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ON THE (NON) EXISTENCE OF VISCOSITY SOLUTIONS OF MULTI–TIME HAMILTON–JACOBI EQUATIONS

A. DAVINI, M. ZAVIDOVIQUE

Abstract. We prove that the multi–time Hamilton–Jacobi equation in general cannot be solved in the viscosity sense, in the non-convex setting, even when the Hamiltonians are in involution.

Introduction

There are two important Hamilton–Jacobi equations associated with a continuous function H defined on the cotangent bundle of a smooth Riemannian manifold M: the stationary equation, that is a nonlinear PDE of the kind

$$H(x, d_x u) = c \qquad \text{in } M,$$

where c is a real constant and the unknown u is a real function defined on M; and the evolutionary equation, which is a time-dependent equation of the form

$$\begin{cases} \partial_t u + H(x, d_x u) = 0 & \text{in } (0, +\infty) \times M \\ u(0, x) = u_0(x), & \text{in } M, \end{cases} \tag{1}$$

where $u_0 : M \to \mathbb{R}$ is an initial datum, and the unknown u is a real function of variables $(t, x) \in [0, +\infty) \times M$.

The first problem that naturally arises is that of finding a good notion of solution. It is in fact well known that such equations do not usually admit classical solutions. For instance, the method of characteristics provides, for smooth initial data and under general assumptions on H, classical solutions for (1) only for small times, until shocks between characteristics occur.

A first idea is to look for Lipschitz functions which solve the equation almost everywhere. However, such a notion of solution is inadequate since it lacks of good uniqueness and stability properties, see for instance [3]. That is why two better notions of weak solutions were introduced.

The first one, introduced by Crandall and Lions in 1983 [11], is that of *viscosity solution* and it applies to a wide range of first and second order nonlinear PDE. It is in some way reminiscent of distributions, in the sense that it uses test functions to drop derivatives, and permits to name solutions functions that are just continuous, regardless of their regularity. This notion has revealed to be extremely powerful and flexible, and it has been generalized since then in many different directions.

The second one, more geometric in nature, is termed *variational solution* and was introduced by Chaperon, Sikorav and Viterbo (see [10] or [9] and references therein) and is more specific of evolutionary equations of the kind of (1). It follows the idea

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of the method of characteristics, taking the graph of the differential of the initial condition and pushing it with the Hamilton flow. When the image is no longer a graph, a natural way to cut the obtained Lagrangian manifold, known as the graph selector, allows to reconstruct a function which is the desired variational solution.

Note that, as can be expected, both those notions often yield almost everywhere solutions (meaning Lipschitz functions satisfying the Hamilton–Jacobi equation almost everywhere). In the case of a Hamiltonian $H(x,p)$ which is convex in the momentum variable p, it was proved by Zhukovskaya (see [24] and also [6, 21]) that both notions provide the same solution. However, they may differ in the general case. Indeed, variational solutions do not necessarily verify the semi–group (or Markov) property. There is a simple way (proposed by Chaperon) to force variational solutions to verify this property and Wei [21] recently established that the obtained functions are the viscosity solutions (see also the work of Roos [20] on this matter).

In this paper we will focus on the solvability of the following system of uncoupled Hamilton–Jacobi equations:

$$\begin{cases} \partial_t u + H(x, d_x u) = 0 & \text{in } (0,+\infty)\times(0,+\infty)\times M \\ \partial_s u + G(x, d_x u) = 0 & \text{in } (0,+\infty)\times(0,+\infty)\times M \\ u(0,0,x) = u_0(x) & \text{on } M. \end{cases}$$

Such a system is known as *multi–time Hamilton–Jacobi equation* and has its roots in economics. There is now a consequent literature concerning this problem, where either one or the other notion of solution has been considered. The issue about existence and uniqueness of viscosity solutions of the multi–time equation was first addressed by Lions and Rochet [18] in the case of convex and coercive Hamiltonians on $\mathbb{R}^n \times \mathbb{R}^n$ only depending on the momentum variable. This work was subsequently generalized in [2] to Hamiltonians depending on both variables of $\mathbb{R}^n \times \mathbb{R}^n$, but still convex and coercive in the momentum. As a counterpart, a commutation hypothesis is necessary (which is always verified in the case of Lions and Rochet), namely the vanishing of the Poisson bracket:

$$\{H, G\} := \partial_p H \cdot \partial_x G - \partial_p G \cdot \partial_x H = 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^n.$$

Those results were then extended to less regular settings in [19].

In the framework of symplectic geometry and variational solutions, Cardin and Viterbo [9] extended these results to a larger class of Hamiltonians that do not satisfy any convexity assumption in the momentum variable. The authors proved that the null Poisson bracket condition entails existence and uniqueness of the variational solutions of the multi–time equation.

Knowing the results of Cardin and Viterbo, and the close link between variational and viscosity solutions, it seems natural to ask the following question: if two regular Hamiltonians (non necessarily convex) H and G satisfy the null Poisson bracket condition, can we solve the multi–time Hamilton–Jacobi in the viscosity sense?

The purpose of this note is to show that the answer to this question is, in general, negative.

The article is organized as follows. In the first section we recall the definitions of viscosity and variational solutions and we present the link between the two through Wei's result. In the second section we give our main result. More precisely, we show that, when the convexity condition on one Hamiltonian is dropped, and replaced

by a concavity condition, the multi–time equation admits a viscosity solution only for very special choices of the initial datum u_0 (see Theorem 2.1 for the precise statement). Given a pair of such Hamiltonians H and G, the corresponding set of admissible initial data is typically empty. An example of this fact is provided in Corollary 2.4.

In the appendix we give the proof of a folklore theorem on existence and uniqueness of viscosity solutions to the evolutionary Hamilton–Jacobi equation. The result is proved under a set of conditions either more general or different than the ones usually assumed in the literature. The techniques and ideas employed are by no means new, however, since they are disseminated in the literature and the adaptation to the case at issue requires some technical work, we felt it might be useful to gather them in a single reference and provide a neat proof of this fact.

1. Preliminaries

Throughout the paper, we will call *Hamiltonian* a continuous function defined on the cotangent bundle of a smooth and compact Riemannian manifold M. We will say that H is *coercive* if

$$\lim_{|p|\to\infty} H(x,p) = +\infty \qquad \text{for every } x \in M. \tag{C}$$

For convenience, we will assume that M is the flat n-dimensional torus $\mathbb{T}^n$, however most results keep holding in the more general setting since they are essentially local in nature.

