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INTRODUCTION TO THE THEORY OF SITUATIONS

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**INTRODUCTION
TO
THE THEORY OF
SITUATIONS**

*Fundamental Concepts
of the Didactics of Mathematics*

Annie Bessot

Université Grenoble-Alpes

To achieve this English translation, I made a first draft with the help of the free online version of the automatic translator DeepL, then Jean-Luc Dorier corrected and/or polished some passages before David Pimm gave the final touch.

Most of the references in this text are in French. The quotations have been translated into English, but the French original are always given in footnote. Brousseau's work has been popularized in English through a collection of texts collected, commented and translated by N. Balacheff, M. Cooper, R. Sutherland and V. Warfield. Published in 1997 by Kluwer Academic Publishers. More recently a translation of the Guy and Nadine Brousseau's work on fractions have been translated by V. Warfield and published in 2014 by Springer. Both publications are listed in the references.

Finally, the project ICMI Awardees Multimedia Online Resources (ICMI AMOR) the unit devoted to Guy Brousseau (<https://www.mathunion.org/icmi/awards/amor/guy-brousseau-unit>) offers an original way to get acquainted with Guy Brousseau's most fundamental ideas in English. The translation into English of this text has been in great part motivated to be added as a reference in this unit.

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FOREWORD

A Ph.D. program (*Diplôme d'Études Approfondies* – DEA) in the didactics of scientific disciplines was created in 1994 at the University of Grenoble–Joseph Fourier. Following Bologna reform,¹ this was included into a second year of a Master's degree in “Computer-based learning environments and didactics” created at the same university.

The following document is based on a handout of a course I gave in this Master's program from 2003 to 2007, as part of the education unit (UE1: “Fundamental concepts of the didactics”) for which I was responsible.

In this course, I presented the theoretical framework developed by Guy Brousseau.

My original title for this course was ‘Introduction to the theory of didactic situations’. In the reworking I have done here, I have removed the adjective ‘didactic’ from the title.

Indeed, quite recently, Brousseau (1997) drew a distinction within the theory of situations between the theory of mathematical situations for didactic use (TMS or *theory of mathematical situations*) and the theory of didactic situations proper (TDS).

I only became aware of it when I was preparing my lecture at the 15th Summer School for Research in Didactics of Mathematics in Clermont-Ferrand. I took it into account when writing the course for this 15th summer school (Bessot, 2011).

The present text is therefore a revision of the 2003 course. This revision was largely motivated by the desire to translate it into English and possibly into other languages. Indeed, in our work with Claire Margolinas for the ICMI AMOR project,² we needed an English reference representing a first approach to Brousseau's work, and this course seems to meet our expectations.

I hope that this work and its various translations will raise awareness of the Theory of Situations, one of the essential foundations of the French Didactics of Mathematics, which has been taken up in other didactics and more widely in Educational Sciences, and, what is more, outside the particular context of France. Ironically, just as I am putting the finishing touches to this foreword, the sad news of Guy Brousseau's death has reached me. It saddens me enormously, but gives added significance to this project.

Finally, I would like to extend my warmest thanks to Jean-Luc Dorier, who initiated AMOR, carefully re-read this course in French, helped me translate it into English and motivated me to have it translated into other languages. A special thank you to David Pimm who accepted to finalise this English version.

Annie Bessot

Grenoble, February 2024

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*Please do not hesitate to contact me if you have any questions about the content of this course.
Thank you very much!*

¹ The Bologna reform also known in France as LMD Licence–Master–Doctorat (Bachelor–Master–Doctorate) is a set of measures designed to bring the French higher education system into line with the European standards of the BMD reform. The founding texts of this reform were published in 2002, but its implementation took several years.

² See: <https://www.mathunion.org/icmi/awards/amor/guy-brousseau-unit>.

At the origin of the theoretical movement under the label ‘Didactics of Mathematics’³ in France is the idea that it is possible to describe and explain in a rational way the teaching phenomena, that are usually more likely to generate empiricism or opinion than reasoned discourse.

One of the concerns widely shared within the French mathematics didactics community is that of establishing an original theoretical framework developing its own concepts [...] There is also a broad consensus on the methodological requirement to have recourse to experimentation in interaction with theory [...]. (Laborde, 1989, p. 47)⁴

This theoretical framework was firstly that of the Theory of Situations, of which Guy Brousseau was the founder, and that of the Conceptual Fields of Gérard Vergnaud, which I will not present here.

The Didactics of Scientific Disciplines, within the Cognitive Sciences, could first be defined as the science of the specific conditions of dissemination of scientific knowledge useful for the functioning of human institutions.

The minimum system of study is the didactic system *stricto sensu*, Teacher, Student, Knowledge: i.e. the *knowledge*-related interactions between *teacher* and *students*, in a situation with a didactic purpose.

Researchers in didactics of mathematics assert the specificity of their discipline compared with other related fields:

In order to produce, improve, reproduce, describe and understand mathematics teaching situations, it has become necessary – and possible – to theorise this teaching activity as an original object of study and not as a simple conjunction of facts that can only be theorised in autonomous fields such as pedagogy, sociology, psychology, mathematics, linguistics or epistemology. (Brousseau, 1986a, p. 2)⁵

The metaphor of the labyrinth can help us to understand better what differentiates the problematic of the didactician from that of the psychologist:

The psychologist [...] studies the behaviour of the rat in the labyrinth; but he knows the structure of the labyrinth, which he himself has designed. The didactician, on the other hand, does not know the structure of the labyrinth in which the student is thrown. He will therefore first have to try logically to explore it. To do so, he may even observe the behaviour of the ‘rat’ (the student!) inside the labyrinth to deduce the structure of the labyrinth [...]. (Chevallard, 1989, p. 36)⁶

³ “A symposium in Bordeaux in 1975 brought together researchers interested in treating this field as a scientific domain, which they suggested designating as ‘experimental epistemology’. At my suggestion, we prefer to assume the somewhat despised label of ‘didactics’ to show our desire to improve education by **means of what we can understand**.” (Brousseau, 1986a, p. 28) (My emphasis in bold.)

⁴ Our translation from the original: “Un des soucis largement partagés au sein de la communauté française de didactique des mathématiques est celui de l’établissement d’un cadre théorique original développant ses propres concepts [...] Un large consensus se fait aussi sur l’exigence méthodologique d’avoir recours à l’expérimentation en interaction avec la théorie [...]”.

⁵ Our translation from the original: “Pour produire, améliorer, reproduire, décrire et comprendre les situations d’enseignement des mathématiques, il est devenu nécessaire – et possible – de théoriser cette activité d’enseignement en tant qu’objet original d’étude et non pas en tant que simple conjonction de faits théorisables uniquement dans des domaines autonomes comme la pédagogie, la sociologie, la psychologie, les mathématiques, la linguistique ou l’épistémologie.”

⁶ Our translation from the original: “Le psychologue [...] étudie le comportement du rat dans le labyrinthe; mais il connaît la structure du labyrinthe, qu’il a lui-même conçue. Le didacticien, en revanche, ne connaît pas la structure du labyrinthe dans lequel l’élève est lancé. Il devra donc d’abord, logiquement, chercher à l’explorer.”

Are theoretical concepts from the field of the didactics of mathematics specific to mathematics? In France, researchers in didactics of other disciplines (e.g. physical education or physics and science) are currently using some of these concepts. My hypothesis is that certain theoretical concepts derived from the didactics of mathematics have a general character: in particular situations, they generate questions and results that are themselves specific to a particular field of knowledge.

The main project of the didactic system, which is to get from an initial state to a final state with regard to the knowledge at stake is what defines the 'teacher' and 'student' positions.

From the point of view of the relation to knowledge, there is a dissymmetry, which is constitutive of the didactic system. We will not say that the student has no relation to knowledge before teaching, but simply that, in the initial state, this relation is poor or not adequate. Without the assumption of this dissymmetry, the didactic system does not need to exist. (Margolinas, 1993, p. 227)⁷

The teacher differs from the student in that he or she is 'supposed to know', but also in that he or she is 'supposed to be able' to anticipate what the student will have to learn.

Moreover, the didactic system has a particular characteristic, since its ultimate goal is to disappear. If the teacher succeeds in his or her mission, he or she must be able to withdraw and *the student must be able to maintain his or her relation to knowledge outside his or her presence.*

1. But what is learning in the theory of situations?

The student's project is to learn.

A widespread ideology assumes a simple transfer link from teaching to learning: the student records what is communicated by the teacher, with perhaps some loss of information. (Laborde, 1989, p. 47)⁸

Many studies have shown the fallacy of this view: learning is not a simple transfer process, nor is it a linear and continuous process.

Understanding student learning requires us to clarify the perspective we take on learning.

Psychological hypothesis (adaptive learning). The subject learns by adapting (assimilation and accommodation) to a milieu which is a producer of contradictions, difficulties and imbalances.

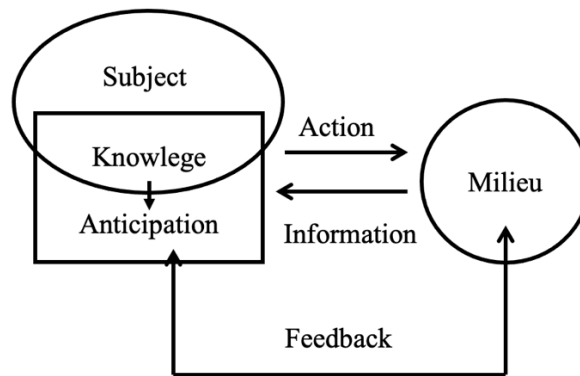
This hypothesis refers to the psychogenetic theory of Piaget (1975).

Pour cela renversant la perspective du psychologue, il pourra même observer le comportement du « rat » (l'élève !) à l'intérieur du labyrinthe pour en déduire la structure du labyrinthe ...".

⁷ Our translation from the original: "Du point de vue de la relation au savoir, il y a une *dissymétrie*, qui est constitutive du système didactique. Nous ne dirons pas que l'élève n'entretient aucune relation au savoir avant l'enseignement, mais simplement que dans l'état initial, cette relation est peu ou pas adéquate. Sans l'hypothèse de cette dissymétrie, le système didactique n'a pas lieu d'être."

