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MEASURING DISCREPANCIES BETWEEN COULOMB AND OTHER GEOTECHNICAL CRITERIA: DRUCKER-PRAGER AND MATSUOKA-NAKAI

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Abstract. *Whereas the Mohr-Coulomb criterion is widely used in geotechnics, the Drucker-Prager is common in Finite Element Methods software. Another smooth criterion developed in soil mechanics is the Matsuoka-Nakai criterion, which is also a linear function of mean stress. Parameters of the Coulomb criterion are meaningful to engineers and their determination in laboratory is well known. In this paper, we present a method to measure differences between Mohr-Coulomb and "Coulomb" fitting by Drucker-Prager or Matsuoka Nakai.*

Using some properties of the stress space and of the deviatoric plane, we are able to measure discrepancies between Coulomb and other criteria. We provides results as discrepancies on the friction angle allowing a quick evaluation of the error.

Stress sign convention Traction stresses are positive, and the principal stresses ordered as follow : $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$.

1 INTRODUCTION

The Mohr-Coulomb criterion is the most common failure criterion encountered in geotechnical engineering. It is robust, and parameters are easily deduced from tests and are meaningful for engineers. However the presence of corners makes it less straightforward to implement. Whereas the Drucker-Prager is easy to implement and allows a fast computation of plastic strain, it is widely criticized from a physical point of view. So others criteria, like Matsuoka-Nakai, have been developed, that fit well the laboratory data and are smooth in the deviatoric plane.

Our aim is to measure discrepancies between Mohr-Coulomb and other geotechnical laws. We kept in mind two goals while developing this methodology. We wanted first to give a precise measurement of the discrepancies. We also wanted to provide clear results that can be understood at a glance by an engineer. So we choose to use the friction angle ϕ as the variable of measure.

2 DIFFERENT CRITERIA FOR A SAME MATERIAL BEHAVIOR

2.1 Criteria

2.1.1 Mohr-Coulomb

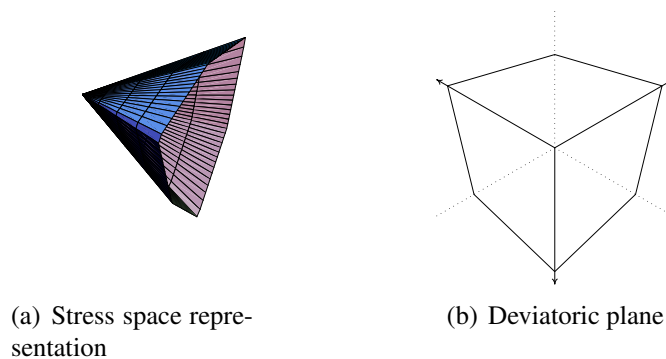


Figure 1: Mohr-Coulomb criterion

Mohr-Coulomb is certainly the oldest and most notorious geotechnical criterion. It is defined as the equation of an intrinsic curve in the Mohr plane. Parameters (cohesion c , and friction ϕ) are meaningful and easy to deduce from laboratory tests.

$$\tau = c + \sigma_n \tan \phi \quad (1)$$

The corresponding yield function can be expressed as a function of the principal stresses :

$$f(\underline{\sigma}) = (\sigma_I - \sigma_{III}) + (\sigma_I + \sigma_{III}) \sin \phi - 2C \cos \phi \quad (2)$$

In the stress space (or Haigh-Westergaard space), the Mohr-Coulomb yield surface is a conical prism. Its deviatoric shape is an irregular hexagon

2.1.2 Drucker-Prager

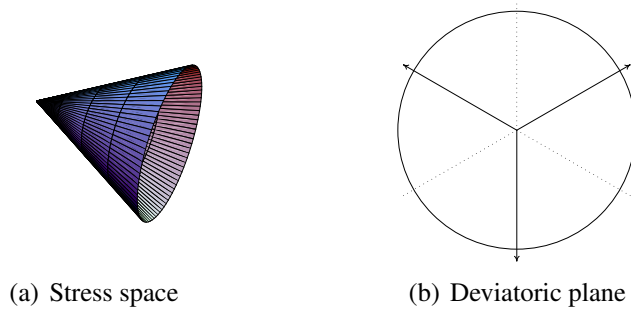


Figure 2: Drucker-Prager

Drucker-Prager [3] is a pressure dependant function of the second invariant of the deviatoric stress ($J_2 = \frac{1}{2}\text{Tr}(\underline{\underline{s}}^2)$). The Drucker-Prager yield surface is a cone in the stress space. In the deviatoric plane, it is a circular criterion.

$$f(\underline{\underline{\sigma}}) = 3\alpha(\sigma_m - H) + \sqrt{J_2} \quad (3)$$

σ_m being the mean stress.

