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# DP-MINIMAL GROUPS

FRANK O. WAGNER

ABSTRACT. A dp-minimal group is virtually nilpotent.

**Definition 1.** A partial type  $\pi(x)$  is dp-minimal if there are no formulas  $\varphi(x, y)$  and  $\psi(x, y)$  and mutually indiscernible sequences  $(a_i)_{i < \omega}$  and  $(b_i)_{i < \omega}$  such that for any  $i, j \in \omega$  the type

$$\pi(x) \cup \{\varphi(x, a_i), \psi(x, b_j)\} \cup \{\neg\varphi(x, a_s) : s \neq i\} \cup \{\neg\psi(x, b_t) : t \neq j\}$$

is consistent.

Equivalently, for any two mutually indiscernible sequences  $(a_i)_{i < \omega}$  and  $(b_i)_{i < \omega}$  and any  $c \models \pi$ , either  $(a_i)_{i < \omega}$  or  $(b_i)_{i < \omega}$  is indiscernible over  $c$ .

Recall that a dp-minimal theory is dependent, so in particular the Baldwin-Saxl condition holds, and any intersection of uniformly definable subgroups is equal to a (uniformly) finite subintersection. Moreover, Pierre Simon has shown [4, Claim in Proof of 4.31] that in a dp-minimal group, for any two definable subgroups  $H$  and  $K$  the intersection  $H \cap K$  has finite index in either  $H$  or in  $K$ . By compactness, for any family of uniformly definable subgroups  $\{H_i : i \in I\}$  there is  $n < \omega$  such that  $H_i \cap H_j$  has index at most  $n$  in either  $H_i$  or in  $H_j$ , for any  $i, j \in I$ .

We shall call a group  $\bigwedge$ -definable if it is an intersection of definable groups. Recall that a group is *virtually P* if it has a subgroup of finite index which is P.

**Remark 2.** If  $G$  is virtually soluble, then its soluble radical  $R(G)$  (the subgroup generated by all normal soluble subgroups) is soluble and definable; if  $G$  is virtually nilpotent, its Fitting subgroup  $F(G)$  (the subgroup generated by all normal nilpotent subgroups) is nilpotent and definable. If  $G$  is soluble, it has a definable characteristic series with abelian quotients; if  $G$  is nilpotent, it has a definable characteristic series with central quotients.

*Proof.* If  $G$  has a soluble subgroup  $H$  of finite index,  $H$  contains a normal soluble subgroup  $N$  of finite index. Now for any normal soluble subgroup  $S$  the group  $NS$

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is again normal soluble. Hence  $G$  has a maximal normal soluble subgroup of finite index, so  $R(G)$  is soluble. It is then definable by [3, Theorem 1.1]. The proof for  $F(G)$  is analogous, replacing soluble by nilpotent.

Now if  $G$  is soluble of derived length  $n + 1$ , define  $c_0(x) = x$  and  $c_{i+1}(x_1, \dots, x_{2^{i+1}}) = [c_i(x_1, \dots, x_{2^i}), c_i(x_{2^i+1}, \dots, x_{2^{i+1}})]$ . Then

$$A = Z(C_G(c_n(x_1, \dots, x_{2^n}) : x_1, \dots, x_{2^n} \in G))$$

is a definable characteristic abelian subgroup containing  $G^{(n)}$ . We divide out by  $A$  and finish by induction. For nilpotency, the claim is obvious, as  $Z(G)$  is clearly definable.  $\square$

**Proposition 3.** *A dp-minimal group  $G$  is virtually soluble.*

*Proof.* By a theorem of Simon [4, Proposition 4.31],  $G$  has a definable characteristic abelian subgroup  $A$  such that  $G/A$  has finite exponent. As  $G/A$  is still dp-minimal, we may assume that  $G$  has finite exponent.

Let  $H \leq G$  be a definable subgroup. The family of its  $G$ -conjugates is a family of uniformly definable subgroups, so there is  $n < \omega$  such that  $H \cap H^g$  has index  $\leq n$  either in  $H$  or in  $H^g$ . Suppose  $|H : H \cap H^g| \leq n$  (the other case is analogous). Then  $|H^g : H^g \cap H^{g^2}| \leq n$ , and  $|H \cap H^g : H \cap H^g \cap H^{g^2}| \leq n$ , whence  $|H : H \cap H^g \cap H^{g^2}| \leq n^2$ . Iterating, we see that  $|H : H \cap \dots \cap H^{g^k}| \leq n^k$ . But for  $k = o(g) - 1$  the subgroup  $N = H \cap \dots \cap H^{g^k}$  is  $g$ -invariant, so  $|H : N| = |H^g : N| \leq n^k$ . But this implies  $|H : H \cap H^g| = |H^g : H \cap H^g| \leq n$ . Now NIP implies that there is  $\ell$  such that the intersection of any finite number of  $G$ -conjugates of  $H$  is an intersection of at most  $\ell$  of them, and has index  $\leq \ell n$  in  $H$ . Hence  $H^{nor} = \bigcap_{g \in G} H^g$  is a normal subgroup of index at most  $\ell n$  in  $H$ , definable over the same parameters as  $H$ .