⁸ Our translation from the original: "Une idéologie très répandue suppose un lien de simple transfert de l'enseignement vers l'apprentissage : l'élève enregistre ce qui est communiqué par l'enseignant avec peut-être quelques pertes d'informations."

Figure 1: *Non-didactic learning situation*



Example. *Non-didactic learning situation: the bicycle (Brousseau, 1988b)*

Situation 1: four-wheeled bicycle, one at the front and three at the back

With these small wheels, the child learns how to pedal and turn the handlebars according to the following implicit model:

$D \rightarrow D$

$G \rightarrow G$

Situation 2: two-wheeled bicycle (the two small wheels at the back are removed)

[...] the child wants to go straight, but the bike leans to the right, so the child wants to go back to the left and turns the handlebars to the left following the implicit pattern acquired previously and [...] falls!

To keep the balance [...], he must first turn the handlebars to the side where he leans, in order to get a push that straightens him, thus following a (momentary) inverted but indispensable pattern.

$G \rightarrow D$

$D \rightarrow G$

The change of schema (implicit model) is a key feature of learning. (Brousseau, 1988b, p. 62)

Didactic hypothesis. An environment without didactic intentions (i.e. not deliberately organised to teach knowledge) is insufficient to induce in a subject all the knowledge that society wishes him or her to acquire.

The teacher must therefore provoke the desired adaptations in the students by a judicious choice of the situations he or she proposes.

The teacher's task is not to get students to learn, but to make sure that they *can learn*. The teacher's task is not to take charge of learning – which remains outside his power – but to create the *conditions for learning* to take place. (Chevallard, 1986, p. 39)⁹

⁹ Our translation from the original: “L’enseignant n’a pas pour mission d’obtenir des élèves qu’ils apprennent. Mais bien de faire en sorte qu’ils *puissent apprendre*. Il a pour tâche, non la prise en charge de l’apprentissage – ce qui demeure par nature hors de son pouvoir – mais la prise en charge de la création de *conditions de possibilité de l’apprentissage*.”.

The consequences of these hypotheses lead to the introduction of a model of the didactic situation: *didactic contract/adidactic situation*.

2. Didactic contract

2.1. The didactic contract represents¹⁰ the implicit rights and duties of the students and the teacher about the objects of mathematical knowledge being taught

What each one (teacher or student) has the right to do or not about a piece of knowledge is based on a set of explicit but mostly implicit rules. Brousseau named this set of rules ‘didactic contract’, which share and limit the responsibilities of each one with regard to a mathematical knowledge being taught.

He introduced this concept in 1978, and again in 1980, to explain the failure of elementary school students *succeeding in all subjects except mathematics* (*elective failure* in mathematics). In this sense, the elective failures would not come from an inability of the students to learn but from didactic contracts specific to such or such piece of mathematical knowledge preventing certain students from entering into a learning process of this knowledge.

During a session aimed at teaching a student a certain knowledge (*didactic situation*), the student interprets the situation presented to him or her, the questions asked, the information provided, the constraints imposed on him or her, according to what the teacher reproduces, consciously or not, in a *repetitive way*¹¹ in his or her teaching practice. We are particularly interested in what is specific to the knowledge taught in these routines. (Brousseau, 1980, p. 127)¹²

Example 1. Set relations at the beginning of secondary school

In the 1970s in France, at the beginning of secondary education (grade 6, 11–12 years old), lessons were given on basic notions of set theory.¹³ A common exercise on these notions was like the following:

Which of the signs \in , \notin , \subset , $\not\subset$ are appropriate?

$\{a\}$ $\{a, b, c\}$

b $\{a, b, c\}$

A study of a questionnaire of this type (Duval & Pluvinage, 1977, cited by Tonnelle, 1979) shows that the behaviour of students for answering these questions was regulated by a choice tree allowing decisions to be made:

if there are braces ‘on both sides’, it is a sign of inclusion;

if there are braces ‘only on the right’, it is a sign of belonging.

Sixth graders could then answer most of the exercises on this subject correctly without *the mathematical meaning* of the set relationships playing a role.

¹⁰ i.e. the didactic contract is a model constructed by the researcher.

¹¹ My emphasis.

¹² Our translation from the original: “Au cours d’une séance ayant pour objet l’enseignement à un élève d’une connaissance déterminée (*situation didactique*), l’élève interprète la situation qui lui est présentée, les questions qui lui sont posées, les informations qui lui sont fournies, les contraintes qui lui sont imposées, en fonction de ce que le maître reproduit, consciemment ou non, *de façon répétitive*¹² dans sa pratique de l’enseignement. Nous nous intéressons plus particulièrement à ce qui, dans ces habitudes, est spécifique des connaissances enseignées.”.

¹³ This type of exercise disappeared in the late 1980s from teaching in France.

When faced with a certain number of drill exercising, the students are led to learn implicit rules (such as the presence or absence of braces) that enable them to answer economically to the question posed.

Example 2. Factoring polynomials in secondary education¹⁴

When asking to factor the polynomial expression $16x^2 - 4$, a secondary school teacher expects that students recognise the opportunity to apply the factoring rule $a^2 - b^2 = (a - b)(a + b)$ and answer $16x^2 - 4 = (4x - 2)(4x + 2)$; whereas, with the polynomial $4x^2 - 36x$, a simple factoring, i.e. $4x^2 - 36x = 4x(x - 9)$, is expected.

The answers: $16x^2 - 4 = 2(8x^2 - 2)$, $16x^2 - 4 = 3(\frac{16}{3}x^2 - \frac{4}{3})$ or $16x^2 - 4 = 16x^2(1 - \frac{1}{4x^2})$ with $x \neq 0$ ¹⁵ are all correct, but will be eliminated or will not have the opportunity to appear, *not because they do not satisfy a previously formulated mathematical condition,¹⁶ but as an act that deviates from a code of conduct.*

The teacher's power in the classroom is not to *forbid* (more precisely, to forbid in a direct way) the answer $16x^2 - 4 = 2(8x^2 - 2)$, but to produce the answer $16x^2 - 4 = (4x - 2)(4x + 2)$. Its power lies less in pointing out the 'wrong answers' than in eliciting the right answer – which implicitly designates the other answers as wrong. (Chevallard, 1985, p. 74)¹⁷

The existence of implicit rules strongly structuring the conduct of the students, and that of the teacher, modifies the status of the error: an error will be an admissible false answer. The error thus arises as a fault on the background of a model respecting the code of conduct: the answer $16x^2 - 4 = 16x^2(1 - \frac{1}{4x^2})$ is true, but not admissible, while the answer $16x^2 - 3 = (4x - 3)(4x + 1)$ is wrong, i.e. *admissible and false*.

What we have shown about factoring is not a pathological case with respect to the teaching of knowledge. We have used it to highlight a general mechanism involved in the didactic communication of knowledge. The study of the didactic contract relating to a piece of knowledge makes it possible to trace the limits of the meaning for the student of the knowledge being taught.

2.2. Didactic contract and negotiations between students and teacher about knowledge

In the course of teaching some piece of knowledge, the rules of communication between students and teachers concerning the objects of knowledge are established, changed, broken and re-established as acquisitions are made, and evolve, through a historical process. These rules do not have a single, fixed form over time, but are the result of a constantly renewed negotiation.

On the one hand, the interactions between the teacher and the taught obey locally stable rules and, on the other, these are not immutable.

This negotiation produces a kind of game whose provisionally stable rules allow the protagonists, and in particular the student, to make decisions with a certain degree of

¹⁴ Based on Tonnelle (1979) and Chevallard (1985).

¹⁵ Which will be an appropriate answer when learning about the limits of polynomial functions.

¹⁶ Like the rules for factoring in $Z[X]$ (set of polynomials with integer coefficients).

¹⁷ Our translation from the original: "Le pouvoir de l'enseignant dans sa classe, ça n'est pas d'*interdire* (plus précisément: d'*interdire de manière directe*) la réponse $16x^2 - 4 = 2(8x^2 - 2)$, mais bien de *produire* la réponse $16x^2 - 4 = (4x - 2)(4x + 2)$. Son pouvoir consiste moins à désigner les « mauvaises réponses », qu'à susciter la bonne réponse – qui désigne implicitement les autres réponses comme mauvaises."

security, necessary to ensure the independent characteristic of appropriation. (Brousseau, 1986b, p. 55)¹⁸

The contract is specific to the knowledge involved and therefore necessarily perishable: contextual knowledge and even institutional knowledge evolve and change, whereas the pedagogical contract tends to be stable. Moments of rupture allow the didactic contract to be revealed experimentally. (Brousseau 1988a, p. 322)¹⁹

This negotiation process is subject to a number of paradoxes. We will only look at one of them here: the teacher does not have the right to tell the student what to do (otherwise he or she is not fulfilling the role as a teacher), and yet he or she has to make the student produce the expected answer (otherwise he or she has not succeeded in the teaching).

Similarly, if the student accepts that the teacher teaches him or her the results, he or she does not establish them himself or herself and therefore does not learn them. If, on the contrary, he or she refuses any information from the teacher, then he or she breaks the didactic relationship.

Brousseau has characterised different negotiation processes to obtain the expected answer from the students (which the teacher knows, but the student does not). The *Topaze*²⁰ effect is one of these forms of negotiation. The teacher tries to make the meaning of the student's answer as rich as possible. If the student fails, he or she adds information that reduces the possibility for the student's answer to be meaningful. At the extreme level, he or she accepts such conditions that the student produces a right answer, but without having been able to invest any meaning.

Example 3. Drawing the axis of symmetry with a ruler and a set square

(See Annex II and III)

The teacher's plan is for the students to use the following geometric properties to justify the drawing of the axis of symmetry:

P₂ (property of orthogonality): the axis of symmetry of a figure is perpendicular to the line joining any two symmetrical points.

P₃ (incidence property): let (A, A') and (B, B') be two pairs of symmetric points: the point of intersection of the lines (A, B) and (A', B'), if it exists, is on the axis of symmetry.

The choice of instruments (ruler without graduation and set square) prevents the use of:

P₁ (property of the middle point): the axis of symmetry of a figure passes through the middle of the segment joining any two symmetrical points.

