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# Simplifying normal functors: an old and a new model of $\lambda$ -calculus

Morgan Rogers ✉ 

Sorbonne Paris Nord University, France

Thomas Seiller ✉ 

CNRS, France

William Troiani ✉ 

Melbourne University, Australia

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## Abstract

The introduction of Linear Logic by Jean-Yves Girard takes its origins in the so-called *normal functors* model of lambda-calculus in which untyped  $\lambda$ -terms are interpreted as ‘normal functors’ between presheaf categories. As a result, we produce a new model in ‘normal functors’ between sets of (possibly infinite) multisets, gaining new insights on Girard’s original construction. We then extend this to an explicit model of intuitionistic Linear Logic, contrasting the result with the weighted relational model.

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## 1 Introduction

What is a model of Linear Logic? Loosely speaking, there are at least two possible answers organised as those which take the *models* as central, with the syntax playing a supporting role, and those which take the *syntax* as central, with the models playing a supporting role. So-called *categorical models* fall in the former class: often, constraints are imposed that are justified by mathematical elegance or convenience rather than logical necessity; foremost of these is the requirement that the exponential modality (!) should be a comonad. Investigating the origins of this requirement leads to a paper by Seely [22] in which he writes ‘Notice that these rules seem to imply that we should regard ! as a functor, indeed a cotriple [...]’, an extremely loose justification. The models we will work with in the present paper, which focus on invariants and dynamics of the logic, are of the latter kind sometimes referred to as “models of cut-elimination”. Although present from the very origins of Linear Logic, this more general syntactic perspective has since remained comparatively unexplored.

**Normal Functors and Linear logic.** In the initial development of Linear Logic [9], an important part was played by the ‘normal functors’ model of untyped  $\lambda$ -calculus [10] where terms are interpreted by finite polynomial functors [15]. This model has been studied through the lens of categorical semantics [24, 13], and plays a fundamental role in the current work around 2-categorical models of Linear Logic [2, 5, 3]. In [10] Girard proved his so-called Normal Form Theorem<sup>1</sup>: an equivalence between *normal functors* and *analytic functors*<sup>2</sup>, by way of a *normal form* common to each type of functor. Exploiting this result, he constructed a model of the untyped  $\lambda$ -calculus which can be understood as a *categorification* of Scott

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<sup>1</sup> Which is a variant of normal form theorems on ordinals obtained within the theory of dilators [7].

<sup>2</sup> We stress here that the notion of analytic functor as introduced by Girard differs from that introduced and studied by Joyal [14].



41 domains [19, 20, 21, 6]: instead of interpreting terms as continuous functions between directed  
 42 complete partially ordered sets, he interprets them as functors preserving certain (co)limits  
 43 (normal functors) between categories which possess the corresponding (co)limits. More  
 44 precisely, a term  $t$  (equipped with a valid context  $\underline{x}$ ) is interpreted as a normal functor  
 45  $\llbracket \underline{x} \mid t \rrbracket : (\text{Set}^A)^n \rightarrow \text{Set}^A$ , where  $A$  is a fixed countably infinite set.

46 However, within this model lurks more structure than that which is reflected in the  
 47 syntax of the untyped  $\lambda$ -calculus. A defining property of normal functors is that they  
 48 are determined by their restriction to *integral* functors. Though this holds for all normal  
 49 functors, the stronger condition that a functor is defined by its restriction to the underlying  
 50 sets  $A \times \dots \times A \rightarrow \text{Set}^A$  only holds for a subclass of normal functors which we call *linear*.  
 51 Restricting to the simply typed  $\lambda$ -calculus (à la Church) then extending the syntax to capture  
 52 linear functors leads to the familiar logical system of Linear Logic.

53 How exactly it is that Linear Logic is modelled by normal functors was never written down  
 54 in Girard’s original paper. And it turns out that the origin of Linear Logic is most of the  
 55 time explained, even by Girard himself, by referencing coherence space semantics [10, 8, 9], a  
 56 special case of domains that is obtained as a *qualitative* version of normal functors.

57 **Contributions.** The starting point of this paper was the realisation that what *is* written  
 58 in Girard’s paper is overcomplicated, both in presentation and content. We remedy the  
 59 former by providing a greatly streamlined account of both Girard’s Normal Form Theorem in  
 60 Section 2 and a restructured presentation of Girard’s model in Section 3. As a consequence,  
 61 we find that one need not consider normal functors at all, as the core mathematical ideas at  
 62 work can be understood by considering much simpler *normal functions* instead. This leads  
 63 to a simplification of the normal functors model in Section 4; in contrast to coherence spaces,  
 64 this model remains *quantitative*. At face value, the simplified model bares similarities to the  
 65 weighted relational model [16], and also to the “weighted Scott domains” model [1, Section  
 66 3]. We show in Section 6 that it is distinct from these, so we declare this new (to the best of  
 67 the authors’ knowledge) model, as one of this paper’s main contributions.

68 The other main contribution is more conceptual. A critical point in Girard’s original  
 69 paper is the use of so-called *analytic functors* (see Definition 4) which admit a presentation  
 70 as a kind of generalized power series. This presentation is exploited to define the operations  
 71 that interpret application and abstraction in Girard’s model in spite of the fact that the  
 72 ambient category in which Girard implicitly works is not cartesian closed. The fact that  
 73 normal functors are naturally isomorphic to analytic functors (see Theorem 7) is thus framed  
 74 as crucial to recovering a functorial, presentation-independent version of his model. This  
 75 idea of functors which can be presented via variants of power series has been elaborated  
 76 upon by work generalising Girard’s ideas to the setting of profunctors and generalised species  
 77 [4, 25]. These developments are of interest in their own right, but they may give one the false  
 78 impression that such a presentation is *necessary* for modelling  $\lambda$ -calculus. Our simplification  
 79 of Girard’s model demonstrates that the conditions for interpreting  $\lambda$ -calculus (resp. Linear  
 80 Logic) are in fact much simpler, requiring only that a function interpreting a  $\lambda$ -term (resp.  
 81 proof) is determined by its restrictions to smaller domains in a systematic way.

82 That our simplification moves our understanding of Girard’s model away from power  
 83 series is a deliberate and positive aspect of this current paper. We remark again that neither  
 84 Girard’s model nor our own is a ‘categorical model’ in the usual sense. We stress this  
 85 because it is usually treated as a deficiency of Girard’s original model, whereas we find this  
 86 assessment, and more generally the associated belief that a model of Linear Logic should  
 87 satisfy the axioms of a categorical model, potentially harmful as it could lead to disregarding  
 88 interesting models. Indeed, had the normal functor model been abandoned due to the failure

of cartesian-closedness, we would likely not have Linear Logic, since Girard himself points to this paper as the origin of the formalism in reprints of this work [10]. In this regard, we judge that the most important feature resulting in the emergence of Linear Logic is the presence of *linearity* in his model, in the sense of a subclass of normal functors having a stronger property than being normal. We exhibit the corresponding class of linear functions appearing in our formalism as part of a model of Intuitionistic Linear Logic in Section 5; an interpretation of Linear Logic in Girard's original model can be deduced from this one.

## 2 Girard's Normal Form Theorem

In this section we survey Girard's model, taking the opportunity to state the main results from his paper [10] in more modern terminology. We will show in the next section how this is used to define a model of  $\lambda$ -calculus.

► **Definition 1.** Let  $A, B$  be fixed sets. A functor  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}^B$  is **normal** if it preserves directed colimits and wide pullbacks. More generally, a functor  $\text{Set}^{A_1 \times \dots \times A_n} \rightarrow \text{Set}^B$  is normal if it is so in each argument, or equivalently, if it is normal as a functor  $\text{Set}^{A_1 \sqcup \dots \sqcup A_n} \rightarrow \text{Set}^B$ .

► **Remark 2.** The collection of normal functors is closed under composition, by inspection. For  $b \in B$ , the evaluation functor  $\text{ev}_b : \text{Set}^B \rightarrow \text{Set}$  is a normal functor. As such, given a functor  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}^B$  we write  $\mathcal{F}_b$  for the composite functor  $\text{ev}_b \circ \mathcal{F}$ , which will be normal whenever  $\mathcal{F}$  is.

Normal functors  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}^B$  have the crucial property that the image of any functor  $F \in \text{Set}^A$  under  $\mathcal{F}$  is determined by *finitary data*, even when  $F$  takes values in infinite sets. To illustrate this point, consider the special case  $A = B = \{*\}$ , so  $\mathcal{F}$  is a normal functor  $\text{Set} \rightarrow \text{Set}$ . Given a set  $X$ , let  $\{X_i\}_{i \in I}$  be the collection of its finite subsets. Then  $X$  can be written as the direct colimit  $\text{colim}_{i \in I} X_i$ . Since  $\mathcal{F}$  preserves filtered colimits and wide pullbacks, we have the following:

$$\mathcal{F}(X) = \mathcal{F}\left(\text{colim}_{i \in I} X_i\right) = \text{colim}_{i \in I} \mathcal{F}(X_i).$$

We can think of the collection  $\{\mathcal{F}(X_i)\}_{i \in I}$  as the collection of approximations of  $\mathcal{F}(X)$  determined by *finite data* from  $X$ . In particular, it follows that  $\mathcal{F}$  is entirely determined by its values on finite sets, and even by its restriction to the Von Neumann integers.

Since  $\mathcal{F}$  preserves pullbacks it preserves monomorphisms, so the colimit is a directed union. Moreover, if  $y \in \mathcal{F}(X)$  and  $X_i, X_j \subseteq X$  are such that  $y \in \mathcal{F}(X_i)$  and  $y \in \mathcal{F}(X_j)$  then,

$$y \in \mathcal{F}(X_i) \cap \mathcal{F}(X_j) = \mathcal{F}(X_i \cap X_j) \tag{1}$$

as  $\mathcal{F}$  preserves pullbacks. This implies that there exists a *minimal finite subset*  $X_k \subseteq X$ , depending on  $y$ , from which  $y$  emerges upon applying  $\mathcal{F}$ . Note that we only needed finite pullbacks here because  $A$  was a singleton, but wide pullbacks are needed as soon as  $A$  is infinite.

The theory presented in the remainder of this section can be thought of as a generalisation of the phenomena just observed. First, we must identify the analogue of finite sets.

► **Definition 3.** Let  $X \in \text{Set}$  be a set and  $F \in \text{Set}^A$  a functor. We introduce the terminology:

- $X$  is an **integer** if it is a Von Neumann integer ( $0 := \emptyset, 1 := \{0\}, \dots, n := \{0, \dots, n-1\}, \dots$ ).
- $F$  is **finite** if for all  $a \in A$  the set  $F(a)$  is finite, and all but finitely many of the  $F(a)$  are equal to  $\emptyset$ .

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132 ■  $F$  is **integral** if it is finite and for all  $a \in A$  the set  $F(a)$  is an integer.  
133 For an arbitrary set  $A$  we denote by  $\text{Int}(A)$  the set of integral functors in  $\text{Set}^A$ .

134 The main reason that we need to restrict to integral functors rather than finite functors  
135 is to provide a set of representatives to serve as indices in the following definition.

136 ► **Definition 4.** A functor  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}^B$  is **analytic** if there exists a family of functors  
137  $\{C_G\}_{G \in \text{Int}(A)}$  in  $\text{Set}^B$  such that for all objects  $F \in \text{Set}^A$  and all morphisms  $\mu : F \rightarrow F'$ :

$$138 \quad \mathcal{F}(F) = \coprod_{G \in \text{Int}(A)} (C_G \times \text{Set}^A(G, F)) \quad \mathcal{F}(\mu) = \coprod_{G \in \text{Int}(A)} (C_G \times \text{Set}^A(G, \mu)).$$

139 Girard presented the formulas in the definition of analytic functors as a kind of power  
140 series, whence the choice of name. To compare normal functors and analytic functors, we  
141 consider ‘normal forms’.

