessarily solution to a classical SDE.) Examples are the estimation of diffusion coefficient [12], of the stickiness parameter [1], of the skewness parameter [24, 25, 27], and of volatility jumps [27, 31].

Diffusions can exhibit a wide variety of threshold-behaviors, some of which cannot be replicated by classical SDE solutions. We characterize these behaviors as follows: A threshold is *sticky* if the diffusion spends a positive amount of time there upon contact, with the duration governed by a *stickiness parameter*; it is *skew* if it induces asymmetric reflection, characterized by a *skewness parameter*; and it is *oscillating* if it exhibits a discontinuity in the diffusion coefficient, leading to regime-switching dynamics. Elementary examples include sticky Brownian motion [11], skew Brownian motion [16], oscillating Brownian motion [23], and reflected Brownian motion. Furthermore, one can define processes that combine these behaviors, such as the skew-sticky Brownian motion (SkS-BM) studied in [35, 37], the oscillating-sticky Brownian motion of [36], and the skew-oscillating-sticky Brownian motion (SOS-BM) defined in Section 3.1. The SOS-BM, which also encompasses the case of sticky-reflected BM (or slowly reflected BM), plays a central role in our analysis.

To motivate this work and in particular the significance of parameter estimation for these processes, we highlight several applications. Such stochastic motions arise in: Quantum mechanics, describing particle behavior near emission sources, *e.g.*, [10], particle dynamics in layered media, *e.g.*, [29, 21], and colloids, *e.g.*, [7, 34]; financial modeling of stock price movements *e.g.*, [3, 9, 28, 31]; and Queuing systems, *e.g.*, [15].

Regarding the problem of local time approximation, existing literature has primarily focused on specific cases. For instance, [25, 27] focus on skew BM and [27, 31] on oscillating BM for more general functions g in (1.1) but less general u_n . All these results are applied to estimate the skewness parameter or the volatility jump. While [26] developed occupation time-based estimators for estimating the diffusion coefficient of oscillating BM. In the case of skew-oscillating BM convergence rates for (1.1) (and more general statistics) were obtained in [27] under the particular scaling $u_n = \sqrt{n}$. No result is available, not even consistency, for $u_n = o(n)$. Instead, [1] proved consistency of local time approximations (1.1) at sticky thresholds for sequences $(u_n)_n$ diverging slower than \sqrt{n} , leaving open the case of general $u_n = o(n)$ and in particular $u_n = \sqrt{n}$. Motivated by numerical simulations, in the last work, the convergence for $o(n)$ is formulated as a conjecture. The sequence $u_n = \sqrt{n}$ achieves the optimal convergence rate for local time approximations at non-sticky thresholds (c.f. [19, Theorem 2.1]). This underscores the need to characterize all modes of convergence at sticky thresholds for $(u_n)_n$, which remain unaddressed in [1]. More specifically, since the normalizing sequence \sqrt{n} is not $o(n^{1/2})$, it is not covered by Theorem 1.2 and Corollary 1.3 in [1].

The main goal of this paper is to fill these gaps and prove that for any diverging normalizing sequence that is $o(n)$ and any bounded integrable function g with $g(0) = 0$, convergence to local time occurs in the case of diffusions presenting an Skew-Oscillating-Sticky (SOS) threshold. This means, in particular, in case $g(0) = 0$, at a sticky threshold the statistic (1.1) has the same limit behavior as for the non-sticky process studied by Jacod [19] (the constants change). We also consider SOS thresholds instead of a purely sticky ones, giving a unified result for this problem and generalizing the works of [26, 27, 31] with respect to the choice of normalizing sequence. The new contribution for purely sticky thresholds relies on new estimates for the sticky kernel (Lemmata 5.6 and 5.7). Regarding the generalization to an SOS-threshold, one needs to consider a different base process, the SOS-Brownian motion (SOS-BM). The SOS-BM is the diffusion that behaves like a re-scaled Brownian motion separately on positive and negative half-line, and

exhibits at 0 an SOS threshold. Its local time process discontinuous at the threshold. Hence, some proofs require to be considered separately on positive and negative half-lines, where the local time process is continuous in space and time.

For simplicity, we assume the threshold is located at 0 and focus on the case of a single SOS threshold. The proofs works by first proving the result for SkS-BM and extend via transforms and equivalence of measure to the general case. Our approach is inspired by, though differs from: [19] where the problem is reduced to BM, [27] where the problem is reduced to skew BM, and [1] where the problem is reduced to sticky BM. This indicates that the threshold behavior must match the one of the base process.

We do not address the problem of finding the rates of convergence for the local time approximation and the associated estimators (as is the case in [19, 27]). Nevertheless, we believe that building on the results of this paper one can obtain rates the same way as in the aforementioned papers for non-sticky thresholds. This is the subject of further work.

Paper Outline

In Section 3, we present the processes we deal with and the main results of the paper: local time approximation and parameter estimation. In Section 4, we reduce the proof of the local time approximation result for SOS-diffusions to the case of SkS-BM. The main result for SkS-BM is proven in Section 6. In Section 5, we establish preliminary results: bounds on the semigroup of SkS-BM and a preliminary local time approximation based on the Tanaka formula. Lengthy proofs for this section are deferred to Appendix D. In Section 7, we prove all results related to parameter estimation.

Regarding the appendix sections: Appendix A derives explicit symmetric stochastic calculus versions of the Itô–Tanaka formula and Girsanov theorem for SOS-diffusions. Appendix B proves an approximation of the occupation time, used in Section 7 to establish consistent parameter estimations. Appendix C explores the scaling properties of the skew sticky semigroup.

2 Notation and notions of convergence

Before stating the main results of this paper let us introduce some preliminary notions. First, we introduce two modes of convergence in which the results are expressed: the *uniform convergence in time, in probability* (or ucp), and the *convergence in conditional probability* (or ccp). We provide sufficient conditions for ucp and ccp convergence to occur which are used throughout this paper. The local time approximation holds in ucp convergence and the estimation results hold in ccp convergence, conditioned on the event that the process has reached the threshold of interest (we assumed to be 0). Second, we recall the definitions of occupation and local time, and elaborate on the distinction between right and symmetric local times. Last, we recall the usual conditions on a filtered probability space that we assume throughout this paper.

For clarity of presentation, let $(A^n)_{n \geq 0}$ be a sequence of processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1 (ucp convergence). We say that $(A^n)_{n \geq 1}$ *converges locally uniformly in time, in probability* or ucp to A^0 if for every $t > 0$, we have that

$$\sup_{s \leq t} |A_s^n - A_s^0| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

We denote this convergence with $A^n \xrightarrow[n \rightarrow \infty]{\text{P-ucp}} A^0$.

Lemma 2.2 (see [20], §2.2.3). *If A^n and A^0 have increasing paths and A^0 is continuous, then*

$$A_t^n \xrightarrow[n \rightarrow \infty]{\text{P}} A_t^0, \forall t \in D, \text{ with } D \text{ dense in } \mathbb{R}^+ \Rightarrow A^n \xrightarrow[n \rightarrow \infty]{\text{P-ucp}} A^0.$$

Definition 2.3 (ccp convergence). We say that $(A^n)_{n \geq 1}$ converges to A^0 in probability, conditionally on \mathcal{L} , if $A^n \rightarrow A^0$ in $\text{P}^{\mathcal{L}}$ -probability. We denote this convergence with

$$A^n \xrightarrow[n \rightarrow \infty]{\text{P}^{\mathcal{L}}} A^0.$$

The next result is direct consequence of Bayes' rule.

Lemma 2.4. *Let \mathcal{L} be an event ($\mathcal{L} \in \mathcal{F}$) such that $\text{P}(\mathcal{L}) > 0$. The sequence A^n converges to A^0 in $\text{P}^{\mathcal{L}}$ -probability if and only if A^n converges to A^0 in probability on \mathcal{L} , i.e., for all $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \text{P}(|A^n - A^0| > \varepsilon; \mathcal{L}) = 0.$$

Let X be a continuous real semimartingale. For every measurable set U of \mathbb{R} , the occupation time of U by X is the process defined by

$$\mathcal{O}_t^U(X) := \int_0^t \mathbb{1}_U(X_s) ds, \quad t \geq 0.$$

We denote by $(\ell^y(X))_{y \in \mathbb{R}}$ the right local time field of X defined in [30, Theorem VI.1.2] via the Tanaka formula. We denote by $(L^y(X))_{y \in \mathbb{R}}$ the symmetric local time field of the process X , defined for all $t \geq 0$ and $y \in \mathbb{R}$ by

$$L_t^y(X) := \frac{\ell_t^y(X) + \ell_t^{y-}(X)}{2}, \quad \text{where } \ell_t^{y-}(X) = \lim_{h \rightarrow 0} \ell_t^{y-h}(X).$$

It is not always the case that the right and symmetric local time fields are equal. An example is the skew BM, see e.g. [5, Theorem 2.1]. Typically, path-wise characterizations of diffusions are expressed in terms of the symmetric local time, see e.g. [33]. Conversely, results such as the Itô–Tanaka formula [30, Theorem VI.1.5], the occupation times formula [30, Corollary VI.1.6], and the representation of martingale diffusions as time-changed Brownian motion [32, Theorem V.47.1], are typically expressed in terms of the right local time.

In this paper we always assume (filtered) probability spaces satisfy the usual conditions of right-continuity and completeness. This means the filtration of such space $\mathcal{P} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \text{P})$ satisfies $\bigcap_{s \geq t} \mathcal{F}_s = \mathcal{F}_t$, for all $t \geq 0$, and that \mathcal{F}_0 contains all P -negligible sets.

3 Main results

Let us first define the process of interest, which is (the weak) solution to the system

$$\begin{cases} dX_t = \nu_t \mathbb{1}_{\{X_t \neq 0\}} dt + \sigma(X_t) \mathbb{1}_{\{X_t \neq 0\}} dB_t + \beta dL_t^0(X), & t \geq 0, \\ \mathbb{1}_{\{X_t = 0\}} dt = \rho dL_t^0(X), & t \geq 0, \end{cases} \quad (3.1)$$

where $\beta \in [-1, 1]$, $\rho \geq 0$, B is a standard BM, $(L_t^y(X); y \in \mathbb{R}, t \geq 0)$ is the symmetric local time field of X , ν is a measurable process and σ a positive measurable function. The equation

$\mathbb{1}_{\{X_t=0\}} dt = \rho dL_t^0(X)$, combined with vanishing coefficients, introduces a delay at 0 called stickiness. This causes the process to spend a positive amount of time at 0, and with intensity proportional to ρ . The term $\beta dL_t^0(X)$ generates a partial reflection effect at 0, called skew, whose preferred direction depends on the sign of β , with limit cases $\beta \in \{-1, 1\}$ corresponding to upper and lower reflection. The parameters ρ and β are accordingly called stickiness and skewness. We also allow σ to be discontinuous at 0, and in this case the threshold is also of oscillating type.

Assume that (ν, σ) are such that (3.1) has a unique (in law) weak solution, that is: There exists a filtered probability space \mathcal{P} and a BM B on $\mathcal{P} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ such that (X, B) jointly solve (3.1). When we wish to indicate that $X_0 = x$ a.s., we use the notation $(\mathcal{P}_x, \mathbb{P}_x)$ for the probability space and measure. To abbreviate the above, we say that:

X is the solution to (3.1) on \mathcal{P}_x .

We also refer to this process as an *SOS-diffusion*. The reason we only consider weak solutions is that in presence of stickiness ($\rho > 0$), no strong solution exist, see [11].

Throughout the paper we also assume the following additional assumptions on the pair (ν, σ) : Condition 3.1. The first assumption, ensures that the driftless version of (3.1) (where $\nu \equiv 0$) admits a unique (in law) weak solution and determines the state-space. The second assumption allows us to reduce the problem for the solution to (3.1) to one for the simple base process SOS-BM defined in the forthcoming (3.3).

Condition 3.1. c1. *The diffusion coefficient σ ensures that the stochastic differential equation*

$$dY_t = \sigma(Y_t) dB_t + \beta dL_t^0(Y),$$

admits a unique-in-law solution (a diffusion) that almost surely does not explode in finite time. Its state space J is either of the form $[0, b)$ (if $\beta = 1$), $(a, 0]$ (if $\beta = -1$), or (a, b) (otherwise), with $-\infty \leq a < 0 < b \leq +\infty$. The coefficient σ is continuously differentiable on $J \setminus \{0\}$, i.e., $\sigma \in C^1(J \setminus \{0\})$.

c2. *The (law of the) solution to (3.1) is locally equivalent to the (law of the) solution to*

$$\begin{cases} dY_t = \frac{\sigma(Y_t)\sigma'(Y_t)}{2} \mathbb{1}_{\{Y_t \neq 0\}} dt + \sigma(Y_t) \mathbb{1}_{\{Y_t \neq 0\}} dB_t + \beta dL_t^0(Y), & t \geq 0, \\ \mathbb{1}_{\{Y_t=0\}} dt = \rho dL_t^0(Y), & t \geq 0, \end{cases} \quad (3.2)$$

with the same σ , β , and ρ as in (3.1). Let us say that X solves (3.1) on \mathcal{P}_x and Y solves (3.2) on $\tilde{\mathcal{P}}_x = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}_x)$. Local equivalence here means that for every $T > 0$, there exists an equivalent probability measure $\tilde{\mathbb{P}}_x^T \sim \tilde{\mathbb{P}}_x$ such that the law of $X|_{[0, T]}$ under \mathbb{P}_x is equivalent to the law of $Y|_{[0, T]}$ under $\tilde{\mathbb{P}}_x^T$.

Remark 3.2. By Lemma A.2, a sufficient condition for **c2** is Novikov's criterion (see e.g. [22, Proposition 3.5.12] and the discussion above):

$$\mathbb{E}_x \left[\exp \left(\frac{1}{2} \int_0^t \left(\frac{\nu_s}{\sigma(X_s)} - \sigma'(X_s) \right)^2 ds \right) \right] < \infty, \quad \forall t \geq 0.$$

In particular, this is satisfied $\nu_s = b(X_s)$ where b bounded and σ uniformly elliptic (that is bounded from above and below by a positive constant).

Of particular interest to our analysis are the following processes, that are special cases of solutions to (3.1).

- The SOS-BM solves (3.1) with $\nu \equiv 0$, $\beta \in [-1, 1]$ and diffusion coefficient $\sigma_0(x) := \mathbb{1}_{\{x \leq 0\}}\sigma_1 + \mathbb{1}_{\{x > 0\}}\sigma_2$, with $\sigma_1, \sigma_2 > 0$. In particular it solves

$$\begin{cases} dX_t = \sigma_0(X_t)\mathbb{1}_{\{X_t \neq 0\}} dB_t + \beta dL_t^0(X), \\ \mathbb{1}_{\{X_t=0\}} dt = \rho dL_t^0(X), \end{cases} \quad (3.3)$$

Its state-space is $J = \mathbb{R}$ if $\beta \in (-1, 1)$ and scale function speed measure pair (s_0, m_0) given by

$$s_0(x) := \frac{x}{a(x)}, \quad m_0(dx) := \frac{a(x)}{\sigma_0^2(x)} dx + \rho\delta_0(dx), \quad (3.4)$$

where $a(x) := (1 + \text{sgn}(x)\beta)$.

If $\beta = 1$ (resp. $\beta = -1$), the state space is $J = [0, +\infty)$ (resp. $J = (-\infty, 0]$) because the process is reflected, and (s_0, m_0) above are supported on J .

- In case $\sigma_0 \equiv 1$, the SOS-BM is an SkS-BM. In case $\beta = 1$ and $\sigma_0 \equiv 1$, the SOS-BM is called *sticky-reflected Brownian motion* and its state-space is $[0, \infty)$.

Clarifications and Conventions. We clarify our choices regarding: (i) parametrization for the pathwise description of a diffusion and (ii) scaling in the scale and speed characterization. These conventions vary across the literature (see [2, Section 2.1] for a discussion on scaling choices). Here, we use the scaling for (s, m) so that the infinitesimal generator of the associated diffusion reads $L = (1/2)D_m D_s$ and the *stickiness parameter* ρ is the local time coefficient in the occupation-local time relation.

Under our convention for (s, m) , for a diffusion on natural scale (that is $s(x) = x$), ρ is also weight of the speed measure's atom at zero, i.e., $\rho = m(\{0\})$, otherwise one should account for the change of scale. Also, the speed measure of standard and sticky BMs are respectively $m(dx) = dx$ and $m(dx) = dx + \rho\delta_0(dx)$. Since we express these relations in terms of symmetric local time, these features remain valid even when 0 is a reflecting barrier. This is not the case if one uses instead the right local time (see, e.g., [11] for sticky BM and sticky-reflected BM).

Comparison with literature: Let us keep denoting our speed measure by m .

- In [6, pp.123–124], the generator reads $L = D_{\tilde{m}} D_s$ with speed measure of sticky BM given by $\tilde{m}(dx) = 2 dx + 2\rho\delta_0(dx) = 2m(dx)$ and our stickiness parameters match.
- In [1], the infinitesimal generator is $L = D_{\tilde{m}} D_s$ with speed measure of sticky BM given by $\tilde{m}(dx) = 2 dx + \tilde{\rho}\delta_0(dx)$ and the occupation-local time relation reads:

$$\int_0^t \mathbb{1}_{\{X_s=0\}} ds = \frac{\tilde{\rho}}{2} L_t^0(X).$$

Hence, $\tilde{m} = 2m$ and the stickiness parameter $\tilde{\rho}$ is twice ours.

3.1 Local time approximation

We now present our main results, expressed in terms of the speed measure of the SOS-BM. The proofs are given in Section 4, under the assumption that Theorem 3.4 holds for the SkS-BM. This theorem is proven in Section 6.

Theorem 3.3. *Let X be the solution to (3.1) on \mathcal{P}_x . We consider an integrable bounded function g with $g(0) = 0$ and a sequence $(u_n)_n$ that diverges slower than n , i.e.,*

$$\lim_{n \rightarrow \infty} u_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{u_n}{n} = 0. \quad (3.5)$$

Then, it holds that

$$\frac{u_n}{n} \sum_{i=1}^{[n]} g(u_n X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{P_x\text{-ucp}} \left(\int_{\mathbb{R}} g \, dm_0 \right) L_t^0(X), \quad (3.6)$$

where m_0 is the measure defined in (3.4) with

$$\sigma_0(y) := \begin{cases} \sigma(0+), & y > 0, \\ \sigma(0-), & y \leq 0. \end{cases}$$

The fact that the speed measure m_0 of the SOS-BM appears in the limit (3.6) reflects the structure of the proof. By local equivalence between the processes described by (3.1) and (3.2) and a Lamperti transform, we reduce the problem to the following result for SOS-BM.

Theorem 3.4. *Let X be the solution to (3.3) on \mathcal{P}_x and let a function g and a sequence $(u_n)_n$ both satisfying the assumptions of Theorem 3.3. Consider a twice differentiable function T such that for some $\varepsilon > 0$*

$$T(0) = 0, \quad T'(0) = 1, \quad \varepsilon \leq T'(y) \leq 1/\varepsilon, \quad |T''(y)| \leq 1/\varepsilon, \quad \forall y \in \mathbb{R}; \quad (3.7)$$

and the functions $(g_n[T])_n$ defined as $g_n[T](y) := g(u_n T(y/u_n))$, for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, it holds that

$$\frac{u_n}{n} \sum_{i=1}^{[n]} g_n[T](u_n X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{P_x\text{-ucp}} \left(\int_{\mathbb{R}} g \, dm_0 \right) L^0(X). \quad (3.8)$$

In the above result, T is a Lamperti-like transform function that allows to transform SOS-BM into some SOS-diffusions that solve equations of the kind (3.2).

