

ON SOME FRAÏSSÉ LIMITS WITH FREE AMALGAMATION

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Abstract : In the first section of this work we discuss some curiosities around stable Kim-forking. In the second section we give a general way of building some known and some new examples of NSOP1 theories as the limit of some Fraïssé class satisfying some strong conditions. These limits will satisfy existence, $\downarrow^K = \downarrow^a$ and $\downarrow^f = \downarrow^M$ over arbitrary sets, and they come with a stationary independence relation. This study is based on the results of Baudish, Ramsey, Chernikov and Kruckman.¹

1 About stable Kim-forking

1.1 In General

We work here in T a complete theory, all sets of parameters and elements are assumed to be in a monster model \mathbb{M} of T . We will write \downarrow^K for Kim-forking independence and \downarrow^f for forking independence.

Definition 1.1.1. *Say a theory T has stable Kim-forking over models if whenever $a \not\downarrow_M^K N$ for $M \subseteq N$ models there is a stable formula $\varphi(x, y) \in \mathcal{L}$ and $n \in N$ such that $a \models \varphi(x, n)$ and $\varphi(x, n)$ Kim-forks over M . Same for stable forking with a $\not\downarrow_M^f N$.*

Definition 1.1.2. *Say a theory T has weak stable Kim-forking over models if whenever $a \not\downarrow_M^K N$ for $M \subseteq N$ models there is a stable formula $\varphi(x, y, m) \in \mathcal{L}_m$ (i.e. the formula with fixed $m \in M$ is stable) and $n \in N$ such that $a \models \varphi(x, n, m)$ and $\varphi(x, n, m)$ Kim-forks over M . Same for weak stable forking with a $\not\downarrow_M^f N$.*

It is known that a theory with stable forking is simple (see [5, Proposition 1.3]). We show that a theory with stable Kim-forking over models is simple and that a theory with weak stable Kim-forking over models is NSOP1 and not necessarily simple. We also show that a theory with weak stable forking over models is simple.

Lemma 1.1.3. *Let $M \subseteq N$ be models of a theory T with weak stable Kim-forking over models. For any type $p = \text{tp}(a/N)$ we have $\downarrow_M^K N$ if and only if for every stable formula $\varphi(x, y, m)$ with $m \in M$ the canonical parameter $Cb(d_p x \varphi(x, y, m))$ is in $\text{acl}^{eq}(M)$.*

Proof. [\implies] If for some stable formula $\varphi(x, y, m)$ with $m \in M$ the canonical parameter $Cb(d_p x \varphi(x, y, m))$ is not in the bounded closure of the basis M , so in particular if c is a representative then $c \notin M$ and we can take an infinite M coheir Morley sequence in $\text{tp}(c/M)$, the type of a over this sequence would then divide since its $\varphi(x, y, m)$ -type would have different definitions, so p Kim-forks over M . This way does not use the hypothesis.

[\impliedby] Conversely assume that p Kim-forks over M and that M contains the canonical parameters. Then there is a stable formula $\varphi(x, y, m)$ with $m \in M$ and $n_0 \in N$ such that $\varphi(x, n_0, m) \in p$ and $\varphi(x, n_0, m)$ Kim-forks over M . So there is an M coheir Morley sequence $(n_i)_{i < \omega}$ such that $\{\varphi(x, n_i, m) : i < \omega\}$ is inconsistent. Now since $Cb(d_p x \varphi(x, y, m)) \in \text{acl}^{eq}(M)$ by indiscernibility we have $\models d_p x \varphi(x, n_i, m)$ for all $i < \omega$, so $\{\varphi(x, n_i, m) : i < \omega\}$ is satisfied by any realisation of a non-forking extension of p to $N(n_i)_{i < \omega}$, a contradiction. \square

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In the same fashion and with a similar proof we have the following :

Lemma 1.1.4. *Let $M \subseteq N$ be models of a theory T with weak stable forking over models. For any type $p = \text{tp}(a/N)$ we have $a \downarrow_M^f N$ if and only if for every stable formula $\varphi(x, y, m)$ with $m \in M$ the canonical parameter $Cb(d_p x \varphi(x, y, m))$ is in $\text{acl}^{eq}(M)$.*

Proposition 1.1.5. *Let T be a theory with weak stable Kim-forking over models, then T is NSOP1.*

Proof. We use here the second point of [16, Theorem 3.9]. Assume that T has weak stable Kim-forking and that we have $(M_i)_{i < \kappa}$ for $\kappa = |T|^+$ a continuous increasing sequence of models of size $|T|$ and a tuple a such that $a \not\downarrow_{M_i}^K M_{i+1}$ for all $i < \kappa$. Write $M = \bigcup_{i < \kappa} M_i$. For every $i < \kappa$ the number of stable formulas with parameters in M_i is at most $|T|$, so since κ is regular there is a smaller index $\alpha_i \geq i$ such that for every stable formula $\varphi(x, y, e)$ with $e \in M_i$, if we write $p = \text{tp}(a/M)$, $Cb(d_p x \varphi(x, y, e)) \in M_{\alpha_i}$. Now let us consider $(\beta_i)_{i < \omega}$ defined by induction by $\beta_0 = 0$ and $\beta_{i+1} = \alpha_{\beta_i}$. Take $\beta = \bigcup_{i < \omega} \beta_i$, we have that $\alpha_\beta = \beta$, so by Lemma 1.1.3 $a \downarrow_{M_\beta}^K M$, which contradicts $a \not\downarrow_{M_\beta}^K M_{\beta+1}$. \square

Proposition 1.1.6. *Let T be a theory with weak stable forking over models, then T is simple.*

Proof. We show that \downarrow^f satisfies local character. Let a be a finite tuple, M a model, $p = \text{tp}(a/M)$. We consider the set $C_0 = \{Cb(d_p x \varphi(x, y)) : \varphi(x, y) \in \mathcal{L} \text{ a stable formula}\}$. Now we take a model $M_0 \subseteq M$ such that $|M_0| = |T|$ and M_0 contains a representative of every element of C_0 . We iterate the process taking $C_{i+1} = \{Cb(d_p x \varphi(x, y, e)) : \varphi(x, y, e) \in \mathcal{L}_e \text{ a stable formula with } e \in M_i\}$ and $M_i \subseteq M_{i+1} \subseteq M$ such that $|M_{i+1}| = |T|$ and M_{i+1} contains a representative of every element of C_{i+1} . Now if we set $M' = \bigcup_{i < \omega} M_i$ by Lemma 1.1.4 $a \downarrow_{M'}^f M$ and M' has size $|T|$. \square

Corollary 1.1.7. *Let T be a theory with stable Kim-forking over models, then T is simple.*

Proof. By Proposition 1.1.5 we know that T is NSOP1, now if T is non-simple we can build an arbitrary long sequence of Kim-forking extensions with an instance of TP2 (as in [4, Proposition 3.6]). Now we can run the same proof as in [5, Proposition 1.3] : we have a stable formula with a dividing chain of infinite length and hence with the tree property, which is not possible. \square

1.2 In a theory with existence

We assume that the theory T has existence, now we can define weak stable Kim-forking over arbitrary sets in a similar fashion.

Definition 1.2.1. *Say a theory T with existence has weak stable Kim-forking if whenever $a \not\downarrow_E^K B$ for $E = \text{acl}(E) \subseteq B$ there is a stable formula $\varphi(x, y, e) \in \mathcal{L}_e$ (i.e. the formula with fixed $e \in E$ is stable) and $b \in B$ such that $a \models \varphi(x, b, e)$ and $\varphi(x, b, e)$ Kim-forks over E .*

Lemma 1.2.2. *Let $E = \text{acl}(E) \subseteq B$ in a theory T with weak stable Kim-forking. For any type $p = \text{tp}(a/B)$ we have $a \downarrow_E^K B$ if and only if for every stable formula $\varphi(x, y, e)$ with $e \in E$ the canonical parameter $Cb(d_p x \varphi(x, y, e))$ is in $\text{acl}^{eq}(E)$.*

Proof. The proof is the same as the one of Lemma 1.1.3, we just replace coheir Morley sequences by Morley sequences over E . \square

An example of such a theory is the theory of parameterized equivalence relations (see [10, Section 6.3] and [3]). This theory has two sorts O for objects and P for parameters, it has existence, and for arbitrary sets $E \subseteq A, B$ we have that :

1. $A \downarrow_E^K B$ iff $A \cap B = E$ and $O(A) \downarrow_{O(E)}^p O(B)$ for every $p \in P(E)$ where \downarrow^p is forking independence in the theory T_∞ of an equivalence relation with an infinite number of classes all infinite.
2. $A \downarrow_E^f B$ iff $A \cap B = E$ and $O(A) \downarrow_{O(E)}^p O(B)$ for every $p \in P(B)$.

The first condition is witnessed by stable formulas (namely equalities). For the second one, the difference between Kim-forking and forking here is that in the first case the parameter is inside of the basis. For a given parameter p the formula $\varphi(x, y) = (x \equiv_p y)$ is stable since it defines an equivalence relation, so Kim-forking is always witnessed by stable formulas in this theory. Now if we do not fix p , the formula $\psi(x, yz) := (x \equiv_z y)$ is unstable, so forking is not always witnessed by stable formulas in this theory (which we already knew from Proposition 1.1.6).

The theory of vector spaces of infinite dimension over a field with weak stable Kim-forking (so in particular over a stable field) and the model companion of the empty theory with a symbol for a binary function (see [18] the characterisation of Kim-forking in this theory) are other examples. A difference with the previous example is that in these two cases there is a non-trivial algebraic closure, and that Kim-forking independence is computed for algebraically closed structures $E = \text{acl}(E) \subseteq A = \text{acl}(A), B = \text{acl}(B)$ and is clearly witnessed by stable formulas in this case, so one could think that something different might happen if we consider arbitrary sets. It is not the case thanks to the two following facts, first in an NSOP1 theory with existence $A \downarrow_E^K B$ if and only if $\text{acl}(EA) \downarrow_{\text{acl}(E)}^K \text{acl}(EB)$ for every sets E, A, B , and secondly :

Remark 1.2.3. [5, Remark 1.2] *Let $\varphi(x, y, e)$ be a stable formula (with some parameters e). If $\theta(v, x, e) \vdash \exists^n x \theta(v, x, e)$ then $\psi(v, y, e) = \exists x (\theta(v, x, e) \wedge \varphi(x, y, e))$ is also stable.*

This means that we can get a stable formula from another one by replacing our variables by something 'bigger' in the sense of the algebraic closure : Assume that $a \not\downarrow_{\text{acl}(E)}^K b$, then $\text{acl}(Ea) \not\downarrow_{\text{acl}(E)}^K \text{acl}(Eb)$, so there is a stable formula $\varphi(x, y, e)$ with $e \in \text{acl}(E)$ and $\bar{a} \in \text{acl}(Ea), \bar{b} \in \text{acl}(Eb)$ such that $\models \varphi(\bar{a}, \bar{b}, e)$ and $\varphi(x, \bar{b}, e)$ Kim-divides over E . We can assume that $\bar{a} \in \text{acl}(ea), \bar{b} \in \text{acl}(eb)$. Then using Remark 1.2.3 we can find a stable formula $\psi(x', y', e)$ such that $\models \psi(a, b, e)$ and $\psi(x, b, e)$ Kim-divides over $\text{acl}(E)$ (for this last condition we take a formula θ isolating the type of \bar{b} over $\text{acl}(E)b$).

Remark 1.2.4. *If T is an NSOP1 theories with existence, weak stable Kim-forking and elimination of hyperimaginaries then T^{eq} has weak canonical bases in the sense of [17, Definition 4.1]. More generally, without the last assumption, if T is an NSOP1 theory with existence, weak stable Kim-forking and elimination of imaginaries, then given some tuples $E = \text{acl}^{\text{eq}}(E) \subseteq B = \text{acl}^{\text{eq}}(B)$ and a there is a smaller subset $E \subseteq B_0 = \text{acl}^{\text{eq}}(B_0) \subseteq B$ such that $a \downarrow_{B_0}^K B$, in the sense that given any $E \subseteq B' = \text{acl}^{\text{eq}}(B') \subseteq B$ such that $a \downarrow_{B'}^K B$ we have that $B_0 \subseteq B'$.*

1.3 Imaginaries :

We adapt some results of the section 2 of [5] where it is shown that if a theory T has stable forking so does T^{eq} to our context, but we cannot get a similar result (at least for the moment).

Lemma 1.3.1. *Let $T = T^{\text{eq}}$ be a theory with existence such that whenever $a \not\downarrow_E^K N$ for N a model and $E \subseteq N$ there is $e \in E, n \in N, \varphi(x, y, e)$ a stable formula such that $\models \varphi(a, n, e)$ and $\varphi(x, n, e)$ Kim-forks over E . Then T has weak stable Kim-forking.*

Proof. Let $a, C \subseteq B$ such that $a \not\downarrow_E^K B$. Let N be a model containing B such that $a \downarrow_B^f N$. So there is a stable formula $\psi(x, y, e)$ such that $\psi(x, n, e)$ Kim-forks over E for some $n \in N$. Then $b' = Cb(d_p x \psi(x, y, e)) \in \text{acl}^{\text{eq}}(B) \setminus \text{acl}^{\text{eq}}(E)$. Let us write $\chi(y, b, e) = d_p x \psi(x, y, e)$, then this formula

is stable since it is equivalent to a generalised $\psi(x, y, e)^{-1}$ formula. Now there is $b \in B$ and $\theta(v, w, e)$ such that $\theta(v, w, e) \vdash \exists^{=n} w \theta(v, w, e)$ and $\models \theta(b, b', e)$. Then $\varphi(y, v, e) = \exists w (\theta(v, w, e) \wedge \chi(y, w, e))$ is also stable by Remark 1.2.3. Now the formula $\forall y (\psi(x, y, e) \leftrightarrow \varphi(y, v, e))$ is stable, $\models \forall y (\psi(a, y, e) \leftrightarrow \varphi(a, b, e))$, and $\forall y (\psi(x, y, e) \leftrightarrow \varphi(x, b, e))$ Kim-forks over E . \square

Proposition 1.3.2. [5, Proposition 2.2] *If T has weak stable Kim-forking and existence, then T^{eq} has weak stable Kim-forking over real parameters, meaning that whenever $a_F \not\downarrow_E^K B$ for $E \subseteq B$ sets of real elements and $a_F \in \mathbb{M}^{eq}$ the F class of a there is a stable formula $\varphi(x, y, e) \in \mathcal{L}_e$ for $e \in E$ and $b \in B$ such that $a \models \varphi(x, b, e)$ and $\varphi(x, b, e)$ Kim-forks over E .*

Proof. Let a, E, B be as in the statement. We choose a saturated model $M \supseteq B$ such that $a_F \in dcl^{eq}(M)$ and there is a representative a of a_F such that $a \downarrow_{a_F}^K M$ (using existence). Then $a \not\downarrow_E^K B$. By assumption there is a stable formula $\varphi(x, y, e)$ for $e \in E$, and a tuple $b \in B$ such that $\models \varphi(a, b, e)$ and $\varphi(x, b, e)$ Kim-forks over E . Write p the $\varphi(x, y, e)$ -type of a over B . Then p has a definition $d_p x \varphi(x, y, e)$ which is an instance of a stable formula. p does not Kim-fork over a_F so its canonical basis d_H is in $acl^{eq}(a_F)$. For some $\chi(w, y, e) \in \mathcal{L}^{eq}$, $\chi(d_H, y)$ defines p . Since $d_p x \varphi(x, y, e) \equiv \chi(d_H, y)$ there is some $\mu(x) \in tp(d_F/E)$ such that the formula $\chi(w, y) \wedge \mu(w)$ is stable. W.l.o.g. w.c.a. that $\chi(w, y)$ is stable.