2.1.3 Matsuoka-Nakai

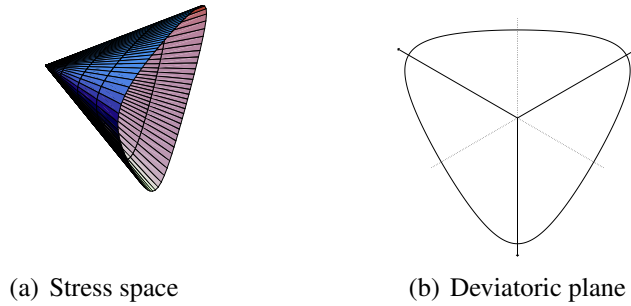


Figure 3: Matsuoka-Nakai

Matsuoka-Nakai [9] was developed initially for cohesionless material, but can be adapted to be used with material with cohesion[5]. The Matsuoka-Nakai yield function is a function of the polynomial invariants of the stress tensor :

$$f(\underline{\underline{\sigma}}) = k_1 I_{III} - I_1 I_{II} \quad (4)$$

Où :

$$\begin{aligned} I_I &= \sigma_I + \sigma_{II} + \sigma_{III} \\ I_{II} &= \sigma_I \sigma_{II} + \sigma_{II} \sigma_{III} + \sigma_{III} \sigma_I \\ I_{III} &= \det \underline{\underline{\sigma}} = \sigma_I \sigma_{II} \sigma_{III} \end{aligned}$$

The deviatoric shape of Matsuoka-Nakai is smooth but non circular.

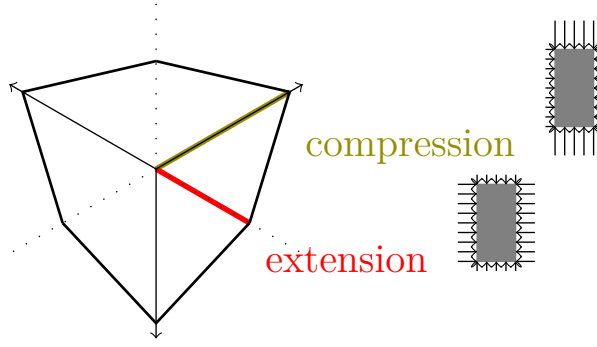


Figure 4: Physical meaning of the deviatoric shape

2.2 Extension ratio

The deviatoric shape of Mohr-Coulomb (irregular) or Matsuoka-Nakai (non circular) has a physical meaning. For a given mean stress, one can observe a significant difference (figure 4) for J_2 , depending if we consider a compression configuration ($\sigma_I = \sigma_{II} > \sigma_{III}$) (configuration of the classical triaxial test and compressive test) or an extension configuration ($\sigma_I > \sigma_{II} = \sigma_{III}$) (extension triaxial test (inverse triaxial test) and traction test)

We can define the extension ratio L_S :

$$L_S = \frac{\sqrt{J_2}(\theta = -\frac{\pi}{6})}{\sqrt{J_2}(\theta = \frac{\pi}{6})} = \frac{(\sigma_I - \sigma_{III})(extension)}{(\sigma_I - \sigma_{III})(compression)} \tag{5}$$

With θ the Lode angle defined at section 3 on the following page.

L_S belongs to $[\frac{1}{2}, 1]$:

- A value of one corresponds to a circle (Von Mises, Drucker-Prager) or a regular hexagon (Tresca). While theoretically possible, a value of L_S greater than one would mean that the traction behaviour would be better than the compressive one.
- A value of $\frac{1}{2}$ corresponds to a triangle in the deviatoric plane. A smaller value would mean the criterion is not convex.

