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ON SOME FRAÏSSÉ LIMITS WITH FREE AMALGAMATION

Yvon BOSSUT

Abstract : In this work a general way is given to construct some examples of NSOP₁ theories as limits of some Fraïssé class satisfying strong conditions. These limits will satisfy existence, that Kim-independence coincides with algebraic independence, and that forking independence is obtained by forcing base monotonicity on Kim-forking. These theories also come with a stationary independence relation. We conclude by asking some questions around stabilizers in NSOP₁ groups. This study is based on results of Baudisch, Ramsey, Chernikov and Kruckman.¹

1 Introduction

NSOP₁ theories have recently been studied as a generalisation of simple theories. The notion of Kim-forking plays a similar role in NSOP₁ theories than forking in simple theories. Kim and Pillay [22] have shown that forking independence in simple theories is characterised by some of its properties (see [25] for example). Chernikov and Ramsey have shown a similar result for Kim-independence in NSOP₁ theories, [12, Theorem 5.8]. The properties of Kim-forking in NSOP₁ theories have recently been studied by Ramsey, Kim, Chernikov, Kaplan ([19], [20] among others). In this work we shall study independence relations in NSOP₁ theories.

At the origin of this work is an interesting construction which appears in [5] : The parameterization of a theory. Given some theory T in a language L , consider the language L_P where we add a new sort P and one variable in P to every symbol of L , and denote by O (for object) the original sort. Then an L_P -structure A consists of its set of parameters $P(A)$ and of its set of objects $O(A)$ which has an L -structure $A_p := O(A)_p$ induced by every parameter $p \in P(A)$. There does not always exist a model companion to the theory T_P expressing that for every parameter p , the L -structure induced by p is a model of T .

Examples of such model companions are given in [12, Section 6.3] and constructed as Fraïssé limits. Ramsey and Chernikov consider a relational language L and a class K of structures with the strong amalgamation property. Then the parameterized class K_P is defined as the class of L_P -structures A such that for every $p \in P(A)$ the L -structure induced by p on A is in K , and show that the limit of this class is NSOP₁. They give some condition on K under which the limit of K_P is non-simple. One example of the theories they build is the theory of parameterized equivalence relations.

Here we shall give a context in which we can parameterize in the same fashion without assuming that the language is relational. The cost of this is to have strong assumptions about the generated structure, namely a set of condition (H) which strengthens the usual Fraïssé properties. We obtain similar results as Ramsey and Chernikov : If some class K satisfies (H) then its limit T is NSOP₁ and we can compute forking independence and Kim-independence in T (Corollary 3.2.10 and Proposition 3.2.15). This is done through the use of some stationary independence relation which comes from the assumption that K is closed under free amalgamation in the categorical sense. This independence relation was already mentioned in [6], and it does not coincide with the notion of Free Amalgamation of [13], which also is a stationary independence relation but requires the algebraic closure to be generically trivial : If A

¹Partially supported by ANR-DFG AAPG2019 GeoMod

and B are independent and algebraically closed then their union is algebraically closed - this will not hold in our case unless the algebraic closure is trivial.

It is also shown that the condition (H) is preserved under some modifications of the language L and the class K : parameterizing K and adding generic structure to K . We also give a condition under which the limit of the parameterized class is non-simple. This context allows us to build model companions of some exotic theories, for example the theory of an abelian group with a unary function f such that $f(0) = 0$. This example is studied to exhibit some behaviour of NSOP₁ groups which we try to explore in the last section. Here we set some definitions and try to find analogous notions to the stabilizer in stability and simplicity theory in the NSOP₁ context.

2 Preliminaries on independence relations

Let $\mathbb{M} \models T$ be a monster model of some complete theory. We introduce the different relations of independence that we will work with. See [1] and [16] on the topic of the axiomatic approach to independence relations.

Definition 2.1. Let b_0, e be tuples of \mathbb{M} and $\varphi(x, b_0)$ be a formula.

1. $\varphi(x, b_0)$ *divides over e* if there is an e -indiscernible sequence $I = (b_i : i < \omega)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent. In that case we say that $\varphi(x, b_0)$ divides over e with respect to I .
2. A partial type $p(x, b)$ *forks over e* if it implies a finite disjunction of formulas each of which divides over e .
3. Let $a \in \mathbb{M}$. We write $a \downarrow_e^f b$ to denote the assertion that $\text{tp}(a/eb_0)$ does not fork over e and $a \downarrow_e^d b_0$ to denote the assertion that $\text{tp}(a/eb_0)$ does not divide over e .
4. A sequence of tuples $(a_i : i < \omega)$ is called *e -Morley* if it is e -indiscernible and if $a_i \downarrow_e^f a_{<i}$ for every $i < \omega$.

Definition 2.2. A set $e \subseteq \mathbb{M}$ is an *extension basis* if $a \downarrow_e^f e$ for every $a \in \mathbb{M}$. A theory T has *existence* if every set is an extension basis. This is equivalent to saying that for every $e \in \mathbb{M}$ and every $p(x) \in S(e)$ there is an e -Morley sequence in p .

Remark 2.3. Every model is an extension basis, simple theories satisfy existence, and it is a conjecture that NSOP₁ theories do as well.

Definition 2.4. Assume that e is an extension basis. Let $b_0 \in \mathbb{M}$ and $\varphi(x, b_0)$ be a partial type.

1. $\varphi(x, b_0)$ *Kim-divides over e* if there is an e -Morley sequence $(b_i : i < \omega)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent. We will say that $\varphi(x, b_0)$ Kim-divides over e with respect to $(b_i : i < \omega)$.
2. $p(x, b)$ *Kim-forks over e* if it implies a finite disjunction of formulas each of which Kim-divides over e .
3. Let $a \in \mathbb{M}$. We write $a \downarrow_e^K b_0$ to denote the assertion that $\text{tp}(a/eb)$ does not Kim-fork over e and $a \downarrow_e^{Kd} b$ to denote the assertion that $\text{tp}(a/eb)$ does not Kim-divide over e .
4. $I = (a_i : i < \omega)$ is *Kim-Morley over e* if it is e -indiscernible and if $a_i \downarrow_e^K a_{<i}$ for every $i < \omega$.

Definition 2.5. We define *algebraic independence* as the relation $A \downarrow_C^a B$ iff $\text{acl}(AC) \cap \text{acl}(BC) = \text{acl}(C)$.

Definition 2.6. We define *M -independence* as the weakest relation implying \downarrow^a and satisfying base monotonicity and closure, i. e. $A \downarrow_C^M B$ iff $\text{acl}(AC') \downarrow_{C'}^a \text{acl}(BC)$ for every $C \subseteq C' \subseteq \text{acl}(BC)$.

This relation was defined by Onshuus in the context of rosy theories (see [24] for reference), but we will not make further use of rosiness because of Remark 3.2.16.

There are a lot of simplicity-like results characterising NSOP₁-theories in terms of properties of Kim-forking. We shall see some of them in Section 3., but for now let us set the following definition :

Definition 2.7. Let T be a theory with existence. T is NSOP₁ if and only if Kim-independence is symmetric, meaning that $a \downarrow_e^K b$ iff $b \downarrow_e^K a$ for all tuples $a, e, b \in \mathbb{M}$.

If we were to remove the hypothesis of existence we would have to restrict ourselves to the study of Kim-forking over models, which is also interesting. However since we will mostly work with existence we will not add the definitions for this case. They can be found in [19].

3 Constructing the limit

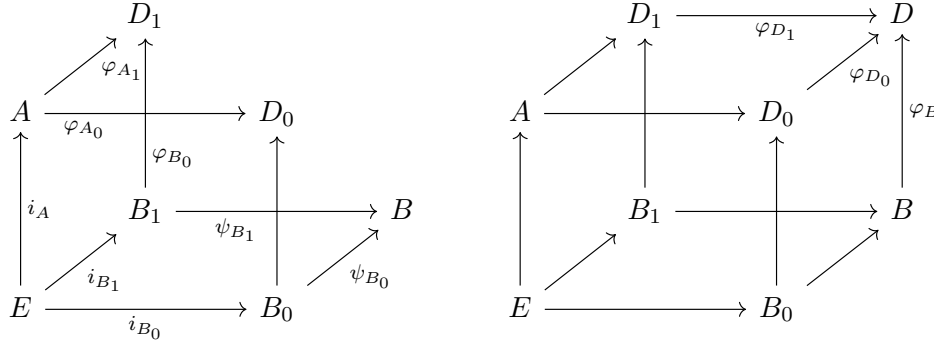
In this section we consider some set of hypotheses (H) on a class K of structures. We begin by showing that if K satisfies (H) then we can construct the limit theory T of K . We show that this limit T is an NSOP₁ theory, has existence, that $\downarrow^K = \downarrow^a$, that $\downarrow^M = \downarrow^f = \downarrow^d$ over arbitrary sets and that it has weak elimination of imaginaries. We give some basic examples of such classes K . The bases of this study are [12, Section 6.3] and [5]. The construction of Baudisch works in a much more general context, but we focus here on the study of independence relations, which appears in his paper in [5, Theorem 4.1] under some stronger hypotheses.

3.1 Hypothesis on the class and basic examples

Definition 3.1.1. We consider a countable language L and a class K of countable L -structures with the following properties, which we will refer to as (H):

1. **Hereditary Property** : If $A \subseteq B \in K$ is a substructure then $A \in K$.
2. **Axiomatisability** : If an L -structure A is not in K there is a quantifier free formula φ and a finite tuple $a \in A$ such that $A \models \varphi(a)$ and $A' \not\models \varphi(a')$ for every $a' \in A' \in K$.
3. **Quantifier Elimination** : For every quantifier free formula $\varphi(x, y)$ satisfied in some structure of K there is a quantifier free formula $\psi(x)$ such that if $a \in A \in K$ and $A \models \varphi(a)$ then there is $A \subseteq B \in K$ an extension and $b \in B$ such that $B \models \varphi(a, b)$.
4. **Free Amalgamation** : If $C \subseteq A, B$ are structures in K (where we allow $C = \emptyset$ when the language does not have any constants) there is $D \in K$ and some embeddings i_A, i_B of A and B respectively to D satisfying $i_A|_C = i_B|_C$, $i_A(A) \cap i_B(B) = i_A(C)$, D is generated by the images of A and B and such that given any morphism of L -structures $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ to $D' \in K$ there is a unique morphism $\varphi : D \rightarrow D'$ such that $\varphi_A = \varphi \circ i_A$ and $\varphi_B = \varphi \circ i_B$. We will write $A \oplus_C B$ for this structure D . We also assume that the elements of K agree on the structure generated by the constants, which is equivalent to the joint embedding property when Free Amalgamation holds.
5. **Generic Extension** : If $A \in K$ and V is a sort there is an extension $A \oplus_{(\emptyset)} \{x\} \in K$ generated by a single element $x \in V$ over A such that for every morphism $\varphi : A \rightarrow A' \in K$ and every element $a' \in V(A')$ there is a unique morphism $\psi : A \oplus_{(\emptyset)} \{x\} \rightarrow A'$ extending φ by sending x to a' .
6. **Algebraically Independent 3-Amalgamation** : Given a commutative diagram in K as on the left below, where the arrows are embeddings, $\varphi_{B_i}(B_i) \cap \varphi_{A_i}(A) = \varphi_{A_i} \circ i_A(E) = \varphi_{B_i} \circ i_{B_i}(E)$

for $i = 0, 1$ and $\psi_{B_0}(B_0) \cap \psi_{B_1}(B_1) = \psi_{B_0} \circ i_{B_0}(E) = \psi_{B_1} \circ i_{B_1}(E)$, we can complete it inside of K to a commutative diagram as on the right (where the arrows are embeddings) such that $\varphi_{D_i}(D_i) \cap \varphi_B(B) = \varphi_{D_i} \circ \varphi_{B_i}(B_i) = \varphi_B \circ \psi_{B_i}(B_i)$ for $i = 0, 1$ and $\varphi_{D_0}(D_0) \cap \varphi_{D_1}(D_1) = \varphi_{D_0} \circ \varphi_{A_0}(A) = \varphi_{D_1} \circ \varphi_{A_1}(A)$ (the blank arrows of the right diagram are those of the left diagram; their names have been omitted for clarity).



Notice that we do not assume that there are countably many isomorphism classes, nor that the structures in K are finitely generated. To check condition 5. for a formula it is enough to check it for a strengthening of this formula that is also realised in K . The notation $A \oplus_{\langle \emptyset \rangle} \{x\}$ can be read as a Free Amalgam over the structure generated by the constants. The notation $\{x\}$ is here to insist on the fact that what is between the braces is considered only as a set of generator. This notation $A \oplus_{\langle \emptyset \rangle} \{D\}$ where $D \in K$ means the Generic Extension with generators indexed on D and should not be confused with $A \oplus_{\langle \emptyset \rangle} D$ which is the Free Amalgam. Also 2. implies that K is closed under unions of countable chains and implies 1.

