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Abstract We propose a new test statistic $\mathrm{MR}_{\gamma,n}$ for detecting a changed segment in the mean, at unknown dates, in a regularly varying sample. Our model supports several alternatives of shifts in the mean, including one change point, constant, epidemic and linear form of a change. Our aim is to detect a short length changed segment ℓ^*, assuming ℓ^*/n to be small as the sample size n is large. $\mathrm{MR}_{\gamma,n}$ is built by taking maximal ratios of weighted moving sums statistics of four sub-samples. An important feature of $\mathrm{MR}_{\gamma,n}$ is to be scale free. We obtain the limiting distribution of ratio statistics under the null hypothesis as well as their consistency under the alternative. These results are extended from i.i.d. samples under H_0 to some dependent samples. To supplement theoretical results, empirical illustrations are provided by generating samples from symmetrized Pareto and Log-Gamma distributions.

Keywords change-point detection · changed segment in the mean · epidemic change · Hölder norm statistics · regularly varying random variables · scan statistics

Mathematics Subject Classification (2010) 62G10 · 60F17

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1 Introduction

For each $n \geq 1$, we consider the model

$$Y_{nk} = \mu_{nk} + X_k, \quad k = 1, 2, \ldots, n, \tag{1}$$

where $(\mu_{nk}) \subset \mathbb{R}$ and $(X_k, k \geq 1)$ is a stationary sequence of zero mean random variables. Throughout the paper, we denote by X a generic random variable identically distributed with X_k. Our aim is to construct a test for the hypothesis

$$H_0 : \quad \mu_{n1} = \cdots = \mu_{nn} = 0 \tag{2}$$

against the alternative

$$H_A : \quad \mu_{nk} = \mu_n(k)\mathbb{1}_{I^*}(k), \quad k = 1, \ldots, n, \tag{3}$$

where $I^* := \{k^* + 1, \ldots, k^* + \ell^*\}$ for some $k^* = k_n^* \geq 0$ and $\ell^* = \ell_n^* \leq n$.

The model (3) includes several types of alternatives that are widely discussed in the literature. In the case where $\mu_n(k) = \mu_n^*, k = k^* + 1, \ldots, k^* + \ell^*$, we have the so called changed segment of length ℓ^* model. The case $\mu_n(k) = f_n(k/n)$, where $f_n(t) = 0$ if $t \notin [t^*, t^* + h^*]$, models epidemic type deviation from the usual state with h^* called the duration of the epidemic state. If the function $\mu_n(k)$ is linearly increasing in the interval $(k^*, k^* + \ell^*]$, then one speaks about the so called smooth-abrupt type change-point model. Note that the special cases $k^* = 0$ or $\ell^* = n - k^*$ in (3) give a simple one change point.

Motivated by the detection of a short length changed segment, we assume throughout that $\ell^* < n/4$.

Statistical change point tests are of interest in many areas of applications, e.g., finance, climate studies, medicine, neuroscience, hydrology, sensor networks, to name a few. There is a huge literature on testing changes of a mean in a sequence of random variables, see, e.g., Basseville and Nikiforov [1], Csörgo and Horváth [7], Brodsky and Darkhovsky [4], Chen and Gupta [5] for some basics on various methods and models. Previous research related to changed segment type models has been done by Levin and Kline [18], Commenges, Seal and Pinatel [6], Yao [29], Gombay [13], Ramanayake and Gupta [27], Račkauskas and Suquet [25], Enikeeva et al. [8], Račkauskas and Wendler [26], to name a few. Of course, formally the changed segment type alternative may be viewed as a special case of multiple change points model. But tests designed to a special type could be more efficient, see Gombay [13] for an example.

In this paper we assume that X is regularly varying with index $a > 0$, denoted $X \in \mathrm{RV}_a$, in the sense that the tail balance condition

$$P(X > x) \sim px^{-a}L(x) \quad \text{and} \quad P(X \leq -x) \sim qx^{-a}L(x), \quad x \to \infty, \tag{4}$$

is satisfied, where L is a slowly varying function, and $p, q \in (0, 1)$, $p + q = 1$.