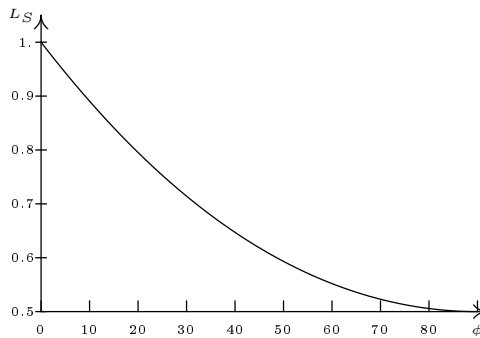


Figure 5: Extension ratio, function of friction angle

The main concern with Drucker-Prager is the value of L_S equal to one. Experimental results better agree with values of Matsuoka-Nakai or Mohr-Coulomb. For the latter, the extension

ratio is a function of the friction angle (figure 5) :

$$L_S = \frac{3 - \sin \phi}{3 + \sin \phi} \quad (6)$$

2.3 Choice of the suited criterion

Mohr-Coulomb is the oldest and most used criterion. Parameters are easy to deduce from standard tests. Moreover, they are really meaningful for engineers. However, corners make it less straightforward to implement. The circular shape of Drucker-Prager, even it is a drawback from a physical point a view, is really a great advantage from a numerical point view. Plastic strain computations thus go down to literal expressions [6], known as radial return. Radial return is both easy to implement and time saving during computations.

Matsuoka-Nakai seems to best fit material behavior, particularly with regard to true triaxial tests. However, closest point projection method [11], used to compute plastic strain, is more complex to implement. Works are done to improve this method [2, 4] or use alternative computing method [7].

We want to define a method that can measure discrepancies between criteria. One thus can evaluate the difference between a model used in a geotechnical software, and the model fitted in a laboratory. As the results have to be well and immediately understood, we choose to express discrepancies as variations of the friction angle. We also try to estimate if Matsuoka and Nakai were right when they affirmed [10], with the introduction of the Spatially Mobilised Plan, their criterion was corresponding to Mohr-Coulomb the same way Tresca was corresponding to Von Mises.

3 Polar decomposition of the yield surface

3.1 Principe

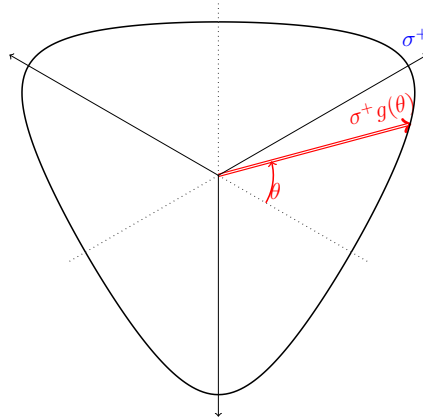


Figure 6: Polar decomposition of the yield surface in the deviatoric plane

For a given mean stress ($\sigma_m = \text{Tr}\underline{\underline{\sigma}}/3$), the yield surface can be reduced to its cross-sectional shape on the deviatoric plane, or π plane. A yield surface can be represented in a unique manner by the mean stress and the deviatoric invariants ($J_2 = \frac{1}{2}\text{Tr}(\underline{\underline{s}}^2)$, $J_3 = \frac{1}{3}\text{Tr}(\underline{\underline{s}}^3)$, with $\underline{\underline{s}} = \underline{\underline{\sigma}} - \sigma_m \underline{\underline{1}}$), but it is more practical to replace the third invariant by the Lode angle, to work in the π plane

$$-\frac{\pi}{6} \leq \theta = \frac{1}{3} \arcsin \left(\frac{-3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^3}} \right) \leq \frac{\pi}{6} \quad (7)$$

The set $(\sqrt{J_2}, \theta)$ defines polar coordinates on one sixth of the deviatoric plane, which is sufficient for an isotropic criterion. Zienkiewicz and Pande [12], introduced what can be called a polar decomposition of the criterion :

$$\sqrt{J_2} = \sigma^+ g_p(\theta) \quad (8)$$

The deviatoric radius : $\sigma^+(\sigma_m) = \sqrt{J_2/\theta=\frac{\pi}{6}}$, gives the yield function in the meridional plane $(\sigma_m, \sqrt{J_2})$, for $\theta = \frac{\pi}{6}$. This value of the Lode angle corresponds to a classical triaxial test, or compression triaxial test ($\sigma_I = \sigma_{II} > \sigma_{III}$). The function $g_p(\theta)$ is the shape function of the yield surface in the deviatoric plane. It is normalized ($g_p(\frac{\pi}{6}) = 1$) and gives directly the value of the extension ratio $g_p(-\frac{\pi}{6}) = L_S$.