Claim : If $q(w) = tp(d_H/E)$, then Kim-forks over E .

Proof. Assume not, now if we consider an E Morley sequence $(b_i)_{i < \omega}$ in $tp(b/E)$ inside of M . We have that $q(w) \cup \{\chi(w, b_i) : i < \omega\}$ is consistent, and hence there is a realisation d'_H of this type. $d_H \equiv_E d'_H$ so there is a sequence $(b'_i)_{i < \omega}$ in M such that $d_H(b'_i)_{i < \omega} \equiv_E d'_H(b_i)_{i < \omega}$. Then $\models \chi(b_i, d_H)$ for all $i < \omega$, so $\varphi(x, b_i, e) \in p$ and $a \models \{\varphi(x, b_i, e) : i < \omega\}$, which contradicts that $\varphi(x, b, e)$ Kim-forks over E , so $\{\varphi(x, b_i, e) : i < \omega\}$ is k -inconsistent for some $k < \omega$. \square

With the claim we can choose $\mu(w) \in q(w)$ such that $\chi(w, b) \wedge \mu(w)$ Kim-forks over E . Then $\chi'(w, y) = \chi(w, b) \wedge \mu(w)$ is stable and Kim-divides over E , witnessed by an E -Morley sequence $(b_i)_{i < \omega}$ in $tp(b/E)$. Since $d_H \in acl^{eq}(a_F)$ there is a formula $\theta(v, w) \in \mathcal{L}^{eq}$ and an $n < \omega$ such that $\models \theta(a_F, d_H)$ and $\theta(v, w) \vdash \exists^{=n} w \theta(v, w)$. Let $\psi(v, y) = \exists w (\theta(v, w) \wedge \chi'(w, y))$. Then this formula is stable (by Remark 1.2.3 and the sequence $(b_i)_{i < \omega}$ witnesses that it Kim-divides over E). \square

Our problem now is that given some set of imaginaries E and some set of representatives E' we cannot move from a stable formula $\varphi(x, y, e')$ with real parameters $e' \in E'$ to a stable formula $\varphi(x, y, e'_R)$ with R a definable equivalence relation such that $e'_R \in acl^{eq}(E)$ in an obvious way.

Some questions about this :

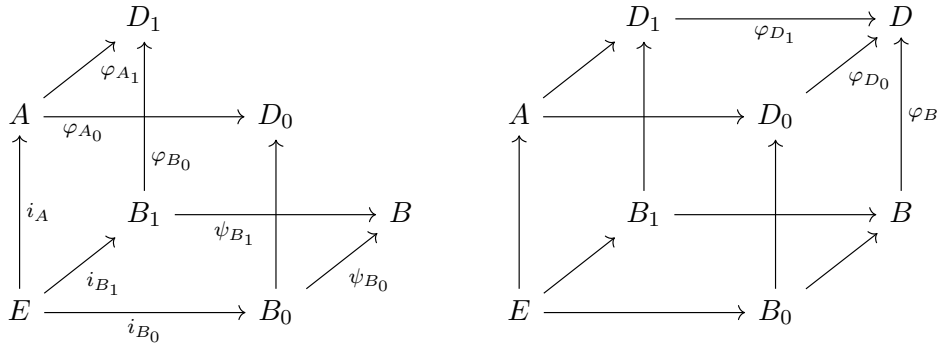
1. Does existence and weak stable Kim-forking over models imply weak stable Kim-forking over arbitrary sets?
2. [5, Proposition 2.3] : If T^{eq} has weak stable Kim-forking over real parameters and existence, then does T^{eq} has weak stable Kim-forking?
3. More specifically, can we prove that if a simple theory T has weak stable forking then T^{eq} also have weak stable forking?
4. Is there an NSOP1 theory that does not have weak stable Kim-forking? Either over models or algebraically closed sets if it has existence.

2 Adding generic structure to a Fraïssé class with free amalgamation

2.1 Properties of the limit

The bases of this study are [10, Section 6.3] and [2]. Baudisch's construction works in a way more general context, but what we focus on here is the study of independence relations, which appears in his paper under some stronger hypothesis (namely in [2, Theorem 4.1]). In this section we consider a countable language L and a class K of countable L -structures with the following properties :

1. (H.P.) : If $A \subseteq B \in K$ is a substructure then $A \in K$.
2. (Free Amalgamation) : If $C \subseteq A, B$ are structures in K (where we allow $C = \emptyset$ when the language does not have any constants) there is $D \in K$ and some embeddings i_A, i_B of A and B respectively to D satisfying $i_A|_C = i_B|_C$, $i_A(A) \cap i_B(B) = i_A(C)$, D is generated by the images of A and B and such that given any morphism of L -structures $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ to $D' \in K$ there is a unique morphism $\varphi : D \rightarrow D'$ such that $\varphi_A = \varphi \cdot i_A$ and $\varphi_B = \varphi \cdot i_B$. We will write $A \oplus_C B$ for this structure D .
3. (Generic extension) : If $A \in K$ and V is a sort there is an extension $A \oplus_{\emptyset} \langle x \rangle \in K$ generated by a single element $x \in V$ over A such that for every morphism $\varphi : A \rightarrow A' \in K$ and every element $a' \in V(A')$ there is a unique morphism $\psi : A \oplus_{\emptyset} \langle x \rangle \rightarrow A'$ extending φ by sending x to a' .
4. K is axiomatisable : if the isomorphism class of an L -structure A is not inside of K there is a quantifier free formula φ and a finite tuple $a \in A$ such that $A \models \varphi(a)$ and $A' \not\models \varphi(a')$ for every $a' \in A' \in K$.
5. Any limit of K has Q.E. : for every quantifier free formula $\varphi(x, y)$ satisfied in some structure of K there is a quantifier free formula $\psi(x)$ such that if $a \in A \in K$ and $A \models \psi(a)$ then there is $A \subseteq B \in K$ an extension and $b \in B$ such that $B \models \varphi(a, b)$.
6. (3 Amalgamation for algebraically independent extension) : Given a commutative diagram in K as on the left, where the arrows are embeddings, $\varphi_{B_i}(B_i) \cap \varphi_{A_i}(A) = \varphi_{A_i} \cdot i_A(E) = \varphi_{B_i} \cdot i_{B_i}(E)$ for $i = 0, 1$ and $\psi_{B_0}(B_0) \cap \psi_{B_1}(B_1) = \psi_{B_0} \cdot i_{B_0}(E) = \psi_{B_1} \cdot i_{B_1}(E)$, we can complete it inside of K into a commutative diagram as on the right (where the arrows are embeddings) such that $\varphi_{D_i}(D_i) \cap \varphi_B(B) = \varphi_{D_i} \cdot \varphi_{B_i}(B_i) = \varphi_B \cdot \psi_{B_i}(B_i)$ for $i = 0, 1$ and $\varphi_{D_0}(D_0) \cap \varphi_{D_1}(D_1) = \varphi_{D_0} \cdot \varphi_{A_0}(A) = \varphi_{D_1} \cdot \varphi_{A_1}(A)$ (the blank arrows of the right diagram are those of the left diagram, i did not write the names for clarity).



Notice that we do not assume (E.C.), i.e. that there are countably many isomorphism class, nor that the structures in K are finitely generated, also to check the condition 5 for a formula it is enough to check it for a strengthening of this formula that is also realised in K . 6 is a strengthening of 3-amalgamation for \perp^a , in the usual 3 amalgamation we would ask that D_i or B are generated by the

image of the two embeddings from A and B_i or from B_0 and B_1 respectively, here however we allow ourselves to take an extension of the generated structure. We will refer to the conditions 1 – 6 as (H) .

We begin by constructing a limit of K and presenting its properties. We assume the continuum hypothesis for the reason that we need an uncountable cardinal κ such that $2^{<\kappa} = \kappa$ and that it is convenient to take \aleph_1 as such.

Definition 2.1.1. *Let K be a class of countable structures in some language L . An L structure \mathcal{M} is K - \aleph_1 -saturated if the collection of its countable substructures correspond up to isomorphism to K and if for every $\mathcal{U}, \mathcal{B} \in K$ countable and every embeddings $f_0 : \mathcal{U} \rightarrow \mathcal{M}$, $f_1 : \mathcal{U} \rightarrow \mathcal{B}$ there is some embedding $g : \mathcal{B} \rightarrow \mathcal{M}$ such that $g \cdot f_1 = f_0$.*

Proposition 2.1.2. *There exist an L structure \mathcal{M} which is K - \aleph_1 -saturated and has cardinal \aleph_1 . When L does not have any constants every structure of age included in K and of cardinal $\leq \aleph_1$ embeds into \mathcal{M} , and \mathcal{M} is unique up to isomorphism. Otherwise the age of \mathcal{M} is the set of structures in K that have the same isomorphism type for the structure generated by the constants, and any such structure of cardinal $\leq \aleph_1$ embeds in \mathcal{M} .*

Proof. Let $(\mathcal{B}_i)_{i < \aleph_1}$ enumerate the elements of K up to isomorphism. We construct by induction an chain $(\mathcal{U}_i)_{i < \aleph_1}$ of L_P -structures in K . For every $i, k < \aleph_1$ let $(f_j^{\mathcal{U}_i, k})_{j < \aleph_1}$ enumerate the embeddings of \mathcal{U}_i in \mathcal{B}_k (eventually if these structure do not agree on the constants there are none). Let $\mathcal{U}_0 = \mathcal{B}_0$. Assume that \mathcal{U}_i is defined for every $i < \lambda$. By (Free Amalgamation) there is a structure \mathcal{U}'_λ extending both $\bigcup_{i < \lambda} \mathcal{U}_i$ and \mathcal{B}_λ . We define a chain $(\mathcal{U}^k)_{k < \lambda}$ of L_P -structures extending \mathcal{U}'_λ , let $\mathcal{U}^0 = \mathcal{U}'_\lambda$. Assume that every \mathcal{U}^k is defined for every $k < \mu$. We define \mathcal{U}^μ as the union of a chain $(\mathcal{U}_i^\mu)_{i < \lambda}$ of L_P -structures, let $\mathcal{U}_0^\mu = \bigcup_{i < \lambda, k < \mu} \mathcal{U}_i^k$. Assume \mathcal{U}_i^μ is defined for every $i < \nu$, we consider the embeddings $f_j^{\mathcal{U}_\mu, \nu} : \mathcal{U}_\mu \rightarrow \mathcal{B}_\nu$ for $j < \lambda$. $\mathcal{U}_{\nu, 0}^\mu := \bigcup_{k < \mu} \mathcal{U}_k^\mu$, if $\mathcal{U}_{\nu, j}^\mu$ is defined for every $j < \epsilon$ let $\mathcal{U}_{\nu, \epsilon}^\mu = \bigcup_{j < \epsilon} \mathcal{U}_{\nu, j}^\mu$ and write $f = f_\epsilon^{\mathcal{U}_\mu, \nu}$ by *S.A.P.* there is a structure $\mathcal{U}_{\nu, \epsilon}^\mu \in K$ extending $\mathcal{U}_{\nu, \epsilon}^\mu$ and an embedding $g = g_\epsilon^{\mathcal{U}_\mu, \nu} : \mathcal{B}_\nu \rightarrow \mathcal{U}_{\nu, \epsilon}^\mu$ such that $g \cdot f : \mathcal{U}_\mu \rightarrow \mathcal{U}_{\nu, \epsilon}^\mu$ is the inclusion. We now set $\mathcal{U}_\lambda = \bigcup_{i < \lambda} \mathcal{U}^i$ and $\mathcal{M} = \bigcup_{i < \aleph_1} \mathcal{U}_i$.

$$\begin{array}{ccc}
 \mathcal{U}_{\nu, \epsilon}^\mu & \xrightarrow{\subseteq} & \mathcal{U}_{\nu, \epsilon}^\mu \\
 \uparrow & & \uparrow g \\
 \subseteq & & \\
 \mathcal{U}_\mu & \xrightarrow{f} & \mathcal{B}_\nu
 \end{array}$$

We now show that \mathcal{M} is K - \aleph_1 -saturated and has *age* inside of K . Let \mathcal{U} be a countable structure of \mathcal{M} , $\mathcal{U} \subseteq \mathcal{U}_i$ for some $i < \aleph_1$ so by *(H.P.)* $\mathcal{U} \in K$. Now lets consider an embedding $f : \mathcal{U} \rightarrow \mathcal{B} \in K$. \mathcal{B} is isomorphic to \mathcal{B}_k for some $k < \aleph_1$ via $\varphi : \mathcal{B} \rightarrow \mathcal{B}_k$. By (Free Amalgamation) we can find some $k' < \aleph_1$ and some embeddings $\psi : \mathcal{B}_k \rightarrow \mathcal{B}_{k'}$ and $f' : \mathcal{U}_i \rightarrow \mathcal{B}_{k'}$ such that $\psi \cdot \varphi \cdot f = f'$. Now $f' : \mathcal{U}_i \rightarrow \mathcal{B}_{k'}$ is equal to $f_l^{\mathcal{U}_i, k'}$ for some $l < \aleph_1$, and if we choose a large enough j , by construction, there is a $h = g_l^{\mathcal{U}_i, k'} : \mathcal{B}_{k'} \rightarrow \mathcal{M}$ such that $h \cdot f'$ is the inclusion $\mathcal{U}_i \subseteq \mathcal{M}$. Then $h \cdot \psi \cdot \varphi$ is the embedding we want.