We refer to [3] for basics on regular variation of random variables. We denote by a_n the $1/n$ quantile of $|X|$, that is

$$a_n := \inf\{x \in \mathbb{R} : P(|X| \leq x) \geq 1 - 1/n\}. \tag{5}$$

It is well known that $a_n = n^{1/a} L_1(n)$ for some slowly varying function L_1.

The test statistics constructed in this paper are motivated by asymptotic properties of the moving sums

$$S_{k,\ell} = Y_{n,k+1} + \cdots + Y_{n,k+\ell}, \quad \ell = 1, 2, \ldots, n; \quad k = 0, 1, \ldots, n-\ell,$$

with weight ℓ^γ, that is

$$T_n^{(\gamma)} := \max_{1 \le \ell \le n} \ell^{-\gamma} \max_{0 \le k \le n-\ell} |S_{k,\ell}| \,. \tag{6}$$

Statistics $T_n^{(\gamma)}$ has been investigated in [25] for $\gamma \le 1/2 - 1/a$ and in [21] for $\gamma > 1/2 - 1/a$. The choice of the parameter γ, constrained by the tail behavior of X, help to find the minimal detectable length of changed segment.

Assuming $n = 4m$, we divide the sample $Y_{n1}, \ldots, Y_{nn}$ into four equal parts and introduce for $\gamma \ge 0$ the maximal ratio statistics

$$\mathrm{MR}_{\gamma,n} := \max \left\{ \frac{T_{n1}}{T_{n3}}, \frac{T_{n3}}{T_{n1}}, \frac{T_{n2}}{T_{n4}}, \frac{T_{n4}}{T_{n2}} \right\}, \tag{7}$$

where

$$T_{nj} = T_{nj}^{(\gamma)} := \max_{1 \le \ell \le m} \ell^{-\gamma} \max_{(j-1)m \le k \le jm-\ell} |S_{k,\ell}| \,, \quad j = 1, \ldots, 4.$$

The idea behind this definition is that under alternative (3) if $\ell^* < n/4$, then the changed segment appears either in one quarter or in two adjacent quarters of the sample only. Then, at least one of the arguments of the max in (7) should be significantly bigger than under H_0. Not surprisingly, we shall see that under H_0, the asymptotic behavior of $\mathrm{MR}_{\gamma,n}$ is determined by the one of $T_n^{(\gamma)}$. The main advantage of $\mathrm{MR}_{\gamma,n}$ is its invariance under any common normalization of the T_{nj}'s by homogeneity. In particular, we do not need to compute or estimate a_n or the standard deviation of X. Another interesting feature is that the independence of the four blocks (under H_0 with i.i.d. X_i's) helps for the explicit computation of the limiting distribution. This can be somewhat extended beyond the i.i.d. setting since, for the linear processes considered in Th. 3, the limiting distribution "forgets" the filter.

Before going on, we have to mention that (7) should be considered as an informal definition of $\mathrm{MR}_{\gamma,n}$ because we deliberately forgot to put an exception in formula for the case where one of the T_{nj} vanishes, which may happen with positive probability for each n if the distribution of X is not continuous.

Section 2 exposes and comments our main theoretical results. In Th. 1, we obtain the limiting distribution under H_0 of $\mathrm{MR}_{\gamma,n}$. The situation is conveniently pictured by a two phases diagram (in the coordinates system (a, γ)). Above the arc of hyperbola with equation $\gamma = 1/2 - 1/a$ (in the "Fréchet phase"), the limiting distribution is related to Fréchet distribution with a very simple formula for the distribution function, while below (in the "Brownian phase"), the limiting distribution is the one of some functional of the Brownian

motion and computable (up to now) only approximately by Monte-Carlo methods. On the boundary between the two phases, that is when $\gamma = 1/2 - 1/a$, the asymptotic behavior in distribution of $\mathrm{MR}_{\gamma,n}$ depends on the slowly varying function L in (4).