Let  $\mathfrak{M}$  be a model, and  $\mathfrak{S}$  the set of  $M$ -definable normal soluble subgroups. Since  $\mathfrak{S}$  is closed under product,  $\bigcup \mathfrak{S}$  is a normal subgroup of  $G$ . If  $G$  is not virtually soluble, there is a minimal  $M$ - $\wedge$ -definable subgroup  $H$  which is not virtually contained in any subgroup in  $\mathfrak{S}$ . Let  $p$  be an f-generic type of  $H$  over  $M$ . Note that by minimality  $H = H^0$ , so  $H$  is connected; if  $H = \bigcap_i H_i$  then  $H = \bigcap_i H_i^{nor}$  as well by connectivity, so  $H$  is an intersection of  $M$ -definable normal subgroups and must itself be normal. By dp-minimality,  $p$  is either generically stable or distal. If  $p$  is distal, then by the generalization of [5, Section 5] to  $\wedge$ -definable normal subgroups of a dp-minimal group,  $H$  is nilpotent. By compactness it is contained in a definable normal nilpotent group of the same class which must be in  $\mathfrak{S}$ , a contradiction.

So  $p$  is generically stable. Let  $(a_i)_{\mathbb{Z}} \hat{\ } (b_i)_{\mathbb{Z}}$  be a Morley sequence in  $p$  over  $M$ .

If  $[a_0, b_0]$  lies in some  $M$ -definable coset of some group  $S \in \mathfrak{S}$ , say  $[a_0, b_0] \in mS$ , then  $b_0, b_1$  and  $b_0 b_1$  all satisfy  $p|Ma_0$  by generic stability and genericity of  $p$ , so

$$[a_0, b_0] = [a_0, b_1] = [a_0, b_0 b_1] = [a_0, b_1][a_0, b_0]^{b_1} \pmod{S}.$$

Hence  $[a_0, b_0]^{b_1} \in S$ , so  $[a_0, b_0] \in S$ . Now  $C_G(a_0/S)$  contains a generic  $b_0$ , so has finite index in  $H$ , which is connected. By compactness  $C_G(a_0/S)$  contains an  $M$ -definable normal supergroup  $K$  of  $H$ . Then  $Z(K/S)$  is  $M$ -definable and contains  $a_0$  which is generic in  $H$ . By connectivity  $H \leq Z(K/S)$ . But  $Z(K/S) \in \mathfrak{S}$ , a contradiction.

In particular  $q = \text{tp}([a_0, b_0]/M)$  is a non-algebraic type over  $M$ ; it is generically stable since  $\text{tp}(a_0, b_0/M)$  is. By [5, Lemma 6.2] the stabilizer  $\text{Stab}(q)$  is infinite; in fact it contains  $c_0 c_1^{-1}$  for some/any independent  $c_0, c_1 \models q$ . Thus the coset  $\text{Stab}(p)c_1$  does not depend on  $c_1 \models q$  and must be  $M$ -definable; moreover it contains all realizations of  $q$ . By the previous paragraph,  $\text{Stab}(q) \notin \mathfrak{S}$ ; as  $q(x)$  implies  $x \in H$  we get  $\text{Stab}(q) \leq H$  and we must have equality by minimality of  $H$ .

Work over  $M$ . By dp-minimality of  $q$ , one of  $(a_i)_{\mathbb{Z}}$  or  $(b_i)_{\mathbb{Z}}$  is indiscernible over  $[a_0, b_0]$ , whence independent of  $[a_0, b_0]$ . In particular  $[a_0, b_0]$  is independent of  $a_0$  or of  $b_0$ ; by symmetry  $[b_0, a_0]$  is independent of  $b_0$  or of  $a_0$ . But  $[b_0, a_0] = [a_0, b_0]^{-1}$ , so it is independent of  $a_0$  and of  $b_0$ . It follows that

$$H = \text{Stab}(q) = \text{Stab}(q|Mb_0) = \text{Stab}([a_0, b_0]/Mb_0) = \text{Stab}(b_0^{-a_0}b_0/Mb_0) = \text{Stab}(b_0^{-a_0}/Mb_0).$$