142 ► **Definition 5.** Let  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}^B$  and  $b \in B$ . Let  $\text{El}(\mathcal{F}_b)$  denote the category of elements  
143 of  $\mathcal{F}_b$  (cf. Remark 2) and  $(F, x)$  an object of this category, so  $F \in \text{Set}^A$  and  $x \in \mathcal{F}(F)(b)$ . A  
144 **form** of  $\mathcal{F}$  with respect to  $(F, x)$  is an object of the slice category  $\text{El}(\mathcal{F}_b)/(F, x)$ . Given a  
145 form  $\eta : (G, y) \rightarrow (F, x)$ , we say:

- 146 ■  $\eta$  is **finite** if  $G$  is finite.
- 147 ■  $\eta$  is **integral** if  $G$  is integral.
- 148 ■  $\eta$  is **normal** if it is an initial object in  $\text{El}(\mathcal{F}_b)/(F, x)$ .

149 With these notions established, we can introduce a third property of functors which  
150 mediates between normal and analytic functors.

151 ► **Definition 6.** A functor  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}^B$  is said to satisfy the **finite normal form**  
152 **property** if for every  $b \in B$  and object  $(F, x)$  in  $\text{El}(\mathcal{F}_b)$  there exists a finite normal form  
153  $\eta : (G, y) \rightarrow (F, x)$ . The functor  $\mathcal{F}$  is said to satisfy the **integral normal form property**  
154 if in the above the form  $\eta$  can be taken to be integral.

155 Girard’s main theorem states that the three properties of functor are equivalent. Notice  
156 that the statement from the original article contains a minor error, the correct statement is  
157 as follows.

158 ► **Theorem 7.** Let  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}^B$  be a functor. The following are equivalent:

- 159 1.  $\mathcal{F}$  is normal.
- 160 2.  $\mathcal{F}$  satisfies the finite normal form property.
- 161 3.  $\mathcal{F}$  is isomorphic to an analytic functor.

162 The proof of Theorem 7 is difficult and requires all of the definitions introduced so far. For  
163 instance, when Girard proves  $(2 \Rightarrow 1)$  he relies on an explicit translation through integral  
164 functors. A full exposition of the proof of Theorem 7 is given in Appendix A.

### 165 **3** $\lambda$ -terms as normal functors

166 In this section, we present Girard’s model of the untyped  $\lambda$ -calculus using normal functors.  
167 In his paper, Girard was concerned with the *minimal data property* that we sketched at  
168 the start of Section 2; this corresponds to the hypothesis that normal functors preserve  
169 wide pullbacks. Accordingly, if one is willing to drop this property then the preservation  
170 of pullbacks may be dropped too! While preservation of pullbacks was important for the  
171 correspondence in Theorem 7, we will not require it at any point in the following results;

172 in other words, it is not essential for modelling  $\lambda$ -calculus. In Section 4 we will present our  
 173 simplification of Girard’s model consisting of normal *functions*, where we do not impose the  
 174 analogue of this condition.

175 In the following, we use the notation  $\text{Norm}(\mathcal{C}, \mathcal{D})$  to denote the collection of normal  
 176 functors from a category  $\mathcal{C}$  to a category  $\mathcal{D}$ . The properties of normal functors enable us to  
 177 “curry” and “uncurry” functions of several variables.

178 ► **Lemma 8.** *For sets  $A, B, C$  there exists a pair of functions  $(-)^+, (-)^-$ :*

$$179 \quad \text{Norm}(\text{Set}^A \times \text{Set}^B, \text{Set}^C) \xrightleftharpoons[(-)^-]{(-)^+} \text{Norm}(\text{Set}^A, \text{Set}^{\text{Int}(B) \times C}) \quad (2)$$

180 such that the composite  $((-)^+)^-$  is the identity.

181 **Proof.** Let  $\mathcal{F} : \text{Set}^A \times \text{Set}^B \rightarrow \text{Set}^C$  be normal. By Theorem 7,  $\mathcal{F}$  is analytic (up to natural  
 182 isomorphism), so that for  $F \in \text{Set}^A$  and  $F' \in \text{Set}^B$  we have isomorphisms,

$$183 \quad \mathcal{F}(F, F') \cong \coprod C_{G, G'} \times \text{Set}^A(G, F) \times \text{Set}^B(G', F'), \quad (3)$$

184 where the coproduct is taken over all  $(G, G') \in \text{Int}(A) \times \text{Int}(B)$ , for some family of functors  
 185  $\{C_{G, G'} \in \text{Set}^C\}_{(G, G') \in \text{Int}(A) \times \text{Int}(B)}$ . For  $(G', c) \in \text{Int}(B) \times C$ , we can thus define:

$$186 \quad \mathcal{F}^+(F)(G', c) := \coprod_{G \in \text{Int}(A)} C_{G, G'}(c) \times \text{Set}^A(G, F). \quad (4)$$

187 Conversely, given a normal functor  $\mathcal{G} : \text{Set}^A \rightarrow \text{Set}^{\text{Int}(B) \times C}$  we define for  $(F, F') \in$   
 188  $\text{Set}^A \times \text{Set}^B$  and  $c \in C$ :

$$189 \quad \mathcal{G}^-(F, F')(c) := \coprod_{G' \in \text{Int}(B)} \mathcal{G}(F)(G', c) \times \text{Set}^B(G', F'). \quad (5)$$

190 The fact that  $((-)^+)^-$  is the identity follows easily. ◀

191 ► **Remark 9.** We can express the operations employed by Girard in terms of  $(-)^-$  and  $(-)^+$ .  
 192 For example, Girard’s **App functor**, [10, Definition 2.20], is defined as follows:

$$193 \quad \text{App} : \text{Set}^{\text{Int}(A) \times B} \times \text{Set}^A \rightarrow \text{Set}^B$$

$$194 \quad (H, F) \mapsto \coprod_{G \in \text{Int}(A)} H(G, -) \times \text{Set}^A(G, F),$$

195 but it can be expressed more succinctly as  $(\text{id}_{\text{Set}^{\text{Int}(A) \times B}})^-$ . Girard presents the fact that App is  
 196 a normal functor as a result [10, Theorem 2.21], but in this form it follows immediately from  
 197 the fact that  $(-)^-$  takes normal functors to normal functors.

198 The model of the untyped (or more precisely, untyped)  $\lambda$ -calculus will interpret terms as  
 199 normal functors between powers of  $\text{Set}^A$  for a suitable choice of  $A$ . As such, let us fix a choice  
 200 of infinite set  $A$  and a bijection  $q : \text{Int}(A) \times A \rightarrow A$ . This bijection induces an equivalence of  
 201 categories  $\bar{q} : \text{Set}^A \rightarrow \text{Set}^{\text{Int}(A) \times A}$ .

202 ► **Remark 10.** In the original presentation, [10, Proposition 3.1],  $A$  is taken to be the initial  
 203 algebra for the endofunctor  $X \mapsto \text{Int}(X) \times X$  on  $\text{Set}$ , which Girard constructs inductively  
 204 and denotes by  $\mathcal{A}_\infty$ . This is countably infinite and equipped with a canonical bijection  
 205  $\text{Int}(\mathcal{A}_\infty) \times \mathcal{A}_\infty \rightarrow \mathcal{A}_\infty$ . Since the universal property of this algebra is not used anywhere, we  
 206 allow for greater flexibility here.

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207 To be more specific, the model we will interpret  $\lambda$ -terms *in context*, so that terms of  
208 several variables are interpreted as functors of an appropriate arity.

209 ► **Definition 11.** A *context* is a sequence of variables  $\underline{x} = \{x_1, \dots, x_n\}$ . A context  $\underline{x}$  is *valid*  
210 for a  $\lambda$ -term  $t$  if the set of free variables of  $t$  is a subset of  $\underline{x}$ .

211 ► **Definition 12.** Let  $t$  be a term and  $\underline{x} = \{x_1, \dots, x_n\}$  a valid context for  $t$ . We define the  
212 functor  $\llbracket \underline{x} \mid t \rrbracket : (\text{Set}^A)^n \rightarrow \text{Set}^A$  by induction on the structure of  $t$ :

- 213 ■ when  $t = x_i$  is a variable,  $\llbracket \underline{x} \mid x_i \rrbracket := \pi_i$ ;
- 214 ■ when  $t = (t_1)t_2$  is an application,  $\llbracket \underline{x} \mid (t_1)t_2 \rrbracket := (\bar{q} \circ \llbracket \underline{x} \mid t_1 \rrbracket)^- \circ (\text{id}_{(\text{Set}^A)^n}, \llbracket \underline{x} \mid t_2 \rrbracket)$ ;
- 215 ■ when  $t = \lambda y.t'$  is an abstraction,  $\llbracket \underline{x} \mid \lambda y.t' \rrbracket := \bar{q}^{-1} \circ (\llbracket \underline{x}, y \mid t' \rrbracket)^+$ .

216 ► **Remark 13.** Although the notation and presentation differs significantly, this definition is  
217 identical in content to [10, The model  $\mathcal{A}_\infty$ ]. In particular, we eliminate the App functor by  
218 observing that for  $H : (\text{Set}^A)^n \rightarrow \text{Set}^{\text{Int}(A) \times A}$  and  $J : (\text{Set}^A)^n \rightarrow \text{Set}^A$  we have:

$$219 \quad \text{App} \circ \langle H, J \rangle = H^- \circ (\text{id}_{(\text{Set}^A)^n}, J). \quad (6)$$

220 Our presentation makes it transparent, by the cancellation of  $\bar{q}$  and  $\bar{q}^{-1}$  and of  $((-)^+)^-$ ,  
221 that the application of an abstracted term corresponds to substitution into the last argument  
222 (see the proof of Theorem 25). Definition 12 therefore gives a model of untyped  $\lambda$ -calculus.  
223 It is moreover a *denotational model*.

224 ► **Definition 14.** Given  $\lambda$ -terms  $t, s$  and a variable  $x$ , we write  $t[x := s]$  for the term given  
225 by *substituting*  $s$  for  $x$  in  $t$ . More formally, we inductively define:

- 226 ■ if  $t$  is a variable then either  $t = x$  in which case  $t[x := s] = s$ , or  $t \neq x$  in which case  
227  $t[x := s] = t$ .
- 228 ■ if  $t = t_1 t_2$  then  $t[x := s] = t_1[x := s] t_2[x := s]$ .
- 229 ■ if  $t = \lambda y.t'$  we may assume by  $\alpha$ -equivalence that  $y \neq x$  and that  $y$  does not occur in  $s$  and  
230 set  $t[x := s] = \lambda y.t'[x := s]$ .

231 Recall that a model of untyped  $\lambda$ -calculus is said to be a **denotational model** if for all  
232  $\lambda$ -terms  $s, t$  in a context  $\underline{x}$ , we have:

$$233 \quad \llbracket \underline{x} \mid (\lambda y.t)s \rrbracket = \llbracket \underline{x} \mid t[y := s] \rrbracket. \quad (7)$$

234 ► **Theorem 15.** Definition 12 yields a denotational model of the  $\lambda$ -calculus.

235 An essential intermediate result in proving Theorem 15 is the Substitution Lemma, which  
236 witnesses the fact that substitution works as expected in terms of the interpreting functors.

237 ► **Lemma 16 (Substitution Lemma).** Let  $t, s$  be  $\lambda$ -terms,  $\underline{x} = \{x_1, \dots, x_n\}$  a collection of  
238 variables and  $y$  a further variable such that  $\underline{x} \cup \{y\}$  is a valid context for  $t$  and  $\underline{x}$  is a valid  
239 context for  $s$ . Then for any  $\underline{F} \in (\text{Set}^A)^n$  we have

$$240 \quad \llbracket \underline{x} \mid t[y := s] \rrbracket(\underline{F}) = \llbracket \underline{x}, y \mid t \rrbracket(\underline{F}, \llbracket \underline{x} \mid s \rrbracket(\underline{F})). \quad (8)$$

241 The proofs of these results are identical to the proofs of the analogous results in the  
242 simpler model, Lemma 24 and Theorem 25, below.

