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# An Automata Theoretic Approach to Observer Design for Switched Linear Systems <sup>★</sup>

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## Abstract

We present an approach for designing asymptotic observers for discrete-time switched linear systems. We first give an automata theoretic characterization of switching signals containing an infinite number of reconstructible sequences, i.e. sequences allowing to estimate the state of the system. We show that such switching signals can be generated by a deterministic Büchi automaton whose construction is given in the paper. Then, we present a methodology to design switched observers. These observers have an internal discrete state variable whose dynamics is given by the transition map of the Büchi automaton. We then present two approaches to design observer gains such that the observer is convergent for all switching signals whose occurrence rate of reconstructible sequences is higher than a tunable threshold. The first approach gives an explicit construction of the observer gains while the second one is based on linear matrix inequalities. For switched systems with invertible state matrices, we show that the proposed observer structure is universal in the sense that it is always possible to design an observer of the proposed form. We use a simple example to illustrate our methodology and then consider a case study in which we design an observer for a multicellular converter.

*Key words:* Switched systems, Observer design, Lyapunov methods, Automata theoretic techniques.

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## 1 Introduction

Switched systems are dynamical systems with several operation modes, each mode being associated with a particular dynamics, typically modeled by a differential equation for continuous-time systems, or by a difference equation for discrete-time systems. A switching signal determines at each time instant which mode is active. Switched systems is a simple yet powerful modeling paradigm that can be found in many applications such as network controlled systems [14, 21, 4], automotive systems [16] or power converters [10, 28]. Since the 1990s, many theoretical contributions have been made to help us analyze or design switched systems, see e.g. [19, 24, 20] and the references therein.

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In this paper, we consider the problem of designing asymptotic observers for discrete-time switched linear systems. For this class of systems, there exists a diversity of observability notions depending on whether the switching signal is known [15, 5, 8] or need to be estimated [6, 13]. In our work, we assume that the switching signal is known. In that context, there have been several works for characterizing observable or reconstructible switching sequences [15, 26], i.e. finite switching sequences that make it possible to estimate the initial or the current state of the system, respectively. Relevant work also includes the characterization of observability for continuous-time switched systems with jumps at the switching instants [30, 25], which can be seen as a generalization of discrete-time switched systems. Finally, when the switching signal is constrained by some automaton, necessary and sufficient conditions for observability have been given in [8, 17].

The design of asymptotic observers itself has also been considered in several papers. Observers that are convergent for arbitrary switching sequences have been designed in [3, 12] using mode dependent observer gains and quadratic or poly-quadratic Lyapunov functions. Note that this design requires that the dynamics in each

mode is observable, since constant switching signals are allowed. The case where individual dynamics are unobservable but some observable (or reconstructible) switching sequences exist is also of interest. To be able to estimate the state of the system, it is necessary that the switching signal contains at least one reconstructible sequence. Actually, to be able to design observers that are robust to unmodeled disturbances or to measurement noises, it is necessary to consider switching signals containing an infinite number of reconstructible sequences. This is the setting considered in this paper.

Our contributions are multiple and can be summarized as follows. Firstly, we provide a formal characterization of the set of switching signals containing an infinite number of reconstructible sequences. More precisely, we show that this set coincides with the language of a Deterministic Büchi Automaton (DBA, see e.g. [7]) whose construction is presented in the paper. We then consider the problem of designing an asymptotic observer. We propose a switched observer with an internal discrete state whose dynamics is given by the transition map of the DBA. Building on our recent work on stability analysis of switched systems driven by DBAs [2, 1], we then establish sufficient conditions to design the observer gains such that the resulting observer is convergent for all switching signals whose occurrence rate of reconstructible sequences is higher than a certain tunable parameter. In the case where all state matrices of the switched system are invertible, we present an explicit construction of suitable observer gains, showing that the proposed observer structure is universal in the sense that one can always find an observer of the proposed form. We also show an alternative design based on Linear Matrix Inequalities (LMIs) which are shown to always admit a feasible solution under the same invertibility assumption. We then show how to extend our approach to take into account additional constraints on the switching signals. Our theoretical contributions are illustrated using a simple example. To show the effectiveness of the proposed methodology, we consider a case study in which we design an observer for a multicellular converter [27, 28].

The most related works in the literature are the following. In [18], a switched observer is presented where the observer gain at some instant  $t$  depends on the sequence of modes at time  $t - L, \dots, t$ . Similar to our construction, this can be seen as a switched observer with an internal discrete state though the discrete dynamics is different from that presented in this paper. In [18], the design of the observer gains is done by solving LMIs. However, contrarily to our approach there is no clear characterization of the cases when the proposed design can be successful. Another approach can be to consider a switched linear system as a time-varying linear system and to apply associated observer design techniques, such as Kalman filters [31]. Alternatively, the design proposed in [25] for observers of continuous-time switched

linear systems with jumps at the switching instants can be adapted to discrete-time systems. The observers designed in [31] and [25] can be shown to be convergent for switching signals containing an infinite number of reconstructible sequences. However, their gains have to be computed online. In comparison, the gains of our observer are computed offline, resulting in reduced requirements for its implementation.

This paper is organized as follows. Section 2 presents the problem under consideration and introduces preliminary results on observability of switched systems and on DBAs. Section 3 gives a construction of a DBA generating switching signals containing an infinite number of reconstructible sequences. Section 4 presents the structure of the switched observers, its robustness with respect to disturbances, and the methods to design its gains. Section 5 shows an illustration on a simple example while Section 6 deals with the application to a multicellular converter.

**Notations:** For a matrix  $M$ ,  $\ker(M)$  denotes its kernel and  $\text{rank}(M)$  denotes its rank. For an invertible matrix  $M$ ,  $M^{-\top}$  denote the transpose of the inverse of  $M$ , i.e.  $M^{-\top} = (M^{-1})^{\top}$ . Given a vector  $v \in \mathbb{R}^n$ ,  $\text{diag}(v)$  denotes the  $n \times n$  diagonal matrix whose diagonal elements are equal to  $v$ . For symmetric matrices  $M_1, M_2$ ,  $M_1 \leq M_2$  and  $M_1 < M_2$  stand for  $M_2 - M_1$  being positive semi-definite and positive definite respectively.

## 2 Problem statement and preliminaries

Let us consider a discrete-time switched linear system described by the following equation:

$$\begin{aligned} x(t+1) &= A_{\theta(t)}x(t), \\ y(t) &= C_{\theta(t)}x(t) \end{aligned} \tag{1}$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system,  $\theta : \mathbb{N} \rightarrow \mathcal{I}$  is the switching signal and  $\mathcal{I}$  is the set of modes,  $\mathcal{I} = \{1, \dots, m\}$ , the state matrices belong to the set  $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathbb{R}^{n \times n}$ . The output vector is  $y(t) \in \mathbb{R}^p$  and the output matrices are in the set  $\mathcal{C} = \{C_1, \dots, C_m\} \subseteq \mathbb{R}^{p \times n}$ .

Our goal in this paper is to propose a general procedure for designing asymptotic observers for system (1) capable of reconstructing the state  $x$  from the knowledge of the switching signal  $\theta$  and of the output  $y$ .

### 2.1 Observability of discrete-time switched systems

By following [24], we recall a definition and a characterization of observability for the switched system (1).

**Definition 1.** *The switched system (1) is observable (resp., reconstructible), if there exist a switching signal  $\theta$  and  $k \in \mathbb{N}$  such that the knowledge of the output sequence  $y(0), \dots, y(k)$  is sufficient to determine the initial condition  $x_0$  (resp., the state  $x(k)$ ). We refer to the finite sequence of modes  $\theta(0), \dots, \theta(k)$  as an observable (resp., reconstructible) sequence.*

We recall that an observable system is also reconstructible, while the opposite is true if all the matrices in  $\mathcal{A}$  are invertible. We define the observability matrix corresponding to a sequence of modes  $i_1, \dots, i_j \in \mathcal{I}$ , as

$$\Omega(i_1, \dots, i_j) = \begin{bmatrix} C_{i_1}^\top & A_{i_1}^\top C_{i_2}^\top & \cdots & A_{i_1}^\top \cdots A_{i_{j-1}}^\top C_{i_j}^\top \end{bmatrix}^\top.$$

Necessary and sufficient conditions for observability and reconstructibility are recalled in the following theorem.

**Theorem 1** ([24, Theorems 4.32 & 4.33]). *A sequence of modes  $i_1, \dots, i_j \in \mathcal{I}$  is observable if and only if*

$$\text{rank}\left(\Omega(i_1, \dots, i_j)\right) = n. \quad (2)$$

*It is reconstructible if and only if*

$$\ker\left(\Omega(i_1, \dots, i_j)\right) \subseteq \ker(A_{i_1} \cdots A_{i_j}). \quad (3)$$

As far as the design of asymptotic observers is concerned, it is sufficient to consider the notion of reconstructibility. However, let us remark that the conditions in (2) and (3) are equivalent if all matrices in  $\mathcal{A}$  are invertible.

## 2.2 Deterministic Büchi automaton

From Definition 1, it is clear that one can reconstruct the state  $x$  of the system whenever the switching signal  $\theta$  contains one reconstructible sequence. However, if one wants to design an observer that is robust to unmodeled disturbances or to measurement noises, it becomes necessary to consider switching signals that contain an infinite number of reconstructible sequences. In Section 3, we shall characterize such switching sequences using the notion of deterministic Büchi automaton introduced below.

Given a finite set  $\Sigma$  called *alphabet*, a *word* is either a finite or an infinite sequence of elements of  $\Sigma$ , i.e.  $w = \sigma_1 \sigma_2 \dots$ , or the *empty word*  $w = \epsilon$ .  $\Sigma^*$  denotes the set of all finite words over the alphabet  $\Sigma$ , including the empty word, while  $\Sigma^\omega$  denotes the set of all infinite words over the alphabet  $\Sigma$ . The *length* of a word  $w \in \Sigma^*$ , denoted by  $|w|$ , is  $n$  if  $w = \sigma_1 \cdots \sigma_n$ , or 0 if  $w = \epsilon$ . For  $w \in \Sigma^*$ , we say that  $w'$  is a *prefix* of  $w$  if it belongs to the set  $P(w) = \{w' \in \Sigma^* \mid \exists w'' \in \Sigma^*, w = w'w''\}$ . Similarly,

we say that  $w'$  is a *suffix* of  $w$  if it belongs to the set  $S(w) = \{w' \in \Sigma^* \mid \exists w'' \in \Sigma^*, w = w''w'\}$ . We stress that, in the previous definitions,  $w'$  and  $w''$  can be the empty word. Let  $S_1, S_2 \subseteq \Sigma^*$ , then the set  $S_1 S_2$  is the set of words consisting of the concatenation of words of  $S_1$  and  $S_2$ , i.e.  $S_1 S_2 = \{w = w_1 w_2 \mid w_1 \in S_1, w_2 \in S_2\}$ . Let  $S \subseteq \Sigma^* \setminus \{\epsilon\}$ , then  $S^\omega$  is the set of words consisting of the concatenation of an infinite sequence of words of  $S$ , i.e.  $S^\omega = \{w_1 w_2 \cdots \mid w_i \in S, i = 1, \dots\}$ .

