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# A numerical scheme accurate at low Mach number based on space velocity enrichment

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## Model

Compressible barotropic Euler equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_{\mathbf{x}} p = 0 \end{cases} \quad \text{where } \rho \mapsto p(\rho) \text{ is a convex function}$$

Properties:

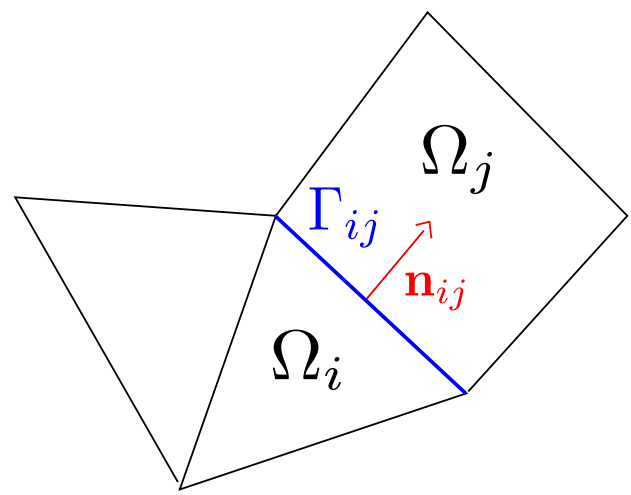
- System of conservative laws :  $\partial_t \mathbf{W} + \operatorname{div}_{\mathbf{x}} \mathbf{f}(\mathbf{W}) = 0$ .
- Hyperbolic system of conservation laws with 3 different eigenvalues  $\lambda_1 = \mathbf{u} \cdot \mathbf{n} - c$  (acoustic),  $\lambda_2 = \mathbf{u} \cdot \mathbf{n}$  (material),  $\lambda_3 = \mathbf{u} \cdot \mathbf{n} + c$  (acoustic) where  $c = \sqrt{p'(\rho)}$  is the sound speed.

## Motivation

Numerical scheme able to compute accurately

- low Mach number flows ( $M = \|\mathbf{u}\|/c \ll 1$ ),
- acoustic waves.

## Collocated finite volume scheme



$$\int_{\Omega_i} \partial_t \mathbf{W} + \operatorname{div}_{\mathbf{x}} \mathbf{f}(\mathbf{W}) = 0 \Rightarrow |\Omega_i| \frac{d}{dt} \mathbf{W}_i + \sum_{\Gamma_{ij} \subset \partial \Omega_i} |\Gamma_{ij}| \underbrace{F(\mathbf{W}_i, \mathbf{W}_j, \mathbf{n}_{ij})}_{\text{numerical flux}} = 0.$$

Numerical flux: **Rusanov** or **Roe**

$$F(\mathbf{W}_i, \mathbf{W}_j, \mathbf{n}_{ij}) = \frac{\mathbf{f}(\mathbf{W}_i) + \mathbf{f}(\mathbf{W}_j)}{2} \cdot \mathbf{n}_{ij} - \star \Delta \rho - \star \Delta(\rho \mathbf{u}).$$

## Two time scales asymptotic analysis

Scaling variables:  $x_0, \rho_0, u_0$

$$\tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{p} = \frac{p}{p(\rho_0)}, \quad \tilde{\mathbf{u}} = \frac{\mathbf{u}}{u_0}, \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{x_0}, \quad \tilde{t} = \frac{t}{x_0/u_0}, \quad \tau = \frac{t}{x_0/c_0} = \frac{\tilde{t}}{u_0/c_0} = \frac{\tilde{t}}{M}.$$

Dimensionless system:

$$\begin{cases} \partial_{\tilde{t}} \tilde{\rho} + \frac{1}{M} \partial_{\tilde{t}} \tilde{\rho} + \operatorname{div}_{\tilde{\mathbf{x}}}(\tilde{\rho} \tilde{\mathbf{u}}) = 0, \\ \partial_{\tilde{t}}(\tilde{\rho} \tilde{\mathbf{u}}) + \frac{1}{M} \partial_{\tilde{t}}(\tilde{\rho} \tilde{\mathbf{u}}) + \operatorname{div}_{\tilde{\mathbf{x}}}(\tilde{\rho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \frac{1}{\gamma M^2} \nabla_{\tilde{\mathbf{x}}} \tilde{p} = 0. \end{cases}$$