In view of practical applications, e.g., in times series context, it is of some interest to investigate the extension of Th. 1 to dependent variables. For the Brownian phase, we prove in Th. 2 that such an extension is valid as soon as $(X_k)_{k\geq 1}$ satisfies a Hölderian functional central limit theorem. This provides a large anthology of weak dependence settings for the extension. For the Fréchet phase, the only results known up to now for the asymptotic distribution of $T_n^{(\gamma)}$ are due to Mikosch and Moser [20]. We obtain in Th. 3 the asymptotic distribution of $\mathrm{MR}_{\gamma,n}$ under H_0 for the $\mathrm{MA}(q)$ model and some regularly varying linear processes investigated in [20]. In both cases, it is noticeable that the limiting distribution is the same as for the i.i.d. setting.

Th. 4 is devoted to the consistency under alternative H_A. It relies on the hypothesis that for some normalizing sequence $(c_n)_{n\geq 1}$, $c_n^{-1} T_n^{(\gamma)} = O_P(1)$ under H_0. This is satisfied in the i.i.d. setting as well as in all the dependence settings involved in Th. 2 and Th. 3. It provides a sufficient condition on ℓ^* and μ_n insuring the convergence to 1 of the power of the test. If all the values of μ_n are equal to some $\mu^* = \mu^*(n)$ inside I^*, Th. 4 provides also a condition separating the (ℓ^*, μ^*) for which the test with $\mathrm{MR}_{\gamma,n}$ is asymptotically consistent.

A simulation study of the test under H_0 with maximal ratio statistics $\mathrm{MR}_{\gamma,n}$ is presented in Section 3. In the i.i.d. settings, we generate samples from the symmetrized Pareto and Log-Gamma distributions to study the effects of various parameters of the model on the empirical power of the test. This is complemented by a short study of the test in the setting of a moving average and an autoregressive processes.

In Section 4 we provide a complete picture of the asymptotic behavior of $T_n^{(\gamma)}$ which includes already known results from [21, 25] and prove some additional results for the boundary case $\gamma = 1/2 - 1/a$. Finally the proofs of theoretical results presented in Section 2 are detailed in Section 5.

2 Theoretical results

2.1 Preliminaries

As already mentioned, the concern about (7) is that the definition of RV_a does not forbid that for some $X \in \mathrm{RV}_a$, the corresponding $T_n^{(\gamma)}$ satisfies $P(T_n^{(\gamma)} = 0) > 0$ for every n. For instance, the symmetric random variable X with distribution given by $P(X = 0) = \frac{1}{3}$ and $P(X > x) = \frac{1}{3}$ if $x \in [0, 1]$, $P(X > x) = \frac{1}{3} x^{-a}$ if $x > 1$, verifies (4) with $p, q = \frac{1}{2}$ and the slowly varying (at infinity) L given by $L(x) = \frac{2}{3} x^a$ for $x \in (0, 1]$, and $L(x) = \frac{2}{3}$ for $x > 1$. Under

H_0, noticing that $T_n^{(\gamma)} = 0$ if and only if all the X_i, $i = 1, \ldots, n$, are null, we obtain, for independent X_i's distributed as X, that $P(T_n^{(\gamma)} = 0) = 3^{-n}$.

To remedy this drawback, we will redefine $\mathrm{MR}_{\gamma,n}$ with the help of the following map M. Let D be the open set of $\mathbb{R}_+^4$

$$D := \{(t_1, t_2, t_3, t_4) \in \mathbb{R}_+^4;\ t_1 t_2 t_3 t_4 \neq 0\}.$$

We denote by M the map $\mathbb{R}_+^4 \to \mathbb{R}_+$ defined by

$$M(t_1, t_2, t_3, t_4) := \begin{cases} \left(\frac{t_1}{t_3}, \frac{t_3}{t_1}, \frac{t_2}{t_4}, \frac{t_4}{t_2}\right) & \text{if } (t_1, t_2, t_3, t_4) \in D, \\ 0 & \text{if } (t_1, \ldots, t_4) \in \mathbb{R}_+^4 \setminus D. \end{cases} \tag{8}$$

M is continuous on D and the set of its discontinuities is the intersection of $\mathbb{R}_+^4$ with the union of the four hyperplanes with equation $t_j = 0$, so is of null Lebesgue measure in $\mathbb{R}_+^4$. Therefore, M preserves the weak convergence of a sequence $(V_n)_{n\geq 1}$ of random vectors in $\mathbb{R}_+^4$ provided that its limit V satisfies $P(V \notin D) = 0$, see Corollary 1 to Th. 5.1 on p. 31 in [2]. Throughout the paper we will refer to this property as "measurable mapping". Another important property of M is homogeneity, that is $M(ct_1, ct_2, ct_3, ct_4) = M(t_1, t_2, t_3, t_4)$ for any $c > 0$.