$$\begin{array}{ccc}
\mathcal{U}_j & \xrightarrow{\subseteq} & \mathcal{M} \\
\uparrow \subseteq & & \uparrow h \\
\mathcal{U}_i & \xrightarrow{f'} & \mathcal{B}_{k'} \\
\uparrow \subseteq & & \uparrow g \\
\mathcal{U}_\mu & \xrightarrow{f} \mathcal{B}_\nu & \xrightarrow{\varphi} \bullet
\end{array}$$

For the second point we proceed like as in Fraïssé theory and only use the property of K -saturation and the fact that we can amalgamate over either the empty set or the structure generated by the constants : if \mathcal{M}' is a structure of age inside of K and of cardinal $\leq \aleph_1$ we fix an enumeration $(m_i)_{i < \aleph_1}$ and we define by induction an increasing sequence of embeddings $f_i : \langle m_{<i} \rangle \rightarrow \mathcal{M}$, and the union $f = \bigcup_{i < \aleph_1} f_i : \mathcal{M} \rightarrow \mathcal{M}'$ is our embedding. We need to amalgamate over the empty set for the first step of our induction : take some $a \in \mathcal{M}$, and consider $\langle a \rangle \subseteq \mathcal{M}$, we can embed $\langle a \rangle$ into $\langle a \rangle \oplus_\emptyset \langle m_0 \rangle$, by saturation we can find some substructure $\langle m'_0 \rangle \subseteq \mathcal{M}$ isomorphic to $\langle m_0 \rangle$ and we take f_0 to be this isomorphism. In a similar fashion, to show uniqueness we construct an isomorphism by back and forth. \square

Definition 2.1.3. Let T be the theory of \mathcal{M} and \mathbb{M} be a monster model of T . When L does not have constants this theory is entirely determined by the data of K by the last part of Proposition 2.1.2, and we will refer to T as the theory of the limit of K . Otherwise we will refer to the theory of \mathcal{M} as a limit of K , in that case each limit correspond to an isomorphism type of the structure generated by the constants.

Remark 2.1.4. *J.E.P.* holds in K if and only if all the structures of K agree on the structure generated by the constants.

Remark 2.1.5. A consequence of the (Free Amalgamation) property is that $dcl(A) = acl(A) = \langle A \rangle$ for every small set $A \subseteq \mathbb{M}$.

Lemma 2.1.6. T has *Q.E.* in L .

Proof. This is clear from the condition 5, if we consider any \aleph_1 -saturated model it is K - \aleph_1 -saturated, so partial isomorphism between finitely generated structures have the back and forth property. \square

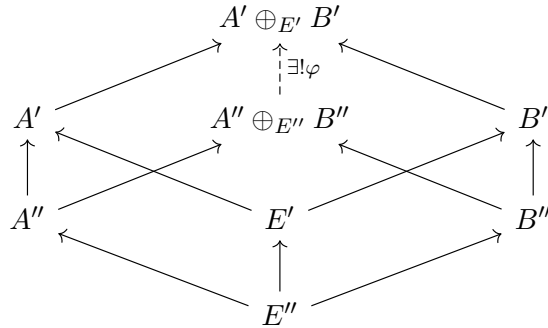
Lemma 2.1.7. The relation \downarrow^a defined on subsets of \mathbb{M} the monster model of a limit by $A \downarrow_E^a B$ if $acl(EA) \cap acl(EB) = acl(E)$ satisfies the independence theorem, i.e. that if $E, A_i, B_i \subseteq \mathbb{M}$ for $i \in \{0, 1\}$, $A_0 \equiv_E A_1$, $A_i \downarrow_E^a B_i$ for every $i \in \{0, 1\}$ and $B_0 \downarrow_E^a B_1$ there is an A such that $A \equiv_{B_i} A_i$ for every $i \in \{0, 1\}$ and $A \downarrow_E^a B_0 B_1$.

Proof. If we consider E, A, B_0 and B_1 as in the statement we have to show that $Diag_{B_0}(\langle A_0 B_0 \rangle) \cup Diag_{B_1}(\langle A_1 B_1 \rangle) \cup \{t(\bar{a}) \notin \langle B_0 B_1 \rangle : \text{for all terms } t \text{ such that } t(\bar{a}) \notin E\}$ is consistent, by compactness is it enough to consider finitely generated structures E, A, B_0 , and in that case we get the consistency by just applying 6. \square

Definition 2.1.8. We define an independence relation on small subsets of the monster model : for $E \subseteq A, B$ say that $A \downarrow_E^{\Gamma} B$ if the structure generated by A and B , written $\langle AB \rangle$ and the inclusions $i_A : A \rightarrow \langle AB \rangle$, $i_B : B \rightarrow \langle AB \rangle$ satisfies the universal property written as $A \oplus_E B$, i.e. every common extension of A and B factorises uniquely through $\langle AB \rangle$.

Lemma 2.1.9. *Full existence : For every $E \subseteq A, B \subseteq \mathbb{M}$ such that $A \cap B = E$ there is $A' \equiv_E A$ such that $A' \downarrow_E^\Gamma B$.*

Proof. We want to construct an L structure D extending B and an embedding $i_A : A \rightarrow D$ such that if $A' := i_A(A)$, $D = \langle A'B \rangle$ and $A' \downarrow_E^\Gamma B$. Consider the set of the L terms with parameters in A, B, E viewed as constant symbols, to each of those terms t we associate a variable x_t . We consider the equivalence relation E of X defined by $E[t(\bar{a}, \bar{b}, \bar{e}), t'(\bar{a}', \bar{b}', \bar{e}')] if and only if for some countable substructures E', A' and B' of E, A and B respectively these two terms computed in the structure $A' \oplus_{E'} B'$ are equal. To show the E is transitive we need to look at the canonical map $A'' \oplus_{E''} B''$ to $A' \oplus_{E'} B'$ where $E'' \subseteq E', A'' \subseteq A', B'' \subseteq B'$ and $E' = A' \cap B'$. We describe it in the following diagram. This map assures us that the system of the $A' \oplus_{E'} B'$ we are considering is directed and the structure we are constructing can also be presented as its limit.$



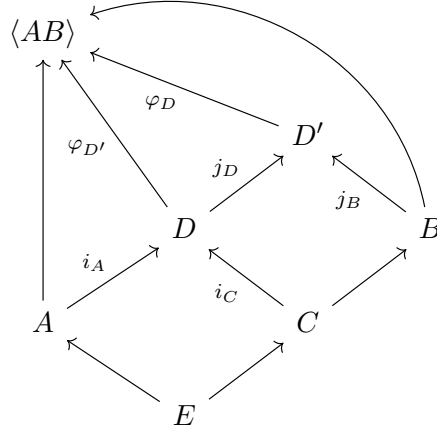
Now we give to X/E the natural structure : for a function symbol f set $f^D([x_{\bar{t}}]) := [x_{f(\bar{t})}]$ and for a relation symbol R set $R^D([x_{\bar{t}}])$ if for some countable substructures E', A' and B' of A, A and B respectively the terms \bar{t} computed inside of $A' \oplus_{E'} B'$ satisfies R . This is well defined by the same argument as previously and we get our structure D . By assumption the diagram of D over B is finitely satisfied in \mathbb{M} , so we can assume that $B \subseteq D \subseteq \mathbb{M}$, and the function that to $a \in A$ associates the E class of x_a is an embedding from A to D over E . So we have two embeddings i_A, i_B from A and B respectively to D , with i_B the inclusion, let $A' = i_A(A)$ (so it is the set of the classes of the terms in A). D is generated by A' and B , now if we have D', φ_A and φ_B as in the statement the map $\varphi : D \rightarrow D'$ which associates to $x_{t(\bar{a}, \bar{b}, \bar{e})}$ the element $t(\varphi_A(\bar{a}), \varphi_B(\bar{b}), \varphi_E(\bar{e}))$ is well defined and is a morphism (by the same argument as previously, we go through $A' \oplus_{E'} B'$), and it is naturally the unique map satisfying the universal statement. \square

The difference with the previous free amalgam notion is that it was defined for the elements of K only, which corresponds to the countable substructures of \mathbb{M} , whereas this notion is defined for small substructures. Also since our class K is axiomatisable every countable substructure of \mathbb{M} is inside of K so there is no problem in considering the structures $A' \oplus_{E'} B'$ for any countable substructures E', A', B' of \mathbb{M} .

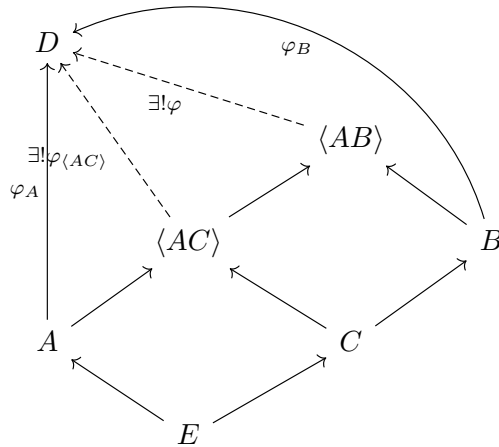
Proposition 2.1.10. *The relation $A \downarrow_E^\Gamma B$ defined for structures $E \subseteq A, B$ is invariant, symmetric, satisfies existence, extension, monotonicity, finite character, stationarity, base monotonicity, transitivity, implies \downarrow^a and implies non dividing (and non-forking by extension).*

Proof. By existence here we mean that for every extension $E \subseteq A$ we have $A \downarrow_E^\Gamma E$. The properties of symmetry, existence and the fact that it implies \downarrow^a are immediate. For monotonicity assume that $A \downarrow_E^\Gamma B$ and consider $E \subseteq C \subseteq B$ a substructure (this is enough by symmetry). We consider the free amalgam D of A and C and the associated embeddings i_A, i_C . Now let us consider the free amalgam of D and B over C and its associated embeddings j_D, j_B . There is a unique morphism $\varphi_D : D \rightarrow \langle AB \rangle$

such that $\varphi_D \cdot i_A = id_A$ and $\varphi_D \cdot i_C = id_C$. There is a unique morphism $\varphi_{D'} : D' \rightarrow \langle AB \rangle$ such that $\varphi_{D'} \cdot j_D = \varphi_D$ and $\varphi_{D'} \cdot j_B = id_B$. So $\varphi_{D'} \cdot j_D \cdot i_A = id_A$, and we get that D' is generated by the images of A and B and that $\varphi_{D'}$ is an isomorphism. We know that D is generated by the images of A and C , and that it is sent to $\langle AC \rangle \subseteq \langle AB \rangle$ by j_D , so $\varphi_{D'} \cdot j_D : D \rightarrow \langle AC \rangle$ is an isomorphism, and $A \downarrow_E^\Gamma C$.



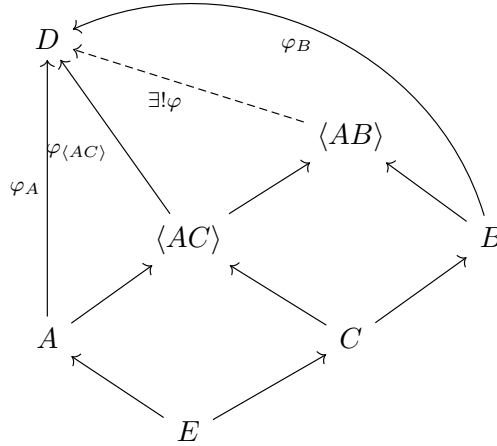
For transitivity, given structures $E \subseteq A, C, C \subseteq B$ such that $A \downarrow_E^\Gamma C$ and $\langle AC \rangle \downarrow_C^\Gamma B$ we want to show that $A \downarrow_E^\Gamma B$. For this we use the universal property Proposition 2.1.10. Consider a structure D and two embeddings φ_A, φ_B that coincide in E from A and B respectively to D . Then the embeddings φ_A and $\varphi_B|_C$ coincide over E , and by assumption there is a unique map $\varphi_{\langle AC \rangle} : \langle AC \rangle \rightarrow D$ such that $\varphi_B|_C = \varphi_{\langle AC \rangle}|_C$ and $\varphi_A = \varphi_{\langle AC \rangle}|_A$. Now we consider the two maps $\varphi_{\langle AC \rangle}$ and φ_B which coincide over C . Since $\langle AC \rangle \downarrow_C^\Gamma B$ there is a unique map $\varphi : \langle \langle AC \rangle B \rangle = \langle AB \rangle \rightarrow D$ such that $\varphi|_B = \varphi_B$ and $\varphi|_{\langle AC \rangle} = \varphi_{\langle AC \rangle}$, so in particular $\varphi|_A = \varphi_A$. This proof consist just in saying that the square formed by two adjacent pushouts forms a pushout.



For extension and stationarity, given structures $E \subseteq A, C, C \subseteq B$ such that $A \downarrow_E^\Gamma C$ we can form the free amalgam of $\langle AB \rangle$ and C over B inside of \mathbb{M} thanks to Lemma 2.1.9, this allows us to find some A' such that $\langle A'C \rangle \cong \langle AC \rangle$ (since the free amalgam defines a unique isomorphism type), so by Q.E. $A' \equiv_B A$ and $\langle A'B \rangle \downarrow_B^\Gamma C$, so $A' \downarrow_E^\Gamma C$ by transitivity, and this gives us a unique type over C for the same reason. Finite character is also a direct consequence of the universal property : assume that $A \not\downarrow_E^K B$, so there is a structure D and two embeddings φ_A, φ_B from A and B respectively to D that coincide in E such that there is no map $\varphi : \langle AB \rangle \rightarrow D$ satisfying $\varphi_A = \varphi|_A$ and $\varphi_B = \varphi|_B$. This is

equivalent to saying that the function that to an element $t(\bar{a}, \bar{b}) \in \langle AB \rangle$ associates $t(\varphi_A(\bar{a}), \varphi_B(\bar{b}))$ is not well defined (since this function is the only candidate) or that it is not a morphism. In either of those cases we can find some finite tuples $\bar{a} \in A$, $\bar{b} \in B$ such that the same thing is happening with the function that to an element $t(\bar{a}', \bar{b}') \in \langle E\bar{a}\bar{b} \rangle$ associates $t(\varphi_A(\bar{a}'), \varphi_B(\bar{b}'))$, so $\bar{a} \not\downarrow_E^K \bar{b}$.