The teacher wants a geometric drawing process (which is not a geometric construction in the sense of Félix Klein, 1846²¹): the teacher does not work on the material object, but rather on the geometric object which we call 'figure'. For him, it is a theoretical problem: the control on

¹⁸ Our translation from the original: "Cette négociation produit une sorte de jeu dont les règles provisoirement stables permettent aux protagonistes et notamment à l'élève de prendre des décisions dans une certaine sécurité, nécessaire pour lui assurer l'indépendance caractéristique de l'appropriation."

¹⁹ Our translation from the original: "Le contrat est spécifique des connaissances en jeu et donc nécessairement périssable : les connaissances et même les savoirs évoluent et se transforment, tandis que le contrat pédagogique a tendance à être stable. Les moments de rupture permettent la mise en évidence expérimentale du contrat didactique."

²⁰ In reference to the play 'Topaze' by the writer Marcel Pagnol, in which a teacher adds more and more information to the point of caricature, so that a student writes the marks of the plural (normally inaudible in oral French) in the sentence: "... des moutonssess étai-hunt réunisse ..." (difficult to transpose into English).

²¹ See Annex I.

the drawing is done by reference to the geometric properties P_2 and P_3 (here) of the geometric figure.

In the previous tasks (1st group of figures), the problems were solved by using P_1 . The implementation of this property was successful. For the students (at least in group 3), it is a practical and material problem: find the middles of two parallel segments. The aim is therefore to produce the material drawing of the axis of symmetry with the help of the measurement, and the control can be done by reference to folding, as proposed by one student, or by checking the precision of the measurements.

What is said on a drawing is always ambiguous: it can be about the geometric figure (which the drawing represents) or about the drawing itself. What the teacher and the students say is not free from this ambiguity.

The teacher tries to get the students to produce an answer corresponding to the implementation of properties P_2 and P_3 : she must disqualify the answers based on P_1 . To this end, and on several occasions (see Annex III: interventions 10, 14, 18, 20, 39, 41, 45, 51, 53, 55, and 89 of the chronicle), she refers to the precision to question the answers based on P_1 and then (see Annex III: intervention 89) to conclude!

There is a paradoxical situation here: the teacher would like the precision to be geometrical and she cannot give the solution. To disqualify the solutions proposed by group 3, she uses an argument of precision, which confirms the students in a problematic of measurement: for them, better precision requires greater care in the material operation of the measurement (see Annex III: group 3 and François). This argument becomes an argument of authority.

The teacher, by trying to get the students to produce the answer she wants, transforms the situation. The last student does not necessarily know that he or she has produced an answer and why: he or she responds according to the teacher's expectations. There is no explanation of why it is right or why it is wrong.

Here, we can refer to the *Topaze* effect: the expected answer could not be produced as a solution to the initial problem under the responsibility of the students. The teacher was forced to restrict the conditions for producing the answer (reducing the student's uncertainty) until the expected answer is obtained. The meaning of this imposed answer becomes, for the students, very limited.

2.3. Teaching and contract's breaks

Any teaching of a new object of knowledge provokes contract's breaks, with respect to old objects of knowledge and the re-negotiation of new contracts: the student's learning comes at the price of these breaks, which the teacher must negotiate.

Example 4. Status of drawings in geometry teaching

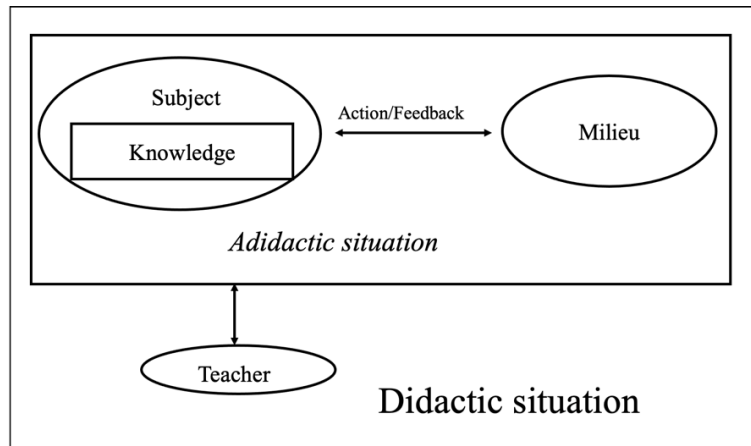
In primary schools (6–11 years old), pupils learn to use drawing instruments to develop graphic skills. Geometric drawings (called triangles, rectangles, etc.) are then material drawings on which one operates, and whose measurements are verified: they are the objects studied.

In secondary school, and more particularly from the class 7 (pupils aged 13–14), a break in this initial contract will have to take place: the student will have to establish proofs, not on the drawing itself, but on the abstract and ideal objects (named triangles, rectangles, etc.) that the material drawing represents (new didactic contract about drawings).

3. Adidactic situation

Referring to the point of view adopted on learning, the student's learning is modelled by an adidactic situation organised by the teacher within a didactic situation (Figure 2).

Figure 2: *A didactic situation and an adidactic situation*



Brousseau (1988a) defines the *milieu* as the student's antagonistic system in the didactic situation. This system modifies the states of knowledge in a way not controlled by the student: the milieu behaves like a non-finalised system. It has a relative character.

It plays a central role in learning, as the cause of adaptations and in teaching, as a reference and epistemological object. (Brousseau, 1988a, p. 321)²²

For Chevallard (1992, p. 90), the elements that constitute the milieu are those that subjectively, i.e. for the subjects of the institution, appear to be self-evident, transparent and unproblematic.

The teacher will try to propose a situation, so that the students construct their relation to the object of knowledge or modify this relation as a response to the demands of a milieu, and not to the teacher's desire. *Such a situation is one in which what one does has a character of necessity in relation to obligations that are neither arbitrary nor didactic, but of the order of knowledge.* The teacher must succeed in getting the student to remove the didactic presuppositions from the situation, so that the resolution of the problem becomes independent of the teacher's desire: the devolution sought by the teacher so that the student learns is thus that of a *non-didactic situation* (on the model of the bicycle situation).

There is a shift in responsibility towards knowledge from the teacher to the student: the student in an adidactic situation becomes responsible for his/her relation to knowledge.

It is important to understand the distinctions between didactic, adidactic and non-didactic 'situations' in Brousseau's "theory of didactic situations": a non-didactic situation specific to a knowledge is a situation without didactic finality for which the relation to the knowledge is elaborated as an economic means of action (cf. the bicycle situation); an adidactic situation is a situation with a didactic purpose (i.e. organised by the teacher) where the subject answers *as if* the situation were non-didactic (i.e. the subject answers independently of the teacher's expectations). *There are then elements in the didactic situation which form an adidactic milieu antagonistic to the student.*

²² Our translation from the original: "Il joue un rôle central dans l'apprentissage, comme cause des adaptations et dans l'enseignement comme référence et objet épistémologique."

Learning is a *modification* of the relation to knowledge produced by the student himself, which the teacher can only provoke through choices (voluntary or involuntary) of values of *variables* of the didactic situation: the *didactic variables*.

Modelling in terms of the didactic situation allows the design of didactic engineering where the conditions to provoke (at best) the student's learning have been 'calculated'. (See the tangram situation further down.)

3.1. What are the conditions for a situation to be experienced as didactic?

At least the following conditions must be met:

- The student can imagine an answer, but this initial answer (*basic procedure based on previous knowledge*) is not the one sought to be taught: if the answer were already known, it would not be a learning situation.

Without a basic strategy the student does not understand the game, even if the instructions are clear. (Brousseau, 1988b, p. 61)²³

- This basic procedure must very quickly prove insufficient or ineffective, so that the student is forced to make accommodations and/or modifications to his or her knowledge system. The student is uncertain about the decisions to be made.

- The targeted knowledge is *a priori* required to move from the basic strategy to the optimal strategy.

- There is a milieu for validation: the milieu allows for feedback.

- The student can try again.

Learning will consist of changing strategies and changing the knowledge associated with them. (Brousseau, 1988b, p. 61)²⁴

3.2. Which elements of the situation are modelled by the milieu?

In the didactic situation, the milieu refers to the system that is antagonistic to the student, i.e. what modifies the states of knowledge in a way that is not controlled by the student: the milieu behaves like a non-finalised system. It has a relative character (Brousseau, 1998).

The general schema of interactions of the student and the milieu in a didactic situation is the one given for a non-didactic learning situation (Figure 1).

Interactions are significant in terms of the ability of the system (subject ↔ milieu) to regain equilibrium following disturbances, or even to evolve if these disturbances are such that it is necessary to do so.

By problem we mean the result of a disturbance in the equilibrium of the subject/milieu relation (...) but this only makes sense if the disturbance can be attested to by the subject and the milieu. A disturbance which the subject cannot perceive, or which the milieu cannot account for in any discernible way, cannot be a source of problems. (Balacheff 1995, p. 226)²⁵

²³ Our translation from the original: "Sans stratégie de base l'élève ne comprend pas le jeu, même si la consigne est claire."

²⁴ Our translation from the original: "L'apprentissage va consister à changer de stratégies et à changer les connaissances qui leur sont associées."

²⁵ Our translation from the original: "Nous désignons par problème le résultat d'une perturbation de l'équilibre de la relation sujet / milieu (...) mais cela n'a de sens que si cette perturbation peut être attestée par le sujet et par le

Feedback is a particular piece of information provided by the milieu: that is, information that is received by the student as a sanction, positive or negative, relating to his or her action and that allows him or her to adjust that action, to accept or reject a hypothesis, to choose between several solutions.

Just as the elements modelled in the milieu are not necessarily material (they may be the subject's old, stabilised, self-evident knowledge), the subject's actions may be 'mental' actions, not visible.

The evolution of the didactic situation in which a student is can be read as a succession of states among permitted or possible states. The milieu models in the didactic situation what the student does *not control*, but which modifies his knowledge. The student's procedures are what *can be observed* in the evolution of the student's knowledge.

There is necessarily uncertainty in an didactic situation.

If there is no longer any uncertainty (in the sense of relevant possible choices) as to the terminal states of the situation, it means that the subject knows the answer, *he or she knows*. It can be said that the situation is then controlled by the student (and that there is no longer a milieu in relation to this didactic situation).

Knowledge *reduces* the uncertainty of the subject by removing possibilities of choice.