Now we redefine $\mathrm{MR}_{\gamma,n}$ by

$$\mathrm{MR}_{\gamma,n} := M\left(T_{n1}, T_{n2}, T_{n3}, T_{n4}\right), \tag{9}$$

with the same T_{ni}'s as in (7).

Next, we gather some notations used in the sequel of the paper. For any sequence $(Z_i)_{i\geq 1}$, we introduce for integers $0 \leq u < v$, the generic notation

$$T^{(\gamma)}(u, v, Z) := \max_{1\leq \ell \leq v-u} \ell^{-\gamma} \max_{u \leq k \leq v-\ell} \left| \sum_{i=k+1}^{k+\ell} Z_i \right|, \tag{10}$$

with the following special abbreviations (n, m, j, q being integers, $1 \leq j \leq 4$)

$$\begin{aligned} T_n^{(\gamma)}(Z) &:= T^{(\gamma)}(0, n, Z), && n \geq 1, \\ T_{n,j}^{(\gamma)}(Z) &:= T^{(\gamma)}((j-1)m, jm, Z), && n = 4m, m \geq 1, \\ T_{n,j,q}^{(\gamma)}(Z) &:= T^{(\gamma)}((j-1)m, jm - q, Z), && n = 4m, m \geq 1, q > 1. \end{aligned} \tag{11}$$

In particular, $T_n^{(\gamma)}(X)$ is equal to $T_n^{(\gamma)}$ under H_0. It is worth noticing that when applied with a finite deterministic sequence $(x_1, \ldots, x_n)$, $T_n^{(\gamma)}(x)$ induces a norm on $\mathbb{R}^n$.

2.2 Asymptotic behavior of $\mathrm{MR}_{\gamma,n}$ under H_0

In the first theorem we consider the case of independent random variables, whereas Th. 2 and Th. 3 are some extensions under dependence.

Theorem 1. *Assume for model* (1) *that* $(X_k)_{k\geq 1}$ *are i.i.d., in* RV_a *with exponent* $a > 1$. *The following statements hold under* (H_0)*:*

a) if $\gamma > \max\{0, 1/2 - 1/a\}$,

$$\lim_{n\to\infty} P(\mathrm{MR}_{\gamma,n} > x) = \begin{cases} 1 & \text{if } x < 1, \\ 4x^a(1+x^a)^{-2} & \text{if } x \geq 1; \end{cases} \tag{12}$$

b) if $a > 2$ *and* $0 \leq \gamma < 1/2 - 1/a$,

$$\lim_{n\to\infty} P(\mathrm{MR}_{\gamma,n} > x) = \begin{cases} 1 & \text{if } x < 1 \\ 1 - \left[1 - 2\int_0^\infty F_\gamma(xy)\,\mathrm{d}F_\gamma(y)\right]^2 & \text{if } x \geq 1, \end{cases} \tag{13}$$

where

$$F_\gamma(y) = P\left(\sup_{0\leq s<t\leq 1} \frac{|W(t) - W(s)|}{|t-s|^\gamma} \leq y\right), \quad y > 0, \tag{14}$$

is the distribution function of the γ*-Hölder norm of a standard Wiener process* $W = (W_t, t \in [0,1])$*;*

c) If $a > 2$ *and* $\gamma = 1/2 - 1/a$, *then* (12) *holds provided*

$$\lim_{t\to\infty} t^a P(|X| > t) = \infty, \tag{15}$$

whereas (13) *holds provided*

$$\lim_{t\to\infty} t^a P(|X| > t) = 0; \tag{16}$$

d) If $a > 2$, $\gamma = 1/2 - 1/a$ *and*

$$\sup_{t>0} t^a P(|X| > t) < \infty,$$

then $\mathrm{MR}_{\gamma,n} = O_P(1)$ *as* $n \to \infty$.