In this section, we focus on the case of a single Hamilton–Jacobi equation, i.e.

$$\partial_t u + H(x, D_x u) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{T}^n, \tag{HJ}$$

where we assume either H or $-H$ to be a coercive Hamiltonian. When H is at least C^1 with Lipschitz derivatives ($C^{1,1}$ in short), it induces a Lipschitz vectorfield, called *Hamiltonian vectorfield*, defined as follows:

$$X_H(x,p) = \big(\partial_p H(x,p), -\partial_x H(x,p)\big) \qquad \text{for all } (x,p)\in\mathbb{T}^n\times\mathbb{R}^n.$$

When well defined (which is the case for coercive Hamiltonians), the flow of this vectorfield will be denoted by (φ^t).

1.1. Viscosity solutions.

Let u be a continuous function in $(0,+\infty)\times\mathbb{T}^n$.

The function u is called a *viscosity subsolution* of (HJ) if, for every function $\varphi \in \mathrm{C}^1((0,+\infty)\times\mathbb{T}^n)$ such that $u-\varphi$ attains a local maximum at $(t,x)\in(0,+\infty)\times\mathbb{T}^n$, we have

$$\partial_t\varphi(x,t) + H\big(x, D_x\varphi(x,t)\big) \leqslant 0. \tag{1.2}$$

Any such test function φ will be called *supertangent* to u at (t,x).

The function u is called a *viscosity supersolution* of (HJ) if, for every function $\varphi \in \mathrm{C}^1((0,+\infty)\times\mathbb{T}^n)$ such that $u-\varphi$ attains a local minimum at $(t,x)\in(0,+\infty)\times\mathbb{T}^n$, we have

$$\partial_t\varphi(x,t) + H\big(x, D_x\varphi(x,t)\big) \geqslant 0. \tag{1.3}$$

Any such test function φ will be called *subtangent* to u at (t,x). Last, u is called a *viscosity solution* of (HJ) if it is both a sub and a supersolution.

It is well known, see for instance [3], that the notions of viscosity sub and supersolutions are local, in the sense that the test function φ needs to be defined only in a neighborhood of the point (t,x). Moreover, up to adding to φ a quadratic term, such

a point can be always assumed to be either a strict maximum or a strict minimum point of $u - \varphi$. In this instance, we will say that φ is a *strict supertangent* (resp., *strict subtangent*) to u at (t, x).

One of the main features of the notion of viscosity solution is that it is extremely stable. Indeed, in rough terms, if $\tilde{u}$ is a small perturbation of u and the test function φ is, for instance, a strict subtangent to u at (t, x), then it will also be a subtangent to $\tilde{u}$ at a point $(\tilde{t}, \tilde{x})$ close to (t, x). It follows that a continuous function obtained as local uniform limit of a sequence of viscosity solutions of (HJ) is still a solution of the same equation. Furthermore, it has the advantage to be a reasonable notion, in the sense that it yields existence and uniqueness results under mild conditions, as it is illustrated by the following now classical result

Theorem 1.1. *Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ such that either H or $-H$ is a coercive Hamiltonian. If $u_0 \in \mathrm{C}(\mathbb{T}^n)$, there exists a unique uniformly continuous function u in $[0, +\infty) \times \mathbb{R}^n$ satisfying $u(0, \cdot) = u_0$ and solving* (HJ) *in the viscosity sense. Furthermore, if u_0 is Lipschitz-continuous in $\mathbb{T}^n$, then u is Lipschitz continuous in $[0, +\infty) \times \mathbb{R}^n$ and its Lipschitz constant only depends on $\|Du_0\|_\infty$ and H.*

The proof of this result can be always reduced to the case when the Hamiltonian is coercive thanks to the following simple remark:

Proposition 1.2. *Let u be a continuous function in $(0, +\infty) \times \mathbb{T}^n$. Then u is a subsolution (respectively, a supersolution) of* (HJ) *if and only if $-u$ is a supersolution (resp., a subsolution) of*

$$\partial_t v - H(x, -D_x v) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{T}^n.$$

Theorem 1.1 is known by PDE specialists, however all the the proofs we have found in literature are given under sets of assumptions somewhat different from the ones herein considered. For completeness, we have provided in the Appendix a proof of Theorem 1.1 in a slightly more general form, namely we deal with the case when $\mathbb{T}^n$ is replaced by $\mathbb{R}^n$, see Theorem A.1. Theorem 1.1 follows from this by taking the lift of H and of the initial datum u_0 to the universal cover of the torus.

In the rest of the paper, we will always assume the initial datum u_0 to be Lipschitz in $\mathbb{T}^n$ and we will denote by $S^t_H u_0(x)$, or simply by $S^t u_0(x)$ when no ambiguity is possible, the unique Lipschitz solution $u(t, x)$ to the evolutionary Hamilton–Jacobi equation (HJ) with Hamiltonian H and taking the initial datum u_0 at $t = 0$.

We end this section by recalling some facts about Tonelli Hamiltonians. We remind that a Hamiltonian H is termed *Tonelli* if it is of class C^2, satisfies the following superlinear growth in p:

$$\frac{H(x, p)}{|p|} \underset{|p| \to \infty}{\longrightarrow} +\infty,$$

and is strictly convex, in the sense that $\partial^2_{pp} H$ is everywhere positive definite as a quadratic form. We will be interested in some regularity properties of solutions to the Hamilton–Jacobi equation. Recall that a function $f : \mathbb{T}^n \to \mathbb{R}$ is said to be locally semi–concave (resp. semi–convex) (with a linear modulus) if it can be locally written as the sum of a smooth function and a concave (resp. convex) function (see [8] for a complete presentation). In particular, a locally semi–concave function has a supertangent at every point. This implies, for example, that the map $x \mapsto |x|$ is not locally semi–concave because of its singularity at 0 which is downward. The following well-known fact holds, (see for example Lemma 4.2 in [4]):

Lemma 1.3. *Let H be a Tonelli Hamiltonian and u_0 an initial datum. Then, for all $t > 0$, the function $S^t u_0$ is locally semi–concave in $\mathbb{T}^n$.*

We also underline for further use that a function is both locally semi–concave and locally semi–convex if and only if it is $C^{1,1}$ (C^1 with Lipschitz derivative, see Theorem 6.1.5 in [16]).