Develop each variable as power of  $M$ :

$$\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}, \tau, M) = \tilde{\psi}^{(0)}(\tilde{\mathbf{x}}, \tilde{t}, \tau) + M \tilde{\psi}^{(1)}(\tilde{\mathbf{x}}, \tilde{t}, \tau) + M^2 \tilde{\psi}^{(2)}(\tilde{\mathbf{x}}, \tilde{t}, \tau) + \mathcal{O}(M^3).$$

Continuous case:  $(\tilde{p}^{(1)}/\gamma, (\tilde{\rho} \tilde{\mathbf{u}})^{(0)}) = (\bar{p}, \bar{\mathbf{u}}) = \mathbf{U}$  satisfies a first order wave system

$$\begin{cases} \partial_{\tau} \bar{p} + \bar{c}_0^2 \operatorname{div}_{\tilde{\mathbf{x}}} \bar{\mathbf{u}} = 0, \\ \partial_{\tau} \bar{\mathbf{u}} + \nabla_{\tilde{\mathbf{x}}} \bar{p} = 0, \end{cases} \quad \text{where } \tau = \tilde{t}/M \xrightarrow{M \rightarrow 0} +\infty.$$

Discrete case: **Rusanov** and **Roe** schemes are asymptotically consistent with a wave system discretization

$$\partial_{\tau} \mathbf{U}_i + \frac{1}{|\tilde{\Omega}_i|} \sum_{\Gamma_{ij} \subset \partial \Omega_i} |\tilde{\Gamma}_{ij}| \left[ A(\mathbf{n}_{ij}) \frac{\mathbf{U}_i + \mathbf{U}_j}{2} + D(\mathbf{n}_{ij})(\mathbf{U}_i - \mathbf{U}_j) \right] = 0,$$

$$\text{with } A(\mathbf{n}) = \begin{bmatrix} 0 & \bar{c}_0^2 \mathbf{n}^T \\ 1 & 0 \end{bmatrix}, \quad D^{\text{Rusanov/Godunov}}(\mathbf{n}) = \frac{\bar{c}_0}{2} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{I}/\mathbf{nn}^T \end{bmatrix}.$$

## Low Mach number limit/long time wave [1, 2]

### Euler system

Well-prepared initial/boundary condition

$$\begin{cases} 0 = \tilde{p}_0^{(0)} = \tilde{p}_0^{(1)} = \tilde{p}_b^{(0)} = \tilde{p}_b^{(1)}, \\ 0 = \operatorname{div}_{\tilde{\mathbf{x}}}(\tilde{\rho} \tilde{\mathbf{u}})_0^{(0)} = \int_{\partial \Omega} (\tilde{\rho} \tilde{\mathbf{u}})_b^{(0)} \cdot \mathbf{n}. \end{cases}$$

Low Mach number limit

$$\begin{cases} \tilde{p} = \bar{p}^{\infty} + \mathcal{O}(M^2), \\ \tilde{\rho} \tilde{\mathbf{u}} = \bar{\mathbf{u}}^{\infty} + \mathcal{O}(M). \end{cases}$$

### Wave system

Initial/boundary condition

$$\begin{cases} 0 = \bar{p}_0 = \bar{p}_b, \\ 0 = \operatorname{div}_{\tilde{\mathbf{x}}} \bar{\mathbf{u}}_0 = \int_{\partial \Omega} \bar{\mathbf{u}}_b \cdot \mathbf{n}. \end{cases}$$

Long time limit

$$\begin{cases} \bar{p}^{\infty} = 0, \\ \operatorname{div}_{\tilde{\mathbf{x}}} \bar{\mathbf{u}}^{\infty} = 0. \end{cases}$$

## Wave structure

$$\partial_{\tau} (\operatorname{curl}_{\tilde{\mathbf{x}}} \bar{\mathbf{u}}) = 0$$