Now, we discuss the extension of Th. 1 to various dependence settings. Let us start with the extension for the Brownian phase, i.e., the convergence (13), assuming $a > 2$ and $0 \leq \gamma \leq 1/2 - 1/a$. The relevant theoretical tool is the Hölderian functional central limit theorem introduced hereafter.

We assume throughout that $\mathsf{E}\, X_k = 0$, $k \geq 1$. With the usual convention $\sum_{k\in\emptyset} := 0$, we denote by $\xi_n = (\xi_n(t), t \in [0,1])$ the polygonal partial sums process defined by

$$\xi_n(t) := \sum_{k=1}^{[nt]} X_k + (nt - [nt])X_{[nt]+1}, \quad t \in [0,1],$$

that is the random polygonal line with vertices $(k/n, \sum_{i=1}^{k} X_i)$, $0 \leq k \leq n$. In what follows, for any $0 < \gamma < 1$ and any segment $[b, c]$ of $\mathbb{R}$, we denote by $\mathcal{H}_\gamma^o[b, c]$ the space of continuous functions f on $[b, c]$ such that

$$\omega_\gamma(f, \delta) := \sup_{\substack{0<t-s\leq\delta \\ s,t\in[b,c]}} \frac{|f(t) - f(s)|}{(t-s)^\gamma} \xrightarrow[\delta\to 0]{} 0.$$

Endowed with the norm $\|f\|_\gamma := |f(b)| + \omega_\gamma(f, c-b)$, $\mathcal{H}_\gamma^o[b, c]$ is a separable Banach space.

We say that $(X_k)_{k\geq 1}$ satisfies the Hölderian functional central limit theorem with exponent γ, in short HFCLT(γ), if for some positive constant $\sigma > 0$, $n^{-1/2}\sigma^{-1}\xi_n$ converges in distribution in $\mathcal{H}_\gamma^o[0, 1]$ to a standard Brownian motion W. This property is possible only for $\gamma < 1/2$ because of the Hölderian regularity of W. The first HFCLT for the i.i.d. case goes back to Lamperti [17]. The final result for this case is: $(X_k)_{k\geq 1}$ satisfies the HFCLT(γ) if and only if $\lim_{t\to\infty} t^a P(|X_1| > t) = 0$, where $1/a = 1/2 - \gamma$, [23, 24]. Several extensions of the HFCLT are known in the setting of dependent X_k's under a moment condition, possibly stronger than above, e.g., $\mathsf{E}\,|X_1|^{a+\varepsilon} < \infty$, and an adequate behavior of some "dependence rate". First results of this type may be found in Hamadouche [14] for strong mixing or associated $(X_k)_{k\geq 1}$. The case of linear processes with short memory is treated in [16]. Nearly non stationary AR(1) case is studied in [19]. HFCLT for strictly stationary martingale difference sequences, stationary strong mixing, ρ-mixing, τ-dependent sequences are investigated by Giraudo [10–12]. A general method to prove an HFCLT is provided by Th. 3 and Cor. 4 in [16].

Now we are in a position to provide the key to the extension of the convergence (13) to the case of dependent X_k's. It is worth noticing that in the following theorem, the assumption of regular variation is not needed.

Theorem 2. *Assume that the sequence of random variables $(X_k)_{k\geq 1}$ satisfies the* HFCLT(γ) *for some $0 < \gamma < \frac{1}{2}$. Then* $\mathrm{MR}_{\gamma,n}$ *verifies the convergence* (13).

When the limiting distribution of the ratio statistics $\mathrm{MR}_{\gamma,n}$ is governed by the Fréchet distribution instead of the HFCLT, the possibility of extensions of Th. 1 to dependence setting seem to be narrower than in Th. 2. This is not surprising since the investigation of the limiting distribution of $T_n^{(\gamma)}$ under dependence is more recent. The first results in this direction and to our best knowledge the only ones up to now, are due to Mikosch and Moser [20] and concern in particular the case of finite moving average MA(q) or more generally linear processes built on an i.i.d. regularly varying noise sequence, which inherit the regular variation property. The authors also treat the case of strictly stationary processes (X_t) with multiplicative noise, that is of the form $X_t = \sigma_t Z_t$ where (Z_t) is an i.i.d. sequence, (σ_t) a strictly stationary sequence of non-negative random variables such that σ_t and Z_t are independent for each t, including the cases of GARCH processes and some stochastic volatility models. In what follows, we present an extension of Th. 1 (a) to the linear processes considered in [20].