### 243 **4** $\lambda$ -terms as normal functions

244 There are several dissatisfying aspects of the model given in the previous sections. Firstly, from  
245 the perspective of category theory it is unnatural to have to choose particular representatives



246 of finite sets (the Von Neumann integers). Moreover, requiring that  $A$  is a set renders  
 247 naturality of transformations between functors vacuous, as *any* collection of  $A$ -indexed  
 248 functions is natural. Lastly, the preservation of wide pullbacks plays no technical role in the  
 249 model *per se*: it instead provides a feature of the model which may or may not be desired.

250 We eliminate these aspects in the present section by constructing a simpler model. We  
 251 dispose of the preservation of wide pullbacks entirely, as well as the superfluous categorical  
 252 structure. In the sequel to this paper [18] we will show that both Girard's model (minus  
 253 wide pullback preservation) and our simplified model are special cases of a family of models.

254 ► **Notation 1.** For a set  $A$ , we denote by  $\mathcal{Q}(A)$  the set of functions  $\underline{a} : A \rightarrow \mathbb{N} \cup \{\infty\}$  and by  
 255  $\mathcal{I}(A)$  the subset consisting of those  $\underline{a}$  such that  $\sum_{a \in A} \underline{a}(a) < \infty$  (that is, those for which all  
 256 values are finite and all but finitely many are 0). The set  $\mathcal{Q}(A)$  admits a partial order  $\leq$   
 257 given by  $\underline{a}_1 \leq \underline{a}_2$  if and only if  $\forall a \in A, \underline{a}_1(a) \leq \underline{a}_2(a)$ .

258 ► **Definition 17.** We say a function  $f : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B)$  is **normal** if it is order-preserving  
 259 and preserves suprema of filtered sets. That is, if  $\{\underline{a}_i\}_{i \in I}$  is a filtered set of elements in  $\mathcal{Q}(A)$ ,  
 260 then  $f(\sup_{i \in I} \{\underline{a}_i\}) = \sup_{i \in I} \{f(\underline{a}_i)\}$ .

261 Observe that  $\mathcal{Q}(A) \times \mathcal{Q}(A') \cong \mathcal{Q}(A \sqcup A')$  and this bijection induces a natural ordering on the  
 262 left-hand side, so we can extend Definition 17 to functions of several variables as we did in  
 263 Definition 1.

264 Since we do not impose the analogue of wide pullback preservation, we do not have a  
 265 presentation of normal functions resembling power series and therefore do not have a direct  
 266 analogue of analytic functors. Nonetheless, we can still break our functions down into finite  
 267 parts in a natural way to obtain a result comparable to Theorem 7 in this simplified context.

268 ► **Theorem 18.** Let  $f : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B)$  be order preserving. Then  $f$  is normal if and only if  
 269 for any pair  $(\underline{a}, b) \in \mathcal{Q}(A) \times B$  we have

$$270 \quad f(\underline{a})(b) = \sup_{\underline{u} \in \mathcal{I}(A)} f(\underline{u})(b) \tau_{\underline{u} \leq \underline{a}} \quad (9)$$

271 where  $\tau_{\underline{u} \leq \underline{a}}$  is equal to 1 if and only if  $\underline{u} \leq \underline{a}$  and is equal to 0 otherwise.

272 **Proof.** Suppose  $f$  is normal and let  $(\underline{a}, b) \in \mathcal{Q}(A) \times B$ . Consider the set  $\mathcal{X}_{\underline{a}} := \{\underline{u} \in \mathcal{I}(A) \mid$   
 273  $\underline{u} \leq \underline{a}\}$ . Then  $\mathcal{X}_{\underline{a}}$  is filtered with respect to the ordering on  $\mathcal{I}(A)$  and  $\sup \mathcal{X}_{\underline{a}} = \underline{a}$ . Since  $f$  is  
 274 normal, we thus have

$$275 \quad f(\underline{a})(b) = f(\sup_{\underline{u} \in \mathcal{X}_{\underline{a}}} \underline{u})(b) = \sup_{\underline{u} \in \mathcal{X}_{\underline{a}}} f(\underline{u})(b) = \sup_{\underline{u} \in \mathcal{I}(A)} f(\underline{u})(b) \tau_{\underline{u} \leq \underline{a}}. \quad (10)$$

276 On the other hand, suppose (9) holds. Let  $\{\underline{a}_i\}_{i \in I}$  be a filtered set. Then for any  $b \in B$  we  
 277 have

$$278 \quad f(\sup_{i \in I} \{\underline{a}_i\})(b) = \sup_{\underline{u} \in \mathcal{I}(A)} \{f(\underline{u})(b) \tau_{\underline{u} \leq \sup_{i \in I} \{\underline{a}_i\}}\} \quad (11)$$

279 Also,

$$280 \quad \sup_{i \in I} \{f(\underline{a}_i)(b)\} = \sup_{i \in I} \left\{ \sup_{\underline{u} \in \mathcal{I}(A)} \{f(\underline{u})(b) \tau_{\underline{u} \leq \underline{a}_i}\} \right\} \quad (12)$$

281 One can verify that the right-hand sides of (11) and (12) are equal by a circle of inequalities,  
 282 exploiting the fact that  $\underline{a} \leq \underline{a}'$  implies  $\tau_{\underline{u} \leq \underline{a}} \leq \tau_{\underline{u} \leq \underline{a}'}$  for all  $\underline{u}$ . ◀

283 As in Section 3, we can “curry” a normal function  $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(C)$  to a function  
 284  $f^+ : \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{I}(B) \times C)$  and dually “uncurry” functions.



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285 ► **Definition 19.** Let  $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(C)$  be arbitrary. We can define a function  
 286  $f^+ : \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{I}(B) \times C)$  as follows.

$$287 \quad f^+(\underline{a})(\underline{u}, c) = f(\underline{a}, \underline{u})(c) \quad (13)$$

288 Conversely, given arbitrary  $g : \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{I}(B) \times C)$  we define  $g^- : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(C)$   
 289 as:

$$290 \quad g^-(\underline{a}, \underline{b})(c) := \sup_{\underline{u} \in \mathcal{I}(B)} g(\underline{a})(\underline{u}, c) \tau_{\underline{u} \leq \underline{b}} \quad (14)$$

291 We note that  $f^+$  is normal when  $f$  is and  $g^-$  is normal when  $g$  is by Theorem 18.

292 The analogue of Lemma 8, that currying then uncurrying yields the identity, is the  
 293 following proposition. We also consider of the effect of uncurrying followed by currying.

294 ► **Proposition 20.** Given  $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(C)$  and  $g : \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{I}(B) \times C)$  which are  
 295 normal, we have  $(f^+)^- = f$  and  $(g^-)^+ \geq g$ .

296 **Proof.** Let  $(\underline{a}, \underline{b}) \in \mathcal{Q}(A) \times \mathcal{Q}(B), c \in C$ . We have:

$$297 \quad (f^+)^-(\underline{a}, \underline{b})(c) = \sup_{\underline{u} \in \mathcal{I}(B)} f^+(\underline{a})(\underline{u}, c) \tau_{\underline{u} \leq \underline{b}} = \sup_{\underline{u} \in \mathcal{I}(B)} f(\underline{a}, \underline{u})(c) \tau_{\underline{u} \leq \underline{b}} = f(\underline{a}, \underline{b})(c).$$

298 On the other hand, for  $\underline{a}, c$  as above and  $\underline{u} \in \mathcal{I}(B)$ ,

$$299 \quad (g^-)^+(\underline{a})(\underline{u}, c) = g^-(\underline{a}, \underline{u})(c) = \sup_{\underline{u}' \in \mathcal{I}(B)} g(\underline{a})(\underline{u}', c) \tau_{\underline{u}' \leq \underline{u}} \geq g(\underline{a})(\underline{u}, c). \quad \blacktriangleleft$$

300 Now fix an infinite set  $A$  and a choice of bijection  $q : \mathcal{I}(A) \times A \rightarrow A$ . There is an induced  
 301 bijection  $\bar{q} : \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{I}(A) \times A)$ .

302 ► **Definition 21.** Let  $\underline{x} = \{x_1, \dots, x_n\}$  be a set of variables and let  $t$  be a  $\lambda$ -term for which  $\underline{x}$   
 303 is a valid context (Definition 11). We associate to each such pair  $(\underline{x}, t)$  a normal function  
 304  $\llbracket \underline{x} \mid t \rrbracket : \mathcal{Q}(A)^n \rightarrow \mathcal{Q}(A)$  inductively on the structure of  $t$ :

- 305 ■ when  $t = x_i$  is a variable,  $\llbracket \underline{x} \mid x_i \rrbracket := \pi_i$ ;
- 306 ■ when  $t = (t_1)t_2$  is an application,  $\llbracket \underline{x} \mid (t_1)t_2 \rrbracket := (\bar{q} \circ \llbracket \underline{x} \mid t_1 \rrbracket)^- \circ (\text{id}_{(\text{Set } A)^n}, \llbracket \underline{x} \mid t_2 \rrbracket)$ ;
- 307 ■ when  $t = \lambda y. t'$  is an abstraction,  $\llbracket \underline{x} \mid \lambda y. t' \rrbracket := \bar{q}^{-1} \circ (\llbracket \underline{x}, y \mid t' \rrbracket)^+$ .

308 ► **Remark 22.** The resemblance between Definition 12 and Definition 21 is evident; the latter  
 309 is obtained from the former by a substitution of simpler ingredients. This demonstrates that  
 310 our model is capturing the conceptual essence of Girard's.

311 ► **Example 23** (Church numeral  $\underline{2}$  in  $\lambda$ -calculus). Consider the term  $(f)(f)x$  in the context  
 312  $(f, x)$ . Its interpretation in our model is as follows after simplifying:

$$313 \quad \llbracket f, x \mid (f)(f)x \rrbracket : \mathcal{Q}(A) \times \mathcal{Q}(A) \rightarrow \mathcal{Q}(A)$$

$$314 \quad (\underline{a}_1, \underline{a}_2) \mapsto \bar{q}^-(\underline{a}_1, \bar{q}^-(\underline{a}_1, \underline{a}_2)).$$

315 The interpretation of the Church numeral  $\underline{2} := \lambda f \lambda x. (f)(f)x$  is obtained by applying  $(-)^+$   
 316 and  $\bar{q}^{-1}$  (twice) but the essence of the interpretation is captured by the above. Beware that  
 317  $\bar{q}^-$  is distinct from  $\bar{q}^{-1}$ !

318 In our model, application is interpreted by introducing a new summand in the domain  
 319 (via  $\bar{q}^-$ ) and then substituting the interpretation of the second term into this new summand.  
 320 So, in the above, we think of the interpretation of  $(f)x$  as the substitution of  $\underline{a}_2$  into the  
 321 new argument of  $\underline{a}_1$  introduced by  $\bar{q}^-$ . Then for  $(f)(f)x$ , this intermediate term  $\bar{q}^-(\underline{a}_1, \underline{a}_2)$   
 322 is substituted into the new argument of  $\underline{a}_1$  introduced by the outermost  $\bar{q}^-$ .

323 Our simplified model remains a denotational model (Definition 14); the proof relies on an  
324 analogue of Lemma 16.

325 ► **Lemma 24** (Substitution Lemma). *Let  $t, s$  be  $\lambda$ -terms and  $\underline{x} = \{x_1, \dots, x_n\}$  be a collection  
326 of variables and  $y$  another variable so that  $\underline{x} \cup \{y\}$  is a valid context for  $t$  and  $\underline{x}$  is a valid  
327 context for  $s$ . Then for any  $\alpha \in \mathcal{Q}(A)^n$  we have*

$$328 \quad \llbracket \underline{x} \mid t[y := s] \rrbracket(\alpha) = \llbracket \underline{x}, y \mid t \rrbracket(\alpha, \llbracket \underline{x} \mid s \rrbracket(\alpha)) \quad (15)$$

329 **Proof.** We proceed by induction on the structure of the term  $t$ . The base case where  $t$  is a  
330 variable is trivial.