3.2 Parameter estimation

Let $t > 0$ be fixed. Assume we observe an SOS-diffusion X defined on \mathcal{P}_x at increasing frequencies n : $X_0 = x, X_{1/n}, \dots, X_{i/n}, \dots, X_{[nt]/n}$. We address two distinct estimation problems. The first is the estimation of the parameters (β, ρ) of an SOS-diffusion when the volatility levels $(\sigma(0-), \sigma(0+))$ are known. The second is the joint estimation of the full parameter set $(\beta, \rho, \sigma(0-), \sigma(0+))$ of an SOS-BM.

Note that with respect to our local time approximation result, the second estimation problem faces an identifiability issue: The volatility gap $\sigma(0+) - \sigma(0-)$ and the stickiness parameter β are confounded in their asymptotic behavior of the statistic (1.1), as noted in [27]. We solve this issue in Proposition 3.6 for the case of SOS-BM by means of additional (occupation times) statistics inspired by quadratic variation estimates for Oscillating BM in [26], paralleling the treatment in [27] (both references consider $\rho = 0$). For this, we need an occupation times approximation result, Lemma 3.7, which extends and corrects a related result for sticky BM of [1].

For any non-trivial function g , let $g_{>0} := \mathbb{1}_{(0,\infty)}g$ and $g_{<0} := \mathbb{1}_{(-\infty,0)}g$. Also, for a process X , let $S_n^{g+}(X)$ and $S_n^{g-}(X)$ be statistics defined, for all n , as

$$S_n^{g+}(X) := \frac{u_n}{n} \frac{\sigma^2(0+)}{\int_{\mathbb{R}} g_{>0}(x) dx} \sum_{i=1}^{[nt]} g_{>0}(u_n X_{\frac{i-1}{n}}),$$

$$S_n^{g-}(X) := \frac{u_n}{n} \frac{\sigma^2(0-)}{\int_{\mathbb{R}} g_{<0}(x) dx} \sum_{i=1}^{[nt]} g_{<0}(u_n X_{\frac{i-1}{n}}).$$

Note that in order to compute these statistics, one needs to know $\sigma(0-)$ and $\sigma(0+)$.

We now present the first estimation result which yields consistent estimators of stickiness and skewness parameters.

Proposition 3.5. *Let X be the SOS-diffusion that solves (3.1) on \mathcal{P}_x , and let a function g and a sequence $(u_n)_n$ both satisfying the assumptions of Theorem 3.3. We consider the following estimators*

$$\hat{\rho}_n(X) := \frac{2}{n} \frac{\sum_{i=1}^{[nt]} \mathbb{1}_{\{X_{\frac{i-1}{n}}=0\}}}{S_n^{g+}(X) + S_n^{g-}(X)}, \quad \hat{\beta}_n(X) := \frac{S_n^{g+}(X) - S_n^{g-}(X)}{S_n^{g+}(X) + S_n^{g-}(X)}.$$

If $\mathcal{L}_t := \{\tau_0^X < t\}$, with τ_0^X the hitting time of 0 by X , then $\hat{\rho}_n(X)$ and $\hat{\beta}_n(X)$ are respectively $\mathbb{P}_x^{\mathcal{L}_t}$ -conditional consistent estimators of ρ and skew β , i.e.,

$$\hat{\rho}_n(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x^{\mathcal{L}_t}} \rho, \quad \text{and} \quad \hat{\beta}_n(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x^{\mathcal{L}_t}} \beta.$$

Let us note that the estimators can only be consistent on \mathcal{L}_t . Indeed, on \mathcal{L}_t^c we have that $\mathcal{O}_t^0(X) = L_t^0(X) = 0$, therefore the estimators are asymptotically 0/0 divisions.

We now state the second estimation result: The joint estimation of $(\rho, \beta, \sigma_-, \sigma_+)$ for SOS-BM.

Proposition 3.6. *Let X be the SOS-BM that solves (3.3) on \mathcal{P}_x and τ_+, τ_- be the following hitting times:*

$$\tau_+ := \inf\{s \geq 0 : X_s > 0\}, \quad \tau_- := \inf\{s \geq 0 : X_s < 0\},$$

$\mathcal{L}_t^+ := \{\tau_+ < t\}$ and $\mathcal{L}_t^- := \{\tau_- < t\}$. Then, for all $t > 0$, in \mathbb{P}_x -probability,

$$\sqrt{\frac{\sum_{i=1}^{[nt]} (X_{\frac{i}{n}}^+ - X_{\frac{i-1}{n}}^+)^2}{\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{(0,\infty)}(X_{\frac{i-1}{n}})}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x^{\mathcal{L}_t^+}} \sigma_+, \quad \sqrt{\frac{\sum_{i=1}^{[nt]} (X_{\frac{i}{n}}^- - X_{\frac{i-1}{n}}^-)^2}{\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{(-\infty,0)}(X_{\frac{i-1}{n}})}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x^{\mathcal{L}_t^-}} \sigma_-,$$

where $X^+ = \max\{0, X\}$ and $X^- = \max\{0, -X\}$.

Both estimation results rely on the following occupation times approximation, whose proof is deferred to Appendix B.

Lemma 3.7. *Let X be the SOS-diffusion solving (3.1) on \mathcal{P}_x and let U be any interval of \mathbb{R} (singletons included). Then, the following convergence holds*

$$\frac{1}{n} \sum_{i=1}^{[n\cdot]} \mathbb{1}_U(X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x\text{-ucp}} \mathcal{O}^U(X) = \int_0^\cdot \mathbb{1}_U(X_s) ds.$$

In particular, for $U = \{0\}$, this reduces to:

$$\frac{1}{n} \sum_{i=1}^{[n\cdot]} \mathbb{1}_{\{X_{\frac{i-1}{n}}=0\}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x\text{-ucp}} \int_0^\cdot \mathbb{1}_{\{0\}}(X_s) ds = \rho L^0(X).$$

Remark 3.8. One can show that Lemma 3.7 holds for all measurable real set U .

4 A natural reduction

We now prove Theorems 3.3 and 3.4, assuming that the latter holds for the SkS-BM with $\beta \in (-1, 1)$. Theorem 3.4 for SkS-BM with $\beta \in (-1, 1)$ is proven in Section 6.

We first prove Theorem 3.4 for SkS-BM with $\beta \in \{-1, 1\}$, completing the picture for SkS-BM. Next, we use this to prove Theorem 3.4 for SOS-BM. And finally, we prove Theorem 3.3. The “reduction” from SOS-diffusions to SkS-BM with $\beta \in [-1, 1]$ relies on the following two transformations

$$T_1(x) := \frac{x}{\sigma_0(x)}, \quad T_2(x) := \int_0^x \frac{\sigma_0(y)}{\sigma(y)} dy, \quad x \in \mathbb{R}. \quad (4.1)$$

The transformation T_2 (along with local equivalence of laws) reduces the problem from an SOS-diffusion to an SOS-BM. The transformation T_1 reduces the problem from an SOS-BM to an SkS-BM.

The reduction to SkS-BM, and in particular to non-reflected SkS-BM (i.e. $\beta \in (-1, 1)$), is chosen for analytical convenience. While equivalent results (to the ones derived in Sections 5 and 6, and in Appendix C) can be obtained directly for SOS-BM (reflected or not), this would require more intricate computations.

4.1 From non-reflected SkS-BM to sticky-reflected BM

Assume Theorem 3.4 holds for non-reflected SkS-BM (i.e. $\beta \in (-1, 1)$). We now show the result also holds for sticky-reflected BM (i.e. $|\beta| = 1$). Without loss of generality, we consider only the case of positive reflection (i.e. $\beta = 1$).

Given any non-reflecting SkS-Brownian motion Y of parameters (ρ, β) , the process $X := |Y|$ is a sticky-reflected BM starting at $|Y_0|$ with the same stickiness parameter ρ . Given functions (g, T) as in the assumptions of Theorem 3.4, let $\tilde{g}(y) := g(|y|)$ and $\tilde{T}(y) := \text{sgn}(y)T(\text{sgn}(y)y)$, $y \in \mathbb{R}$. Observe that $\tilde{T}(y) = T(|y|)$ if $y > 0$, and $\tilde{T} = -T(|y|)$ if $y \leq 0$. This yields the identity

$$\frac{u_n}{n} \sum_{i=1}^{[n]} g_n[T](u_n X_{\frac{i-1}{n}}) = \frac{u_n}{n} \sum_{i=1}^{[n]} \tilde{g}_n[\tilde{T}](u_n Y_{\frac{i-1}{n}}).$$

Since g and T satisfy the hypothesis of Theorem 3.4 (g bounded, integrable, vanishing at 0, and T twice differentiable so that (3.7) holds), so do the \tilde{g} and \tilde{T} . By Theorem 3.4 for the process Y , the right hand side of the above identity converges in ucp to $(\int_{\mathbb{R}} \tilde{g} dm^Y) L^0(Y)$, where since we consider symmetric local times $L^0(Y) = L^0(|Y|) = L^0(X)$ and

$$\int_{\mathbb{R}} \tilde{g} dm^Y = \int_{-\infty}^{+\infty} g(|x|)(1 + \text{sgn}(x)\beta) dx = \int_{-\infty}^{+\infty} g(|x|) dx = 2 \int_0^{+\infty} g(x) dx = \int_{\mathbb{R}} g dm^X,$$

with $m^Y(dx) := (1 + \text{sgn}(x)\beta) dx + \rho\delta_0(dx)$ and $m_0(dx) = m^X(dx) := 2\mathbb{1}_{(0+\infty)}(x) dx + \rho\delta_0(dx)$ (see (3.4)). Therefore, the convergence (3.8) holds for the process X .

Observe that the above convergence entails convergence in law: for every $t > 0$

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}) - \left(\int_{\mathbb{R}} g dm_0 \right) L_t^0(X) \xrightarrow[n \rightarrow \infty]{\text{law}} 0. \quad (4.2)$$

Now, we deduce that the convergence holds for any sticky-reflected BM. Let X be a sticky-reflected BM on \mathcal{P}_x (recall that the space is an element of the (weak) solution). By uniqueness

in law, X has the same law as the sticky-reflected BM constructed above from a non-reflected SkS-BM. Hence (4.2) holds.

Since the limit is constant, convergence in law and convergence in probability are equivalent (see e.g. [4, Theorem 25.3]). Therefore, we have that

$$\frac{u_n}{n} \sum_{i=1}^{\lfloor nt \rfloor} g_n[T](u_n X_{\frac{i-1}{n}}) - \left(\int_{\mathbb{R}} g \, dm_0 \right) L_t^0(X) \xrightarrow[n \rightarrow \infty]{P_x \rightarrow \infty} 0,$$

for all $t > 0$. By Lemma 2.2, the proof is complete for sticky-reflected BM.

4.2 From SOS-SM to SkS-BM and viceversa, proof of Theorem 3.4

Let X be a SOS-BM of parameters $(\rho, \beta, \sigma_-, \sigma_+)$.

Lemma 4.1. *The process $T_1(X)$ is an SkS-BM of parameters $(\rho^{(1)}, \beta^{(1)})$, where*

$$\rho^{(1)} := \rho \frac{2\sigma_- \sigma_+}{\sigma_-(1+\beta) + \sigma_+(1-\beta)}, \quad \beta^{(1)} := \frac{\sigma_-(1+\beta) - \sigma_+(1-\beta)}{\sigma_-(1+\beta) + \sigma_+(1-\beta)}.$$

Proof. The function T_1 is difference of two convex functions, invertible, and $xf(x) > 0$ for all $x \neq 0$. From Lemma A.1, the process $Y = T_1(X)$ solves

$$\begin{cases} Y_t = Y_0 + \int_0^t \mathbb{1}_{\{Y_s \neq 0\}} \, dW_s + \frac{(\sigma_- + \sigma_+)\beta + (\sigma_- - \sigma_+)}{(\sigma_- - \sigma_+)\beta + (\sigma_- + \sigma_+)} L_t^0(Y), \\ \int_0^t \mathbb{1}_{\{Y_s = 0\}} \, ds = \rho \frac{2\sigma_- \sigma_+}{(\sigma_- - \sigma_+)\beta + (\sigma_- + \sigma_+)} L_t^0(Y), \end{cases}$$

which proves the result. \square

Remark 4.2. 1. Similarly, if $\beta \in (-1, 1)$, one can prove that with a similar transformation $\tilde{T}_1 := [x \mapsto x/a(x)]$ with a is defined in (3.4), the process $\tilde{T}_1(X)$ is an oscillating sticky BM, i.e., an SOS-BM with skewness parameter $\beta = 0$, of stickiness and diffusion coefficients $(\rho^{(2)}, \sigma^{(2)})$ given by

$$\rho^{(2)} := \rho, \quad \sigma^{(2)}(x) := \frac{\sigma_-}{1-\beta} \mathbb{1}_{[0,+\infty)}(x) + \frac{\sigma_+}{1+\beta} \mathbb{1}_{(-\infty,0)}(x), \quad \text{for all } x \in \mathbb{R}.$$

2. For $\rho = 0$, we recover the interplay between skew and oscillating BM (without stickiness), see [16, 27].

We are now ready to address the proof of Theorem 3.4 for SOS-BM by the assumption that it holds for SkS-BM for any skewness parameter $\beta \in [-1, 1]$.

Proof of Theorem 3.4. Let T be a twice differentiable function which satisfies (3.7), and let T_1 be the function in (4.1). For all $c > 0$, consider the functions S_c and ϕ_c , defined as

$$S_c(y) = \frac{T(cy)}{c}, \quad \phi_c(y) = cy, \quad \text{for all } y \in \mathbb{R}.$$

Note that these functions are both sign-preserving and $C^2(\mathbb{R})$, and S_c further satisfies (3.7).

We consider the process $Y := T_1(X)$. By Lemma 4.1, this process is an SkS-BM of parameters $(\rho^{(1)}, \beta^{(1)})$ and

$$L^0(Y) = \frac{(1 + \beta)\sigma_- + (1 - \beta)\sigma_+}{2\sigma_+\sigma_-} L^0(Y).$$

Rewriting the statistics in terms of Y instead of X , we obtain that

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}) = \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T] \left(u_n \sigma_0(Y_{\frac{i-1}{n}}) Y_{\frac{i-1}{n}} \right).$$

Therefore,

$$\begin{aligned} & \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}) \\ &= \frac{u_n}{n} \sum_{i=1}^{[nt]} (g_{<0} \circ \phi_{\sigma_-})_n [S_{\sigma_-}] \left(u_n Y_{\frac{i-1}{n}} \right) + \frac{u_n}{n} \sum_{i=1}^{[nt]} (g_{>0} \circ \phi_{\sigma_+})_n [S_{\sigma_+}] \left(u_n Y_{\frac{i-1}{n}} \right), \end{aligned}$$

where $g_{>0} = \mathbb{1}_{(0, \infty)} g$ and $g_{<0} = \mathbb{1}_{(-\infty, 0)} g$.

Applying Theorem 3.4 for SkS-BM and the functions $S_{\sigma_+}, S_{\sigma_-}$, we get

$$\begin{aligned} \frac{u_n}{n} \sum_{i=1}^{[n]} g_n[T](u_n X_{\frac{i-1}{n}}) & \xrightarrow[n \rightarrow \infty]{\text{P}_x\text{-ucp}} (1 - \beta^{(1)}) \int g_{<0} \circ \phi_{\sigma_-}(y) dy \\ & + (1 + \beta^{(1)}) \int g_{>0} \circ \phi_{\sigma_+}(y) dy \Big) L^0(Y). \end{aligned}$$

A change of variables in the integrals, the relation between the local times, and the expression of $\beta^{(1)}$ yield

$$\frac{u_n}{n} \sum_{i=1}^{[n]} g_n[T](u_n X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\text{P}_x\text{-ucp}} \left(\frac{1 - \beta}{\sigma_-^2} \int_{\mathbb{R}} g_{<0}(y) dy + \frac{1 + \beta}{\sigma_+^2} \int_{\mathbb{R}} g_{>0}(y) dy \right) L^0(X). \quad (4.3)$$

Observe that the right-hand side of (4.3) is $(\int_{\mathbb{R}} g dm_0) L^0(X)$. This completes the proof. \square

4.3 Proof of Theorem 3.3

The proof relies on the local equivalence in law between the solutions to (3.1) and (3.2). In the entire proof, let $T > 0$ be fixed.

Let us first prove the statement for Y solution to (3.2) on $\tilde{\mathcal{P}}_x$.

First, we suppose that σ, σ' , and $1/\sigma$ are all bounded by $1/\varepsilon$ for some $\varepsilon > 0$. In this case, since the function σ/σ_0 is continuously differentiable, the transformation T_2 defined in (4.1) satisfies the condition (3.7). Also, its inverse T_2^{-1} satisfies $T_2^{-1}(0) = 0$, $(T_2^{-1})'(0) = 1$,

$$\varepsilon^2 \leq (T_2^{-1})'(y) \leq \frac{1}{\varepsilon^2}, \quad \text{and} \quad |(T_2^{-1})''(y)| \leq \frac{1}{\varepsilon^4},$$

for all $y \in \mathbb{R}$.

Defining the process $Z := T_2(Y)$ and applying the Itô–Tanaka formula (Lemma A.1), we see that it satisfies $\tilde{\mathcal{P}}_x$ -a.s. the system of equations

$$\begin{cases} dZ_t = \sigma_0(Z_t) \mathbb{1}_{\{Z_t \neq 0\}} dB_t + \beta dL_t^0(Z), & t \in [0, T], \\ \mathbb{1}_{\{Z_t = 0\}} dt = \rho dL_t^0(Y) = \rho dL_t^0(Z), & t \in [0, T]. \end{cases}$$

This qualifies Z as an SOS-BM of speed measure m_0 (see (3.4)).

Direct application of Theorem 3.4 and the a.s. identity $L^0(Y) = L^0(Z)$ yield that

$$\frac{u_n}{n} \sum_{i=1}^{[n]} g(u_n Y_{\frac{i-1}{n}}) = \frac{u_n}{n} \sum_{i=1}^{[n]} g_n[T_2^{-1}](u_n Z_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\tilde{P}_x\text{-ucp}} \left(\int_{\mathbb{R}} g \, dm_0 \right) L^0(Y).$$

For the general case ($\sigma, \sigma', 1/\sigma$ unbounded), consider an increasing sequence of compacts $(K_m)_m$ of the state space J of Y such that $0 \in K_m$, for all $m \in \mathbb{N}$, and $\bigcup_m K_m = J$.

- In case $J = [0, r)$: let $K_m = [0, r_m]$ where $r_m = m$ if $r = \infty$; otherwise $r_m = r - \frac{1}{m}$.
- In case $J = (l, r)$: let $K_m = [l_m, r_m]$ where $l_m = -m$ if $l = -\infty$; otherwise $l_m = l + \frac{1}{m}$, and r_m as above.