A *deterministic Büchi automaton* (DBA, see e.g. [7]) is a 5-tuple  $\mathcal{B} = (Q, \Sigma, \delta, q_0, F)$ , where  $Q$  is a finite set of *states*,  $\Sigma$  is a finite *alphabet*,  $\delta : Q \times \Sigma \rightarrow Q$  is the *transition function*,  $q_0$  is the *initial state*,  $F$  is the set of *accepting states*. A run associated with a finite or an infinite word  $\sigma_1 \sigma_2 \cdots \in \Sigma^* \cup \Sigma^\omega$  is a sequence of states  $q_0, q_1, q_2, \dots$  in  $Q$  starting from the initial state  $q_0$  and such that  $q_{i+1} = \delta(q_i, \sigma_{i+1})$  for all  $i = 0, 1, \dots$ ; a run  $q_0, q_1, \dots$  associated with an infinite word  $\sigma_1 \sigma_2 \cdots \in \Sigma^\omega$  is said to be *accepting* if  $q_i \in F$  for infinitely many indices  $i \in \mathbb{N}$ . The language of  $\mathcal{B}$ , denoted by  $\text{Lang}(\mathcal{B})$ , is the set of all words in  $\Sigma^\omega$  which have an associated accepting run in  $\mathcal{B}$ . It is well known that  $\text{Lang}(\mathcal{B})$  is an  $\omega$ -regular language, see e.g. [7]. Given  $q', q'' \in Q$  and  $\sigma_1 \cdots \sigma_n \in \Sigma^*$ , we write  $q' \xrightarrow{\sigma_1 \cdots \sigma_n} q''$  if  $q''$  is obtained as a concatenation of transitions from  $q'$ , namely  $q'' = \delta(\cdots \delta(\delta(q', \sigma_1), \sigma_2), \dots), \sigma_n)$ .

Let  $\mathcal{B}$  be a DBA over the alphabet  $\Sigma = \mathcal{I} = \{1, \dots, m\}$ . Let us consider a switching signal  $\theta : \mathbb{N} \rightarrow \mathcal{I}$  for system (1). With an abuse of notation, we identify  $\theta$  with the infinite word  $w \in \mathcal{I}^\omega$  given by  $w = \theta(0)\theta(1)\dots$ . For  $\theta \in \text{Lang}(\mathcal{B})$ , we denote by  $q_0, q_1, \dots$  the associated accepting run. Then, following [2], we define the sequence of *return instants*  $(\tau_k^{\theta, \mathcal{B}})_{k \in \mathbb{N}}$  by  $\tau_0^{\theta, \mathcal{B}} = 0$  and, for  $k \in \mathbb{N}$ ,

$$\tau_{k+1}^{\theta, \mathcal{B}} = \min\{t > \tau_k^{\theta, \mathcal{B}} \mid q_t \in F\},$$

and the *return index*  $\kappa^{\theta, \mathcal{B}} : \mathbb{N} \rightarrow \mathbb{N}$  by

$$\kappa^{\theta, \mathcal{B}}(t) = \max\{k \in \mathbb{N} \mid \tau_k^{\theta, \mathcal{B}} \leq t\}.$$

Intuitively,  $\tau_k^{\theta, \mathcal{B}}$  is the first instant where the run associated with  $\theta$  has visited the set  $F$  of accepting states  $k$  times, and  $\kappa^{\theta, \mathcal{B}}(t)$  is the number of times the run associated with  $\theta$  has visited  $F$  up to time  $t$ . Since  $\theta \in \text{Lang}(\mathcal{B})$ , the set  $F$  will be visited an infinite number of times, so  $\tau_k^{\theta, \mathcal{B}}$  is well defined for every  $k \in \mathbb{N}$  and  $\lim_{t \rightarrow \infty} \kappa^{\theta, \mathcal{B}}(t) = \infty$ . We also define the *accepting rate* by

$$\gamma^{\theta, \mathcal{B}} = \liminf_{t \rightarrow \infty} \frac{\kappa^{\theta, \mathcal{B}}(t)}{t}.$$

### 3 A DBA for reconstructible switching signals

In this section, we present a construction of a deterministic Büchi automaton whose language consists of switching signals that contain an infinite number of reconstructible sequences.

#### 3.1 Construction of the automaton

Using Theorem 1, one can efficiently compute reconstructible sequences up to a specific length for a switched linear system given by  $(\mathcal{A}, \mathcal{C})$  and described by equation (1). The set of reconstructible sequences of length  $j \geq 1$  is defined by

$$\mathcal{O}_j = \left\{ \sigma_1 \cdots \sigma_j \in \mathcal{I}^* \mid \begin{array}{l} \ker(\Omega(\sigma_1, \dots, \sigma_j)) \subseteq \\ \ker(A_{\sigma_1} \cdots A_{\sigma_j}) \end{array} \right\}$$

Moreover, the set of reconstructible sequences of length up to  $k \geq 1$  is given by

$$\mathcal{O}^{[k]} = \bigcup_{j=1, \dots, k} \mathcal{O}_j.$$

It is easy to see that if  $w$  is a reconstructible sequence, then any sequence containing  $w$  as a subsequence is also reconstructible. Therefore it is useful to define the following reduced set of reconstructible sequences:

$$\mathcal{O}'^{[k]} = \left\{ w \in \mathcal{O}^{[k]} \mid \begin{array}{l} \nexists w_1, w_2 \in \mathcal{I}^*, w' \in \mathcal{O}^{[k]} \setminus \{w\} \\ \text{s.t. } w = w_1 w' w_2 \end{array} \right\}.$$

Intuitively,  $\mathcal{O}'^{[k]}$  consists of minimal reconstructible sequences, i.e. reconstructible sequences that do not contain reconstructible subsequences. It is easy to see that for any  $k_1 \leq k_2$ ,  $\mathcal{O}'^{[k_1]} \subseteq \mathcal{O}'^{[k_2]}$ .

Let us consider  $k \geq 1$ , such that  $\mathcal{O}'^{[k]} \neq \emptyset$ . Our goal is to generate words containing an infinite number of reconstructible sequences in  $\mathcal{O}'^{[k]}$ . To this aim, we provide Algorithm 1 which outputs a DBA  $\mathcal{B}_k = (Q, \mathcal{I}, \delta, q_0, F)$  generating the language  $(\mathcal{I}^* \mathcal{O}'^{[k]})^\omega$ . The main idea is to build an automaton which visits an accepting state each time a reconstructible sequence occurs. We note that in this automaton the states are words in  $\mathcal{I}^*$ , i.e.  $Q \subseteq \mathcal{I}^*$ .

Algorithm 1 is comprised of three parts:

- In the first part, from step 2 to step 9, we check if there are reconstructible sequences of length 1. If it is the case we create a self transition to the accepting state, otherwise we add a new state and we define a transition from the accepting state to the newly added state.

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#### Algorithm 1 Construction of DBA $\mathcal{B}_k$

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**Inputs:** alphabet  $\mathcal{I}$ , reconstructible sequences  $\mathcal{O}'^{[k]}$   
**Output:** DBA  $\mathcal{B}_k$  such that  $\text{Lang}(\mathcal{B}_k) = (\mathcal{I}^* \mathcal{O}'^{[k]})^\omega$

- 1:  $q_0 := \epsilon, F := \{\epsilon\}, Q := \{\epsilon\}$
- 2: **for**  $\sigma \in \mathcal{I}$  **do**
- 3:     **if**  $\sigma \in \mathcal{O}'^{[k]}$  **then**
- 4:          $\delta(\epsilon, \sigma) := \epsilon$
- 5:     **else**
- 6:          $Q := Q \cup \{\sigma\}$
- 7:          $\delta(\epsilon, \sigma) := \sigma$
- 8:     **end if**
- 9: **end for**
- 10: **for**  $\sigma_1 \cdots \sigma_l \in \mathcal{O}'^{[k]}, l \geq 2$  **do**
- 11:     **if**  $l = 2$  **then**
- 12:          $\delta(\sigma_1, \sigma_2) := \epsilon$
- 13:     **else**
- 14:         **for**  $j \in \{2, \dots, l-1\}$  **do**
- 15:              $Q := Q \cup \{\sigma_1 \cdots \sigma_j\}$
- 16:              $\delta(\sigma_1 \cdots \sigma_{j-1}, \sigma_j) := \sigma_1 \cdots \sigma_j$
- 17:         **end for**
- 18:          $\delta(\sigma_1 \cdots \sigma_{l-1}, \sigma_l) := \epsilon$
- 19:     **end if**
- 20: **end for**
- 21:  $\mathcal{W} := \{(w, \sigma) \in Q \times \mathcal{I} \mid (w, \sigma) \notin \delta^{-1}(Q)\}$
- 22: **for**  $(w, \sigma) \in \mathcal{W}$  **do**
- 23:     **if**  $\exists w_s \in \mathcal{S}(w)$  s.t.  $w_s \sigma \in \mathcal{O}'^{[k]}$  **then**
- 24:          $\delta(w, \sigma) := \epsilon$
- 25:     **else**
- 26:          $\delta(w, \sigma) := \arg \max_{v \in Q \cap \mathcal{S}(w\sigma)} |v|$
- 27:     **end if**
- 28: **end for**

---

- In the second part, from step 10 to step 20, we consider reconstructible sequences  $w$  in  $\mathcal{O}'^{[k]}$  of length  $l \geq 2$ . We add to the set  $Q$  all the prefixes of  $w$ , except  $w$  itself, and we define the corresponding transitions between consecutive prefixes. The last transition, which happens when the reconstructible sequence  $w$  has occurred, leads to the accepting state.
- Finally, in the third part, from step 21 to step 28, we focus on the couples  $(w, \sigma) \in Q \times \mathcal{I}$  for which no transition is defined yet. In this case, if there exists a suffix  $w_s$  of  $w$  such that  $w_s \sigma \in \mathcal{O}'^{[k]}$ , then we add a transition from  $w$  to the accepting state. Otherwise, we look for the longest suffix  $v$  of  $w\sigma$  such that  $v \in Q$  and we define a transition toward this state.

**Remark 1.** *The construction of the DBA  $\mathcal{B}_k$  mostly relies on the set  $\mathcal{O}'^{[k]}$ . Actually, it is readily seen that the number of states of  $\mathcal{B}_k$  is upper bounded by  $m + kc_k$ , where  $c_k$  denotes the number of elements of  $\mathcal{O}'^{[k]}$ . Note that  $c_k$  may grow unbounded with  $k$ . A trivial bound can be given by  $c_k \leq m^k$ , however this bound appears to be conservative in practice, as will be seen later on numerical examples. The derivation of tight bounds on the cardinality of  $\mathcal{O}'^{[k]}$  remains an open question. Also relevant is the minimal value of  $k$  for which  $\mathcal{O}'^{[k]} \neq \emptyset$ . If all matrices in  $\mathcal{A}$  are invertible, it is easy to see from (3) that*

$\mathcal{O}'^{[k]} = \emptyset$  for all  $k < \frac{n}{p}$ . Moreover, it follows from Theorem 3 in [33] that there exist observable switched systems such that  $\mathcal{O}'^{[k]} = \emptyset$  for all  $k \leq \frac{n(n+1)}{2} - 1$ . Conversely, it can be obtained from the proof of Theorem 1 in [32] that if a switched system is observable then  $\mathcal{O}'^{[k]} \neq \emptyset$  for all  $k \geq \frac{n(n^2-1)}{2}$ .

**Example 1.** Consider the switched linear system with two modes, i.e.  $\mathcal{I} = \{1, 2\}$ , defined by the set of matrices  $\mathcal{A} = \{A_1, A_2\}, \mathcal{C} = \{C_1, C_2\}$ , where

$$A_1 = I_3, \quad A_2 = 1.5 \times \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$C_1 = (1 \ 0 \ 0)$  and  $C_2 = (0 \ 1 \ 1)$ . For  $k = 3$ , we get that

$$\mathcal{O}'^{[k]} = \{121, 122, 212, 221\}.$$

Applying Algorithm 1, we obtain the DBA represented in Figure 1 where transitions are represented with arrows of different types depending on which phase of the algorithm they have been added. Let us comment on the influence of  $k$  on our construction. It is not hard to check that, for  $k \geq 3$ ,

$$\mathcal{O}'^{[k]} = \{121, 122, 221\} \cup \{21^l 2 \mid l = 1, \dots, k-2\}.$$

So the cardinalities of  $\mathcal{O}'^{[k]}$  and of the set of states of  $\mathcal{B}_k$  are  $k+1$  and  $k+3$ , respectively. This corresponds to a linear growth far from the exponential growth bounds discussed in Remark 1.