331 **Say  $t = (t_1)t_2$  is an application.** First, for  $(\alpha, \underline{a}) \in \mathcal{Q}(A)^n \times \mathcal{Q}(A)$ , we have the following.  
332 Note that we suppress the contexts to ease notation.

$$333 \quad \llbracket (t_1)t_2 \rrbracket(\alpha, \underline{a}) = (\bar{q} \llbracket t_1 \rrbracket)^-((\alpha, \underline{a}), \llbracket t_2 \rrbracket(\alpha, \underline{a})) \quad (16)$$

334 On the other hand,

$$335 \quad \begin{aligned} \llbracket (t_1[y := s])(t_2[y := s]) \rrbracket(\alpha) &= (\bar{q} \llbracket t_1[y := s] \rrbracket)^-(\alpha, \llbracket t_2[y := s] \rrbracket(\alpha)) \\ 336 &= (\bar{q} \llbracket t_1 \rrbracket)^-((\alpha, \llbracket s \rrbracket(\alpha)), \llbracket t_2 \rrbracket(\alpha, \llbracket s \rrbracket(\alpha))) \end{aligned}$$

337 where in the final line we have used the inductive hypothesis.

338 **Say  $t = \lambda y'.t'$  is an abstraction.** We have, for  $(\alpha, \underline{a}) \in \mathcal{Q}(A)^n \times \mathcal{Q}(A)$ :

$$339 \quad \llbracket \underline{x}, y \mid \lambda y'.t' \rrbracket(\alpha, \underline{a}) = \bar{q}^{-1} \llbracket \underline{x}, y, y' \mid t' \rrbracket^+(\alpha, \underline{a}) \quad (17)$$

340 On the other hand, we have for  $\alpha \in \mathcal{Q}(A)^n$  and  $c \in A$  the following (assume  $q^{-1}(c) = (\underline{c}', c'')$ ).

$$341 \quad \begin{aligned} \llbracket \underline{x}, y \mid \lambda y'.t'[y := s] \rrbracket(\alpha)(c) &= (\bar{q}^{-1} \llbracket \underline{x}, y, y' \mid t'[y := s] \rrbracket^+)(\alpha)(c) \\ 342 &= \llbracket \underline{x}, y, y' \mid t'[y := s] \rrbracket^+(\alpha)(\underline{c}', c'') \\ 343 &= \sup_{u \in \mathcal{I}(A)^n} \llbracket \underline{x}, y, y' \mid t'[y := s] \rrbracket(u, \underline{c}')(c'') \tau_{u \leq \alpha} \\ 344 &= \sup_{u \in \mathcal{I}(A)^n} \llbracket \underline{x}, y, y' \mid t' \rrbracket(u, \llbracket \underline{x} \mid s \rrbracket(u), \underline{c}')(c'') \tau_{u \leq \alpha} \\ 345 &= \llbracket \underline{x}, y, y' \mid t' \rrbracket^+(\alpha, \llbracket \underline{x} \mid s \rrbracket(\alpha), \underline{c}')(c'') \\ 346 &= \bar{q}^{-1} \llbracket \underline{x}, y, y' \mid t' \rrbracket^+(\alpha, \llbracket \underline{x} \mid s \rrbracket)(c) \end{aligned}$$

347 where we have used the inductive hypothesis in the fourth line. ◀

348 ► **Theorem 25.** *Definition 21 gives a denotational model of the  $\lambda$ -calculus.*

349 **Proof.** By the Substitution Lemma we have for  $\alpha \in \mathcal{Q}(A)^n$ :

$$350 \quad \llbracket \underline{x} \mid t[y := s] \rrbracket(\alpha) = \llbracket \underline{x}, y \mid t \rrbracket(\alpha, \llbracket \underline{x} \mid s \rrbracket(\alpha)) \quad (18)$$

351 On the other hand, we have

$$352 \quad \begin{aligned} \llbracket \underline{x} \mid (\lambda y.t)s \rrbracket(\alpha) &= (\bar{q} \bar{q}^{-1} \llbracket \underline{x}, y \mid t \rrbracket^+)^-(\text{id}, \llbracket \underline{x} \mid s \rrbracket)(\alpha) \\ 353 &= \llbracket \underline{x}, y \mid t \rrbracket(\alpha, \llbracket \underline{x} \mid s \rrbracket(\alpha)) \end{aligned}$$

354 which concludes the proof. ◀

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$$\begin{array}{cccc}
\frac{}{X \vdash X} \text{(ax)} & \frac{\begin{array}{c} \vdots \pi_1 \\ \Gamma \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ \Delta, A, \Delta' \vdash B \end{array}}{\Gamma, \Delta, \Delta' \vdash B} \text{(cut)} & \frac{\begin{array}{c} \vdots \pi' \\ \Gamma, A, B, \Delta \vdash C \end{array}}{\Gamma, B, A, \Delta \vdash C} \text{(ex)} \\
\text{(a) Axiom} & \text{(b) Cut} & \text{(c) Exchange} \\
\\
\frac{\begin{array}{c} \vdots \pi' \\ \Gamma, A, B, \Delta \vdash C \end{array}}{\Gamma, A \otimes B, \Delta, \vdash C} \text{(L}\otimes\text{)} & \frac{\begin{array}{c} \vdots \pi_1 \\ \Gamma \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ \Delta \vdash B \end{array}}{\Gamma, \Delta \vdash A \otimes B} \text{(R}\otimes\text{)} & \frac{\begin{array}{c} \vdots \pi' \\ \Gamma, A, \Gamma' \vdash \Delta \end{array}}{\Gamma, !A, \Gamma' \vdash \Delta} \text{(der)} & \frac{\begin{array}{c} \vdots \pi' \\ !\Gamma \vdash A \end{array}}{!\Gamma \vdash !A} \text{(prom)} \\
\text{(d) Left tensor} & \text{(e) Right tensor} & \text{(f) Dereliction} & \text{(g) Promotion} \\
\\
\frac{\begin{array}{c} \vdots \pi' \\ \Gamma, A, \Delta \vdash B \end{array}}{\Gamma, \Delta \vdash A \multimap B} \text{(R}\multimap\text{)} & \frac{\begin{array}{c} \vdots \pi_1 \\ \Gamma \vdash A \end{array} \quad \begin{array}{c} \vdots \pi_2 \\ B, \Delta \vdash C \end{array}}{\Gamma, A \multimap B, \Delta \vdash C} \text{(L}\multimap\text{)} & \frac{\begin{array}{c} \vdots \pi' \\ \Gamma, !A, !A \vdash B \end{array}}{\Gamma, !A \vdash B} \text{(ctr)} & \frac{\begin{array}{c} \vdots \pi' \\ \Gamma \vdash B \end{array}}{\Gamma, !A \vdash B} \text{(weak)} \\
\text{(h) Right implication} & \text{(i) Left implication} & \text{(j) Contraction} & \text{(k) Weakening}
\end{array}$$

■ **Figure 1** Rules for linear logic sequent calculus

### 5 Linear proofs as linear functions

This model of the untyped  $\lambda$ -calculus can easily be extended to a model of the simply-typed  $\lambda$ -calculus by allowing the chosen set  $A$  to vary. In fact, the situation is much better than that: we can extend our model to a model of Intuitionistic Linear Logic!

We recall the deduction rules for Intuitionistic Linear Logic (ILL henceforth) in Figure 1. Since the cut-elimination rules (which we need for Theorem 34) are more cumbersome, we refer the reader to [17, Section 3] for these. For our purposes, a **model** of ILL in a monoidal category<sup>3</sup>  $(\mathcal{C}, \boxtimes, I)$  consists of an object  $\underline{A}$  of  $\mathcal{C}$  for each formula  $A$  and a morphism  $\llbracket \pi \rrbracket : \underline{A}_1 \boxtimes \cdots \boxtimes \underline{A}_n \rightarrow \underline{B}$  of  $\mathcal{C}$  for each proof  $\pi$  of a sequent  $A_1, \dots, A_n \vdash B$  constructed using the deduction rules of Figure 1, such that  $\llbracket \pi \rrbracket = \llbracket \pi' \rrbracket$  whenever  $\pi \sim \pi'$  are two proofs which are equivalent up to cut-elimination. (Note that the empty monoidal product when  $n = 0$  is interpreted as the monoidal unit  $I$ .)

In order to construct the underlying category of our model, we need some further definitions. Recall from Section 4 that for a set  $A$ ,  $\mathcal{Q}(A)$  consists of all functions  $f : A \rightarrow \overline{\mathbb{N}}$ . Considering  $\overline{\mathbb{N}}$  as a set equipped with the operation of natural number addition (extended in the intuitive way to include  $\infty$ ), the set  $\mathcal{Q}(A)$  with its point-wise addition inherits a commutative monoid structure.

► **Definition 26.** Given an element  $a \in A$ , let  $\delta_a \in \mathcal{Q}(A)$  be the function for which  $\delta_a(a')$  evaluates to 1 if  $a = a'$  and to 0 otherwise. We say a function  $f : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B)$  is **linear** if

$$f(\underline{a})(b) = \sum_{a \in A} \underline{a}(a) f(\delta_a)(b).$$

More generally, given sets  $A_1, \dots, A_n, B$ , a function  $f : \prod_{i=1}^n \mathcal{Q}(A_i) \rightarrow \mathcal{Q}(B)$  is said to be **multilinear** if it is linear in each argument. We denote the set of such functions  $\text{Lin}(\prod_{i=1}^n \mathcal{Q}(A_i), \mathcal{Q}(B))$ .

<sup>3</sup> Formally we could take  $\mathcal{C}$  to be a multicategory here, but we will not exploit this level of generality.

378 ▶ **Remark 27.** Whereas a normal function  $f : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B)$  is determined by its restriction  
 379 to the domain  $\mathcal{I}(A) \rightarrow \mathcal{Q}(B)$ , if  $f$  is *linear* then it is determined by its restriction to the  
 380 domain  $A \rightarrow \mathcal{Q}(B)$  (after identifying  $a \in A$  with  $\delta_a$ ).

381 To understand multilinearity, given a function  $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(C)$  which is linear  
 382 in the second argument, for any  $\underline{a} \in \mathcal{Q}(A)$  and  $\underline{b} \in \mathcal{Q}(B)$  we have

$$383 \quad f(\underline{a}, \underline{b}) = f\left(\underline{a}, \sum_{b \in B} \underline{b}(b) \cdot \delta_a\right) = \sum_{b \in B} \underline{b}(b) \cdot f(\underline{a}, \delta_b). \quad (19)$$

384 We can actually “curry” and “uncurry” multilinear functions using the presentation expressed  
 385 in (19). Unlike currying for normal functions, this linear currying is a bijection.

386 ▶ **Proposition 28.** *There is a bijection,*

$$387 \quad \text{Lin}(\mathcal{Q}(A) \times \mathcal{Q}(B), \mathcal{Q}(C)) \xrightleftharpoons[(-)^\dagger]{(-)^\times} \text{Lin}(\mathcal{Q}(A), \mathcal{Q}(B \times C)) \quad (20)$$

388 **Proof.** We define  $f^\times : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B \times C)$  as follows for  $\underline{a} \in \mathcal{Q}(A)$  and  $(b, c) \in B \times C$ :

$$389 \quad f^\times(\underline{a})(b, c) = f(\underline{a}, \delta_b)(c) \quad (21)$$

390 Conversely, given a linear function  $g : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B \times C)$  we define  $g^\dagger : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(C)$   
 391 as follows for  $(\underline{a}, \underline{b}) \in \mathcal{Q}(A) \times \mathcal{Q}(B), c \in C$ :

$$392 \quad g^\dagger(\underline{a}, \underline{b})(c) = \sum_{b \in B} \underline{b}(b) \cdot g(\underline{a})(b, c) \quad (22)$$

393 Clearly, if  $f : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(C)$  is linear in its second argument, then  $(f^\times)^\dagger = f$ .  
 394 Conversely, for any  $g : \mathcal{Q}(A) \rightarrow \mathcal{Q}(B \times C)$  we have  $(g^\dagger)^\times = g$ . ◀

395 ▶ **Example 29.** Taking  $A = B$  and  $C = \{*\}$  in Proposition 28, we find that  $(\text{id}_{\mathcal{Q}(A)})^\dagger$  is the  
 396 ‘scalar product’ map,

$$397 \quad (\text{id}_{\mathcal{Q}(A)})^\dagger(\underline{a}, \underline{a}') = \sum_{a \in A} \underline{a}(a) \cdot \underline{a}'(a) \in \overline{\mathbb{N}}$$

398 which is the linear extension of  $(a, a') \mapsto \delta_a(a')$ .