The questions that the subject/milieu system (*inseparable from the notion of the didactic situation*) makes it possible to ask is therefore the following:

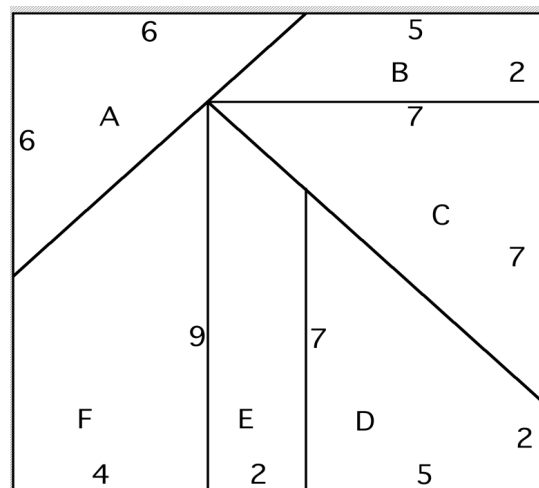
What, in a didactic situation, can cause (*a priori*) the modification of the student's states of knowledge or explain them (*a posteriori*)?

3.2. Example. *A priori* analysis of a didactic situation: "Enlargement of a tangram" (Adapted from G. & N. Brousseau, 1987)

Material

- 6 to 8 cardboards (20 cm × 15 cm) with the same tangram printed (Figure 3).

Figure 3: Representation of the tangram



Legend: The numbers on the tangram picture indicate the measurements in cm of the tangram pieces.

milieu. Une perturbation que le sujet ne pourrait percevoir, ou dont le milieu ne pourrait rendre compte de façon discernable ne pourra être source de problèmes.”.

- Sheets of graph paper.
- A ruler with graduations in cm for each student.

Instruction for the students

“Here are some tangrams. You are going to make similar tangrams, larger than the models, according to the following rule: the segment that measures four centimetres on the model must measure seven centimetres on your reproduction.

I give each team a tangram. Each student has to make one or two pieces. When you have finished, you must be able to reconstruct the same figures as with the model.”

Phase 1. The children are divided into teams of four or five and are first asked to discuss and decide on a common procedure to enlarge the pieces of the tangram.

Phase 2. After this short discussion in each team, they split up to make their piece(s).

Phase 3. Students group together again to put together their enlarged pieces.

The teacher posts an enlarged representation of the complete tangram on the board.

A priori analysis

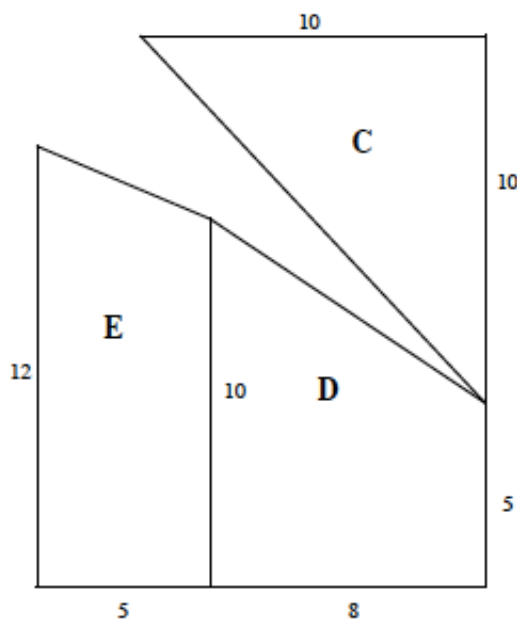
- *Two non-successful strategies*

Both strategies are based on knowledge about natural integers and enlargements: to enlarge, we add or multiply by an integer. *The meaning of multiplication by an integer is that of reiterated addition.*

Strategy 1: Add three to the measures of the sides at ‘right angles’.

The number of sides of each piece is kept as well as the right angles: three is added to the measure of each side at the right angles.

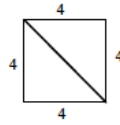
Here is the result of this procedure on the ‘enlarged’ pieces C, D and E of the tangram:



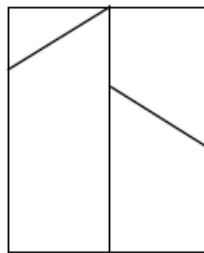
Feedback from the material milieu: the pieces do not fit together in a *very visible* way.

- Domain of validity of the procedure 1.

The domain of validity in the strict sense of this procedure (i.e. which respects the geometric shape of the pieces) is the set of tangrams where all pieces are composed of right-angled isosceles triangles of sides four, for example:

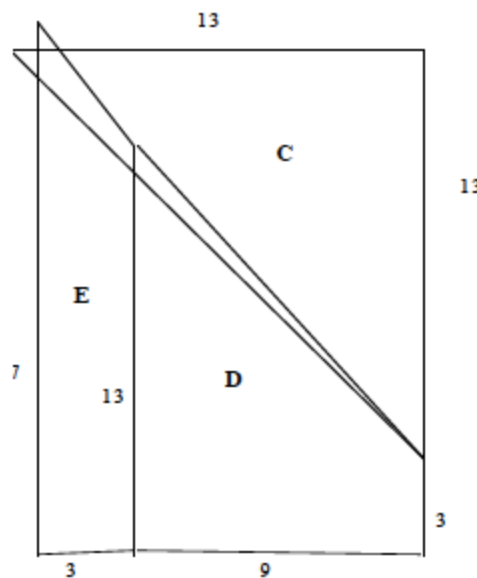


If we define the domain of validity in relation to the material milieu, and thus consider the enlargement to be valid if the pieces of the tangram fit together well, then the domain of validity is wider, it also includes 'rectangular' tangrams resulting from the putting together of 'right-angled' pieces (with at most one oblique side) and such that the rectangle has its sides subdivided in the same number of pieces. Three is then added a same number of times to each side of the square.



Strategy 2: Multiply each measure by two and subtract one

Result of this procedure on the 'enlarged' pieces C, D and E of the tangram:



Feedback from the material milieu: the pieces do not fit together.

But, for this procedure, this misfit is *perceptually questionable*. The tangram almost fits and the students may think that the wrong arrangement is the result of clumsy cutting. It may then be necessary to use a different kind of feedback, based on the relation between the measures of the original tangram and the measures of the 'image' tangram.

For example, for pieces E, D and C, we have in the initial tangram the relation that characterises the fact that they fit together: $2 + 5 = 7$.

What about the ‘image’ tangram?

$$2 \rightarrow 2 \times 2 - 1 = 3$$

$$5 \rightarrow 5 \times 2 - 1 = 9$$

$$7 \rightarrow 7 \times 2 - 1 = 13$$

and $3 + 9 \neq 13$. This criterion attests to the non-fitting of the pieces in the ‘image’ tangram of this procedure and allows it to be rejected unambiguously.

- Knowledge targeted in this situation by the failure of these procedures?

The meaning of multiplication by a natural number is that of reiterated addition. The meaning of multiplication by a rational number must be constructed against this primary meaning (primary because of the necessary precedence of the construction of integers over that of rational numbers).

The stake of this situation is that the students explicitly reject the procedures involving integers (we often speak of the *additive model* to talk about these enlargement procedures) and at least *implicitly* construct a rule of rejection that can be formulated as follows: ‘if $a + b = c$ in the original tangram, then $f(c) = f(a + b) = f(a) + f(b)$ in the enlarged tangram’ (otherwise the pieces do not fit!).

The rejection of this additive model then becomes constitutive of the meaning of multiplication by a rational number. The linear application, solution of this problem, is the multiplication by $\frac{7}{4}$, i.e. a rational application.

After a long learning process, this first sense of multiplication of numbers by a rational will be that of the image of a rational-measure by a rational-linear application.

This first sense of the multiplication of numbers by a rational will be, at the end of a long learning process (Brousseau & Brousseau, 1987), that of the image of a rational-measure by a rational-linear application.

- *Can this situation be experienced by the students as didactic?*

- The teacher can limit his or her interventions to those that are neutral with regard to the knowledge at stake in this situation: he or she can simply encourage and observe the facts, without making any particular demands.

- There is a basic procedure based on the knowledge of integers: indeed, knowledge about integers is primary to any knowledge about other numbers, i.e. the construction of any set of numbers (such as rational numbers) is based on the existence of the set of integers. This basic procedure uses the operations defined on integers.

- There is a milieu for validation of the procedure, i.e. the possibility for students to know whether their procedure is correct or not without the teacher intervening: the milieu is composed, for each student, of all the ‘enlarged’ pieces of the tangram produced independently by himself or herself and his or her partners (phase 2).

- This basic procedure is insufficient, since the procedure in question necessarily involves a rational number: the result of the basic procedure (made explicit in phase 1) is that the pieces do not connect when the team members regroup and put them together (phase 3).

4. Didactic variables: definitions and examples

The following definitions are those given by Guy Brousseau in two (already old²⁶) texts of the second French research summer school in didactics of mathematics. In the text “Didactic engineering: from a problem to the a priori study of a didactic situation”, he analyses a situation (fundamental for subtraction) in terms of variables:

A field of problems can be generated from a situation by changing the values of certain variables which, in turn, change the characteristics of the solving strategies (cost, validity, complexity, ..., etc.). [...]

Only those changes that affect the hierarchy of strategies are to be considered (relevant variables), and among the relevant variables, those that a teacher can manipulate are particularly interesting: they are the didactic variables. (Brousseau, 1982a)²⁷

These variables are relevant at a given age in so far as they command different behaviour. They will be didactic variables in so far as, by acting upon them, one can provoke adaptations and regulations: learning. (Brousseau, 1982b)²⁸

Example. Didactic variables of the tangram situation

$$\bullet V_1 = (n, p)$$

n and p are the integer numbers that define the proportionality ratio.

The relevant values are divided into four categories: $p = kn$ (k integer); $p = kn + \frac{n}{2}$ (k integer, and n even); $\frac{p}{n}$ decimal (but not half an integer); $\frac{p}{n}$ non-decimal rational.

For example, here are pairs of integers illustrating each category: (4, 8); (4, 6); (4, 7); (3, 7).

An enlargement from 4 to 8 involves the addition $4 + 4$. The student remains in the additive model and the natural numbers.

An enlargement from 4 to 6 is often analysed as “ x plus half of x ”. We thus have an intermediate model between the additive model and the linear model, which is ‘closer’ to the additive model, because of the special status of ‘half’, especially since in this case the half is also an integer.