Using this idea, Patrick Bernard proved the following result (Proposition 4.10 in [4]):

Proposition 1.4. *Let u_0 be a semi–convex function in $\mathbb{T}^n$ and H a Tonelli Hamiltonian. Then, for $t > 0$ small enough, $S^t u_0$ is $C^{1,1}$.*

Finally, we will need the following beautiful result due to Fathi (Theorem 6.4.1 in [16]):

Theorem 1.5. *Let $v : \mathbb{T}^n \to \mathbb{R}$ be a C^1 function and H be a Tonelli Hamiltonian whose flow will be denoted (φ^t). Let L be the graph of the differential of v. If there is an increasing and diverging sequence $(t_n)_{n\in\mathbb{N}}$ such that $\varphi^{t_n}(L)$ is a graph above $\mathbb{T}^n$ for all $n \in \mathbb{N}$, then L is invariant by (φ^t). Moreover, it follows that v is $C^{1,1}$ and that there exists a constant $\alpha[0] \in \mathbb{R}$, only depending on H, such that v is a strong solution of $H(x, D_x v) = \alpha[0]$ in $\mathbb{T}^n$. In particular*

$$S^t v(x) = v(x) - t\alpha[0] \qquad \text{for all } (t, x) \in [0, +\infty) \times \mathbb{T}^n.$$

The constant $\alpha[0]$ appearing above is called *critical value* for H. It is characterized as the only constant a for which an equation of the kind $H(x, D_x v) = a$ in $\mathbb{T}^n$ admits viscosity solutions, see [17]. The viscosity solutions of the corresponding critical equation were used by Fathi as central objects in the weak KAM and Aubry-Mather theory (see [16, 15, 4] for introductions), hence they are also known as *weak KAM solutions*.

1.2. **Variational solutions and their link with viscosity solutions.** In this section we give a very brief overview of the notion of variational solutions. For a more detailed account we refer to [6, 21] and to the references therein.

We will assume here that $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C^2 Hamiltonian with compact levels, meaning that $H^{-1}(\{a\})$ is compact for all $a \in \mathbb{R}$. This ensures that H has what is called finite propagation speed and that all the notions that will be used are well defined. Moreover, its Hamiltonian flow (φ^t) is complete (indeed, H remains constant along the trajectories of the flow).

Let us consider a smooth function $u_0 : \mathbb{T}^n \to \mathbb{R}$, and we will denote by $L_{u_0} \subset \mathbb{T}^n \times \mathbb{R}^n$ the graph of the differential of u_0. It is a smooth Lagrangian submanifold of $\mathbb{T}^n \times \mathbb{R}^n$ for its standard symplectic structure. As it is commonly done, we introduce the Hamiltonian

$$\mathcal{H} : \mathbb{R} \times \mathbb{T}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad (t, x, \tau, p) \mapsto \tau + H(x, p),$$

so that solving the evolutionary Hamilton–Jacobi equation is equivalent to solving $\mathcal{H}(t, x, D_{(t,x)}U) = 0$. We also define

$$\Gamma_{u_0} = \{(0, x, -H(x, p), p), \quad (x, p) \in L_{u_0}\}.$$

Let us denote by (Φ^t) the Hamiltonian flow generated by $X_{\mathcal{H}} = (\partial_{(\tau,p)}\mathcal{H}, -\partial_{(t,x)}\mathcal{H})$. For every fixed $T > 0$ we set

$$L_{\mathcal{H},u_0} = \bigcup_{t\in[0,T]} \Phi^t(\Gamma_{u_0}).$$

It is then known that $L_{\mathcal{H},u_0}$ admits a *generating function quadratic at infinity*, that is a function

$$S : [0,T] \times \mathbb{T}^n \times \mathbb{R}^k \to \mathbb{R}, \quad (t,x,\eta) \mapsto S(t,x,\eta),$$

for some integer k, such that S coincides with a quadratic form outside a compact set, 0 is a regular value of the mapping $(t,x,\eta) \mapsto \partial_\eta S(t,x,\eta)$ and

$$L_{\mathcal{H},u_0} = \big\{ \big(t,x,\partial_{(t,x)} S(t,x,\eta)\big) \,:\, \partial_\eta S(t,x,\eta) = 0 \big\}.$$

Moreover, the construction of S is done in such a way that if $\partial_\eta S(0,x,\eta) = 0$ then $S(0,x,\eta) = u_0(x)$. Then, define

$$V^t u_0(x) := \inf_{[\sigma]=A} \max_{\eta \in \sigma} S(t,x,\eta),$$

where A generates some homology group depending on the topology of the sublevels of S. The function $(t,x) \mapsto V^t u_0(x)$ is the variational solution. This construction can also be generalized to Lipschitz initial data by limit arguments.

If H is convex in p, then $S^t u_0(x) = V^t u_0(x)$ for any Lipschitz continuous initial datum u_0. However, for general Hamiltonians, this is not always the case since $V^t \circ V^s u_0$ may differ from $V^{t+s} u_0$. This is what is commonly referred to as the lack of the *semi–group property*. Recently, Wei [21, 22] established the following link (which is formulated in a more general way in her thesis):

Theorem 1.6 ([21])**.** *For any Lipschitz function $u_0 : \mathbb{T}^n \to \mathbb{R}$, the following holds:*

$$S^t u_0(x) = \lim_{n \to +\infty} (V^{t/n})^n u_0(x) \qquad \textit{for all } (t,x) \in (0,+\infty) \times \mathbb{T}^n.$$

2. The multi–time Hamilton–Jacobi equation

We will only be interested in systems of two equations. Therefore, in the following, H and G will be two smooth and proper Hamiltonians on $\mathbb{T}^n \times \mathbb{R}^n$. Note that no convexity assumptions are taken in the momentum variable. Given a Lipschitz function $u_0 : \mathbb{T}^n \to \mathbb{R}$, we are looking for a real function $u(t,s,x)$ defined on $[0,+\infty) \times [0,+\infty) \times \mathbb{T}^n$ which solves

$$\begin{cases} \partial_t u + H(x, D_x u) = 0 & \text{in } (0,+\infty) \times (0,+\infty) \times \mathbb{T}^n \\ \partial_s u + G(x, D_x u) = 0 & \text{in } (0,+\infty) \times (0,+\infty) \times \mathbb{T}^n \\ u(0,0,x) = u(x) & \text{on } \mathbb{T}^n \end{cases} \tag{2.1}$$

in the viscosity sense. As first remarked in [2], a necessary condition for viscosity solutions of (2.1) to exist, for any Lipschitz initial datum, is that H and G commute, in the sense that their Poisson bracket vanishes, i.e.

$$\{H,G\} := \partial_p H \cdot \partial_x G - \partial_p G \cdot \partial_x H = 0 \qquad \text{in } \mathbb{T}^n \times \mathbb{R}^n.$$