Hodge-Helmholtz decomposition:  $\bar{\mathbf{u}} = \bar{\mathbf{u}}^{\varphi} + \bar{\mathbf{u}}^{\Psi}$  where  $\begin{cases} \operatorname{curl}_{\tilde{\mathbf{x}}} \bar{\mathbf{u}}^{\varphi} = 0, \\ \operatorname{div}_{\tilde{\mathbf{x}}} \bar{\mathbf{u}}^{\Psi} = 0. \end{cases}$

Preserving a curl  $\leftrightarrow$  preserving  $\bar{\mathbf{u}}^{\Psi}$ .

## Triangular mesh [3]

Discrete Hodge-Helmholtz decomposition:  $d\mathbb{P}_0(\Omega) \ni \bar{\mathbf{u}}_h = \bar{\mathbf{u}}_h^{\varphi} + \bar{\mathbf{u}}_h^{\Psi}$  where

- $\bar{\mathbf{u}}_h^{\varphi} = \nabla_{\tilde{\mathbf{x}}} \varphi$  with  $\varphi \in \mathbb{C}\mathbb{R}(\Omega)$ ,
- $\bar{\mathbf{u}}_h^{\Psi} = \nabla_{\tilde{\mathbf{x}}}^{\perp} \Psi$  with  $\Psi \in \mathbb{P}_1(\Omega)$ .

Structure preserved on wave system with **Godunov**' numerical flux:

- **Periodic domain**:  $(\nabla_{\tilde{\mathbf{x}}}^{\perp})^* \bar{\mathbf{u}}_h$  is preserved over time,
- **Bounded domain**:  $(\bar{p}_b, \bar{\mathbf{u}}_h^{\Psi}(\bar{\mathbf{u}}_0))$  is preserved over time and the relative entropy is decreasing.

## Quadrangular mesh: velocity enrichment

$$\nabla_{\tilde{\mathbf{x}}}^{\perp} \mathbb{Q}_1, \nabla_{\tilde{\mathbf{x}}} \mathbb{R}a\mathbb{T}u \subset d\mathbb{S}_0 := \operatorname{span} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x - 1/2 \\ -y + 1/2 \end{pmatrix} \right)$$

Discrete Hodge-Helmholtz decomposition:  $d\mathbb{S}_0(\Omega) \ni \bar{\mathbf{u}}_h = \bar{\mathbf{u}}_h^{\varphi} + \bar{\mathbf{u}}_h^{\Psi}$  where