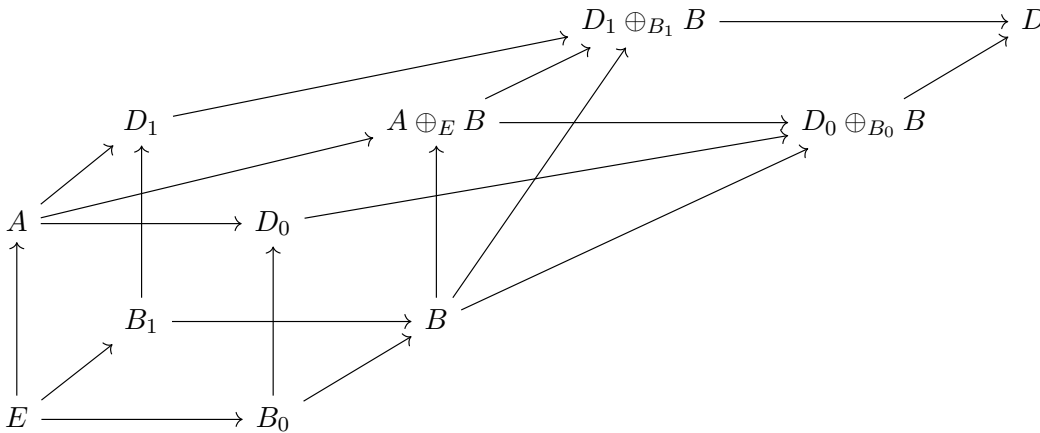
6. is a strengthening of 3-amalgamation for \downarrow^a ; the fact that the images of A and B_1 are disjoint over the image of E in D_1 should be read as 'A and B are algebraically independent inside of D_1 ' (which is written $A \downarrow_E^{a, D_1} B_1$ with the notations of [4], see this reference for some use of independence relation in categories). In the usual 3-amalgamation we would ask that D_i or B are generated by the image of the two embeddings from A and B_i or from B_0 and B_1 respectively. Here however we allow ourselves to take an extension of the generated structure which will be useful when adding structure in Section 4.2.

Some examples of classes K satisfying (H) are one based stable theories that can be presented as Fraïssé limits : The class of countable vector spaces over a finite (or countable) field (which is in the language), the class of countable abelian groups, the class of countable sets with no structure, and the class of countable graphs. In all those cases the conditions 1. – 5. are easy to check, and the condition 6. is a consequence of the following proposition.

Lemma 3.1.2. Suppose some class K satisfies 1. – 5. and that free independence and algebraic independence coincide, i. e. that for every $E \subseteq A, B \in K$ such that $A, B \subseteq D \in K$, the canonical map $\varphi : A \oplus_E B \rightarrow D$ is an embedding if and only if $A \cap B = E$ (the examples previously quoted satisfy this). Then condition 6. holds.

Proof. The hypotheses imply in particular that algebraic independence $\downarrow^{a, D}$ satisfies base monotonicity in the structures of K (it is shown later in Proposition 3.2.9 that free independence always satisfies base monotonicity, and it is clear that $\downarrow^{a, D}$ satisfies transitivity). Given E, A, B_0, B_1, D_0, D_1 and a diagram satisfying the assumptions of 6. consider the following diagram, where $D := [D_0 \oplus_{B_0} B] \oplus_{A \oplus_E B} [D_1 \oplus_{B_1} B]$. We want to show that $D_0 \cap D_1 = A$, $D_0 \cap B = B_0$ and $D_1 \cap B = B_1$ inside of D (i. e.

$D_0 \downarrow_A^{a,D} D_1$, $D_0 \downarrow_{B_0}^{a,D} B$ and $D_1 \downarrow_{B_1}^{a,D} B$). We abuse notation and write $\langle AB \rangle$ for the substructure of D generated by the images of A and B , and for substructures F, G, H of D such that $F \cap G = H$, we will indifferently write $\langle FG \rangle$ and $F \oplus_H G$ for the generated structure.



The maps from $A \oplus_E B$ to $D_i \oplus_{B_i} B$ come from the fact that A and B are free over E in $D_i \oplus_{B_i} B$: In fact if some element of $D_i \oplus_{B_i} B$ is in the image of both A and B , then it is also in the image of D_i , so it is in the image of B_i , and by assumption B_i and A are free over E in D_i . This diagram is commutative and all arrows are embeddings. If some element of D is in the image of D_i and B then it is in the image of B_i , so $D_i \downarrow_{B_i}^{a,D} B$ for $i \in \{0, 1\}$. If some element of D is in the image of D_0 and D_1 then it is in the image of $A \oplus_E B$ which is just the substructure generated by the images of A and B . Since $D_i \downarrow_{B_i}^{a,D} B$ by base monotonicity $D_i \downarrow_{\langle AB_i \rangle}^{a,D} \langle AB \rangle$, so it is enough to show that $\langle AB_0 \rangle \downarrow_A^{a,D} \langle AB_1 \rangle$. Now $A \downarrow_E^{a,D} B$ since $D_0 \downarrow_{B_0}^{a,D} B$ and $A \downarrow_E^{a,D} B_0$. By base monotonicity $\langle AB_0 \rangle \downarrow_{B_0}^{a,D} \langle B_0 B_1 \rangle$, since $B_0 \downarrow_E^{a,D} B_1$ transitivity yields $\langle AB_0 \rangle \downarrow_E^{a,D} B_1$, so $\langle AB_0 \rangle \downarrow_A^{a,D} \langle AB_1 \rangle$ by base monotonicity and 6. holds. \square

3.2 Properties of the limit theory

We begin by constructing the limit theory T of K and then we prove some of its properties. We shall assume the continuum hypothesis as we need an uncountable regular cardinal κ such that $2^{<\kappa} = \kappa$, and it is convenient to take \aleph_1 as such.

Definition 3.2.1. Given an L -structure M and a cardinal κ the κ -age of M is the class of the substructures of M of cardinal $< \kappa$.

Definition 3.2.2. Let K be a class of countable structures in some countable language L . An L -structure \mathcal{M} is K - \aleph_1 -saturated if its \aleph_1 -age is K and if for every $\mathcal{U}, \mathcal{B} \in K$ countable and any embeddings $f_0 : \mathcal{U} \rightarrow \mathcal{M}$, $f_1 : \mathcal{U} \rightarrow \mathcal{B}$ there is some embedding $g : \mathcal{B} \rightarrow \mathcal{M}$ such that $g \circ f_1 = f_0$.

Proposition 3.2.3. There exist an L -structure \mathcal{M} which is K - \aleph_1 -saturated and has cardinal \aleph_1 . Every structure of \aleph_1 -age included in K and of cardinal $\leq \aleph_1$ embeds into \mathcal{M} , and \mathcal{M} is unique up to isomorphism.

Proof. Let $(\mathcal{B}_i)_{i < \aleph_1}$ enumerate the elements of K up to isomorphism. We construct by induction an chain $(\mathcal{U}_i)_{i < \aleph_1}$ of L_P -structures in K . Let $\mathcal{U}_0 = \mathcal{B}_0$. If \mathcal{U}_i has been defined, for every $k < \aleph_1$ let $(f_j^{\mathcal{U}_i, k})_{j < \aleph_1}$ enumerate the embeddings of \mathcal{U}_i into \mathcal{B}_k . We begin by showing that we can extend a countable number of partial embeddings in K .

Claim : If $\mathcal{U} \in K$, $B_i \in K$ for $i < \omega$ and $f_i : \mathcal{U} \rightarrow B_i$ are partial embeddings, then there is an extension $\mathcal{U} \subseteq \mathcal{U}' \in K$ and embeddings $g_i : B_i \rightarrow \mathcal{U}'$ such that $g_i \circ f_i$ is the identity on its domain for all $i < \omega$.

Proof : We build an increasing sequence $(\mathcal{U}'_i)_{i < \omega}$ of extensions of \mathcal{U} . Let $E_i \subseteq \mathcal{U}$ be the domain of f_i for $i < \omega$. Consider the extensions $E_0 \subseteq \mathcal{U}$ and $f_0 : E_0 \rightarrow B_0$. By Free Amalgamation we can define $\mathcal{U}'_0 = \mathcal{U} \oplus_{E_0} B_0 \in K$, and we set g_0 to be the natural embedding $B_0 \rightarrow \mathcal{U}'_0$. Assume that \mathcal{U}'_{i-1} and g_{i-1} are defined for $i-1 < \omega$. Consider the extensions $E_i \subseteq \mathcal{U}'_{i-1}$ and $f_i : E_i \rightarrow B_i$. We can define $\mathcal{U}'_i = \mathcal{U}'_{i-1} \oplus_{E_i} B_i$, and we set g_i to be the natural embedding $B_i \rightarrow \mathcal{U}'_i$. Now let $\mathcal{U}' := \bigcup_{i < \omega} \mathcal{U}'_i$. Since K is closed under unions of countable chains this is a structure in K and it clearly satisfies the condition. \square

We can now construct the chain. Assume that \mathcal{U}_j has been defined for $j < i < \aleph_1$. Let $\mathcal{U}' := \bigcup_{j < i} \mathcal{U}_j$.

Consider the sequence of embeddings $(f_l^{\mathcal{U}_j, k})_{j, k, l < i}$ seen as partial embeddings on $\mathcal{U}' \supseteq \mathcal{U}_j$. By the claim there is an extension $\mathcal{U}_i \supseteq \mathcal{U}'$ and a sequence of embeddings $(g_l^{\mathcal{U}_j, k})_{j, k, l < i} : \mathcal{B}_k \rightarrow \mathcal{U}_i$ such that $g_l^{\mathcal{U}_j, k} \circ f_l^{\mathcal{U}_j, k}$ is the identity on its domain \mathcal{U}_j for every $j, k, l < i$. We define $\mathcal{M} := \bigcup_{i < \aleph_1} \mathcal{U}_i$.

Now we show that \mathcal{M} is K - \aleph_1 -saturated. Let \mathcal{U} be a countable structure of \mathcal{M} . By isomorphic correction we can assume that the map f_0 of the saturation property is the inclusion. Then $\mathcal{U} \subseteq \mathcal{U}_j$ for some $i < \aleph_1$ so $\mathcal{U} \in K$ by Hereditariness thus \aleph_1 -age(\mathcal{U}) $\subseteq K$. Now consider an embedding $f : \mathcal{U} \rightarrow \mathcal{B} \in K$. \mathcal{B} is isomorphic to \mathcal{B}_k for some $k < \aleph_1$ via $\varphi : \mathcal{B} \rightarrow \mathcal{B}_k$. By Free Amalgamation there is some $k' < \aleph_1$ and some embeddings $g : \mathcal{B}_k \rightarrow \mathcal{B}_{k'}$ and $f' : \mathcal{U}_j \rightarrow \mathcal{B}_{k'}$ such that $g \circ \varphi \circ f = f'$ ($\mathcal{B}_{k'}$ is isomorphic to $\mathcal{U}_i \oplus_{\mathcal{U}} \mathcal{B}_k$). Now $f' : \mathcal{U}_j \rightarrow \mathcal{B}_{k'}$ is equal to $f_l^{\mathcal{U}_j, k'}$ for some $l < \aleph_1$. Set $i = \max\{j, k, l\}$, by construction, there is a $h = g_l^{\mathcal{U}_j, k'} : \mathcal{B}_{k'} \rightarrow \mathcal{U}_{i+1}$ such that $h \circ f'$ is the inclusion $\mathcal{U}_j \subseteq \mathcal{U}_{i+1}$. Then $h \circ g \circ \varphi$ is the embedding we want.

$$\begin{array}{ccccc}
\mathcal{U}_i & \xrightarrow{\subseteq} & \mathcal{U}_{i+1} & \subseteq & \mathcal{M} \\
\uparrow & & & & \uparrow h \\
\mathcal{U}_j & \xrightarrow{f'} & \mathcal{B}_{k'} & & \\
\uparrow & & & & \uparrow g \\
\mathcal{U} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{\varphi} & \mathcal{B}_k
\end{array}$$

For the second point we proceed as in Fraïssé theory and only use the property of \aleph_1 -saturation and the fact that we can amalgamate over the structure generated by the constants : if \mathcal{M}' is a structure of age inside of K and of cardinal $\leq \aleph_1$ we fix an enumeration $(m_i)_{i < \aleph_1}$ of \mathcal{M}' and define by induction an increasing sequence of embeddings $f_i : \langle m_{< i} \rangle \rightarrow \mathcal{M}$, then the union $f = \bigcup_{i < \aleph_1} f_i : \mathcal{M} \rightarrow \mathcal{M}'$ is our

embedding. Let f_0 be the embedding of the constants from \mathcal{M}' to \mathcal{M} (possibly f_0 is the empty set). If f_j is defined for $j < i < \aleph_1$, let $f' := \bigcup_{j < i} f_j : \mathcal{B}' \subseteq \mathcal{M}' \rightarrow \mathcal{M}$. Let \mathcal{B} be the image of \mathcal{M} and $g' : \mathcal{B} \rightarrow \mathcal{B}'$

be the inverse map of f' , we can compose g' with the inclusion $\mathcal{B}' \subseteq \langle m_{< i} \rangle$. Since $\langle m_{< i} \rangle \in K$ we can use the \aleph_1 -saturation of \mathcal{M} to find a map $f_i : \langle m_{< i} \rangle \rightarrow \mathcal{M}$ such that $f_i \circ g' = id_{\mathcal{B}}$, which implies that $f_i|_{\mathcal{B}'} = f$ by pre-composing with f , i. e. that f_i extends f_j for $j < i$. In a similar fashion, to show uniqueness we construct an isomorphism by back and forth. \square

Let T be the theory of \mathcal{M} constructed in Proposition 3.2.3, we refer to T as the *limit theory* of K .