This can be derived by differentiating the equation and by using the method of characteristics (for smooth initial data), see for instance Appendix C in [14]. When the Hamiltonians are convex and coercive in the momentum variable, this condition is sufficient as well, as it is proved in [2]. In the more general case when the Hamiltonians are proper but not necessarily convex in the momentum, it is proved in [9] that the null Poisson bracket condition implies the existence of variational solutions to the multi–time equation. Knowing the links between variational and viscosity solutions (Theorem 1.6), it seems natural to wonder if the vanishing of $\{H,G\}$ also

implies the existence of solutions to (2.1) in the viscosity sense. Using the semi–group property and Theorem 1.1, it is easily seen that this is equivalent to requiring that, for any Lipschitz $u_0 : \mathbb{T}^n \to \mathbb{R}$, the following hold:

$$u(t,s,\cdot) = S_H^t \circ S_G^s u_0 = S_G^s \circ S_H^t u_0 \qquad \text{for every } s,t > 0.$$

We will prove the following theorem:

Theorem 2.1. *Let H and G be two commuting Hamiltonians on $\mathbb{T}^n \times \mathbb{R}^n$ such that both H and $-G$ are Tonelli. Then there exists a solution to the multi–time equation (2.1) with Lipschitz initial datum $u_0 : \mathbb{T}^n \to \mathbb{R}$ if and only if u_0 is $C^{1,1}$ and the graph of its differential is invariant by both the Hamiltonian flows of H and G. Moreover, in this case, there are two constants c_H and c_G such that*

$$H(x, D_x u_0) = c_H \quad \text{and} \quad G(x, D_x u_0) = c_G, \qquad \textit{for every } x \in \mathbb{T}^n,$$

which means that

$$S_H^t \circ S_G^s u_0 = S_G^s \circ S_H^t u_0 = u_0 - t c_H - s\, c_G \qquad \textit{for every } s,t > 0.$$

Remark 2.2. Note that $c_H = \alpha_H[0]$ and $c_G = -\alpha_{-G}[0]$, where $\alpha_H[0]$ and $\alpha_{-G}[0]$ are the critical values associated with H and $-G$.

Before proving this theorem, we establish a simple lemma:

Lemma 2.3. *Let $G : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ such that either G or $-G$ is a coercive Hamiltonian. Let us set*

$$\overline{G}(x,p) := -G(x,-p) \qquad \textit{for every } (x,p) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Then, for any Lipschitz function $u_0 : \mathbb{T}^n \to \mathbb{R}$, the following holds :

$$S_G^s u_0 = -S_{\overline{G}}^s(-u_0) \qquad \textit{for every } s > 0.$$

In particular, if $-G$ is Tonelli, then $S_G^s u$ is locally semi–convex for any $s > 0$.

Proof. The first assertion follows from Proposition 1.2, while the second is a consequence of Lemma 1.3. □

In the next proof, recall that if $w : \mathbb{T}^n \to \mathbb{R}$ is any C^1 function, we denote by $L_w \subset \mathbb{T}^n \times \mathbb{R}^n$ the graph of its differential.

Proof of Theorem 2.1. Let u_0 be an initial condition for which there exists a solution to the multi–time equation. We have seen that in this case

$$S_G^s \circ S_H^t u_0 = S_H^t \circ S_G^s u_0 \qquad \text{for every } s,t > 0.$$

In particular, as the left hand side is locally semi–convex by Lemma 2.3 and the right hand side is locally semi–concave, by Lemma 1.3, we deduce that for all $s,t > 0$, the function $S_G^s \circ S_H^t u_0 = S_H^t \circ S_G^s u_0$ is $C^{1,1}$.

Let $s > 0$ be fixed. For $\varepsilon > 0$, set $v_\varepsilon = S_G^s \circ S_H^\varepsilon u_0 \in C^{1,1}$. It follows from [16, Proposition 4.11.1 and Theorem 6.2.2] that

$$\varphi_H^t(L_{v_\varepsilon}) = L_{S_H^t v_\varepsilon} \qquad \text{for every } t > 0.$$

In particular, we may let t go to infinity and use Fathi's theorem 1.5, yielding the existence of c_H such that

$$S_H^t v_\varepsilon = v_\varepsilon - t c_H \qquad \text{for every } t \geqslant 0.$$

Now letting $\varepsilon \to 0^+$ and passing to the limit, we deduce that

$$S_H^t \circ S_G^s u_0 = S_G^s u_0 - t c_H \qquad \text{for every } t \geqslant 0. \tag{2.2}$$

By continuity, the above identity holds for $s = 0$ as well.

Let us now fix $t > 0$. The same argument applied to $\overline{H}$, $\overline{G}$ and $-u_0$ yields the existence of a constance c_G such that

$$S_G^s \circ S_H^t u_0 = S_H^t u_0 - s c_G \qquad \text{for every } s \geqslant 0. \tag{2.3}$$

By continuity, the above identity holds for $t = 0$ as well. Putting (2.2) and (2.3) together we get

$$S_H^t \circ S_G^s u_0 = u_0 - s c_G - t c_H = S_G^s \circ S_H^t u_0 \qquad \text{for every } s, t \geqslant 0,$$

finally showing that u_0 is $C^{1,1}$ and a strong solution of the stationary equations associated with H and with G. □

Even if for a Tonelli Hamiltonian H there are always weak KAM solutions, they may all fail to be $C^{1,1}$. In this instance, the previous theorem implies that there are no global solutions to the multi–time equation for H and $-H$ (nor in fact for H and $-G$ for any Tonelli Hamiltonian G).

We furnish an example to conclude.

Corollary 2.4. *There exists a Tonelli Hamiltonian $H : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that the multi–time equation* (2.1) *with $G := -H$ does not admit viscosity solutions, for any Lipschitz initial datum u_0.*

Proof. According to Theorem 2.1, we just need to exhibit a Tonelli Hamiltonian H that does not admit weak KAM solutions of class $C^{1,1}$. The example that follows is classical, see for instance [16, Section 4.14].