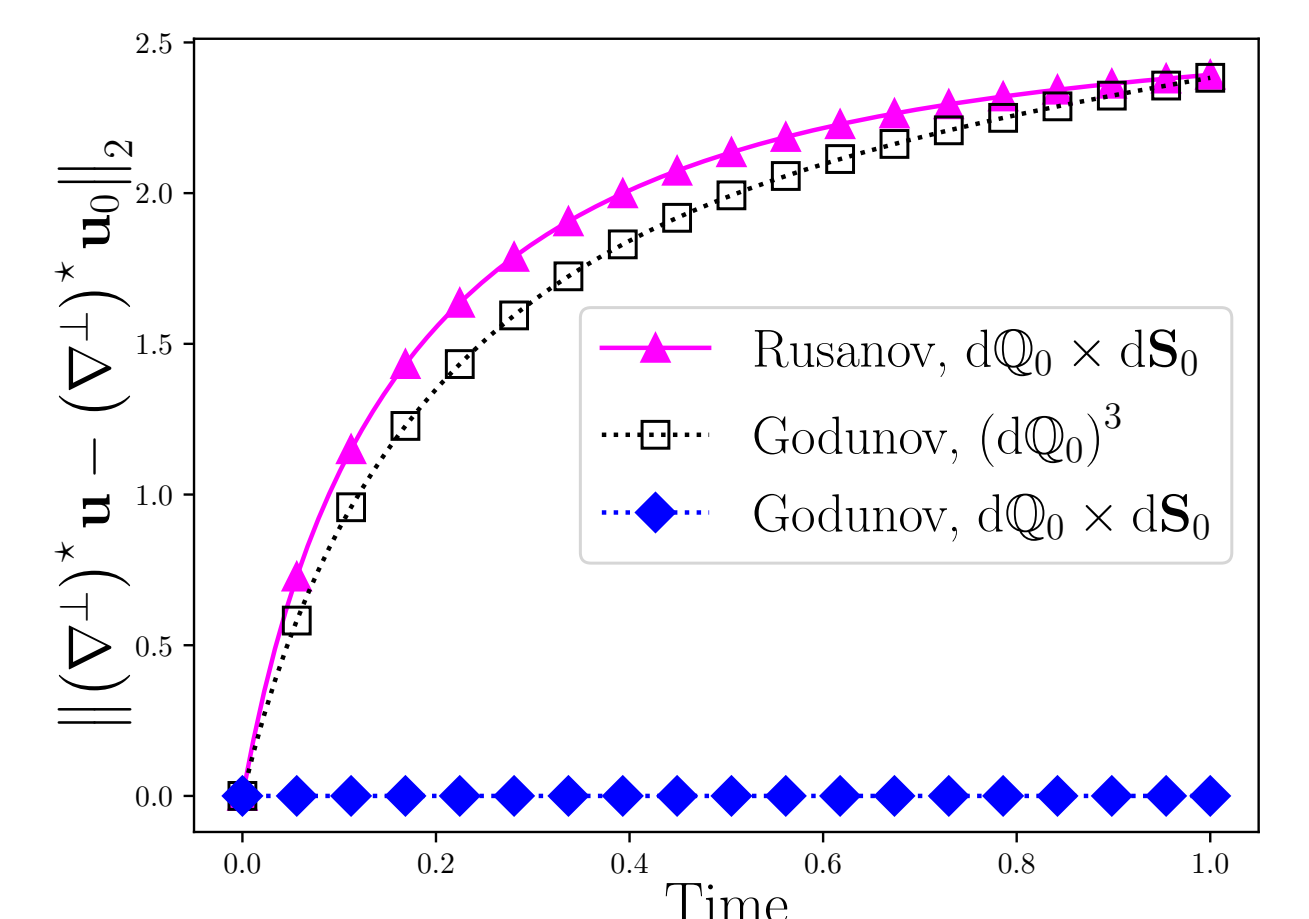
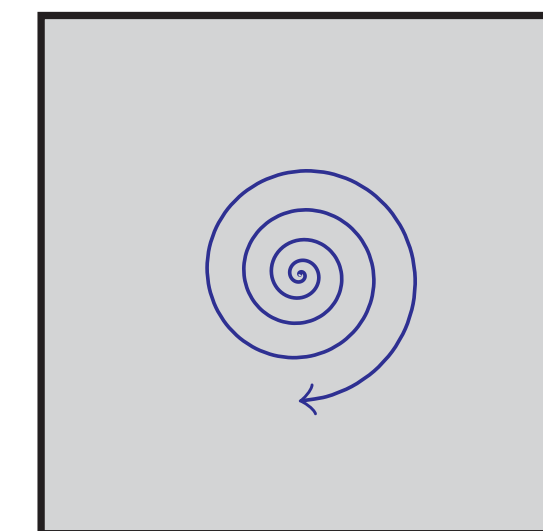
- $\bar{\mathbf{u}}_h^{\varphi} = \nabla_{\tilde{\mathbf{x}}} \varphi$  with  $\varphi \in \mathbb{R}a\mathbb{T}u(\Omega)$ ,
- $\bar{\mathbf{u}}_h^{\Psi} = \nabla_{\tilde{\mathbf{x}}}^{\perp} \Psi$  with  $\Psi \in \mathbb{Q}_1(\Omega)$ .