For all m , let τ_m be the exit time of K_m by Y . By the boundary behavior of Y , we have that $\tau_m \rightarrow \infty$ almost surely as $m \rightarrow \infty$. Therefore, by localization, we fall in the previous case ($\sigma, 1/\sigma, \sigma'$ bounded). More precisely for all $t > 0$, $\varepsilon > 0$ $m \in \mathbb{N}$,

$$\begin{aligned} & \tilde{P}_x \left(\sup_{s \leq t} \left| \frac{u_n}{n} \sum_{i=1}^{[ns]} g(u_n Y_{\frac{i-1}{n}}) - \left(\int_{\mathbb{R}} g \, dm_0 \right) L_s^0(Y) \right| \geq \varepsilon \right) \\ & \leq \tilde{P}_x \left(\sup_{s \leq t} \left| \frac{u_n}{n} \sum_{i=1}^{[ns]} g(u_n Y_{\frac{i-1}{n}}) - \left(\int_{\mathbb{R}} g \, dm_0 \right) L_s^0(Y) \right| \geq \varepsilon, \tau_m \geq t \right) + \tilde{P}_x(\tau_m < t). \end{aligned}$$

For every m , the summand of the right hand side of the latter inequality vanishes as $n \rightarrow \infty$. Therefore, letting $m \rightarrow \infty$ yields that

$$\frac{u_n}{n} \sum_{i=1}^{[n]} g(u_n Y_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\tilde{P}_x\text{-ucp}} \left(\int_{\mathbb{R}} g \, dm_0 \right) L^0(Y).$$

By local equivalence of the laws of X and Y one can show that

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g(u_n X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{P_x} \left(\int_{\mathbb{R}} g \, dm_0 \right) L_t^0(X), \quad \text{for all } t \in [0, T].$$

Since $T > 0$ was chosen arbitrarily, by Lemma 2.2, the convergence is also P_x -ucp. This completes the proof.

5 Preliminary results on SkS-BM

In this section we present several results on SkS-BM that are key for our derivations. First, we present some bounds on SkS semigroup. Then, we prove a preliminary local time approximation based on a sequence of test functions \hat{g}_n and present an asymptotic of \hat{g}_n . The proofs of the bounds and of the asymptotic are deferred to Appendix D.

Given $\rho \geq 0$ and $\beta \in (-1, 1)$, let $X^{(\rho, \beta)}$ be the SkS-BM, that is, the unique (in law) weak solution to the following system, which is a special case to (3.1) (it is (3.3) with $\sigma_0 \equiv 1$):

$$\begin{cases} dX_t^{(\rho, \beta)} = \mathbb{1}_{\{X_t^{(\rho, \beta)} \neq 0\}} dB_t + \beta dL_t^0(X^{(\rho, \beta)}), \\ \mathbb{1}_{\{X_t^{(\rho, \beta)} = 0\}} dt = \rho dL_t^0(X^{(\rho, \beta)}), \end{cases} \quad (5.1)$$

where B is a standard BM and $L^0(X^{\rho,\beta})$ is the local time of $X^{\rho,\beta}$ at zero.

Let us introduce the following notations.

Notation 5.1. Let

- $m_{(\rho,\beta)}$ be the speed measure of $X^{\rho,\beta}$, which is special case of (3.4) and admits the decomposition

$$m_{(\rho,\beta)}(dx) = m_{(0,\beta)}(dx) + \rho \delta_0(dx).$$

The measure $m_{(0,\beta)}$ is absolutely continuous with respect to Lebesgue measure λ ;

- $p_{(\rho,\beta)}(t, x, y)$ be the transition density of $X^{\rho,\beta}$ with respect to $m_{(\rho,\beta)}$, whose existence follows from [17, p. 151]. We provide it explicitly in the next section.
- If $X^{\rho,\beta}$ is the solution to (5.1) on \mathcal{P}_x , then the transition semigroup $(P_t^{(\rho,\beta)})_{t \geq 0}$ reads

$$P_t^{(\rho,\beta)} f(x) := \mathbb{E}_x [f(X_t^{(\rho,\beta)})] = \int_{\mathbb{R}} p_{(\rho,\beta)}(t, x, y) f(y) m_{(\rho,\beta)}(dy),$$

for all $t > 0$, $x \in \mathbb{R}$, and every suitable f .

Notation 5.2. For every measure m , m -integrable function f , measurable function h , and $\gamma \geq 0$, let $m(f) := \int_{\mathbb{R}} f(x) m(dx)$, $m^{(\gamma)}(h) := \int_{\mathbb{R}} |x|^\gamma |h(x)| m(dx)$.

5.1 The transition density of SkS-BM

The probability transition kernel $p_{(\rho,\beta)}$ of the (ρ, β) -SkS-BM with respect to $m_{(\rho,\beta)}$ was computed in [35, Theorem 2.4]:

$$p_{(\rho,\beta)}(t, x, y) = \frac{1}{a(y)} (u_1(t, x, y) - u_2(t, x, y)) + v_\rho(t, x, y), \quad (5.2)$$

for all $t > 0$ and $x, y \in \mathbb{R}$, where a is the function defined in (3.4), and u_1, u_2, v_ρ are defined for all $t > 0$, $x, y \in \mathbb{R}$ by

$$\begin{cases} u_1(t, x, y) = (2\pi t)^{-1/2} e^{-(x-y)^2/2t}, \\ u_2(t, x, y) = (2\pi t)^{-1/2} e^{-(|x|+|y|)^2/2t}, \\ v_\rho(t, x, y) = \frac{1}{\rho} e^{2(|x|+|y|)/\rho+2t/\rho^2} \operatorname{erfc}\left(\frac{|x|+|y|}{\sqrt{2t}} + \frac{\sqrt{2t}}{\rho}\right), \quad \forall \rho > 0, \\ v_0(t, x, y) = u_2(t, x, y). \end{cases} \quad (5.3)$$

Remark 5.3. 1. For $\rho > 0$ and $xy \leq 0$, the expression of the probability transition kernel simplifies to $p_{(\rho,\beta)}(t, x, y) = v_\rho(t, x, y)$.

2. The term $v_\rho(t, x, y)$ does not appear in the probability transition kernel of the standard BM and skew BM. The probability transition kernel of the skew BM for $\beta \neq 0$ and of the standard BM with respect to their speed measure $(m_{(0,\beta)}(dx) = a(x) dx$ and $m_{(0,0)}(dx) = dx)$, are respectively

$$p_{(0,\beta)}(t, x, y) = \frac{1}{a(y)} (u_1(t, x, y) - u_2(t, x, y)) + u_2(t, x, y)$$

$$\text{and } p_{(0,0)} = u_1(t, x, y).$$

5.2 Bounds on the SkS-BM semigroup

Lemma 5.4. *There exists a constant $K > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}$,*

$$p_{(\rho, \beta)}(t, x, y) \leq K u_1(t, x, y),$$

where K does not depend on ρ . In particular there exists a constant $K > 0$ (not depending on ρ) such that

$$v_\rho(t, x, y) \leq K u_1(t, x, y),$$

for all $t > 0, \rho \geq 0$ and $x, y \in \mathbb{R}$.

We now present bounds on the SkS-BM semigroup. These play a central part in proving the main results of this paper. The proofs are deferred to Appendix D.

Lemma 5.5. *There exists a constant $K > 0$ such that*

$$|P_t^{(\rho, \beta)} h(x)| \leq K \frac{\rho \sqrt{2t}}{\rho |x|/2 + 2t} |h(0)| + \frac{K}{\sqrt{t}} m_{(0, \beta)}(|h|) \leq \frac{K}{\sqrt{t}} m_{(\rho, \beta)}(|h|),$$

for all $h \in L^1(\mathbb{R})$, $t > 0$, and $x \in \mathbb{R}$.

Lemma 5.6. *For every $\gamma \geq 0$, there exists a constant $K_\gamma > 0$, that does not depend on (ρ, β) , such that*

$$\left| P_t^{(\rho, \beta)} h(x) - m_{(\rho, \beta)}(h) p_{(\rho, \beta)}(t, x, 0) \right| \leq K_\gamma \frac{1}{t} \left(m_{(0, \beta)}^{(1)}(h) + \frac{m_{(0, \beta)}^{(1)}(h)}{1 + |x/\sqrt{t}|^\gamma} + \frac{m_{(0, \beta)}^{(\gamma)}(h)}{1 + |x|^\gamma} \right),$$

for all $h \in L^1(\mathbb{R})$, $t > 0$, and $x \in \mathbb{R}$.

We now present the last semigroup estimates we need for our main proof (of Theorem 3.3). The result is expressed in terms of the following functional, which aggregates the action of the semigroup over the sampling data. This functional arises naturally when we consider moments of our statistics. It is defined for every suitable (e.g. bounded or positive) measurable function h , $t > 0$, $n \in \mathbb{N}$, and $x \in \mathbb{R}$, as

$$\gamma_n^{(\rho, \beta)}[h](t, x) := \sum_{i=2}^{[nt]} \mathbb{E}_x \left(h(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right) = \sum_{i=2}^{[nt]} P_{i-1}^{(\sqrt{\rho}, \beta)} h(\sqrt{n} x),$$

where the last equality is consequence of the time-space scaling of $X^{(\rho, \beta)}$, see Corollary C.4.

Lemma 5.7. (i) *There exists a positive constant $K > 0$ such that, for every $t > 0$, $n \in \mathbb{N}$, and suitable function h :*

$$|\gamma_n^{(\rho, \beta)}[h](x, t)| \leq K m_{(\rho \sqrt{n}, \beta)}(|h|) \sqrt{nt},$$

(ii) *Assume that $m_{(\sqrt{n}\rho, \beta)}(h) = 0$, for all $n \in \mathbb{N}$. Then, there exists a positive constant $K > 0$ such that, for every $t > 0$, $n \in \mathbb{N}$:*

$$\gamma_n^{(\rho, \beta)}[h](x, t) \leq K m_{(0, \beta)}(h) (1 + \max(0, \log(nt))).$$

5.3 A preliminary local time approximation

In this section we establish a preliminary local time approximation result based on the Tanaka formula. We prove also a property of this approximation that will prove useful for our further derivations.

The statistic used for our first approximation of the local time is the rescaled mean absolute displacement of the process. Given the parameters $\rho \geq 0$ and $\beta \in (-1, 1)$, the statistics is constructed from the sequence of functions

$$\widehat{g}_n(x) := \mathbb{E}_x \left[|X_1^{(\rho\sqrt{n}, \beta)}| - |x| \right], \quad n \in \mathbb{N}, \quad (5.4)$$

where $X^{(\rho\sqrt{n}, \beta)}$ is the SkS-BM on \mathcal{P}_x for all $x \in \mathbb{R}$. Recall that the state space is \mathbb{R} , since $\beta \in (-1, 1)$.

Proposition 5.8. *Let $X^{(\rho, \beta)}$ be an SkS-BM on \mathcal{P}_x . The following convergence holds:*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[n]} \widehat{g}_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x\text{-ucp}} L^0(X^{(\rho, \beta)}).$$

Proof. Since the process $(X_t^{(\rho, \beta)})_{t \geq 0}$ is a semimartingale, the (symmetric) Tanaka formula (see e.g. Itô–Tanaka formula in Lemma A.1) ensures that

$$|X_t^{(\rho, \beta)}| - |x| = \int_0^t \operatorname{sgn}(X_s^{(\rho, \beta)}) \mathbb{1}_{\{X_s \neq 0\}} dB_s + L_t^0(X^{(\rho, \beta)}).$$

Since $\int_0^t \operatorname{sgn}(X_s^{(\rho, \beta)}) \mathbb{1}_{\{X_s \neq 0\}} dB_s$ is a martingale,

$$\mathbb{E}_x \left[|X_t^{(\rho, \beta)}| - |X_s^{(\rho, \beta)}| \right] = \mathbb{E}_x \left[L_t^0(X^{(\rho, \beta)}) - L_s^0(X^{(\rho, \beta)}) \right].$$

Also, from the scaling property (Corollary C.3),

$$\frac{1}{\sqrt{n}} \widehat{g}_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) = \mathbb{E}_x \left[\left| X_{\frac{i}{n}}^{(\rho, \beta)} \right| - \left| X_{\frac{i-1}{n}}^{(\rho, \beta)} \right| \middle| \mathcal{F}_{\frac{i-1}{n}} \right].$$

Thus, from [18, Lemma 2.14],

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \widehat{g}_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) = \sum_{i=1}^{[nt]} \mathbb{E}_x \left[L_{\frac{i}{n}}^0(X^{(\rho, \beta)}) - L_{\frac{i-1}{n}}^0(X^{(\rho, \beta)}) \middle| \mathcal{F}_{\frac{i-1}{n}} \right] \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} L_t^0(X^{(\rho, \beta)})$$

and, from Lemma 2.2, the convergence is locally uniform in time, in probability (ucp). \square

Proposition 5.9. *It holds that $m_{(\rho\sqrt{n}, \beta)}(\widehat{g}_n) \rightarrow 1$ as $n \rightarrow \infty$.*

6 Proof of Theorem 3.4 for SkS-BM

Throughout this section, let $X^{(\rho, \beta)}$ be a SkS-BM (i.e., solution to (5.1)) on \mathcal{P}_x , and $m_{(\rho, \beta)}$ be its speed measure (see (3.4)). Recall the state space $J = \mathbb{R}$ since $\beta \in (-1, 1)$.

We now prove Theorem 3.4 for SkS-BM, i.e., that if $(u_n)_n$ is a sequence that diverges slower than n , T is a twice differentiable functions that satisfies (3.7), g is an integrable and bounded

function with $g(0) = 0$, and $g_n[T] := g_n[T](y) := g(u_n T(y/u_n))$, for all (n, y) ; then the following convergence holds:

$$\frac{u_n}{n} \sum_{i=1}^{\lfloor n \rfloor} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-ucp}} \left(\int_{\mathbb{R}} g \, dm_{(\rho, \beta)} \right) L^0(X^{(\rho, \beta)}). \quad (6.1)$$

For this, we consider separately two different modes of convergence for the normalizing sequence $(u_n)_n$.

- m1.** The mode $(u_n)_n : \log n/u_n \rightarrow 0, u_n \rightarrow \infty$. We prove this in Section 6.2 via rescaling to the case $u_n = n^{1/2}$ which is considered in Section 6.1.
- m2.** The mode $(u_n)_n : u_n^2/n \rightarrow 0, u_n \rightarrow \infty$. We prove this in Section 6.3 using properties of the local time of the process, namely its continuity on each half-plane and its time-scaling (Lemma C.2).

Note that any sequence $(u_n)_n$ that satisfies (3.5) falls necessarily in at least one of these modes. Note also that:

- The mode n^α with $\alpha \in [1/2, 1)$ is **m1** but not **m2**. In fact, **m1** includes the case $u_n = n^\alpha$ with $\alpha \in (0, 1)$. Both modes include $u_n = n^\alpha$ with $\alpha \in (0, 1/2)$.
- The mode $\log n$ is **m2** but not **m1**.

The proof of the mode **m1** relies on an intermediate result that quantifies the asymptotic behavior of the statistic for more general sequences of test functions than $n \mapsto g_n[T]$ and for the specific normalizing sequence $u_n = n^{1/2}$. This intermediate result, which is stated and proved in the next Section 6.1, relies on our new estimates on the sticky semigroup (Lemma 5.6) and of its action (Lemma 5.7(ii)), as well as on a linearization which leads to consider Proposition 5.8. We refer to the subsections containing the proofs for more insight.

Comments on Related Literature. The modes **m1–m2** are an improvement with respect to [1, Theorem 1.2] for sticky BM, which considers only **m2** and does not deal with skew-oscillating thresholds. The modes **m1–m2** are an improvement with respect to the results for skew BM in [25] and skew-oscillating BM in [27], where only the normalizing sequence $u_n = n^{1/2}$ is considered and do not deal with sticky thresholds. Therefore, **m1** is new in the context of sticky thresholds and **m2** is new in the context of skew-oscillating thresholds.

The ideas of the proofs and their main steps are detailed in the respective subsections.

6.1 A first result for $u_n = \sqrt{n}$

For convenience, we fix x , and we let $X^{(\rho, \beta)}$ be solution to (5.1) on \mathcal{P}_x . We use Notation 5.2 and introduce the sequence of functionals $(B_n)_n$ defined for all bounded measurable f , and $n \in \mathbb{N}$ as

$$\begin{aligned} B_n[f](x) &:= \frac{f^2(\sqrt{n}x)}{n} + \frac{m_{(\rho\sqrt{n}, \beta)}(f^2)}{\sqrt{n}} \\ &+ \frac{m_{(0, \beta)}^{(1)}(f) (1 + \log(n)) f(\sqrt{n}x)}{n} + \frac{(1 + \log(n)) m_{(\rho\sqrt{n}, \beta)}(|f|)}{\sqrt{n}}. \end{aligned}$$

The main result of this section, Proposition 6.3, works under the following condition on the sequence of test functions $(g_n)_n$.

Condition 6.1. *The real sequence $n \mapsto B_n[g_n](x)$ vanishes as $n \rightarrow \infty$.*

Remark 6.2. If two sequences of functions $(f_n)_n, (g_n)_n$ satisfy Condition 6.1, then the sequence $(f_n + g_n)_n$ also satisfies Condition 6.1.

We can now present the result, which reads as follows.

Proposition 6.3. *Let $(g_n)_n$ be a sequence of functions that satisfies Condition 6.1 and $\lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta)}(g_n) = M$. Then, for all $t \geq 0$,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \xrightarrow[n \rightarrow \infty]{P_x} ML_t^0(X^{(\rho, \beta)}). \quad (6.2)$$

Also, if $\sup_n \{m_{(\sqrt{n}\rho, \beta)}(|g_n|)\} < \infty$, then the above convergence is P_x -ucp.

The proof relies on a linearization of the statistic in (6.2) with respect to the approximation from Proposition 5.8. More specifically, instead of considering the sequence $(g_n)_n$, one considers the sequence $(c_n \hat{g}_n)_n$ with $(c_n)_n$ a suitable sequence converging to M . We then establish that the difference between the statistic with g_n and the one constructed from its approximation $c_n \hat{g}_n$ is bounded in $L^2(\mathbb{P}_x)$ by $KB_n[g_n - c_n \hat{g}_n](x)$, where $K > 0$ is a constant. (See Lemma 6.5.) Therefore, Condition 6.1 on $(g_n)_n$ guarantees that also $(g_n - c_n \hat{g}_n)_n$ satisfies Condition 6.1 and so, the two corresponding statistics have the same limit in probability.

Before proving Proposition 6.3, we provide a couple of preliminary results that are useful to the proof. The first one is not directly applied in this paper, but contains steps used in proofs of subsequent sections, like Proposition 6.8 and the one of Lemma 6.12.

Lemma 6.4. *Let $(g_n)_n$ be a sequence of functions such that*

$$\lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta)}(|g_n|) + g_n(\sqrt{n}x)/\sqrt{n} = 0.$$

Then, as $n \rightarrow \infty$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \rightarrow 0,$$

locally uniformly in time, in $L^1(\mathbb{P}_x)$.

Proof. We observe that

$$\mathbb{E}_x \left(\sup_{s \in [0, t]} \left| \sum_{i=1}^{\lfloor ns \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right| \right) \leq |g_n(x\sqrt{n})| + \gamma_n[|g_n|](x, t).$$

Thus, from Item (i) in Lemma 5.7,

$$\mathbb{E}_x \left(\sup_{s \in [0, t]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right| \right) \leq \frac{|g_n(x\sqrt{n})|}{\sqrt{n}} + Km_{(\sqrt{n}\rho, \beta)}(|g_n|)\sqrt{t},$$

which vanishes as $n \rightarrow \infty$. This completes the proof. \square

The next result is an adaptation to this context of [19, Lemma 4.2].