Consider the pendulum equation, corresponding to

$$H(x,p) = \frac{1}{2}|p|^2 + \cos(2\pi x), \text{ on } \mathbb{T}^1 \times \mathbb{R}.$$

The Hamiltonian vector-field is then given by $X_H(x,p) = \big(p, 2\pi \sin(2\pi x)\big)$. It possesses 2 fixed points, $(\frac{1}{2}, 0)$ which is an elliptic fixed point, and $(0,0)$ which is hyperbolic. The stable and unstable manifold are given by the two separatrices which equations are $p = \pm\sqrt{2\big(1 - \cos(2\pi x)\big)} = \pm 2\sin(\pi x)$. Those separatrices bound an elliptic island filled with periodic trajectories. In this case, it is known that $\alpha[0] = 1$. Moreover, the only weak KAM solution (up to addition of a constant) is given by

$$u(x) = \begin{cases} \displaystyle\int_0^x 2\sin(\pi t)dt = \frac{2}{\pi}\big(1 - \cos(\pi x)\big), & \forall x \in \big[0, \frac{1}{2}\big], \\ \displaystyle\frac{2}{\pi} - \int_{\frac{1}{2}}^x 2\sin(\pi t)dt = \frac{2}{\pi}\big(1 + \cos(\pi x)\big), & \forall x \in \big(\frac{1}{2}, 1\big), \end{cases}$$

which is not of class $C^{1,1}$. Indeed, the graph of the derivative of any other weak KAM solution v must stay on the union of the two separatrices, the fact that v is semi–concave means that this graph may only "jump downwards", therefore it may possess only one discontinuity, and in order for v to be 1-periodic, this jump must occur at $1/2$. Hence $v' = u'$, that is $v - u$ is constant. □

It is proved in [23, 13] that two commuting Tonelli Hamiltonians have the same weak KAM solutions, therefore the previous theorem is consistent with this fact, in the sense that it is normal that whenever H has a $C^{1,1}$ weak KAM solution, it will be the same for G.

Let us conclude with one last speculation. The previous results, in order to be global, are stated for two Tonelli Hamiltonians. However, when a single Hamiltonian is convex in the fibers in some areas (say for x in an open set U of the torus), and concave in others (say for x in an open set V of the torus), the previous discussions show that variational solutions of the evolutionary equation will be either semi–concave (resp. semi–convex) in U (resp. V) for small times, i.e. as long as the method of characteristics applies and the latter do not leave those open sets. This need not remain true in the long time. On the other hand, due to the semi–group property and the finite speed propagation, the viscosity solutions always remain semi–concave in U (resp. semi–convex in V). These two different kinds of regularity might give a new insight as to detecting when variational solutions differ from viscosity solutions and thereby do not verify the semi–group property. We also mention the recent work [5] on very similar issues.

Appendix A.

In this Appendix, we want to give a proof of Theorem 1.1 in a slightly more general setting. In what follows we will denote by $\mathrm{UC}(X)$ and $\mathrm{BUC}(X)$ the space of uniformly continuous and bounded uniformly continuous real functions on the metric space X, respectively.

We consider the following evolutionary Hamilton–Jacobi equation:

$$\partial_t u + F(x, D_x u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^n, \tag{A}$$

where the Hamiltonian $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions:

(F1) $\quad F \in \mathrm{BUC}(\mathbb{R}^n \times B_R) \qquad$ for every $R > 0$;

(F2) $\quad \inf_{x \in \mathbb{R}^n} F(x, p) \to +\infty \qquad$ as $|p| \to +\infty$.

We will prove the following result:

Theorem A.1. *Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Hamiltonian satisfying (F1)-(F2) and let $u_0 \in \mathrm{UC}(\mathbb{R}^n)$. Then there exists a unique function $u \in \mathrm{UC}([0, +\infty) \times \mathbb{R}^n)$ with $u(0, \cdot) = u_0$ that solves* (A) *in the viscosity sense. If u_0 is Lipschitz in $\mathbb{R}^n$, then u is Lipschitz continuous in $[0, +\infty) \times \mathbb{R}^n$ and its Lipschitz constant only depends on $\|Du_0\|_\infty$ and F.*

We need to generalize the notions of viscosity sub and supersolution given at the beginning of Section 1.1 to the case of discontinuous functions. Let Ω be an open subset of $(0, +\infty) \times \mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be a possibly discontinuous locally bounded function. We define the *upper* and *lower semicontinuous envelope* of u, denoted by u^* and u_* respectively, as follows:

$$u^*(t, x) := \inf_{r>0} \sup_{(s,y) \in B_r((t,x)) \cap \Omega} u(s, y), \qquad u_*(t, x) := \sup_{r>0} \inf_{(s,y) \in B_r((t,x)) \cap \Omega} u(s, y),$$

for every $(t, x) \in \Omega$. We say that u^* is a *(non–continuous) viscosity subsolution* of (A) in Ω if (1.2) holds for any supertangent φ to u^* at the point $(t, x) \in \Omega$. The gradient of any such supertangent φ at the point (t, x) is called *supergradient* of u^* at (t, x). We will denote by $D^+ u^*(t, x)$ the set consisting of all supergradients of u^* at (t, x). Analogously, we say that u_* is a supersolution of (A) in Ω if (1.3) holds for any subtangent φ to u_* at the point $(t, x) \in \Omega$.

Given a function $f : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$, we introduce the following inequality depending on a pair of constants $\alpha > 0$ and $M_\alpha > 0$:

$$f(t,x) \geqslant f(0,x) - (\alpha + M_\alpha t) \qquad \text{for all } (t,x) \in [0,+\infty) \times \mathbb{R}^n. \tag{C}$$

We will need the following comparison result between discontinuous sub and supersolutions.

Proposition A.2. *Let v and u be, respectively, an upper and lower semicontinuous function on $[0,+\infty) \times \mathbb{R}^n$ satisfying condition* (C) *for a pair of constants $\hat{\alpha} > 0$ and $M_{\hat{\alpha}} > 0$. Let us assume that v is a subsolution and u is a supersolution of* (A)*, and that either one of the following conditions hold:*

(a) *for every $\alpha > 0$, there exists $M_\alpha > 0$ such that* (C) *holds with $f := v$;*

(b) *v is continuous on $[0,+\infty) \times \mathbb{R}^n$.*

Then

$$\sup_{[0,+\infty)\times\mathbb{R}^n} \big(v - u\big) \leqslant \sup_{\mathbb{R}^n} \big(v(0,\cdot) - u(0,\cdot)\big).$$

Remark A.3. If the supersolution u does not satisfy condition (C), then the comparison principle stated above does not hold, no matter how regular the subsolution v is, see for instance Example 4.9 in [7]. We remark that condition (C) is always fulfilled by u and v when the latter are absolutely continuous in $[0,+\infty) \times \mathbb{R}^n$. In particular, Proposition A.2 entails uniqueness of the solution of (A) in $\mathrm{UC}\left([0,+\infty) \times \mathbb{R}^n\right)$.