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## Numerical results on quadrangular mesh

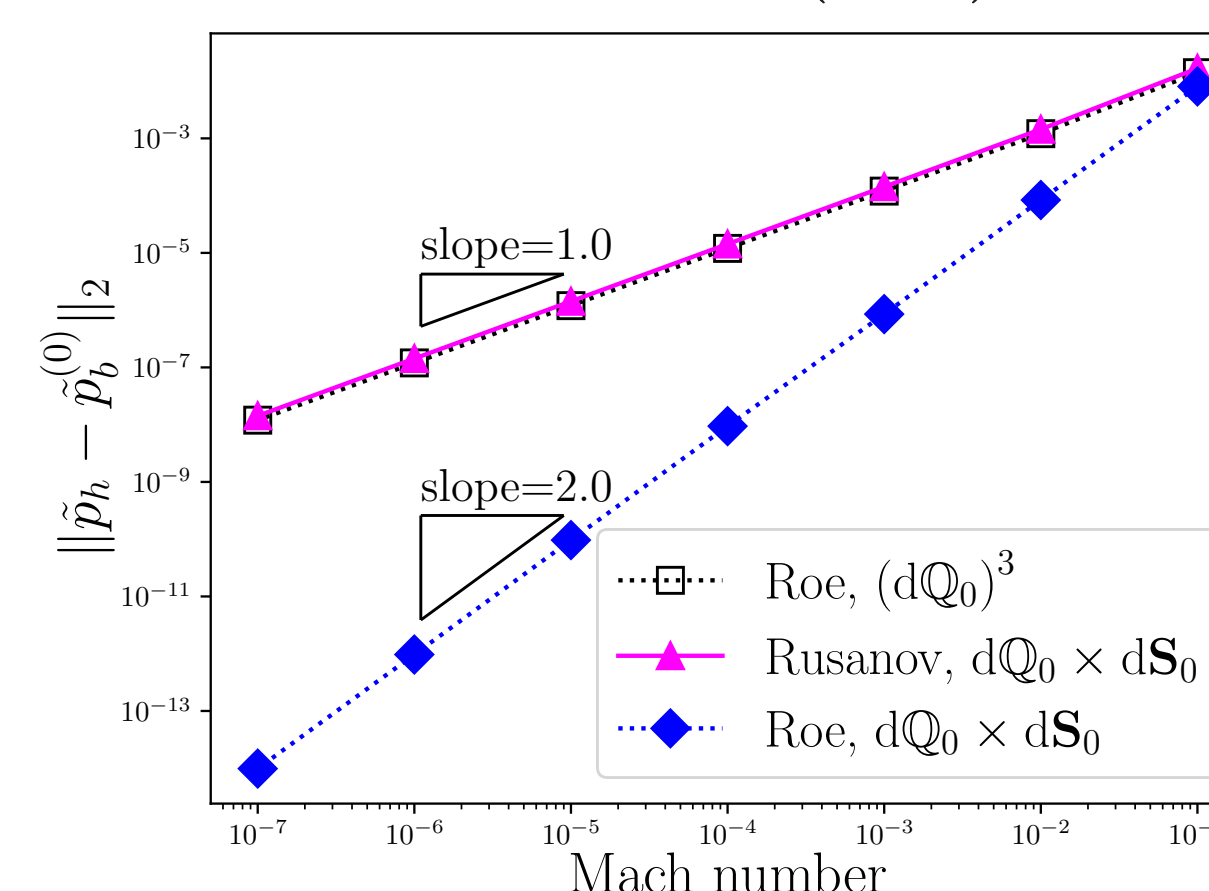
Wave system: preservation of the adjoint curl