But  $H = \text{Stab}(p) = \text{Stab}(p|Mb_0)$  by generic stability of the generic type  $p$ . Then  $H$  is the *right* stabilizer of  $p^{-1}$ . So for two independent realizations  $a, a' \models p$  we have that

$$\text{tp}(a^{-1}a'/Ma') = p^{-1}a'|Ma' = p^{-1}|Ma' \quad \text{and} \quad \text{tp}(a^{-1}a'/Ma) = a^{-1}p|Ma = p|Ma.$$

In particular  $p = \text{tp}(a^{-1}a'/M) = p^{-1}$ , and  $H$  is the left and right stabilizer of  $p$ . Similarly, if  $a \models p|Mb_0$  and  $c \models \text{tp}(b_0^{-a_0}/Mb_0)$  with  $a \perp_{Mb_0} c$ , then

$$p|Mb_0 = \text{tp}(ac/Mb_0) = \text{tp}(b_0^{-a_0}/Mb_0).$$

Since  $b_0^{-1} \models d_p x \exists z \in G x = y^z$ , there is  $m' \in M$  such that  $p \vdash x \in m'^G$ . As  $H = p \cdot p$  is normal in  $G$ , we have that  $H = m'^G \cdot m'^G$  is  $M$ -definable. Again by definability of  $p$  there is  $m \in M$  such that  $p \vdash x \in m^H$ . (In fact it is easy to see that we can choose  $m = m'$ .)

It follows that  $m \in H(M) \setminus \bigcup \mathfrak{S}(M)$ ; in particular  $H(M)/\bigcup \mathfrak{S}(M)$  is nontrivial. Note that if there is  $S \in \mathfrak{S}$  such that  $m^2 \in S$ , then every realization of  $p$  has order two modulo  $S$ . But for two independent realizations  $a, a' \models p$  we have  $aa' \models p$ , whence  $a^2, a'^2, (aa')^2 \in S$  and

$$aa' = (aa')^{-1} = a'^{-1}a^{-1} = a'a \quad \text{mod } S.$$

So  $H/S$  is generically commutative and  $[a_0, b_0] \in S$ ; as above  $H \leq \bigcup \mathfrak{S}$ , a contradiction. It follows that  $m^2 \notin \bigcup \mathfrak{S}$ .

Now  $m \neq m^{-1} \models p^{-1} = p$ , so there is  $h \in H$  with  $m^{-1} = m^h$ . Since  $m \in M$  we can choose  $h \in H(M)$ . Clearly  $h \notin \bigcup \mathfrak{S}$ , and  $m \in C_H(h^2) \setminus C_H(h)$ .

Consider any  $c \in H(M) \setminus \bigcup \mathfrak{S}(M)$ , and put  $q' = \text{tp}(c^{a_0}/M)$ , a generically stable type. By [5, Lemma 6.2] for  $T = \text{Stab}(q')$  and any  $t \models q'$  the coset  $Tt$  does not depend on  $t$ , is  $M$ -definable and contains  $q'$ .

If  $T$  were soluble, then for generic  $a \models p \upharpoonright Ma_0$  we have  $\text{tp}(a_0a/M) = p = \text{tp}(a_0/M)$ , so  $q'^a = q' \subseteq (Tt)^a \cap Tt$ , whence  $[a, t^{-1}] \in T$  (recall that  $T$  is a stabilizer, whence connected, and thus normal). Hence  $C_H(Tt/T)$  is a generic  $M$ -definable subgroup of  $H$ , and must equal  $H$  by connectivity. Thus  $t$  is central in  $H/T$ , as is  $t^G$ , and there is a definable  $S \in \mathfrak{S}$  containing  $Tt$ . Then  $c \in S$ , a contradiction.

Since  $T \leq H$  we get  $T = H$  by minimality. But then  $q' = p$  as before, and  $c \in m^H$ . So  $H(M)/\bigcup \mathfrak{S}(M)$  is a group of order at least 3, finite exponent, and a single non-trivial conjugacy class, a contradiction.  $\square$

**Proposition 4.** *Let  $G$  be a soluble  $dp$ -minimal group. Then  $G$  is virtually nilpotent.*

*Proof.* By [5, Theorem 5.21] we may assume that  $G$  has a generically stable principal  $f$ -generic type. Since  $G$  is soluble, it has a definable normal series with abelian quotients. So by induction on the derived length we may assume that there is a normal definable nilpotent subgroup  $N$  such that  $G/N$  is finite-by-abelian. Let  $F/N$  be the finite subgroup such that  $G/F$  is abelian. Then  $\bar{n}^g \in \bar{n}F$  for every  $\bar{n} \in G/N$ , and  $|G : C_G(\bar{n})| \leq |F|$ . Since  $G$  is NIP, the intersection of all  $C_G(\bar{n})$  is a finite subintersection  $G^c$ . We replace  $G$  by  $G^c$  and assume that  $G/N$  is abelian.