For the proof of Proposition A.2, we will need to regularize the subsolution v through the following *sup–convolution in time*, see [1]: for every fixed $\delta > 0$, define

$$v^\delta(t,x) := \sup_{s \geqslant 0} \left\{ v(s,x) - \frac{|t-s|}{\delta} \right\}, \qquad (t,x) \in [0,+\infty) \times \mathbb{R}^n. \tag{A.1}$$

For every $(t,x) \in [0,+\infty) \times \mathbb{R}^n$, we denote by $\mathcal{M}^\delta(t,x)$ the set of $s \geqslant 0$ that realize the supremum in the definition of $v^\delta(t,x)$. We set

$$\Omega^\delta := \mathrm{int} \left\{ (t,x) \in (0,+\infty) \times \mathbb{R}^n \,:\, v^\delta(t,x) > v(0,x) - \frac{t}{\delta}, \right\},$$

that is, Ω^δ is the largest open subset of $(0,+\infty) \times \mathbb{R}^n$ made up of points (t,x) for which $0 \notin \mathcal{M}^\delta(t,x)$. Last, we set $\mu := \max\{1, \sup_{\mathbb{R}^n \times \mathbb{R}^n} -F\}$, which is finite by assumption (F2). The following holds:

Proposition A.4. *Let us assume that v is upper semicontinuous on $[0,+\infty) \times \mathbb{R}^n$ and a non-continuous subsolution of* (A)*. Then for every $\delta < 1/\mu$ we have:*

(i) $v(t,x) \leqslant v^\delta(t,x) \leqslant v(0,x) + \mu t$ *for every $(t,x) \in [0,+\infty) \times \mathbb{R}^n$.*

(ii) For every $(t,x) \in [0,+\infty) \times \mathbb{R}^n$ the set $\mathcal{M}^\delta(t,x)$ is nonempty and contained in $[0, 2t/(1-\delta\mu)]$. In particular, v^δ is upper semicontinuous in $[0,+\infty) \times \mathbb{R}^n$.

(iii) If $(t_0,x_0) \in \Omega^\delta$, then

$$D^+ v^\delta(t_0,x_0) \subseteq D^+ v(s_0,x_0) \qquad \text{for every } s_0 \in \mathcal{M}^\delta(t_0,x_0).$$

In particular, v^δ is an upper semicontinuous subsolution of (A) *in Ω^δ.*

(iv) The function $v^\delta(\cdot,x)$ is $1/\delta$–Lipschitz continuous in $[0,+\infty)$ for every fixed $x \in \mathbb{R}^n$. Moreover, there exists a positive constant L_δ such that

$$|D_x v^\delta(t,x)| \leqslant L_\delta \qquad \text{for a.e. } (t,x) \in \Omega^\delta.$$

(v) Let us assume that v satisfies (C) *for some positive constants α and M_α. Then for every $\delta > 0$ small enough*

$$\left[2\alpha\,\frac{\delta}{1-\delta M_\alpha}, +\infty\right) \times \mathbb{R}^n \subset \Omega^\delta$$

Proof. (i) The first inequality is obvious. To prove the second inequality, we first observe that v satisfies

$$\partial_t v \leqslant -F(x, Dv) \leqslant \mu \qquad \text{in } (0,+\infty)\times\mathbb{R}^n$$

in the viscosity sense. We infer

$$v(s,x) \leqslant v(\tau,x) + \mu(s-\tau) \qquad \text{for every } x\in\mathbb{R}^n \text{ and } 0<\tau<s,$$

in particular, by letting $\tau \to 0^+$ and by using the upper semicontinuity of v, we get

$$v(s,x) \leqslant v(0,x) + \mu s \qquad \text{for every } (s,x)\in[0,+\infty)\times\mathbb{R}^n. \tag{A.2}$$

For $\delta < 1/\mu$ we get

$$v^\delta(t,x) \leqslant v(0,x) + \sup_{s\geqslant 0}\left\{\mu s - \frac{|t-s|}{\delta}\right\} = v(0,x) + \mu t.$$

(ii) Let us fix $(t,x)\in[0,+\infty)\times\mathbb{R}^n$. The fact that $\mathcal{M}^\delta(t,x)$ is nonempty follows from (A.2) and the upper semicontinuity of v. Pick $s\in\mathcal{M}^\delta(t,x)$. From the definition of $v^\delta(t,x)$ and (A.2) we have

$$v(0,x) - \frac{t}{\delta} \leqslant v^\delta(t,x) = v(s,x) - \frac{|s-t|}{\delta} \leqslant v(0,x) + \mu s - \frac{|s-t|}{\delta}.$$

A straightforward computation shows that $s \leqslant 2t/(1-\delta\mu)$. The upper semicontinuity of v^δ readily follows from this and from the upper semicontinuity of v.

(iii) First note that $(t_0,x_0)\in\Omega^\delta$ implies $\mathcal{M}^\delta(t_0,x_0)\subset(0,+\infty)$. To prove the first assertion, it suffices to observe that, if φ is a supertangent to v^δ at $(t_0,x_0)\in\Omega^\delta$, then the function $(t,x)\mapsto\varphi(t+(t_0-s_0),x)$ is a supertangent to v at (s_0,x_0) for every fixed $s_0\in\mathcal{M}^\delta(t_0,x_0)$. Since v^δ is upper semicontinuous by *(ii)* and F is time–independent, we infer that v^δ is an upper semicontinuous subsolution of (A) in Ω^δ.

(iv) The fact v^δ is $1/\delta$–Lipschitz in t is apparent by definition. By using *(iii)* we get

$$F(x, Dv^\delta) \leqslant -\partial_t v^\delta \leqslant \frac{1}{\delta} \qquad \text{in } \Omega^\delta,$$

and the estimate on $|D_x v^\delta|$ in Ω^δ follows from the coercivity of F, see Lemma 2.5 in [3].

(v) Pick $\delta < 1/M_\alpha$ and set $T_\delta := 2\delta\alpha/(1-\delta M_\alpha)$. Then every $(t,x)\in(T_\delta/2,+\infty)\times\mathbb{R}^n$ satisfies

$$v(0,x) - \frac{t}{\delta} < v(0,x) - (M_\alpha t + \alpha) \leqslant v(t,x) \leqslant v^\delta(t,x),$$

in particular $[T_\delta,+\infty)\times\mathbb{R}^n\subset\Omega^\delta$, as it was to be shown. □

Proof of Proposition A.2. We assume $\sup_{\mathbb{R}^n}\big(v(0,\cdot)-u(0,\cdot)\big) < +\infty$, being the statement otherwise trivial. Furthermore, up to adding a suitable constant to v, we can assume that $\sup_{\mathbb{R}^n}\big(v(0,\cdot)-u(0,\cdot)\big) = 0$. The assertion is thus reduced to proving that $v-u\leqslant 0$ in $[0,+\infty)\times\mathbb{R}^n$. The proof is divided in two steps.