399 At this point we already have enough structure to interpret the formulas of Linear Logic.

400 ▶ **Definition 30.** *We choose, for each atomic formula  $X$ , a set which we denote  $\underline{X}$ . For a  
 401 formula, we define the interpretation inductively via the rules:*

$$402 \quad \underline{A} \otimes \underline{B} = \underline{A} \multimap \underline{B} = \underline{A} \times \underline{B}, \quad !\underline{A} = \mathcal{I}(A). \quad (23)$$

403 Rather than the category of sets, we take the context for these interpretations to be  
 404 the Kleisli category of  $\mathcal{Q}$ . Indeed,  $\mathcal{Q}$  becomes a monad on  $\text{Set}$  when equipped with unit  
 405 transformation  $\delta : A \rightarrow \mathcal{Q}(A)$  mapping  $a$  to  $\delta_a$  and multiplication  $\mu : \mathcal{Q}(\mathcal{Q}(A)) \rightarrow \mathcal{Q}(A)$   
 406 given by viewing elements on each side as extended multisets and taking the disjoint union.  
 407 Morphisms  $A \rightarrow B$  in the Kleisli category are functions  $A \rightarrow \mathcal{Q}(B)$ , which in turn correspond  
 408 to linear functions  $\mathcal{Q}(A) \rightarrow \mathcal{Q}(B)$ . As such, we will interpret a proof  $\pi$  of a sequent  
 409  $A_1, \dots, A_n \vdash B$  as a multilinear function  $\mathcal{Q}(A_1) \times \dots \times \mathcal{Q}(A_n) \rightarrow \mathcal{Q}(B)$ , which correspond to  
 410 linear maps  $\mathcal{Q}(A_1 \times \dots \times A_n) \rightarrow \mathcal{Q}(B)$ . Thus cartesian products of sets induces the monoidal  
 411 product operation on the Kleisli category.

412 To interpret proofs, we need a little more structure. Let  $d_A : \mathcal{Q}(\mathcal{I}(A)) \rightarrow \mathcal{Q}(A)$  be  
 413 the map sending  $\delta_a$  to  $\sum_{a \in A} \underline{a}(a) \delta_a$ , extended linearly. Let  $p_A : \mathcal{Q}(A) \rightarrow \mathcal{Q}(\mathcal{I}(A))$  be the  
 414 morphism that maps  $\delta_a$  to  $\delta_{\delta_a}$ , extended linearly. We will also employ the linear extension  
 415 of the diagonal map, which we denote  $\Delta_A : \mathcal{Q}(\mathcal{I}(A)) \rightarrow \mathcal{Q}(\mathcal{I}(A) \times \mathcal{I}(A))$ , and the swap map  
 416  $s_{A,B} : \mathcal{Q}(A) \times \mathcal{Q}(B) \rightarrow \mathcal{Q}(B) \times \mathcal{Q}(A)$ .

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417 ► **Definition 31.** We construct the interpretation  $\llbracket \pi \rrbracket$  of a proof  $\pi$  by induction on the  
 418 structure of  $\pi$ , with reference to Figure 1. Throughout, when a composition symbol carries a  
 419 subscript, this indicates the formula corresponding to the argument at which to compose.

- 420 ■ if  $\pi$  consists of a single axiom rule (Figure 1a), then  $\llbracket \pi \rrbracket := \text{id}_{\mathcal{Q}(\underline{X})}$ ;
- 421 ■ if  $\pi$  ends with a cut rule (Figure 1b), then  $\llbracket \pi \rrbracket := \llbracket \pi_2 \rrbracket \circ_A \llbracket \pi_1 \rrbracket$ ;
- 422 ■ if  $\pi$  ends with an exchange rule (Figure 1c), then  $\llbracket \pi \rrbracket := \llbracket \pi' \rrbracket \circ_{A,B} s_{\underline{B},\underline{A}}$ ;
- 423 ■ if  $\pi$  ends with a left tensor rule (Figure 1d), then  $\llbracket \pi \rrbracket := \llbracket \pi' \rrbracket$  up to identifying multilinear  
 424 maps out of  $\mathcal{Q}(A) \times \mathcal{Q}(B)$  with linear maps out of  $\mathcal{Q}(A \times B)$ ;
- 425 ■ if  $\pi$  ends with a right tensor rule (Figure 1e), then  $\llbracket \pi \rrbracket(a, b) := \llbracket \pi_1 \rrbracket(a) \times \llbracket \pi_2 \rrbracket(b)$ ;
- 426 ■ if  $\pi$  ends with a right linear arrow rule (Figure 1h), then  $\llbracket \pi \rrbracket := \llbracket \pi' \rrbracket^*$ ;
- 427 ■ if  $\pi$  ends with a left linear arrow rule (Figure 1i), then for  $a \in A$ ,  $\llbracket \pi \rrbracket(\alpha, \underline{a}, \underline{b}, \beta) :=$   
 428  $\llbracket \pi_1 \rrbracket^\dagger(\alpha, \underline{a}) \cdot \llbracket \pi_2 \rrbracket(\underline{b}, \beta)$  (this is the linear version of application);
- 429 ■ if  $\pi$  ends with a dereliction rule (Figure 1f), then  $\llbracket \pi \rrbracket := \llbracket \pi' \rrbracket \circ_A d_A$ ;
- 430 ■ if  $\pi$  ends with a promotion rule (Figure 1g), then  $\llbracket \pi \rrbracket := p_A \circ \llbracket \pi' \rrbracket$ ;
- 431 ■ if  $\pi$  ends with a contraction rule (Figure 1j), then  $\llbracket \pi \rrbracket := \llbracket \pi' \rrbracket \circ_{!A,!A} \Delta_A$ ;
- 432 ■ if  $\pi$  ends with a weakening rule (Figure 1k), then  $\llbracket \pi \rrbracket(\underline{a}_1, \dots, \underline{a}_n, \underline{a}) := \llbracket \pi' \rrbracket(\underline{a}_1, \dots, \underline{a}_n)$ .

433 ► **Example 32** (Church numeral  $\underline{2}_A$  in Linear Logic). Consider the Church numeral  $\underline{2}_A$  (without  
 434 the penultimate right implication rules).

$$\begin{array}{c}
 \frac{}{A \vdash A} \text{(ax)} \quad \frac{}{A \vdash A} \text{(ax)} \\
 \frac{}{A \vdash A} \text{(ax)} \quad \frac{\frac{}{A \vdash A} \text{(ax)} \quad \frac{}{A \vdash A} \text{(ax)}}{A, A \multimap A \vdash A} \text{(L } \multimap \text{)} \\
 \frac{\frac{}{A, A \multimap A, A \multimap A \vdash A} \text{(L } \multimap \text{)}}{A, !(A \multimap A), A \multimap A \vdash A} \text{(der)} \\
 \frac{\frac{}{A, !(A \multimap A), !(A \multimap A) \vdash A} \text{(der)}}{A, !(A \multimap A) \vdash A} \text{(ctr)}
 \end{array}$$

436 Recall that by definition,  $\underline{A \multimap A} = A \times A$  (where on the right-hand side we drop the underline  
 437 on the  $A$  for convenience). Thus we can write  $f \in !(A \multimap A) = \mathcal{I}(A \times A)$  as  $f = \sum_{i=1}^n c_i(a_i, b_i)$   
 438 with  $a_i, b_i \in A$  and  $c_i \in \mathbb{N}$ . With this notation,  $d_{A \times A}(\delta_f) = \sum_{i=1}^n c_i \delta_{(a_i, b_i)}$ , and hence the  
 439 interpretation of the above proof is the function  $\mathcal{Q}(A) \times \mathcal{Q}(\mathcal{I}(A \times A)) \rightarrow \mathcal{Q}(A)$  obtained as  
 440 the linear extension of:

$$\begin{array}{c}
 441 \quad A \times \mathcal{I}(A \times A) \rightarrow \mathcal{Q}(A) \\
 442 \quad (a, f) \mapsto \left( \sum_{i=1}^n c_i \cdot \delta_a(a_i) \right) \cdot \left( \sum_{i,j=1}^n c_i \cdot c_j \cdot \delta_{b_i}(a_j) \right) \cdot \left( \sum_{j=1}^n c_j \cdot \delta_{b_j} \right).
 \end{array}$$

443 ► **Remark 33.** Let us take a moment to reflect on the relationship between normal functions  
 444 and linear functions in light of the interpretations of terms and proofs in Definitions 30 and  
 445 31. Since linear functions are determined by their restriction along the inclusion  $\delta : A \rightarrow \mathcal{Q}(A)$   
 446 and normal functions are determined by their restriction to  $\mathcal{I}(A) \subseteq \mathcal{Q}(A)$ , one might expect to  
 447 be able to identify normal functions  $\mathcal{Q}(A) \rightarrow \mathcal{Q}(B)$  with linear functions  $\mathcal{Q}(\mathcal{I}(A)) \rightarrow \mathcal{Q}(B)$ ,  
 448 which would be the interpretation of a proof of  $!A \vdash B$ . While any normal function can indeed  
 449 be presented as a linear function in this way, not every such linear function is normal, since  
 450 it need not respect the ordering on  $\mathcal{I}(A)$ ; we must apply the plus and minus constructions  
 451 to each argument to recover a normal function. Nonetheless, this presentation of ‘ordinary’  
 452 functions as linear functions out of a modified domain is, according to Girard, the original  
 453 inspiration for Linear Logic.

454 The dereliction operation  $d$  is the linear extension of the inclusion  $\mathcal{I}(A) \rightarrow \mathcal{Q}(A)$  whose  
 455 normal extension is the identity  $\mathcal{Q}(A) \rightarrow \mathcal{Q}(A)$ . Thus the interpretation of the simplest proof  
 456 of  $!A \vdash A$ , namely

$$457 \frac{}{A \vdash A} \text{(ax)} \quad \frac{}{!A \vdash A} \text{(der)}$$

458 is precisely the linear function corresponding to Girard's interpretation of a variable in  $\lambda$ -  
 459 calculus, and similarly for the simplest proof of more general variables. Similarly, promotion  
 460 is chosen so that application of terms corresponds to the interpretation of promotion followed  
 461 by cut:

$$462 \frac{\frac{\vdots \pi_1}{! \Gamma \vdash A} \text{(prom)} \quad \frac{\vdots \pi_2}{! \Delta, !A, !\Delta' \vdash B}}{! \Gamma, !\Delta, !\Delta' \vdash B} \text{(cut)}$$

463 ► **Theorem 34.** *Definition 31 gives a model of Intuitionistic Linear Logic. That is, if  $\pi_1$*   
 464 *and  $\pi_2$  are (cut)-equivalent proofs, then  $\llbracket \pi_1 \rrbracket = \llbracket \pi_2 \rrbracket$ .*

465 **Proof.** We go through each (cut)-elimination rule methodically and prove invariance of the  
 466 interpretations under these transformations.

467 The interesting cases are (prom)/(der) and (R  $\multimap$ )/(L  $\multimap$ ). First we consider (prom)/(der).  
 468 Say  $\pi$  is on the left of the cut and  $\pi'$  is on the right. The two interpretations are respectively

$$469 \llbracket \pi' \rrbracket \circ_A d_A \circ_{!A} p_A \circ \llbracket \pi \rrbracket, \quad \llbracket \pi' \rrbracket \circ_A \llbracket \pi \rrbracket \quad (24)$$

470 So it suffices to show that  $d_A \circ p_A = \text{id}_{\mathcal{Q}(\mathcal{I}(A))}$ . It suffices to check this on elements of the  
 471 form  $\delta_a$ , and indeed  $d_A(p_A(\delta_a)) = d_A(\delta_{\delta_a}) = \delta_a$  is the identity, as required.