Remark 3.2.4. A consequence of the Free Amalgamation property is that $dcl(A) = acl(A) = \langle A \rangle$ for every set $A \subseteq \mathbb{M}$.

Lemma 3.2.5. T has Quantifier Elimination in L .

Proof. This is clear from the condition 3. : If we consider any \aleph_1 -saturated model it is K - \aleph_1 -saturated, so partial isomorphisms between finitely generated structures have the back and forth property. \square

Let now \mathbb{M} be a monster model of T .

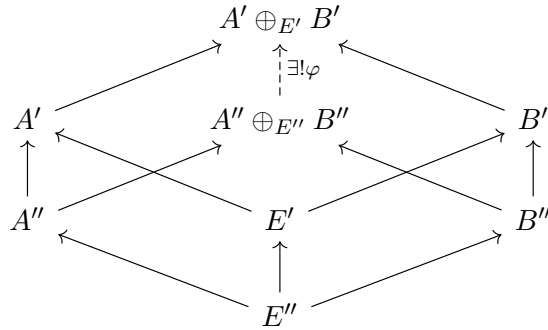
Lemma 3.2.6. The relation of algebraic independence \downarrow^a on subsets of \mathbb{M} satisfies the independence theorem : If $E, A_i, B_i \subseteq \mathbb{M}$ for $i \in \{0, 1\}$, $A_0 \equiv_E A_1$, $A_i \downarrow_E^a B_i$ for every $i \in \{0, 1\}$ and $B_0 \downarrow_E^a B_1$ there is an A such that $A \equiv_{B_i} A_i$ for $i \in \{0, 1\}$ and $A \downarrow_E^a B_0 B_1$.

Proof. Consider E, A_i, B_i for $i = 0, 1$ as in the statement, \bar{x}_A a tuple of variable indexed on A_0 (and A_1 thought the isomorphism φ induced by $A_0 \equiv_E A_1$) where x_a is the variable associated to $a \in A_0$ and $\varphi(a) \in A_1$. We extend it into $\bar{x}_A \bar{y}_0$ a tuple of variable indexed on $\langle A_0 B_0 \rangle$ and $\bar{x}_A \bar{y}_1$ a tuple of variable indexed on $\langle A B_1 \rangle$ disjoint with \bar{y}_0 . Let $q_i(\bar{x}_A \bar{y}_i) := \text{qftp}(\langle A_i B_i \rangle / B_i)$ for $i = 0, 1$. We show that $q_0(\bar{x}_A \bar{y}_0) \cup q_1(\bar{x}_A \bar{y}_1) \cup \{t(\bar{x}_a) \notin \langle B_0 B_1 \rangle\}$: For all terms t and tuples $\bar{a} \in A_0$ such that $t(\bar{a}) \notin E$ is consistent. By compactness is it enough to consider finitely generated structures E, A_i, B_i , and in that case we get consistency by just applying 6. \square

Definition 3.2.7. We define the Γ -independence relation on small subsets of the monster model : for $E \subseteq A, B$ say that $A \downarrow_E^\Gamma B$ if the structure $\langle AB \rangle$ and the inclusions $i_A : A \rightarrow \langle AB \rangle$, $i_B : B \rightarrow \langle AB \rangle$ satisfy the universal property that every common extension of A and B in \mathbb{M} factorises uniquely through $\langle AB \rangle$.

Lemma 3.2.8. Full existence : For every $E \subseteq A, B \subseteq \mathbb{M}$ such that $A \cap B = E$ there is $A' \equiv_E A$ such that $A' \downarrow_E^\Gamma B$.

Proof. We want to construct an L -structure D extending B and an embedding $i_A : A \rightarrow D$ such that for $A' := i_A(A)$, $D = \langle A'B \rangle$ and $A' \downarrow_E^\Gamma B$. For countable substructures $E' \subseteq E$, $A' \subseteq A$, $B' \subseteq B$ and $E' = A' \cap B'$ we consider $A' \oplus_{E'} B' \in K$. The D we want to build is the inductive limit of these structures, we show that they indeed form an inductive system. Let $E'' \subseteq E' \subseteq E$, $A'' \subseteq A' \subseteq A$, $B'' \subseteq B' \subseteq B$ be countable substructures such that $E' = A' \cap B'$ and $E'' = A'' \cap B''$. Then there is a canonical map from $A'' \oplus_{E''} B''$ to $A' \oplus_{E'} B'$ where $E'' \subseteq E'$, $A'' \subseteq A'$, $B'' \subseteq B'$ and $E' = A' \cap B'$. We describe it in the following diagram. This map assures us that the system of the $A' \oplus_{E'} B'$ we are considering is directed and that we can define the L -structure D as its direct limit. Also $\text{Diag}(D/B)$ is consistent with T because any formula that appears in it is satisfied in some of the $A' \oplus_{E'} B' \in K$



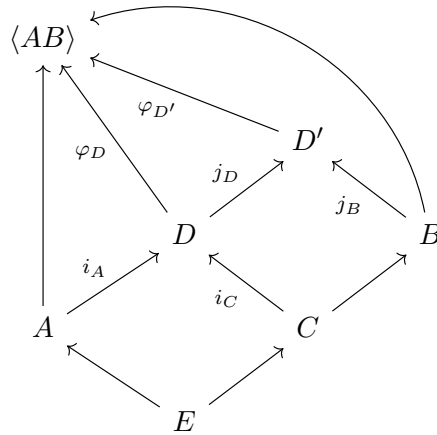
Now we show that D satisfies the right property : Let $D' \subseteq \mathbb{M}$, and $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ form a commutative square over E . For every $E'' \subseteq E' \subseteq E$, $A'' \subseteq A' \subseteq A$, $B'' \subseteq B' \subseteq B$ countable substructures such that $E' = A' \cap B'$ and $E'' = A'' \cap B''$ the restricted morphism $\varphi_{E|E''} \subseteq \varphi_{E|E'}$, $\varphi_{A|A''} \subseteq \varphi_{A|A'}$, $\varphi_{B|B''} \subseteq \varphi_{B|B'}$ factorises uniquely via $\varphi_{A'' \oplus_{E''} B''}$ and $\varphi_{A' \oplus_{E'} B'}$ respectively trough

$A'' \oplus_{E''} B''$ and $A' \oplus_{E'} B'$ respectively. By uniqueness if φ is the canonical map from $A'' \oplus_{E''} B''$ to $A' \oplus_{E'} B'$ then $\varphi_{A' \oplus_{E'} B'} \circ \varphi = \varphi_{A'' \oplus_{E''} B''}$, so the system of the $\varphi_{A' \oplus_{E'} B'}$ is also directed and we can consider its limit $\varphi_{A \oplus_E B}$, which is the map we want. \square

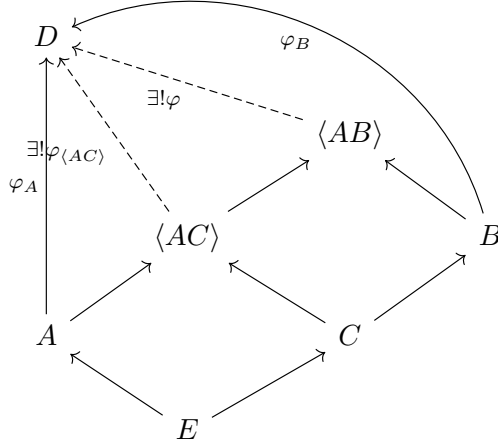
The difference with the previous free amalgam notion is that it was defined for the elements of K only, which corresponds to the countable substructures of \mathbb{M} , whereas this notion is defined for small substructures. Also since our class K is axiomatisable every countable substructure of \mathbb{M} is inside of K so there is no problem in considering the structures $A' \oplus_{E'} B'$ for any countable substructures E', A', B' of \mathbb{M} .

Proposition 3.2.9. The relation $A \downarrow_E^\Gamma B$ defined for structures $E \subseteq A, B$ is invariant, symmetric, satisfies existence, extension, monotonicity, finite character, stationarity, base monotonicity and transitivity.

Proof. By existence here we mean that $A \downarrow_E^\Gamma E$ for every extension $E \subseteq A$. The properties of symmetry, existence and the fact that it implies \downarrow^a are immediate. Stationarity is a consequence of Q. E. : Up to isomorphism over B there is a unique structure $D = A \oplus_E B$ that satisfies $A \downarrow_E^\Gamma B$, which means a unique type over B . For monotonicity assume that $A \downarrow_E^\Gamma B$ and consider $E \subseteq C \subseteq B$ a substructure (this is enough by symmetry). We consider the free amalgam D of A and C and the associated embeddings i_A, i_C . Now let us consider the free amalgam D' of D and B over C and its associated embeddings j_D, j_B . There is a unique morphism $\varphi_D : D \rightarrow \langle AB \rangle$ such that $\varphi_D \circ i_A = id_A$ and $\varphi_D \circ i_C = id_C$ and $\varphi_{D'} \circ j_B = id_B$. Since $\langle AB \rangle = A \oplus_E B$ there is a unique morphism $\psi_{D'} : D' \rightarrow \langle AB \rangle$ such that $\psi_{D'}$ restricted to A and B respectively is $j_D \circ i_A$ and j_B respectively. From this we get that $\varphi_{D'} \circ \psi_{D'} = id_{D'}$ and $\psi_{D'} \circ \varphi_{D'} = id_D$, so $\varphi_{D'}$ is an isomorphism. There is a unique morphism $\varphi_{D'} : D' \rightarrow \langle AB \rangle$ such that $\varphi_{D'} \circ j_D = \varphi_D$. So $\varphi_{D'} \circ j_D \circ i_A = id_A$, and we get that D' is generated by the images of A and B and that $\varphi_{D'}$ is an isomorphism. We know that D is generated by the images of A and C , and that it is sent to $\langle AC \rangle \subseteq \langle AB \rangle$ by j_D , so $\varphi_{D'} \circ j_D : D \rightarrow \langle AC \rangle$ is an isomorphism by a similar argument, so $A \downarrow_E^\Gamma C$.



For transitivity, given structures $E \subseteq A, C, C \subseteq B$ such that $A \downarrow_E^\Gamma C$ and $\langle AC \rangle \downarrow_C^\Gamma B$ we want to show that $A \downarrow_E^\Gamma B$. Consider a structure D and two embeddings φ_A, φ_B that coincide in E from A and B respectively to D . Then the embeddings φ_A and $\varphi_B|_C$ coincide over E , and by assumption there is a unique map $\varphi_{\langle AC \rangle} : \langle AC \rangle \rightarrow D$ such that $\varphi_B|_C = \varphi_{\langle AC \rangle}|_C$ and $\varphi_A = \varphi_{\langle AC \rangle}|_A$. Now we consider the two maps $\varphi_{\langle AC \rangle}$ and φ_B which coincide over C . Since $\langle AC \rangle \downarrow_C^\Gamma B$ there is a unique map $\varphi : \langle \langle AC \rangle B \rangle = \langle AB \rangle \rightarrow D$ such that $\varphi|_B = \varphi_B$ and $\varphi|_{\langle AC \rangle} = \varphi_{\langle AC \rangle}$, so in particular $\varphi|_A = \varphi_A$. This proof consist just in saying that the square formed by two adjacent pushouts forms a pushout.