For base monotonicity, given structures $E \subseteq A, C$, $C \subseteq B$ such that $A \downarrow_E^\Gamma B$ we want to show that $\langle AC \rangle \downarrow_C^\Gamma B$. We take the same notations as for transitivity. As previously $A \downarrow_E^\Gamma C$ (by monotonicity this time). For this we use the universal property Proposition 2.1.10. Consider a structure D and two embeddings $\varphi_{\langle AC \rangle}, \varphi_B$ from $\langle AC \rangle$ and B respectively to D that coincide in C . Then the embeddings $\varphi_{\langle AC \rangle}|_A$ and φ_B coincide over E , by assumption there is a unique map $\varphi : \langle AB \rangle = \langle \langle AC \rangle B \rangle \rightarrow D$ such that $\varphi|_A = \varphi_{\langle AC \rangle}|_A$ and $\varphi|_B = \varphi_B$. Then we also have that $\varphi|_C = \varphi_B|_C = \varphi_{\langle AC \rangle}|_C$, so since $A \downarrow_E^\Gamma C$ we have that $\varphi|_{\langle AC \rangle} = \varphi_{\langle AC \rangle}$. This proof consist just in saying that if the square formed by two adjacent squares is a pushout and the left one also is then the right one is also a pushout.



Now we want to show that $A \downarrow_E^\Gamma B$ implies that $A \downarrow_E^d B$. Let $(B_i)_{i < \omega}$ be an E -indiscernible sequence. We begin by showing that we can assume that for every $i < \omega$ $B_i \downarrow_E^a B_{<i}$, then the proof is straightforward using Proposition 2.2.1.12. Begin by adding a negative part to the sequence $(B_k)_{k \in \mathbb{Z}}$, by extension we can assume that $A \downarrow_E^\Gamma \langle (B_k)_{k \in \mathbb{Z}} \rangle$. Now set $E' = \langle (B_k)_{k < 0} \rangle$ the structure generated by the negative part. Consider the structures $B'_i = \langle E'B_i \rangle$ for $i < \omega$. They form an E' -indiscernible sequence, define $A' = \langle E'A \rangle$, by base monotonicity we have $A' \downarrow_{E'}^\Gamma \langle (B'_i)_{i < \omega} \rangle$. Now if $b_i \in B'_i$ is also in $\langle B'_{<i} \rangle$ then it has to be in E' by indiscernibility of $(B_k)_{k \in \mathbb{Z}}$ (this is the point of adding the negative part : algebraic relations happening in the sequence are already happening inside of the negative part). So, replacing A by A' , E by E' and $(B_i)_{i < \omega}$ by $(B'_i)_{i < \omega}$ we get our first point.

Now assume that $B_i \downarrow_E^a B_{<i}$ for every $i < \omega$. Since $B_0 \equiv_E B_i$ these is some A_i such that $AB_0 \equiv_E A_i B_i$. We define by induction on $i < \omega$ a sequence of structures A'_i such that $A'_i \downarrow_E^a B_{\leq i}$ and for all $j \leq i$ $AB_0 \equiv_E A'_i B_j$. Let $A_0 = A'_0 = A$. If A'_i has been defined we have $A'_i \downarrow_E^a B_{\leq i}$, $A_{i+1} \downarrow_E^a B_{i+1}$, $B_{i+1} \downarrow_E^a B_{\leq i}$ and $A'_i \equiv_E A_i$, so by Proposition 2.2.1.12 we can find a suiting A'_{i+1} . By compactness there is an A' such that $A'B_i \equiv_E AB_0$, so $A \downarrow_E^d B$. \square

Corollary 2.1.11. *T has existence, is NSOP1 and $\downarrow^K = \downarrow^a$ over arbitrary sets.*

Proof. Existence is a direct consequence of Proposition 2.1.10. We use the Kim-Pillay theorem for Kim-forking [9, Theorem 5.1], by [18, Lemma 2.7] \downarrow^a satisfies extension, existence, monotonicity, symmetry, strong finite character, and witnessing. By Lemma 2.1.7 it satisfies the amalgamation theorem, and it is clear that it satisfies transitivity. We now show that it satisfies the local character

condition. Let \bar{a} be a finite tuple, κ an uncountable regular cardinal, $(A_i)_{i < \kappa}$ a continuous increasing sequence of structure of cardinal smaller than κ and $A_\kappa = \bigcup A_i$. We want to show that there is an $i < \kappa$ such that $\langle A_i \bar{a} \rangle \downarrow_{A_i}^a A_\kappa$. We choose some $i_0 < \kappa$. If $\langle A_{i_0} \bar{a} \rangle \not\downarrow_{A_{i_0}}^a A_\kappa$ the set $\langle A_{i_0} \bar{a} \rangle \cap A_\kappa$ is non empty and has cardinal smaller than κ , so it is contained in A_{i_1} for some $i_1 > i_0$. We iterate this process to define a sequence $(i_j)_{j < \omega}$ and set $i = \bigcup_{j < \omega} i_j$ which is smaller than κ by regularity. Now if $a \in \langle A_i \bar{a} \rangle \cup A_\kappa$ it is in $\langle A_{i_j} \bar{a} \rangle \cup A_\kappa$ for some $j < \omega$ so by construction it is inside of $A_{i_{j+1}} \subseteq A_i$. \square

Remark 2.1.12. By [17, Theorem 2.7] we know that when applying the amalgamation theorem to E, A_0, A_1, B_0, B_1 (as in the statement of the amalgamation theorem) for \downarrow^K in an NSOP1 theory with existence we can take A such that $AB_i \downarrow_{B_i} B_0 B_1$ for $i = 0, 1$. In our setting by Lemma 2.1.7 and (3 Amalgamation for algebraically independent extension) we also have $AB_0 \downarrow_A^K AB_1$, however i do not know if this is true in general. Also notice that in order to get this result we only need 1 – 5 and Lemma 2.1.7 to hold and not the full generality of (3 Amalgamation for algebraically independent extension).

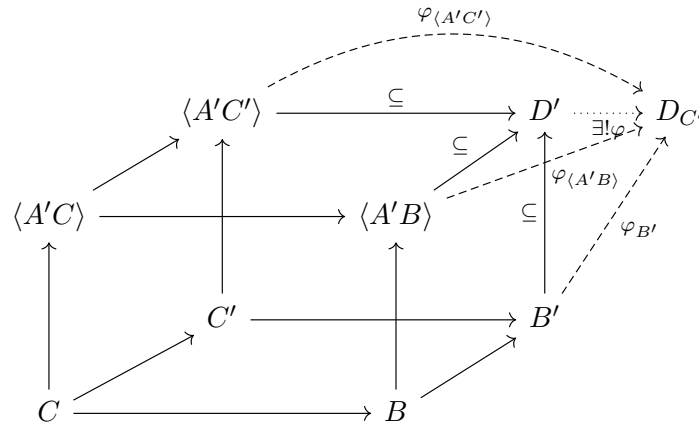
The following proposition is a general fact for theories such that algebraic independence satisfies the independence theorem, a proof can be found in [18, Proposition 3.22]. This fact is part of some more general results about deducting weak elimination of imaginaries from the properties of some independence relations (see [12, Section 4.4] for example).

Proposition 2.1.13. *The theory T has weak elimination of imaginaries.*

Definition 2.1.14. We define M -independence \downarrow^M by “forcing base monotonicity” on $\downarrow^a : A \downarrow_E^M B$ if $A \downarrow_C^a B$ for every $E \subseteq C \subseteq B$.

Proposition 2.1.15. *In \mathbb{M} we have that \downarrow^M satisfies extension, so $\downarrow^f = \downarrow^d = \downarrow^M$ over arbitrary sets.*

Proof. Consider $E \subseteq A, B \subseteq \mathbb{M}$ such that $A \downarrow_E^M B$ and $B \subseteq B'$. We can find A' such that $A' \equiv_B A$ and $\langle A'B \rangle \downarrow_B^\Gamma B'$, let us write $D' = \langle A'B' \rangle$. We show that $A' \downarrow_E^M B'$. Consider $C' \subseteq B'$ and define $C = C' \cap B$, we want to show that $\langle A'C' \rangle \cap B' = C'$. By 6, since $\langle A'C \rangle \cap C' = C$, $\langle A'C \rangle \cap B = C$ (because $A \downarrow_E^M B$) and $B \cap C' = C$, the following diagram with the solid arrows, where $\langle A'C' \rangle$ is the structure generated by A' and C' in D' , can be completed with the dashed arrows and a structure $D_{C'}$ such that $\varphi_{B'}(B') \cap \varphi_{\langle A'C' \rangle}(\langle A'C' \rangle) = \varphi_{B'}(C') = \varphi_{\langle A'C' \rangle}(C')$.



There is a unique map φ such that $\varphi|_{\langle A'B \rangle} = \varphi_{\langle A'B \rangle}$ and $\varphi|_{B'} = \varphi_{B'}$. The two maps $\varphi_{\langle A'C' \rangle}$ and $\varphi|_{\langle A'C' \rangle}$ coincide on the sets C' and $\langle A'C' \rangle$, and these sets generates $\langle A'C' \rangle$, so $\varphi|_{\langle A'C' \rangle} = \varphi_{\langle A'C' \rangle}$. If

we have some equality between two terms $t(a', c') = t'(b')$ for $a' \in A'$, $c' \in C'$ and $b' \in B'$ inside of D' then by applying φ we get that $\varphi_{\langle A' C' \rangle}(t(a', c')) = \varphi_{B'}(t'(b'))$, so by 6 there is a $c \in C'$ such that $\varphi_{\langle A' C' \rangle}(c) = \varphi_{\langle A' C' \rangle}(t(a', c'))$ and $\varphi_{B'}(c) = \varphi_{B'}(t'(b'))$. Since the maps $\varphi_{B'}$ and $\varphi_{\langle A' C' \rangle}$ are embeddings we get that $t(a', c') = t'(b') \in B'$, so $\langle A' C' \rangle \cup B' = C'$.

Now for the second part of the statement it is always true that $\downarrow^d \implies \downarrow^M$. To show that $\downarrow^M \implies \downarrow^f$ it is sufficient to show that $\downarrow^M \implies \downarrow^d$ by extension, and for this we can run the same proof as the one of $\downarrow^\Gamma \implies \downarrow^d$ in Proposition 2.1.10. \square

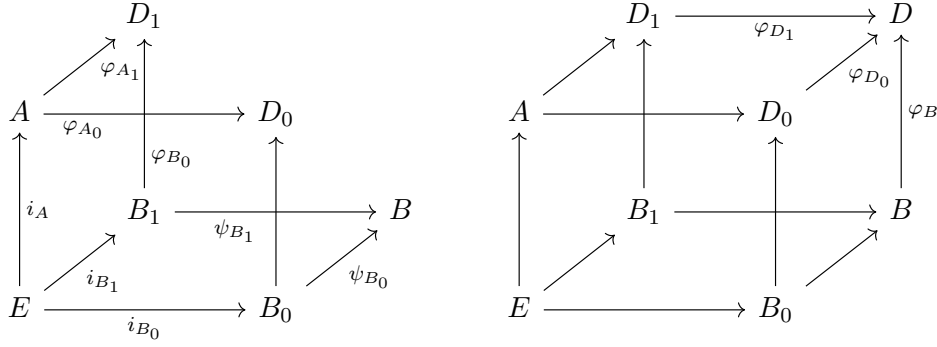
2.2 The parameterized class

2.2.1 The base case

In this section we consider a Fraïssé class K_0 and a limit T_0 satisfying the conditions (H), we show that we can 'parameterize' this class to get a new class K of structures in a new language that also satisfies (H), and that in some cases the structure we get by constructing a limit of K is not simple, and so strictly NSOP1 by Corollary 2.1.11. We need the assumption that any of the structures of K_0 can be embedded together, or equivalently that the structures of K_0 agree on what the isomorphism type of structure generated by the constants is. This can be done by restricting K_0 to a smaller class.

Let L_0 be a countable language and K_0 a Fraïssé class of finite L_0 -structures satisfying (H), so :

1. (H.P.) : If $A \subseteq B \in K_0$ is a substructure then $A \in K_0$
2. (Free Amalgamation) : If $C \subseteq A, B$ are substructures, there is $D \in K_0$ and embeddings i_A, i_B of A and B respectively to D satisfying $i_A|_C = i_B|_C$, $i_A(A) \cap i_B(B) = i_A(C)$, D is generated by the images of A and B and such that given any morphism of L -structures $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ to $D' \in K_0$ there is a unique morphism $\varphi : D \rightarrow D'$ such that $\varphi_A = \varphi \cdot i_A$ and $\varphi_B = \varphi \cdot i_B$. We will write $A \oplus_C B$ for this structure D .
3. (Generic extension) : If $A \in K_0$ and V is a sort there is an extension $A \oplus_{\emptyset} \langle x \rangle \in K_0$ generated by a single element $x \in V$ over A such that for every morphism $\varphi : A \rightarrow A' \in K_0$ and every element $a' \in V(A')$ there is a unique morphism $\psi : A \oplus_{\emptyset} \langle x \rangle \rightarrow A'$ extending φ by sending x to a' .
4. K_0 is axiomatisable : if the isomorphism class of an L -structure A is not inside of K there is a formula φ and a finite tuple $a \in A$ such that $A \models \varphi(a)$ and $A' \not\models \varphi(a')$ for every $a' \in A' \in K$.
5. Any limit of K_0 has Q.E. : for every quantifier free formula $\varphi(x, y)$ satisfied in some structure of K_0 there is a quantifier free formula $\psi(x)$ such that if $a \in A \in K_0$ and $A \models \psi(a)$ then there is $A \subseteq B \in K_0$ an extension and $b \in B$ such that $B \models \varphi(a, b)$.
6. (3 Amalgamation for algebraically independent extension). Given a commutative diagram in K_0 as on the left, where the arrows are embeddings, $\varphi_{B_i}(B_i) \cap \varphi_{A_i}(A) = \varphi_{A_i} \cdot i_A(E) = \varphi_{B_i} \cdot i_{B_i}(E)$ for $i = 0, 1$ and $\psi_{B_0}(B_0) \cap \psi_{B_1}(B_1) = \psi_{B_0} \cdot i_{B_0}(E) = \psi_{B_1} \cdot i_{B_1}(E)$, we can complete it inside of K_0 into a commutative diagram as on the right (where the arrows are embeddings) such that $\varphi_{D_i}(D_i) \cap \varphi_B(B) = \varphi_{D_i} \cdot \varphi_{B_i}(B_i) = \varphi_B \cdot \psi_{B_i}(B_i)$ for $i = 0, 1$ and $\varphi_{D_0}(D_0) \cap \varphi_{D_1}(D_1) = \varphi_{D_0} \cdot \varphi_{A_0}(A) = \varphi_{D_1} \cdot \varphi_{A_1}(A)$.