Let $(X_t, t \in \mathbb{Z})$ be the linear process defined by

$$X_t = \sum_{j=1}^{\infty} \psi_j \epsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{17}$$

where $(\epsilon_t)_{t\in\mathbb{Z}}$ is an i.i.d. zero mean noise sequence of regularly varying random variables with index $a > 1$.

Recall that a sufficient condition in order that the series in (17) converges almost surely and $X_t \in \mathrm{RV}_a$, see [22, Lemma A.3] or [20, Lemma 3.1], is

$$\sum_{j=0}^{\infty} |\psi_i|^p < \infty, \quad \text{where} \quad p = \begin{cases} 2 & \text{for } a > 2, \\ a - \delta & \text{for some } \delta > 0, \text{ for } a \leq 2. \end{cases} \tag{18}$$

Theorem 3. *Under the null hypothesis* (2) *with* $\gamma > \max(0, 1/2 - 1/a)$*, the following hold:*

i) if (X_t) *is a moving average of order* $q > 1$*, that is* $\psi_q \neq 0$ *and for every* $j > q$*,* $\psi_j = 0$*, then convergence* (12) *is verified;*

ii) if $(X_t)_{t\in\mathbb{Z}}$ *is the linear process defined by* (17) *with the coefficients* ψ_j *satisfying*

$$\sum_{j=1}^{\infty} |\psi_j| < \infty, \tag{19}$$

then the convergence (12) *is verified.*

We notice that for $a > 1$, (19) implies (18), so the series in (17) converges almost surely and $X_t \in \mathrm{RV}_a$.

2.3 Asymptotic consistency

Under alternative hypothesis the following consistency type result holds. It is worth noticing here that this consistency theorem is valid in the i.i.d. setting of Th. 1 as well as under the dependence settings considered in Th. 2 and Th. 3. For the notation $T_n^{(\gamma)}(X)$, see the abbreviations (11).

Theorem 4. *Let* $\gamma \geq 0$*. Assume* $X \in \mathrm{RV}_a$*,* $a > 1$*. Then for model* (1)*, under alternative hypothesis* (3)*, with* $\ell^* < n/4$*, the following hold:*

i) if for some normalizing sequence $(c_n)_{n\geq 1}$*,* $c_n^{-1} T_n^{(\gamma)}(X) = O_P(1)$ *and*

$$\frac{1}{\ell^{*\gamma} c_n} \left| \sum_{i \in I^*} \mu_{ni} \right| \xrightarrow[n\to\infty]{} \infty, \tag{20}$$

then

$$\mathrm{MR}_{n,\gamma} \xrightarrow[n\to\infty]{\mathrm{P}} \infty, \tag{21}$$

so the test is asymptotically consistent.
In the case where all the μ_{ni} *for* $i \in I^*$ *are equal to some* $\mu^* = \mu^*(n)$*, that is* $\mu = \mu^* \mathbb{1}_{I^*}$*,* (20) *may be replaced by*

$$c_n^{-1} \ell^{*(1-\gamma)} \mu^* \xrightarrow[n\to\infty]{} \infty; \tag{22}$$

ii) assume that $\mu = \mu^* \mathbb{1}_{I^*}$ *and*

$$V_n := c_n^{-1} \left(T_{n1}^{(\gamma)}(X), \ldots, T_{n4}^{(\gamma)}(X) \right) \xrightarrow[n\to\infty]{\mathcal{D}} V, \text{ with } P(V \in D) = 1. \quad (23)$$

If, moreover,

$$c_n^{-1} \ell^{*(1-\gamma)} \mu^* \xrightarrow[n\to\infty]{} 0, \quad (24)$$

then $\mathrm{MR}_{\gamma,n} = O_P(1)$*, so the test is not asymptotically consistent.*