For extension and stationarity, given structures $E \subseteq A, C$, $C \subseteq B$ such that $A \downarrow_E^\Gamma C$ we can form the free amalgam of $\langle AB \rangle$ and C over B inside of \mathbb{M} thanks to Lemma 3.2.8. This allows us to find some A' such that $\langle A'C \rangle \cong \langle AC \rangle$ (since the free amalgam defines a unique isomorphism type), so by Q. E. $A' \equiv_B A$ and $\langle A'B \rangle \downarrow_B^\Gamma C$, so $A' \downarrow_E^\Gamma C$ by transitivity, and this gives us a unique type over C for the same reason. Finite character is also a direct consequence of the universal property : assume that $A \not\downarrow_E^K B$, so there is a structure D and two embeddings φ_A, φ_B from A and B respectively to D that coincide in E such that there is no map $\varphi : \langle AB \rangle \rightarrow D$ satisfying $\varphi_A = \varphi|_A$ and $\varphi_B = \varphi|_B$. This is equivalent to saying that the function that to an element $t(\bar{a}, \bar{b}) \in \langle AB \rangle$ associates $t(\varphi_A(\bar{a}), \varphi_B(\bar{b}))$ is not well defined (since this function is the only candidate) or that it is not a morphism. In either of those cases we can find some finite tuples $\bar{a} \in A, \bar{b} \in B$ such that the same thing is happening with the function that to an element $t(\bar{a}', \bar{b}') \in \langle E\bar{a}\bar{b} \rangle$ associates $t(\varphi_A(\bar{a}'), \varphi_B(\bar{b}'))$, so $\bar{a} \not\downarrow_E^K \bar{b}$.

For base monotonicity, given structures $E \subseteq A, C$ and $C \subseteq B$ such that $A \downarrow_E^\Gamma B$ we want to show that $\langle AC \rangle \downarrow_C^\Gamma B$. We take the same notations as for transitivity. As previously $A \downarrow_E^\Gamma C$ (by monotonicity this time). Consider a structure D and two embeddings $\varphi_{\langle AC \rangle}, \varphi_B$ from $\langle AC \rangle$ and B respectively to D that coincide on C . Then the embeddings $\varphi_{\langle AC \rangle}|_A$ and φ_B coincide over E . By assumption there is a unique map $\varphi : \langle AB \rangle = \langle \langle AC \rangle B \rangle \rightarrow D$ such that $\varphi|_A = \varphi_{\langle AC \rangle}|_A$ and $\varphi|_B = \varphi_B$. Then we also have that $\varphi|_C = \varphi_B|_C = \varphi_{\langle AC \rangle}|_C$, so $\varphi|_{\langle AC \rangle} = \varphi_{\langle AC \rangle}$ since $A \downarrow_E^\Gamma C$. This proof consists just in saying that if the square formed by two adjacent squares is a pushout and the left one also is then the right one is also a pushout. \square

Corollary 3.2.10. For every $E \subseteq A, B$ if $A \downarrow_E^\Gamma B$ then $A \downarrow_E^f B$, so T has existence. Also T is NSOP₁ and for every $E \subseteq A, B$ we have $A \downarrow_E^K B$ if and only if $A \downarrow_E^a B$.

Proof. We show that $A \downarrow_E^\Gamma B$ implies $A \downarrow_E^d B$. Assume that $A \downarrow_E^\Gamma B$ and let $(B_i)_{i < \omega}$ be an E -indiscernible sequence with $B_0 = B$. By extension we can find $A' \equiv_{B_0} A$ such that $A' \downarrow_{B_0}^\Gamma B_{<\omega}$. By transitivity and invariance $A' \downarrow_E^\Gamma B_{<\omega}$. By indiscernibility $B_i \equiv_E B_0$ for every $i < \omega$. Since $B_0 \downarrow_E^\Gamma A'$ and $B_i \downarrow_E^\Gamma A'$ it follows from stationarity that $A'B_i \equiv_E A'B_0 \equiv_E AB_0$ for every $i < \omega$. So $A \downarrow_E^d B$. Since \downarrow^Γ satisfies extension we get that if $A \downarrow_E^\Gamma B$ then $A \downarrow_E^f B$, and existence for \downarrow^f follows from existence for \downarrow^Γ .

Now that we know that T has existence we can consider Kim-independence over arbitrary sets. We use the Kim-Pillay theorem for Kim-forking [11, Theorem 5.1]. By [23, Lemma 2.7] \downarrow^a satisfies extension, existence, monotonicity, symmetry, strong finite character, and witnessing. By Lemma 3.2.6 it satisfies the amalgamation theorem, and it is clear that it satisfies transitivity. We now show that it satisfies the local character condition. Let \bar{a} be a finite tuple, κ an uncountable regular cardinal, $(A_i)_{i < \kappa}$ a

continuous increasing sequence of structures of cardinal smaller than κ and $A_\kappa = \bigcup_{i < \kappa} A_i$. We want to show that there is an $i < \kappa$ such that $\langle A_i \bar{a} \rangle \downarrow_{A_i}^a A_\kappa$. We choose some $i_0 < \kappa$. If $\langle A_{i_0} \bar{a} \rangle \not\downarrow_{A_{i_0}}^a A_\kappa$ the set $\langle A_{i_0} \bar{a} \rangle \cap A_\kappa$ is non empty and has cardinal smaller than κ , so it is contained in A_{i_1} for some $i_1 > i_0$. We iterate this process to define a sequence $(i_j)_{j < \omega}$ and set $i = \bigcup_{j < \omega} i_j$ which is smaller than κ by regularity. Now if $a \in \langle A_i \bar{a} \rangle \cup A_\kappa$ it is in $\langle A_{i_j} \bar{a} \rangle \cup A_\kappa$ for some $j < \omega$ so by construction it is inside of $A_{i_{j+1}} \subseteq A_i$. \square

Remark 3.2.11. In the case where the limit T of the class K is a stable theory $\downarrow^K = \downarrow^f = \downarrow^a$ and $\downarrow^a = \downarrow^\Gamma$ since both of these relations are stationary and satisfy existence. Also the Kim-Pillay criterion for simple theories applied to the stationary relation \downarrow^Γ tells us that \downarrow^Γ satisfies local character if and only if T is stable.

Remark 3.2.12. By [21, Theorem 2.7] we know that when applying the amalgamation theorem to E, A_0, A_1, B_0, B_1 (as in the statement of the amalgamation theorem) for \downarrow^K in an NSOP₁ theory with existence we can take A such that $AB_i \downarrow_{B_i} B_0 B_1$ for $i = 0, 1$. In our setting by Lemma 3.2.6 and (Algebraically Independent 3-Amalgamation) we also have $AB_0 \downarrow_A^K AB_1$. However I do not know if this is true in every NSOP₁ theory with existence, see Section 7.2 later on this topic. Also notice that in order to get this result we only need 1. – 5. and Lemma 3.2.6 to hold and not the full generality of Algebraically Independent 3-Amalgamation.

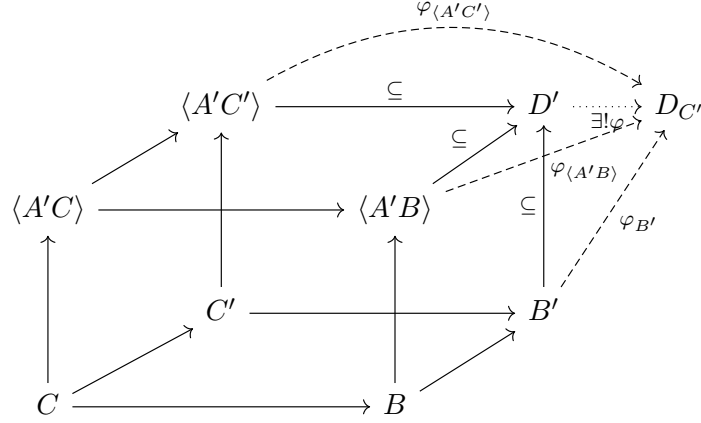
Remark 3.2.13. The proof of Corollary 3.2.10 can be generalised to show that in any theory T with some stationary independence relation \downarrow satisfying Proposition 3.2.9 and such that \downarrow^a satisfies the independence theorem over arbitrary sets, \downarrow implies \downarrow^f , so that T is NSOP₁, has existence and $\downarrow^a = \downarrow^K$ over arbitrary sets.

The following proposition is a general fact for theories such that algebraic independence satisfies the amalgamation theorem, a proof can be found in [23, Proposition 3.22]. This fact is part of some more general results about deducing weak elimination of imaginaries from the properties of some independence relations (see [14, Section 4.4] for example).

Proposition 3.2.14. The theory T has weak elimination of imaginaries.

Proposition 3.2.15. For any $E \subseteq A, B$ we have $A \downarrow_E^f B$ iff $A \downarrow_E^d B$ iff $A \downarrow_E^M B$.

Proof. We begin by showing that in \mathbb{M} the relation \downarrow^M satisfies extension. Consider $E \subseteq A, B \subseteq \mathbb{M}$ such that $A \downarrow_E^M B$ and $B \subseteq B'$. We can find A' such that $A' \equiv_B A$ and $\langle A'B \rangle \downarrow_B^\Gamma B'$, let us write $D' = \langle A'B' \rangle$. We show that $A' \downarrow_E^M B'$. Consider $C' \subseteq B'$ and define $C = C' \cap B$, we want to show that $\langle A'C' \rangle \cap B' = C'$. By 6, since $\langle A'C \rangle \cap C' = C$, $\langle A'C \rangle \cap B = C$ (because $A \downarrow_E^M B$) and $B \cap C' = C$, the following diagram with the solid arrows, where $\langle A'C' \rangle$ is the structure generated by A' and C' in D' , can be completed with the dashed arrows and a structure $D_{C'}$ such that $\varphi_{B'}(B') \cap \varphi_{\langle A'C' \rangle}(\langle A'C' \rangle) = \varphi_{B'}(C') = \varphi_{\langle A'C' \rangle}(C')$.



There is a unique map φ such that $\varphi|_{\langle A'B \rangle} = \varphi_{\langle A'B \rangle}$ and $\varphi|_{B'} = \varphi_{B'}$. The two maps $\varphi_{\langle A'C' \rangle}$ and $\varphi|_{\langle A'C' \rangle}$ coincide on the sets C' and $\langle A'B \rangle$, and these sets generate $\langle A'C' \rangle$, so $\varphi|_{\langle A'C' \rangle} = \varphi_{\langle A'C' \rangle}$. If $t(a', c') = t'(b')$ for two terms $t, t', a' \in A', c' \in C'$ and $b' \in B'$ inside of D' then by applying φ we get that $\varphi_{\langle A'C' \rangle}(t(a', c')) = \varphi_{B'}(t'(b'))$, so by 6. there is a $c \in C'$ such that $\varphi_{\langle A'C' \rangle}(c) = \varphi_{\langle A'C' \rangle}(t(a', c'))$ and $\varphi_{B'}(c) = \varphi_{B'}(t'(b'))$. Since the maps $\varphi_{B'}$ and $\varphi_{\langle A'C' \rangle}$ are embeddings we get that $t(a', c') = t'(b') \in B'$, so $\langle A'C' \rangle \cup B' = C'$. So \downarrow^M satisfies extension.

It is always true that $\downarrow^d \implies \downarrow^M$. To show that $\downarrow^M \implies \downarrow^f$ it is sufficient to show that $\downarrow^M \implies \downarrow^d$ by extension. For this assume that $A \downarrow_E^M B_0$ and consider an indiscernible sequence $I = (B_i)_{i < \omega}$. We begin by showing that we can assume that $B_i \downarrow_E^a B_{<i}$ for every $i < \omega$, then the proof is straightforward using Lemma 3.2.6. Begin by adding a copy of ω at the beginning of the sequence $(B_k)_{k \in \omega + \omega} = I' \frown I$, by extension we can assume that $A \downarrow_E^M \langle (B_k)_{k \in \omega + \omega} \rangle$. Now set $E' = \langle (B_k)_{k < 0} \rangle$ the structure generated by I' . Consider the structures $B'_i = \langle E' B_i \rangle$ for $i < \omega$. They form an E' -indiscernible sequence. Define $A' = \langle E' A \rangle$, by base monotonicity $A' \downarrow_{E'}^M \langle (B'_i)_{i < \omega} \rangle$. Now, if $b_i \in B'_i$ is also in $\langle B'_{<i} \rangle$ then it has to be in E' by indiscernibility of $(B_k)_{k \in \omega + \omega}$ (this is the point of adding the negative part : algebraic relations happening in the sequence are already happening inside the negative part). So, replacing A by A' , E by E' and $(B_i)_{i < \omega}$ by $(B'_i)_{i < \omega}$ we get the hypothesis we wanted.

Now assume that $B_i \downarrow_E^a B_{<i}$ for every $i < \omega$. Since $B_0 \equiv_E B_i$ there is some A_i such that $AB_0 \equiv_E A_i B_i$. We define by induction on $i < \omega$ a sequence of structures A'_i such that $A'_i \downarrow_E^a B_{\leq i}$ and for all $j \leq i$ $AB_0 \equiv_E A'_j B_j$. Let $A_0 = A'_0 = A$. If A'_i has been defined then $A'_i \downarrow_E^a B_{\leq i}$, $A_{i+1} \downarrow_E^a B_{i+1}$, $B_{i+1} \downarrow_E^a B_{\leq i}$ and $A'_i \equiv_E A_i$, so by Lemma 3.2.6 we can find a suitable A'_{i+1} . By compactness there is some A'' such that $A'' B_i \equiv_E AB_0$ for all $i < \omega$. So if $A \downarrow_E^M B$ then $A \downarrow_E^d B$, by extension $A \downarrow_E^f B$ also holds, so these three notions are equivalent. \square

Remark 3.2.16. We get from the previous proposition that in T thorn-forking coincides with forking (see [3] for an introduction to thorn forking in terms of independence relations), so rosiness of T (which is defined as thorn-forking satisfying local character) coincides with simplicity of T .