Write T_0 the theory of some fixed limit of K_0 and let \mathbb{M}_0 be a monster model of T_0 . By the previous section we know that T_0 has Q.E., satisfies that $acl(A) = dcl(A) = \langle A \rangle$ and that \perp^a satisfies the independence theorem over arbitrary sets, which implies that T_0 is NSOP1 and that $\perp^K = \perp^a$ over arbitrary sets in \mathbb{M}_0 .

Definition 2.2.1.1. Let L be a new language built from L_0 by adding a new sort P (we will refer to the sorts of L_0 as the object sorts and write them O) and replacing every symbol of constant c in L_0 of sort O by a function symbol $c_- : P \rightarrow O$, every function symbol $f : \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ by and function symbol $f_- : P \times \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ and similarly for relation symbols. For an L structure A and $p \in P(A)$ we write A_p to denote the L_0 -structure induced by p on $O(A)$. For a set of objects A and a tuple of parameters \bar{p} we will write $\langle A \rangle_{\bar{p}}$ for the structure generated by (A, \bar{p}) , we will also write $\langle A \rangle$ for the structure generated by a set A (eventually containing parameters and objects), so $\langle A \rangle = \langle O(A) \rangle_{P(A)}$.

Definition 2.2.1.2. Let K be the class of countable L -structures A such that $A_p \in K_0$ for every $p \in P(A)$.

Let us set some notations : we will write $A \oplus_{\emptyset} \langle \bar{x} \rangle$ for the structure freely generated over $A \subseteq \mathbb{M}_0$ by a tuple \bar{x} of distinct elements. Such an extension exists by the (Generic extension) assumption. For an L_0 -formula φ and a parameter p we will write φ_p for the L -formula parameterized by p .

Remark 2.2.1.3. The class K is axiomatisable : if an L -structure A is not in K then for some parameter $p \in P(A)$ there is some $p \in A$ such that $A_p \notin K_0$, so there is an L_0 -formula φ such that $A_p \models \varphi$ and such that no structures in K_0 satisfy φ . So $A \models \varphi_p$ and no structure in K satisfy this formula.

Lemma 2.2.1.4. The class K satisfies the condition of (Free Amalgamation).

Proof. Consider the structures $E \subseteq A, B \in K$. We will construct an L -structure in K satisfying the (Free Amalgamation) universal property. Let us write $E = (O(E), P(E))$, $A = (O_A, P_A)$ and $B = (O_B, P_B)$. We begin by defining $P(D) := (P(A) \setminus P(E)) \dot{\cup} (P(B) \setminus P(E))$ which will be the parameter sort of the free amalgam in K . Let $(p_i)_{i < \omega}$ enumerate $P(D)$. We define a sequence $((D_i^k)_{i < \omega})_{k < \omega}$ of tuples of countable L_0 -structures indexed by P_D by induction. Let $D_{-1}^0 := A \cup B$.

We define $D_{p_i}^0 := (A_{p_i} \oplus_{E_{p_i}} B_{p_i}) \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \setminus (A \cup B) \rangle$ if $p_i \in P(E)$, $D_{p_i}^0 := A_{p_i} \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \setminus A \rangle$ if $p_i \in P(A) \setminus P(E)$, $D_{p_i}^0 := B_{p_i} \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \setminus B \rangle$ if $p_i \in P(B) \setminus P(E)$, and finally $D_{p_i}^0 := E_0 \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \rangle$ if $p_i \in P(D) \setminus (P(A) \cup P(B))$. Then set $D^0 = \bigcup_{i < \omega} D_{p_i}^0$. If D^k is defined for some $1 \leq k < \omega$ we set $D_{p_0}^{k+1} = D_{p_0}^k \oplus_{\emptyset} \langle D^k \setminus D_{p_0}^k \rangle$, similarly if $D_{p_i}^k$ is defined for some $1 \leq i < \omega$ we set $D_{p_{i+1}}^{k+1} = D_{p_{i+1}}^k \oplus_{\emptyset} \langle D_i^{k+1} \setminus D_{p_i}^k \rangle$.

Every D_p^k is an L_0 -structure for the parameter p . We set $O(D) = \bigcup_{k < \omega} D^k$, this set is countable and has an L_0 -structure for the parameter p given by $D_p = \bigcup_{k < \omega} D_p^k \in K_0$, so $D \in K$. This defines an L structure $D = (O(D), P(D)) \in K$ which extends A and B , and such that $A \cap B = E$ inside of D . Also if A and B are generated by $(\bar{a}, P(A))$ and $(\bar{b}, P(B))$ respectively then $(\overline{ab}, P(A), P(B))$ generates D .

Now that we constructed the structure D we show that it satisfies the right property. Consider $D' \in K$, $\varphi_A : A \rightarrow D'$ and $\varphi_B : B \rightarrow D'$ as in the statement, and write φ_E their restriction to E . The uniqueness of φ is clear from the fact that D is generated by the images of A and B . To prove existence we construct an increasing sequence of tuples of morphism of L_0 -structures indexed on $P(D) : ((\varphi_{p_i}^k : D_{p_i}^k \rightarrow D'_{\varphi_{P(p_i)}})_{i < \omega})_{k < \omega}$. Let us begin by defining φ on the parameters, let $\varphi(p_i) = \varphi_E(p_i)$ if $p_i \in P(E)$, $\varphi(p_i) = \varphi_A(p_i)$ if $p_i \in P(A) \setminus P(E)$ and $\varphi(p_i) = \varphi_B(p_i)$ if $p_i \in P(B) \setminus P(E)$.

For $k = 0$, if $p_i \in P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_{P(p_i)}}$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_{P(p_i)}}$ and $\varphi_B(p, -) : B_p \rightarrow D'_{\varphi_{P(p_i)}}$ and sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$. If $p_i \in P(A) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_{P(p_i)}}$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_{P(p_i)}}$ and sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$, similarly if $p_i \in P(B) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_{P(p_i)}}$ that extends $\varphi_B(p, -) : B_{p_i} \rightarrow D'_{\varphi_{P(p_i)}}$ and sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$.

Assume that for every $i < \omega$ the functions $\varphi_p^k : D_p^k \rightarrow D'_{\varphi(p)}$ have been defined. Define $\varphi_{p_0}^{k+1} : D_{p_0}^{k+1} \rightarrow D'$ as the unique morphism extending $\varphi_{p_0}^k : D_{p_0}^k \rightarrow D'$ by sending $x \in D_{p_i}^k$ to $\varphi_{p_i}^k(x)$ for every $i < \omega$. We just repeat the same steps for the other parameters. Once all these functions are constructed we consider $\varphi = \bigcup_{k < \omega} \varphi_p^k$ for every $p \in P(D)$. This function defines a map in K from D to D' such that its restrictions to A and B are φ_A and φ_B respectively, and it is a morphism of L_0 -structure since $\varphi = \bigcup_{k < \omega} \varphi_p^k : D_p \rightarrow D'_{\varphi(p)}$ is a morphism of L_0 -structure for every parameter $p \in P(D)$. \square

Remark 2.2.1.5. *This means that K seen as a category has pushouts for pairs of monomorphisms. The proof also implies that the construction of D does not depend (up to isomorphism) on the choices of the bases of A and B and on the enumeration of the parameters : given the same construction for a different enumeration $D^* = \bigcup_{i < \omega} D_p^{*,k}$ we can show that D^* also has this universal property with a similar proof.*

Remark 2.2.1.6. *From the proof of Lemma 2.2.1.4 it is clear how to show that K satisfies the (Generic extension property), we begin with a sort among the object sorts, we fix some enumeration $(p_i)_{i < \omega}$ of the parameters, we begin by adding one object x in the right sort and consider the associated L_0 -structure $A_{p_0}^0 = A \oplus_{\emptyset} \langle x \rangle$, then $A_{p_{i+1}}^0 = A \oplus_{\emptyset} \langle A_{p_i}^0 \setminus A \rangle$, $A_{p_{i+1}}^{k+1} = A_{p_{i+1}}^k \oplus_{\emptyset} \langle A_{p_i}^{k+1} \setminus A_{p_{i+1}}^k \rangle$ and so on. When adding a generic parameter p we consider the L_0 -structure E_0 it generates, i.e. the structure generated by the constants, and form $A \oplus_{\emptyset} \langle E_0, p \rangle$ the same way but with adding a parameter p before the enumeration $(p_i)_{i < \omega}$ of $P(A)$ for which we consider the structure freely generated by A over E_0 , $E_0 \oplus_{\emptyset} \langle A \rangle$. We can indeed send p to any parameter because any parameter agree with p on the structure generated by the constants.*

A substructure of a finitely generated structure is not necessarily finitely generated : an L_P structure with only one parameter is the same as an L_0 -structure, so depending on T_0 (if it is uniformly locally finite for example) by removing all parameters except one we can find examples of this fact. With the properties of K we have proven so far we can construct a limit \mathcal{M} with Proposition 2.1.2 and consider its theory T and \mathbb{M} a monster model of it.

Lemma 2.2.1.7. *K satisfies 4 : for every quantifier free formula $\varphi(x, y)$ satisfied in some structure of K there is a quantifier free formula $\psi(x)$ such that if $a \in A \in K$ and $A \models \psi(a)$ then there is $A \subseteq B \in K$ an extension and $b \in B$ such that $B \models \varphi(a, b)$.*

Such a formula $\varphi(x, y)$ can be decomposed the following way : take $x = x', \bar{p}, y = y' \bar{q}$ with x', y' in the object sorts and \bar{p}, \bar{q} in the parameters, then $\varphi(x, y)$ is implied by a formula of the form $\bigwedge_{i < n} \varphi_{i, p_i}(x', y') \wedge \bigwedge_{j < m} \varphi'_{j, q_j}(x', y')$ where the φ_i, φ'_j are quantifier free L_0 -formulas and the parameters p_i and q_j are all distinct. By assumption if ψ_i is the formula associated to $\varphi_i(x', y')$ then the formula $\bigwedge_{i < n} \psi_{i, p_i}(x') \wedge \bigwedge_{i \neq i' < n} (p_i \neq p_{i'})$ satisfies our condition : if $(a', \bar{p}') \in A' \in K$ satisfies it by assumption there are L_0 -structures $(B'_i)_{i < n}$ and $b'_i \in B'_i$ such that $A'_{p_i} \subseteq B'_i \models \varphi_i(a', b'_i)$. For $j < m$ we can find some $b'_{q_j} \in B_{q_j} \in K_0$ containing the objects of A' such that $B_{q_j} \models \varphi'_j(a', b'_{q_j})$. We can build B' an L structure extending A' with parameters $P(A') \cup \{q_j : j < m\}$ by amalgamating all of these L_0 -structure freely over $A \cup b'$ (as in the proof of Lemma 2.2.1.4), we then have that $B' \models \varphi(a', b')$. Once this condition is satisfied we can show with the same proof as in Lemma 2.1.6 that T has Q.E. in L .

Proposition 2.2.1.8. *T proves the following statements :*

1. *If \bar{a} is a finite tuple of objects and $\pi_i(\bar{x})$ is a finitary type in T_0 for every $i \leq n$ such that $\pi_i \vdash x_j \neq x_{j'}$ for $j \neq j'$ there exist an infinite number of tuples of parameters $(p_i)_{i \leq n}$ such that $\models \pi_i(\bar{a}_{p_i})$ for every $i \leq n$.*
2. *If $\bar{p} = p_0, \dots, p_n$ is a finite tuple of parameters and $\pi_i(\bar{x}, \bar{b}_{p_i})$ is a type in \mathcal{M}_{p_i} over a tuple of objects \bar{b} such that the induced type in any of the x_i is non algebraic and that $\pi_i \vdash x_j \neq x_{j'}$ for $j \neq j'$ then there is a tuple of objects \bar{a} such that $\models \pi_i(\bar{a}_{p_i}, \bar{b}_{p_i})$ for every $i \leq n$.*

Proof. Consider the structure \mathcal{M} from Proposition 2.1.2, a finite tuple of object $\bar{a} \in \mathcal{M}$ and a finite set of types $\pi_i(\bar{x})$ as in the first point of the statement. We can build a structure $A \in K$ containing \bar{a} and with parameters $(p_i)_{i \leq n}$ such that $\models \pi_i(\bar{a}_{p_i})$ for every $i \leq n$. Then by saturation we can embed this structure A inside of \mathcal{M} over \bar{a} and find the parameters. The second point is similar : consider the structure B generated by \bar{b}, \bar{p} and types $\pi_i(\bar{x}, \bar{b}_{p_i})$ as in the statement. We can extend B into $B' \in K$ that contains a tuple of objects \bar{a}' such that $\bar{a}' \bar{b}$ has the right type for the parameter p_i for every i and then embed B' into \mathcal{M} over B to find the right objects. \square

Corollary 2.2.1.9. *The theory T has I.P. if there is two distinct non algebraic types in T_0 and is non-simple if the generated structure in K_0 is non trivial, in the sense that we can find $c \in \langle ab \rangle \setminus (\langle a \rangle \cup \langle b \rangle)$ for some $a, b \in \mathbb{M}_0$.*

Proof. For I.P. consider two distinct T -types π_i in \bar{x} for $i = 0, 1$. Now if we consider a sequence $(p_i)_{i < \omega}$ of distinct parameters in \mathbb{M} then by 2.2.1.8 for every $I \subseteq \omega$ we can find \bar{a}_I such that $\models \pi_0(\bar{a}_I)$ if and only if $i \in I$.