472 Next we consider (R  $\multimap$ )/(L  $\multimap$ ). One of the interpretations involves  $(\llbracket \zeta \rrbracket)^\dagger$  for some  
 473 proof  $\zeta$  where the other involves simply  $\llbracket \zeta \rrbracket$ . These are equal by Proposition 28 and the  
 474 result follows. ◀

## 475 6 Comparison to other models

476 By replacing the set  $\overline{\mathbb{N}}$  in the definition of  $\mathcal{Q}$  with the two-element poset  $0 < 1$ , we recover  
 477 the **power set monad**  $\mathcal{P}$  on Set and many of the components of the models have parallels  
 478 in that setting, producing a model resembling the relational model of Linear Logic. One can  
 479 generalise powerset algebras to **qualitative domains**, which consist of a set  $X$  along with a  
 480 set  $|X|$  of subsets of  $X$ , which contains the empty set, and is closed under arbitrary subset  
 481 and also filtered colimits. Girard took this approach in [10, 8], where he showed that in fact  
 482 **binary** qualitative domains also suffice for the construction of a model of  $\lambda$ -calculus. It is  
 483 easy to show that binary qualitative domains are exactly coherent spaces. Investigating the  
 484 true origins of Linear Logic was in fact the original motivation for the current paper.

485 More generally, we could replace  $\overline{\mathbb{N}}$  with a continuous semiring and recover the components  
 486 of weighted relational models [16]. A significant difference between our model and those is  
 487 that ours is *not* functorial in the usual sense, as applying a dereliction rule followed by a  
 488 promotion rule to an axiom yields a proof whose denotation is *not* the identity morphism on  
 489  $\mathcal{Q}(\mathcal{I}(A))$ . This can be seen easily by the following calculation:

$$490 p_A d_A(\delta_a) = p_A \left( \sum_{a \in A} \underline{a}(a) \cdot \delta_a \right) = \sum_{a, a' \in A} \underline{a}(a) \delta_a(a') \cdot \delta_{\delta_{a'}} = \sum_{a \in A} \underline{a}(a) \cdot \delta_{\delta_a} \quad (25)$$

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491 which is *not* equal to  $\delta_a$ . In this sense our model lies outside of the accepted categorical  
492 definition of a model of Linear Logic [17].

493 We note however that ‘non-functorial’ denotational models already exist in the literature.  
494 In particular the interpretation of Linear Logic proofs in Geometry of Interaction constructions  
495 [11, 12, 23] do not interpret the proof  $\pi$  obtained from applying a dereliction and a promotion  
496 to an axiom as an identity. Indeed, using the notations of the first geometry of interaction<sup>4</sup>,  
497 the interpretation of that proof is equal to

$$498 \quad \pi^\bullet = \begin{pmatrix} 0 & t(1 \otimes p^*) \\ (1 \otimes p)t^* & 0 \end{pmatrix}$$

499 which does not act as an identity, i.e. if applied to an argument  $a$ , this yields

$$500 \quad \text{Ex}(\pi^\bullet, a) = t(1 \otimes p^*)a(1 \otimes p)t^*$$

501 which is different from  $a$  in general.

502 We adopt the attitude that the research program which realises Linear Logic models as  
503 categorical structures differs in principle from ours as we put the syntactic systems themselves  
504 as primary, and the models themselves as secondary, in the sense that we have not imposed  
505 convenient categorical hypotheses which are not justified by the logic: most notably,  $!$  is not  
506 a comonad here. We wish to emphasise the fact that a perspective prioritising categorical  
507 aesthetics would have discarded our model, and we believe this would have been a mistake.

508 The merit of our model is that it puts on firm footing exactly how  $\lambda$ -terms may be  
509 thought of as structure preserving functions, and it does so in a down-to-earth way that we  
510 intend to generalize in future.

---

<sup>4</sup> As a very quick recap, the first geometry of interaction interprets proofs as matrices whose coefficients are linear operators acting on the Hilbert space  $\ell^2(\mathbf{N})$  of square-summable sequences of complex numbers. The operators  $p$  and  $q$  are the linear operators induced by applying the injections  $n \mapsto 2n$  and  $n \mapsto 2n+1$  on the standard natural basis. The operator  $t$  implements associativity of the tensor product (which encodes the tensor of  $a$  and  $b$  as  $p^*ap + q^*bq$ ): it is induced by the following bijection applied to the standard basis:  $2n \mapsto 4n$ ,  $4n+1 \mapsto 4n+2$ ,  $4n+3 \mapsto 2n+1$ .



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## A Proof of Girard's Normal Form Theorem

574

575 Clearly, every integral functor is finite. Conversely, every finite functor is isomorphic to an  
 576 integral functor. It follows that the finite normal form property is equivalent to the integral  
 577 normal form property. Moreover, this holds even when  $A$  is an arbitrary category, even  
 578 though this case was not considered in Girard's original paper [10].

579

We now show that if a functor  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}$  admits the finite normal form property  
 580 then it is isomorphic to an analytic functor. This result can be thought of as recovering  
 581 the functor  $\mathcal{F}$  from its collection of normal forms. In short, given a functor  $F \in \text{Set}^A$  and  
 582 an element  $x \in \mathcal{F}(F)$ , a normal form  $\eta : (G, y) \rightarrow (F, x)$  will induce the data of a triple  
 583  $(G, \eta, y') \in \coprod_{G \in \text{Int}(A)} (\text{Set}^A(G, F) \times C_G)$  where  $y'$  is equivalent to  $y$  under an appropriate  
 584 equivalence relation. To finish the proof, we must define the equivalence relation defining the  
 585 classes which form  $C_G$ . This will require an alternate classification of when an integral form  
 586 is normal without reference to its codomain.

587

► **Lemma 35.** *Let  $\eta : (G, y) \rightarrow (F, x)$  be an integral form (not necessarily normal) and say  
 588  $\mathcal{F}$  satisfies the integral normal form property. Then  $\eta$  is normal if and only if  $\text{id}_G : (G, y) \rightarrow$   
 589  $(G, y)$  is.*

590

**Proof.** Let  $\eta' : (G, y') \rightarrow (F, x)$  be an integral normal form associated to  $(F, x)$ . Then by  
 591 normality there exists a morphism  $\gamma : G \rightarrow G$  so that the following is a commutative diagram  
 592 in  $\text{El}(\mathcal{F})$ .

593

$$\begin{array}{ccc} (G, y') & \xrightarrow{\eta'} & (F, x) \\ \gamma' \downarrow \cong & \nearrow \eta & \\ (G, y) & & \end{array} \quad (26)$$

594

Since  $\text{id}$  is normal, there exists a section  $\gamma'$  rendering (26) commutative.

595

Since  $\gamma\gamma' = \text{id}_G$  and  $\eta$  is normal, it follows that  $\eta'$  is normal. On the other hand, say  $\eta$  is  
 596 normal. Let  $\epsilon : (H, w) \rightarrow (G, y)$  be arbitrary. Consider the composition  $\eta\epsilon$ . By normality of  
 597  $\eta$ , there exists a unique  $\gamma : (G, y) \rightarrow (H, w)$  so that the following diagram commutes:

598

$$\begin{array}{ccc} & & (F, x) \\ & \nearrow \eta & \uparrow \eta\epsilon \\ (G, y) & \xrightarrow{\gamma} & (H, w) \end{array} \quad (27)$$

599

If  $\gamma'$  was another such map, then  $\eta\epsilon\gamma = \eta\epsilon\gamma'$  so by normality of  $\eta$  we have that  $\gamma = \gamma'$ . ◀

600

► **Lemma 36.** *If a functor  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}$  satisfies the finite normal form property, then  $\mathcal{F}$   
 601 is isomorphic to an analytic functor.*

602

**Proof.** The main step in the proof will be to define for each  $G \in \text{Int}(A)$  a set  $C_G$  and for  
 603 each  $F \in \text{Set}^A$  a bijection

604

$$h_F : \mathcal{F}(F) \rightarrow \coprod_{G \in \text{Int}(A)} (\text{Set}^A(G, F) \times C_G) \quad (28)$$

605

In fact, in the current setting where  $A$  admits only identity morphisms, this will complete  
 606 the proof.

607

For any element  $(F, x)$  of  $\text{El}(\mathcal{F})$  there is some finite normal form  $\eta : (G, y) \rightarrow (F, x)$ ,  
 608 isomorphic to an integral normal form. Thus, it suffices to consider the case where  $\mathcal{F}$  satisfies  
 609 the *integral* normal form property.

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610 An integral normal form  $\eta : (G, y) \rightarrow (F, x)$  is *not* uniquely determined by  $(F, x)$ , however,  
 611 given another integral normal form  $\eta' : (G', y') \rightarrow (F, x)$  we have that  $G' \cong G$  by normality  
 612 and thus  $G' = G$  by integrality. So at least the domain of the object is uniquely determined  
 613 by  $(F, x)$ .

614 Let  $X_G$  denote the elements  $y \in \mathcal{F}(G)$  for which  $\text{id}_G : (G, y) \rightarrow (G, y)$  is normal, since  $\mathcal{F}$   
 615 satisfies the integral form property, there is always at least one such  $y$ . Let  $C_G$  denote a set  
 616 of choices of representatives of the isomorphism classes of  $X_G$ .

617 Thus, to each  $x \in \mathcal{F}(F)$  we have associated an integral normal form  $\eta : (G, y) \rightarrow (F, x)$   
 618 and fixed particular choices so that this map  $h_F(x) = (G, \eta, y)$  is a bijection.  $\blacktriangleleft$

619 The converse to Lemma 36 also holds, which we now move onto proving.

620 In general, if  $\mu : H \rightarrow G$  is a natural transformation and  $\eta : (G, y) \rightarrow (F, x)$  is a normal  
 621 form, then the composite  $\eta\mu$  is need *not* be a normal form. However, if  $\mathcal{F}$  satisfies the finite  
 622 normal form property the normal forms *can* be carried through natural transformations.  
 623 This is the content of the next Lemma.

624 **► Lemma 37.** *Let  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}$  be a functor satisfying the normal form property. Then  
 625 if  $\eta : (G, y) \rightarrow (F, x)$  is a normal form and  $\mu : G \rightarrow H$  is a natural transformation, then  
 626  $\mu\eta : (G, y) \rightarrow (H, \mathcal{F}(\mu)(x))$  is a normal form.*

627 **Proof.** Let  $\epsilon : (K, z) \rightarrow (H, \mathcal{F}(\mu)(x))$  be an arbitrary form. We show that there exists a  
 628 unique morphism  $(G, y) \rightarrow (K, z)$  in the category  $\text{El}(\mathcal{F})/(H, \mathcal{F}(\mu)(x))$ . Since  $\mathcal{F}$  satisfies  
 629 the normal form property there exists some normal form  $\gamma : (L, w) \rightarrow (H, \mathcal{F}(\mu)(x))$ . It is  
 630 convenient to draw this situation out in the category  $\text{El}(\mathcal{F})$ , ignore the dashed arrows for  
 631 now.

$$\begin{array}{ccc}
 (G, y) & \xrightarrow{\eta} & (F, x) \\
 \uparrow \gamma' & & \downarrow \mu \\
 (L, w) & \xrightarrow{\gamma} & (H, \mathcal{F}(\mu)(x)) \\
 \downarrow \beta & \nearrow \epsilon & \\
 (K, z) & & 
 \end{array} \tag{29}$$

633 Since  $\mu\eta : (G, y) \rightarrow (H, \mathcal{F}(\mu)(x))$  is a form with respect to  $(H, \mathcal{F}(\mu)(x))$  we have by  
 634 initiality of  $\gamma : (L, w) \rightarrow (H, \mathcal{F}(\mu)(x))$  that there exists a morphism  $\gamma : (L, w) \rightarrow (G, y)$   
 635 fitting into (29).