### 3.2 Language characterization

In this subsection, we prove that  $\text{Lang}(\mathcal{B}_k) = (\mathcal{I}^* \mathcal{O}'^{[k]})^\omega$ . For that purpose, we make use of the following three lemmas.

**Lemma 1.** Let  $\mathcal{B}_k$  be the Büchi automaton constructed using Algorithm 1 for some  $k \geq 1$ , then  $\mathcal{B}_k$  is deterministic and any state of  $\mathcal{B}_k$  is reachable from any other state.

*Proof.* Algorithm 1 univocally builds the application  $\delta$  for each pair of  $(w, \sigma) \in Q \times \mathcal{I}$ , hence the ensuing Büchi automaton is deterministic. Also, it is apparent from the couples of lines 6, 7 and 15, 16 that, whenever a new state is added to the set  $Q$  a chain of transitions from the accepting state to such a state is also created. As a consequence, all the states of  $\mathcal{B}_k$  are reachable from the accepting state. Furthermore, the last transition of any such chain, see steps 4, 12 and 18, brings the run back to the accepting state.  $\square$

**Lemma 2.** Let  $\mathcal{B}_k$  be the DBA constructed using Algorithm 1 for some  $k \geq 1$ . Then the following statements hold true:

- (1) If  $w', w'' \in Q, w \in \mathcal{I}^*$  are such that  $w' \xrightarrow{w} w''$  then  $w'' \in \mathcal{S}(w'w)$ ;

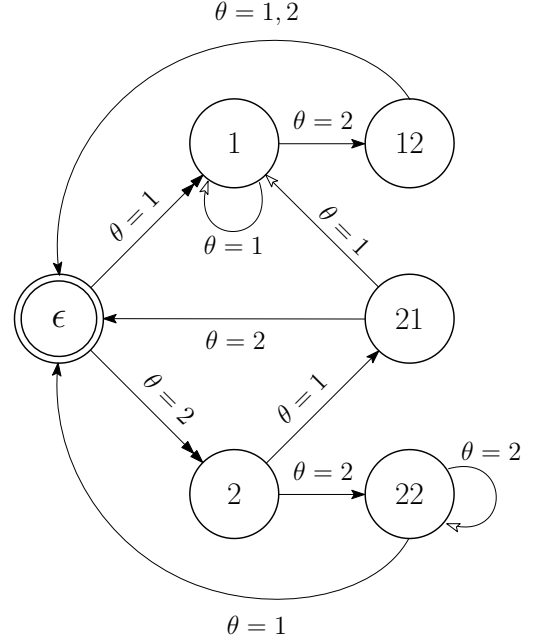


Fig. 1. The generated DBA  $\mathcal{B}_k$  corresponding to  $\mathcal{O}'^{[k]} = \{121, 122, 212, 221\}$  for  $k = 3$ . The transitions built in part 1, 2 and 3 of Algorithm 1 are represented respectively by arrows with double head, single filled head and single empty head.

- (2) If  $w \in Q, \sigma \in \mathcal{I}$  are such that  $\delta(w, \sigma) = \epsilon$  then there exists  $w' \in \mathcal{S}(w)$  such that  $w'\sigma \in \mathcal{O}'^{[k]}$ .

*Proof.* Note that the transition map  $\delta$  defined by the algorithm satisfies  $\delta(v, \sigma) \in \mathcal{S}(v\sigma)$  for every  $v \in Q$  and  $\sigma \in \mathcal{I}$ . Furthermore, it is straightforward to see that, for every  $v, v', v'' \in \mathcal{I}^*$  with  $v'' \in \mathcal{S}(v')$ , it holds  $\mathcal{S}(v''v) \subseteq \mathcal{S}(v'v)$ . Now, let  $w, w', w''$  be as in Item (1). We write  $w = \sigma_1 \dots \sigma_p$  and define  $w_0 = w'$  and, recursively for  $k = 1, \dots, p$ ,  $w_k = \delta(w_{k-1}, \sigma_k)$ . We then have

$$\begin{aligned} w'' \in \mathcal{S}(w_{p-1}\sigma_p) &\subseteq \mathcal{S}(w_{p-2}\sigma_{p-1}\sigma_p) \\ &\subseteq \dots \subseteq \mathcal{S}(w'\sigma_1 \dots \sigma_p), \end{aligned}$$

concluding the proof of Item (1).

Let us prove Item (2). Note that in steps 4, 12, 18 and 24  $\delta(w, \sigma) = \epsilon$  and the conclusion of Item (2) holds. In steps 7 and 16 it is apparent that  $\delta(w, \sigma) \neq \epsilon$ . We now show that this is also the case for step 26. Note that, in this step,  $\sigma$  can not belong to  $\mathcal{O}'^{[k]}$ , thus, by step 6,  $\sigma \in Q$ . Therefore  $\delta(w, \sigma) \neq \epsilon$  in step 26.  $\square$

**Lemma 3.** Let  $\mathcal{B}_k$  be the DBA constructed using Algorithm 1 for some  $k \geq 1$ . Let  $w \in Q, w_o \in \mathcal{O}'^{[k]}$ , then there exists  $w^* \in \mathcal{P}(w_o)$  such that  $w \xrightarrow{w^*} \epsilon$ .

*Proof.* Write  $w_o = \sigma_1 \dots \sigma_p$ . For  $p = 1$  the result follows from steps 4 and 24 of the algorithm. Let us then consider

the case  $p \geq 2$ . For  $k = 1, \dots, p$ , let us define  $w_k$  such that  $w \xrightarrow{\sigma_1 \cdots \sigma_k} w_k$ . To prove the lemma, it is enough to show that if  $w_k \neq \epsilon$  for  $k = 1, \dots, p-1$  then  $w_p = \epsilon$ . We first claim that, if  $w_k \neq \epsilon$  for  $k = 1, \dots, p-1$  then  $w_k = w'_k \sigma_1 \cdots \sigma_k$  for some  $w'_k \in S(w)$ . In order to prove the claim, we proceed by induction on  $k$ . It follows from steps 7, 16 and 26 that either  $w_1 = w\sigma_1$ , and thus the claim holds with  $w'_1 = w$ , or  $w_1 = \arg \max_{z \in Q \cap S(w\sigma_1)} |z|$ . In

the latter case, since  $w_1 \neq \epsilon$  by assumption, it follows that  $w_1 = w'_1 \sigma_1$  for some  $w'_1 \in S(w)$ . Hence the claim is proved for  $k = 1$ .

Assume now that the claim holds true for some  $k \in \{1, \dots, p-2\}$ , hence we can write  $w_k = w'_k \sigma_1 \cdots \sigma_k$  for some  $w'_k \in S(w)$ . Let us prove the claim for  $k+1$ . It follows from steps 16 and 26 that either  $w_{k+1} = w_k \sigma_{k+1} = w'_k \sigma_1 \cdots \sigma_{k+1}$ , in which case the claim holds true with  $w'_{k+1} = w'_k$ , or  $w_{k+1} = \arg \max_{z \in Q \cap S(w_k \sigma_{k+1})} |z|$ .

In the second case, since  $w_o \in \mathcal{O}^{[k]}$  we have, by step 15, that  $\sigma_1 \cdots \sigma_{k+1} \in Q$ . By construction  $\sigma_1 \cdots \sigma_{k+1} \in S(w_k \sigma_{k+1})$ , from which the claim follows for  $k+1$ .

From the claim we have that  $w_{p-1} = w'_{p-1} \sigma_1 \cdots \sigma_{p-1}$ . By step 24 we then have  $w_p = \epsilon$ .  $\square$

We can now state the main result of this subsection:

**Theorem 2.** *Let  $\mathcal{B}_k$  be the DBA constructed using Algorithm 1 for some  $k \geq 1$ , then  $\text{Lang}(\mathcal{B}_k) = (\mathcal{I}^* \mathcal{O}^{[k]})^\omega$ .*

*Proof.* Note that  $(\mathcal{I}^* \mathcal{O}^{[k]})^\omega \subseteq \mathcal{I}^\omega$  and  $\mathcal{B}_k$  is deterministic by Lemma 1, hence for any sequence in  $(\mathcal{I}^* \mathcal{O}^{[k]})^\omega$  there exists a (unique) corresponding run in  $\mathcal{B}_k$ .

First, we show that  $(\mathcal{I}^* \mathcal{O}^{[k]})^\omega \subseteq \text{Lang}(\mathcal{B}_k)$ . Let  $w = w_1 w_1^o w_2 w_2^o \cdots \in (\mathcal{I}^* \mathcal{O}^{[k]})^\omega$ , where  $w_i \in \mathcal{I}^*$ ,  $w_i^o \in \mathcal{O}^{[k]}$ ,  $i \geq 1$ . Let us define some elements of the run in  $\mathcal{B}_k$  corresponding to the sequence  $w$  as  $w_{(0)} = \epsilon$  and, recursively,  $w_{(i-1)} \xrightarrow{w_i w_i^o} w_{(i)}$  for all  $i \geq 1$ . Let us define  $v_{(i)}$  such that  $w_{(i-1)} \xrightarrow{w_i} v_{(i)}$ , then, by Lemma 3, there exists  $w_i^* \in P(w_i^o)$  such that  $v_{(i)} \xrightarrow{w_i^*} \epsilon$ . Now, in the infinite run corresponding to  $w$ , the state  $\epsilon$  is visited at least once between each couple of states  $w_{(i-1)}$  and  $w_{(i)}$ . Therefore the accepting state is visited an infinite number of times in the run. This proves the first inclusion.

Let us focus on the inclusion  $(\mathcal{I}^* \mathcal{O}^{[k]})^\omega \supseteq \text{Lang}(\mathcal{B}_k)$ . Let  $w \in \text{Lang}(\mathcal{B}_k)$ . Then we can write  $w = w_1 w_2 \cdots$  where  $w_i \in \mathcal{I}^*$  are such that  $\epsilon \xrightarrow{w_1} \epsilon \xrightarrow{w_2} \epsilon \cdots$ . Let us consider  $w_i$  in the previous decomposition of  $w$ . We write  $w_i = \sigma_1 \cdots \sigma_l$  for some  $l \geq 1$ , and consider the word  $w'_i$  such that  $\epsilon \xrightarrow{\sigma_1 \cdots \sigma_{l-1}} w'_i \xrightarrow{\sigma_l} \epsilon$ . By applying Item (2) in Lemma 2 we obtain the existence of  $w''_i \in S(w'_i)$  such that  $w''_i \sigma_l \in \mathcal{O}^{[k]}$ . By applying

Item (1) in Lemma 2 we deduce that  $w'_i \in S(\sigma_1 \cdots \sigma_{l-1})$ , hence  $w''_i \sigma_l \in S(\sigma_1 \cdots \sigma_l) = S(w_i)$ . We thus get  $w''_i \sigma_l \in S(w_i) \cap \mathcal{O}^{[k]}$ , that is  $w_i \in \mathcal{I}^* \mathcal{O}^{[k]}$ . Since this is true for all  $w_i$  with  $i \geq 1$  in  $w = w_1 w_2 \cdots$ , we obtain  $w \in (\mathcal{I}^* \mathcal{O}^{[k]})^\omega$ , concluding the proof of the inclusion  $(\mathcal{I}^* \mathcal{O}^{[k]})^\omega \supseteq \text{Lang}(\mathcal{B}_k)$ .  $\square$

Let us remark that, since for any  $k_1 \leq k_2$ ,  $\mathcal{O}^{[k_1]} \subseteq \mathcal{O}^{[k_2]}$ , it follows from Theorem 2 that  $\text{Lang}(\mathcal{B}_{k_1}) \subseteq \text{Lang}(\mathcal{B}_{k_2})$ . Hence, considering longer reconstructible sequences produces a DBA accepting more switching signals.