In the previous proposition we insist on the fact that we are considering two algebraically closed sets A, B containing the basis E ; the reason for this is that in NSOP_1 theories it is not always true that $\downarrow^f = \downarrow^d$ over arbitrary sets. A counterexample is given in [15, Section 3.2] : if T_f^\emptyset is the model completion of the empty theory in a language containing only a binary function and T_f^{eq} its imaginary expansion, then they give some real tuple a , small model M and some unordered pair d such that $a \downarrow_M^d Md$ and $a \not\downarrow_M^d a \text{acl}^{eq}(Md)$.

4 The parameterized class

In this section we consider a Fraïssé class K_0 in a language L_0 satisfying (H) , its limit theory T_0 and a second Fraïssé class K_P in a language L_P satisfying (H) which is one sorted and its limit theory T_P . We show that we can 'parameterize' the class K_0 by K_P to get a new class K of structures in a new language that also satisfies (H) , and that in some cases the limit theory of K is not simple, and so strictly NSOP₁ by Corollary 3.2.10.

Let \mathbb{M}_0 and \mathbb{M}_P be monster models of T_0 and T_P respectively. By the previous section we know that T_0 and T_P have Q.E., satisfy $acl(A) = dcl(A) = \langle A \rangle$ and that \perp^a satisfies the independence theorem over arbitrary sets, which implies that T_0 and T_P are NSOP₁ and that $\perp^K = \perp^a$ over arbitrary sets in \mathbb{M}_0 and \mathbb{M}_P .

Definition 4.1. Let L be a new language built from L_0 by adding a new sort P equipped with an L_P structure. We will refer to P as the parameter sort, we will refer to the sorts of L_0 as the object sorts, denoted O . We replace every symbol of constant c in L_0 of sort O by a function symbol $\bar{c} : P \rightarrow O$, every function symbol $f : \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ by a function symbol $\bar{f} : P \times \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ and every relation symbol $R \subseteq \prod_{i < n} O_i$ by a relation symbol $\bar{R} \subseteq P \times \prod_{i < n} O_i$. For an L -structure A and $p \in P(A)$ we write A_p to denote the L_0 -structure induced by p on $O(A)$. For a set of objects A and a tuple of parameters \bar{p} we will write $\langle A \rangle_{\bar{p}}$ for the structure generated by (A, \bar{p}) , we will also write $\langle A \rangle$ for the structure generated by a set A (possibly containing parameters and objects), so $\langle A \rangle = \langle O(A) \rangle_{\langle P(A) \rangle}$.

Definition 4.2. Let K be the class of countable L -structures A such that $A_p \in K_0$ for every $p \in P(A)$ and $P(A) \in K_P$.

Let us fix some notations : We will write $A \oplus_{\langle \emptyset \rangle} \{\bar{x}\}$ for the L_0 -structure freely generated over $A \subseteq \mathbb{M}_0$ by a tuple \bar{x} of distinct elements in O . Such an extension exists by Generic Extension. For a symbol of function $f \in L$ we will sometimes write $f_p(\bar{x})$ instead of $\bar{f}(p, \bar{x})$, and similarly for relation symbols. For an L_0 -formula φ and a parameter p we will write φ_p for the L -formula parameterized by p .

Remark 4.3. We will most often consider the case where L_P is the empty language and so K_P is the class of countable sets. It might be easier to read the proofs with the case $K_P = =$ in mind.

Remark 4.4. The class K is axiomatisable : if an L -structure A is not in K then either for some parameter $p \in P(A)$ there is some $p \in A$ such that $A_p \notin K_0$, so there is an L_0 -formula φ such that $A_p \models \varphi$ and such that no structures in K_0 satisfy φ . So $A \models \varphi_p$ and no structure in K satisfy this formula. Else $P(A) \notin K_P$, which is then also expressed by a formula by condition 2. in K_P . Also K clearly satisfies the Heredity Property.

Lemma 4.5. The class K satisfies Free Amalgamation.

Proof. Consider some structures $E \subseteq A, B \in K$. We construct an L -structure in K satisfying the Free Amalgamation property. Let us write $E = (O(E), P(E))$, $A = (O_A, P_A)$ and $B = (O_B, P_B)$. We begin by forming the free amalgam $P(D) := P(A) \oplus_{P(E)} P(B) \in K_P$ which will be the parameter sort of the free amalgam in K .

Let $(p_i)_{i < \omega}$ enumerate $P(D)$. We define a sequence $((D_{p_i}^k)_{i < \omega})_{k < \omega}$ of tuples of countable L_0 -structures indexed by $P(D)$ by induction such that the sequence $(D_p^k)_{k < \omega}$ is a chain of L_0 -structures for any $p \in P(D)$ and that for any $k < \omega$ the sequence $(D_{p_i}^k)_{i < \omega}$ is an increasing sequence of sets. Let $D_{p_{-1}}^0 := A \cup (B \setminus A)$ and let E_0 be the structure generated by the constants inside of K_0 .

We define

$$\begin{aligned}
D_{p_i}^0 &:= (A_{p_i} \oplus_{E_{p_i}} B_{p_i}) \oplus_{\langle \emptyset \rangle} \{D_{p_{i-1}}^0 \setminus (D_{-1}^0)\} \text{ if } p_i \in P(E), \\
D_{p_i}^0 &:= A_{p_i} \oplus_{\langle \emptyset \rangle} \{D_{p_{i-1}}^0 \setminus A\} \text{ if } p_i \in P(A) \setminus P(E), \\
D_{p_i}^0 &:= B_{p_i} \oplus_{\langle \emptyset \rangle} \{D_{p_{i-1}}^0 \setminus B\} \text{ if } p_i \in P(B) \setminus P(E), \\
D_{p_i}^0 &:= E_0 \oplus_{\langle \emptyset \rangle} \{D_{p_{i-1}}^0\} \text{ if } p_i \in P(D) \setminus (P(A) \cup P(B)).
\end{aligned}$$

Set $D^0 = \bigcup_{i < \omega} D_{p_i}^0$. If D^k is defined for some $1 \leq k < \omega$ we set $D_{p_0}^{k+1} = D_{p_0}^k \oplus_{\langle \emptyset \rangle} \{D^k \setminus D_{p_0}^k\}$, similarly if $D_{p_i}^k$ is defined for some $1 \leq i < \omega$ we set $D_{p_{i+1}}^{k+1} = D_{p_{i+1}}^k \oplus_{\langle \emptyset \rangle} \{D_i^{k+1} \setminus D_{p_i}^k\}$.

Every D_p^k is an L_0 -structure for the parameter p . We set $O(D) = \bigcup_{k < \omega} D^k$, this set is countable and has an L_0 -structure for the parameter $p \in P(D)$ given by $D_p = \bigcup_{k < \omega} D_p^k \in K_0$, so $D \in K$. This defines an L -structure $D = (O(D), P(D)) \in K$ which extends A and B , and such that $A \cap B = E$ inside of D . Also if A and B are generated by $(\bar{a}, P(A))$ and $(\bar{b}, P(B))$ respectively $(\overline{ab}, P(A), P(B))$ generates D .

Now that we constructed the structure D we show that it satisfies the amalgamation property. Consider $D' \in K$, $\varphi_A : A \rightarrow D'$ and $\varphi_B : B \rightarrow D'$ as in the statement, and write φ_E their restriction to E . The uniqueness of φ is clear from the fact that D is generated by the images of A and B . For existence we define φ on the parameters : By free amalgamation in K_P there is a unique morphism φ_P of K_P -structures from $P(D)$ to $P(D')$ such that the diagram commutes. We then construct an increasing sequence of tuples of morphism of L_0 -structures indexed on $P(D)$: $((\varphi_{p_i}^k : D_{p_i}^k \rightarrow D'_{\varphi_P(p_i)})_{i < \omega})_{k < \omega}$.

For $k = 0$, if $p_i \in P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_P(p_i)}$ and $\varphi_B(p, -) : B_p \rightarrow D'_{\varphi_P(p_i)}$ and sends any element $d \in D_{p_{i-1}}^0 \setminus D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$. If $p_i \in P(A) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_P(p_i)}$ and sends any element $d \in D_{p_{i-1}}^0 \setminus A$ to $\varphi_{p_{i-1}}^0(d)$, similarly if $p_i \in P(B) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that extends $\varphi_B(p, -) : B_{p_i} \rightarrow D'_{\varphi_P(p_i)}$ and sends any element $d \in D_{p_{i-1}}^0 \setminus B$ to $\varphi_{p_{i-1}}^0(d)$, and finally if $p_i \in P(D) \setminus (P(A) \cup P(B))$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$.

Assume that for every $i < \omega$ the functions $\varphi_{p_i}^k : D_{p_i}^k \rightarrow D'_{\varphi_P(p_i)}$ have been defined. Define $\varphi_{p_0}^{k+1} : D_{p_0}^{k+1} \rightarrow D'_{\varphi_P(p_0)}$ as the unique morphism extending $\varphi_{p_0}^k : D_{p_0}^k \rightarrow D'_{\varphi_P(p_0)}$ by sending $x \in D_{p_i}^k$ to $\varphi_{p_i}^k(x)$ for every $i < \omega$. We just repeat the same steps for the other parameters. Once all these functions are constructed we consider $\varphi = \bigcup_{k < \omega} \varphi_p^k$ for every $p \in P(D)$. This function defines a map in K from D to D' such that its restrictions to A and B are φ_A and φ_B respectively, and it is a morphism of L_0 -structure since $\varphi = \bigcup_{k < \omega} \varphi_p^k : D_p \rightarrow D'_{\varphi_P(p)}$ is a morphism of L_0 -structure for every parameter $p \in P(D)$ and $\varphi_P : P(D) \rightarrow P(D')$ is a morphism of L_P -structure. \square

Remark 4.6. This means that K seen as a category has pushouts for pairs of monomorphisms. The proof also implies that the construction of D does not depend (up to isomorphism) on the choices of the bases of A and B and on the enumeration of the parameters : given the same construction for a different enumeration $D^* = \bigcup_{i < \omega} D_{p_i}^{*,k}$, by the same proof D^* also has the universal property.

Remark 4.7. From the proof of Lemma 4.5 it is clear how to show that K satisfies Generic Extension. If we add an element from a sort among the object sorts, we fix some enumeration $(p_i)_{i < \omega}$ of the

parameters of the L -structure A . Begin by adding one object x in the right sort and consider the associated L_0 -structure $A_{p_0}^0 = A_{p_0} \oplus_{\langle \emptyset \rangle} \{x\}$, then $A_{p_{i+1}}^0 = A_{p_{i+1}} \oplus_{\langle \emptyset \rangle} \{A_{p_i}^0 \setminus A\}$, $A_{p_{i+1}}^{k+1} = A_{p_{i+1}}^k \oplus_{\langle \emptyset \rangle} \{A_{p_i}^{k+1} \setminus A_{p_{i+1}}^k\}$ and so on. Else if we add a parameter p we consider the L_0 -structure E_0 it generates, i. e. the structure generated by the constants in the class K_0 , and form $A \oplus_{\langle \emptyset \rangle} \langle E_0, p \rangle$ the free product in K . We can indeed send p to any parameter because any parameter agree with p on the structure generated by the constants.

A substructure of a finitely generated structure is not necessarily finitely generated : If we consider the examples of parameterized vector space over a finite field with $L_P = =$, the structure $E = \langle p_1 p_2 \rangle$ freely generated by two parameters is infinite however the substructure $(O(E), p_1)$ where we removed one parameter is not finitely generated. With the properties of K that we have proven so far we can construct the limit \mathcal{M} with Proposition 3.2.3, consider its theory T and \mathbb{M} a monster model of it.

Lemma 4.8. K satisfies condition 3. : For every quantifier-free formula $\varphi(\bar{x}, \bar{a})$ satisfied in some structure of K there is a quantifier-free formula $\psi(\bar{x})$ such that if $\bar{a} \in A \in K$ and $A \models \varphi(\bar{a})$ then there is $A \subseteq B \in K$ an extension and $\bar{b} \in B$ such that $B \models \varphi(\bar{a}, \bar{b})$.