636 The morphism  $\eta\gamma : (L, w) \rightarrow (F, x)$  induces the morphism  $\gamma'$  and composing this with the  
 637 morphism  $\beta$  (which is induce by initiality of  $\gamma : (L, w) \rightarrow (H, \mathcal{F}(\mu)(x))$ ) induces a morphism  
 638  $(G, y) \rightarrow (K, z)$  which is the unique morphism rending the full diagram commutative. Thus  
 639  $\mu\eta : (G, y) \rightarrow (H, \mathcal{F}(\mu)(x))$  is initial.  $\blacktriangleleft$

640 **► Lemma 38.** *Let  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}$  be analytic. Then  $\mathcal{F}$  satisfies the normal form property.*

641 **Proof.** Let  $F \in \text{Set}^A$  be arbitrary and consider an element  $(G, \eta, y)$  of  $\mathcal{F}(F) = \coprod_{G' \in \text{Int}(A)} (\text{Set}^A(G', F) \times$   
 642  $C_{G'})$ . We can then consider the set

$$\mathcal{F}(G) = \coprod_{G' \in \text{Int}(A)} \text{Set}^A(G', G) \times C_{G'}.$$

644 A particular element of this set is  $(G, \text{id}_G, y)$ . We show that  $\eta : (G, (G, \text{id}_G, y)) \rightarrow (F, (G, \eta, y))$   
 645 is normal.

646 Say  $\epsilon : (H, (G', \eta', y')) \rightarrow (F, (G, \eta, y))$  is a form, then

$$\mathcal{F}(\epsilon)(G', \eta', y') = (G, \eta, y) \tag{30}$$

648 We unpack the definition of the function  $\mathcal{F}(\epsilon) = \coprod_{G \in \text{Int}(A)} (\text{Set}^A(G, \epsilon) \times C_G)$ . This function  
 649 makes the following Diagram commute, where the vertical morphisms are canonical inclusion  
 650 maps.

$$\begin{array}{ccc}
 \coprod_{G \in \text{Int}(A)} (\text{Set}^A(G, H) \times C_G) & \xrightarrow{\mathcal{F}(\mu)} & \coprod_{G \in \text{Int}(A)} (\text{Set}^A(G, F)) \\
 \uparrow & & \uparrow \\
 \text{Set}^A(G, H) \times C_G & \xrightarrow{- \circ \epsilon \times \text{id}_{C_G}} & \text{Set}^A(G, F) \times C_G
 \end{array} \tag{31}$$

652 So (30) implies  $((\_ \circ \epsilon) \times \text{id})(\eta', y') = (\eta, y)$ . We thus have:

$$653 \quad G' = G, \quad \epsilon \eta' = \eta, \quad y' = y \tag{32}$$

654 Thus, the domain of the morphism  $\epsilon : (H, (G', \eta', y')) \rightarrow (F, (G, \eta, y))$  is equal to  $(H, (G, \eta', y))$ .  
 655 We need a unique morphism  $(G, (G, \text{id}_G, y)) \rightarrow (H, (G, \eta', y))$ . Clearly  $\eta'$  is such a morphism,  
 656 and it is the unique such because for any morphism  $\mu : G \rightarrow G$  we have  $(\text{Set}^A(G, \mu) \times C_G)(\mu) =$   
 657  $\mu$ , and so  $\eta'$  is the unique morphism  $\mu$  determined by the condition  $(\text{Set}^A(G, \mu) \times C_G)(\mu) =$   
 658  $\eta'$ . ◀

659 Everything so far also holds in the setting where  $A$  is an arbitrary category, even though  
 660 the assumption was made in [10] that  $A$  is a set.

661 ▶ **Lemma 39.** *Any functor  $F \in \text{Set}^A$  is the colimit of finite functors in  $\text{Set}^A$ .*

662 Lemma 39 is useful for proving that certain subobjects are finite. In short, one can prove  
 663 a set  $Y$  is finite by defining a surjective function  $f : X \rightarrow Y$  where  $X$  is finite. This suggest a  
 664 relaxing of the finite normal form condition to the *saturated form condition*, which is to say  
 665 that every appropriate pair  $(F, x)$  admits a saturated form.

666 ▶ **Definition 40.** *A form  $\eta : (G, y) \rightarrow (F, x)$  is **saturated** if any other form  $\epsilon : (H, z) \rightarrow (G, y)$   
 667 is an epimorphism.*

668 ▶ **Lemma 41.** *If  $\mathcal{F}$  is normal, then every saturated form is finite.*

669 **Proof.** Let  $\eta : (G, y) \rightarrow (F, x)$  be a saturated form. We have by Lemma 39 that  $G$  is the  
 670 colimit of its finite subobjects, so we write  $G \cong \text{Colim}\{G_i\}_{i \in I}$ . Hence,  $\mathcal{F}(G) \cong \mathcal{F} \text{Colim}\{G_i\} \cong$   
 671  $\text{Colim}\{\mathcal{F}(G_i)\}$ , using normality.

672 Thus, we can view  $y$  as an element of  $\text{Colim}\{\mathcal{F}(G_i)\}$  and consider  $i \in I$  along with  
 673  $y' \in \mathcal{F}(G_i)$  which maps onto  $y \in \text{Colim}\{\mathcal{F}(G_i)\}$  under the corresponding morphism of the  
 674 colimit. We thus have a commutative diagram.

$$\begin{array}{ccc}
 \mathcal{F}(G) & \xrightarrow{\cong} & \text{Colim}\{\mathcal{F}(G_i)\} \\
 & \swarrow & \uparrow \\
 & & \mathcal{F}(G_i)
 \end{array} \tag{33}$$

676 Thus,  $(G_i, y') \rightarrow (G, y)$  is a form which is surjective by saturation of  $\eta$ . Since  $G_i$  is finite,  
 677 this implies  $G$  is finite. ◀

678 The final preliminary lemma required states that morphisms out of saturated normal  
 679 forms are unique, in an appropriate sense. The proof of this lemma will use the fact that any  
 680 functor preserving pullbacks preserves equalisers.

681 ▶ **Lemma 42.** *Let  $\eta : (G, y) \rightarrow (F, x)$  be saturated and  $\eta' : (G, y) \rightarrow (F, x)$  an arbitrary form.  
 682 Then  $\eta = \eta'$ .*

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683 **Proof.** Consider the equaliser  $\text{Eq}(\mathcal{F}\eta, \mathcal{F}\eta')$ . Since  $\mathcal{F}\eta(y) = \mathcal{F}\eta'(y)$  we have that  $y \in$   
 684  $\text{Eq}(\mathcal{F}\eta, \mathcal{F}\eta')$ . Since  $\text{Eq}(\mathcal{F}\eta, \mathcal{F}\eta') \cong \mathcal{F}\text{Eq}(\eta, \eta')$  it follows that  $(\text{Eq}(\eta, \eta'), y) \rightarrow (G, y)$  is a  
 685 form, which in fact is surjective by saturation of  $\eta$ . It follows that  $\eta = \eta'$ .  $\blacktriangleleft$

686 **► Lemma 43.** *If  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}$  is normal then it satisfies the normal form property.*

687 **Proof.** Let  $(F, x)$  be a pair consisting of a functor  $F \in \text{Set}^A$  and an element  $x \in \mathcal{F}(F)$ .  
 688 Consider all the saturated forms with codomain  $(F, x)$  and take the pullback of this entire  
 689 diagram. We use the labelling as given by (34).

$$\begin{array}{ccc}
 & (S_i, y_i) & \\
 \eta_i \nearrow & & \searrow \sigma_i \\
 \text{PullBack} & & (F, x) \\
 \eta_j \searrow & & \nearrow \sigma_j \\
 & (S_j, y_j) &
 \end{array}
 \quad (34)$$

691 There exists  $y \in \mathcal{F}(\text{PullBack})$  so that  $\mathcal{F}\eta_i(y) = y_i$  for all  $i$ . We consider a saturated form  
 692  $\epsilon : (G, z) \rightarrow (\text{PullBack}, y)$ . We claim that this is a normal form with respect to  $(F, x)$ .

693 Assume there is a form  $\gamma : (H, w) \rightarrow (F, x)$  and consider a saturated form  $\gamma' : (H', w') \rightarrow$   
 694  $(H, w)$ . A saturated form is one such that any form *into* it is surjective. Thus  $\gamma\gamma' : (H', w') \rightarrow$   
 695  $(F, x)$  is saturated as  $\gamma' : (H', w') \rightarrow (H, w)$  is.

696 It follows that  $(H, w) = (S_i, y_i)$  for some  $i$ . Thus we have a morphism  $\eta_i\epsilon : (G, z) \rightarrow$   
 697  $(S_i, y_i) = (H, w)$ . It follows from Lemma 42 that this is the unique morphism in the  
 698 appropriate sense. This completes the proof.  $\blacktriangleleft$

699 The remaining result to be proved for is the converse to Lemma 43.

700 **► Lemma 44.** *A functor  $\mathcal{F} : \text{Set}^A \rightarrow \text{Set}$  satisfying the finite normal form property is normal.*

701 **Proof.** We must show that  $\mathcal{F}$  preserves direct colimits and wide pullbacks.

702  $\mathcal{F}$  **preserves direct colimits:** consider a direct system, that is, assume there exists a  
 703 collection of objects  $\{F_i\}_{i \in I}$  fo  $\text{Set}^A$ , where  $I$  is a set equipped with a partial order  $<$ , along  
 704 with a collection of morphisms  $\{\alpha_{ij} : F_i \rightarrow F_j\}_{i, j \in I}$  subject to the following conditions

- 705 ■  $\forall i, j \in I, \exists k \in I$  such that  $\alpha_{ik} : F_i \rightarrow F_k$ , and  $\alpha_{jk} : F_j \rightarrow F_k$  exist.
- 706 ■  $\forall i, j, k \in I, \alpha_{jk}\alpha_{ij} = \alpha_{ik}$
- 707 ■  $\forall i \in I \alpha_{ii} = \text{id}_{F_i}$

708 Let  $C$  denote the direct colimit of this direct system in the category  $\text{Set}^A$  and let  $\{\mu_i : F_i \rightarrow C\}$   
 709 denote the associated morphisms into  $C$ . Consider also the direct colimit

$$(C', \{g_i : \mathcal{F}(F_i) \rightarrow C'\}_{i \in I}) \quad (35)$$

711 of the direct system given by  $(\{\mathcal{F}(F_i)\}_{i \in I}, \{\mathcal{F}(\alpha_{ij}) : \mathcal{F}(F_i) \rightarrow \mathcal{F}(F_j)\}_{i, j \in I})$  in the category  
 712  $\text{Set}$ .

713 By the universal property of  $C'$ , there exists a unique function

$$f : C' \rightarrow \mathcal{F}(C) \quad (36)$$

715 so that for all  $i \in I$  the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{F}(F_i) & & \\
 g_i \downarrow & \searrow \mathcal{F}(\mu_i) & \\
 C' & \xrightarrow{f} & \mathcal{F}(C)
 \end{array}
 \quad (37)$$

717 We need to prove that  $f$  is an isomorphism (ie, a bijection). We do this by proving that it is  
 718 injective and surjective.

719 First we prove surjectivity. Let  $z \in \mathcal{F}(C)$ . By the finite normal form property, there  
 720 exists a finite normal form  $\epsilon : (G, w) \rightarrow (C, z)$ . Now, for each  $a \in A$  there is a function

$$721 \quad \epsilon_a : G(a) \rightarrow C(a) \quad (38)$$

722 hence, there exists some  $i \in I$  and function  $\epsilon'_a : G(a) \rightarrow F_i(a)$  through which the function  $\epsilon_a$   
 723 factors. Since  $G$  is finite, and the colimit is direct, there exists an  $i \in I$  such that for each  
 724  $a \in A$  there is a morphism  $G(a) \rightarrow F_i(a)$ , which we also call  $\epsilon'_a$ , which makes the following  
 725 diagram commute.

$$726 \quad \begin{array}{ccc} G(a) & \xrightarrow{\epsilon'_a} & F_i(a) \\ & \searrow \epsilon_a & \downarrow \\ & & C(a) \end{array} \quad (39)$$

727 We claim the collection  $\epsilon' := \{\epsilon'_a : G(a) \rightarrow F_i(a)\}$  is a natural transformation, however since  
 728  $A$  is discrete (ie, has no non-identity morphisms), there is no condition to check, so this is  
 729 vacuously satisfied.