Now for simplicity let $t(x, y)$ be a L_0 -term such that $t(a, b) \notin \langle a \rangle \cup \langle b \rangle$ and consider a tree of parameters all distinct $(p_\mu)_{\nu \in \omega < \omega}$. Lets now consider a tree of distinct objects $(a_\mu)_{\mu \in \omega < \omega}$ and the formulas $\varphi(x, y, a_{\mu \smallfrown i}, p_\mu) := t_{p_\mu}(x, y) = a_{\mu \smallfrown i}$ along this tree. Clearly $\{\varphi(x, y, a_{\mu \smallfrown i}, p_\mu) : i < \omega\}$ is 2-inconsistent for every $\mu \in \omega^{< \omega}$, and by 2.2.1.8 every path is consistent. \square

Remark 2.2.1.10. *The relation \perp^Γ in \mathbb{M} does not necessarily satisfy local character (otherwise it would satisfy all conditions of the Kim-Pillay theorem) : take some distinct objects $\bar{o} = o_1, o_2, o_3$ and a set $P = (p_i)_{i < \kappa}$ of parameters for $\kappa = |T_P|^+$ such that $o_3 \in \langle o_1 o_2 \rangle_{p_i}$ for every $i < \kappa$ and $o_j \notin \langle P \rangle$ for every $j \in \{0, 1, 2\}$. Then there is no subset P_0 of cardinal $|T|$ of P such that $\bar{o} \perp_{P_0}^\Gamma P$.*

Remark 2.2.1.11. In the case where we can find in K_0 some $c = t(a, b) \in \langle ab \rangle \setminus (\langle a \rangle \cup \langle b \rangle)$ then the relation \downarrow^a in \mathbb{M} does not admit 4-amalgamation : take 3 distinct parameters p_0, p_1, p_2 and distinct objects b_0, b_1, b_2 such that $b_i, b_j \notin \langle p_k \rangle$ for every i, j, k such that $\{i, j, k\} = \{0, 1, 2\}$ and such that $t_{p_0}(b_1, b_2) = t_{p_1}(b_0, b_2)$, $t_{p_1}(b_0, b_2) = t_{p_2}(b_0, b_1)$ and $t_{p_0}(b_1, b_2) \neq t_{p_2}(b_1, b_2)$. Then the type $\text{tp}(p_0/b_1b_2) \cup \text{tp}(p_1/b_0b_2) \cup \text{tp}(p_2/b_0b_1)$ is inconsistent.

Since the theory T_0 has Q.E., is 'one based for \downarrow^K , and since the algebraic closure and the generated structure coincide in \mathbb{M}_0 we can represent non-Kim-forking extension as commutative squares in the following way : the arrows are embeddings, the structure D is generated by the images $f(A)$ and $g(B)$, and $f(A) \cap g(B) = f \cdot j(E) = g \cdot i(E)$. Then the isomorphism type of D over B gives us a type over D which correspond to a non-forking extension of $\text{tp}(A/E)$. We will refer to such squares as strong amalgams. However for the (3 Amalgamation for algebraically independent extension) the squares we are considering consists in an extension of such a square, meaning that we compose with an embedding from D to some $D' \in K_0$.

$$\begin{array}{ccc} A & \xrightarrow{f} & D \\ \uparrow j & & \uparrow g \\ E & \xrightarrow{i} & B \end{array}$$

Proposition 2.2.1.12. K satisfies (3 Amalgamation for algebraically independent extension).

Proof. Consider the following diagram in K , which is as in the definition of (3 Amalgamation for algebraically independent extension) and where for simplicity we take the embeddings to be inclusions. We want to complete it, for this we begin by choosing a completion for every parameter. Fix an enumeration $(p_i)_{i < \omega}$ of $P(D) := P(D_0) \dot{\cup} [P(D_1) \setminus P(A)] \dot{\cup} [P(B) \setminus P(B_0) \dot{\cup} P(B_1)]$, we will take this set as the set of parameters of our structure D . Looking at the L_0 -structure indexed by any of the parameters $p_j \in P(E)$ we can find an L_0 -structure $D_{p_j}^{*,0}$ as in the right diagram.

$$\begin{array}{ccc} & & D_1 \\ & \nearrow \varphi_{A_1} & \uparrow \\ A & \xrightarrow{\varphi_{A_0}} & D_0 \\ \uparrow i_A & & \downarrow \varphi_{B_0} \\ E & \xrightarrow{i_{B_0}} & B_0 \\ & \nearrow i_{B_1} & \uparrow \psi_{B_1} \\ & & B_1 \end{array} \quad \begin{array}{ccc} & & D_{1,p} \\ & \nearrow & \xrightarrow{\varphi_{D_1}} \\ A_{p_j} & \xrightarrow{} & D_{0,p_j} \\ \uparrow & & \downarrow \varphi_{D_0} \\ E_{p_j} & \xrightarrow{} & B_{1,p_j} \\ & \nearrow & \uparrow \\ & & B_{p_j} \\ & \xrightarrow{} & \uparrow \varphi_B \\ & & D_{p_j}^{*,0} \\ & \xrightarrow{} & B_{0,p_j} \end{array}$$

These sets $D_{p_j}^{*,0}$ for $p_j \in P(E)$ are also equipped with the L -structure on the sets $\varphi_B(B)$ and $\varphi_{D_i}(D_i)$ for $i = 0, 1$ induced by the embeddings. We now define their equivalent for the parameters of D_i and B . Let $p_j \in P(A) \setminus P(E)$. We consider the L_0 -structure $D_{p_j}^{*,0} = [D_{0,p_j} \oplus_{A_{p_j}} D_{1,p_j}] \oplus_{\emptyset} \langle B \setminus (B_0 \cup B_1) \rangle$ with the natural mappings from D_0 , D_1 and B to $D_{p_j}^{*,0}$. Similarly if $p_j \in P(B_i) \setminus P(E)$ set $D_{p_j}^{*,0} = [D_0 \oplus_{B_0} B] \oplus_{\emptyset} \langle D_1 \setminus (A \cup B_1) \rangle$ with the natural mappings from D_0 , D_1 and B to $D_{p_j}^{*,0}$. If $p_j \in P(D_0) \setminus (P(A) \cup P(B_0))$, set $D_{p_j}^{*,0} = D_{0,p_j} \oplus_{\emptyset} \langle (B \setminus B_0) \cup (D_1 \setminus A) \rangle$ with the natural mappings from D_0 , D_1 and B to $D_{p_j}^{*,0}$ and similarly for $p_j \in P(D_1) \setminus (P(A) \cup P(B_1))$. If $p_j \in P(B) \setminus (P(B_0) \cup P(B_1))$, set $D_{p_j}^{*,0} = B_{p_j} \oplus_{\emptyset} \langle (D_0 \setminus B_0) \cup (D_1 \setminus B_1) \rangle$ with the natural mappings from D_0 , D_1 and B to $D_{p_j}^{*,0}$.

Now that we have all of our L_0 -structures we will just put them together in a canonical way : we define an increasing sequence of tuples of L_0 -structures $((D_{p_j}^k)_{j < \omega})_{k < \omega}$. Let $D_{p_0}^0 = D_{p_0}^{*,0}$, if $D_{p_j}^0$ has been defined for some $j < \omega$ let $D_{p_{j+1}}^0 = D_{p_{j+1}}^{*,0} \oplus_{\emptyset} \langle D_{p_j}^0 \setminus (D_0 \cup D_1 \cup B) \rangle$ where we assimilate D_0 , D_1 and B with their image in $D_{p_j}^0$. Once $D_{p_j}^0$ has been defined for every $j < \omega$ we set $D^0 = \bigcup_{i < \omega} D_{p_j}^0$. If D^k is defined we let $D_{p_0}^{k+1} = D_{p_0}^k \oplus_{\emptyset} \langle D^k \setminus D_{p_0}^k \rangle$. Now assume that $D_{p_j}^{k+1}$ is defined for some $k < \omega, j < \omega$, we set $D_{p_{j+1}}^{k+1} = D_{p_{j+1}}^k \oplus_{\emptyset} \langle D_{p_j}^{k+1} \setminus D_{p_{j+1}}^k \rangle$. Then take $D = \bigcup_{k < \omega} D_{p_j}^k$ with the L_0 -structure induced for every $j < \omega$, this defines an L -structure, the natural mappings we had from D_0 , D_1 and B to D are embeddings for the L -structure we put on D and they satisfy the conditions. \square

Finally we can state that the class K satisfies the conditions of (H), and so that T is an NSOP1 theory with existence such that $\downarrow^K = \downarrow^a$ over arbitrary sets.

2.2.2 Some structure on the parameters

In this subsection we describe how we can generalise the previous part by having some structure on the parameters, the proof are essentially similar so we wont give as much details. Let K_0 be a class of L_0 structures as in the previous section : it satisfies (H) and there is only one isomorphism type for the structure generated by the constants. Let us consider an other class K_P of one-sorted L_P -structures satisfying (H).

Definition 2.2.2.1. *Let L be a new language built from L_0 and L_P by adding a new sort P (we will refer to the sorts of L_0 as the object sorts and write them O) to L_0 which we equip with L_P , and replacing every symbol of constant c in L_0 of sort O by a function symbol $c_- : P \rightarrow O$, every function symbol $f : \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ by and function symbol $f_- : P \times \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ and similarly for relation symbols. For an L structure A and $p \in P(A)$ we write A_p to denote the L_0 -structure induced by p on $O(A)$. For a set of objects A and a tuple of parameters \bar{p} we will write $\langle A \rangle_{\bar{p}}$ for the structure generated by (A, \bar{p}) , we will also write $\langle A \rangle$ for the structure generated by a set A (eventually containing parameters and objects), so $\langle A \rangle = \langle O(A) \rangle_{P(A)}$.*

Definition 2.2.2.2. *Let K be the class of countable L -structures A such that $P(A) \in K_P$ and $A_p \in K_0$ for every $p \in P(A)$. In particular the previous section corresponds to the case of K_P being the class of countable sets with the equality.*

Remark 2.2.2.3. *The class K is axiomatisable : if an L -structure A is not in K then either for some parameter $p \in P(A)$ there is some $p \in A$ such that $A_p \notin K_0$, in which case the same argument as in Remark 2.2.1.3 applies, or $P(A) \notin K_P$, which is then also expressed by a formula by 4 in K_P . Also K clearly satisfies (H.P.).*

Lemma 2.2.2.4. *K satisfies 4 : for every quantifier free formula $\varphi(x, y)$ satisfied in some structure of K there is a quantifier free formula $\psi(x)$ such that if $a \in A \in K$ and $A \models \psi(a)$ then there is $A \subseteq B \in K$ an extension and $b \in B$ such that $B \models \varphi(a, b)$.*

Proof. Such a formula $\varphi(x, y)$ can be decomposed the following way : take $x = x', \bar{p}$, $y = y', \bar{q}$ with x', y' in the object sorts and \bar{p}, \bar{q} in the parameters, then $\varphi(x, y)$ is implied by a formula of the form $\varphi(\bar{p}, \bar{q}) \wedge \bigwedge_{i < n} \varphi_{i, p_i}(x', y') \wedge \bigwedge_{j < m} \varphi'_{j, q_j}(x', y')$ where φ a quantifier free L_P -formula, the φ_i, φ'_j are quantifier free L_0 -formulas and the parameters p_i and q_j are all distinct. By assumption if ψ is the formula associated to φ by 5 in K_P and ψ_i is the formula associated to $\varphi_i(x', y')$ by 5 in K_0 then the formula $\psi(\bar{p}) \wedge \bigwedge_{i < n} \psi_{i, p_i}(x')$ satisfies our condition : if $(a', \bar{p}') \in A' \in K$ satisfies it by assumption there is $\bar{q} \in B'_P \supseteq P(A')$ an L_P structure such that $B'_P \models \varphi(\bar{p}, \bar{q})$. We also have L_0 -structures $(B'_i)_{i < n}$ and $b'_i \in B'_i$ such that $A'_i \subseteq B'_i \models \varphi_i(a', b'_i)$.

Now we construct some extension B' of A' in K from all of these pieces. We set $P(B') = B'_p$, for every $j < m$ we choose some L_0 -structure $B_{q_j} \in K_0$ containing $O(A) \cup b'$ such that $B_{q_j} \models \varphi'_j(a', b'_{q_j})$, for any other $q \in P(B')$ we choose some L_0 -structure $B_q \in K_0$ containing $O(A) \cup b'$ and then we build $O(B')$ by amalgamating all of these L_0 -structure over $A \cup b'$, we get that $B' \models \varphi(a', b')$. \square

Lemma 2.2.2.5. *The class K satisfies the conditions of (Free Amalgamation) and (Generic extension).*

Proof. Consider the structures $E \subseteq A, B \in K$. We will construct an L -structure in K satisfying the (Free Amalgamation) universal property. Let us write $E = (O(E), P(E))$, $A = (O_A, P_A)$ and $B = (O_B, P_B)$. We begin by forming the free amalgam $P(D) := P(A) \oplus_{P(E)} P(B) \in K_P$ which will be the parameter sort of the free amalgam in K .

Let $(p_i)_{i < \omega}$ enumerate $P(D)$. We define a sequence $((D_i^k)_{i < \omega})_{k < \omega}$ of tuples of countable L_0 -structures indexed by P_D by induction. Let $D_{-1}^0 := A \cup B$ and let E_0 be the structure generated by the constants inside of K_0 (which is unique).

We define $D_{p_i}^0 := (A_{p_i} \oplus_{E_{p_i}} B_{p_i}) \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \setminus (A \cup B) \rangle$ if $p_i \in P(E)$, $D_{p_i}^0 := A_{p_i} \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \setminus A \rangle$ if $p_i \in P(A) \setminus P(E)$, $D_{p_i}^0 := B_{p_i} \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \setminus B \rangle$ if $p_i \in P(B) \setminus P(E)$, and finally $D_{p_i}^0 := E_0 \oplus_{\emptyset} \langle D_{p_{i-1}}^0 \rangle$ if $p_i \in P(D) \setminus (P(A) \cup P(B))$. Then set $D^0 = \bigcup_{i < \omega} D_{p_i}^0$. If D^k is defined for some $1 \leq k < \omega$ we set $D_{p_0}^{k+1} = D_{p_0}^k \oplus_{\emptyset} \langle D^k \setminus D_{p_0}^k \rangle$, similarly if $D_{p_i}^k$ is defined for some $1 \leq i < \omega$ we set $D_{p_{i+1}}^{k+1} = D_{p_{i+1}}^k \oplus_{\emptyset} \langle D_i^{k+1} \setminus D_{p_i}^k \rangle$.