3.2 Application to the criteria

Mohr-Coulomb can be rewritten as a function of Lode angle:

$$f(\underline{\sigma}) = \sigma_m \sin \phi + \sqrt{J_2} \left(\cos \theta - \frac{\sin \theta \sin \phi}{\sqrt{3}} \right) \quad (9)$$

We can thus express the deviatoric radius, shape function and extension ratio :

$$\sigma_{MC}^+ = \frac{2\sqrt{3} \sin \phi (H - \sigma_m)}{3 - \sin \phi} \quad (10)$$

$$g_p^{MC}(\theta) = \frac{3 - \sin \phi}{2(\sqrt{3} \cos \theta - \sin \theta \sin \phi)} \quad (11)$$

$$L_S^{MC} = \frac{3 - \sin \phi}{3 + \sin \phi} \quad (12)$$

As Drucker-Prager is circular, it is independant from the Lode angle :

$$\sigma_{DP}^+ = 3\alpha (H - \sigma_m) \quad (13)$$

$$g_p^{DP}(\theta) = 1 \quad (14)$$

$$L_S^{DP} = 1 \quad (15)$$

It is not possible to get immediately an explicit expression of the shape function of Matsuoka-Nakai (this will be done when studying the correpondance with Mohr-Coulomb). We can however rewrite it as e function of the Lode angle:

$$f(\underline{\sigma}) = -J_2 (k_1 - 3) \sigma_m + (k_1 - 9) \sigma_m^3 - \frac{2}{3} \frac{J_2^{\frac{3}{2}} \sin 3\theta}{\sqrt{3}} k_1 \quad (16)$$

We can thus express the deviatoric radius and the extension ratio :

$$\sigma_{MN}^+ = -\frac{\sqrt{3} \sigma_m \left(k_1 + 3 \left(-3 + \sqrt{9 - 10k_1 + k_1^2} \right) \right)}{4k_1} \quad (17)$$

$$L_S^{MN} = \frac{4k_1}{2k_1 + 3\sqrt{2}\sqrt{9 + (k_1 - 3)\sqrt{(k_1 - 9)(k_1 - 1)} - 8k_1 + k_1^2}} \quad (18)$$

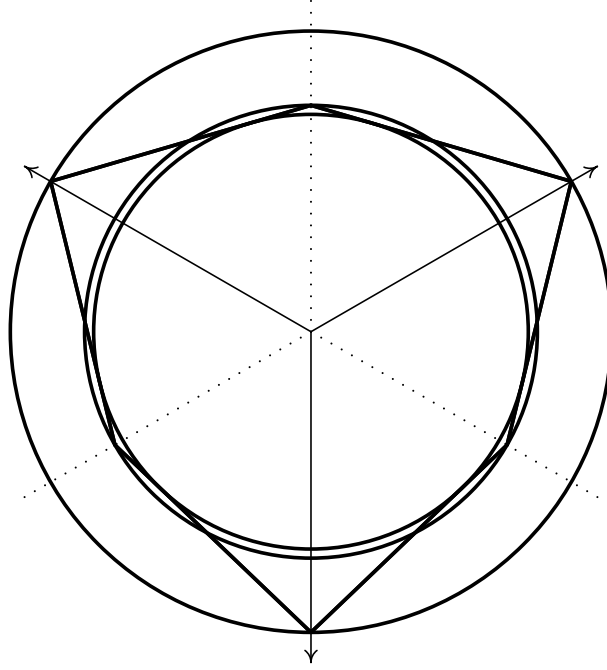


Figure 7: Correspondences between Mohr-Coulomb and Drucker-Prager

3.3 Correspondence between criteria

It is possible to make correspondences between Mohr-Coulomb and the other criteria. The parameters of the latter have to be rewritten as function of Mohr-Coulomb parameters : cohesion c and friction angle ϕ .

3.3.1 Drucker-Prager and Mohr-Coulomb

First step is to use the same cohesive pressure :

$$H = \frac{C}{\tan \phi} \quad (19)$$

Three ways can be used to fit Drucker-Prager and Mohr-Coulomb: the two first are correspondences on the corners (compression or extension), the third is an inscribed Drucker-Prager (meaning a Drucker-Prager yield value always lower or equal than Mohr-Coulomb) :

Compression fit Criteria will be equal in the case of a standard triaxial test and a compressive test, thus the value of the Lode angle :