Lemma 6.5. Let $(g_n)_n$ be a sequence of functions that satisfies Condition 6.1 and for all $n \in \mathbb{N}$: $m_{(\sqrt{n}\rho, \beta)}(g_n) = 0$. Then,

$$\lim_{n \rightarrow \infty} \mathbb{E}_x \left(\left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right|^2 \right) = 0.$$

Proof. It holds that

$$\begin{aligned} \mathbb{E}_x \left(\left| \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right|^2 \right) &\leq g_n^2(\sqrt{nx}) + \gamma_n[g_n^2](t, x) \\ &\quad + 2 \sup_{y \in \mathbb{R}; s \leq t} \{|\gamma_n[g_n](s, y)|\} \left(g_n(\sqrt{nx}) + \gamma_n[g_n](t - 1/n, x) \right). \end{aligned}$$

From Lemma 5.7, for some $K_t > 0$,

$$\begin{aligned} \sup_{y \in \mathbb{R}; s \leq t} \{|\gamma_n[g_n](s, y)|\} &\leq K_t m_{(0, \beta)}^{(1)}(g_n) (1 + \log(n)), \\ \gamma_n[g_n](t, x) &\leq K_t m_{(\sqrt{n}\rho, \beta)}(|g_n|) \sqrt{n}, \\ \gamma_n[g_n^2](t, x) &\leq K_t m_{(\sqrt{n}\rho, \beta)}(g_n^2) \sqrt{n}. \end{aligned}$$

Therefore,

$$\mathbb{E}_x \left(\left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right|^2 \right) \leq K_t B_n[g_n](x),$$

which converges to 0 as $n \rightarrow \infty$. \square

Lemma 6.6. The sequence of functions $(\hat{g}_n)_n$ defined in (5.4) satisfies Condition 6.1. Moreover,

$$\lim_{n \rightarrow \infty} \hat{g}_n(0) = 0.$$

Proof. In this proof, with abuse of notation, we make vary the probability space \mathcal{P}_y , $y \in \mathbb{R}$, to underline when the process we are considering starts at y . By [32, p. 277] or [35, above Proposition 2.2], for all n , there exists a skew BM $X^{(0, \beta)}$ defined on an extension of the probability space such that $X^{(\rho, \beta)} = (X_{\gamma_n(t)}^{(0, \beta)})_{t \geq 0}$, where the time change γ_n is (the right-inverse of $t \mapsto t + c\rho\sqrt{n} L_t^0(X^{(0, \beta)})$, with c a non-negative constant and it is) strictly increasing, a.s. continuous, and $0 < \gamma_n(t) \leq t$, for all $t \geq 0$. By the above we have that

$$0 \leq \hat{g}_n(y) = \mathbb{E}_y \left(|X_1^{(\rho\sqrt{n}, \beta)}| - |y| \right) = \mathbb{E}_y \left(|X_{\gamma_n(1)}^{(0, \beta)}| - |y| \right) \leq \mathbb{E}_y \left(|X_1^{(0, \beta)}| - |y| \right).$$

Recall the scale function s of $X^{(0, \beta)}$, defined in (3.4). We observe that $s(X^{(0, \beta)})$ is a martingale and $z \mapsto |s^{-1}(z)|$ is convex. This qualifies the process $|X^{(0, \beta)}| = |s^{-1} \circ s(X^{(0, \beta)})|$ as a supermartingale. This, combined with Lemma 5.4, yields

$$0 \leq \hat{g}_n(y) \leq K \mathbb{E}_y (|B_1| - |y|) =: K \hat{g}(y).$$

The function \hat{g} satisfies Condition 6.1 with $\rho = 0$ in $B_n[\hat{g}](x)$ (see the proof of [19, Theorem 4.1]). To show that $(\hat{g}_n)_n$ satisfies Condition 6.1 for $\rho \neq 0$, it remains to prove that

$$\lim_{n \rightarrow \infty} \hat{g}_n^2(0) \rightarrow 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} (1 + \log n) \hat{g}_n(0) \rightarrow 0.$$

Indeed, since $0 < a(y) \leq 2$,

$$\begin{aligned}\widehat{g}_n(0) &= \mathbb{E}_0 \left(|X_1^{(\rho\sqrt{n}, \beta)}| \right) = \int_{\mathbb{R}} |y| p_{(\rho\sqrt{n}, \beta)}(1, 0, y) m_{(\rho\sqrt{n}, \beta)}(dy) \\ &\leq 2 \int_{\mathbb{R}} |y| v_{\rho\sqrt{n}}(1, 0, y) dy \\ &= \frac{2}{\rho\sqrt{n}} \int_{\mathbb{R}} |y| e^{2|y|/\rho\sqrt{n}+2/\rho^2 n} \operatorname{erfc} \left(\frac{|y|}{\sqrt{2}} + \frac{\sqrt{2}}{\rho\sqrt{n}} \right) dy.\end{aligned}$$

Therefore, $((1 + \log n)\widehat{g}_n(0)) \rightarrow 0$ as $n \rightarrow \infty$. Similarly,

$$\begin{aligned}(\widehat{g}_n(0))^2 &\leq \mathbb{E}_0 \left(|X_1^{(\rho\sqrt{n}, \beta)}|^2 \right) \leq 2 \int_{\mathbb{R}} |y|^2 v_{\rho\sqrt{n}}(1, 0, y) dy \\ &= \frac{2}{\rho\sqrt{n}} \int_{\mathbb{R}} |y|^2 e^{2|y|/\rho\sqrt{n}+2/\rho^2 n} \operatorname{erfc} \left(\frac{|y|}{\sqrt{2}} + \frac{\sqrt{2}}{\rho\sqrt{n}} \right) dy.\end{aligned}$$

Therefore, $\widehat{g}_n^2(0) \rightarrow 0$ as $n \rightarrow \infty$ and Condition 6.1 is satisfied. This finishes the proof. \square

We are now ready to address the main proof of this section.

Proof of Proposition 6.3. Let $(\widehat{g}_n)_{n \in \mathbb{N}}, (h_n)_{n \in \mathbb{N}}$ be the sequences of functions defined for all $y \in \mathbb{R}, n \in \mathbb{N}$ by

$$h_n(y) = g_n(y) - \frac{m_{(\sqrt{n}\rho, \beta)}(g_n)}{m_{(\sqrt{n}\rho, \beta)}(\widehat{g}_n)} \widehat{g}_n(y), \quad \widehat{g}_n(y) = \mathbb{E}_y \left(|X_1^{(\rho\sqrt{n}, \beta)}| - |y| \right).$$

Note that \widehat{g}_n was already defined in (5.4).

From Proposition 5.9, Remark 6.2, and Lemma 6.6, the sequence $(h_n)_n$ satisfies Condition 6.1. Also, for all $n \in \mathbb{N}$, $m_{(\sqrt{n}\rho, \beta)}(h_n) = 0$. Thus, from Lemma 6.5,

$$\mathbb{E}_x \left(\left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} h_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right|^2 \right) \xrightarrow{n \rightarrow \infty} 0, \quad \text{for all } t > 0.$$

Convergences in probability follows, i.e.,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} h_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 0, \quad \text{for all } t > 0. \quad (6.3)$$

This, combined with Propositions 5.8 and 5.9, proves (6.2).

It remains to prove that the convergence is \mathbb{P}_x -ucp under the additional assumption that $\sup_n \{m_{(\sqrt{n}\rho, \beta)}(|g_n|)\} < \infty$. If $(g_n)_n$ are all positive, the processes $\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} g_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho, \beta)})$ are non-decreasing and have a continuous limit, \mathbb{P}_x -almost surely. Thus, by Lemma 2.2, the convergence is locally uniform in time, in probability (ucp). More precisely, for an arbitrary $(g_n)_n$ satisfying the conditions of Proposition 6.3, let $g_n = g_n^+ - g_n^-$, where $g_n^+(y) = \max\{g_n(y), 0\}$ and $g_n^-(y) = \max\{-g_n(y), 0\}$. Since $(g_n^+)_n$ and $(g_n^-)_n$ are both sequences of positive functions, if $\lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta)}(g_n^+) =: M_+$ and $\lim_{n \rightarrow \infty} m_{(\sqrt{n}\rho, \beta)}(g_n^-) =: M_-$, then

$$\begin{aligned}\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n \cdot \rfloor} g_n^+(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}_x \text{-ucp}} M_+ L^0(X^{(\rho, \beta)}) \\ \text{and} \quad \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n \cdot \rfloor} g_n^-(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}_x \text{-ucp}} M_- L^0(X^{(\rho, \beta)}).\end{aligned}$$

From this, one easily deduces that the convergence (6.3) is also ucp. This completes the proof. \square

6.2 Proof of the mode [m1](#)

The statement in this case is the following.

Proposition 6.7. *The convergence (6.1) holds under the additional assumption that*

$$\log n/u_n \xrightarrow[n \rightarrow \infty]{} 0.$$

The proof of this mode reduces to applying Proposition 6.3 after a proper re-scaling of the test function. More precisely, we first prove Proposition 6.8 which allows to pass from considering $u_n = \sqrt{n}$ (for which Proposition 6.3 holds) to general u_n . The proof of Proposition 6.8 is based on showing that the asymptotic behavior of the same statistics associated with a sequence $(g_n)_n$ is similar to that of the statistics with $g_n \mathbb{1}_{(-r,r)}$, $r > 0$, instead of g_n . Next, in the proof of Proposition 6.7, we show that the sequence $(g_n[T])_n$ satisfies the assumptions of Proposition 6.8.

Proposition 6.8. *Let $(g_n)_n$ be a sequence of functions and $(u_n)_n$ a sequence of real numbers such that*

1. *as $n \rightarrow \infty$: $u_n \rightarrow \infty$, $\log n/u_n \rightarrow 0$, and $u_n/n \rightarrow 0$,*
2. *$(g_n)_n$ a sequence of functions such that*

$$\begin{aligned} \sup \lambda(|g_n|) &< \infty, & \sup_{n,y} |g_n(y)| &< \infty, \\ \lim_{n \rightarrow \infty} m_{(0,\beta)}(g_n) &=: M, & \lim_q \limsup_n \int_{|y|>q} |g_n(y)| dy &= 0. \end{aligned}$$

Then,

$$\frac{u_n}{n} \sum_{i=1}^{[n]} g_n(u_n X_{\frac{i-1}{n}}^{(\rho,\beta)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-ucp}} ML^0(X^{(\rho,\beta)}).$$

Proof. Let $(k_n)_n$ be the sequence of functions defined for all $y \in \mathbb{R}$ by $k_n(y) = \frac{u_n}{\sqrt{n}} g_n\left(\frac{u_n}{\sqrt{n}} y\right)$. Assume first that $\sup_n \lambda^{(1)}(g_n) < \infty$. Then, for some $K > 0$,

$$\begin{aligned} |k_n(x)| &\leq K \frac{u_n}{\sqrt{n}}, & m_{(0,\beta)}(k_n) &= m_{(0,\beta)}(g_n), & m_{(0,\beta)}(|k_n|) &= m_{(0,\beta)}(|g_n|) \leq K, \\ m_{(0,\beta)}^{(1)}(k_n) &= \frac{\sqrt{n}}{u_n} m_{(0,\beta)}^{(1)}(g_n) \leq \frac{\sqrt{n}}{u_n} K, & \lambda(k_n^2) &= \frac{u_n}{\sqrt{n}} \lambda(g_n^2) \leq \frac{u_n}{\sqrt{n}} K. \end{aligned}$$

From the hypotheses made on $(u_n)_n$ and $(g_n)_n$, we have also that $(k_n)_n$ satisfies Condition 6.1. Thus, from Proposition 6.3,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[n]} k_n(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho,\beta)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-ucp}} ML^0(X^{(\rho,\beta)}). \quad (6.4)$$

This completes the proof under the condition $\sup_n \lambda^{(1)}(g_n) < \infty$.

Consider now $\sup_n \lambda^{(1)}(g_n) = \infty$. For all $r \geq 1$ consider $h_{n,r} := g_n \mathbb{1}_{(-r,r)}$. Then, $m_{(0,\beta)}^{(1)}(h_{n,r}) \leq r m_{(0,\beta)}(|g_n|) < \infty$. Moreover, for all r , $m_{(0,\beta)}(h_{n,r}) \rightarrow \alpha_r$ as $n \rightarrow \infty$ (up to considering a subsequence) for α_r such that $\alpha_r \rightarrow M$ as $r \rightarrow \infty$. Hence, from (6.4),

$$\frac{u_n}{n} \sum_{i=1}^{[n]} h_{n,r}(u_n X_{\frac{i-1}{n}}^{(\rho,\beta)}) \xrightarrow[n \rightarrow \infty]{P_x\text{-ucp}} \frac{\alpha_r}{2} L^0(X^{(\rho,\beta)}).$$

Since $\alpha_r \rightarrow M$ as $r \rightarrow \infty$, it remains to show that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}_x \left(\sup_{s \in [0, t]} \left| \frac{u_n}{n} \sum_{i=1}^{[ns]} g_n(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) - \frac{u_n}{n} \sum_{i=1}^{[ns]} h_{n,r}(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right| \right) = 0. \quad (6.5)$$

If $\bar{h}_{n,r}(y) = g_n(y) - h_{n,r}(y) = g_n(y) \mathbb{1}_{\{|y| > r\}}$, then the expectation in (6.5) is bounded by

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\left| \bar{h}_{n,r}(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right| \right) \leq \frac{u_n}{n} \bar{h}_{n,r}(u_n x) + \frac{1}{\sqrt{n}} \gamma_n[\tilde{h}_{n,r}](x, t),$$

where $\tilde{h}_{n,r}(y) := \frac{u_n}{\sqrt{n}} \bar{h}_{n,r}(\frac{u_n}{\sqrt{n}} y) = |k_n|(y) \mathbb{1}_{\{|y| > r\sqrt{n}/u_n\}}$ and $\gamma_n[\tilde{h}_{n,r}](y, t)$ defined in Notation 5.2. Since $|k_n|$ is bounded by Ku_n/\sqrt{n} ,

$$\lim_{n \rightarrow \infty} \frac{u_n}{n} \bar{h}_{n,r}(u_n x) = 0.$$

It remains to check that

$$\limsup_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \gamma_n[\tilde{h}_{n,r}](x, t) = 0.$$

Indeed, since $\tilde{h}_{n,r}(0) = 0$, Lemma 5.7 yields

$$\frac{1}{\sqrt{n}} \gamma_n[\tilde{h}_{n,r}](x, t) \leq Km_{(0, \beta)}(|\tilde{h}_{n,r}|) \sqrt{t},$$

which converges to 0 as $r \rightarrow \infty$. □

Proof of Proposition 6.7. We observe that

$$\begin{aligned} \sup_{n, y} |g_n[T](y)| &\leq \|g\|_\infty, & \lambda(|g_n[T]|) &\leq \lambda(|g|) \left\| \frac{1}{T'} \right\|_\infty \leq \frac{\lambda(|g|)}{\varepsilon}, \\ \lim_{n \rightarrow \infty} m_{(0, \beta)}(g_n[T]) &= m_{(0, \beta)}(g). \end{aligned}$$

Also,

$$\begin{aligned} \int_{|y| > q} |g_n[T](y)| \, dy &= \int_{|y| > \frac{q}{u_n}} |g(u_n T(y))| u_n \, dx \\ &\leq \frac{1}{\varepsilon} \int_{u_n |y| > q} |g(u_n y)| u_n \, dy \leq \frac{1}{\varepsilon} \int_{|y| > q} |g(y)| \, dy, \end{aligned}$$

which converges to 0 as $q \rightarrow \infty$. Since $g(0) = 0$ and $m_{(0, \beta)}$ is equal to λ up to a multiplicative bounded weight, the conditions of Proposition 6.8 are satisfied for $(g_n)_n = (g_n[T])_n$. The proof is thus completed. □

6.3 Proof of the mode m_2

The object of this section is to prove the next Proposition 6.9. The proof relies on reduction of the problem of convergence for the discrete empirical statistic

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta)})$$

to one for the continuous quantity

$$\int_{\mathbb{R}} g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy.$$

We show that (i) the continuous quantity converges to the desired limit (Lemma 6.10) and that (ii) discrete and continuous quantities have the same limit. For the former, we use For the latter, we use the functionals introduced in Lemma 6.11, the limit therein, and the bound from Lemma 6.12. Note that Lemma 6.11 consider only Lipschitz functions vanishing in the vicinity of the origin, with whom we approximate in L^1 the test function g in the proof of Proposition 6.9.

The strategy of our proof is inspired by the ones of [1, Theorem 3.1] for sticky BM and [19] for standard BM, as the semigroups of sticky BM and SOS-BM scale similarly in time, space, and stickiness parameter. The main particularity of SOS-BM is the discontinuous character of the local time field in space at 0, caused by the skew-oscillating behavior of the process. Therefore, whenever we use continuity arguments (as in Lemma 6.10), we need to handle the behavior on each half-line separately. Moreover, we provide a detailed proof to complete and revise some steps in [1].

Proposition 6.9. *The convergence (6.1) holds under the additional assumption that $(u_n^2)_n$ satisfies (3.5), that is:*

$$u_n^2/n \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad u_n \xrightarrow{n \rightarrow \infty} \infty. \quad (6.6)$$

For this, we use the following preliminary results.

Lemma 6.10. *Assume $X^{(\rho,\beta)}$, \mathcal{P}_x , u_n , g , and T as in the foreword of this section. The following convergence holds*

$$\int_{\mathbb{R}} g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy \xrightarrow[n \rightarrow \infty]{\mathcal{P}_x} m_{(\rho,\beta)}(g) L_t^0(X^{(\rho,\beta)}).$$

Proof. We first prove that the right and left local times $(\ell_t^y(X^{(\rho,\beta)}); t \geq 0, y \in \mathbb{R})$ and $(\ell_t^{y-}(X^{(\rho,\beta)}); t \geq 0, y \in \mathbb{R})$ are both almost surely (t, y) -jointly continuous on each half-plane and

$$\ell_t^0(X^{(\rho,\beta)}) = (1 + \beta) L_t^0(X^{(\rho,\beta)}), \quad \ell_t^{0-}(X^{(\rho,\beta)}) = (1 - \beta) L_t^0(X^{(\rho,\beta)}). \quad (6.7)$$

For this we reduce to the case of skew BM. Indeed, as seen in Lemma 6.6, there exists a skew BM $X^{(0,\beta)}$ defined on an extension of the probability space such that $X^{(\rho,\beta)} = (X_{\gamma(t)}^{(0,\beta)})_{t \geq 0}$, where γ a a.s. continuous strictly increasing time-change. By [30, Exercice VI.1.27], we have that $\ell_t^0(X^{(\rho,\beta)}) = \ell_{\gamma(t)}^0(X^{(0,\beta)})$ for all $t \geq 0$.

By [30, Theorem VI.1.6], the right local time field $(t, y) \rightarrow \ell_t^y(X^{(0,\beta)})$ and symmetric local time field $(t, y) \rightarrow L_t^y(X^{(0,\beta)})$ are both almost surely (t, y) -jointly continuous on each half-plane $(\mathbb{R}_+ \times (-\infty, 0))$, $(\mathbb{R}_+ \times [0, \infty))$. The relation between right and symmetric local times of $X^{(0,\beta)}$ at the skew threshold 0 is given by the identities (see e.g. [5, Theorem 2.1])

$$\ell_t^0(X^{(0,\beta)}) = (1 + \beta) L_t^0(X^{(0,\beta)}), \quad \ell_t^{0-}(X^{(0,\beta)}) = (1 - \beta) L_t^0(X^{(0,\beta)}).$$

Since γ^{-1} is a.s. continuous, the first step of the proof is demonstrated.