Every D_p^k is an L_0 -structure for the parameter p . We set $O(D) = \bigcup_{k < \omega} D^k$, this set is countable and has an L_0 -structure for the parameter p given by $D_p = \bigcup_{k < \omega} D_p^k \in K_0$, so $D \in K$. This defines an L structure $D = (O(D), P(D)) \in K$ which extends A and B , and such that $A \cap B = E$ inside of D . Also if A and B are generated by $(\bar{a}, P(A))$ and $(\bar{b}, P(B))$ respectively then $(\bar{a}\bar{b}, P(A), P(B))$ generates D .

Now that we constructed the structure D we show that it satisfies the right property. Consider $D' \in K$, $\varphi_A : A \rightarrow D'$ and $\varphi_B : B \rightarrow D'$ as in the statement, and write φ_E their restriction to E . The uniqueness of φ is clear from the fact that D is generated by the images of A and B . By free amalgamation in K_P there is a unique morphism φ_P of K_P -structures from $P(D)$ to $P(D')$ such that the diagram commutes. For existence we construct an increasing sequence of tuples of morphism of L_0 -structures indexed on $P(D) : ((\varphi_{p_i}^k : D_{p_i}^k \rightarrow D'_{\varphi_P(p_i)})_{i < \omega})_{k < \omega}$.

For $k = 0$, if $p_i \in P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_P(p_i)}$ and $\varphi_B(p, -) : B_p \rightarrow D'_{\varphi_P(p_i)}$ and sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$. If $p_i \in P(A) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_P(p_i)}$ and sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$, similarly if $p_i \in P(B) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that extends $\varphi_B(p, -) : B_{p_i} \rightarrow D'_{\varphi_P(p_i)}$ and sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$, and finally if $p_i \in P(D) \setminus (P(A) \cup P(B))$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$.

Assume that for every $i < \omega$ the functions $\varphi_p^k : D_p^k \rightarrow D'$ have been defined. Define $\varphi_{p_0}^{k+1} : D_{p_0}^{k+1} \rightarrow D'$ as the unique morphism extending $\varphi_{p_0}^k : D_{p_0}^k \rightarrow D'$ by sending $x \in D_{p_i}^k$ to $\varphi_{p_i}^k(x)$ for every $i < \omega$. We just repeat the same steps for the other parameters. Once all these functions are constructed we consider $\varphi = \bigcup_{k < \omega} \varphi_p^k$ for every $p \in P(D)$. This function defines a map in K from D to D' such that its restrictions to A and B are φ_A and φ_B respectively, and it is a morphism of L_0 -structure

since $\varphi = \bigcup_{k < \omega} \varphi_p^k : D_p \rightarrow D'_{\varphi(p)}$ is a morphism of L_0 -structure for every parameter $p \in P(D)$ and $\varphi_P : P(D) \rightarrow P(D')$ is a morphism of L_P -structure.

The proof of (Generic extension) carries over similarly. \square

Proposition 2.2.2.6. *K satisfies (3 Amalgamation for algebraically independent extension).*

Proof. The proof carries over verbatim, we form the amalgam for the parameter sort to find $P(D)$, then for every $p \in P(D)$ we choose some L_0 -structures D_p^0 , using (3 Amalgamation for algebraically independent extension) in K_0 when the parameter p is in the basis, and we put all of these in common by amalgamating them freely as we did in Lemma 2.2.2.5. \square

2.3 Examples and non-examples

One example of a class K_0 satisfying (H) is the class of countable vector spaces over a finite (or countable) field (which is in the language). A trivial one is the class of countable sets with no structure, and an other one is the class of countable graphs (or countable sets with a binary relation). In the case where a limit T_0 of the class K_0 is a stable theory we get that $\perp^a = \perp^\Gamma$. We will describe now how adding generic structure to the class K_0 might affect the conditions (H) and some properties of the theory.

2.3.1 Adding a generic predicate

Definition 2.3.1.1. *Consider a new symbol U for a unary predicate in some sort S , we define $L_U = L \cup \{U\}$ and the class K_U of L_U -structures consisting of a structure A in K_0 equipped with a subset $U(A) \subseteq S(A)$.*

Proof. The conditions 1 – 4 are satisfied in K_U , we just take $U(A \oplus_E B) := U(A) \cup U(B)$ and $U(A \oplus_\emptyset \langle x \rangle) := U(A)$, similarly for 6. For 5 consider a finitely generated structure $\langle a, b \rangle \in K_U$ and a quantifier free L_U formula φ such that $\models \varphi(a, b)$. W.l.o.g. w.c.a. that $\varphi(x, y) = \varphi'(x, y) \wedge \bigwedge_{i < n} U(t_i(x, y)) \wedge \bigwedge_{i < n} \neg U(t'_i(x, y))$ where $\varphi'(x, y)$ is an L_0 -formula and the t_i and t'_i are L_0 -terms. By taking a strengthening of φ we can assume that $\varphi'(x, y) \vdash \bigwedge_{j < n} t'_i(x, y) \neq t_j(x, y)$ for every $i < n$. Now we consider the formula $\psi'(x)$ associated to the L_0 -formula $\varphi'(x, y) \wedge \bigwedge_{i, j < n} (t'_i(x, y) \neq t_j(x, y))$ by 5 in K_0 . If for some $a' \in A' \in K_R$ we have that $A' \models \psi'(a')$ then by assumption we can find some $b' \in B'$ an extension of A' in K_0 such that $B' \models \varphi'(a', b') \wedge \bigwedge_{i < n} (t'_i(a', b') \notin \langle t_j(a', b') \rangle_{j < n})$. Define the subset U on B' to be $U(A') \cup \{t_j(a', b')\}_{j < n}$, we get that $B' \models \neg U(t'_i(a', b'))$ for all $i < n$, so $B' \models \varphi(a', b')$, and B' is an extension of A' in K_U . \square

Remark 2.3.1.2. *If there is some $a, b \in A \in K$ and $c = t(a, b) \in \langle ab \rangle \setminus (\langle a \rangle \cup \langle b \rangle)$ the theory T_U of a limit of K_U is non stable, in fact the formula $U(t(x, y))$ would then have the order property.*

Definition 2.3.1.3. *Let $L_R = L \cup \{R\}$ a new unary predicate symbol. We define the class K_R of L_R -structures consisting of a structure A in K_0 equipped with a substructure $R(A)$.*

Proposition 2.3.1.4. *If the structures of K_0 are uniformly locally finite the class K_R satisfies 1 – 5. However if the generated structure in K_0 is non trivial the condition (6) need not to hold.*

Proof. The conditions 1–4 are still true by the same arguments as before, we will need some assumption on K_0 to get 5. We take $P(A \oplus_E B) := \langle P(A)P(B) \rangle$ and $P(A \oplus_\emptyset \langle x \rangle) := P(A)$. For 5 consider a finitely generated structure $\langle a, b \rangle \in K_R$ and a quantifier free L_R formula φ such that $\langle a, b \rangle \models \varphi(a, b)$. W.l.o.g.

w.c.a. that $\varphi(x, y) = \varphi'(x, y) \wedge \bigwedge_{i < n} R(t_i(x, y)) \wedge \bigwedge_{i < n} \neg R(t'_i(x, y))$ where $\varphi'(x, y)$ is an L_0 -formula, for all $c \in R(\langle ab \rangle)$ there is some $i < n$ such that $c = t_i(a, b)$ and for all $c \in \langle ab \rangle \setminus R(\langle ab \rangle)$ there is some $i < n$ such that $c = t'_i(a, b)$. Since our theory is uniformly locally finite the condition $t'_i(x, y) \notin \langle (t_j(x, y))_{j < n} \rangle$ is definable. Now we consider the formula $\psi'(x)$ associated to the L_0 -formula $\varphi'(x, y) \wedge \bigwedge_{i < n} (t'_i(x, y) \notin \langle (t_j)_{j < n} \rangle)$ by 5 in K_0 , and then the formula $\psi(x) := \psi'(x) \wedge \bigwedge_{k < m} R(t_{i_k}(x)) \wedge \bigwedge_{k < m} \neg R(t'_{i_k}(x))$, where $t_{i_k}(x)$ enumerates the terms t such that $R(t(a))$ and $t'_{i_k}(x)$ enumerates the terms t' such that $\neg R(t'(a))$. Now if for some $a' \in A' \in K_R$ we have that $A' \models \psi(a')$ then by assumption we can find some b' such that $\langle a'b' \rangle \models \varphi'(a', b') \wedge \bigwedge_{i < n} (t'_i(a', b') \notin \langle t_j(a', b') \rangle_{j < n})$. Define the substructure R on $\langle a'b' \rangle$ to be $\langle t_j(a', b') \rangle_{j < n}$, we get that $\langle a'b' \rangle \models \neg R(t'_i(a', b'))$ for all $i < n$, so $\langle a'b' \rangle \models \varphi(a', b')$, and this property still holds in $B' := A' \oplus_{\langle a' \rangle} \langle a'b' \rangle$, which is an extension of A as an L_R -structure.

Now about 6 : By 6 in the class K_0 we can find some structure $D \in K_0$ and some embeddings as in the statement. We want to define $R(D)$. We know that in any case $\langle R(D_0) \cup R(D_1) \cup R(B) \rangle \subseteq R(D)$, however it might happen that for some $b \in (B \setminus R(B))$ we have that $b \in \langle R(D_0) \cup R(D_1) \cup R(B) \rangle$, and in that case the L_0 -embedding φ_B can not be an L_R -embedding. \square

To give a simple example of this last fact, let us look at some vector space over a finite field with some a predicate for a subspace. Take some independent vectors a_0, a_1, b_0, b_1 as in the statement of the independence theorem for \downarrow^a such that $\neg R(a_0 + b_0)$, $R(a_1 - b_1)$ and $R(b_0 + b_1)$ are satisfied. Then if a satisfies $\text{tp}(a_i/b_i)$ for $i = 0, 1$ we have that $R(a + b_0)$ is satisfied, which is a contradiction. In that case a limit of K_R still exist, and it might be NSOP1, however in that case we have that $\downarrow^K \neq \downarrow^a$ in this limit.

Remark 2.3.1.5. *If we assume that the class K is uniformly locally finite we can restrict it to the class of finite structures in K and have a Fraïssé class in the usual sense, to each isomorphism type of the structure generated by the constants correspond a limit, and this limit is ω -categorical.*

As we have seen in the previous section the condition 6 puts a strong restriction in what we can add to the structure, and some classes with an NSOP1 limit can satisfy the conditions 1 to 5 and not satisfy 6. For example the class of finite equivalence relation, with limit the (stable) theory of the equivalence relation with an infinite number of classes all infinite (which we will write T_∞), or its parameterized version (see [15] or [3]) which is strictly NSOP1, satisfies existence and that $\downarrow^{K^M} = \downarrow^f = \downarrow^d$ over arbitrary sets.

Definition 2.3.1.6. *Consider a new symbol E for a binary predicate in some sort S , we define the language $L_E = L \cup \{V, E, p\}$ where V is a new sort, E a binary relation on S , and p a unary map from S to V . Define the class K_E of L_E -structures consisting of a structure A in K_0 equipped with an equivalence relation $E^A \subseteq S(A)^2$, a new sort $V(A)$ and a map $p : S(A) \rightarrow V(A)$ realizing the quotient.*

Proposition 2.3.1.7. *The class K_E satisfies (H).*

Proof. If we do not add the quotient to the structure the conditions 1 – 5 are satisfied but condition 6 is not, also if we ask for the map p to be surjective the condition 3 will not hold. It is easy to see that the conditions 1 – 5 are satisfied in K_E : the only structure on the quotient is equality and we can do 'in parallel' free amalgamation in K and in the quotient. For 6 we proceed similarly by forming the amalgam in K on one end and the amalgam of the quotient on the other end. \square

This construction is the same as adding a new sort V with no structure other than equality and then some function from S to V .

2.3.2 Adding generic functions

Definition 2.3.2.1. Consider a new symbol f for a binary function, we define $L_f = L \cup \{f\}$ and the class K_f of L_f -structures consisting of a structure A in K_0 equipped with a binary function $f_A : A \times A \rightarrow A$.

Proposition 2.3.2.2. The class K_f satisfies (H).

Proof. The conditions 1 and 4 are clearly satisfied in K_f . For the condition 3 we construct some L_f -structure D using the properties of K . Assume we have our two extensions of L_f -structures $E \subseteq A, B$. We define an increasing sequence of L_0 -structures $(D_i)_{i < \omega}$ equipped with a partial binary function $f_{i+1} : D_i^2 \rightarrow D_{i+1}$. Let $D_0 := A \oplus_E B$ and $f_0 = f_E$. Now set $D_1 := D_0 \oplus_{\emptyset} \langle D_0^2 \setminus (A^2 \cup B^2) \rangle$ and consider $f_1 : D_0^2 \rightarrow D_1$ defined the following way : if $(x, y) \in A^2$, $f_1(x, y) := f_A(x, y)$, if $(x, y) \in B^2$, $f_1(x, y) := f_B(x, y)$ and else (x, y) is sent to the element of D_1 with the corresponding label. We iterate this process : if D_i, f_{i+1} are defined for $i > 0$ we let $D_{i+1} := D_i \oplus_{\emptyset} \langle D_i^2 \setminus (D_{i-1}^2) \rangle$, and f_{i+1} extends f_i by sending $(x, y) \in D_i^2 \setminus (D_{i-1}^2)$ to the element of D_{i+1} with the corresponding label. We then set $D := \bigcup_{i < \omega} D_i$ and $f := \bigcup_{i < \omega} f_i : D^2 \rightarrow D$. This structure satisfies the universal property : it is generated by the images of A and B as an L_f -structure, given D' and embeddings $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ as in the statement we can define an increasing sequence of morphism of L_0 -structures $\varphi_i : D_i \rightarrow D'$ in a canonical way and show that its union is a morphism of L_f -structure. The construction of $A \oplus_{\emptyset} \langle x \rangle$ follows the same method.