Step 1. We first assume v Lipschitz continuous in $[0,+\infty)\times\mathbb{R}^n$, i.e. there exists $L>0$ such that

$$|v(t,x)-v(s,y)|\leqslant L(|x-y|+|t-s|)\qquad \text{for every } (t,x),(s,y)\in[0,+\infty)\times\mathbb{R}^n.$$

Let us suppose by contradiction that there exists $(\hat t,\hat x)\in(0,T)\times\mathbb{R}^n$, for some $T>0$, such that

$$v(\hat t,\hat x)-u(\hat t,\hat x)=\gamma>0.$$

Let us introduce a pair of functions χ, $\Phi:([0,T)\times\mathbb{R}^n)^2\to\mathbb{R}$ defined as

$$\begin{aligned}\chi(t,x,s,y) &:= \frac{1}{2\varepsilon}\big(|x-y|^2+|t-s|^2\big)+\beta\langle x\rangle+\frac{\eta}{T-t},\\ \Phi(t,x,s,y) &:= v(t,x)-u(s,y)-\chi(t,x,s,y),\end{aligned}$$

where $\langle x\rangle:=\sqrt{1+|x|^2}$, η is a positive constant that will be suitably chosen, and ε, β are positive parameters that will be sent to 0. We choose $\eta>0$ and $\beta>0$ small enough such that

$$\beta\langle\hat x\rangle+\frac{\eta}{T-\hat t}\leqslant\frac{\gamma}{2}.$$

Then

$$\sup_{([0,T)\times\mathbb{R}^n)^2}\Phi\geqslant\Phi(\hat t,\hat x,\hat t,\hat x)=\gamma-\left(\beta\langle\hat x\rangle+\frac{\eta}{T-\hat t}\right)\geqslant\frac{\gamma}{2}$$

We proceed to show that such a supremum is attained. By hypothesis,

$$u(s,y)\geqslant u(0,y)-(\hat\alpha+M_{\hat\alpha}s)\qquad\text{for every } (s,y)\in[0,+\infty)\times\mathbb{R}^n$$

for a pair of constants $\hat\alpha>0$ and $M_{\hat\alpha}>0$. Then

$$\Phi(t,x,s,y)\leqslant L(t+|x-y|)+(\hat\alpha+M_{\hat\alpha}s)-\frac{1}{2\varepsilon}\left(|x-y|^2+|t-s|^2\right)-\beta\langle x\rangle-\frac{\eta}{T-t}$$

If $(t,x,s,y)\in[0,T)\times\mathbb{R}^n$ satisfies $\Phi(t,x,s,y)>0$, then

$$\left(\frac{|x-y|}{2\varepsilon}-L\right)|x-y|+\frac{|t-s|^2}{2\varepsilon}+\beta\langle x\rangle+\frac{\eta}{T-t}\leqslant\hat\alpha+(M_{\hat\alpha}+L)T=:C. \tag{A.3}$$

This readily implies that there exists $(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon)\in([0,T)\times\mathbb{R}^n)^2$ such that

$$\Phi(t_\varepsilon,x_\varepsilon,s_\varepsilon,y_\varepsilon)=\sup_{([0,T)\times\mathbb{R}^n)^2}\Phi\geqslant\frac{\gamma}{2} \tag{A.4}$$

This maximum point also depends on the parameters η and β, but such dependence is omitted to ease notations. We infer from (A.3) that

$$\langle x_\varepsilon\rangle\leqslant\frac{C}{\beta},\quad t_\varepsilon\leqslant T-\frac{\eta}{C}.$$

By Proposition 3.7 in [12], we also know that

$$\frac{|t_\varepsilon-s_\varepsilon|^2}{\varepsilon}\to0,\quad \frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon}\to0\qquad\text{as } \varepsilon\to0^+.$$

We infer that $(t_\varepsilon,x_\varepsilon)$ and $(s_\varepsilon,y_\varepsilon)$ converge, along a subsequence as $\varepsilon\to0^+$, to a point $(t_0,x_0)\in[0,T-\eta/C]\times\mathbb{R}^n$ satisfying, cf. Proposition 3.7 in [12],

$$v(t_0,x_0)-u(t_0,x_0)-\beta\langle x_0\rangle-\frac{\eta}{T-t_0}=\Phi(t_0,x_0,t_0,x_0)=\sup_{(t,x)\in[0,T)\times\mathbb{R}^n}\Phi(t,x,t,x)\geqslant\frac{\gamma}{2}.$$

Since $v(0,\cdot) - u(0,\cdot) \leqslant 0$ on $\mathbb{R}^n$, we derive that $t_0 > 0$, hence the points $(t_\varepsilon, x_\varepsilon)$, $(s_\varepsilon, y_\varepsilon)$ definitively belong to an open neighborhood $I \times U$ of (t_0, x_0) compactly contained in $(0,T) \times \mathbb{R}^n$.

We can now exploit the fact that $\varphi(t,x) := \chi(t,x,s_\varepsilon,y_\varepsilon)$ is a supertangent to v at $(t_\varepsilon, x_\varepsilon) \in (0,T) \times \mathbb{R}^n$ and $\psi(s,y) := -\chi(t_\varepsilon, x_\varepsilon, s, y)$ is a subtangent to u at $(s_\varepsilon, y_\varepsilon) \in (0,T) \times \mathbb{R}^n$. Since v is an upper semicontinuous subsolution and u is a lower semicontinuous supersolution of (A), we get

$$\partial_t \varphi(t_\varepsilon, x_\varepsilon) - \partial_s \psi(s_\varepsilon, y_\varepsilon) \leqslant F\big(y_\varepsilon, D\psi(s_\varepsilon, y_\varepsilon)\big) - F\big(x_\varepsilon, D\varphi(t_\varepsilon, x_\varepsilon)\big) \tag{A.5}$$

An easy computation shows that

$$\partial_t \varphi(t_\varepsilon, x_\varepsilon) - \partial_s \psi(s_\varepsilon, y_\varepsilon) = \frac{\eta}{(T - t_\varepsilon)^2} > \frac{\eta}{T^2},$$

while

$$|D\varphi(t_\varepsilon, x_\varepsilon) - D\psi(s_\varepsilon, y_\varepsilon)| \leqslant \beta.$$

Moreover, by the fact that v is L–Lipschitz and φ is a supertangent to v at $(t_\varepsilon, x_\varepsilon)$, we get

$$|D\varphi(t_\varepsilon, x_\varepsilon)| = \left| D\psi(s_\varepsilon, y_\varepsilon) + \beta \frac{x_\varepsilon}{\langle x_\varepsilon \rangle} \right| \leqslant L,$$

yielding in particular that $|D\psi(s_\varepsilon, y_\varepsilon)| \leqslant L + \beta$. We now use the fact that F is uniformly continuous in $U \times B_{L+1}$ in view of (F1): by sending ε and β to 0 in (A.5) and by making use of the above estimates, we get $\eta/T^2 \leqslant 0$, in contrast with the choice of $\eta > 0$.