730 Note: even in the case where  $A$  is an arbitrary category, we still obtain naturality, it is  
 731 inherited from naturality of the morphisms involved in the following diagram:

$$732 \quad \begin{array}{ccccc} G & \longrightarrow & F_{i'} & \xrightarrow{\alpha_{i'i}} & F_i \\ & \searrow & & \uparrow \alpha_{ji} & \\ & & & & F_j \end{array} \quad (40)$$

733 We have constructed a natural transformation  $\epsilon' : G \rightarrow F_i$  so that the following diagram  
 734 commutes.

$$735 \quad \begin{array}{ccc} G & \xrightarrow{\epsilon'} & F_i \\ & \searrow \epsilon & \downarrow \\ & & C \end{array} \quad (41)$$

736 Let  $z'$  denote  $\mathcal{F}(\epsilon')(w)$ . We have commutativity of the following diagram

$$737 \quad \begin{array}{ccc} \mathcal{F}F_i & & \\ g_i \downarrow & \searrow \mathcal{F}\mu_i & \\ C' & \xrightarrow{f} & \mathcal{F}(C) \end{array} \quad (42)$$

738 Hence,  $g_i(z')$  is an element of  $C'$  such that  $f(g_i(z')) = z$ , establishing surjectivity.

739 Now we prove injectivity. Let  $x_1, x_2 \in C'$  be such that  $f(x_1) = f(x_2)$ . Let  $z$  denote  
 740 this element of  $\mathcal{F}(C)$ . The functions  $\{g_i\}_{i \in I}$  form a surjective family over  $C'$  and so there  
 741 exists  $i, i' \in I$  and  $x'_1 \in \mathcal{F}(F_i), x'_2 \in \mathcal{F}(F_{i'})$  so that  $g_i(x'_1) = x_1, g_{i'}(x'_2) = x_2$ . In fact, since the  
 742 diagram the colimit is over is direct, we can assume without loss of generality that  $i = i'$ .

743 Turning our consideration to  $z$ , which is an element of  $\mathcal{F}(C)$ , we choose a normal form  
 744  $\epsilon : (G, y) \rightarrow (C, z)$ . We have already seen in the proof of surjectivity how from this we  
 745 obtain a  $j \in I$  along with a natural transformation  $\epsilon' : G \rightarrow F_j$  so that the following diagram  
 746 commutes.

$$747 \quad \begin{array}{ccc} G & \xrightarrow{\epsilon'} & F_j \\ & \searrow \epsilon & \downarrow \mu_j \\ & & C \end{array} \quad (43)$$



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748 We have that  $\mathcal{F}(\mu_i)(x'_1) = \mathcal{F}(\mu_i)(x'_2) = z$ . So, by initiality of  $(G, y)$  there exists unique  
 749 morphisms  $\gamma_1, \gamma_2 : G \rightarrow F_i$  so that the following diagram commutes

$$750 \quad \begin{array}{ccc} G & & \\ \gamma_2 \downarrow & \searrow \epsilon & \\ \gamma_1 \downarrow & & \\ F_i & \xrightarrow{\mu_i} & C \end{array} \quad (44)$$

751 and so that  $\mathcal{F}(\gamma_1)(y) = x_1$  and  $\mathcal{F}(\gamma_2)(y) = x_2$ .

752 Combining (43) and (44) we obtain commutativity of the following diagram.

$$753 \quad \begin{array}{ccc} G & \xrightarrow{\epsilon'} & F_j \\ \gamma_2 \downarrow & \searrow \gamma_1 & \downarrow \mu_j \\ F_i & \xrightarrow{\mu_i} & C \end{array} \quad (45)$$

754 Now, let  $a \in A$  be an arbitrary element of  $A$  and consider (45) with everything evaluated at  
 755  $a$ , this gives a commuting diagram in  $\text{Set}$ . We notice that if  $G(a)$  is non-empty, then there  
 756 exists a pair of elements  $d, d' \in F_i(a)$  so that  $\mu_{ia}(d) = \mu_{ia}(d')$  and so there exists some  $k \in I$   
 757 such that  $\alpha_{ika} : F_i(a) \rightarrow F_k(a)$  so that  $\alpha_{ika}(d) = \alpha_{ika}(d')$ . By finiteness of  $G$  (in particular,  
 758 since all but finitely many  $a \in A$  are such that  $G(a)$  is non-empty) there thus exists  $k \in I$   
 759 and  $\alpha_{ik} : F_i \rightarrow F_k$  so that for all  $a \in A$  there exists  $d, d' \in F_i(a)$  so that  $\alpha_{ika}(d) = \alpha_{ika}(d')$ .  
 760 Lastly, by the first property above of direct colimits we may assume  $k = j$ . The result is the  
 761 following commutative diagram in  $\text{Set}^A$ .

$$762 \quad \begin{array}{ccc} & & F_k \\ & \nearrow \alpha_{ik} & \downarrow \mu_j \\ F_i & \xrightarrow{\mu_i} & C \end{array} \quad (46)$$

763 Finally, we can consider the following commuting diagram in  $\text{Set}$ .

$$764 \quad \begin{array}{ccc} \mathcal{F}G & & \\ \mathcal{F}\gamma_1 \downarrow & \searrow \mathcal{F}\epsilon' & \\ \mathcal{F}\gamma_2 \downarrow & & \\ \mathcal{F}F_i & \xrightarrow{\mathcal{F}\alpha_{ij}} & \mathcal{F}F_j \end{array} \quad (47)$$

765 Thus,  $\mathcal{F}\alpha_{ij}\mathcal{F}\gamma_1(y) = \mathcal{F}\alpha_{ij}\mathcal{F}\gamma_2(y)$ , ie,  $\mathcal{F}\alpha_{ij}(x'_1) = \mathcal{F}\alpha_{ij}(x'_2)$ , ie,  $x_1 = x_2$ . This establishes  
 766 injectivity.

$$767 \quad \begin{array}{ccc} \mathcal{F}P & \xrightarrow{f} & P' \\ & \searrow \mathcal{F}\pi_i & \downarrow \mu_i \\ & & \mathcal{F}(F_i) \end{array} \quad (48)$$

768 We must show that  $f$  is a bijection. First we show surjectivity. Let  $z \in P'$ . For each  $i$  we  
 769 consider  $\mathcal{F}(\pi_i)(z)$ , which we denote by  $z_i$ . Since  $\mathcal{F}$  satisfies the normal form property, there  
 770 exists a normal form

$$771 \quad \eta_i : (G_i, w_i) \rightarrow (F_i, z_i) \quad (49)$$

772 By Lemma 37 the compositions

$$773 \quad \alpha_i \eta_i : (G_i, w_i) \rightarrow (H, \mathcal{F}(\alpha_i)(w_i)) \quad (50)$$

774 are normal forms with respect to  $(G, \mathcal{F}(\alpha_i)(w_i))$  (note,  $\mathcal{F}(\alpha_i)(w_i)$  is independent of  $i$ ).

775 Hence by essential uniqueness of initial objects, we can assume without loss of generality  
776 that for all pairs  $i, j \in I$  we have  $G_i = G_j$ , denote this common element by  $G$ .

777 By the universal property of the pullback, there exists a natural transformation  $\gamma : G \rightarrow P$   
778 rendering the following Diagram commutative.

$$779 \quad \begin{array}{ccc} G & \xrightarrow{\eta_i} & F_i \\ \gamma \downarrow & \nearrow \pi_i & \\ P & & \end{array} \quad (51)$$

780 We notice also that the collection of elements  $\{w_i\}_{i \in I}$  induces an element  $w \in \mathcal{F}G$  so that for  
781 all  $i \in I$  we have  $\mathcal{F}(\eta_i)(w) = z_i$ .

782 We claim that

$$783 \quad f\mathcal{F}(\gamma)(w) = z \quad (52)$$

784 It suffices to show the following for all  $i \in I$ .

$$785 \quad \pi_{\mathcal{F}(F_i)}(z) = \pi_{\mathcal{F}(F_i)}f\mathcal{F}(\gamma)(w) \quad (53)$$

786 This holds by the following calculation.

$$787 \quad \pi_{\mathcal{F}(F_i)}f\mathcal{F}(\gamma)(w) = \mathcal{F}(\pi_i)\mathcal{F}(\gamma)(w) \quad (54)$$

$$788 \quad = \mathcal{F}(\eta_i)(w) \quad (55)$$

$$789 \quad = z_i \quad (56)$$

$$790 \quad = \pi_{\mathcal{F}(F_i)}(z) \quad (57)$$

791 Now we prove injectivity. Let  $x_1, x_2 \in \mathcal{F}P$  be such that  $f(x_1) = f(x_2)$ . By the normal  
792 form property, there is a normal form  $\chi_1 : (X_1, x'_1) \rightarrow (P, x_1)$  with respect to  $(P, x_1)$  and  
793 a normal form  $\chi_2 : (X_2, x'_2) \rightarrow (P, x_2)$  with respect to  $(P, x_1)$ . Let  $i \in I$  be arbitrary and  
794 consider the composition of these normal forms with the natural transformation  $\pi_i$ :

$$795 \quad (X_1, x'_1) \xrightarrow{\chi_1} (P, x_1) \xrightarrow{\pi_i} (F_i, \mathcal{F}(\pi_i\chi_1)(x'_1)) \quad (58)$$

$$(X_2, x'_2) \xrightarrow{\chi_2} (P, x_2) \xrightarrow{\pi_i} (F_i, \mathcal{F}(\pi_i\chi_2)(x'_2))$$

796 Now, by commutativity of (48) we have

$$797 \quad \mathcal{F}(\pi_i\chi_1)(x'_1) = \mathcal{F}(\pi_i)(x_1) = \mu_i f(x_1) = \mu_i f(x_2) = \mathcal{F}(\pi_i\chi_2)(x'_2) \quad (59)$$

798 Let  $w$  denote this common element.

799 This implies that (58) are both objects of the same comma category,  $\text{El}(\mathcal{F})/(F_i, w)$ , and  
800 in fact these are both normal by Lemma 37. We can thus assume without loss of generality  
801 that  $X_1 = X_2, x'_1 = x'_2$ , we let  $X, x$  respectively denote these common elements. Thus, our  
802 hypothesis is: for all  $i \in I$  we have

$$803 \quad \pi_i\chi_1 = \pi_i\chi_2 \quad (60)$$

804 It now remains to show

$$805 \quad \mathcal{F}(\chi_1)(x) = \mathcal{F}(\chi_2)(x) \quad (61)$$

806 We do this by proving  $\chi_1 = \chi_2$ .

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807 First, notice that for  $k = 1, 2$  and all  $i \in I$  we have  $\mathcal{F}(\pi_i \chi_k) = \mu_i f \mathcal{F}(\chi_k)(x)$ . Thus, by  
 808 Lemma 37 the following are normal forms:

809 
$$\pi_i \chi_k : (X, x) \rightarrow (F_i, \mu_i f \mathcal{F}(\chi_k)(x)) \tag{62}$$

810 By uniqueness of normal forms, it follows that  $\pi_i \chi_1 = \pi_i \chi_2$  for all  $i \in I$ . Let  $\xi_i$  denote this  
 811 common morphism. We now have that both  $\chi_1$  and  $\chi_2$  are morphisms  $X \rightarrow P$  rendering the  
 812 following diagram commutative for all  $i, j \in I$ .

813 
$$\begin{array}{ccccc} X & & \xrightarrow{\xi_i} & & F_i \\ & \searrow^{\chi_1, \chi_2} & & \searrow^{\pi_i} & \\ & P & & & \\ & \downarrow^{\pi_j} & & \downarrow^{\alpha_i} & \\ & F_j & \xrightarrow{\alpha_j} & & H \end{array} \tag{63}$$

814 It follows from the universal property of the pullback that  $\chi_1 = \chi_2$ . ◀