Proof. Consider a quantifier free formula $\varphi(\bar{x}\bar{v}, \bar{y}\bar{w})$ satisfied by some $\bar{a}\bar{p}\bar{b}\bar{q} \in A \in K$ with \bar{x}, \bar{y} in the object sorts and \bar{v}, \bar{w} in the parameters. We can assume that $\varphi(\bar{x}\bar{v}, \bar{y}\bar{w})$ is of the form $\bigwedge_{i,j} E_i(f_i, g_i) \wedge \bigwedge_i \bar{R}_i(v_i, n_i, m_i) \wedge \bigwedge_i \bar{R}'_i(w_i, n'_i, m'_i) \wedge \bigwedge_i \bar{R}''_i(w'_i, n''_i, m''_i)$ where R_i, R'_i are relations symbol or negated relations symbols from L_0 , n_i, n'_i, m_i, m'_i are terms in $\bar{x}\bar{v}, \bar{y}\bar{w}$, E_i are relations symbols or negated relation symbols form L_P and w'_i, f_i, g_i are terms in \bar{v}, \bar{w} .

We consider a strengthening of φ satisfied by $\bar{a}\bar{p}\bar{b}\bar{q}$. Let $(h_i)_{i < m}$ enumerate the interpretation of all the sub-terms of the variable terms appearing in φ when we fix $\bar{v}\bar{w}$ to be $\bar{p}\bar{q}$. Let $(c_i)_{i < n}$ enumerate the interpretation of all the sub-terms of the object terms appearing in φ when we fix $\bar{x}\bar{v}\bar{y}\bar{w}$ to be $\bar{a}\bar{p}\bar{b}\bar{q}$. Let $\bar{k} = (k_i)_{i < m}$ and $\bar{z} = (z_i)_{i < n}$ be tuples of variable of the same length. Enlarging $\bar{a}\bar{p}$ we assume that it contains every term c_i and h_i such that $c_i \in \langle \bar{a}\bar{p} \rangle$ and $h_i \in \langle \bar{p} \rangle$, let $\bar{x}\bar{v}$ be the corresponding tuples of variables. We replace the tuple $\bar{b}\bar{q}$ by $\bar{b}\bar{c}\bar{q}\bar{h}$, write $\varphi = \varphi(\bar{x}\bar{v}, \bar{y}\bar{z}\bar{w}\bar{k})$. We add to φ the formulas that expresses the fact that k_i is a sub-terms, meaning that if $h_i = t_i(\bar{p}, \bar{q}, h_{<i})$ for t_i an L_P -term we add $k_i = t_i(\bar{x}, \bar{y}, k_{<i})$ to φ and replace the occurrences of $t_i(\bar{x}, \bar{y}, k_{<i})$ by occurrences of k_i . Similarly we add to φ the formulas that expresses the fact that z_i is a sub-terms, meaning that if $c_i = \bar{t}_i(\rho, \bar{a}, \bar{b}, c_{<i})$ for \bar{t}_i an L_0 -term and $\rho \in \bar{p}\bar{b}\bar{h}$ we add $z_i = \bar{t}_i(\nu, \bar{x}, \bar{y}, z_{<i})$ to φ where $\nu \in \bar{v}\bar{w}\bar{k}$ is the variable corresponding to ρ and replace the occurrences of $\bar{t}_i(\nu, \bar{x}, \bar{y}, z_{<i})$ by occurrences of z_i . After this the object terms that appears in φ only use one parameter.

For every parameter variable $\nu \in \bar{v}\bar{w}\bar{k}$ consider the L_0 -formula $\varphi_\nu(\bar{x}, \bar{y}\bar{z})$ that specifies every equality $z_j = \bar{t}_j(\nu, \bar{x}, \bar{y}, z_{<j})$ (we consider the previous equalities and only mention those where the parameter ν is used), the relations $R_{j,\nu}$ that holds between the elements of $\bar{a}\bar{b}\bar{c}$ (so here we look at the L_0 -structure induced by ν), and we include the instances of equality and inequality among these relations.

We consider the formula φ_P that specifies every equality $k_i = t_i(\bar{x}, \bar{y}, k_{<i})$ (we consider the previous equalities and only mention those in the parameter sort), the relations E_j that holds between the elements of $\bar{p}\bar{p}\bar{h}$, and we include the instances of equality and inequality among these relations. We can write the formula φ as $\varphi(\bar{x}\bar{v}, \bar{y}\bar{z}\bar{w}\bar{k}) = \varphi_P(\bar{v}, \bar{w}\bar{k}) \wedge \bigwedge_{\nu \in \bar{v}\bar{w}\bar{k}} \varphi_\nu(\bar{x}, \bar{y}\bar{z})$.

We can consider ψ_P which is the L_P -formula associated to φ_P by 3. in K_P and ψ_ν the L_0 -formula associated to φ_ν by 3. in K_0 for every $\nu \in \bar{v}\bar{w}\bar{k}$. Let $\psi(\bar{x}, \bar{v}) := \psi_P(\bar{v}) \wedge \bigwedge_{\nu \in \bar{v}\bar{w}\bar{k}} \varphi_\nu(\bar{x})$. It satisfies the condition, and we conclude as in Lemma 4.8 : If $\psi(\bar{x}, \bar{v})$ is satisfied by $\bar{a}'\bar{p}' \in A' \in K$ we can extend $P(A)$ in K_P thanks to ψ_P , choose an extension for every parameter variable $\bar{v}\bar{w}\bar{k}$, and amalgamate these extension in an L -structure extending A . \square

Proposition 4.9. T proves the following statements :

1. If \bar{a} is a n -tuple of objects, $\pi_i(\bar{x})$ is a n -type in T_0 for every $i \leq n$ such that $\pi_i \vdash x_j \neq x_{j'}$ for $j \neq j'$ and $q(\bar{v})$ in a type in T_P , then there exist an infinite number of tuples of parameters $(p_i)_{i \leq n}$ such that $\models \pi_i(\bar{a}_{p_i})$ for every $i \leq n$ and $\models q((p_i)_{i \leq n})$.
2. If $\bar{p} = p_0, \dots, p_n$ is a n -tuple of parameters and $\pi_i(\bar{x}, \bar{b}_{p_i})$ is a type in \mathcal{M}_{p_i} over a n -tuple of objects \bar{b} such that the induced type in any of the x_i is non-algebraic and that $\pi_i \vdash x_j \neq x_{j'}$ for $j \neq j'$ then there is a tuple of objects \bar{a} such that $\models \pi_i(\bar{a}_{p_i}, \bar{b}_{p_i})$ for every $i \leq n$.

Proof. Consider the limit \mathcal{M} of K built in Proposition 3.2.3, a finite tuple of objects $\bar{a} \in \mathcal{M}$ and a finite set of types $\pi_i(\bar{x})$ as in the first point of the statement. We can build a structure $A \in K$ containing \bar{a} and with parameters $(p_i)_{i \leq n}$ such that $\models \pi_i(\bar{a}_{p_i})$ for every $i \leq n$ and $\models q((p_i)_{i \leq n})$. Then by saturation we can embed this structure A inside of \mathcal{M} over \bar{a} and find the parameters. The second point is similar : consider the structure B generated by \bar{b}, \bar{p} and types $\pi_i(\bar{x}, \bar{b}_{p_i})$ as in the statement. We can extend B into $B' \in K$ that contains a tuple of objects \bar{a}' such that $\bar{a}'\bar{b}$ has the right type for the parameter p_i for every i and then embed B' into \mathcal{M} over B to find the right objects. \square

Definition 4.10. A formula $\varphi(x, y)$ has the *independence property*, I.P. for short, if there is $(a_i)_{i < \omega}$ and $(b_I)_{I \subseteq \omega}$ two sequences of tuples such that $\models \varphi(a_i, b_I)$ if and only if $i \in I$. A theory T has I.P. if some formula in it has I.P. .

Corollary 4.11. The theory T has I.P. if there are two distinct non-algebraic types in T_0 , and is non-simple if T_0 is non-trivial, in the sense that there are $a, b \in \mathbb{M}_0$ such that $\langle a, b \rangle \setminus (\langle a \rangle \cup \langle b \rangle) \neq \emptyset$.

Proof. For I.P. consider two distinct T_0 -types π_i in \bar{x} for $i = 0, 1$. Now if we consider a sequence $(p_i)_{i < \omega}$ of distinct parameters in \mathbb{M} , then by 4.9 for every $I \subseteq \omega$ we can find \bar{a}_I such that $\models \pi_{0, p_i}(\bar{a}_I)$ if and only if $i \in I$.

Now for simplicity let $t(x, y)$ be a L_0 -term such that $t(a, b) \notin \langle a \rangle \cup \langle b \rangle$ and consider a tree of parameters all distinct $(p_\mu)_{\nu \in \omega < \omega}$. Lets now consider a tree of distinct objects $(a_\mu)_{\mu \in \omega < \omega}$ and the formulas $\varphi(x, y, a_{\mu \smallfrown i}, p_\mu) := t_{p_\mu}(x, y) = a_{\mu \smallfrown i}$ along this tree. Clearly $\{\varphi(x, y, a_{\mu \smallfrown i}, p_\mu) : i < \omega\}$ is 2-inconsistent for every $\mu \in \omega^{< \omega}$, and by 4.9 every path is consistent. \square

Remark 4.12. The relation \downarrow^Γ in \mathbb{M} does not necessarily satisfy local character (otherwise it would satisfy all conditions of the Kim-Pillay theorem) : Take some distinct objects $\bar{o} = o_1, o_2, o_3$ and a set $P = (p_i)_{i < \kappa}$ of parameters for $\kappa = |T_P|^+$ such that $o_3 \in \langle o_1 o_2 \rangle_{p_i}$ for every $i < \kappa$ and $o_j \notin \langle P \rangle$ for every $j \in \{0, 1, 2\}$. Then there is no subset P_0 of cardinal $|T|$ of P such that $\bar{o} \downarrow_{P_0}^\Gamma P$.

Remark 4.13. In the case where we can find in K_0 some $c = t(a, b) \in \langle ab \rangle \setminus (\langle a \rangle \cup \langle b \rangle)$ the relation \downarrow^a in \mathbb{M} does not admit 4-amalgamation : Take 3 distinct parameters p_0, p_1, p_2 and distinct objects b_0, b_1, b_2 such that $b_i, b_j \notin \langle p_k \rangle$ for every i, j, k such that $\{i, j, k\} = \{0, 1, 2\}$ and such that $t_{p_0}(b_1, b_2) = t_{p_1}(b_0, b_2)$, $t_{p_1}(b_0, b_2) = t_{p_2}(b_0, b_1)$ and $t_{p_0}(b_1, b_2) \neq t_{p_2}(b_1, b_2)$. Then the type $\text{tp}(p_0/b_1 b_2) \cup \text{tp}(p_1/b_0 b_2) \cup \text{tp}(p_2/b_0 b_1)$ is inconsistent.

Since the theory T_0 has Q.E., Kim-independence and algebraic independence coincide, and since the algebraic closure and the generated structure coincide in \mathbb{M}_0 we can represent non-Kim-forking extensions as commutative squares in the following way : The arrows are embeddings, the structure D is generated by the images $f(A)$ and $g(B)$, and $f(A) \cap g(B) = f \circ j(E) = g \circ i(E)$. Then the isomorphism type of D over B gives us a type over D which corresponds to a non-forking extension of $\text{tp}(A/E)$. We will refer to such squares as strong amalgams. However for the (Algebraically Independent 3-Amalgamation) the squares we are considering consist in an extension of such a square, meaning that we compose with an embedding from D to some $D' \in K_0$.

$$\begin{array}{ccc}
A & \xrightarrow{f} & D \\
\uparrow j & & \uparrow g \\
E & \xrightarrow{i} & B
\end{array}$$

Proposition 4.14. K satisfies Algebraically Independent 3-Amalgamation.