From the a.s. piece-wise continuity of $(t, y) \rightarrow \ell_t^y(X^{(\rho,\beta)})$, for all $t \geq 0$ and $y > 0$,

$$|\ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) - \ell_t^y(X^{(\rho,\beta)})| \xrightarrow[n \rightarrow \infty]{\mathcal{P}_x \text{-a.s.}} 0 \quad \text{and} \quad |\ell_{[nt]/n}^{-y/u_n}(X^{(\rho,\beta)}) - \ell_t^{y-}(X^{(\rho,\beta)})| \xrightarrow[n \rightarrow \infty]{\mathcal{P}_x \text{-a.s.}} 0. \quad (6.8)$$

Let us fix $t > 0$ and an event Ω_0 of full probability such that the convergences (6.8) hold for all $\omega \in \Omega_0$. We now prove that, on Ω_0 ,

$$\int_0^\infty g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy \xrightarrow{n \rightarrow \infty} \ell_t^0(X^{(\rho,\beta)}) \int_0^\infty g(y) dy$$

and

$$\int_{-\infty}^0 g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy \xrightarrow{n \rightarrow \infty} \ell_t^{0-}(X^{(\rho,\beta)}) \int_{-\infty}^0 g(y) dy.$$

We focus on the first convergence. The second convergence can be proven with similar arguments. We observe that

$$\begin{aligned} \int_0^\infty g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy - \ell_t^0(X^{(\rho,\beta)}) \int_0^\infty g(y) dy \\ = \int_0^\infty g_n[T](y) \left(\ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) - \ell_t^0(X^{(\rho,\beta)}) \right) dy \\ + \ell_t^0(X^{(\rho,\beta)}) \int_0^\infty (g_n[T](y) - g(y)) dy. \end{aligned} \quad (6.9)$$

Regarding the second additive term of the right-hand-side of (6.9), note that

$$\int_0^{+\infty} g_n[T](y) dy = \int_0^{+\infty} g(y) \frac{1}{T'(T^{-1}(y/u_n))} dy. \quad (6.10)$$

And, since T' is lower and upper bounded by a positive constant, $T'(0) = 1$, and g is bounded, by dominated convergence we have

$$\int_0^{+\infty} |g_n[T](y) - g(y)| dy \leq \int_0^{+\infty} |g(y)| \left| \frac{1}{T'(T^{-1}(y/u_n))} - 1 \right| dy \xrightarrow{n \rightarrow \infty} 0.$$

Let us consider the first additive term of the right-hand-side of (6.9). By the non-explosive nature of the process, it holds a.s. that for every $s > 0$

$$-\infty < \inf_{u \leq s} X_u < \sup_{u \leq s} X_u < \infty.$$

Note that $\ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) \leq \ell_t^{y/u_n}(X^{(\rho,\beta)}) = 0$ for all $y > u_n \sup_{u \leq t} X_u^{(\rho,\beta)}$ and by continuity in space $K := \sup_{y \in [0, +\infty)} \ell_t^{y/u_n}(X^{(\rho,\beta)}) = \sup_{y \in [0, \sup_{u \leq t} X_u^{(\rho,\beta)}]} \ell_t^y(X^{(\rho,\beta)}) < \infty$. For all $y \geq 0$,

$$\begin{aligned} & \left| g_n[T](y) \left(\ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) - \ell_t^0(X^{(\rho,\beta)}) \right) \right| \\ & \leq |g_n[T](y)| \left(\ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) + \ell_t^0(X^{(\rho,\beta)}) \right) \leq (K + \ell_t^0(X^{(\rho,\beta)})) |g_n[T](y)|. \end{aligned}$$

By (6.10) and lower boundedness of $|T'|$, the above expression is integrable, and $\int_0^\infty |g_n[T](y)| dy \leq \|1/T'\|_\infty \int_0^\infty |g(y)| dy < +\infty$. Moreover, for every $y \in (0, +\infty)$ we have

$$\left| g_n[T](y) \left(\ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) - \ell_t^0(X^{(\rho,\beta)}) \right) \right| \leq \|g\|_\infty \left| \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) - \ell_t^0(X^{(\rho,\beta)}) \right|,$$

which converges to 0 by joint continuity of the local time. By dominated convergence,

$$\int_0^\infty g_n[T](y) \left(\ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) - \ell_t^0(X^{(\rho,\beta)}) \right) dy$$

vanishes as $n \rightarrow \infty$.

We have thus obtained the almost sure convergence

$$\int_0^\infty g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy \xrightarrow[n \rightarrow \infty]{P_x \text{-a.s.}} \left(\int_0^\infty g(y) dy \right) \ell_t^0(X^{(\rho,\beta)}).$$

An identical argument applied to the interval $(-\infty, 0)$ yields

$$\int_{-\infty}^0 g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy \xrightarrow[n \rightarrow \infty]{P_x \text{-a.s.}} \left(\int_{-\infty}^0 g(y) dy \right) \ell_t^{0-}(X^{(\rho,\beta)}).$$

Adding these two results together, we conclude that the P_x -almost sure limit of the sequence of integrals $n \mapsto \int_{\mathbb{R}} g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho,\beta)}) dy$, as $n \rightarrow \infty$ is

$$\left(\int_{-\infty}^0 g(y) dy \right) \ell_t^{0-}(X^{(\rho,\beta)}) + \left(\int_0^\infty g(y) dy \right) \ell_t^0(X^{(\rho,\beta)}).$$

By the relation between right and symmetric local times (6.7), the proof is complete. \square

Lemma 6.11. *Let k be a bounded Lipschitz function with compact support that vanishes on an open interval around 0, $(u_n)_n$ be a sequence that satisfies (6.6), T a function that satisfies (3.7) and $(k_n[T])_n, (F_n[k])_n$ be the sequences of functions defined for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ as*

$$k_n[T](x) := k(u_n T(x/u_n)), \quad F_n[k](x) := \int_0^1 \left(P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](x) - k_n[T](x) \right) ds.$$

The sequence $\left(m_{(u_n \rho, \beta)}(F_n[k]) \right)_n$ vanishes as $n \rightarrow \infty$.

Proof. For all $\delta > 0$, let $U_\delta = \{x \in \mathbb{R} : |x| \in (\delta, 1/\delta)\}$. We choose $\delta > 0$ such that for all $x \notin U_\delta$, $k(x) = 0$. We observe that there exists some $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $|x| \notin U_{\delta/2}$, $k_n[T](x) = 0$.

In what follows, we consider separately the absolutely continuous and singular parts of $m_{(u_n \rho, \beta)}$. By Fubini's theorem,

$$\begin{aligned} \left| m_{(\cdot, \beta)}(F_n[k]) \right| &= \left| \int_{\mathbb{R}} \left(\int_0^1 \left(P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](x) - k_n[T](x) \right) ds \right) m_{(\cdot, \beta)}(dx) \right| \\ &\leq \int_0^1 \left(\int_{\mathbb{R}} \left| P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](x) - k_n[T](x) \right| m_{(\cdot, \beta)}(dx) \right) ds. \end{aligned}$$

Also,

$$|F_n[k](0)| \leq \int_0^1 \left| P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](0) - k_n[T](0) \right| ds.$$

Lemma 5.4 yields

$$\begin{aligned} &\left| P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](x) - k_n[T](x) \right| \\ &\leq \int_{\mathbb{R}} |k_n[T](y) - k_n[T](x)| p_{(u_n \rho, \beta)}(su_n^2/n, x, y) m_{(u_n \rho, \beta)}(dy) \\ &\leq K \int_{\mathbb{R}} |k_n[T](y) - k_n[T](x)| u_1(su_n^2/n, x, y) dy \\ &\quad + K |k_n[T](x)| u_1(su_n^2/n, x, 0). \end{aligned} \tag{6.11}$$

Regarding the second additive term of the right-hand-side of (6.11), by the convergence $\lim_{n \rightarrow \infty} u_n^2/n = 0$ and boundedness of k (and of k_n), it follows that

$$\lim_{n \rightarrow \infty} K |k_n[T](x)| u_1(su_n^2/n, x, 0) = 0, \quad \text{for all } x \in \mathbb{R} \text{ and } s \in [0, 1].$$

It also follows that there exists some $n_1 \geq n_0$ such that for all $n \geq n_1$ and $s \in [0, 1]$,

$$\begin{aligned} K |k_n[T](x)| u_1(su_n^2/n, x, 0) &\leq K \|k\|_\infty u_1(su_n^2/n, x, 0) \mathbb{1}_{U_{\delta/2}} \\ &\leq \|k\|_\infty \mathbb{1}_{U_{\delta/2}}(x) \sup_{\substack{n \geq n_1 \\ x \in U_{\delta/2}}} u_1(su_n^2/n, x, 0) \\ &\leq K \|k\|_\infty u_1(u_{n_1}^2/n_1, 0, \delta/2) \mathbb{1}_{U_{\delta/2}}(x), \end{aligned}$$

which is $L^1(\mathbb{R})$. Thus, from dominated convergence,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} K |k_n[T](x)| u_1(su_n^2/n, x, 0) m_{(\rho, \beta)}(dx) = 0.$$

Regarding the second additive term, if Z is a Gaussian $\mathcal{N}(0, 1)$ random variable, by assumptions on k , for every $n \geq n_0$ and $s \in [0, 1]$,

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} |k_n[T](y) - k_n[T](x)| u_1(su_n^2/n, x, y) dy m_{(\cdot, \beta)}(dx) \\ &= \mathbb{E} \left(\int_{\mathbb{R}} \left| k_n[T] \left(x + \frac{u_n \sqrt{s}}{\sqrt{n}} Z \right) - k_n[T](x) \right| m_{(\cdot, \beta)}(dx) \right) \\ &= \mathbb{E} \left(\int_{\mathbb{R}} \left(\mathbb{1}_{U_{\delta/2}}(x) + \mathbb{1}_{U_{\delta/2}} \left(x + \frac{u_n \sqrt{s}}{\sqrt{n}} Z \right) \right) \left| k_n[T] \left(x + \frac{u_n \sqrt{s}}{\sqrt{n}} Z \right) - k_n[T](x) \right| m_{(\cdot, \beta)}(dx) \right) \\ &\leq \|k\|_{\text{Lip.}} \|T\|_{\text{Lip.}} \frac{u_n}{\sqrt{n}} \mathbb{E} \left(|Z| \int_{\mathbb{R}} \left(\mathbb{1}_{U_{\delta/2}}(x) + \mathbb{1}_{U_{\delta/2}} \left(x + \frac{u_n \sqrt{s}}{\sqrt{n}} Z \right) \right) m_{(\cdot, \beta)}(dx) \right) \\ &\leq \|k\|_{\text{Lip.}} \|T\|_{\text{Lip.}} \frac{u_n}{\sqrt{n}} \frac{8}{\delta} \mathbb{E}(|Z|), \end{aligned}$$

which, since $E(|Z|) < \infty$, converges to 0 as $n \rightarrow \infty$.

Since the bounds do not depend on the variable s , from the dominated convergence theorem on $L^1([0, 1])$, we obtain

$$m_{(\cdot, \beta)}(F_n[k]) \xrightarrow[n \rightarrow \infty]{} 0.$$

Regarding the singular part of the measures, from (5.2),

$$\begin{aligned} \rho u_n \left| P_{su_n^2/n}^{(u_n \rho, \beta)} k_n[T](0) - k_n[T](0) \right| &\leq \rho u_n K \int_{\mathbb{R}} |k_n[T](y) - k_n[T](0)| v_{u_n \rho}(su_n^2/n, 0, y) dy \\ &= \rho u_n K \int_{U_{\delta/2}} |k_n[T](y)| v_{u_n \rho}(su_n^2/n, 0, y) dy. \end{aligned}$$

Thus, by symmetry of $y \mapsto v_\rho(t, 0, y)$, and the fact that $\text{erfc}(y) \leq K_{\text{Mills}} e^{-y^2}/y$ (see [14, p. 98]),

$$\begin{aligned} &\rho u_n \int_{U_{\delta/2}} |k_n[T](y)| v_{u_n \rho}(su_n^2/n, 0, y) dy \\ &\leq 2 \|k\|_\infty \int_{\delta/2}^{2/\delta} \exp \left(2 \frac{|y|}{u_n \rho} + 2 \frac{u_n^2 s}{n u_n^2 \rho^2} \right) \text{erfc} \left(\frac{|y|}{\sqrt{2su_n^2/n}} + \frac{\sqrt{2su_n^2/n}}{u_n \rho} \right) dy \end{aligned}$$

$$\begin{aligned}
&\leq 2 \|k\|_\infty K_{\text{Mills}} \int_{\delta/2}^{2/\delta} \frac{u_n \rho \sqrt{2s/n}}{|y| \rho + 2su_n/n} e^{-\frac{y^2 n}{2su_n^2}} dy \\
&\leq 4 \|k\|_\infty K_{\text{Mills}} \frac{\rho \sqrt{2u_n^2/n}}{\delta \rho} \left(\frac{2}{\delta} - \frac{\delta}{2} \right),
\end{aligned}$$

which converges to 0 as $n \rightarrow \infty$.

Again, since the bounds do not depend on s , from Lebesgue's dominated convergence theorem on $L^1([0, 1])$,

$$\lim_{n \rightarrow \infty} (\rho u_n) F_n[k](0) = 0.$$

This finishes the proof. \square

Lemma 6.12. *Let $X^{(\rho, \beta)}$ be the (ρ, β) -SkS-BM on \mathcal{P}_x . Then, there exists a positive constant $K > 0$, such that for all $t > 0$ and integrable function g ,*

$$\mathbb{E}_x \left(\sup_{s \leq t} \left| \sum_{i=1}^{[ns]} g(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right| \right) \leq |g(u_n x)| + K \left(\rho |g(0)| + \frac{1}{u_n} m_{(0, \beta)}(|g|) \right) n \sqrt{2t}.$$

Proof. We observe that

$$\int_{\mathbb{R}} |g(u_n y)| m_{(\rho, \beta)}(dy) = \rho |g(0)| + \frac{1}{u_n} \int_{\mathbb{R}} |g(y)| m_{(0, \beta)}(dy) = \rho |g(0)| + \frac{1}{u_n} m_{(0, \beta)}(|g|).$$

From Lemma 5.5 and the fact that $\sum_{i=1}^{[nt]} i^{-1/2} \leq 2\sqrt{nt}$, there exists a positive constant $K > 0$ that does not depend on (t, g) such that

$$\begin{aligned}
\mathbb{E}_x \left(\sup_{s \leq t} \left| \sum_{i=1}^{[ns]} g(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right| \right) &\leq \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\left| g(u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) \right| \right) \\
&= |g(u_n x)| + \sum_{i=2}^{[nt]} P_{\frac{i-1}{n}}^{(\rho, \beta)} |g|(x) \\
&\leq |g(u_n x)| + K \left(\rho |g(0)| + \frac{1}{u_n} m_{(0, \beta)}(|g|) \right) \sum_{i=2}^{[nt]} \frac{\sqrt{n}}{\sqrt{i}} \\
&\leq |g(u_n x)| + K \left(\rho |g(0)| + \frac{1}{u_n} m_{(0, \beta)}(|g|) \right) n \sqrt{2t}.
\end{aligned}$$

This finishes the proof. \square

We are now ready to address the proof of Proposition 6.9.

Proof of Proposition 6.9. For simplicity, in this proof we fix $t > 0$. We observe that $X^{(\rho, \beta)}$ is a semimartingale with quadratic variation $d \langle X^{(\rho, \beta)} \rangle_s = \mathbb{1}_{\{X_s^{(\rho, \beta)} \neq 0\}} ds$. By the occupation times formula [30, Corollary VI.1.6], for any non-negative measurable function f , it holds that

$$\int_0^t f(X_s^{(\rho, \beta)}) \mathbb{1}_{\{X_s^{(\rho, \beta)} \neq 0\}} ds = \int_{\mathbb{R}} f(y) \mathbb{1}_{\{y \neq 0\}} \ell_t^y(X^{(\rho, \beta)}) dy.$$

In particular, if $f(0) = 0$, then the formula becomes

$$\int_0^t f(X_s^{(\rho, \beta)}) ds = \int_{\mathbb{R}} f(y) \ell_t^y(X^{(\rho, \beta)}) dy. \quad (6.12)$$

Applying consecutive change of variables and (6.12) yields

$$\begin{aligned} \frac{u_n}{n} \int_0^{[nt]} g_n[T](u_n X_{s/n}^{(\rho, \beta)}) ds &= u_n \int_0^{[nt]/n} g_n[T](u_n X_s^{(\rho, \beta)}) ds \\ &= u_n \int_{\mathbb{R}} g_n[T](u_n y) \ell_{[nt]/n}^y(X^{(\rho, \beta)}) dy = \int_{\mathbb{R}} g_n[T](y) \ell_{[nt]/n}^{y/u_n}(X^{(\rho, \beta)}) dy. \end{aligned}$$

And, from Lemma 6.10, we have

$$\int_{\mathbb{R}} g_n[T](x) \ell_{[nt]/n}^{x/u_n}(X^{(\rho, \beta)}) dx \xrightarrow[n \rightarrow \infty]{P_x} m_{(\rho, \beta)}(g) L_t^0(X^{(\rho, \beta)}).$$

By these assertions, to complete the proof, it suffices to show that

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) - \frac{u_n}{n} \int_0^{[nt]} g_n[T](u_n X_{s/n}^{(\rho, \beta)}) ds \xrightarrow[n \rightarrow \infty]{P_x} 0. \quad (6.13)$$

Indeed, once (6.13) is proven, we obtain

$$\frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) \xrightarrow[n \rightarrow \infty]{P_x} m_{(\rho, \beta)}(g) L_t^0(X^{(\rho, \beta)}).$$

With the same arguments, based on Lemma 2.2, as in the closure of the proof of Proposition 6.3, the P_x -ucp convergence is proven.

Let us now prove (6.13). First, let us note that

$$\begin{aligned} \frac{u_n}{n} \sum_{i=1}^{[nt]} g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) - \frac{u_n}{n} \int_0^{[nt]} g_n[T](u_n X_{s/n}^{(\rho, \beta)}) ds \\ = \frac{u_n}{n} \sum_{i=1}^{[nt]} \int_0^1 \left(g_n[T](u_n X_{\frac{i-1}{n}}^{(\rho, \beta)}) - g_n[T](u_n X_{\frac{i-1}{n} + \frac{s}{n}}^{(\rho, \beta)}) \right) ds \\ + \frac{u_n}{n} \int_0^{nt-[nt]} \left(g_n[T](u_n X_{\frac{[nt]}{n}}^{(\rho, \beta)}) - g_n[T](u_n X_{\frac{[nt]}{n} + \frac{s}{n}}^{(\rho, \beta)}) \right) ds. \end{aligned} \quad (6.14)$$

The second additive term of the right-hand-side of (6.14) is bounded by $2 \|g\|_{\infty} u_n/n$, therefore

$$\frac{u_n}{n} \int_0^{nt-[nt]} \left(g_n[T](X_{\frac{[nt]}{n}}^{(\rho, \beta)}) - g_n[T](X_{\frac{[nt]}{n} + \frac{s}{n}}^{(\rho, \beta)}) \right) ds \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

By [13, Lemma 9], and since g is bounded, to show that the first additive term of the right-hand-side of (6.14) converges to 0 in probability, it suffices to prove that

$$D_t^{(n)} := \frac{u_n}{n} \sum_{i=1}^{[nt]} \int_0^1 \mathbb{E}_x \left(g_n[T](X_{\frac{i-1}{n}}^{(\rho, \beta)}) - g_n[T](X_{\frac{i-1}{n} + \frac{s}{n}}^{(\rho, \beta)}) \middle| \mathcal{F}_{\frac{i-1}{n}} \right) ds \xrightarrow[n \rightarrow \infty]{P_x} 0.$$

Using the notation of Lemma 6.11, $D_t^{(n)}$ reads

$$D_t^{(n)} = \frac{u_n}{n} \sum_{i=1}^{[nt]} F_n[g](X_{\frac{i-1}{n}}^{(\rho, \beta)}).$$

By Lemma 6.12, we have the bound

$$\mathbb{E}_x \left(|D_t^{(n)}| \right) \leq \frac{u_n}{n} |F_n[g](u_n x)| + \frac{u_n}{n} \left(\rho |F_n[g](0)| + \frac{1}{u_n} m_{(0,\beta)}(|F_n[g]|) \right) n\sqrt{2t},$$

where $|F_n[g](u_n x)| \leq 2 \|g\|_\infty$.