## 4 Asymptotic observer design

In this section, we present an approach for designing asymptotic observers for system (1). The proposed design results in a switched observer where the switching is driven by the DBA  $\mathcal{B}_k$  presented in the previous section. We first introduce the structure of the observer and then propose two approaches for designing the gains of the observer.

### 4.1 Observer structure

Let us consider the system (1) and let  $\mathcal{B}_k = (Q, \mathcal{I}, \delta, q_0, F)$  be the DBA constructed using Algorithm 1, for some  $k \geq 1$  such that  $\mathcal{O}^{[k]} \neq \emptyset$ . We consider the following switched observer where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimate of the state of (1) and  $q(t) \in Q$  is an internal discrete state:

$$\begin{aligned} q(t+1) &= \delta(q(t), \theta(t)), \quad q(0) = q_0, \\ \hat{x}(t+1) &= A_{\theta(t)} \hat{x}(t) + L_{(q(t), \theta(t))} (y(t) - C_{\theta(t)} \hat{x}(t)). \end{aligned} \quad (4)$$

Let us remark that the dynamics of  $q$  is given by the transition function  $\delta$  of the DBA  $\mathcal{B}_k$ . The structure of the observer being given by (4), it remains to design the observer gains  $L_{(w, \sigma)}$ , for all  $w \in Q$  and  $\sigma \in \mathcal{I}$ .

Let  $e(t) = x(t) - \hat{x}(t)$  be the estimation error of the observer, then the dynamics of  $e(t)$  is given by:

$$e(t+1) = (A_{\theta(t)} - L_{(q(t), \theta(t))} C_{\theta(t)}) e(t).$$

We note that, given an initial condition  $e_0 \in \mathbb{R}^n$ , and a switching signal  $\theta$ , the trajectory with  $e(0) = e_0$  is unique and denoted by  $\mathbf{e}(\cdot, e_0, \theta)$ . In order to guarantee the convergence of the asymptotic observer, one needs to prove that  $\mathbf{e}(t, e_0, \theta)$  goes to 0 as  $t$  tends to infinity. For this purpose, we will make use of the following Lyapunov-type result.

**Proposition 1.** *Let us assume that there exist a function  $V : Q \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ , and scalars  $\alpha_1, \alpha_2, \rho > 0$ , and  $\lambda \in$*

$(0, 1)$ , such that for all  $e \in \mathbb{R}^n$ , the following inequalities hold:

$$\alpha_1 \|e\| \leq V(w, e) \leq \alpha_2 \|e\|, \quad \forall w \in Q, \quad (5)$$

$$V(w', (A_\sigma - L_{(w, \sigma)} C_\sigma) e) \leq \rho V(w, e), \quad (6)$$

$$\forall w \in Q, \sigma \in \mathcal{I}, \text{ s.t. } w' = \delta(w, \sigma) \neq \epsilon,$$

$$V(w', (A_\sigma - L_{(w, \sigma)} C_\sigma) e) \leq \rho \lambda V(w, e), \quad (7)$$

$$\forall w \in Q, \sigma \in \mathcal{I}, \text{ s.t. } w' = \delta(w, \sigma) = \epsilon.$$

Then, there exists  $C \geq 1$  such that for all  $e_0 \in \mathbb{R}^n$ , and for all  $\theta \in \text{Lang}(\mathcal{B}_k)$ :

$$\forall t \in \mathbb{N}, \|\mathbf{e}(t, e_0, \theta)\| \leq C \rho^t \lambda^{\kappa^{\theta, \mathcal{B}_k}(t)} \|e_0\|. \quad (8)$$

Moreover, whenever the accepting rate satisfies  $\gamma^{\theta, \mathcal{B}_k} > \frac{\ln(\rho)}{-\ln(\lambda)}$ , we have

$$\lim_{t \rightarrow \infty} \|\mathbf{e}(t, e_0, \theta)\| = 0. \quad (9)$$

*Proof.* The proof of (8) follows the same rationale as in [2, Theorem 1], with a slight modification that is the scaling of the set of state matrices by  $\rho > 0$ . The proof of (9) is as in [1, Corollary 1].  $\square$

Equation (8) relates the convergence rate of the asymptotic observer to the return index  $\kappa^{\theta, \mathcal{B}_k}(t)$  associated with the DBA  $\mathcal{B}_k$ , which essentially counts the number of reconstructible sequences (i.e. elements of  $\mathcal{O}^{[k]}$ ) that occur up to time  $t$ . Note that  $\rho$  may be greater than 1. In that case, in order to guarantee the convergence of the observer, one must consider switching signals where reconstructible sequences occur sufficiently often. This occurrence rate is measured by the accepting rate  $\gamma^{\theta, \mathcal{B}_k}$ . If  $\gamma^{\theta, \mathcal{B}_k}$  is sufficiently large, we get from (9) that the observer asymptotically converges. In general the set of switching signals  $\theta$  satisfying  $\gamma^{\theta, \mathcal{B}_k} > \frac{\ln(\rho)}{-\ln(\lambda)}$  is not an  $\omega$ -regular language anymore but belongs to the class of quantitative languages, studied in [11].

**Remark 2.** Proposition 1 shows convergence of the observer for all switching signals  $\theta$  such that  $\gamma^{\theta, \mathcal{B}_k} > \frac{\ln(\rho)}{-\ln(\lambda)}$ . We briefly discuss how this inequality can be verified for important classes of switching signals. Firstly, let us assume that the switching signal is a stochastic process generated by a finite state Markov chain  $\mathcal{M}_C$ . Then, by the ergodic theorem (see e.g. Theorem 1.10.2 in [22]) applied to the Markov chain obtained by the synchronous product of  $\mathcal{M}_C$  and  $\mathcal{B}_k$ , one can compute a value  $\gamma^*$  such that  $\gamma^{\theta, \mathcal{B}_k} = \gamma^*$ , almost surely. Secondly, let us assume that the switching signal is non-stochastic but constrained by some automaton  $\mathcal{M}_A$  (see e.g. [23]). Let us remark that this case includes classical minimal or maximal dwell time constraints. Using algorithms from model checking [7], one can check

if  $\text{Lang}(\mathcal{M}_A) \subseteq \text{Lang}(\mathcal{B}_k)$  where  $\text{Lang}(\mathcal{M}_A)$  denotes the set of switching signals generated by automaton  $\mathcal{M}_A$ . Then, by using the techniques for quantitative languages developed in [11] applied to the synchronous product of  $\mathcal{M}_A$  and  $\mathcal{B}_k$ , it is possible to compute the value  $\gamma^* = \inf\{\gamma^{\theta, \mathcal{B}_k} \mid \theta \in \text{Lang}(\mathcal{M}_A)\}$ . Hence, we can see that for two broad classes of switching signals, there exist effective procedures to verify whether the condition  $\gamma^{\theta, \mathcal{B}_k} > \frac{\ln(\rho)}{-\ln(\lambda)}$  holds.

**Remark 3.** The observer design proposed in this section can be readily adapted to the case where the dynamics (1) contains a known input (e.g. a control term). That is, if

$$x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t),$$

one may replace the switched observer in (4) by

$$\hat{x}(t+1) = A_{\theta(t)}\hat{x}(t) + L_{(q(t), \theta(t))}(y(t) - C_{\theta(t)}\hat{x}(t)) + B_{\theta(t)}u(t),$$

leading to the same error dynamics as before.

## 4.2 Robustness properties

In this subsection, we discuss robustness properties of the proposed observer. We consider a perturbed version of system (1):

$$x(t+1) = A_{\theta(t)}x(t) + d_1(t),$$

$$y(t) = C_{\theta(t)}x(t) + d_2(t)$$

where  $d_1(t) \in \mathbb{R}^n$  and  $d_2(t) \in \mathbb{R}^p$  represent disturbances on the dynamics and on the measurements of (1), respectively. We consider observer (4) and denote  $\tilde{e}(t) = x(t) - \hat{x}(t)$  the estimation error. Then, the dynamics of the estimation error of the observer is given by:

$$\tilde{e}(t+1) = (A_{\theta(t)} - L_{(q(t), \theta(t))}C_{\theta(t)})\tilde{e}(t) + d(t)$$

where  $d(t) = d_1(t) - L_{(q(t), \theta(t))}d_2(t)$ . We note that, given an initial condition  $e_0 \in \mathbb{R}^n$ , a switching signal  $\theta$  and a disturbance  $d$ , the trajectory with  $e(0) = e_0$  is unique and denoted by  $\tilde{\mathbf{e}}(\cdot, e_0, \theta, d)$ .

We will establish a robustness result for a class of switching signals. Let  $T_0 \geq 0$  and  $\gamma > 0$  and let  $\mathcal{S}_{T_0, \gamma}(\mathcal{B}_k)$  be the set of switching signals  $\theta \in \text{Lang}(\mathcal{B}_k)$  such that

$$\forall l_1 \geq l_2, \tau_{l_1}^{\theta, \mathcal{B}_k} - \tau_{l_2}^{\theta, \mathcal{B}_k} \leq T_0 + \frac{l_1 - l_2}{\gamma}. \quad (10)$$

Let us remark that for all  $\theta \in \mathcal{S}_{T_0, \gamma}(\mathcal{B}_k)$  it holds that  $\gamma^{\theta, \mathcal{B}_k} \geq \gamma$ , but the converse is not true. Indeed, (10) requires that the deviation of the return instants to the accepting rate is uniformly bounded.



**Proposition 2.** *Let us assume that there exist a function  $V : Q \times \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ , and scalars  $\alpha_1, \alpha_2, \alpha_3 > 0, \rho > 0$ , and  $\lambda \in (0, 1)$ , such that for all  $e \in \mathbb{R}^n$ , the inequalities (5), (6), (7) hold and for all  $w \in Q$  and  $d \in \mathbb{R}^n$*

$$V(w, e + d) \leq V(w, e) + \alpha_3 \|d\|. \quad (11)$$

Let  $T_0 \geq 0$  and  $\gamma > \frac{\ln(\rho)}{-\ln(\lambda)}$ , then, there exists  $C_1, C_2 > 0$  such that for all  $e_0 \in \mathbb{R}^n$ , for all  $\theta \in \mathcal{S}_{T_0, \gamma}(\mathcal{B}_k)$  and for all  $d : \mathbb{N} \rightarrow \mathbb{R}^n$ , for all  $t \geq 1$ ,

$$\|\tilde{e}(t, e_0, \theta, d)\| \leq C_1 \tilde{\rho}^t \|e_0\| + C_2 \max_{0 \leq s < t} \|d(s)\| \quad (12)$$

where  $\tilde{\rho} = \rho \lambda^\gamma < 1$ .