Proof. Consider the following diagram in K , which is as in the definition of (Algebraically Independent 3-Amalgamation) and where for simplicity we take the embeddings to be inclusions. We want to complete it. First by condition 6. in K_P we can complete the parameter part of this diagram with an L_P -structure $P(D) \in K_P$ which is going to be the parameter sort of our structure D . After this we begin by choosing a completion for every parameter. Fix an enumeration $(p_i)_{i < \omega}$ of $P(D)$. Looking at the L_0 -structure indexed by any of the parameters $p_j \in P(E)$ we can find an L_0 -structure $D_{p_j}^{*,0}$ as in the right diagram. We begin by filling the cube for the L_0 structures induced by each of the parameters.

$$\begin{array}{ccc}
& & D_1 \\
& \nearrow \varphi_{A_1} & \uparrow \\
A & \xrightarrow{\varphi_{A_0}} & D_0 \\
& \searrow \varphi_{B_0} & \uparrow \\
& & B_1 \xrightarrow{\psi_{B_1}} B \\
\uparrow i_A & & \uparrow \psi_{B_0} \\
E & \xrightarrow{i_{B_0}} & B_0 \\
& \nearrow i_{B_1} & \\
& &
\end{array}
\qquad
\begin{array}{ccc}
& & D_{1,p_j} \xrightarrow{\varphi_{D_1}} D_{p_j}^{*,0} \\
& \nearrow & \uparrow \varphi_{D_0} \\
A_{p_j} & \xrightarrow{} & D_{0,p_j} \\
& \searrow & \uparrow \\
& & B_{1,p_j} \xrightarrow{} B_{p_j} \\
\uparrow & & \uparrow \varphi_B \\
E_{p_j} & \xrightarrow{} & B_{0,p_j} \\
& \nearrow & \\
& &
\end{array}$$

These sets $D_{p_j}^{*,0}$ for $p_j \in P(E)$ are also equipped with the L -structure on the sets $\varphi_B(B)$ and $\varphi_{D_i}(D_i)$ for $i = 0, 1$ induced by the embeddings. We construct the L_0 -structure $D_{p_j}^{*,0}$ for $j \in P(D) \setminus P(E)$ and consider the natural mappings from D_0 , D_1 and B to $D_{p_j}^{*,0}$.

$$\begin{aligned}
&\text{If } p_j \in P(A) \setminus P(E) \text{ set } D_{p_j}^{*,0} = [D_{0,p_j} \oplus_{A_{p_j}} D_{1,p_j}] \oplus_{\langle \emptyset \rangle} \{B \setminus (B_0 \cup B_1)\} \\
&\text{If } p_j \in P(B_i) \setminus P(E) \text{ set } D_{p_j}^{*,0} = [D_0 \oplus_{B_0} B] \oplus_{\langle \emptyset \rangle} \{D_1 \setminus (A \cup B_1)\} \\
&\text{If } p_j \in P(D_0) \setminus (P(A) \cup P(B_0)) \text{ set } D_{p_j}^{*,0} = D_{0,p_j} \oplus_{\langle \emptyset \rangle} \{(B \setminus B_0) \cup (D_1 \setminus A)\} \\
&\text{If } p_j \in P(B) \setminus (P(B_0) \cup P(B_1)) \text{ set } D_{p_j}^{*,0} = B_{p_j} \oplus_{\langle \emptyset \rangle} \{(D_0 \setminus B_0) \cup (D_1 \setminus B_1)\} \\
&\text{If } p_j \in P(D) \setminus (P(D_0 \cup P(D_1) \cup P(B))) \text{ set } D_{p_j}^{*,0} = E_0 \oplus_{\langle \emptyset \rangle} \{(D_0 \cup D_1 \cup B)\}
\end{aligned}$$

In this last union we assimilate elements that have the same predecessors in the diagram. Now that we have all of our L_0 -structures we will just put them together in a canonical way : we define an increasing sequence of tuples of L_0 -structures $((D_{p_j}^k)_{j < \omega})_{k < \omega}$. Let $D_{p_0}^0 = D_{p_0}^{*,0}$, if $D_{p_j}^0$ has been defined for some $j < \omega$ let $D_{p_{j+1}}^0 = D_{p_{j+1}}^{*,0} \oplus_{\langle \emptyset \rangle} \langle D_{p_j}^0 \setminus (D_0 \cup D_1 \cup B) \rangle$ where we assimilate D_0 , D_1 and B with their image in $D_{p_j}^0$. Once $D_{p_j}^0$ has been defined for every $j < \omega$ we set $D^0 = \bigcup_{i < \omega} D_{p_j}^0$. If D^k is defined we let $D_{p_0}^{k+1} = D_{p_0}^k \oplus_{\langle \emptyset \rangle} \langle D^k \setminus D_{p_0}^k \rangle$. Now assume that $D_{p_j}^{k+1}$ is defined for some $k < \omega, j < \omega$, we set $D_{p_{j+1}}^{k+1} = D_{p_{j+1}}^k \oplus_{\langle \emptyset \rangle} \langle D_{p_j}^{k+1} \setminus D_{p_{j+1}}^k \rangle$. Then take $D = \bigcup_{k < \omega} D_{p_j}^k$ with the L_0 -structure induced for every $j < \omega$, this defines an L -structure, the natural mappings we had from D_0 , D_1 and B to D are embeddings for the L -structure we put on D and they satisfy the conditions. \square

Finally we can state that the class K satisfies the conditions of (H) . By Corollary 3.2.10 we know that T is an NSOP₁ theory with existence such that $\downarrow^K = \downarrow^a$ over arbitrary sets.

5 Examples and non-examples

Let K be a class of countable L -structures satisfying (H) . We describe now how adding generic structure to the class K might affect the conditions (H) and some properties of the theory.

5.1 Adding a generic predicate

Definition 5.1.1. Consider a new symbol U for a unary predicate in some sort S , we define $L_U = L \cup \{U\}$ and the class K_U of L_U -structures consisting of a structure A in K equipped with a subset $U(A) \subseteq S(A)$.

Proposition 5.1.2. The class K_U satisfies (H) .

Proof. The conditions 1., 2. are satisfied in K_U . For condition 4. we just take $U(A \oplus_E B) := U(A) \cup U(B)$ where $A \oplus_E B$ is the free amalgam in K_0 . For condition 5. set $U(A \oplus_{(\emptyset)} \langle x \rangle) := U(A)$. Similarly for condition 6. we find an amalgam D using condition 6. in K and set $U(D)$ to be $U(B_0) \cup U(B_1) \cup U(B)$. For condition 3. consider a finitely generated structure $\langle a, b \rangle \in K_U$ and a quantifier free L_U formula φ such that $\models \varphi(a, b)$. W. l. o. g. we can assume that $\varphi(x, y) = \varphi'(x, y) \wedge \bigwedge_{i < n} U(t_i(x, y)) \wedge \bigwedge_{i < n} \neg U(t'_i(x, y))$ where $\varphi'(x, y)$ is an L_0 -formula and the t_i and t'_i are L_0 -terms. By taking a strengthening of φ we can assume that $\varphi'(x, y) \vdash \bigwedge_{j < n} t'_i(x, y) \neq t_j(x, y)$ for every $i < n$. Now we consider the formula $\psi'(x)$ associated to the L_0 -formula $\varphi'(x, y) \wedge \bigwedge_{i, j < n} (t'_i(x, y) \neq t_j(x, y))$ by condition 5. in K_0 . If $A' \models \psi(a')$ for some $a' \in A' \in K_R$ then by assumption we can find some $b' \in B'$ an extension of A' in K_0 such that $B' \models \varphi'(a', b') \wedge \bigwedge_{i < n} (t'_i(a', b') \notin \langle t_j(a', b') \rangle_{j < n})$. Define the subset U on B' to be $U(A') \cup \{t_j(a', b')\}_{j < n}$, we get that $B' \models \neg U(t'_i(a', b'))$ for all $i < n$, so $B' \models \varphi(a', b')$, and B' is an extension of A' in K_U . \square

Remark 5.1.3. If there are some $a, b \in A \in K$ and a term $t(x, y)$ such that $t(a, b) \in \langle ab \rangle \setminus (\langle a \rangle \cup \langle b \rangle)$ the limit theory T_U of K_U is non stable, in fact the formula $U(t(x, y))$ has the order property.

Definition 5.1.4. Let $L_R = L \cup \{R\}$ be a new unary predicate symbol. We define the class K_R of L_R -structures consisting of a structure A in K equipped with a substructure $R(A)$.

Proposition 5.1.5. If the structures of K are uniformly locally finite the class K_R satisfies condition 1. – 5. However if the generated structure in K is non-trivial the condition 6. need not to hold.

Proof. The conditions 1. 2. 4. 5. are still true by the same arguments as before, we will need the local finiteness assumption on K to get condition 3. We define $P(A \oplus_E B) := \langle P(A)P(B) \rangle$ and $P(A \oplus_{(\emptyset)} \langle x \rangle) := P(A)$. For condition 5. consider a finitely generated structure $\langle a, b \rangle \in K_R$ and a quantifier free L_R formula φ such that $\langle a, b \rangle \models \varphi(a, b)$. W. l. o. g. we can assume that $\varphi(x, y) = \varphi'(x, y) \wedge \bigwedge_{i < n} R(t_i(x, y)) \wedge \bigwedge_{i < n} \neg R(t'_i(x, y))$ where $\varphi'(x, y)$ is an L_0 -formula, for all $c \in R(\langle ab \rangle)$ there is some $i < n$ such that $c = t_i(a, b)$ and for all $c \in \langle ab \rangle \setminus R(\langle ab \rangle)$ there is some $i < n$ such that $c = t'_i(a, b)$. Since our theory is uniformly locally finite the condition $t'_i(x, y) \notin \langle (t_j(x, y))_{j < n} \rangle$ is definable. Now we consider the formula $\psi'(x)$ associated to the L_0 -formula $\varphi'(x, y) \wedge \bigwedge_{i < n} (t'_i(x, y) \notin \langle (t_j)_{j < n} \rangle)$ by condition 3. in K , and then the formula $\psi(x) := \psi'(x) \wedge \bigwedge_{k < m} R(t_{i_k}(x)) \wedge \bigwedge_{k < m} \neg R(t'_{i_k}(x))$, where $t_{i_k}(x)$ enumerates the terms t such that $R(t(a))$ and $t'_{i_k}(x)$ enumerates the terms t' such that $\neg R(t'(a))$.

If $A' \models \psi(a')$ for some $a' \in A' \in K_R$ then by assumption we can find some b' such that $\langle a'b' \rangle \models \varphi'(a', b') \wedge \bigwedge_{i < n} (t'_i(a', b') \notin \langle t_j(a', b') \rangle_{j < n})$. Define the substructure R on $\langle a'b' \rangle$ to be $\langle t_j(a', b') \rangle_{j < n}$. We obtain that $\langle a'b' \rangle \models \neg R(t'_i(a', b'))$ for all $i < n$, so $\langle a'b' \rangle \models \varphi(a', b')$, and this property still holds in $B' := A' \oplus_{\langle a' \rangle} \langle a'b' \rangle$, which is an extension of A as an L_R -structure.

Now about condition 6. : By condition 6. in the class K we can find some structure $D \in K$ and some embeddings as in the statement. We want to define $R(D)$. We know that in any case $\langle R(D_0) \cup R(D_1) \cup R(B) \rangle \subseteq R(D)$, however it might happen that $b \in \langle R(D_0) \cup R(D_1) \cup R(B) \rangle$ for some $b \in (B \setminus R(B))$, and in that case the L_0 -embedding φ_B can not be an L_R -embedding. \square

To give a simple example of this last fact, let us look at some vector space over a finite field with a predicate for a subspace. Take some independent vectors a_0, a_1, b_0, b_1 as in the statement of the independence theorem for \downarrow^a such that $\neg R(a_0 + b_0)$, $R(a_1 - b_1)$ and $R(b_0 + b_1)$ are satisfied. Then $R(a + b_0)$ is satisfied if a satisfies $\text{tp}(a_i/b_i)$ for $i = 0, 1$, which is a contradiction. In that case the limit of K_R still exist, and it might be NSOP_1 , however in that case $\downarrow^K \neq \downarrow^a$ in this limit.

Remark 5.1.6. If we assume that the class K is uniformly locally finite we can restrict it to the class of finite structures in K and have a Fraïssé class in the usual sense, and its limit is ω -categorical.

As we have seen in the previous section condition 6. puts a strong restriction on what we can add to the structure, and some classes with an NSOP_1 limit can satisfy the conditions 1. – 5. and not satisfy condition 6. For example the class of finite equivalence relations, with limit the (stable) theory of the equivalence relation with an infinite number of classes all infinite (which we will write T_∞), or its parameterized version (see [19] or [7]) which is strictly NSOP_1 , satisfies existence and that $\downarrow^{K^M} = \downarrow^f = \downarrow^d$ over arbitrary sets.

Definition 5.1.7. Consider a new symbol E for a binary predicate in some sort S , we define the language $L_E = L \cup \{V, E, p\}$ where V is a new sort, E a binary relation on S , and p a unary map from S to V . Define the class K_E of L_E -structures consisting of a structure A in K_0 equipped with an equivalence relation $E^A \subseteq S(A)^2$, a new sort $V(A)$ and a map $p : S(A) \rightarrow V(A)$ realizing the quotient.

Proposition 5.1.8. The class K_E satisfies (H).

Proof. If we do not add the quotient to the structure conditions 1. – 5. are satisfied but condition 6. is not, also if we ask for the map p to be surjective the condition 3. will not hold. It is easy to see that conditions 1. – 5. are satisfied in K_E : The only structure on the quotient is equality and we can do 'in parallel' free amalgamation in K and in the quotient. For 6. we proceed similarly by forming the amalgam in K on one end and the amalgam of the quotient on the other end. \square

This construction is the same as adding a new sort V with no structure other than equality and then some function from S to V .