An enlargement from 4 to 7 does not allow a simple passage, even if it is possible to use divisions by 2: [the enlargement from 4 to 7 could be interpreted as “2 times x minus half of half of x ” or “ x plus half of x plus half of half of x ”].

There is an information gap (see further down) when you have to ‘enlarge from 3 to 7’, because no additive relation is possible. One is forced to leave the additive model and leave the integers.

The values of V_1 involve the meaning of multiplication by a rational, whether or not they allow the transition from multiplication by an integer (additive model, reiterated addition) to multiplication by a rational (multiplicative model, image by a linear application).

²⁶ As these two texts have never been officially published, so we do not give here the pages of the quotations, which we have copied from our personal documents.

²⁷ Our translation from the original: “Un champ de problèmes peut être engendré à partir d’une situation par la modification des valeurs de certaines variables qui, à leur tour, font changer les caractéristiques des stratégies de solution (coût, validité, complexité...etc.) [...] Seules les modifications qui affectent la hiérarchie des stratégies sont à considérer (variables pertinentes) et parmi les variables pertinentes, celles que peut manipuler un professeur sont particulièrement intéressantes : ce sont les *variables didactiques*.”

²⁸ Our translation from the original: “Ces variables sont pertinentes à un âge donné dans la mesure où elles commandent des comportements différents. Ce seront des variables didactiques dans la mesure où en agissant sur elles, on pourra provoquer des adaptations et des régulations : des apprentissages.”

- $V_2 = \text{organization of interactions between students}$

The values of this variable are an n -tuple describing the different types of phases over time: individual work, co-operation, concertation, confrontation, etc. In the tangram situation, it is described as concertation, individual work, confrontation. The implementation of the last two phases allows for a strong feedback from the milieu on the strategy decided in the first phase. If we reduce the phases to one (co-operation to build the enlarged tangram), we remove from the milieu its property of a milieu for validation and the adidactic character of the situation.

- $V_3 = \text{tangram configuration}$

The values are also n -tuples describing the respective sub-divisions of the sides (by the same number or not), the measures of the pieces, etc. The relevant values of this variable are those that relate to the validity domain of the basic strategies. Here, the measures are chosen in such a way that the ‘add 3’ procedure (which is assumed to be the initial strategy for the vast majority of students) leads to the construction of a pieces that do not fit together in a perceptually obvious way.

- V_1 is on the side of the *knowledge at stake* (multiplication by a non-decimal rational, the number as image by a linear application).
- V_2 is constitutive of the *adidactic* character of the situation.
- V_3 is related to the *students’ knowledge*, as it is based upon hypotheses about the students’ basic strategies.

Other ‘variables’ or constraints seem to play a role in the situation: for example, the material available to the students, grid paper and a ruler. They are integrated into the situation to facilitate the construction of the pieces of the enlarged tangram and the measurement of lengths. They are therefore linked to measurement. Will these elements of the situation be given the status of a didactic variable? In order to answer, it is useful to return to the knowledge at stake, i.e. the transition from the additive model to the linear model. Measurement is not a knowledge stake in this situation. These elements may intervene in the situation, but we make the hypothesis (which can be argued) that their presence or not does not fundamentally change it.

5. Analysis of learning in an ‘ordinary’ teaching situation

The tangram situation is a situation resulting from a didactic engineering built by Guy and Nadine Brousseau (1987). The notion of adidactic situation is also a model to analyse the student’s learning in an *ordinary teaching situation*. This model leads to the following questions. In which adidactic situation(s) might the students find themselves? This raises the question: what can they learn in this teaching situation (search for possible milieus)? For what knowledge?

There are adidactic phases in all teaching, usually beyond the control of the teacher. Chevallard introduced the notion of *didactic time* to designate the specific time of the teaching institution, a time marked by the gap between the moment of teaching and the moment of learning. There is a fiction of a homogeneous didactic time in teaching. In his thesis, Alain Mercier (1992) shows that the official introduction of new objects of knowledge modifies the relation to objects that are already there, naturalised, transparent. There is then a devolution of responsibility to the student concerning these objects of naturalised knowledge: as ancient objects, the student has a responsibility to know them. This can then be an adidactic phase for these old objects (old compared with the time of their teaching) and therefore a time of learning with hindsight.

6. Different status of knowledge, different types of didactic situations

With reference to the mathematician's activity, Régine Douady (1986) uses the term *tool-object* dialectic to refer to the process of change in the status of concepts, a process that necessarily occurs in the student's activity when faced with a problem. She distinguishes three statuses for the same concept: that of object, that of implicit tool and that of explicit tool.

Brousseau distinguishes three functions (and therefore three statuses) of knowledge: action, formulation and validation. These are the types of *situations* that make it possible to characterise the functions of an object of knowledge.

In (didactic) *situations of action*, a subject (a student) develops implicit knowledge as a means of acting on a milieu (for action): this milieu provides him or her with information and feedback in return for his/her actions.

In (didactic) *situations of formulation*, the subject (the student) himself or herself makes explicit the implicit model of his or her actions. For this formulation to make sense to him or her, it must itself be a means of action on a milieu that provides him or her with information and feedback: this formulation must make it possible to obtain or make others obtain a result. Situations of communication between groups of students can be an example of such situations.

Finally, in (didactic) *situations of validation*, empirical validation from the milieu becomes insufficient: the subject, in order to convince an opponent, must develop intellectual proofs. For example, in mathematics, explicit statements about the situation become assertions whose correctness and relevance must be proven according to common rules, in order to make them a theorem known to all.

7. Fundamental situation

7.1. Introduction

In didactic phases (modelled by an didactic situation), the student has to solve a problem for which he or she is responsible. The central question is: how can we ensure that the problem posed is *relevant* to the knowledge at stake? How does the problem posed relate to the *raison d'être* of the object of knowledge, which is the stake of teaching? What meaning does it give to knowledge?

These are epistemological issues.

Epistemological hypothesis

We will also assume that for all knowledge there is a family of situations that can give it a correct meaning. (Brousseau, 1986a, p. 316)²⁹

Correct in relation to the history of this concept, in relation to the social context, in relation to the scientific community.

For any knowledge, it is possible to construct a formal game, *communicable without using this knowledge*,³⁰ and yet it determines the optimal strategy. (Brousseau, 1988a, pp. 314–315)³¹

²⁹ Our translation from the original: “Nous admettrons aussi qu’il existe pour toute connaissance une famille de situations susceptibles de lui donner un sens correct.”

³⁰ I underline.

³¹ Our translation from the original: “Pour toute connaissance, il est possible de construire un jeu formel, communicable sans utiliser cette connaissance, et dont elle détermine pourtant la stratégie optimale.”

A *fundamental situation* of a piece of knowledge is a *model* of a family of *non-didactic* situations specific to this piece of knowledge.

Modelling a teaching situation consists in producing a specific set of targeted element of knowledge [...] (Brousseau, 1986a, p. 334)³²

This model is fundamental:

- With regard to knowledge: the game must be such that the piece of knowledge appears in the chosen form, as the solution, or as the means of establishing the optimal strategy. [...]
- With regard to the teaching activity: the ‘game’ must make it possible to represent all the situations observed in the classroom [...] even the least 'satisfactory' ones, as long as they succeed in getting the students to learn a form of the targeted element of knowledge. It must be able to generate all variants, even the most degenerate. These will be obtained by choosing certain variables characteristic of the situation. (Brousseau, 1986a, pp. 334–335)³³

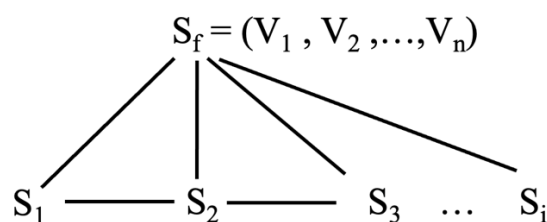
A fundamental situation must at least allow (by playing with the values of the didactic variables):

- an effective genesis of a piece of knowledge, i.e. a genesis of situations representative of the different meanings of a piece of knowledge (of the different occasions of use of this piece of knowledge);
- a ‘rereading’ of this genesis according to the logic of the organisation of knowledge.

A ‘fundamental’ situation can be represented by a finite set of variables, relevant to the meaning of the piece of knowledge at stake in the teaching. Conversely, by giving values to these variables, particular situations are generated which give the piece of knowledge a particular meaning.

The following diagram (Figure 4) illustrates our point:

Figure 4: *Fundamental situation S_f*



7.2. Example. *A priori* analysis of a fundamental situation “The race to n game” (after Brousseau, 1978)

Game 1: ‘The race to 20’

Rules of the game

³² Our translation from the original: “Modéliser une situation d’enseignement consiste à produire un jeu spécifique du savoir visé [...]”.

³³ Our translation from the original: “- Au regard de la connaissance : le jeu doit être tel que la connaissance apparaisse sous la forme choisie, comme la solution, ou comme le moyen d’établir la stratégie optimale. [...] - Au regard de l’activité d’enseignement : le ‘jeu’ doit permettre de représenter toutes les situations observées dans des classes [...] même les moins ‘satisfaisantes’, dès lors qu’elles parviennent à faire apprendre à des élèves une forme de savoir visé. Il doit pouvoir engendrer toutes les variantes même les plus dégénérées. Elles seront obtenues par le choix de certaines variables caractéristiques de cette situation.”.

The game involves two opponents who take turns saying a number. The aim is for each player to say 20 first.

The first player to play has the right to say 1 or 2. A number can only be said if it is obtained by adding 1 or 2 to the number the opponent has just said.

Play a few rounds and formulate a winning strategy (i.e. one that allows you to win no matter what your opponent does).

Answer: Whoever plays first and says 2, then 5, 8, 11, 14, 17, 20 (which he or she can say no matter what the opponent says) wins.

Comment: very quickly we ‘know’ that the one who says 17 has won: the race to 20 becomes the race to 17. We can therefore repeat the reasoning. In fact, the winning sequence is found ‘by descending’: 20, 17, 14, etc.

The following games are played with the same teams of opponents

Game 2: ‘The race to 27’

The aim is to be the first to say 27. Whoever starts playing has the right to say a non-zero integer less than or equal to 4. A number can only be said if it is obtained by adding a non-zero number less than or equal to 4 to the number that the opponent has just said.