*Proof.* Let  $e_0 \in \mathbb{R}^n$ ,  $\theta \in \mathcal{S}_{T_0, \gamma}(\mathcal{B}_k)$ ,  $d : \mathbb{N} \rightarrow \mathbb{R}^n$ , we denote  $\tilde{e}(\cdot) = \tilde{e}(\cdot, e_0, \theta, d)$ . Let  $q : \mathbb{N} \rightarrow Q$  be given by (4), let the function  $W : \mathbb{N} \rightarrow \mathbb{R}_0^+$  be given by  $W(t) = V(q(t), \tilde{e}(t))$  for all  $t \in \mathbb{N}$ . For better readability we denote  $\tau_l = \tau_l^{\theta, \mathcal{B}_k}$ ,  $\kappa(t) = \kappa^{\theta, \mathcal{B}_k}(t)$ , and  $b_t = \max_{0 \leq s < t} \|d(s)\|$ . Let  $t \geq 1$ , for all  $l \geq 1$ , such that  $\tau_l \leq t$ , we get from (7) and (11)

$$W(\tau_l) \leq \lambda \rho W(\tau_l - 1) + \alpha_3 b_t.$$

Then, from (6), it follows that

$$\begin{aligned} W(\tau_l) &\leq \lambda \rho \left( \rho^{(\tau_l - \tau_{l-1} - 1)} W(\tau_{l-1}) + \alpha_3 b_t \sum_{i=0}^{\tau_l - \tau_{l-1} - 2} \rho^i \right) \\ &\quad + \alpha_3 b_t \\ &\leq \lambda \rho^{(\tau_l - \tau_{l-1})} W(\tau_{l-1}) + \alpha_3 b_t \sum_{i=0}^{\tau_l - \tau_{l-1} - 1} \rho^i. \end{aligned}$$

Let us assume that  $\rho > 1$ , then we get

$$W(\tau_l) \leq \lambda \rho^{(\tau_l - \tau_{l-1})} W(\tau_{l-1}) + \frac{\alpha_3 b_t}{\rho - 1} \rho^{(\tau_l - \tau_{l-1})}.$$

Then, by induction, we get

$$W(\tau_l) \leq \lambda^l \rho^{\tau_l} W(\tau_0) + \frac{\alpha_3 b_t}{\rho - 1} \sum_{i=0}^{l-1} \lambda^i \rho^{(\tau_l - \tau_{l-1} - i)}.$$

From (10), we have  $\tau_l - \tau_{l-1-i} \leq T_0 + \frac{i+1}{\gamma}$ . Hence,

$$\begin{aligned} W(\tau_l) &\leq \lambda^l \rho^{\tau_l} W(\tau_0) + \frac{\alpha_3 b_t}{\rho - 1} \sum_{i=0}^{l-1} \lambda^i \rho^{(T_0 + \frac{i+1}{\gamma})} \\ &\leq \lambda^l \rho^{\tau_l} W(\tau_0) + \frac{\alpha_3 b_t}{\rho - 1} \rho^{T_0 + \frac{1}{\gamma}} \frac{1}{1 - \lambda \rho^{\frac{1}{\gamma}}} \end{aligned} \quad (13)$$

since by assumption  $\lambda \rho^{\frac{1}{\gamma}} < 1$ . Then, from (6), we have

$$W(t) \leq \rho^{t - \tau_{\kappa(t)}} W(\tau_{\kappa(t)}) + \frac{\alpha_3 b_t}{\rho - 1} \rho^{t - \tau_{\kappa(t)}}.$$

Then, from (13) with  $l = \kappa(t)$ , it follows that

$$\begin{aligned} W(t) &\leq \rho^t \lambda^{\kappa(t)} W(\tau_0) \\ &\quad + \left( \frac{\alpha_3 b_t}{\rho - 1} \rho^{T_0 + \frac{1}{\gamma}} \frac{1}{1 - \lambda \rho^{\frac{1}{\gamma}}} + \frac{\alpha_3 b_t}{\rho - 1} \right) \rho^{t - \tau_{\kappa(t)}}. \end{aligned}$$

Then, from (10), we can show that  $t - \tau_{\kappa(t)} \leq T_0 + \frac{1}{\gamma}$  and that  $\kappa(t) \geq \gamma t - \gamma T_0 - 1$ . This yields

$$\begin{aligned} W(t) &\leq \rho^t \lambda^{\gamma t - \gamma T_0 - 1} W(\tau_0) \\ &\quad + \left( \frac{\alpha_3 b_t}{\rho - 1} \rho^{T_0 + \frac{1}{\gamma}} \frac{1}{1 - \lambda \rho^{\frac{1}{\gamma}}} + \frac{\alpha_3 b_t}{\rho - 1} \right) \rho^{T_0 + \frac{1}{\gamma}}. \end{aligned}$$

Hence, there exist  $C'_1, C'_2 > 0$  such that for all  $t \geq 1$

$$W(t) \leq C'_1 \rho^t W(\tau_0) + C'_2 b_t. \quad (14)$$

Note that, following similar steps, the same result can be proved when  $\rho \leq 1$ . Then, (12) follows directly from (5) and (14).  $\square$

Hence, Proposition 2 shows that the proposed observer is robust to unmodeled dynamics and to measurement noise. We emphasize that the class of switching signals for which the robustness property holds is slightly more restrictive than that in Proposition 1. However, it should be noticed that this restriction allows us to guarantee not only robustness but also exponential convergence of the proposed observer.

#### 4.3 Explicit design of observer gains

In this subsection, we present a particular design of the observer gains  $L_{(w, \sigma)}$ ,  $w \in Q$  and  $\sigma \in \mathcal{I}$ , and of the associated Lyapunov function  $V$ . The proposed design requires the following property which is assumed to hold throughout the subsection:

**Assumption 1.**  $A_\sigma$  is invertible for all  $\sigma \in \mathcal{I}$ .

Let us fix  $\rho > \rho_e(\mathcal{A})$  where  $\rho_e(\mathcal{A})$  denotes the ellipsoid norm approximation of the joint spectral radius of  $\mathcal{A}$  [9]:

$$\rho_e(\mathcal{A}) = \inf \left\{ \rho \geq 0 \left| \begin{array}{l} \exists M > 0, M^\top = M, \\ \forall A \in \mathcal{A}, A^\top M A \leq \rho^2 M \end{array} \right. \right\}.$$

Then, by definition, there exists  $M \in \mathbb{R}^{n \times n}$  symmetric positive definite such that

$$A_\sigma^\top M A_\sigma \leq \rho^2 M, \quad \forall \sigma \in \mathcal{I}. \quad (15)$$

Let us also consider an arbitrary  $\lambda \in (0, 1)$ .

**Remark 4.** *The results presented in this section can be extended to the case  $\rho = \rho_\epsilon(\mathcal{A})$  if there exists a symmetric positive definite matrix  $M$  such that (15) holds.*

We start with a preliminary result:

**Lemma 4.** *There exists  $\gamma > 0$  such that for all reconstructible sequences  $\sigma_1 \cdots \sigma_l \in \mathcal{O}^{[k]}$*

$$W_{\sigma_1 \cdots \sigma_l}^\top W_{\sigma_1 \cdots \sigma_l} \geq \gamma M,$$

where  $W_{\sigma_1 \cdots \sigma_l} = \text{diag}([1, \frac{1}{\rho}, \dots, \frac{1}{\rho^{l-1}}])\Omega(\sigma_1, \dots, \sigma_l)$ .

*Proof.* Since all matrices in  $\mathcal{A}$  are invertible, any reconstructible sequence is also observable. Then, from Theorem 1, we get that for any  $\sigma_1 \cdots \sigma_l \in \mathcal{O}^{[k]}$ ,  $\text{rank}(W_{\sigma_1 \cdots \sigma_l}) = n$ , and therefore  $W_{\sigma_1 \cdots \sigma_l}^\top W_{\sigma_1 \cdots \sigma_l}$  is symmetric positive definite. Then, the result follows directly from the fact that the set  $\mathcal{O}^{[k]}$  is finite.  $\square$

In the following, we provide an explicit construction of observer gains  $L_{(w, \sigma)}$  and of a Lyapunov function  $V$  under the form

$$V(w, e) = \sqrt{e^\top P_w e}.$$

Let us associate a matrix  $P_w \in \mathbb{R}^{n \times n}$  with every word  $w \in \mathcal{I}^*$ , defined recursively as follows:

$$P_\epsilon = M \quad (16)$$

$$P_{w\sigma} = \rho^2 A_\sigma^{-\top} P_w A_\sigma^{-1} + \frac{\rho^2}{\gamma \lambda^2} A_\sigma^{-\top} C_\sigma^\top C_\sigma A_\sigma^{-1}, \quad (17)$$

$$\forall w \in \mathcal{I}^*, \sigma \in \mathcal{I}.$$

Let us prove several instrumental results related to properties of the matrices  $P_w$ .

**Lemma 5.** *For all  $w \in \mathcal{I}^*$ ,  $P_w > 0$ .*

*Proof.* From (17), it follows that

$$P_{w\sigma} \geq \rho^2 A_\sigma^{-\top} P_w A_\sigma^{-1}, \quad \forall w \in \mathcal{I}^*, \sigma \in \mathcal{I}.$$

Then, since  $P_\epsilon = M > 0$  and  $A_\sigma$  is invertible for all  $\sigma \in \mathcal{I}$  by Assumption 1, it follows by induction that  $P_w > 0$  for all  $w \in \mathcal{I}^*$ .  $\square$

**Lemma 6.** *For all  $w \in \mathcal{I}^*$ ,  $\sigma \in \mathcal{I}$ ,  $P_w \leq P_{w\sigma}$ .*

*Proof.* From (17), we get that for all  $\sigma \in \mathcal{I}$ ,

$$P_\sigma \geq \rho^2 A_\sigma^{-\top} P_\epsilon A_\sigma^{-1}.$$

From (15), we get that

$$M \leq \rho^2 A_\sigma^{-\top} M A_\sigma^{-1}. \quad \square$$

Then, from (16) and the two previous inequalities, it follows that  $P_\epsilon \leq P_\sigma$ . Hence, the property holds for  $w = \epsilon$  and hence for all words of length 0. Then, we proceed by induction. Let us assume that the property holds for all words of a certain length  $l \in \mathbb{N}$ , and let us consider a word  $w$  of length  $l + 1$ . Then, there exists  $w_l \in \mathcal{I}^*$ ,  $|w_l| = l$  and  $\sigma_{l+1} \in \mathcal{I}$  such that  $w = w_l \sigma_{l+1}$ . Then, from (17)

$$P_{\sigma w} = P_{\sigma w_l \sigma_{l+1}} \\ = \rho^2 A_{\sigma_{l+1}}^{-\top} P_{\sigma w_l} A_{\sigma_{l+1}}^{-1} + \frac{\rho^2}{\gamma \lambda^2} A_{\sigma_{l+1}}^{-\top} C_{\sigma_{l+1}}^\top C_{\sigma_{l+1}} A_{\sigma_{l+1}}^{-1}.$$

Since  $|w_l| = l$ , we get by the induction assumption  $P_{\sigma w_l} \geq P_{w_l}$  and therefore

$$P_{\sigma w} \geq \rho^2 A_{\sigma_{l+1}}^{-\top} P_{w_l} A_{\sigma_{l+1}}^{-1} + \frac{\rho^2}{\gamma \lambda^2} A_{\sigma_{l+1}}^{-\top} C_{\sigma_{l+1}}^\top C_{\sigma_{l+1}} A_{\sigma_{l+1}}^{-1}.$$

Then, from (17), we also get

$$P_w = P_{w_l \sigma_{l+1}} \\ = \rho^2 A_{\sigma_{l+1}}^{-\top} P_{w_l} A_{\sigma_{l+1}}^{-1} + \frac{\rho^2}{\gamma \lambda^2} A_{\sigma_{l+1}}^{-\top} C_{\sigma_{l+1}}^\top C_{\sigma_{l+1}} A_{\sigma_{l+1}}^{-1}.$$