First assume g is Lipschitz continuous, with compact support and vanishes on an open interval around 0. Then, by Lemma 6.11, the upper bound of the above inequality vanishes as $n \rightarrow \infty$. And, in this case,

$$D_t^{(n)} \xrightarrow[n \rightarrow \infty]{L^1(P_x)} 0,$$

which implies convergence in probability.

Now assume the general form of g . We now approximate g by functions which satisfy the assumptions of Lemma 6.11. Since g is bounded and integrable, for every $p > 0$, there exists a Lipschitz-continuous function with compact support $k^{(p)}$ that vanishes in the vicinity of 0 such that $m_{(\rho,\beta)}(|g - k^{(p)}|) < 1/p$ (see the proof of [19, Lemma 4.5]). Moreover, by positive Fubini's theorem, triangle inequality, the vanishing nature of g and $k^{(p)}$ at 0, and the symmetry of the SkS kernel $p_{(\rho,\beta)}(t, x, y) = p_{(\rho,\beta)}(t, y, x)$ (see [17, p.149]), we have that

$$\begin{aligned} m_{(\rho,\beta)}(|P_t^{(\sqrt{n}\rho,\beta)} g - P_t^{(\sqrt{n}\rho,\beta)} k^{(p)}|) & \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |g(y) - k^{(p)}(y)| p_{(\sqrt{n}\rho,\beta)}(t, x, y) m_{(\sqrt{n}\rho,\beta)}(dy) m_{(\rho,\beta)}(dx) \\ & \leq \int_{\mathbb{R}} |g(y) - k^{(p)}(y)| \left(\int_{\mathbb{R}} p_{(\sqrt{n}\rho,\beta)}(t, y, x) m_{(\sqrt{n}\rho,\beta)}(dx) \right) m_{(\rho,\beta)}(dy) \\ & = \int_{\mathbb{R}} |g(y) - k^{(p)}(y)| m_{(\rho,\beta)}(dy) = m_{(\rho,\beta)}(|g - k^{(p)}|) < 1/p. \end{aligned} \quad (6.15)$$

Similarly, since T was assumed to satisfy $\varepsilon \leq T'(y) \leq 1/\varepsilon$, for all $y \in \mathbb{R}$, by a suitable change of variables we obtain for all $t > 0$, $m_{(\rho,\beta)}(|g_n[T] - k_n^{(p)}[T]|) < \|1/T'\|_\infty (1/p) \leq 1/(p\varepsilon)$. This, combined similar arguments as in (6.15), yields that

$$m_{(\rho,\beta)}(|P_t^{(\sqrt{n}\rho,\beta)} g_n[T] - P_t^{(\sqrt{n}\rho,\beta)} k_n^{(p)}[T]|) < 1/(p\varepsilon).$$

By the above, Fubini's theorem ensures

$$\begin{aligned} m_{(\rho,\beta)}(|F_n[g] - F_n[k^{(p)}]|) & \leq \int_0^1 m_{(\rho,\beta)}(|P_{su_n^2/n}^{(u_n\rho,\beta)} g_n[T] - P_{su_n^2/n}^{(u_n\rho,\beta)} k_n^{(p)}[T]|) ds \\ & \quad + \int_0^1 m_{(\rho,\beta)}(|g_n[T] - k_n^{(p)}[T]|) ds \leq 2/(p\varepsilon). \end{aligned}$$

Also, from (5.2)–(5.3),

$$\begin{aligned} (\rho u_n) p_{(u_n\rho,\beta)}(tu_n^2/n, 0, y) & = (\rho u_n) v_{u_n\rho}(tu_n^2/n, 0, y) \\ & = e^{\frac{2|y|}{u_n\rho} + \frac{2t}{n\rho^2}} \operatorname{erfc} \left(\frac{|y|\sqrt{n}}{\sqrt{2}tu_n} + \frac{\sqrt{2t}}{\sqrt{n}\rho} \right) \xrightarrow{n \rightarrow \infty} C > 0. \end{aligned}$$

Combining the above, by dominated convergence

$$\limsup_{n \rightarrow \infty} u_n |F_n[g](0) - F_n[k^{(p)}](0)|$$

$$\begin{aligned}
&\leq \limsup_{n \rightarrow \infty} \int_0^1 u_n \left| P_{su_n^2/n}^{(u_n \rho, \beta)} g_n[T](0) - P_{su_n^2/n}^{(u_n \rho, \beta)} k_n^{(p)}[T](0) \right| ds \\
&\leq \limsup_{n \rightarrow \infty} \int_0^1 \int_{\mathbb{R}} \left| g_n[T](y) - k_n^{(p)}[T](y) \right| u_n p_{u_n \rho}(su_n^2/n, 0, y) m_{(u_n \rho, \beta)}(dy) ds \\
&\leq (C/\rho) m_{(\rho, \beta)}(|g_n[T] - k_n^{(p)}[T]|) \leq C/(\rho p \varepsilon).
\end{aligned}$$

By Lemma 6.12, and the above bounds,

$$\mathbb{E}_x \left(|D_t^{(n)}| \right) \leq \frac{u_n}{n} |F_n[g](u_n x)| + \sqrt{2t} \left(u_n \rho |F_n[k^{(p)}](0)| + m_{(0, \beta)}(|F_n[k^{(p)}]|) + \frac{C}{p\varepsilon} + \frac{2}{p\varepsilon} \right).$$

Thus, by Lemma 6.11, there exists a constant C' such that for all $p > 0$, we have

$$\limsup_{n \rightarrow \infty} \mathbb{E}_x \left(|D_t^{(n)}| \right) \leq C'/p \implies \lim_{n \rightarrow \infty} \mathbb{E}_x \left(|D_t^{(n)}| \right) = 0,$$

the implication is due to arbitrariness of $p > 0$. Thus, we have proven (6.13) and this completes the proof. \square

7 Proofs of Propositions 3.5 and 3.6

This section develops the proofs for our parameter estimation results, combining occupation time approximations with local time convergence to deduce consistent estimators for the stickiness and skewness parameters.

Proof of Proposition 3.5. We consider the following statistics

$$V_n^{(0)} := \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{\{X_{\frac{i-1}{n}}=0\}}, \quad V_n^{(+)} := \frac{u_n}{n} \sum_{i=1}^{[nt]} g_{>0}(u_n X_{\frac{i-1}{n}}), \quad V_n^{(-)} := \frac{u_n}{n} \sum_{i=1}^{[nt]} g_{<0}(u_n X_{\frac{i-1}{n}}),$$

and

$$W_n^{(+)} := \frac{S_n^{g^+}(X) + S_n^{g^-}(X)}{2}, \quad W_n^{(-)} := \frac{S_n^{g^+}(X) - S_n^{g^-}(X)}{2}.$$

By (3.1) and Lemma 3.7,

$$V_n^{(0)} \xrightarrow[n \rightarrow \infty]{P_x} \rho L_t^0(X). \tag{7.1}$$

By Theorem 3.3,

$$\begin{aligned}
\frac{u_n}{n} \sum_{i=1}^{[nt]} g_{>0}(u_n X_{\frac{i-1}{n}}) &\xrightarrow[n \rightarrow \infty]{P_x} \frac{1 + \beta}{\sigma^2(0+)} \left(\int_0^\infty g(x) dx \right) L_t^0(X), \\
\frac{u_n}{n} \sum_{i=1}^{[nt]} g_{<0}(u_n X_{\frac{i-1}{n}}) &\xrightarrow[n \rightarrow \infty]{P_x} \frac{1 - \beta}{\sigma^2(0-)} \left(\int_{-\infty}^0 g(x) dx \right) L_t^0(X).
\end{aligned}$$

Thus, it holds that

$$S_n^{g^+}(X) \xrightarrow[n \rightarrow \infty]{P_x} (1 + \beta) L_t^0(X), \quad \text{and} \quad S_n^{g^-}(X) \xrightarrow[n \rightarrow \infty]{P_x} (1 - \beta) L_t^0(X),$$

and consequently that

$$W_n^{(+)} \xrightarrow[n \rightarrow \infty]{P_x} L_t^0(X), \quad W_n^{(-)} \xrightarrow[n \rightarrow \infty]{P_x} \beta L_t^0(X). \tag{7.2}$$

We consider the event $\mathcal{L} = \{\tau_0^X < t\}$. We observe that on $(\mathcal{L})^c$ all the aforementioned statistics converge to 0. On $\mathcal{L} = \{\tau_0^X < t\}$ they converge to non-zero random quantities. Thus, from (7.2),

$$\beta_n(X) = W_n^{(-)} / W_n^{(+)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \beta.$$

Also, from (7.1) and (7.2),

$$\rho_n(X) = V_n^{(0)} / W_n^{(+)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \rho.$$

This finishes the proof. \square

Proof of Proposition 3.6. Let X be the SOS-BM such that (X, B) solve the system (3.3) for a standard BM B . From the Itô–Tanaka formula, $X^+ = \max\{X, 0\}$ solves

$$\begin{cases} dX_t^+ = \sigma_0(X_t) \mathbb{1}_{\{X_t > 0\}} dB_t + \frac{\beta+1}{2} dL_t^0(X), \\ \mathbb{1}_{\{X_t=0\}} dt = \rho dL_t^0(X). \end{cases}$$

It is thus a semimartingale with explicit Doob–Meyer decomposition, and quadratic variation given by

$$d\langle X^+ \rangle_t = \sigma_0^2(X_t) \mathbb{1}_{\{X_t > 0\}} dt, \quad \langle X^+ \rangle_t = \int_0^t \sigma_0^2(X_s) \mathbb{1}_{\{X_s > 0\}} ds = \sigma_+^2 \int_0^t \mathbb{1}_{\{X_s > 0\}} ds.$$

By [30, Definition I.2.3],

$$\sum_{i=1}^{[nt]} (X_{\frac{i}{n}}^+ - X_{\frac{i-1}{n}}^+)^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \langle X^+ \rangle_t = \sigma_+^2 \int_0^t \mathbb{1}_{\{X_s > 0\}} ds.$$

With similar arguments, one can show that

$$\sum_{i=1}^{[nt]} (X_{\frac{i}{n}}^- - X_{\frac{i-1}{n}}^-)^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \langle X^- \rangle_t = \sigma_-^2 \int_0^t \mathbb{1}_{\{X_s > 0\}} ds.$$

The above convergences and Lemma 3.7 yield the results. \square

A Singular symmetric Itô calculus

In this section we derive explicit version of classical stochastic calculus results for the semimartingale X that solves (3.1). Namely, the Itô–Tanaka formula and Girsanov theorem. The first is used in Section 4 for extending the local time approximation from SkS-BM to X . The second justifies essentially Remark 3.2.

Lemma A.1 (Singular Itô–Tanaka formula). *Let X be a solution to (3.1) on \mathcal{P}_x . If f is a difference of two convex functions, such that $f' \in C^1(\mathbb{R} \setminus \{0\})$, then $f(X) = (f(X_t))_{t \geq 0}$ satisfies*

$$\begin{cases} f(X_t) = f(X_0) + \int_0^t \left(f'(X_s) \nu_s + \frac{1}{2} f''(X_s) \sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds \\ \quad + \int_0^t f'(X_s) \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \frac{\Sigma f'(0) \beta + \Delta f'(0)}{2} L_t^0(X), \\ \int_0^t \mathbb{1}_{\{X_s=0\}} ds = \rho L_t^0(X), \end{cases}$$

where $\Delta f'(0) = f'(0+) - f'(0-)$ and $\Sigma f'(0) = f'(0+) + f'(0-)$. If f is further assumed invertible, it holds that

$$L_t^0(f(X)) = \frac{\Delta f'(0)\beta + \Sigma f'(0)}{2} L_t^0(X).$$

Proof. The process X is a semimartingale as,

$$X_t = X_0 + \int_0^t \nu_s \mathbb{1}_{\{X_s \neq 0\}} ds + \int_0^t \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \beta L_t^0(X),$$

where

1. $\left(\int_0^t \nu_s \mathbb{1}_{\{X_s \neq 0\}} ds + \beta L_t^0(X)\right)_{t \geq 0}$ is a process of bounded variation,
2. $\left(\int_0^t \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s\right)_{t \geq 0}$ is a local martingale.

Thus, from [30, Theorem VI.1.5],

$$f(X_t) = f(X_0) + \int_0^t \frac{f'(X_s+) + f'(X_s-)}{2} dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^y(X) f''(dy), \quad (\text{A.1})$$

where f' is the right derivative of f , $f'(x-) := \lim_{h \rightarrow 0} f'(x-h)$, and $f''(dx)$ is the measure defined such that for every $g \in C_c^1$,

$$\int_{\mathbb{R}} f'g dx = - \int_{\mathbb{R}} g f''(dx).$$

Since $f \in C^1(\mathbb{R} \setminus \{0\})$, $f''(dx) = f''(x) \mathbb{1}_{\mathbb{R} \setminus \{0\}}(x) dx + \Delta f'(0) \delta_0(dx)$ and from the occupation times formula

$$\frac{1}{2} \int_{\mathbb{R}} L_t^y(X) f''(dy) = \frac{\Delta f'(0)}{2} L_t^0(X) + \frac{1}{2} \int_0^t f''(X_s) \mathbb{1}_{\{X_s \neq 0\}} \sigma(X_s)^2 ds. \quad (\text{A.2})$$

From (A.1) and (A.2),

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \left(f'(X_s) \nu_s + \frac{1}{2} f''(X_s) \sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds \\ &\quad + \int_0^t f'(X_s) \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + \frac{\Sigma f'(0)\beta + \Delta f'(0)}{2} L_t^0(X). \end{aligned} \quad (\text{A.3})$$

This proves the first statement.

Assume now f is invertible and $f(0) = 0$. To complete the proof, it suffices to verify the relationship between the local times. Applying (A.3) to the function $|f|$ yields

$$\begin{aligned} |f(X_t)| &= |f(X_0)| + \int_0^t \operatorname{sgn}(f(X_s)) \left(f'(X_s) \nu_s + \frac{1}{2} f''(X_s) \sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds \\ &\quad + \int_0^t \operatorname{sgn}(f(X_s)) f'(X_s) \sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s \\ &\quad + \frac{\Delta f'(0)\beta + \Sigma f'(0)}{2} L_t^0(X). \end{aligned} \quad (\text{A.4})$$

The Itô–Tanaka formula also yields

$$|f(X_t)| = |f(X_0)| + \int_0^t \operatorname{sgn}(f(X_s)) df(X_s) + L_t^0(f(X)).$$

Thus, since $\text{sgn}(f(0)) = \text{sgn}(0) = 0$,

$$\begin{aligned} |f(X_t)| &= |f(X_0)| + \int_0^t \text{sgn}(f(X_s)) \left(f'(X_s)\nu_s + \frac{1}{2}f''(X_s)\sigma(X_s)^2 \right) \mathbb{1}_{\{X_s \neq 0\}} ds \\ &\quad + \int_0^t \text{sgn}(f(X_s)) f'(X_s)\sigma(X_s) \mathbb{1}_{\{X_s \neq 0\}} dB_s + L_t^0(f(X)). \end{aligned} \quad (\text{A.5})$$

From (A.4) and (A.5),

$$L_t^0(f(X)) = \frac{\Delta f'(0)\beta + \Sigma f'(0)}{2} L_t^0(X).$$

The proof is thus completed. \square

Lemma A.2 (Singular Girsanov theorem). *Let X be a solution to (3.1) on \mathcal{P}_x , $T > 0$ be some fixed time-horizon, θ be a process that satisfies $\mathbb{P}_x(\int_0^T (\theta_s)^2 ds < \infty) = 1$, and $\mathcal{E}(\theta)$ be the process*

$$\mathcal{E}_t(\theta) = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right), \quad t \geq 0.$$

Consider \mathbb{Q}_x^T be the probability measure on (Ω, \mathcal{F}_T) such that $d\mathbb{Q}_x^T \propto \mathcal{E}_T(\theta) d\mathbb{P}_x^T$, where \mathbb{P}_x^T is the restriction of \mathbb{P}_x on (Ω, \mathcal{F}_T) . If $(\mathcal{E}_t(\theta); t \in [0, T])$ is a \mathbb{P}_x -martingale, then the process $\tilde{B} = \left(B_t - \int_0^t \theta_s ds\right)_{t \in [0, T]}$ is a standard BM on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{Q}_x^T)$. Moreover, under \mathbb{Q}_x^T the process X solves the system

$$\begin{cases} d\tilde{X}_t = \left(\nu_t + \theta_t \sigma(\tilde{X}_t)\right) \mathbb{1}_{\{\tilde{X}_t \neq 0\}} dt + \sigma(\tilde{X}_t) \mathbb{1}_{\{\tilde{X}_t \neq 0\}} d\tilde{B}_t + \beta dL_t^0(\tilde{X}), \\ \mathbb{1}_{\{\tilde{X}_t = 0\}} dt = \rho dL_t^0(\tilde{X}), \end{cases} \quad (\text{A.6})$$

for all $t \in [0, T]$.

Proof. By [22, Theorem 3.5.1], the process $(\tilde{B}_t)_{t \in [0, T]}$ is a standard BM under \mathbb{Q}_x^T and the probability measures \mathbb{P}_x^T and \mathbb{Q}_x^T are equivalent. By substitution,

$$dX_t = (\nu_t + \theta_t \sigma(X_t)) \mathbb{1}_{\{X_t \neq 0\}} dt + \sigma(X_t) \mathbb{1}_{\{X_t \neq 0\}} d\tilde{B}_t + \beta dL_t^0(X).$$

Thus, under \mathbb{Q}_x^T , X solves (A.6). \square

B Proof of Lemma 3.7

Without loss of generality, we can always assume that the interval U is contained in J . By Lemma 2.2, it suffices to prove for every $t > 0$ that

$$\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{1}_U(X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \mathcal{O}_t^U(X).$$

Let $t > 0$ be fixed.

Let us first assume X is an SkS-BM on \mathcal{P}_x .

We need some additional notation For all $\varepsilon > 0$, let

$$U_\varepsilon := \{y \in \mathbb{R} : \text{dist}(y, U) < \varepsilon\} \quad \text{and} \quad U_{-\varepsilon} := \mathbb{R} \setminus \{y \in \mathbb{R} : \text{dist}(y, U^c) \leq \varepsilon\}.$$

Define for every $\varepsilon > 0$ the pair of functions $(\psi_\varepsilon, \varphi_\varepsilon)$ as

$$\begin{aligned}\psi_\varepsilon(y) &= \mathbb{1}_{U_{-\varepsilon}}(y) + \left(\frac{1}{\varepsilon} \text{dist}(y, U^c)\right) \mathbb{1}_{U \setminus U_{-\varepsilon}}(y), \quad \forall x \in \mathbb{R}, \\ \varphi_\varepsilon(y) &= \mathbb{1}_U(y) + \left(1 - \frac{1}{\varepsilon} \text{dist}(y, U)\right) \mathbb{1}_{U_\varepsilon \setminus U}(y), \quad \forall y \in \mathbb{R}.\end{aligned}$$

Note that for every $\varepsilon > 0$,

$$\mathbb{1}_{U_{-\varepsilon}}(y) \leq \psi_\varepsilon(y) \leq \mathbb{1}_U(y) \leq \varphi_\varepsilon(y) \leq \mathbb{1}_{U_\varepsilon}.$$

Step 1: U a closed interval with non-empty interior. Without loss of generality we take $U = [l, r]$, $l \neq r$. For every $\varepsilon \in (0, (r-l)/2)$, we have $U_\varepsilon = (l - \varepsilon, r + \varepsilon)$ and $U_{-\varepsilon} = (l + \varepsilon, r - \varepsilon)$, $U \setminus U_{-\varepsilon} = [l, l + \varepsilon) \cup (r - \varepsilon, r]$ and $U_\varepsilon \setminus U = (l - \varepsilon, l) \cup (r, r + \varepsilon)$. Without loss of generality, we can suppose that 0 does not belong to $\text{Int}(U \setminus U_{-\varepsilon}) \cup (U_\varepsilon \setminus U)$, otherwise, we consider a smaller ε .