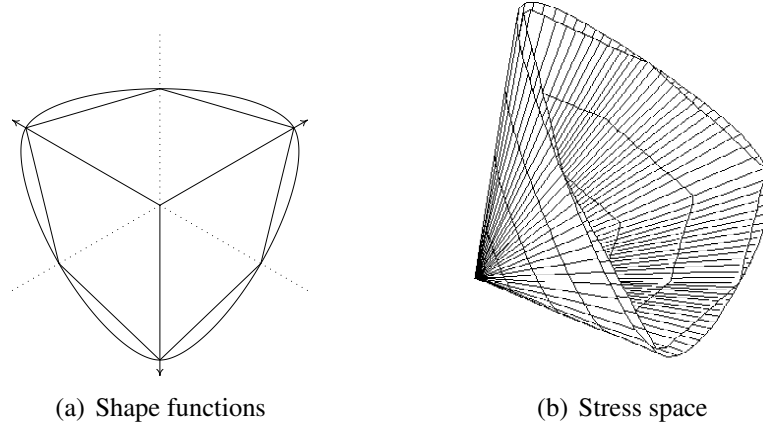
$$\theta_c^c = \frac{\pi}{6} \quad (20)$$

Drucker-Prager coefficient is then :

$$\alpha^c = \frac{2 \sin \phi}{\sqrt{3} (3 - \sin \phi)} \quad (21)$$

Extension fit Criteria will be equal during an inverse triaxial test or during a traction test, the Lode angle of the correspondence is then :

$$\theta_c^e = -\frac{\pi}{6} \quad (22)$$


 Figure 8: Matsuoka-Nakai fitted on Mohr-Coulomb ($\phi = 35$)

Drucker-Prager coefficient is then :

$$\alpha^e = \frac{2 \sin \phi}{\sqrt{3} (3 + \sin \phi)} \quad (23)$$

Inscribed circle fit Drucker-Prager is fitted so that in the deviatoric plane, its circle is inscribed in the Mohr-Coulomb hexagon. The value of the Lode angle of the point of coincidence of criteria depends of the friction angle :

$$\tan \theta_c^i = -\frac{\sin \phi}{\sqrt{3}} \quad (24)$$

Drucker-Prager coefficient is then :

$$\alpha^i = \frac{\sin \phi}{\sqrt{3} \sqrt{3 + \sin^2 \phi}} \quad (25)$$

3.4 Matsuoka-Nakai and Mohr-Coulomb

Matsuoka-Nakai can fit Mohr-Coulomb, on all corners of the hexagon. Thus the criteria are the same on the extension corner ($\theta = -\frac{\pi}{6}$) and on the compression corner ($\theta = \frac{\pi}{6}$). We use then the following parameter :

$$k_1 = \frac{(3 - \sin \phi) (3 + \sin \phi)}{(1 - \sin \phi) (1 + \sin \phi)} \quad (26)$$

A consequence of this coincidence on all the corners, is that the deviatoric radius and the extension ratio are exactly the same as those of Mohr-Coulomb. Moreover Matsuoka-Nakai is then exactly equal to the general yield function of Maiolino [8] fitting Mohr-Coulomb:

$$f(\underline{\sigma}) = \frac{3}{2} \sqrt{3} (1 - L_S) J_3 + (L_S^2 + 1 - L_S) \sigma^+ J_2 - \sigma^{+3} L_S^2 \quad (27)$$

With L_S and σ^+ those of Mohr-Coulomb. From those equation, using polar decomposition properties, we can deduce a third degree equation whose solution is the shape function :

$$g_p^3(\theta) \sin 3\theta (-1 + L_S) + (1 - L_S + L_S^2) g_p^2(\theta) - L_S^2 = 0 \quad (28)$$

Numerically, the shape function of Bigoni and Piccolroaz [1] is solution of this equation, with $\beta = 0$ et $\gamma = \cos \left(3 \arccos \left(\frac{\sqrt{3}}{2\sqrt{1-L_S+L_S^2}} \right) \right)$:

$$g_p(\theta) = \frac{\cos \left(\beta \frac{\pi}{6} - \frac{\arccos(-\gamma)}{3} \right)}{\cos \left(\beta \frac{\pi}{6} - \frac{\arccos(-\gamma \sin 3\theta)}{3} \right)} \quad (29)$$