Formulating a winning strategy

To get to 27 first, what number do you need to say just before?

(Similar reasoning to 17 in the race to 20.) If I say 26, my opponent can add 1 and say 27; if I say 25, my opponent can add 2 and say 27; if I say 24, my opponent can add 3 and say 27; if I say 23, my opponent can add 4 and say 27; if I say 22, whatever my opponent adds – 1, 2, 3, or 4, I will add the complement to 5: $1 + 4, 2 + 3, \dots$: I will therefore say 27 first.

Winning strategy: whoever plays first and says 2 wins. Winning sequence: 27, 22, 17, 12, 7, 2.

Game 3: ‘The race to 24’

The aim is to be the first to say 24. The first player to play has the right to say a non-zero integer less than or equal to 3. A number can only be said if it is obtained by adding a non-zero number less than or equal to 3 to the number that the opponent has just said.

Can you win using the strategy you proposed after game 2?

Answer: No, whoever plays second and says 4 can never say first wins!

Winning sequence: 24, 20, 16, 12, 8, 4.

Game 4: ‘The race to 5929’

The aim is to be the first to say 5929. The first to play is allowed to say a non-zero integer less than or equal to 2. A number can only be said if it is obtained by adding a number less than or equal to 2 to the number the opponent has just said.

Answer: The game becomes: who wins? Should you start or play second? Saying what number?

If one starts to reason as in the previous games, one searches for the last number one has to say to be sure to be able to say 5929 first. This last number is 5926 which is at the ‘right’ distance from 5929, i.e. at distance 3: if my opponent adds 1, I add 2 (and I say 5929 first) – if he or she adds 2, I add 1 (and I still say 5929 first). The game of the race to 5929, becomes the game of the race to 5926.

5929, 5926, 5923, ...: this list is obtained by repeated subtractions of 3.

It becomes very exhausting to find the whole winning sequence: for economy, we will only look for some of the integers of the winning list to get as quickly as possible to the smallest integer. If we subtract from 5929, 1000 times 3, we obtain an integer from the list: 2929. To find an integer from the list smaller than 1000, we can subtract from 2929, for example, 800 times 3 that gives 529. And so on until one gets 1.

Example: $5929 - 3 \times 1000 - 3 \times 800 - 3 \times 100 - 3 \times 70 - 3 \times 6 = 1$

Hence the answer: whoever plays first and says, 1 is sure to be able to win.

A general situation of the games of the race to n

The game involves two opponents who take turns saying a number. The aim is for each of the opponents to say n first.

The first player to play has the right to say a non-zero integer less than p . A number can only be said if it is obtained by adding a non-zero integer less than p to the number the opponent has just said [n and p are natural numbers with $n > p$].

This situation generates the four sets described above by giving values to the variables n and p .

What mathematical knowledge provides an economical and optimal resolution tool for the n -race games?

The Euclidean division of n by the integer $(p + 1)$: $n = (p + 1)q + r$ with $0 \leq r \leq (p + 1)$.³⁴

The ‘meaning’ of this division (in the race to n games) is the repeated subtraction of $(p + 1)$ from n : the number of repeated subtractions to arrive at the smallest integer is the quotient of this division, the smallest integer to which one gets is the remainder. The number $(p + 1)$ that is subtracted repeatedly is the divisor.

For game 1, dividing 20 by 3 gives 2 as the remainder and to get 2 you have to subtract 6 times 3.

For game 4, dividing 5929 by 3 gives 1 and the number of subtractions of 3 is $1000 + 800 + 100 + 70 + 6 = 1976$.

What could be the didactic variables of the n -race problem for learning this mathematical knowledge?

V_1 : n multiple of $(p + 1)$ or not.

If n is not a multiple of $(p + 1)$, then start – the winning sequence is an arithmetic sequence of ratio $(p + 1)$ and the first term is the remainder.

If n is a multiple of $(p + 1)$, do not start – the winning sequence is an arithmetic sequence of ratio $(p + 1)$ [see passage from games 1, 2 to 3].

V_2 (size of n compared with p): n small compared with p versus n large compared with p

If n is small compared with p , writing all the integers of the winning sequence is possible: the strategy of repeated subtraction of $(p + 1)$ is an optimal strategy competing with Euclidean division which does not give the list!

If n is very large compared with p , the strategy of repeated subtractions becomes very costly. It is then only possible to reach some of the integers in the winning sequence. The nature of the game then changes: to be sure to be able to win, should we start or play second? By saying which number?

The strategy of repeated subtractions must be adapted and transformed into a strategy that makes it possible to (re)find the meaning of Euclidean division: one seeks to subtract from n the largest possible multiple of $(p + 1)$. [Passage from games 1, 2, 3 to game 4: this is an example of an information jump.]

Conclusion: the race to n game as a fundamental situation

If we model the race to n games by a general situation, we obtain a fundamental situation of the Euclidean division whose meaning is that of the repeated subtraction.

Notion of information jump:

The information jump consists of, after finding a fundamental situation making a concept ‘work’, to choose first the values of its variables in such a way that students' prior knowledge allows for the development of strategies and explore the characteristics of the situation, constraints, objectives, rules..., then without changing the rules of the game, to change the values of the variables in order to increase drastically the complexity of the

³⁴ This point is entirely debatable: would it not rather be a fundamental situation of modulo $(p + 1)$ congruence? Research in progress ...

task. New strategies must then be established which require building new elements of knowledge. (Brousseau, 1986a, p. 23)³⁵

8. Supplements: conceptions, obstacles

For some knowledge, there are fundamental situations that can be implemented directly at the right moment:

But let us suppose that there are some pieces of knowledge for which the above conditions are not fulfilled. There are no situations that are sufficiently accessible, sufficiently effective and insufficiently small numbers to allow students of any age to gain access from the outset, by adaptation, to a form of knowledge that can be considered correct and definitive: it is necessary to accept stages in learning. The knowledge taught by adaptation in the first stage will be temporarily not only approximate, but also partly *false and inadequate*. (Brousseau, 1986a, pp. 316–317)³⁶

The alternative is directly to teach knowledge that conforms to the requirements of the scientific community, i.e. the text of knowledge. In this way, the student is not expected to make sense of the knowledge: knowledge is not an answer to an uncertain problem for which the student accepts responsibility.

The teacher thus has the choice between teaching formal and meaningless knowledge or teaching more or less false knowledge that will have to be rectified. (Brousseau, 1986a, p. 317)³⁷

We can then assume that the same subject has contradictory relations with the same object of knowledge (from the point of view of the person observing the procedures), the coherence of which must be sought in the particular situation in which the subject is placed: the notion of conception is an answer to this problem of modelling the subject's knowledge.

In particular, the notion of conception makes it possible to answer the following question: what do the students know (and which explain their difficulties, their errors)? In what situations have they learned it (what is the area of validity)?

The notion of didactic contract supplements this model to take into account the constraints of the didactic communication of knowledge.

Postulate: error is a testimony of knowledge

This postulate takes on a strong meaning in the framework of the Piagetian hypothesis on learning as an adaptation to a milieu that allows the subject to get aware of a contradiction and how to overcome it:

³⁵ Our translation from the original: “Le saut informationnel consiste, après avoir trouvé une situation fondamentale faisant « fonctionner » une notion, à choisir d’abord les valeurs de ses variables de telle manière que les connaissances antérieures des élèves permettent d’élaborer des stratégies efficaces...puis, sans modifier les règles du jeu, à changer les valeurs des variables de façon à rendre beaucoup plus grande la complexité de la tâche à accomplir. De nouvelles stratégies doivent être établies qui demandent la construction de nouvelles connaissances.”

³⁶ Our translation from the original: “Mais supposons qu’il existe des connaissances pour lesquelles les conditions ci-dessus ne sont pas réalisées: il n’existe pas de situations suffisamment accessibles, suffisamment efficaces et en nombre suffisamment petit pour permettre à des élèves d’âge quelconque d’accéder d’emblée, par adaptation, à une forme de savoir que l’on puisse considérer comme correcte et définitive : il faut accepter des étapes dans l’apprentissage. Le savoir enseigné par adaptation dans la première étape sera provisoirement, non seulement approximatif, mais aussi en partie *faux et inadéquat*.”

³⁷ Our translation from the original: “Le professeur a donc le choix entre enseigner un savoir formel et dénué de sens ou enseigner un savoir plus ou moins faux qu’il faudra rectifier.”

The error is not only the effect of ignorance, uncertainty, chance as believed in empiricist or behaviourist theories of learning, but the effect of a previous knowledge, which had its values, its successes, but which, now, turns out to be false, or simply inadequate. (Brousseau, 1983, p. 171)³⁸

Brousseau, as early as the 1970s, stated that some of this knowledge is necessary for learning: the student's trajectory should pass through the (temporary) construction of erroneous knowledge, because the awareness of this erroneous character would be constitutive of the meaning of the knowledge whose construction is aimed at. Following Bachelard (1938, pp. 13–22), Brousseau calls these obligatory points of passage *epistemological obstacles*:

(...) a piece of knowledge, like an obstacle, is always the result of the student's interaction with his milieu and more precisely with a situation that makes this knowledge 'interesting'. (Brousseau, 1998, p. 123)³⁹

Three types of obstacles are usually distinguished according to their origin.

1. *Ontogenetic* obstacle linked to the psychogenetic development of the subject.
2. *Didactic* obstacle linked to the didactic transposition of knowledge: it is an obstacle which can be avoided without any consequence for the construction of knowledge, which can disappear by acting on the teaching situations.
3. *Epistemological* obstacle linked to the historical development of the concept.⁴⁰

[...] an epistemological obstacle is constitutive of completed knowledge, in the sense that its rejection must inevitably be explained, and consequently it leaves traces, sometimes profound, in the system of knowledge [...] (Brousseau 1983, p. 195).⁴¹

Duroux (1983) gives criteria for defining an epistemological obstacle.