Since any continuous function on $[0, t]$ is Riemann integrable, and the functions $[t \mapsto \psi_\varepsilon(X_t)]$ and $[t \mapsto \varphi_\varepsilon(X_t)]$ are a.s. continuous, it holds a.s. that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{[nt]} \psi_\varepsilon(X_{\frac{i-1}{n}}) = \int_0^t \psi_\varepsilon(X_s) ds, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{[nt]} \varphi_\varepsilon(X_{\frac{i-1}{n}}) = \int_0^t \varphi_\varepsilon(X_s) ds.$$

From simple computations, we have that

$$\begin{aligned}\limsup_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_U(X_{\frac{i-1}{n}}) - \int_0^t \mathbb{1}_U(X_s) ds \right) \\ \leq \limsup_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^{[nt]} \varphi_\varepsilon(X_{\frac{i-1}{n}}) - \int_0^t \mathbb{1}_U(X_s) ds \right) \leq \int_0^t \varphi_\varepsilon(X_s) ds - \int_0^t \mathbb{1}_U(X_s) ds \\ \leq \int_0^t \left(1 - \frac{1}{\varepsilon} \text{dist}(X_s, U)\right) \mathbb{1}_{U_\varepsilon \setminus U}(X_s) ds \leq \int_0^t \mathbb{1}_{U_\varepsilon \setminus U}(X_s) ds.\end{aligned}\tag{B.1}$$

On the other hand,

$$\begin{aligned}\liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_U(X_{\frac{i-1}{n}}) - \int_0^t \mathbb{1}_U(X_s) ds \right) \\ \geq \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^{[nt]} \psi_\varepsilon(X_{\frac{i-1}{n}}) - \int_0^t \mathbb{1}_U(X_s) ds \right) \\ \geq \int_0^t \psi_\varepsilon(X_s) ds - \int_0^t \mathbb{1}_U(X_s) ds \geq -\frac{1}{\varepsilon} \int_0^t \text{dist}(X_s, U^c) \mathbb{1}_{U \setminus U_{-\varepsilon}}(X_s) ds \\ \geq -\int_0^t \mathbb{1}_{\text{Int}(U \setminus U_{-\varepsilon})}(X_s) ds.\end{aligned}\tag{B.2}$$

Lemma 5.5 ensures that

$$\begin{aligned}\mathbb{E}_x \left(\int_0^t \mathbb{1}_{\text{Int}(U \setminus U_{-\varepsilon}) \cup (U_\varepsilon \setminus U)}(X_s) ds \right) &\leq \int_0^t \mathbb{E}_x \left(\mathbb{1}_{(U \setminus U_{-\varepsilon}) \cup (U_\varepsilon \setminus U)}(X_s) \right) ds \\ &\leq K \lambda(\text{Int}(U \setminus U_{-\varepsilon}) \cup (U_\varepsilon \setminus U)) \int_0^t s^{-1/2} ds \leq 4K\varepsilon\sqrt{t},\end{aligned}$$

for some constant $K \in (0, \infty)$ that depends on ρ, β . The right-hand side of the latter inequality vanishes as $\varepsilon \rightarrow 0$ which implies convergence probability. Let us consider a subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$ that vanishes for $k \rightarrow \infty$. By (B.1)-(B.2) we have that a.s.

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \int_0^t \mathbb{1}_{\text{Int}(U \setminus U_{-\varepsilon_k})}(X_s) \, ds \leq \liminf_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_U(X_{\frac{i-1}{n}}) - \int_0^t \mathbb{1}_U(X_s) \, ds \right) \\ &\leq \limsup_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_U(X_{\frac{i-1}{n}}) - \int_0^t \mathbb{1}_U(X_s) \, ds \right) \leq \lim_{k \rightarrow \infty} \int_0^t \mathbb{1}_{U_{\varepsilon_k} \setminus U}(X_s) \, ds = 0. \end{aligned}$$

From this, we conclude that \mathbb{P}_x -a.s.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_U(X_{\frac{i-1}{n}}) = \int_0^t \mathbb{1}_U(X_s) \, ds. \quad (\text{B.3})$$

Step 2: U is a degenerate closed interval $U = \{a\}$. Assume $U = \{a\}$ and consider the sets $U^{(0)} := [a - \varepsilon, a + \varepsilon]$, $U^{(1)} := [a - \varepsilon, a]$, and $U^{(2)} := [a, a + \varepsilon]$. We observe that $\mathbb{1}_U + \mathbb{1}_{U^{(0)}} = \mathbb{1}_{U^{(1)}} + \mathbb{1}_{U^{(2)}}$. Since (B.3) holds for each of $U^{(0)}$, $U^{(1)}$, and $U^{(2)}$, it also holds \mathbb{P}_x -a.s.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{\{X_{\frac{i-1}{n}}=a\}} = \int_0^t \mathbb{1}_{\{X_s=a\}} \, ds.$$

This completes the proof for the case $U = \{a\}$.

Step 3: U is half-open or open interval. Let U half-open and $a \in \bar{U} \setminus U$. From the previous case, since \bar{U} is closed, we get

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_U(X_{\frac{i-1}{n}}) = \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{\bar{U}}(X_{\frac{i-1}{n}}) - \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{\{X_{\frac{i-1}{n}}=a\}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \mathcal{O}_t^{\bar{U}}(X) - \mathcal{O}_t^{\{a\}}(X) = \mathcal{O}_t^U(X).$$

With a similar argument the convergence also holds for an open interval U .

As in the proof of Theorem 3.3, we deduce the general case (SOS-diffusion) from SkS-BM.

First let Y be solution to (3.2) with $\sigma, \sigma', 1/\sigma$ bounded. The transformations T_1 and T_2 defined in Section 4, are all bijective, continuous, with continuous inverses and we have seen that $Z := T_1 \circ T_2(Y)$ is an SkS-BM. Since $T_1 \circ T_2$ is bijective, continuous, with continuous inverse,

- if U is an open (resp. closed) interval, so is $(T_1 \circ T_2)^{-1}(U) = T_2^{-1} \circ T_1^{-1}(U)$,
- $\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_U(Y_{\frac{i-1}{n}}) = \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{T_2^{-1} \circ T_1^{-1}(U)}(Z_{\frac{i-1}{n}})$,
- and $\mathcal{O}_t^U(Y) = \mathcal{O}_t^{(T_1 \circ T_2)^{-1}(U)}(Z)$.

By a localization argument, as in the proof of Theorem 3.3, we can remove the boundedness assumption on $\sigma, \sigma', 1/\sigma$. Local equivalence in law Condition 3.1-c2 yields the conclusion for X SOS-diffusion.

The proof is thus completed.

C Time scaling of SkS-BM

The occupation time of the positive half-line $[0, \infty)$ by the process X is defined as

$$\mathcal{O}_t^+(X) = \int_0^t \mathbb{1}_{\{X_s \geq 0\}} ds, \quad t \geq 0.$$

In case the process is an SkS-BM (solution to (5.1) with $\beta \in (-1, 1)$) the following results hold.

Lemma C.1. *The joint law of the (ρ, β) -SkS-BM on \mathcal{P}_x with its local time and its occupation time of the positive half-line is*

$$\begin{aligned} & \mathbb{P}_x(X_t \in dy, L_t^0(X) \in dl, \mathcal{O}_t^+(X) \in do) \\ &= g(t, x - y) \delta_0(dl) \left(\delta_t(do) \mathbb{1}_{\{x, y > 0\}} + \delta_0(do) \mathbb{1}_{\{x, y < 0\}} \right) dy \\ & \quad + h(o - \rho l, al + x_+ + y_+) h(t - o, (1 - a)l - x_- - y_-) \mathbb{1}_{\{\ell_\rho < o < t\}} m(dy) dl do, \end{aligned}$$

where for any real number x , $x_+ = x \vee 0$ and $x_- = -(x \wedge 0)$, and (f, g) are the functions defined as

$$g(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad h(t, x) := \frac{|x|}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}, \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}.$$

Proof. See [35, Theorem 3.3] and [8]. □

Lemma C.2 (Scaling property). *For any $\rho > 0$, $\beta \in (0, 1)$ and $x \in \mathbb{R}$ let $X^{(\rho, \beta)}$ solve the equation for (ρ, β) -SkS-BM (5.1) on $\mathcal{P}_x^{(\rho, \beta)} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x^{(\rho, \beta)})$. Then,*

$$\begin{aligned} & \text{law}_{\mathbb{P}_x^{(\rho, \beta)}} \left(X_{ct}^{(\rho, \beta)}, L_{ct}^0(X^{(\rho, \beta)}), \mathcal{O}_{ct}^+(X^{(\rho, \beta)}); t \geq 0 \right) \\ &= \text{law}_{\mathbb{P}_x^{(\rho/\sqrt{c}, \beta)}} \left(\sqrt{c} X_t^{(\rho/\sqrt{c}, \beta)}, \sqrt{c} L_t^0(X^{(\rho/\sqrt{c}, \beta)}), c \mathcal{O}_t^+(X^{(\rho/\sqrt{c}, \beta)}); t \geq 0 \right), \end{aligned}$$

where $\mathbb{P}_x^{(\rho, \beta)}(X_0^{(\rho/\sqrt{c}, \beta)} = \sqrt{c}x) = 1$ and

$$(L^0(X^{(\rho, \beta)}), \mathcal{O}^+(X^{(\rho, \beta)})), (L^0(X^{(\rho/\sqrt{c}, \beta)}), \mathcal{O}^+(X^{(\rho/\sqrt{c}, \beta)}))$$

are the local times at 0, occupation time of \mathbb{R}_+ pairs of $X^{(\rho, \beta)}$ and $X^{(\rho/\sqrt{c}, \beta)}$ respectively.

Proof. We observe that

$$\begin{aligned} & \mathbb{P}_x^{(\rho, \beta)} \left(X_{ct}^{(\rho, \beta)} \in dy, L_{ct}^0(X^{(\rho, \beta)}) \in dl, \mathcal{O}_{ct}^+(X^{(\rho, \beta)}) \in do \right) \\ &= g(ct, x - y) \delta_0(dl) \left(\delta_{ct}(do) \mathbb{1}_{\{x, y > 0\}} + \delta_0(do) \mathbb{1}_{\{x, y < 0\}} \right) dy \\ & \quad + h(o - \rho l, al + x_+ + y_+) h(ct - o, (1 - a)l - x_- - y_-) \mathbb{1}_{\{\ell_\rho < o < t\}} m(dy) dl do. \end{aligned}$$

In particular,

$$\begin{aligned} & g(ct, x - y) \delta_0(dl) \left(\delta_{ct}(do) \mathbb{1}_{\{x, y > 0\}} + \delta_0(do) \mathbb{1}_{\{x, y < 0\}} \right) dy \\ &= g \left(t, \frac{x}{\sqrt{c}} - \frac{y}{\sqrt{c}} \right) \delta_0 \left(\frac{dl}{\sqrt{c}} \right) \left(\delta_t \left(\frac{do}{c} \right) \mathbb{1}_{\{x, y > 0\}} + \delta_0 \left(\frac{do}{c} \right) \mathbb{1}_{\{x, y < 0\}} \right) \frac{dy}{\sqrt{c}}, \end{aligned}$$

and since for all $c > 0$, $t > 0$ and $x \in \mathbb{R}$, $h(ct, x) = c^{-1}h(t, x/\sqrt{c})$, $a(cx) = a(x)$, $(cx)_+ = cx_+$, $(cx)_- = cx_-$

$$\begin{aligned}
& h(o - \rho\ell, a\ell + x_+ + y_+) h(ct - o, (1-a)\ell - x_- - y_-) \mathbb{1}_{\{\ell\rho < o < ct\}} a(y) \, dy \, d\ell \, do \\
&= h\left(c\left(\frac{o}{c} - \frac{\rho}{\sqrt{c}}\frac{\ell}{\sqrt{c}}\right), a\ell + x_+ + y_+\right) \\
&\quad \cdot h\left(c\left(t - \frac{o}{c}\right), (1-a)\ell - x_- - y_-\right) \mathbb{1}_{\{\frac{\ell\rho}{c} < \frac{o}{c} < t\}} a(y) \, dy \, d\ell \, do \\
&= h\left(\frac{o}{c} - \frac{\rho}{\sqrt{c}}\frac{\ell}{\sqrt{c}}, a\frac{\ell}{\sqrt{c}} + \left(\frac{x}{\sqrt{c}}\right)_+ + \left(\frac{y}{\sqrt{c}}\right)_+\right) \\
&\quad \cdot h\left(t - \frac{o}{c}, (1-a)\frac{\ell}{\sqrt{c}} - \left(\frac{x}{\sqrt{c}}\right)_- - \left(\frac{y}{\sqrt{c}}\right)_-\right) \mathbb{1}_{\{\frac{\ell\rho}{c} < \frac{o}{c} < t\}} a(y/\sqrt{c}) \left(\frac{dy}{\sqrt{c}}\right) \left(\frac{d\ell}{\sqrt{c}}\right) \left(\frac{do}{c}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
& P_x^{(\rho, \beta)}(X_{ct}^{(\rho, \beta)} \in dy, L_{ct}^0(X^{(\rho, \beta)}) \in d\ell, \mathcal{O}_{ct}^+(X^{(\rho, \beta)}) \in do) \\
&= P_{x/\sqrt{c}}^{(\rho/\sqrt{c}, \beta)}\left(X_t^{(\rho/\sqrt{c}, \beta)} \in \frac{dy}{\sqrt{c}}, L_t^0(X^{(\rho/\sqrt{c}, \beta)}) \in \frac{d\ell}{\sqrt{c}}, \mathcal{O}_t^+(X^{(\rho/\sqrt{c}, \beta)}) \in \frac{do}{c}\right),
\end{aligned}$$

which is the desired result. \square

Corollary C.3 (Density scaling). *Let $p_{(\rho, \beta)}$ be the probability transition kernel of the (ρ, β) -SkS-BM with respect to its speed measure $m_{(\rho, \beta)}$. Then, for any measurable $h : \mathbb{R}^2 \rightarrow \mathbb{R}$,*

$$\int_{\mathbb{R}} h(x, y) p_{(\rho, \beta)}(ct, x, y) m_{(\rho, \beta)}(dy) = \int_{\mathbb{R}} h(x, \sqrt{c}y) p_{(\rho/\sqrt{c}, \beta)}\left(t, \frac{x}{\sqrt{c}}, y\right) m_{(\rho/\sqrt{c}, \beta)}(dy).$$

Proof. From the proof of Proposition C.2,

$$\begin{aligned}
& \int_{\mathbb{R}} h(x, y) p_{(\rho, \beta)}(ct, x, y) m_{(\rho, \beta)}(dy) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} h(x, y) P_x(X_{ct}^{(\rho, \beta)} \in dy, L_{ct}^0(X^{(\rho, \beta)}) \in d\ell, \mathcal{O}_{ct}^+(X^{(\rho, \beta)}) \in do) \\
&= \int_{\mathbb{R}} \iint_{\mathbb{R}_+^2} h(x, y) P_{\frac{x}{\sqrt{c}}}\left(X_t^{(\frac{\rho}{\sqrt{c}}, \beta)} \in \frac{dy}{\sqrt{c}}, L_t^0(X^{(\frac{\rho}{\sqrt{c}}, \beta)}) \in \frac{d\ell}{\sqrt{c}}, \mathcal{O}_t^+(X^{(\frac{\rho}{\sqrt{c}}, \beta)}) \in \frac{do}{c}\right) \\
&= \int_{\mathbb{R}} h(x, y) p_{(\frac{\rho}{\sqrt{c}}, \beta)}\left(t, \frac{x}{\sqrt{c}}, \frac{y}{\sqrt{c}}\right) d\left(\frac{y}{\sqrt{c}}\right) = \int_{\mathbb{R}} h(x, \sqrt{c}y) p_{(\frac{\rho}{\sqrt{c}}, \beta)}\left(t, \frac{x}{\sqrt{c}}, y\right) dy,
\end{aligned}$$

which completes the proof. \square

Corollary C.4 (Semigroup scaling). *Let X be the (ρ, β) -SkS-BM and $(P_t^{(\rho, \beta)})_{t \geq 0}$ the associated semigroup. Then,*

$$P_t^{(\rho\sqrt{n}, \beta)} h(x\sqrt{n}) = E_x\left(h\left(\sqrt{n}X_{\frac{t}{n}}^{(\rho, \beta)}\right)\right).$$

Proof. It is a special case of Corollary C.3. \square

D Proofs of Section 5

In this section we prove all postponed proofs of Section 5. More precisely, we prove: Lemmata 5.4, 5.5, 5.6, and 5.7 and Proposition 5.9. For reader's convenience, we recall the statement of every result before presenting its proof.

Lemma 5.4. (*Restatement*) There exists a constant $K > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}$,

$$p_{(\rho,\beta)}(t, x, y) \leq K u_1(t, x, y),$$

where K does not depend on ρ . In particular there exists a constant $K > 0$ (not depending on ρ) such that for all $t > 0, \rho \geq 0$ and $x, y \in \mathbb{R}$,

$$v_\rho(t, x, y) \leq K u_1(t, x, y).$$

Proof. Let u_1, u_2, v_ρ be the functions defined in (5.2)–(5.3). We first observe that

$$\frac{1}{a(y)} (u_1(t, x, y) - u_2(t, x, y)) \leq \frac{1}{a(y)} u_1(t, x, y) \leq K u_1(t, x, y), \quad (\text{D.1})$$

for some positive constant K , since $a(y) \geq 1 - |\beta|$. From the Mills ratio (see [14, p. 98]), $\operatorname{erfc}(x) \sim e^{-x^2}/x$ and thus there is a constant $K_{\text{Mills}} > 0$ such that $\operatorname{erfc}(x) \leq K_{\text{Mills}} e^{-x^2}/x$. Thus, if $\rho > 0$,

$$\begin{aligned} v_\rho(t, x, y) &= \frac{1}{\rho} e^{2(|x|+|y|)/\rho+2t/\rho^2} \operatorname{erfc}\left(\frac{|x|+|y|}{\sqrt{2t}} + \frac{\sqrt{2t}}{\rho}\right) \\ &\leq K_{\text{Mills}} \frac{1}{\rho} \frac{\rho\sqrt{2t}}{\rho(|x|+|y|)+2t} e^{-(|x|+|y|)^2/2t} \leq \sqrt{\pi} K_{\text{Mills}} u_1(t, x, y). \end{aligned} \quad (\text{D.2})$$

If $\rho = 0$, from (5.3),

$$v_0(t, x, y) = u_2(t, x, y) \leq u_1(t, x, y). \quad (\text{D.3})$$

Combining (D.1), (D.2) and (D.3), completes the proof. \square

Lemma 5.5. (*Restatement*) There exists a constant $K > 0$ such that

$$|P_t^{(\rho,\beta)} h(x)| \leq K \frac{\rho\sqrt{2t}}{\rho|x|/2+2t} |h(0)| + \frac{K}{\sqrt{t}} m_{(0,\beta)}(|h|) \leq \frac{K}{\sqrt{t}} m_{(\rho,\beta)}(|h|),$$

for all $h \in L^1(\mathbb{R})$, $t > 0$, and $x \in \mathbb{R}$.