The condition 6 is easy to check, we just apply 6 inside of K_0 , find the embeddings, take $f_D|_{D_i^2}$ to be the image of f_{D_i} by φ_{D_i} , similarly for B , and we can set f to be anything outside of D_i^2 and B^2 . Now for condition 5 consider a structure $\langle ab \rangle \in K_f$ and some qf- L_f -formula φ such that $\langle ab \rangle \models \varphi(a, b)$. By replacing every instance of f in φ by a new variable and separating the instances of f applied only to elements of $\langle a \rangle$ from the other ones we can reshape $\varphi(x, y)$ in the following form :

$$\varphi'(x, x_{\leq m}, y, y_{\leq n}) \wedge \bigwedge_{i \leq m} (x_i = f(s_i(x, x_{< i}), s'_i(x, x_{< i}))) \wedge \bigwedge_{i \leq n} (y_i = f(t_i(x, x_{\leq m}, y, y_{< i}), t'_i(x, x_{\leq m}, y, y_{< i})))$$

φ' is an L_0 -qf-formula and the t_i, s_i, t'_i, s'_i are L_0 -terms. By strengthening the formula φ we can assume that, if we define by induction $x_k(a) = f(s_k(a, x_{< k}(a)), s'_k(a, x_{< k}(a)))$ and

$$y_k(a, b) := f(t_i(a, x_{\leq m}(a), b, y_{< i}(a, b)), t'_i(a, x_{\leq m}(a), b, y_{< i}(a, b)))$$

, and if $y_k(a, b) \neq y_{k'}(a, b)$, then $\varphi'(x, x_{\leq m}, y, y_{\leq n})$ specifies that $(t_k(x, x_{\leq m}, y, y_{< k}), t'_k(x, x_{\leq m}, y, y_{< k})) \neq (t_k(x, x_{\leq m}, y, y_{< k}), t'_k(x, x_{\leq m}, y, y_{< k}))$, and similarly for x_k, x'_k and $x_k, y_{k'}$.

Now consider the L_0 -formula $\psi'(x, x_{\leq m})$ associated by 5 in K_0 to $\varphi'(x, x_{\leq m}, y, y_{\leq n})$ and then the L_f -formula $\psi'(x) := \psi'(x, x_{\leq m}) \wedge \bigwedge_{i \leq m} (x_i = f(s_i(x, x_{< i}), s'_i(x, x_{< i})))$. Assume that for some $a' \in A'$ we

have that $A' \models \psi(a')$. By assumption there is an L_0 -extension $A' \subseteq B'$ and $b', b'_{\leq n} \in B'$ such that $B' \models \varphi'(a', a'_{\leq m}, b', b'_{\leq n}) \wedge \bigwedge_{i \leq m} (a'_i = f(s_i(a', a'_{< i}), s'_i(a', a'_{< i})))$. We just need to define a binary function $f_{B'}$ on B' extending $f_{A'}$ such that $b'_k = f(t_i(a', a'_{\leq m}, b', b'_{< i}), t'_i(a', a'_{\leq m}, b', b'_{< i}))$ for every $k \leq n$. We begin by setting these values, this is possible thanks to the strengthening we had, and we can let $f_{B'}$ be whatever outside of this, we write B'_f for the L_f -structure defined on B' , then $B'_f \models \varphi(a', b')$. \square

Remark 2.3.2.3. Adding a generic binary function assures us that the theory of any limit is strictly NSOP1, as proven in [18, Section 3]. Also we get that the generated structure is non-trivial. With the same construction we can add generic functions of any arity.

Definition 2.3.2.4. We define $L_\pi = L \cup \{\pi, \pi^{-1}\}$ new symbols of unary functions and the class K_π of L_π -structures consisting of a structure A in K_0 equipped with a bijection π_A of inverse function π_A^{-1} .

Proposition 2.3.2.5. The class K_π satisfies (H).

Proof. The conditions 1 and 4 are clearly satisfied in K_π , for the conditions 5 and 6 we can proceed as previously. Now we describe what to pick as the free amalgam for 2. Let $E \subseteq A, B$ be two extensions in K_π .

In the same fashion as previously we construct an increasing sequence of L_0 -structures $(D_i)_{i < \omega}$, and an increasing sequence of subsets $H_i \subseteq D_i$ and two increasing sequences of functions $\pi_i, \pi_i^{-1} : H_i \rightarrow D_i$ such that $D_i \subseteq H_{i+1}$ and $\pi_{i+1} \cdot \pi_i^{-1} = \pi_{i+1}^{-1} \cdot \pi_i = id_{H_i}$ for every $i < \omega$. Let $D_0 := A \oplus_E B$, $H_0 = A \cup B$, and $\pi_0 = \pi_A \cup \pi_B$, $\pi_0^{-1} = \pi_A^{-1} \cup \pi_B^{-1}$. $D_1 := D_0 \oplus_\emptyset \langle [D_0 \setminus (A \cup B)]_{k \in \mathbb{Z} \setminus \{0\}} \rangle$, i.e., we consider the L_0 -structure freely generated over D_0 by copies of $D_0 \setminus (A \cup B)$ indexed on $\mathbb{Z} \setminus \{0\}$, we see the element of $D_0 \setminus (A \cup B)$ as those corresponding to the copy of index 0. Set $H_1 = D_0 \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} [D_0 \setminus (A \cup B)]_k$.

We define π_1 as the extension of π_0 that send an element of $[D_0 \setminus (A \cup B)]_k$ to its copy of index $k + 1$, i.e. it sends $t(a, b)_k$ to $t(a, b)_{k+1}$, similarly for π_0^{-1} which sends an element of $[D_0 \setminus (A \cup B)]_k$ to its copy of index $k - 1$. Assume that D_{i+1}, H_i and π_i, π_i^{-1} have been defined for $i > 0$. We set $D_{i+1} := D_i \oplus_\emptyset \langle [D_i \setminus D_{i-1}]_{i \in \mathbb{Z} \setminus \{0\}} \rangle$, $H_{i+1} := D_i \cup \bigcup_{k < \omega} [D_i \setminus D_{i-1}]_k$ and take π_{i+1} to be the function

extending π_i that sends an element of $[D_i \setminus D_{i-1}]_k$ to its copy of index $k + 1$, similarly for π_0^{-1} which sends an element of $[D_i \setminus D_{i-1}]_k$ to its copy of index $k - 1$. Once this is done we take $D := \bigcup_{i < \omega} D_i$,

$\pi_D := \bigcup_{i < \omega} \pi_i$ and $\pi_D^{-1} := \bigcup_{i < \omega} \pi_i^{-1}$. This defines an L_π -structure of bijection π_D , and D satisfies the

universal property : it is generated by the images of A and B as an L_π -structure, given D' and embeddings $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ as in the statement we can define an increasing sequence of morphism of L_0 -structures $\varphi_i : D_i \rightarrow D'$ in a canonical way and show that its union is a morphism of L_π -structure. We take $\varphi_i : D_0 \rightarrow D'$ to be the only morphism of L_0 -structure such that $\varphi_i|_A = \varphi_A$ and $\varphi_i|_B = \varphi_B$. If $\varphi_i : D_0 \rightarrow D'$ is defined we take $\varphi_{i+1} : D_{i+1} \rightarrow D'$ to be its unique morphism extending φ_i that sends $\pi_D^k(t)$ to $\pi_{D'}^k(\varphi(t))$ for every $t \in D_i$. Set $\varphi := \bigcup_{i < \omega} \varphi_i$, it is a morphism of L_0 -structure and by construction it commutes with π , so it is a morphism of L_π -structures. The construction of $A \oplus_\emptyset \langle x \rangle$ follows the same method. \square

Definition 2.3.2.6. We define $L_\sigma = L \cup \{\sigma, \sigma^{-1}\}$ new symbols of unary functions and the class K_σ of L_σ -structures consisting of a structure A in K_0 equipped with an automorphism σ_A of inverse function σ_A^{-1} .

In K_σ the condition 1–4 are easy to check : we use the universal property to show that if $E \subseteq A, B \in K_\sigma$ then there is an (unique) automorphism $\sigma_{A \oplus_E B}$ of $A \oplus_E B$ extending σ_A and σ_B and $(A \oplus_E B, \sigma_{A \oplus_E B})$ satisfies the universal property. For 3 we consider the L_0 -structure $A \oplus_\emptyset \langle (x_i)_{i \in \mathbb{Z}} \rangle$ equipped with the unique automorphism that extends σ_A and sends x_k to x_{k+1} for every $k \in \mathbb{Z}$. As for the generic substructure we need the structures of K_0 to be uniformly locally finite for 5 to hold, in this case the proof follows the same method. Also the condition 6 need not to hold even in this case for similar reasons as in K_R . A limit of K_σ still exist, and it might be NSOP1, however in that case we have that $\downarrow^K \neq \downarrow^a$ in this limit.

2.3.3 Some questions

1. Can we give a more general context that could fit in the case of classes satisfying (H) and the case of Fraïssé classes satisfying S.A.P. in a relational language from [10, Section 6.3], allowing \downarrow^K to be different from \downarrow^a in T^{eq} , by relaxing 6 for example.

2. Let K be a class satisfying (H) and T be the theory of its limit. Is a group definable in the limit T_P of the parameterization of K (without any structure on the parameters) necessarily definably isomorphic to a product of groups definable in T ?

2.3.4 Some strictly NSOP1 theories with strong amalgamation

The theory T of a limit of some K satisfying (H) is not a free amalgamation theory in the sense of [11], our strong independence relation \perp^Γ does not in general satisfies the closure condition (which is defined as : if $C \subseteq A, B$ are algebraically closed and $A \perp_C B$ then $A \cup B$ is algebraically closed). The proof of Corollary 2.1.11 can be generalised to show that in any theory T with some stationary independence relation \perp satisfying Proposition 2.1.10 and such that \perp^a satisfies the independence theorem over arbitrary sets, \perp implies \perp^f , so that T is NSOP1, has existence and $\perp^a = \perp^K$ over arbitrary sets.

Remark 2.3.4.1. *There are other known examples of NSOP1 theories admitting a strong independence relation satisfying Proposition 2.1.10 :*

1. *The theory of parameterized equivalence relation T_{peq} (see [15]), with the relation \perp^Γ defined as : for $C \subseteq A, B$, $A \perp_C^\Gamma B$ if $C = A \cap B$ and for every parameter p in A or B , $O(A) \perp_{O(C)}^p O(B)$. This correspond to the symmetrization of forking independence, since there are no algebraic closure in this theory the condition of closure mentioned before is satisfied and T_{peq} is a free amalgamation theory in the sense of [11].*
2. *The theory of ω -free PAC fields of characteristic 0 (see [8] and [7]) with the the relation \perp^{III} defined as : for $C \subseteq A, B$, $A \perp_C^{III} B$ if $A \perp_C^{lin} B$ and $\text{acl}(AB) = AB$. This relation is named strong independence in [6, Definition 4.5], written as here in [8, Section 3.6], but this definition is not the same as the strong independence of [7, Definition 1.2].*
3. *The theory of vector spaces of infinite dimension over an ACF with a non degenerate bilinear form with the relation \perp^Γ (see [1], [13] or [3] for the definition of \perp^Γ). This is not a free amalgamation theory because of the closure condition.*
4. *The theory ACFG of an ACF with a generic multiplicative subgroup, see strong independence in [14, Section 2.2] and the remark following [14, Section 4.4].*
5. *The model companion of the empty theory in a language with a binary function T_f^\emptyset , with the relation \perp^\otimes defined in [18, Section 3.2]. On this topic see the remark following [19, Definition 4.3].*

Definition 2.3.4.2. *Let $(G, +)$ be a group definable over E in a model M of an NSOP1 theory with existence. For $E \subseteq B$ a set of parameters we say that an element $a \in G$ is Kim-generic in G over B if for every $b \in G$ such that $a \perp_B^K b$ we have $a + b \perp_B^K b$.*

Remark 2.3.4.3. *We consider \mathbb{M} a monster model of the theory of parameterized vector space over a countable field F . For every $\bar{p} \in \mathbb{O}$ finite tuple consider the group $\mathbb{M}_{\bar{p}} = \prod_{i < n} \mathbb{M}_{p_i}$ which definable over \bar{p} . Then if $\bar{p} \in B \subseteq \mathbb{M}$ and if $\bar{a} \in \mathbb{M}_{\bar{p}}$ is such that $\bar{a} \perp_{\emptyset}^\Gamma B$, then, for every $\bar{b} \in \mathbb{M}_{\bar{p}}$ such that $\bar{a} \perp_B^K \bar{b}$ we have $\bar{a} +_{\bar{p}} \bar{b} \perp_B^K \bar{b}$ (where $\bar{a} +_{\bar{p}} \bar{b}$ is the tuple $(a_i +_{p_i} b_i)_{i < n}$), so \bar{a} is Kim-generic over B for the group $\mathbb{M}_{\bar{p}}$.*

Proof. Let \bar{a} be as in the statement, write $\bar{p}' = P(B)$ and \bar{b} be such that $\bar{a} \perp_B^K \bar{b}$. Assume that there is some $a' \in \langle B, \bar{a} +_{\bar{p}} \bar{b} \rangle \cap \langle B, \bar{b} \rangle \setminus B$. We can show by induction on the length of a term $t(B, \bar{a} +_{\bar{p}} \bar{b})$ that by removing elements of $O(B)$ at every level of the term we can get a term only in $\bar{a} +_{\bar{p}} \bar{b}$ and the parameters of B , i.e. we can assume that $a' = t(\bar{a} +_{\bar{p}} \bar{b}, \bar{p}')$. With the same argument by subtracting $-_{p_i} b_i$ at every level of the term we get that $t(a, \bar{p}) \in \langle B, b \rangle$. In fact we can show that there is a term $\tilde{t}(\bar{b}, \bar{p}')$ such that

$t(\bar{a} +_{\bar{p}} \bar{b}, \bar{p}') = t(\bar{a}, \bar{p}') +_{p'} \tilde{t}(\bar{b}, \bar{p}')$ for some $p' \in \bar{p}'$. By $\bar{a} \downarrow_B^K \bar{b}$ this implies that $t(a, \bar{p}) \in B$, but since $\bar{a} \downarrow_{\emptyset}^I B$ by base monotonicity of $\Gamma \bar{a}\bar{p}' \downarrow_{\bar{p}'}^I B$ and the only terms in a with parameters from B being inside of B are the ones being equal to some element $t'(\bar{p}') \in \langle \bar{p}' \rangle = \langle (0_p)_{p \in \bar{p}} \rangle_{\bar{p}}$. Using the universal property and the morphism $\bar{a} \rightarrow \bar{a} +_{\bar{p}} \bar{b}$ over B this implies that $a' = t(a +_p b, \bar{p}) = t'(\bar{p}') \in B$, which is a contradiction. \square

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