- 1- It is a **knowledge** that functions as such on a set of situations and for certain values of the variables of these situations. [...]
- 2- The obstacle is knowledge which, in trying to adapt to other situations or other values of variables, will cause specific, identifiable and analysable errors.
- 3- The obstacle is a stable knowledge. [...]
- 4- The obstacle can therefore only be overcome in specific situations of rejection and will be constitutive of knowledge [...] The very return to the obstacle conception will be an integral part of the new knowledge. (p. 54)⁴²

³⁸ Our translation from the original: "L'erreur n'est pas seulement l'effet de l'ignorance, de l'incertitude, du hasard que l'on croit dans les théories empiristes ou behavioristes de l'apprentissage, mais l'effet d'une connaissance antérieure, qui avait son intérêt, ses succès, mais qui, maintenant, se révèle fautive, ou simplement inadaptée."

³⁹ Our translation from the original: "(...) une connaissance, comme un obstacle, est toujours le fruit d'une interaction de l'élève avec son milieu et plus précisément avec une situation qui rend cette connaissance 'intéressante'."

⁴⁰ The notion of epistemological obstacle was introduced by Gaston Bachelard (1938) in "La formation de l'esprit scientifique".

⁴¹ Our translation from the original: "[...] un obstacle épistémologique est constitutif de la connaissance achevée, en ce sens que son rejet doit être incontournable explicité, et par conséquent il laisse des traces, parfois profondes, dans le système des connaissances [...]"

⁴² Our translation from the original: "1- Il s'agit d'une connaissance qui fonctionne comme telle sur un ensemble de situations et pour certaines valeurs des variables de ces situations.... 2- L'obstacle est une connaissance qui, en tentant de s'adapter à d'autres situations ou à d'autres valeurs des variables, va provoquer des erreurs spécifiques, repérables, analysables. 3- L'obstacle est une connaissance stable. [...] 4- L'obstacle ne pourra donc être franchi que dans des situations spécifiques de rejet et sera constitutif du savoir [...] Le retour même sur la conception obstacle sera partie intégrante du nouveau savoir."

Example. Integer obstacle

Example 1. Decimals/Integers

For the student, the properties of natural numbers are those of numbers in general, of all numbers. However, the inclusion of the set of natural numbers in a superset such as the rational or the decimal numbers, while bringing out new properties, makes some others disappear: they are no longer true for all numbers, or even for none of them: multiplying can become smaller, a decimal no longer has a successor.

[...]

The teacher cannot adequately warn the student of this break, because neither the culture, and in particular the tradition, nor the didactic engineering have yet produced the necessary instruments (exercises, warnings, concepts, remarks, paradoxes, ...). This leads to misunderstandings and misinterpretations on the part of the teacher and mistakes on the part of the student. These misconceptions persist because they are attached to a certain way of understanding the properties of natural numbers, and the effects of the break can be observed for many years.

More important is the mechanism of this obstacle: it is not the piece of knowledge taught that is lacking – usually teachers provide for this inconvenience by trying to maintain a misunderstood but correct discourse – it is the student's personal instruments of understanding. He no longer understands, because what should be changed are precisely the means of what he used to call ‘understanding’. (from Brousseau, 1998, Chapter 6)⁴³

Example 2. Tangram situation to make students meet the obstacle of natural numbers (see above).

9. Devolution/institutionalisation (institutional knowledge/situational knowledge)

The main roles of the teacher are characterised by the following double movement: devolution process and institutionalisation process.

9.1. Devolution

To do this, the teacher first of all does the opposite of the researcher: he or she seeks to re-contextualise and re-personalise the knowledge to be taught; he or she *looks for problems that will give meaning to the knowledge to be taught*, so that the student's activity ‘resembles’ sometimes at that of the researcher. There is a devolution to the student of a responsibility towards knowledge, there is a devolution of an adidactic situation.

⁴³ Our translation from the original: “Pour l'élève, les propriétés des naturels sont celles des nombres en général, de tous les nombres. Or, le plongement de l'ensemble des naturels dans un sur-ensemble comme les rationnels ou les décimaux, en même temps qu'il fait apparaître des propriétés nouvelles en fait disparaître certaines autres : elles ne sont plus vraies pour tous les nombres, ou même elles ne le sont plus pour aucun : multiplier peut rapetisser, un décimal n'a plus de successeur ...

L'enseignant ne peut pas avertir convenablement l'élève de cette rupture, car, ni la culture, et en particulier la tradition, ni l'ingénierie didactique n'ont encore produit les instruments nécessaires (exercices, avertissements, concepts, remarques, paradoxes, ...). Cette situation conduit l'enseignant à provoquer des quiproquos et des malentendus et l'élève à commettre des erreurs. Ces conceptions fausses persistent car elles sont attachées à une certaine manière de comprendre les propriétés des nombres naturels, et on peut observer les effets de la rupture pendant de nombreuses années.

Plus important encore est le mécanisme de cet obstacle : ce sont, non pas les connaissances enseignées qui sont en défaut – en général les enseignants pourvoient à cet inconvénient en essayant de se maintenir dans un discours incompris mais correct – ce sont les instruments personnels de la compréhension de l'élève. Il ne comprend plus, parce que ce qui devrait être changé ce sont justement les moyens de ce qu'il appelait ‘comprendre’ jusque-là.”

The devolution process (of an didactic situation) is described by André Rouchier (1991) in his Ph.D. thesis as a process that allows the conversion of institutional knowledge to be taught into situational knowledge for the student (personalised, contextualised, temporalised).

9.2. Institutionalisation

But if this phase has worked well, when the student has found solutions to the problems posed, the student does not know that he or she has produced knowledge that he or she will be able to use on other occasions. In order to transform the students' answers and knowledge into institutional knowledge, the students, with the help of the teacher, will have to re-decontextualise, re-depersonalise the situational knowledge they have produced, in order to recognise in what they have done something that has a universal character, a reusable cultural knowledge.

In short, the process of institutionalisation is an inverse process to that of devolution which converts situational knowledge from the student into reusable institutional knowledge (de-personalised, de-contextualised, de-temporalised).

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ANNEX I MATHEMATICAL VERSUS PRACTICAL CONSTRUCTION

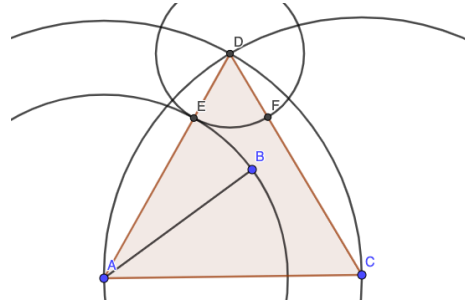
Mathematical construction

Instruments are ‘*ideal*’ instruments. A construction problem in mathematics is first and foremost a problem of *constructability*.

Euclid’s compass is an ‘instrument’ for drawing a circle with a centre point and passing through a point.

A compass does not ‘carry’ distances. It is shown that it can carry distances by the constructability of a segment [CF] of the same length as a given segment [AB].

Be A, B and C three non-aligned (base) points. Draw an equilateral triangle CAD.⁴⁴ Draw the circle of centre A through B; the point of intersection of this circle with the line (AD) is called E. Draw the circle with centre D through E; the point of intersection of this circle with the line (CD) is called F. The segment [CF] answers the question. (Euclid’s solution with modern notation.)



With regard to this construction, we can repeat what Chevallard and Jullien (1991) say about the construction of the middle point of a segment:

The interest of this construction can only be understood if we take into account the fact that Euclid’s viewpoint is, above all, that of constructability: he shows by a logical sequence of propositions – allowing himself to use only propositions that have already been established – that the middle point of a segment is constructible with a ruler and a compass. (p. 65)⁴⁵

A ruler is an ‘instrument’ with a single straight edge, assumed to be unlimited and without graduations or marks.

The solution to a geometric construction problem can be analysed according to the following criteria:

1. it must provide a proof of the existence of the ‘object’ to be built;
2. it must provide a proof of its constructability;
3. it must provide an algorithm for the construction;
4. eventually, it can provide the geometrographic construction.⁴⁶

⁴⁴ Draw the circle with centre A through C. Draw the circle with centre C through A. One of the points of intersection is called D.

⁴⁵ Our translation from the original: “On ne peut saisir l’intérêt de cette construction que si l’on prend en compte le fait que le point de vue d’Euclide est, avant tout, celui de la constructibilité : il montre par un enchaînement logique de propositions – en ne se permettant donc d’utiliser que des propositions déjà établies – que le milieu d’un segment est constructible à la règle et au compas.”

⁴⁶ “[...] geometrography is interested in the *construction processes* themselves, proposing to compare them in order to find the simplest ones, according to determined criteria. [...] the solutions proposed by mathematicians are (often) bad solutions. (Our translation from the original: “[...] la géométrie s’intéresse aux *procédés* de construction eux-mêmes, en se proposant de les comparer afin de trouver les plus simples, selon des critères déterminés. [...] les solutions proposées par les mathématiciens sont (souvent de mauvaises solutions).)” (Chevallard & Jullien, 1991, p. 66, p. 68).

These requirements are obviously ‘nested’ within each other, i.e. the satisfaction of the last one leads to the satisfaction of the other three, the satisfaction of the penultimate one to the satisfaction of the previous two, and so on. (Chevallard & Jullien, 1991, p. 68)⁴⁷

Practical construction

From a practical point of view, the problem is to find drawing procedures using material instruments in order to obtain a satisfactory drawing according to the measures. For example, to find the axes of symmetry of an ellipse with a centre O, one needs to draw the inscribed and circumscribed circles with a compass pointed at O – by modifying its aperture little by little – *but with such precision that it is not possible to graphically (or perceptively) contest the drawing made ...* An approximate drawing procedure is therefore acceptable as far as that it gives a graphical solution with sufficient precision.

The split between the two viewpoints is evident when dealing with a case of non-constructability.

For the designer, for example, there is always an algorithm of construction, exact or approximate, requiring such and such instruments, which enables him to carry out his project. The notion of non-constructability – and, consequently, that of constructability – becomes, as a result, obsolete from his viewpoint. It has little *graphical* relevance; it nevertheless retains all its *mathematical* relevance. (Chevallard & Jullien, 1991, p. 76)⁴⁸

An example of a non-constructability problem with ruler and compass:

The exact division of the circle into equal parts with a ruler and a compass was formerly only possible for the numbers $2n$, 3, 5 and their various multiples. Gauss added other cases by showing the possibility of division into p parts, when p is an integer of the form

$$p = 2^{2^n} + 1$$

and the impossibility of division in all other cases.