Then,  $P_{\sigma w} \geq P_w$ . By induction, the property holds for all  $w \in \mathcal{I}^*$ .  $\square$

**Lemma 7.** *For all  $w \in \mathcal{O}^{[k]}$ ,  $M \leq \lambda^2 P_w$ .*

*Proof.* Let  $w \in \mathcal{O}^{[k]}$ ,  $w = \sigma_1 \cdots \sigma_l$ , then from (17)

$$P_w = P_{\sigma_1 \cdots \sigma_l} \\ = \rho^{2l} A_{\sigma_l}^{-\top} \cdots A_{\sigma_1}^{-\top} M A_{\sigma_1}^{-1} \cdots A_{\sigma_l}^{-1} \\ + \frac{\rho^{2l}}{\gamma \lambda^2} A_{\sigma_l}^{-\top} \cdots A_{\sigma_1}^{-\top} C_{\sigma_1}^\top C_{\sigma_1} A_{\sigma_1}^{-1} \cdots A_{\sigma_l}^{-1} \\ + \frac{\rho^{2(l-1)}}{\gamma \lambda^2} A_{\sigma_l}^{-\top} \cdots A_{\sigma_2}^{-\top} C_{\sigma_2}^\top C_{\sigma_2} A_{\sigma_2}^{-1} \cdots A_{\sigma_l}^{-1} \\ + \cdots + \frac{\rho^2}{\gamma \lambda^2} A_{\sigma_l}^{-\top} C_{\sigma_l}^\top C_{\sigma_l} A_{\sigma_l}^{-1}.$$

Then, it follows that

$$P_w \geq \frac{\rho^{2l}}{\gamma \lambda^2} A_{\sigma_l}^{-\top} \cdots A_{\sigma_1}^{-\top} W_{\sigma_1 \cdots \sigma_l}^\top W_{\sigma_1 \cdots \sigma_l} A_{\sigma_1}^{-1} \cdots A_{\sigma_l}^{-1}.$$

Then, from Lemma 4 and (15), we get

$$P_w \geq \frac{\rho^{2l}}{\lambda^2} A_{\sigma_l}^{-\top} \cdots A_{\sigma_1}^{-\top} M A_{\sigma_1}^{-1} \cdots A_{\sigma_l}^{-1} \geq \frac{1}{\lambda^2} M. \quad \square$$

Let us now define the matrices  $L_{(w,\sigma)} \in \mathbb{R}^{n \times p}$ , for all  $w \in \mathcal{I}^*$ ,  $\sigma \in \mathcal{I}$  as follows:

$$L_{(w,\sigma)} = A_\sigma P_w^{-1} C_\sigma^\top (\gamma \lambda^2 I_p + C_\sigma P_w^{-1} C_\sigma^\top)^{-1}. \quad (18)$$

Then, the following fundamental property holds:

**Lemma 8.** For all  $w \in \mathcal{I}^*$ ,  $\sigma \in \mathcal{I}$ ,

$$(A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w\sigma} (A_\sigma - L_{(w,\sigma)} C_\sigma) \leq \rho^2 P_w.$$

*Proof.* From (17), we get that for all  $w \in \mathcal{I}^*$ ,  $\sigma \in \mathcal{I}$ ,

$$\frac{1}{\rho^2} A_\sigma^\top P_{w\sigma} A_\sigma = P_w + \frac{1}{\gamma \lambda^2} C_\sigma^\top C_\sigma.$$

Then, it follows that

$$\begin{aligned} \rho^2 A_\sigma^{-1} P_{w\sigma}^{-1} A_\sigma^{-\top} &= \left( P_w + \frac{1}{\gamma \lambda^2} C_\sigma^\top C_\sigma \right)^{-1} \\ &= P_w^{-1} - P_w^{-1} C_\sigma^\top (\gamma \lambda^2 I_p + C_\sigma P_w^{-1} C_\sigma^\top)^{-1} C_\sigma P_w^{-1} \\ &= P_w^{-1} - A_\sigma^{-1} L_{(w,\sigma)} C_\sigma P_w^{-1} \end{aligned} \quad (19)$$

$$= P_w^{-1} - A_\sigma^{-1} L_{(w,\sigma)} C_\sigma P_w^{-1} \quad (20)$$

where (19) is obtained by the Woodbury matrix identity and (20) is obtained by (18). Multiplying both sides by  $A_\sigma$  on the left and by  $P_w$  on the right, we obtain

$$A_\sigma - L_{(w,\sigma)} C_\sigma = \rho^2 P_{w\sigma}^{-1} A_\sigma^{-\top} P_w.$$

Therefore,

$$(A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w\sigma} (A_\sigma - L_{(w,\sigma)} C_\sigma) = \rho^4 P_w A_\sigma^{-1} P_{w\sigma}^{-1} A_\sigma^{-\top} P_w. \quad (21)$$

Note that from (19), the following inequality holds:

$$\rho^2 A_\sigma^{-1} P_{w\sigma}^{-1} A_\sigma^{-\top} \leq P_w^{-1},$$

which together with (21) gives:

$$(A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w\sigma} (A_\sigma - L_{(w,\sigma)} C_\sigma) \leq \rho^2 P_w. \quad \square$$

We are now in position of stating the main result of this subsection:

**Theorem 3.** Let  $\mathcal{B}_k$  be the DBA constructed using Algorithm 1 for some  $k \geq 1$ , let  $\rho > \rho_e(\mathcal{A})$  and  $\lambda \in (0, 1)$ . Under Assumption 1, for all  $w \in \mathcal{Q}$  and  $\sigma \in \mathcal{I}$  let the

matrices  $P_w$  and  $L_{(w,\sigma)}$  be defined as in (16), (17) and (18). Then,  $P_w > 0$ , for all  $w \in \mathcal{Q}$ , and

$$(A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w'} (A_\sigma - L_{(w,\sigma)} C_\sigma) \leq \rho^2 P_w, \quad (22)$$

$$\forall w \in \mathcal{Q}, \sigma \in \mathcal{I}, \text{ s.t. } w' = \delta(w, \sigma) \neq \epsilon,$$

$$(A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w'} (A_\sigma - L_{(w,\sigma)} C_\sigma) \leq \rho^2 \lambda^2 P_w, \quad (23)$$

$$\forall w \in \mathcal{Q}, \sigma \in \mathcal{I}, \text{ s.t. } w' = \delta(w, \sigma) = \epsilon.$$

In particular, the function  $V(w, e) = \sqrt{e^\top P_w e}$  satisfies inequalities (5), (6) and (7).

*Proof.* The fact that  $P_w > 0$ , for all  $w \in \mathcal{Q}$ , follows from Lemma 5. Let us consider  $w \in \mathcal{Q}$ ,  $\sigma \in \mathcal{I}$ , such that  $w' = \delta(w, \sigma) \neq \epsilon$ . Then, from Item (1) of Lemma 2, there exist  $w_1, w_2 \in \mathcal{I}^*$  (possibly with  $w_1 = \epsilon$  or  $w_2 = \epsilon$ ) such that  $w = w_1 w_2$  and  $w' = w_2 \sigma$ . Then, from Lemma 6,

$$P_{w'} = P_{w_2 \sigma} \leq P_{w_1 w_2 \sigma} = P_{w\sigma}.$$

Then,

$$\begin{aligned} (A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w'} (A_\sigma - L_{(w,\sigma)} C_\sigma) \\ \leq (A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w\sigma} (A_\sigma - L_{(w,\sigma)} C_\sigma) \\ \leq \rho^2 P_w \end{aligned}$$

where the last inequality comes from Lemma 8.

Now consider  $w \in \mathcal{Q}$ ,  $\sigma \in \mathcal{I}$ , such that  $w' = \delta(w, \sigma) = \epsilon$ . Then, from Item (2) of Lemma 2, there exist  $w_1, w_2 \in \mathcal{I}^*$  (possibly with  $w_1 = \epsilon$  or  $w_2 = \epsilon$ ) such that  $w = w_1 w_2$  and  $w_2 \sigma \in \mathcal{O}^{[k]}$ . From Lemma 7 and Lemma 6, we get

$$P_{w'} = P_\epsilon = M \leq \lambda^2 P_{w_2 \sigma} \leq \lambda^2 P_{w_1 w_2 \sigma} = \lambda^2 P_{w\sigma}.$$

Then,

$$\begin{aligned} (A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w'} (A_\sigma - L_{(w,\sigma)} C_\sigma) \\ \leq \lambda^2 (A_\sigma - L_{(w,\sigma)} C_\sigma)^\top P_{w\sigma} (A_\sigma - L_{(w,\sigma)} C_\sigma) \\ \leq \rho^2 \lambda^2 P_w \end{aligned}$$

where the last inequality comes from Lemma 8.

The fact that the function  $V$  satisfies inequalities (5), (6) and (7) is then a direct consequence of  $P_w > 0$ , (22) and (23), respectively.  $\square$

Hence, in this section, we have shown that when all matrices in  $\mathcal{A}$  are invertible, for all  $\rho > \rho_e(\mathcal{A})$  and  $\lambda \in (0, 1)$  it is possible to design observer gains such that conditions of Proposition 1 are satisfied. In particular, for a given  $\gamma^* > 0$ , chosen for instance according to the approach described in Remark 2, one can appropriately

choose the parameter  $\lambda$  to achieve any arbitrary convergence rate for the error dynamics for all switching signal  $\theta$  such that  $\gamma^{\theta, \mathcal{B}_k} \geq \gamma^*$ . However, it should be noted that the faster the convergence rate, the larger the magnitude of the observer gains is.

#### 4.4 LMI-based design of observer gains

In this subsection, we present an alternative design of the observer gains based on solving a set of linear matrix inequalities.

**Proposition 3.** *Let  $\mathcal{B}_k$  be the DBA constructed using Algorithm 1 for some  $k \geq 1$ , let  $\rho > 0$  and  $\lambda \in (0, 1)$ . Let us assume that there exist matrices  $P_w \in \mathbb{R}^{n \times n}$ ,  $Y_{(w, \sigma)} \in \mathbb{R}^{n \times p}$ , for  $w \in Q$ ,  $\sigma \in \mathcal{I}$ , such that the following LMIs hold:*

$$P_w > 0, \quad \forall w \in Q, \quad (24)$$

$$\begin{pmatrix} P_{w'} & P_{w'} A_\sigma - Y_{(w, \sigma)} C_\sigma \\ \star & \rho^2 P_w \end{pmatrix} \geq 0 \quad (25)$$

$$\forall w \in Q, \sigma \in \mathcal{I}, \text{ s.t. } w' = \delta(w, \sigma) \neq \epsilon,$$

$$\begin{pmatrix} P_{w'} & P_{w'} A_\sigma - Y_{(w, \sigma)} C_\sigma \\ \star & \rho^2 \lambda^2 P_w \end{pmatrix} \geq 0 \quad (26)$$

$$\forall w \in Q, \sigma \in \mathcal{I}, \text{ s.t. } w' = \delta(w, \sigma) = \epsilon.$$

Then, the function  $V(w, e) = \sqrt{e^\top P_w e}$  satisfies inequalities (5), (6) and (7) with observer gains

$$L_{(w, \sigma)} = P_{\delta(w, \sigma)}^{-1} Y_{(w, \sigma)}, \quad w \in Q, \sigma \in \mathcal{I}. \quad (27)$$

Moreover, under Assumption 1, LMIs (24), (25), (26) have a feasible solution for all  $\rho > \rho_e(\mathcal{A})$  and  $\lambda \in (0, 1)$ .