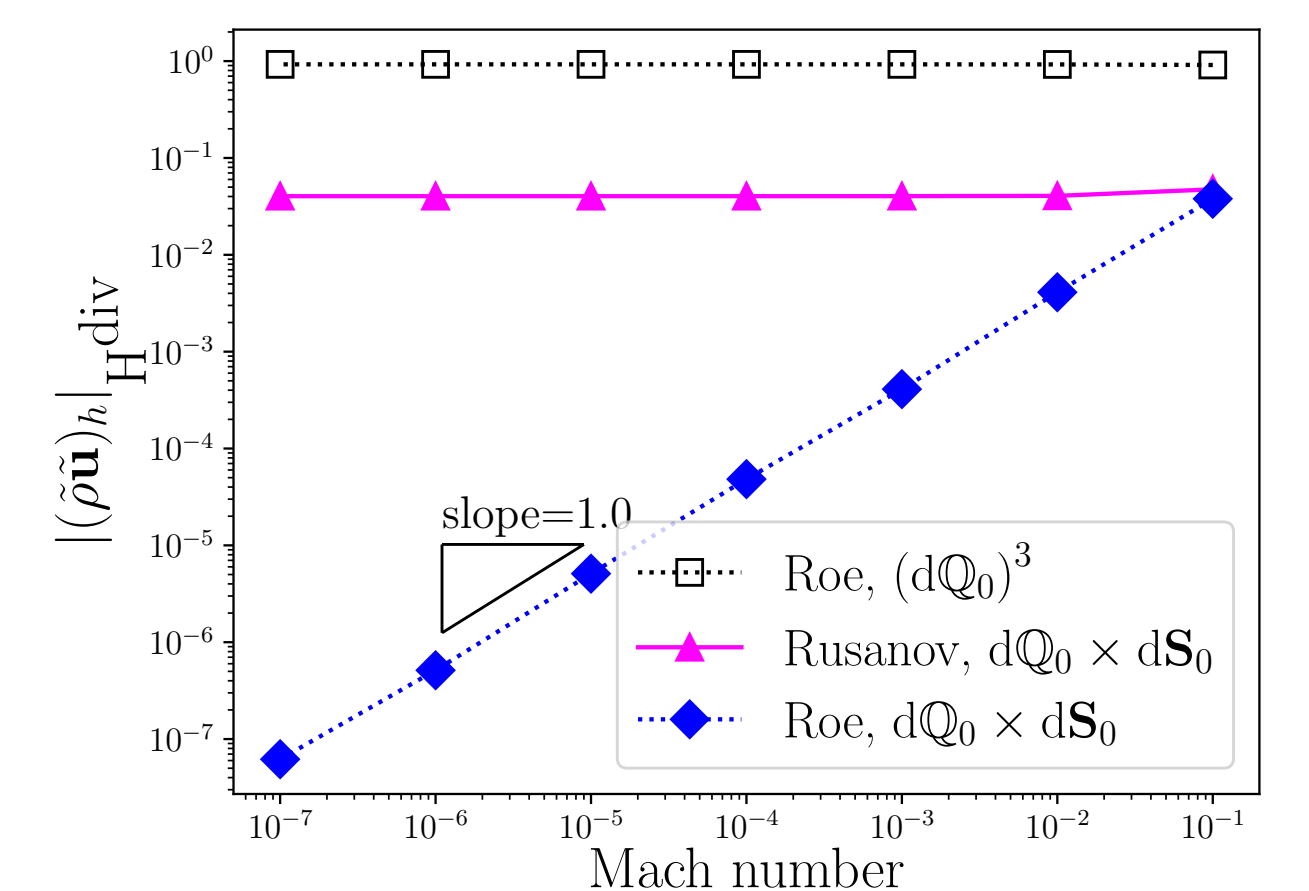
Euler cylinder scattering:

$$\begin{cases} \rho_b = 1 \\ \mathbf{u}_b^x = M \times c_b \\ \mathbf{u}_b^y = 0 \end{cases} \Rightarrow \text{Cylinder}$$

$$\tilde{p} - \tilde{p}_b^{(0)} \stackrel{?}{=} \mathcal{O}(M^2)$$



$$\operatorname{div}_{\tilde{\mathbf{x}}}(\tilde{\rho} \tilde{\mathbf{u}}) \stackrel{?}{=} \mathcal{O}(M)$$



## Conclusion

- Low Mach accuracy is equivalent to curl preserving for wave system.
- Triangular mesh+**Godunov** flux is curl preserving.
- Quadrangular+**Godunov**+change velocity basis is curl preserving.

## References

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- [3] S. Dellacherie, P. Omnes, and F. Rieper. The influence of cell geometry on the Godunov scheme applied to the linear wave equation. *Journal of Computational Physics*, 229(14):5315–5338, 2010.