Let  $\mathfrak{M}$  be an  $\omega$ -saturated model. Recall that  $G$  has a characteristic  $\emptyset$ -definable abelian subgroup  $A$  such that  $G/A$  has finite exponent. Put

$$\mathfrak{Z} = \{Z \leq G : Z = Z_n(G_Z) \text{ for some normal } M\text{-definable } G_Z \leq G \text{ of finite index}\}.$$

Note that  $Z_n(G_Z)Z_m(G_{Z'}) \leq Z_{n+m}((G_Z \cap G_{Z'})ZZ')$ , so  $\mathfrak{Z}$  is closed under product, and consists of normal  $\emptyset$ -definable groups. We want to show that  $N \leq Z$  for some  $Z \in \mathfrak{Z}$ . So suppose not.

Let  $H$  be a minimal  $M$ - $\wedge$ -definable subgroup not virtually  $\wedge$ -contained in  $\bigcup \mathfrak{Z}$ . If  $A$  is not virtually contained in  $\bigcup \mathfrak{Z}$  we choose  $H \leq A$ . Note that  $H$  is connected, and  $G$ -invariant, as  $G/C_G(H)$  has finite exponent, say  $k$ . Dividing out by some  $Z \in \mathfrak{Z}$  we may assume that  $H$  is abelian.

Consider a principal generic element  $g$  of  $G$  over  $M$ , and a  $(g, M)$ -definable subgroup  $K_g$  of  $G$ . Then for an independent  $g' \equiv_M g$  we either have  $M_g \lesssim M_{g'}$  or  $M_{g'} \lesssim M_g$ . But by generic stability of  $\text{tp}(g/M)$  we have  $g, g' \equiv_M g', g$ , so  $M_g$  and  $M_{g'}$  are (uniformly) commensurable. For any  $g'' \equiv_M g$  we choose  $g' \equiv_M g$  independent of  $g, g''$  over  $M$ , so  $M_g$  and  $M_{g'}$  are both uniformly commensurable to  $M_{g''}$ , and hence uniformly commensurable. It follows that the intersection  $\bigcap_{g' \equiv_M g} M_{g'}$  equals a finite subintersection, has finite index in  $M_g$  and is  $M$ -definable.

In particular, for any  $Z \in \mathfrak{Z}$  the centraliser  $C_G(g/Z)$  has an  $M$ -definable subgroup  $K$  of finite index normal in  $G$  which centralises modulo  $Z$  some normal generic subgroup

$G_K \leq G$ . Then  $K \cap G_K$  has finite index in  $C_G(g/Z)$ , and is contained in  $\mathfrak{F}$ . Moreover, the subgroup  $[g, H]$  is  $(g, M)$ - $\wedge$ -definable (as  $H$  is abelian) and connected (since  $H$  is), whence  $M$ - $\wedge$ -definable; it must either equal  $H$  or be virtually contained in  $\bigcup \mathfrak{F}$ , whence contained in  $\bigcup \mathfrak{F}$  by connectedness. The latter case implies that  $H \leq \bigcup \mathfrak{F}$  again by connectedness, a contradiction.

Let  $R$  be the ring of quasi-endomorphisms modulo equivalence of  $H/\bigcup \mathfrak{F}$  generated by  $G$ , and their inverses (where they exist), in the monster model. Note that by the above, the map  $r_g : x \mapsto [g, x]$  has kernel virtually contained in  $\bigcup \mathfrak{F}$  and is surjective, whence invertible in  $R$ . Composing with conjugation by an independent principal generic  $g'$ , we see that  $f_{g,g'} : x \mapsto x^{-gg'} x^{g'}$  is also invertible in  $R$ .

Now let  $I$  be a maximal ideal in  $R$ , so  $R/I$  is a field. Since conjugation by  $g$  has order dividing  $k$ , there are only  $k$  different images in  $R/I$  for conjugation by realizations of  $p$ . But since  $f_{g,g'}$  is invertible, it cannot be in  $I$ . As there is a unique type of two independent realizations of  $p$ , which is realized by  $gg'$  and  $g$ , independent generics have a different image in  $R/I$ , a contradiction.

This finishes the proof.  $\square$

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