4 Calculating discrepancies

4.1 Method of measure

We can notice that K the slope of the Mohr-Coulomb criterion in the meridian plane $(\sigma_m, \sqrt{J_2})$ (standard triaxial test conditions) is a direct function of the friction angle :

$$K(\phi) = -\frac{\partial \sigma_m}{\partial \sqrt{J_2}_{\theta=\frac{\pi}{6}}} = \frac{\sigma^+}{(H - \sigma_m)} = \frac{2\sqrt{3} \sin \phi}{3 - \sin \phi} \quad (30)$$

This value can be computed for every point on the Mohr-Coulomb yield surface, using the polar decomposition (8) :

$$K(\phi) = \frac{\sqrt{J_2}}{(H - \sigma_m) g_p^{MC}(\theta)} \quad (31)$$

For every point $(\theta, \sqrt{J_2})$ we can define an instant friction angle ϕ_i (and K_i), that is defined by the Mohr-Coulomb surface this point belongs to (cohesive pressure H , friction angle ϕ_i (associated shape function $g_{p\phi_i}^{MC}(\theta)$)). If this point is located on a yield surface (deviatoric radius σ^+ , shape function $g_p(\theta)$), we can use the polar decomposition to introduce those values in the expression of K :

$$K(\phi_i) = \frac{\sigma^+ g_p(\theta)}{(H - \sigma_m) g_{p\phi_i}^{MC}(\theta)} \quad (32)$$

We can then calculate the value of the instant friction angle :

$$\phi_i(\theta) = \arcsin \frac{\sqrt{3} \sigma^+ g_p(\theta) \cos \theta}{3\sqrt{3}(H - \sigma_m) + \sigma^+ g_p(\theta) \sin \theta} \quad (33)$$

For this given point we can measure the discrepancy between the criterion, and the Mohr-Coulomb criterion (friction angle ϕ) it is fitting :

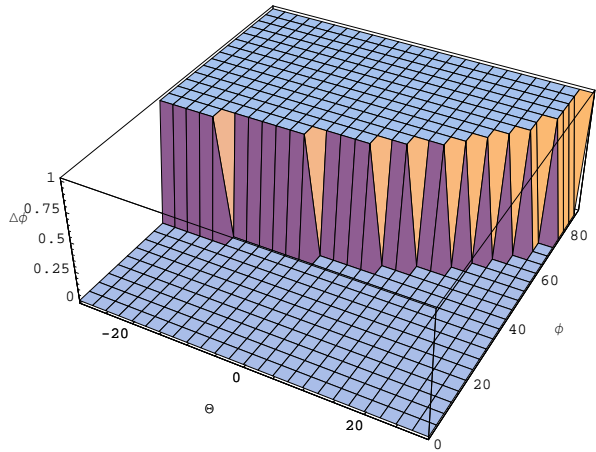
$$\Delta_\phi = \phi_i - \phi \quad (34)$$

We can measure the discrepancies as a variation of the friction angle between the intended friction angle, and the instant friction angle (that is the real friction angle for a given Lode angle). This quantity is a function of the Lode angle and of ϕ .

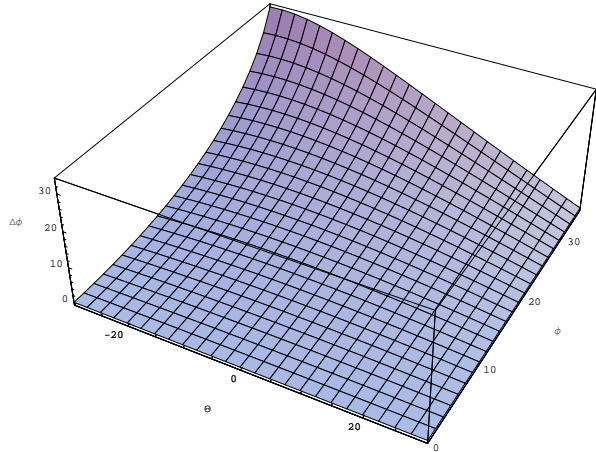
4.2 Drucker-Prager

We give the value of the instant friction angle for a Drucker-Prager:

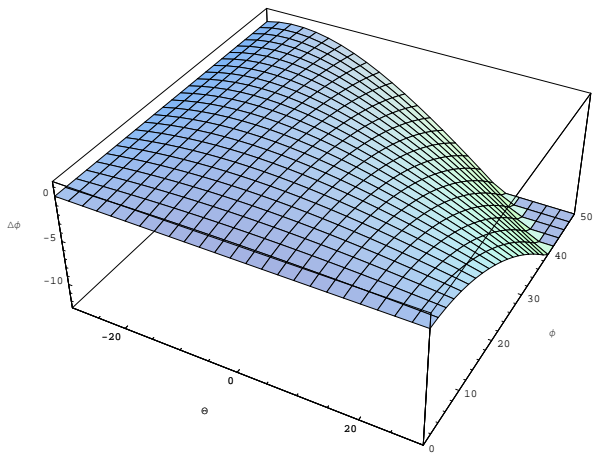
$$\phi_i(\theta) = \arcsin \frac{3\alpha \cos \theta}{1 + \sqrt{3}\alpha \sin \theta} \quad (35)$$



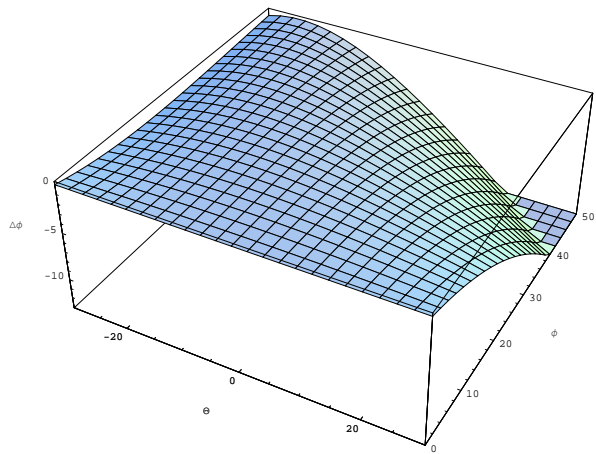
(a) Compression fit : impossible area ($\sin \phi_i > 1$)



(b) Compression fit



(c) Extension fit



(d) Inscribed criterion

Figure 9: Discrepancy of the friction angle for Drucker-Prager: ($\Delta_\phi = \phi_i - \phi$)

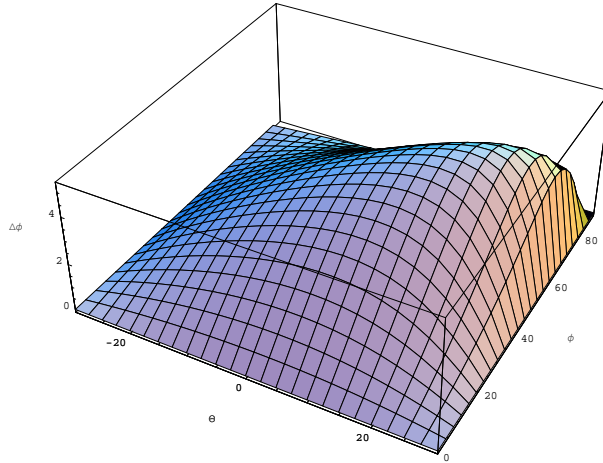


Figure 10: Discrepancies of the friction angle for Matsuoka-Nakai: ($\Delta_\phi = \phi_i - \phi$)

Very important variations of the friction angle can be found when the Lode angle is not the value of fit, as we can see on figure 9 on the previous page. For the compression fit, discrepancies became so important that it cannot be calculated as $\frac{3\alpha \cos \theta}{1 + \sqrt{3}\alpha \sin \theta}$ is greater than one. Even for the incirbed fit, that is considered as better, variations of the friction angle can easily be greater than 10 degrees.

4.3 Matsuoka-Nakai

Matsuoka-Nakai is a good fit of Mohr-Coulomb, because discrepancies of the friction angle are almost always lower than 5 degrees (maximum being equal to 5.3 degrees).

5 Conclusion

While commonly used for numerical reasons, Drucker-Prager induces too great discrepancies of the friction angle. The overestimation of the friction angle made when using Matsuoka-Nakai is very small. We can thus say that it is right to affirm that Matsuoka-Nakai is a smooth Mohr-Coulomb. The same cannot be said about Drucker-Prager, because the friction angle is the core physical notion of the Mohr-Coulomb criterion. We can just say that Drucker-Prager is a circular criterion sensitive to mean stress, maybe a pressure sensitive Von Mises ... Our results are not only theoretical. They can be useful for field engineers also. Discrepancies on Drucker-Prager are so important, that we think it cannot be used without an exhaustive analysis of the Lode angle, to measure and correct the error. Discrepancies on the friction angle induced by Matsuoka-Nakai are so small, that it can be used with confidence, the numerical error can easily be integrated in the boundaries of the parametric study of the friction angle.

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