*Proof.* Considering  $L_{(w, \sigma)}$  given by (27), we get by Schur complement that (25), (26) are equivalent to (22) and (23). This, together with (24) implies that the function  $V$  satisfies inequalities (5), (6) and (7).

Conversely, under Assumption 1, for  $\rho > \rho_e(\mathcal{A})$  and  $\lambda \in (0, 1)$ , let the matrices  $P_w$  and  $L_{(w, \sigma)}$  be defined as in (16), (17) and (18). Then, let  $Y_{(w, \sigma)} = P_{\delta(w, \sigma)} L_{(w, \sigma)}$ . We get from Theorem 3 and using Schur complement that (24), (25), (26) are satisfied.  $\square$

From Theorem 3 and Proposition 3, it appears that under Assumption 1, it is always possible to design gains for the switched observer (4), either using the explicit construction or by solving LMIs. In this sense, the proposed observer structure given by (4) is universal. Let us remark that the observer gains need not be computed online as opposed to approaches such as [25] or based on Kalman filters [31], resulting in much simpler implementations in practice.

**Remark 5.** *Let us briefly discuss the effect of  $k$  on the complexity and the performances of the observer. The number of observer gains is proportional to the number of states of  $\mathcal{B}_k$ , which increases with  $k$ , see Remark 1. The gains are synthesized offline using the approaches described in Theorem 3 or in Proposition 3, which have linear and polynomial complexity with respect to the number of gains to be computed, respectively. As for online computations, though the required storage is proportional to the number of gains, the runtime complexity of the observer does not depend on  $k$ . As for the performance, we already mentioned at the end of Subsection 4.3, that for any given  $k$ , the convergence rate of the observer could be arbitrarily tuned using the design parameter  $\lambda$ . Nevertheless, as mentioned at the end of Section 3,  $\text{Lang}(\mathcal{B}_k)$  generally grows with  $k$ , so larger values of  $k$  may result in observers that converge for a larger class of switching signals.*

#### 4.5 Extension to switched systems with constrained switching

In this subsection, we briefly explain how our approach can be adapted when the switched system (1) is subject to switching constraints. Let us assume that switching signals  $\theta$  belong to a subset  $S$  of  $\mathcal{I}^\omega$ . We start by remarking that we can use the exact same approach as described in the previous subsections, since the conclusions of Proposition 1 hold for all  $\theta \in S \cap \text{Lang}(\mathcal{B}_k)$ . However, we can use the additional information provided by  $S$  to design observers with lower complexity or higher performance.

Firstly, due to switching constraints, not all reconstructible sequences may appear in the switching signals. We denote by  $S_k$  the set of subsequences of length up to  $k$  of sequences in  $S$ . Formally, let

$$S_k = \left\{ w \in \mathcal{I}^* \mid \begin{array}{l} |w| \leq k, \exists w_1 \in \mathcal{I}^*, w_2 \in \mathcal{I}^\omega, \\ w_1 w w_2 \in S \end{array} \right\}.$$

Then, it is sufficient to consider the set of reconstructible sequences given by  $\tilde{O}'^{[k]} = O'^{[k]} \cap S_k$ . Then, we can design an observer using the same approach as before, simply building the automaton  $\tilde{\mathcal{B}}_k$  from the set of reconstructible sequences  $\tilde{O}'^{[k]}$  instead of  $O'^{[k]}$ .

Additionally, if  $S$  is itself generated by a DBA  $\mathcal{B}$ , i.e.  $S = \text{Lang}(\mathcal{B})$ , then we can use the following approach. We start by computing a DBA  $\tilde{\mathcal{B}}_k$  such that  $\text{Lang}(\tilde{\mathcal{B}}_k) = \text{Lang}(\mathcal{B}_k) \cap S$ . This is always possible, following the approach described in [7, Lemma 4.59 and Theorem 4.56]. One can then adapt the approach described in Subsection 4.4 with the DBA  $\tilde{\mathcal{B}}_k$  instead of  $\mathcal{B}_k$ . In that case, LMIs similar to (25) and (26) need to hold for all transitions to non-accepting states and to accepting states, respectively.

## 5 Numerical example

Let us consider Example 1 again and let  $\mathcal{B}_k$  be the DBA constructed using Algorithm 1 for  $k = 3$  as shown in Figure 1. Then, we solve the LMIs (24), (25), (26) for  $\rho = 1.5, \lambda = 0.1$ , to synthesize the following observer gains<sup>1</sup>:

$$\begin{aligned} L_{(\epsilon,1)} &= (1 \ -0.26 \ 0.01)^\top, & L_{(\epsilon,2)} &= (0.82 \ 0.68 \ -0.15)^\top \\ L_{(1,1)} &= (1 \ 6.12 \ -4.1)^\top & L_{(1,2)} &= (0.76 \ 0.74 \ 0.02)^\top \\ L_{(2,1)} &= (1 \ -1.05 \ 0.07)^\top & L_{(2,2)} &= (0.97 \ 0.53 \ -0.43)^\top \\ L_{(12,1)} &= (1 \ -1.03 \ -0.11)^\top & L_{(12,2)} &= (0.14 \ 1.36 \ -1.29)^\top \\ L_{(21,1)} &= (1 \ 0 \ 0)^\top & L_{(21,2)} &= (1.38 \ 0.12 \ 0)^\top \\ L_{(22,1)} &= (1 \ -0.98 \ 0.98)^\top & L_{(22,2)} &= (0.88 \ 0.62 \ 0.04)^\top \end{aligned}$$

For the same values of  $\rho$  and  $\lambda$ , we also computed the observer gains using the explicit design presented in Subsection 4.3 (which is feasible thanks to Remark 4). For the chosen value of  $\rho$  and  $\lambda$ , we have  $\frac{\ln(\rho)}{-\ln(\lambda)} = 0.18$ .

We first consider a switching signal consisting of successions of a random reconstructible sequence in  $\mathcal{O}^{[k]}$  and of a random mode in  $\mathcal{I}$ . Therefore, we have by construction that the accepting rate of this switching signal satisfies  $\gamma^{\theta, \mathcal{B}_k} \geq 0.25 > 0.18$ . Then, from Proposition 1, it follows that the observer asymptotically converges. For both types of observers, this is confirmed by the simulations shown in Figure 2. This figure shows the switching signal  $\theta$ , the discrete state of the switched observer  $q$  given by the discrete dynamics in (4). The return instants (i.e. instants where  $q$  visits the accepting state  $\epsilon$ ) are indicated with red circles. We also show the evolution in logarithmic scale of the norm of the estimation error  $e$  for both type of gains. We can check that the observer converges faster with the explicit design than with the LMI-based design. Actually, the convergence rate of the observer designed using LMIs is quite close to the theoretical guarantees provided by Proposition 1, which essentially predicts a contraction of the estimation error by a factor  $\lambda$  between each return instants.

Now we consider the case when the system is driven by a switching signal with low accepting rate. For this purpose, let us consider a periodic switching signal constructed as follows: between instants  $t = 0$  and  $t = 30$  mode 2 is activated, and between  $t = 31$  and  $t = 40$  mode 1 is activated and the rest is constructed using periodicity. The discrete state of the observer  $q(t)$  visits the accepting state twice over each period. Therefore, this switching signal satisfies  $\gamma^{\theta, \mathcal{B}_k} = 0.05 < 0.18$ . Simulations for both types of observers are shown in Figure 3.

<sup>1</sup> The Matlab scripts of the numerical example and of the case study are available at the following repository: <https://github.com/georgesazanz/w-regular-observer>

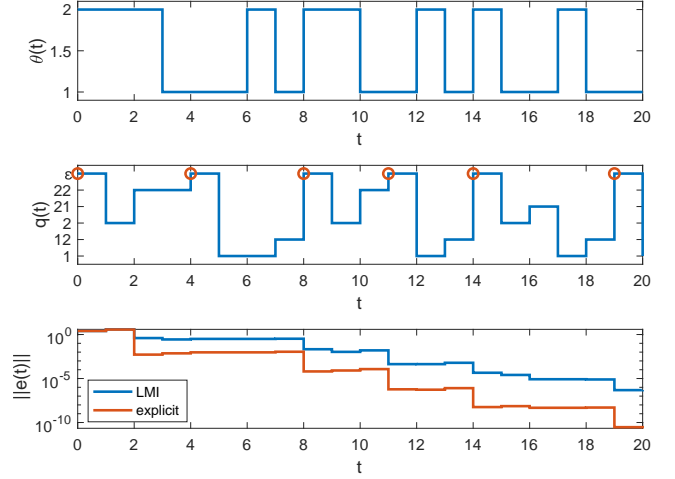


Fig. 2. Simulations for the case  $\gamma^{\theta, \mathcal{B}_k} > \frac{\ln(\rho)}{-\ln(\lambda)}$ : top - switching signal  $\theta$ ; center - discrete state of the switched observer  $q$ , red circles indicate return instants; bottom - norm of the estimation error  $e$  in logarithmic scale for both type of gains.

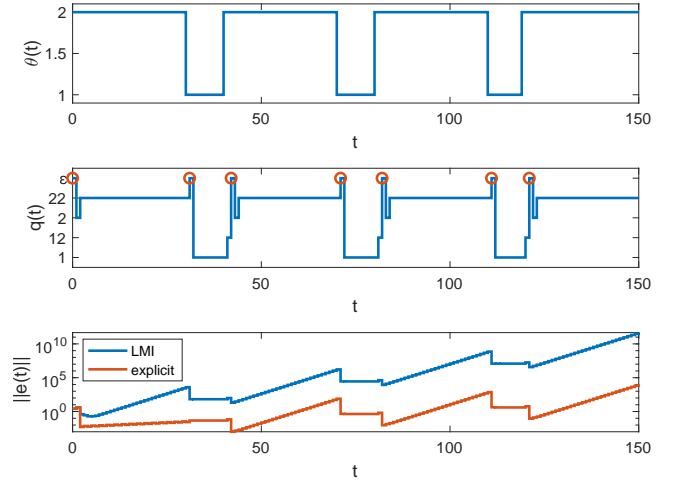


Fig. 3. Simulations for the case  $\gamma^{\theta, \mathcal{B}_k} < \frac{\ln(\rho)}{-\ln(\lambda)}$ : top - switching signal  $\theta$ ; center - discrete state of the switched observer  $q$ , red circles indicate return instants; bottom - norm of the estimation error  $e$  in logarithmic scale for both type of gains.

One can check on these simulations that the observer diverges for both types of gains. This shows, as expected, that the occurrence of reconstructible sequences must be frequent enough in order to make the observer converge.

Let us study the conservativeness of the obtained theoretical bound on the accepting rate guaranteeing convergence. For that purpose, we aim to estimate numerically the following quantity:

$$l^* = \inf \left\{ l \mid \gamma^{\theta, \mathcal{B}_k} \geq l \implies \lim_{t \rightarrow \infty} \|e(t)\| = 0 \right\}.$$

From Proposition 1, we can conclude that  $l^* \leq \frac{\ln(\rho)}{-\ln(\lambda)} = 0.18$ . To estimate  $l^*$  numerically, we run simulations

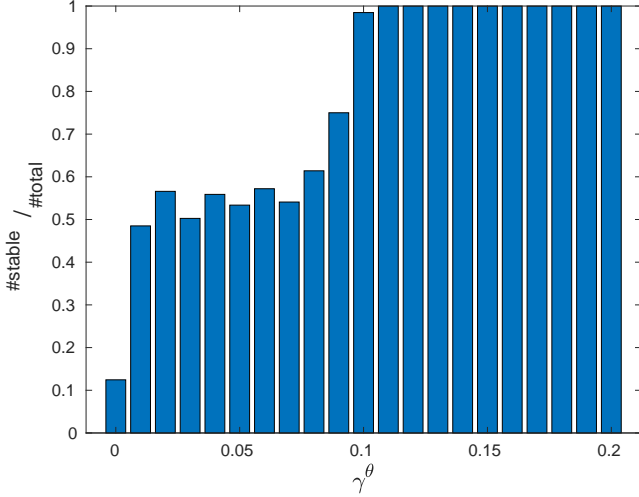


Fig. 4. Ratio of stable simulations for  $0 \leq \gamma^{\theta, \mathcal{B}_k} \leq 0.2$

for several switching signals generated by a two state Markov chain, corresponding to 10201 different values of the transition probabilities. For each signal we checked numerically the stability of the error dynamics for the observer obtained in Proposition 3 and we computed the corresponding accepting rate. The results are presented in Figure 4. The y-axis in Figure 4 represents the ratio of simulations where the estimation error goes to zero over the total number of simulations for an interval of  $\gamma^{\theta, \mathcal{B}_k}$ . We can conclude from these simulations that  $l^* \geq 0.10$ , note that this lower bound is likely to be conservative since considering only signals generated by Markov chains is a restriction with respect to the assumptions of Proposition 1. In any case, we can conclude that  $l^* \in [0.10, 0.18]$ .