Practice can derive no benefit from these results; Gauss’s considerations have a purely theoretical significance [...]. (Klein, 1846, p. 10)

Problem. How can we characterise construction with a ruler and a compass in French geometry teaching?

In the construction of the axis of symmetry, what the teacher expects is neither a ‘mathematical’ construction (in the sense of Klein or Euclid), nor a practical construction. The instruments are material and one must produce a material drawing (as in a ‘practical’ construction), but any drawing must be backed up by a geometric property (as in a ‘mathematical’ construction): *the problem of existence does not arise.*

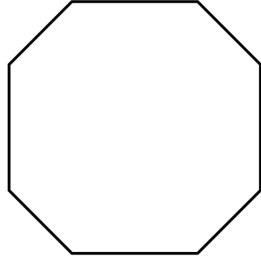
⁴⁷ Our translation from the original: “1. Elle doit fournir une preuve de l’existence de « l’objet » à construire ; 2. Elle doit fournir une preuve de sa constructibilité ; 3. Elle doit fournir un algorithme de construction ; 4. Éventuellement, elle peut fournir la construction géométrographique. Ces exigences sont évidemment « emboîtées » les unes dans les autres, c’est-à-dire que la satisfaction de la dernière entraîne la satisfaction des trois autres, la satisfaction de l’avant-dernière celle des deux précédentes, et ainsi de suite.”

⁴⁸ Our translation from the original: “Pour le dessinateur il existe toujours un algorithme de construction, exact ou approché, nécessitant tels ou tels instruments, lui permettant de réaliser son projet. La notion de non constructibilité – et, par suite, celle de constructibilité – devient, en conséquence, caduque de son point de vue. Elle n’a guère de pertinence *graphique* ; elle conserve pourtant toute sa pertinence *mathématique*.”

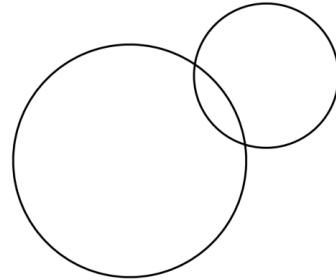
ANNEX II
DRAWING SYMMETRY AXES ON FIGURES (GRENIER, 1984)

For each of the following figures, draw all the axes of Symmetry with the indicated instruments only:

1st group: With a ruler including graduation and a set square

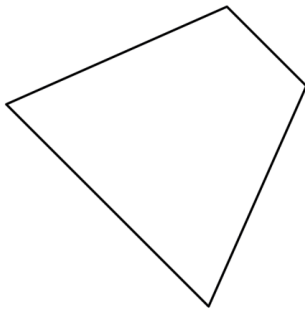


Octagon

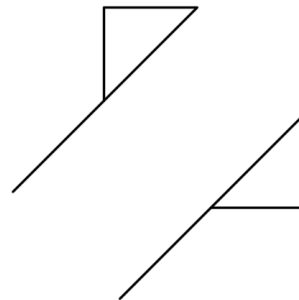


2 intersecting circles without their centres

2nd group: with a ruler without graduations and a set square

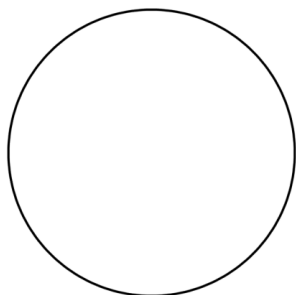


Isosceles trapezoid

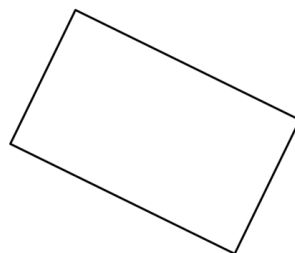


2 flags

3rd group: ruler without graduation and compass



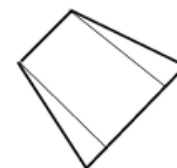
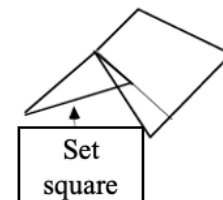
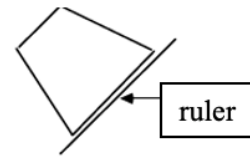
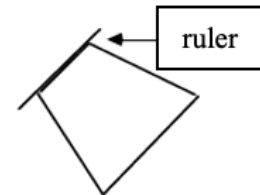
Circle without its centre



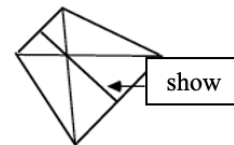
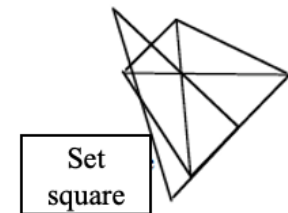
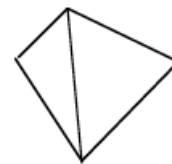
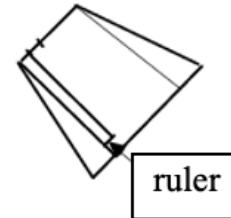
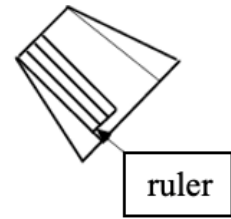
Rectangle

ANNEX III
CHRONICLE “SUMMARY OF THE TRACING ACTIVITY WITH INSTRUMENTS
ABOUT THE ISOCELES TRAPEZOID” – EXTRACT FROM (GRENIER, 1989)

1 Teacher=T: Let's move on to figure 3, the trapezoid. (T pastes a paper with the figure on the board.)
2 T: Group 3 please, shhh (the class is noisy), so there you have the rule without graduation and the set square.
3 Olivier (group 3) takes both instruments and uses the ruler
4 T: Give it to me. (*T takes the square.*)
5 Olivier: So, we've already taken the ruler /.../ we put it down.
6 *T takes the ruler*: Shh, please.
7 Olivier: We put it down and then we put a dot [on the ruler].
8 T: You have marked on the ruler a graduation to measure the length of the segment.
9 Olivier: And then here it's the same. (*Put the ruler on the large base and measure.*)
10 T: Here, it's the same. Well, is it very precise?
11 XX: No.
12 T: In your opinion?
13 XX: It can be erased on the ruler.
14 T: It can be erased on the ruler. It's not very precise. Then say how you did it.
15 Olivier: We saw that it was about half. It was double that ...
16 T: What expression did you say? Did you hear?
17 X Yes.
18 T: Is it very precise? (*he laughs*)
19 XX: No.
20 T: You see! So maybe, because time is running out, we'll carry on here because you're not very precise, we want to be precise.
21 T (*to XF*): Do you have anything to propose? XF: come to the board.
22 T (*to Olivier*): Thank you, Olivier.
23 T (*to XF*): Give it to me if you are embarrassed.
24 XF *gives the ruler and keeps the set square*
25 T: Please speak very loudly.
26 XF: We put our set square on the corner.
27 T: Yes, at the top of the angle,
28 XF: Yes, and we made little dotted lines all the way down. (*XF draws*)
29 T: OK, so a perpendicular?
30 XF: Yes.
31 XF: We do the same on the other side. (*XF draws*)
32 T: Same thing on the other side. Two perpendiculars.
33 XF *gives the set square back and takes the ruler from T.*
34 XF: Then it seems that it measures one centimetre.
35 T: Ah, it seems that ...
36 XF: Well, that's what they said,
37 T: That's what they said and you take their word for it.
38 *XF laughs*
39 T: So, is the 'it seems' very precise?
40 *François protests*
41 T: No, we're not accusing you, François! But let's discuss it. Is it very precise?
42 X: No.
43 T: You thought it was one centimetre. Go on. We're going to listen to you for two seconds.
44 XF: Then they turned the ruler to find out how many centimetres there were and they divided it in two.
45 T: Right. So, you're a bit like Olivier. Is that more accurate than ...
46 *XF goes back to his place and gives François the instruments*



47 T: Well, François? Come on!
 48 François comes to the board, takes the ruler.
 49 T: You have the set square too.
 50 François puts the ruler down and marks each transfer of the section of the ruler with a small line.
 51 T: So, you mark? Is what you're doing very precise?
 52 François: Yes .
 53 T: That makes the length. Well, we agree, so I think that many people have done it, but it's not very precise because you have the thickness, which comes into play here.
 54 François continues to report the distance with his ruler.
 55 T: You are not more precise than Olivier, I'm sorry.
 56 François goes back to his place very disappointed.
 57 A student from group 2 comes to the board, takes the ruler.
 58 Group 2: Let's draw this first.
 59 T: So, there you go, you draw it. So?
 60 T: Who can say what this line is called, what you called it?
 61 Group 2: We draw a ...
 62 T: A diagonal, eh?
 63 X: Perpendicular.
 64 T: One diagonal, then the other. Hush.
 65 Group 2 traces the other.
 66 Group 2: So that gives me the centre.
 67 T: That gives you a point;
 68 Group 2: With the square (takes the set square, place the square on the point and give the ruler back).
 69 T writes on the board.
 70 T: This point, you consider ...
 71 Group 2: That it is in the middle.
 72 T: That it's on? (noises)
 73 Group 2: That gives us this. (shows the segment)
 74 T: So, what is it?
 75 Group 2: It's the line of symmetry.
 76 T: This gives us the axis of symmetry. So, you have the point, then the perpendicular. (T writes on the board)



77 T: And that gives you the axis of symmetry. This point that you chose, well you say at the start that it's on the axis of symmetry.
 78 Group 2: Yes.
 79 T: OK. Does anyone have another construction? Is that OK? (students: yes)
 80 X: We could have folded the sheet and drawn. (laughter from other students)
 81 T: No, it's with the instruments. We'll move on to the next page, which is 'flags'.
 82 Yann: But sir, there was only one point!
 83 T: Yes, one point and the perpendicular (hubbub) that was enough for him. I'm not saying it's exactly like that ...
 84 Yann: Yes, you need at least two points!
 85 T: Shhh! ... silence ... I think Yann has a problem and we have to help him. One point is not enough, says Yann.
 86 Yann: Yes.
 87 T: Go ahead, it's your problem.
 88 X: Well, he's stupid because ... (laughs)
 89 T: Oh no, no! You place the set square along the side and you slide it until the point is on the other side of the set square (she mimics on the board), so you can draw, it's precise.
 90 Yann: Yes.