Step 2. Let us prove the result in the general case. Let $\alpha > 0$ and $M_\alpha > 0$ such that condition (C) holds with $f := v$. Set $T_\delta := 2\delta\alpha/(1 - \delta M_\alpha)$. According to Proposition A.4, for $\delta > 0$ small enough the function $(t,x) \mapsto v^\delta(T_\delta + t, x)$ is Lipschitz continuous in $[0,+\infty) \times \mathbb{R}^n$ and is a subsolution of (A) in $(0,+\infty) \times \mathbb{R}^n$, hence we can apply Step 1 to infer

$$v^\delta(T_\delta + t, x) - u(t,x) \leqslant \sup_{\mathbb{R}^n} \Big(v^\delta(T_\delta, \cdot) - u(0,\cdot) \Big) \tag{A.6}$$

for any fixed $(t,x) \in (0,+\infty) \times \mathbb{R}^n$. If condition (a) holds, namely if the constant $\alpha > 0$ in (C) with $f := v$ can be chosen arbitrarily small, we argue as follows: by Proposition A.4 we know that

$$v(t,x) - \frac{T_\delta}{\delta} \leqslant v^\delta(t,x) - \frac{T_\delta}{\delta} \leqslant v^\delta(T_\delta + t, x) \leqslant v(0,x) + \mu(T_\delta + t),$$

so

$$v(t,x) - u(t,x) \leqslant \sup_{\mathbb{R}^n} \big(v(0,\cdot) - u(0,\cdot) \big) + \frac{2\alpha}{1 - \delta M_\alpha}(1 + \mu\delta).$$

The assertion follows by sending $\delta \to 0^+$ and since $\alpha > 0$ was arbitrarily chosen.

If condition (b) holds, namely if v is continuous in $[0,+\infty) \times \mathbb{R}^n$, we exploit Proposition A.4-(i) to derive from (A.6) that

$$v(T_\delta + t, x) - u(t,x) \leqslant \sup_{\mathbb{R}^n} \big(v(0,\cdot) - u(0,\cdot) \big) + \mu T_\delta.$$

The assertion follows by sending $\delta \to 0^+$ since $T_\delta \to 0$ and v is continuous on $[0,+\infty) \times \mathbb{R}^n$.

□

The proof of Theorem A.1 is standard. It follows from Proposition A.2 by making use of Perron's method.

Proof of Theorem A.1. We only need to show the existence of a solution, for the uniqueness comes from Proposition A.2.

Let us first assume $u_0 \in \mathrm{Lip}(\mathbb{R}^n)$. In view of the assumptions (F1)-(F2), there exists a positive constant M such that $M > \sup\{|F(x,p)| \,:\, |p| \leqslant \|Du_0\|_\infty\}$. Then the functions

$$\underline{u}(t,x) := u_0(x) - Mt, \qquad \overline{u}(t,x) := u_0(x) + Mt, \qquad (t,x) \in [0,+\infty) \times \mathbb{R}^n$$

are, respectively, a sub and a supersolution to (A). We denote by $\mathscr{S}$ the family of continuous functions on $[0,+\infty) \times \mathbb{R}^n$ that are subsolutions of (A) and we set, for every $(t,x) \in [0,+\infty) \times \mathbb{R}^n$,

$$u(t,x) = \sup\left\{v(t,x) \,:\, v \in \mathscr{S},\ \underline{u} \leqslant v \leqslant \overline{u} \quad \text{on } [0,+\infty) \times \mathbb{R}^n\right\}.$$

Then u^* and u_* are, respectively, a sub and a supersolution of (A), see Theorem 2.14, Chapter V in [1], and satisfy

$$u_0(x) - Mt \leqslant u_*(t,x) \leqslant u^*(t,x) \leqslant u_0(x) + Mt \qquad \text{on } [0,+\infty) \times \mathbb{R}^n.$$

We can therefore apply the comparison principle stated in Proposition A.2 to infer that $u^* \leqslant u_*$ on $[0,+\infty) \times \mathbb{R}^n$. Since the other inequality is obvious by definition, we get that u is continuous on $[0,+\infty) \times \mathbb{R}^n$, solves (A) in the viscosity sense and takes the initial datum u_0 at $t = 0$. It is left to show the Lipschitz continuity of u. To this aim, note that, for every fixed $h \in \mathbb{R}^n$, the function $u_h(t,x) := u(t+h,x)$ is a continuous solution to (A) with initial condition $u_h(0,x) = u(h,x)$. From the fact that $|u(t+h,x) - u(0,x)| \leqslant M(t+h)$ for every $(t,x) \in [0,+\infty) \times \mathbb{R}^n$ one easily see that u_h satisfies condition (C) with $\alpha := 2Mh$ and $M_\alpha := M$. By applying Proposition A.2 to the continuous solutions u_h, u, we get

$$\|u(t+h,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant \|u(h,\cdot) - u(0,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant M\,h.$$

In other words, the function u is M-Lipschitz in t. Using the fact that u is a subsolution of (A), we get that u satisfies

$$F(x, D_x u) \leqslant -\partial_t u \leqslant M \qquad \text{in } (0,+\infty) \times \mathbb{R}^n$$

in the viscosity sense. By the coercivity of F, we conclude that $u(t,\cdot)$ is Lipschitz for every $t > 0$, with a constant only depending on u_0 and F, see for instance Lemma 2.5 in [3].

If $u_0 \in \mathrm{UC}(\mathbb{R}^n)$, we can find, for instance by sup-convolution, see [1], a sequence of Lipschitz functions $u_0^k : \mathbb{R}^n \to \mathbb{R}$ that uniformly converge to u_0 on $\mathbb{R}^n$. Let us denote by $u^k : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ the corresponding Lipschitz solution of (A) with initial datum u_0^k. By the comparison principle we have

$$\|u^m - u^k\|_{L^\infty([0,+\infty)\times\mathbb{R}^n)} \leqslant \|u_0^m - u_0^k\|_{L^\infty(\mathbb{R}^n)},$$

that is, $(u^k)_k$ is a Cauchy sequence in $[0,+\infty) \times \mathbb{R}^n$ with respect to the sup–norm. Hence the Lipschitz functions u_k uniformly converge to a function u on $[0,+\infty)\times\mathbb{R}^n$, which is therefore uniformly continuous. By the stability of the notion of viscosity solution, we conclude that u is a solution of (A) with initial datum u_0. □

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