## 6 Case study: the multicellular converter

Let us consider the circuit of the multicellular converter taken from [27, 28] and shown in Figure 5. This system is an example of switched dynamical systems due to several commutation cells in the circuit. The dynamics of this converter is described by the following equations:

$$\begin{aligned} \dot{I} &= -\frac{R}{L}I + \frac{E}{L}S_n - \sum_{j=1}^{n-1} \frac{V_{c_j}}{L}(S_{j+1} - S_j) \\ \dot{V}_{c_j} &= \frac{I}{c_j}(S_{j+1} - S_j), \quad j = 1, \dots, n-1 \end{aligned}$$

where  $c_j$  and  $V_{c_j}$  are the capacitance and the voltage of the  $j$ -th capacitor respectively,  $I$  is the current passing through the load consisting of the resistor  $R$  and the inductor  $L$ ,  $E$  is the voltage of the source.  $S_j \in \{0, 1\}$  is a binary signal corresponding to the  $j$ -th commutation cell. When  $S_j = 1$ , the upper switch of the  $j$ -th cell is “on” and the lower switch is “off” and vice versa in the other case.

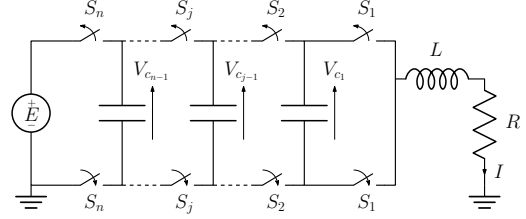


Fig. 5. Multicellular converter with an inductive load.

We consider the bijection between the set  $\{0, 1\}^n$  and the set  $\{1, \dots, 2^n\}$  which maps the binary vector  $(S_1, \dots, S_n)$  to  $\theta = 1 + \sum_{j=0}^{n-1} 2^j S_{j+1}$ . Writing  $x = (V_{c_1}, \dots, V_{c_{n-1}}, I)^\top \in \mathbb{R}^n$ ,  $u = E$ , and assuming only  $y = I$  is measured, the system dynamics can be described by a continuous time switched system of the form:

$$\begin{aligned} \dot{x}(t) &= A_{\theta(t)}x(t) + B_{\theta(t)}u(t), \\ y(t) &= C_{\theta(t)}x(t) \end{aligned} \quad (28)$$

For numerical experiments, we consider  $n = 3$  commutation cells, with the following parameter values  $E = 1500$  V,  $c_1 = c_2 = 40$   $\mu$ F,  $R = 10$   $\Omega$ ,  $L = 0.5$  mH. For such numerical values, the state matrices are given by

$$\begin{aligned} A_1 &= 10^3 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -20 \end{pmatrix}, & A_2 &= 10^3 \times \begin{pmatrix} 0 & 0 & -25 \\ 0 & 0 & 0 \\ 2 & 0 & -20 \end{pmatrix} \\ A_3 &= 10^3 \times \begin{pmatrix} 0 & 0 & 25 \\ 0 & 0 & -25 \\ -2 & 2 & -20 \end{pmatrix}, & A_4 &= 10^3 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -25 \\ 0 & 2 & -20 \end{pmatrix} \\ A_5 &= 10^3 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 25 \\ 0 & -2 & -20 \end{pmatrix}, & A_6 &= 10^3 \times \begin{pmatrix} 0 & 0 & -25 \\ 0 & 0 & 25 \\ 2 & -2 & -20 \end{pmatrix} \\ A_7 &= 10^3 \times \begin{pmatrix} 0 & 0 & 25 \\ 0 & 0 & 0 \\ -2 & 0 & -20 \end{pmatrix}, & A_8 &= 10^3 \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -20 \end{pmatrix} \end{aligned}$$

while the output matrices are  $C_i = (0 \ 0 \ 1)$ , for all  $i = 1, \dots, 8$ . In the following, we consider a sampled version of (28) with a sampling period  $T = 0.3$  ms.

Let us remark that the dynamics in all modes are unobservable. Hence, it is not possible to build an observer that is convergent for arbitrary switching signals. Additionally, the system is subject to the following constraint on the switching signal [27, 28]: only one commutation cell can be switched at a time. This constraint translates to some adjacency relations between modes described by the graph in Figure 6. When a mode change occurs in the switching signal  $\theta$ , the new mode must be adjacent to the previous one. Our goal is to design an observer for this switched system with constrained switching. The approach presented in [27, 28] is similar to that of [25] and requires computing the observer gains online.

As described in Subsection 4.5, we compute the set  $\tilde{\mathcal{O}}^{[k]}$  of reconstructible sequences satisfying the switching constraints, which, for  $k = 3$ , contains 48 sequences. We also checked numerically that the cardinality of this set satisfies  $|\tilde{\mathcal{O}}^{[k]}| = 48(k-2)$  for  $k = 3, 4, 5, 6, 7$ . This suggests that this set increases linearly with  $k$ . Then, we use Algorithm 1 to build the corresponding Büchi automaton

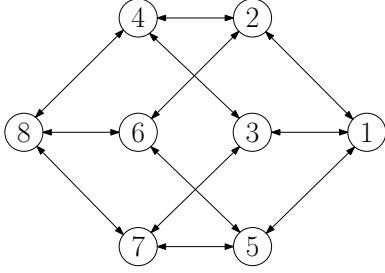


Fig. 6. Adjacency relations between modes describing the constraints on the switching signal  $\theta$ . When a mode change occurs in the switching signal  $\theta$ , the new mode must be adjacent to the previous one.

$\mathcal{B}_3$ . As (28) possesses a nonstrict common quadratic Lyapunov function (the energy of the system) and thanks to Remark 4, we can apply the explicit design of observer gains presented in Subsection 4.3 for  $\rho = 1$  and  $\lambda = 0.1$ . Let us remark that switching sequences satisfying adjacency constraints can be generated using a DBA  $\mathcal{B}$  (see e.g. [29]). Then, as explained in Subsection 4.5, we can further compute the DBA  $\tilde{\mathcal{B}}_3$  shown in Figure 7 and such that  $\text{Lang}(\tilde{\mathcal{B}}_3) = \text{Lang}(\mathcal{B}_3) \cap \text{Lang}(\mathcal{B})$ . Then, we solve the corresponding LMIs (24), (25), (26) with  $\rho = 1$  and  $\lambda = 0.1$  to design observer gains.

Numerical simulations are shown in Figure 8. The figure shows the switching signal  $\theta$ ; the return instants (i.e. instants where  $q$  visits an accepting state) are indicated with red circles. We also show the evolution in logarithmic scale of the norm of the estimation error  $e$  for both designs. We can see again that the observer based on

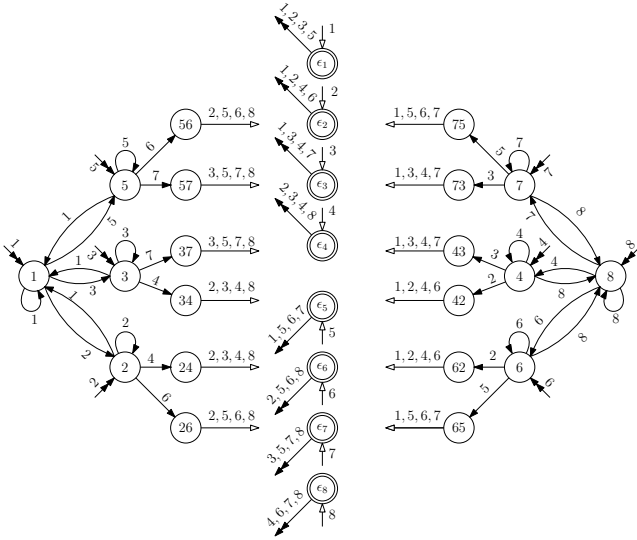


Fig. 7. DBA  $\tilde{\mathcal{B}}_3$  that generates switching sequences satisfying the adjacency constraints shown in Figure 6 and with an infinite number of reconstructible subsequences in  $\mathcal{O}^{[k]}$ . Arrows with double head and labeled by  $\sigma$  lead to state  $\sigma$ , arrows with empty head and labeled by  $\sigma$  lead to accepting state  $\epsilon_\sigma$ .

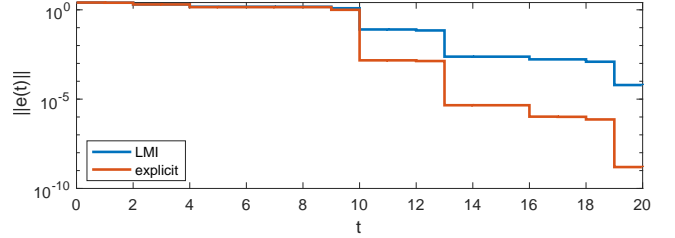
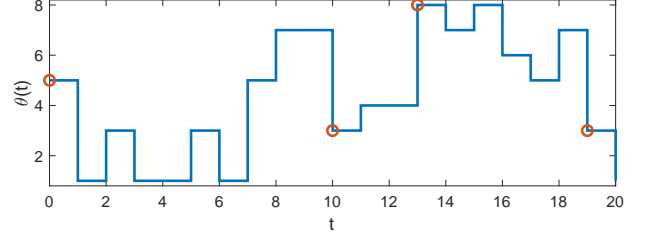


Fig. 8. Simulations for the multicellular converter: top - switching signal  $\theta$ , red circles indicate return instants; bottom - norm of the estimation error  $e$  in logarithmic scale for both type of gains.

the explicit design converges faster than the one based on LMIs whose convergence rate is more consistent with the theoretical guarantees provided by Proposition 1.

We end this section by mentioning that in [27, 28], they consider a sampling period  $T = 1$  ms. In that case, the state matrices of the sampled system are numerically close to singularity and thus Assumption 1 is not robustly satisfied. This results in the explicit design of Subsection 4.3 to be numerically ill-conditioned. In practice, due to numerical errors, the ensuing observer is divergent. However, it is important to remark that even in that case, the observer based on the LMI design still works smoothly.

## 7 Conclusion

In this paper, we presented an approach to design observers for discrete-time switched linear systems. Our approach combines automata theory, Lyapunov techniques and LMI-based design to synthesize switched observers. The most important feature of the proposed observer structure is that it is universal as we show that it is always possible to design observer gains that make this observer convergent. We have shown the effectiveness of our approach in a case study on multicellular converters.

In future work, we aim at investigating the duality between observability and controllability to design, using a similar approach, switched controllers to stabilize discrete-time switched linear systems.

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