5.2 Adding generic functions

Definition 5.2.1. Consider a new symbol f for a binary function, we define $L_f = L \cup \{f\}$ and the class K_f of L_f -structures consisting of a structure A in K_0 equipped with a binary function $f_A : A \times A \rightarrow A$ such that the structure generated by the constants is some specified L_f -structure (in order to have the joint embedding property and uniqueness of the limit the structures need to agree on the constants).

Proposition 5.2.2. The class K_f satisfies (H).

Proof. Conditions 1. and 2. are clearly satisfied in K_f . For condition 4. we construct some L_f -structure D using 4. and 5. in K . Consider $E \subseteq A, B$ two extensions of L_f -structures. We define an increasing sequence of L_0 -structures $(D_i)_{i < \omega}$ equipped with a partial binary function $f_{i+1} : D_i^2 \rightarrow D_{i+1}$. Let $D_0 := A \oplus_E B$ and $f_0 = f_E$. Now set $D_1 := D_0 \oplus_{\langle \emptyset \rangle} \{D_0^2 \setminus (A^2 \cup B^2)\}$ and consider $f_1 : D_0^2 \rightarrow D_1$ defined the following way : if $(x, y) \in A^2$, $f_1(x, y) := f_A(x, y)$, if $(x, y) \in B^2$, $f_1(x, y) := f_B(x, y)$ and else (x, y) is sent to the element of D_1 with the corresponding label. We iterate this process : if D_i, f_{i+1} are defined for $i > 0$ we let $D_{i+1} := D_i \oplus_{\langle \emptyset \rangle} \{D_i^2 \setminus (D_{i-1}^2)\}$, and f_{i+1} extends f_i by sending $(x, y) \in D_i^2 \setminus (D_{i-1}^2)$ to the element of D_{i+1} with the corresponding label. We then set $D := \bigcup_{i < \omega} D_i$ and $f := \bigcup_{i < \omega} f_i : D^2 \rightarrow D$. This structure satisfies the universal property : it is generated by the images of A and B as an L_f -structure, given D' and embeddings $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ as in the statement we can define an increasing sequence of morphisms of L_0 -structures $\varphi_i : D_i \rightarrow D'$ in a canonical way and show that its union is a morphism of L_f -structure. The construction of $A \oplus_{\langle \emptyset \rangle} \{x\}$ in K_f follows the same method.

Condition 6. is easy to check, we just apply 6. inside of K_0 , find the embeddings, take $f_D|_{D_i^2}$ to be the image of f_{D_i} by φ_{D_i} , similarly for B , and we can set f to be anything outside of D_i^2 and B^2 . Now for condition 3. consider a structure $\langle ab \rangle \in K_f$ and some quantifier-free L_f -formula φ such that $\langle ab \rangle \models \varphi(a, b)$. By replacing every instance of f in φ by a new variable and separating the instances of f applied only to elements of $\langle a \rangle$ from the other ones we can reshape $\varphi(x, y)$ in the following form :

$$\varphi'(x, x_{\leq m}, y, y_{\leq n}) \wedge \bigwedge_{i \leq m} (x_i = f(s_i(x, x_{< i}), s'_i(x, x_{< i}))) \wedge \bigwedge_{i \leq n} (y_i = f(t_i(x, x_{\leq m}, y, y_{< i}), t'_i(x, x_{\leq m}, y, y_{< i})))$$

φ' is an L_0 quantifier-free formula and the t_i, s_i, t'_i, s'_i are L_0 -terms. By strengthening the formula φ we can assume that, if we define by induction $x_k(a) = f(s_k(a, x_{< k}(a)), s'_k(a, x_{< k}(a)))$ and

$$y_k(a, b) := f(t_i(a, x_{\leq m}(a), b, y_{< i}(a, b)), t'_i(a, x_{\leq m}(a), b, y_{< i}(a, b)))$$

, and if $y_k(a, b) \neq y_{k'}(a, b)$, then $\varphi'(x, x_{\leq m}, y, y_{\leq n})$ specifies that $(t_k(x, x_{\leq m}, y, y_{< k}), t'_k(x, x_{\leq m}, y, y_{< k})) \neq (t_k(x, x_{\leq m}, y, y_{< k}), t'_k(x, x_{\leq m}, y, y_{< k}))$, and similarly for x_k, x'_k and $x_k, y_{k'}$.

Now consider the L_0 -formula $\psi'(x, x_{\leq m})$ associated by condition 3. in K_0 to $\varphi'(x, x_{\leq m}, y, y_{\leq n})$ and then the L_f -formula $\psi'(x) := \psi'(x, x_{\leq m}) \wedge \bigwedge_{i \leq m} (x_i = f(s_i(x, x_{< i}), s'_i(x, x_{< i})))$. Assume that $A' \models \psi(a')$

for some $a' \in A'$. By assumption there is an L_0 -extension $A' \subseteq B'$ and $b', b'_{\leq n} \in B'$ such that $B' \models \varphi'(a', a'_{\leq m}, b', b'_{\leq n}) \wedge \bigwedge_{i \leq m} (a'_i = f(s_i(a', a'_{< i}), s'_i(a', a'_{< i})))$. We just need to define a binary function $f_{B'}$ on B' extending $f_{A'}$ such that $b'_k = f(t_i(a', a'_{\leq m}, b', b'_{< i}), t'_i(a', a'_{\leq m}, b', b'_{< i}))$ for every $k \leq n$. We begin by setting these values, this is possible thanks to the strengthening we had, and we can let $f_{B'}$ be whatever outside of this, we write B'_f for the L_f -structure defined on B' , then $B'_f \models \varphi(a', b')$. \square

Remark 5.2.3. Adding a generic binary function assures us that the theory of any limit is strictly NSOP₁, as proven in [23, Section 3]. Also we get that the generated structure is non-trivial. With the same construction we can add generic functions of any arity.

Definition 5.2.4. We define $L_\pi = L \cup \{\pi, \pi^{-1}\}$ new symbols of unary functions and the class K_π of L_π -structures consisting of a structure A in K_0 equipped with a bijection π_A of inverse function π_A^{-1} such that the structure generated by the constants is some specified L_π -structure.

Proposition 5.2.5. The class K_π satisfies (H).

Proof. Conditions 1. and 2. are clearly satisfied in K_π , for the conditions 3. ,5. and 6. we can proceed as previously. Now we describe what to pick as the free amalgam for condition 4. . Let $E \subseteq A, B$ be two extensions in K_π .

In the same fashion as previously we construct an increasing sequence of L_0 -structures $(D_i)_{i < \omega}$, and an increasing sequence of subsets $H_i \subseteq D_i$ and two increasing sequences of functions $\pi_i, \pi_i^{-1} : H_i \rightarrow D_i$ such that $D_i \subseteq H_{i+1}$ and $\pi_{i+1} \circ \pi_i^{-1} = \pi_{i+1}^{-1} \circ \pi_i = id_{H_i}$ for every $i < \omega$. Let $D_0 := A \oplus_E B$, $H_0 = A \cup B$, and $\pi_0 = \pi_A \cup \pi_B$, $\pi_0^{-1} = \pi_A^{-1} \cup \pi_B^{-1}$. $D_1 := D_0 \oplus_{\langle \emptyset \rangle} \{[D_0 \setminus (A \cup B)]_{k \in \mathbb{Z} \setminus \{0\}}\}$, i.e., we consider the L_0 -structure freely generated over D_0 by copies of $D_0 \setminus (A \cup B)$ indexed on $\mathbb{Z} \setminus \{0\}$, we see the element of $D_0 \setminus (A \cup B)$ as those corresponding to the copy of index 0. Set $H_1 = D_0 \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} [D_0 \setminus (A \cup B)]_k$.

We define π_1 as the extension of π_0 that send an element of $[D_0 \setminus (A \cup B)]_k$ to its copy of index $k + 1$, i. e. it sends $t(a, b)_k$ to $t(a, b)_{k+1}$, similarly for π_0^{-1} which sends an element of $[D_0 \setminus (A \cup B)]_k$ to its copy of index $k - 1$. Assume that D_{i+1}, H_i and π_i, π_i^{-1} have been defined for $i > 0$. We set $D_{i+1} := D_i \oplus_{\langle \emptyset \rangle} \{[D_i \setminus D_{i-1}]_{i \in \mathbb{Z} \setminus \{0\}}\}$, $H_{i+1} := D_i \cup \bigcup_{k < \omega} [D_i \setminus D_{i-1}]_k$ and take π_{i+1} to be the function

extending π_i that sends an element of $[D_i \setminus D_{i-1}]_k$ to its copy of index $k + 1$, similarly for π_0^{-1} which sends an element of $[D_i \setminus D_{i-1}]_k$ to its copy of index $k - 1$. Once this is done we take $D := \bigcup_{i < \omega} D_i$,

$\pi_D := \bigcup_{i < \omega} \pi_i$ and $\pi_D^{-1} := \bigcup_{i < \omega} \pi_i^{-1}$. This defines an L_π -structure of bijection π_D , and D satisfies the

universal property : it is generated by the images of A and B as an L_π -structure, given D' and embeddings $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ as in the statement we can define an increasing sequence of morphism of L_0 -structures $\varphi_i : D_i \rightarrow D'$ in a canonical way and show that its union is a morphism of L_π -structure. We take $\varphi_i : D_0 \rightarrow D'$ to be the only morphism of L_0 -structure such that $\varphi_i|_A = \varphi_A$ and $\varphi_i|_B = \varphi_B$. If $\varphi_i : D_0 \rightarrow D'$ is defined we take $\varphi_{i+1} : D_{i+1} \rightarrow D'$ to be its unique morphism extending φ_i that sends $\pi_D^k(t)$ to $\pi_{D'}^k(\varphi(t))$ for every $t \in D_i$. Set $\varphi := \bigcup_{i < \omega} \varphi_i$, it is a morphism of L_0 -structures and by construction it commutes with π , so it is a morphism of L_π -structures. The construction of $A \oplus_{\langle \emptyset \rangle} \{x\}$ in K_π follows the same method. \square

Definition 5.2.6. We define $L_\sigma = L \cup \{\sigma, \sigma^{-1}\}$ new symbols of unary functions and the class K_σ of L_σ -structures consisting of a structure A in K equipped with an automorphism σ_A and its inverse function σ_A^{-1} .

In K_σ condition 1. – 5. are easy to check : we use the Free Amalgam property to show that if $E \subseteq A, B \in K_\sigma$ then there is an (unique) automorphism $\sigma_{A \oplus_E B}$ of $A \oplus_E B$ extending σ_A and σ_B and $(A \oplus_E B, \sigma_{A \oplus_E B})$ satisfies the universal property. For condition 4. we consider the L_0 -structure $A \oplus_{\langle \emptyset \rangle} \langle (x_i)_{i \in \mathbb{Z}} \rangle$ equipped with the unique automorphism that extends σ_A and sends x_k to x_{k+1} for every $k \in \mathbb{Z}$. As for the generic substructure we need the structures of K to be uniformly locally finite for condition 5. to hold, in this case the proof follows the same method. Also condition 6. need not to hold even in this case for similar reasons as in K_R . The limit of K_σ still exist, and it might be $NSOP_1$, however in that case $\downarrow^K \neq \downarrow^a$ in the limit.

To sum up this section, the theories we are building are model companion of theories which are made from basic one-based stable theories, like the theory of a \mathbb{F}_2 -vector space, to which we added some basic structure (equivalence relation, bijection,..) that does not interact. This is slightly more general than the model companion of the empty theory, and it allows us to give some minimalist example of strictly $NSOP_1$ theory : The model companion of the theory of an abelian group with a unary function f such that $f(0) = 0$.

6 Some strictly NSOP₁ theories with strong amalgamation

The theory T of the limit of some K satisfying (H) is not a free amalgamation theory in the sense of [13], our strong independence relation \perp^Γ does not in general satisfies the closure condition (which is defined as : if $C \subseteq A, B$ are algebraically closed and $A \perp_C B$ then $A \cup B$ is algebraically closed), however the relation \perp^Γ is interesting.

Remark 6.1. There are other known examples of NSOP₁ theories admitting a strong independence relation satisfying Proposition 3.2.9, which could motivate the study of such theories :

1. The theory of ω -free PAC fields of characteristic 0 (see [10] and [9]) with the the relation \perp^{III} defined as : for $C \subseteq A, B$, $A \perp_C^{III} B$ if $A \perp_C^{lin} B$ and $acl(AB) = AB$. This relation is named strong independence in [8, Definition 4.5], written as here in [10, Section 3.6], but this definition is not the same as the strong independence of [9, Definition 1.2].
2. The theory of vector spaces of infinite dimension over an ACF with a non degenerate bilinear form with the relation \perp^Γ (see [2], [17] or [7] for the definition of \perp^Γ). This is not a free amalgamation theory because of the closure condition.
3. The theory ACFG of an ACF with a generic multiplicative subgroup, see strong independence in [18, Section 2.2] and the remark following [18, Section 4.4].

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