Proof. By definition of the probability transition kernel,

$$P_t^{(\rho,\beta)} h(x) = \int_{\mathbb{R}} h(y) p_{(\rho,\beta)}(t, x, y) m_{(\rho,\beta)}(dy).$$

This, Lemma 5.4, and the identity $p_{(\rho,\beta)}(t, x, 0) = v_\rho(t, x, 0)$, yield that

$$\begin{aligned} |P_t^{(\rho,\beta)} h(x)| &\leq \int_{\mathbb{R}} |h(y)| p_{(\rho,\beta)}(t, x, y) m_{(\rho,\beta)}(dy) \\ &= \rho |h(0)| p_{(\rho,\beta)}(t, x, 0) + \int_{\mathbb{R}} |h(y)| p_{(\rho,\beta)}(t, x, y) m_{(0,\beta)}(dy) \\ &\leq \rho K u_1(t, x, 0) |h(0)| + K \|u_1(t, x, \cdot)\|_\infty m_{(0,\beta)}(|h|) \\ &\leq \frac{K}{\sqrt{2\pi t}} (\rho |h(0)| + m_{(0,\beta)}(|h|)), \end{aligned}$$

which proves the results. \square

Lemma 5.6. (*Restatement*) For every $\gamma \geq 0$, there exists a constant $K_\gamma > 0$, that does not depend on (ρ, β) , such that

$$\left| P_t^{(\rho, \beta)} h(x) - m_{(\rho, \beta)}(h) p_{(\rho, \beta)}(t, x, 0) \right| \leq K_\gamma \frac{1}{t} \left(m_{(0, \beta)}^{(1)}(h) + \frac{m_{(0, \beta)}^{(1)}(h)}{1 + |x/\sqrt{t}|^\gamma} + \frac{m_{(0, \beta)}^{(\gamma)}(h)}{1 + |x|^\gamma} \right),$$

for all $h \in L^1(\mathbb{R})$, $t > 0$, and $x \in \mathbb{R}$.

Proof. We observe that

$$\begin{aligned} & \left| P^{(\rho, \beta)} h(x) - m_{(\rho, \beta)}(h) p_{(\rho, \beta)}(t, x, 0) \right| \\ &= \left| \int_{\mathbb{R}} h(y) \left(p_{(\rho, \beta)}(t, x, y) - p_{(\rho, \beta)}(t, x, 0) \right) m_{(0, \beta)}(dy) \right| \\ &\leq \left| \int_{\mathbb{R}} h(y) (u_1(t, x, y) - u_1(t, x, 0)) dy \right| + \left| \int_{\mathbb{R}} h(y) (u_2(t, x, y) - u_2(t, x, 0)) dy \right| \\ &\quad + \left| \int_{\mathbb{R}} h(y) (v_\rho(t, x, y) - v_\rho(t, x, 0)) m_{(\cdot, \beta)}(dy) \right|. \end{aligned} \quad (\text{D.4})$$

For every $\gamma \geq 0$, there exists a constant $K_\gamma'' > 0$ such that, for every $x \in \mathbb{R}$ and $t > 0$,

$$e^{-x^2/t} \leq e^{-x^2/2t} \leq K_\gamma'' \frac{1}{1 + |x/\sqrt{t}|^\gamma}. \quad (\text{D.5})$$

From [19, Lemma 3.1], for every $\gamma \geq 0$, there exists a $K_\gamma' > 0$ such that

$$\left| \int_{\mathbb{R}} h(y) [u_1(t, x, y) - u_1(t, x, 0)] dy \right| \leq \frac{K_\gamma'}{t} \left(\lambda^{(1)}(h) + \frac{\lambda^{(1)}(h)}{1 + |x/\sqrt{t}|^\gamma} + \frac{\lambda^{(\gamma)}(h)}{1 + |x|^\gamma} \right). \quad (\text{D.6})$$

For the second additive term on the right-hand-side of (D.4), an integration by parts yields that

$$\int_{\mathbb{R}} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy = \int_0^1 \int_{\mathbb{R}} h(\zeta) \zeta \frac{\partial}{\partial y} u_2(t, x, y) \Big|_{y=\zeta\theta} d\zeta d\theta.$$

Since there exists a constant $c > 0$ such that $|xe^{-x^2}| \leq c$, for all $x \in \mathbb{R}$, we have that

$$\left| \frac{\partial}{\partial y} u_2(t, x, y) \right| = \left| \operatorname{sgn}(y) \frac{1}{t\sqrt{\pi}} \frac{|y| + |x|}{\sqrt{2t}} e^{-\frac{(|x|+|y|)^2}{2t}} \right| \leq \frac{c/\sqrt{\pi}}{t}, \quad \text{for all } y \in \mathbb{R}.$$

Setting $c' = c/\sqrt{\pi}$ yields

$$\begin{aligned} & \left| \int_{\mathbb{R}} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy \right| \\ &\leq \int_0^1 \int_{\mathbb{R}} \left| h(\zeta) \zeta \frac{\partial}{\partial y} u_2(t, x, y) \Big|_{y=\zeta\theta} \right| d\zeta d\theta \leq c' \frac{\lambda^{(1)}(h)}{t}. \end{aligned} \quad (\text{D.7})$$

We now bound the third additive term at the right-hand-side of (D.4) and first consider the case $\rho = 0$, where for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^2$, $v_0(t, x, t) = u_2(t, x, y)$. We observe that

$$\int_{\mathbb{R}_+} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy = \int_0^1 \int_{\mathbb{R}_+} h(\zeta) \zeta \frac{\partial}{\partial y} u_2(t, x, y) \Big|_{y=\zeta\theta} d\zeta d\theta,$$

$$\int_{\mathbb{R}_-} h(y) [u_2(t, x, y) - u_2(t, x, 0)] dy = \int_0^1 \int_{\mathbb{R}_-} h(\zeta) \zeta \frac{\partial}{\partial y} u_2(t, x, y) \Big|_{y=\zeta\theta} d\zeta d\theta.$$

Applying the same arguments as for the second additive term on each half-line yields the bound

$$\left| \int_{\mathbb{R}} h(y) (v_0(t, x, y) - v_0(t, x, 0)) m_{(\cdot, \beta)}(dy) \right| \leq c' \frac{m_{(\cdot, \beta)}^{(1)}(h)}{t}. \quad (\text{D.8})$$

For the case $\rho > 0$, let M be the function defined for all $t \geq 0$ and $x, y \in \mathbb{R}$ by

$$M(t, x, y) := \frac{v_\rho(t, x, y)}{u_2(t, x, y)} = \frac{\sqrt{\pi}}{\sqrt{2t}} \frac{2t}{\rho} f\left(\frac{|x| + |y| + 2t/\rho}{\sqrt{2t}}\right),$$

where $f(z) := e^{z^2} \operatorname{erfc}(z)$. This yields that

$$\begin{aligned} v_\rho(t, x, y) &= u_2(t, x, y) M(t, x, y) \\ &= u_2(t, x, y) (M(t, x, y) - M(t, x, 0) + M(t, x, 0)), \end{aligned}$$

and therefore

$$\begin{aligned} v_\rho(t, x, y) - v_\rho(t, x, 0) &= u_2(t, x, y) (M(t, x, y) - M(t, x, 0)) \\ &\quad + (u_2(t, x, y) - u_2(t, x, 0)) M(t, x, 0). \end{aligned} \quad (\text{D.9})$$

We observe that f is decreasing and $|z|f(|z|) \in [0, 1/\sqrt{\pi}]$. Hence, for all $t > 0$ and $x, y \in \mathbb{R}$, we have

$$0 \leq M(t, x, y) \leq M(t, x, 0) \leq M(t, 0, 0) = \frac{\sqrt{\pi}}{\sqrt{2t}} \frac{2t}{\rho} f\left(\frac{2t/\rho}{\sqrt{2t}}\right) \leq 1.$$

This and (D.8) yield

$$\left| \int_{\mathbb{R}} h(y) [u_2(t, x, y) - u_2(t, x, 0)] |M(t, x, 0)| m_{(0, \beta)}(dy) \right| \leq c' \frac{m_{(0, \beta)}^{(1)}(h)}{t}. \quad (\text{D.10})$$

Moreover,

$$0 \leq M(t, x, 0) - M(t, x, y) \leq \frac{\sqrt{\pi}}{\sqrt{2t}} \frac{2t}{\rho} \int_0^{|y|/\sqrt{2t}} \left| f' \left(\frac{|x| + 2t/\rho}{\sqrt{2t}} + \zeta \right) \right| d\zeta.$$

Since $|f'(\zeta)| = \frac{2}{\sqrt{\pi}} (1 - \sqrt{\pi} \zeta f(\zeta))$ is decreasing,

$$\begin{aligned} 0 \leq M(t, x, 0) - M(t, x, y) &\leq |y| \frac{\sqrt{\pi}}{\rho} \left| f' \left(\frac{2t}{\rho \sqrt{2t}} \right) \right| \\ &= |y| \frac{2}{\rho} \left(1 - \sqrt{\pi} \frac{2t}{\rho \sqrt{2t}} f \left(\frac{2t}{\rho \sqrt{2t}} \right) \right). \end{aligned}$$

Note that the function $[x \mapsto x (1 - \sqrt{\pi} x e^{x^2} \operatorname{erfc}(x))]$: $\mathbb{R} \mapsto \mathbb{R}$ takes values in of $[0, 1/4]$, so

$$0 \leq M(t, x, 0) - M(t, x, y) \leq \frac{|y|}{2\sqrt{2t}}.$$

Therefore, we have that

$$\begin{aligned}
& \left| \int_{\mathbb{R}} h(y) [u_2(t, x, y)] [M(t, x, y) - M(t, x, 0)] m_{(0,\beta)}(dy) \right| \\
& \leq \frac{1}{2\sqrt{2t}} \int_{\mathbb{R}} |y| |h(y)| |u_2(t, x, y)| m_{(0,\beta)}(dy) \\
& \leq \frac{1}{4t\sqrt{\pi}} m_{(0,\beta)}^{(1)}(h) e^{-\frac{x^2}{2t}} \leq \frac{1}{t} \frac{K''_{\gamma}}{4\sqrt{\pi}} m_{(0,\beta)}^{(1)}(h) \frac{1}{1 + |x/\sqrt{t}|^{\gamma}},
\end{aligned} \tag{D.11}$$

where the last inequality comes from (D.5). From (D.9), (D.10), and (D.11),

$$\begin{aligned}
& \left| \int_{\mathbb{R}} h(y) (v_{\rho}(t, x, y) - v_{\rho}(t, x, 0)) m_{(0,\beta)}(dy) \right| \\
& \leq \frac{1}{t} \left(c' + \frac{K''_{\gamma}}{4\sqrt{\pi}} \right) \left(m_{(0,\beta)}^{(1)}(h) + \frac{m_{(0,\beta)}^{(1)}(h)}{1 + |x/\sqrt{t}|^{\gamma}} \right).
\end{aligned} \tag{D.12}$$

From (D.4), (D.6), (D.7), (D.8), (D.12), and since that for all $\gamma \geq 0$,

$$(1 - |\beta|)\lambda^{(\gamma)}(h) \leq m_{(0,\beta)}^{(\gamma)}(h) \leq (1 + |\beta|)\lambda^{(\gamma)}(h),$$

the desired bound holds for

$$K_{\gamma} = \frac{2}{1 - |\beta|} (K'_{\gamma} + c') + \left(c' + \frac{K''_{\gamma}}{4\sqrt{\pi}} \right).$$

This completes the proof of Lemma 5.6. \square

Lemma 5.7. (*Restatement*) For every $n \in \mathbb{N}$, $t > 0$, and every suitable function h , let $\gamma_n^{(\rho,\beta)}[h](x, t)$ be the functional defined in Notation 5.2.

(i) There exists a positive constant K , not depending on t, n, h such that

$$|\gamma_n^{(\rho,\beta)}[h](x, t)| \leq K m_{(\rho\sqrt{n},\beta)}(|h|) \sqrt{nt}$$

(ii) If $m_{(\sqrt{n}\rho,\beta)}(h) = 0$ then there exists a positive constant K not depending on t, n, h such that

$$\gamma_n^{(\rho,\beta)}[h](x, t) \leq K m_{(0,\beta)}(h) (1 + \max(0, \log(nt))).$$

Proof. From Corollary C.4, which is a consequence of the scaling property,

$$\gamma_n^{(\rho,\beta)}[h](x, t) = \sum_{i=2}^{[nt]} \mathbb{E}_x \left[h(\sqrt{n} X_{\frac{i-1}{n}}^{(\rho,\beta)}) \right] = \sum_{i=2}^{[nt]} P_{i-1}^{(\sqrt{n}\rho,\beta)} h(\sqrt{nx}).$$

Thus, from Lemma 5.5 and since $\sum_{i=1}^{[nt]-1} i^{-\frac{1}{2}} \leq 2\sqrt{nt}$,

$$\begin{aligned}
& \left| \gamma_n^{(\rho,\beta)}[h](x, t) \right| \leq \sum_{i=2}^{[nt]} \left| P_{i-1}^{(\sqrt{n}\rho,\beta)} h(\sqrt{nx}) \right| \\
& \leq K \sum_{i=1}^{[nt]-1} \frac{1}{\sqrt{i}} m_{(\rho\sqrt{n},\beta)}(|h|) \leq 2K m_{(\rho\sqrt{n},\beta)}(|h|) \sqrt{nt},
\end{aligned}$$

which completes the proof of the first statement.

When $m_{(\sqrt{n}\rho, \beta)}(h) = 0$, from Lemma 5.6 (with $\gamma = 1$) and the fact that $\sum_{i=1}^{\lfloor nt \rfloor} i^{-1} \leq 1 + \log(nt)$ we obtain

$$\begin{aligned} |\gamma_n^{(\rho, \beta)}[h](x, t)| &\leq \sum_{i=2}^{\lfloor nt \rfloor} |P_{i-1}^{(\rho, \beta)} h(\sqrt{nx})| \\ &\leq Km_{(0, \beta)}^{(1)}(h) \sum_{i=2}^{\lfloor nt \rfloor} \frac{1}{i} \leq Km_{(0, \beta)}^{(1)}(h)(1 + \max(0, \log(nt))), \end{aligned}$$

which proves the second and last statement. The proof is thus completed. \square

Proposition 5.9. (*Restatement*) It holds that $m_{(\rho\sqrt{n}, \beta)}(\hat{g}_n) \rightarrow 1$ as $n \rightarrow \infty$.

Proof. In the proof we use the analytic expression of the probability transition kernel of SkS-BM $(t, x, y) \mapsto p_{(\rho, \beta)}(t, x, y)$ defined in terms of the functions u_1 , u_2 , and v_ρ (see (5.2)–(5.3)).

For simplicity, let $m_n := m_{(\rho\sqrt{n}, \beta)}$. Note also that u_1 is the Gaussian kernel and $u_2(t, x, y) = u_1(t, |x|, -|y|)$, for all (t, x, y) . Moreover, for every $t > 0$, the following identities hold:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (y - x) u_1(t, x, y) dy dx = \frac{1}{2}, \quad \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} x u_2(t, x, y) dy dx = \frac{1}{4}. \quad (\text{D.13})$$

We first consider the case $\rho = 0$, so that

$$m_{(\rho\sqrt{n}, \beta)} = m_{(0, \beta)}, \quad \hat{g}_n = \hat{g} : x \mapsto \mathbb{E}_x \left(|X_1^{(0, \beta)}| - |x| \right).$$

From (5.2)–(5.3),

$$\begin{aligned} m_n(\hat{g}_n) &= \int_{\mathbb{R}} \hat{g}(x) m_{(0, \beta)}(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(0, \beta)}(1, x, y) m_{(0, \beta)}(dy) m_{(0, \beta)}(dx) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_1(1, x, y) a(x) dy dx - \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) a(x) dy dx \\ &\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) a(x) a(y) dy dx. \end{aligned}$$

By (D.13), and since for all $t > 0$ and $x, y \in \mathbb{R}$, $u_2(t, x, y) = u_2(t, y, x)$,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_1(1, x, y) a(x) dy dx &= 1, \\ \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) a(x) dy dx &= 0, \\ \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) u_2(1, x, y) a(x) a(y) dy dx &= 0. \end{aligned} \quad (\text{D.14})$$

Thus, $m_n(\hat{g}_n) = 1$.

We now suppose that $\rho > 0$. Then,

$$\begin{aligned} m_n(\hat{g}_n) &= \int_{\mathbb{R}} \hat{g}_n(x) m_n(dx) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(\sqrt{n}\rho, \beta)}(1, x, y) m_n(dy) m_n(dx). \end{aligned}$$

Since m_n is defined for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ by $m_n(dx) = a(x)dx + \rho\sqrt{n}\delta_0(dx)$,

$$m_n(\hat{g}_n) = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(\sqrt{n}\rho, \beta)}(1, x, y) a(x) a(y) dy dx + \sqrt{n}\rho \left(\int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta)}(1, 0, y) a(y) dy - \int_{\mathbb{R}} |x| p_{(\sqrt{n}\rho, \beta)}(1, x, 0) a(x) dx \right). \quad (\text{D.15})$$

We now show that the last additive term of the right-hand-side of (D.15) vanishes. Indeed, we observe that $\int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta)}(t, 0, y) a(y) dy < \infty$ and that for all $t > 0$ and $x, y \in \mathbb{R}$, $p_{(\sqrt{n}\rho, \beta)}(t, x, y) = p_{(\sqrt{n}\rho, \beta)}(t, y, x)$. Thus,

$$\int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta)}(t, 0, y) a(y) dy = \int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta)}(t, y, 0) a(y) dy$$

and

$$\sqrt{n}\rho \left(\int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta)}(1, 0, y) a(y) dy - \int_{\mathbb{R}} |y| p_{(\sqrt{n}\rho, \beta)}(1, y, 0) a(y) dy \right) = 0.$$

For the first additive term of the right-hand-side of (D.15), from (5.2),

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) p_{(\sqrt{n}\rho, \beta)}(1, x, y) a(x) a(y) dy dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) (u_1(1, x, y) - u_2(1, x, y)) a(x) dy dx \\ & \quad + \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) v_{\sqrt{n}\rho}(1, x, y) a(x) a(y) dy dx. \end{aligned}$$

We observe that $v_{\sqrt{n}\rho}(t, x, y)$ vanishes as $n \rightarrow \infty$. Also, from Lemma 5.4,

$$(|y| - |x|) v_{\sqrt{n}\rho}(t, x, y) a(x) a(y) \leq K (|y| + |x|) u_1(t, x, y)$$

which is an integrable function with respect to $(x, y) \in \mathbb{R}^2$. Thus, from Lebesgue's convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) v_{\sqrt{n}\rho}(1, x, y) a(x) a(y) dy dx = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} m_n(\hat{g}_n) = \int_{\mathbb{R}} \int_{\mathbb{R}} (|y| - |x|) (u_1(1, x, y) - u_2(1, x, y)) a(x) dy dx,$$

which by (D.14) equals 1. This proves the result. \square

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