Some remarks on the performance of the test are in order here. To simplify the discussion, we restrict ourselves to the case where all μ_{ni} are equal to the same $\mu^* = \mu^*(n) \neq 0$ for $i \in I^*$ and the X_k's are i.i.d.. An examination of the assumptions in Th. 1 show that in each case, $c_n^{-1} T_n^{(\gamma)}(X) = O_P(1)$ with

$$c_n = \max(a_n, n^{1/2-\gamma}). \quad (25)$$

Then, under the assumptions of Th.4,

$$\mathrm{MR}_{\gamma,n} \begin{cases} \xrightarrow[n\to\infty]{\mathrm{P}} \infty & \text{if } |\mu^*| \ell^{*1-\gamma} c_n^{-1} \to \infty, \\ = O_P(1) & \text{if } |\mu^*| \ell^{*1-\gamma} c_n^{-1} \to 0. \end{cases} \quad (26)$$

In practice, one is interested in the ability of the test to detect an epidemics when it is present. Intuitively, this detectability increases with the length ℓ^* and the amplitude $|\mu^*|$.

But even an epidemics of length 1 is detectable if the corresponding amplitude is significantly higher than $\max_{1\le i\le n} |X_i|$. This is clear when $c_n = a_n$ for n large since then, under the choices of γ by Th. 1, (26) provides the condition $|\mu^*| a_n^{-1} \to \infty$ and we know that $\max_{1\le i\le n} |X_i| = O_P(a_n)$ when $X_1 \in \mathrm{RV}_a$. With the choices of γ allowed by the case where $c_n = n^{1/2-\gamma}$ for n large, this detection is possible only with $\gamma = 1/2 - 1/a$ because then (26) provides the condition $|\mu^*| n^{-1/2+\gamma} = |\mu^*| n^{-1/a} \to \infty$, and when $\sup_{x\ge 1} L(x) < \infty$, $a_n = O(n^{1/a})$. Clearly, the same detectability conditions hold for an epidemic of length ℓ^* not depending on n.

On the opposite side, epidemics with a small amplitude $|\mu^*|$ should require to be detected larger lengths and this can be quantified by (26). To fix the idea, let us focus on the special case where $|\mu^*| > 0$ is constant (i.e., not depending on n). In this setting, the detectability condition provided by (26) becomes

$$l^{*1-\gamma} c_n^{-1} \xrightarrow[n\to\infty]{} \infty.$$

Let us remark at once that the choice $\gamma \ge 1$ has to be discarded here because it implies $l^{*1-\gamma} c_n^{-1} \to 0$. Now, it is easily seen that the detectability condition when $c_n = a_n$ ultimately reduces to

$$a_n^{1/(1-\gamma)} = o(\ell^*) \quad \text{with} \quad \begin{cases} 0 < \gamma < 1 & \text{if } 1 < a \le 2, \\ \frac{1}{2} - \frac{1}{a} < \gamma < 1 & \text{if } a > 2, \\ \gamma = \frac{1}{2} - \frac{1}{a} & \text{if } a > 2 \text{ and } \lim_{x\to\infty} L(x) = \infty. \end{cases}$$

In particular, when $a > 2$, the minimal length detectable when $c_n = a_n$ ultimately is obtained from the third subcase above and satisfies

$$a_n^{\frac{2a}{2+a}} = n^{\frac{2}{2+a}} L_1(n)^{\frac{2a}{2+a}} = o(\ell^*),$$

noticing that when $\lim_{x\to\infty} L(x) = \infty$, $L_1(n) \to \infty$.

Next, it is easily seen that the detectability condition when $c_n = n^{1/2-\gamma}$ ultimately writes

$$n^{\frac{1-2\gamma}{2-2\gamma}} = o(\ell^*).$$

Noticing that $\frac{1-2\gamma}{2-2\gamma}$ is a decreasing function of γ on $[0, 1/2 - 1/a]$, we can conclude that the minimal length detectable for the choices of γ allowed when $c_n = n^{1/2-\gamma}$ ultimately satisfies

$$n^{\frac{2}{2+a}} = o(\ell^*).$$

The simulations presented in Section 3 show that when $a > 2$, the empirical power of the test is maximal for γ close to $1/2 - 1/a$. This observation is in accordance with the above result which shows that when $a > 2$, the minimal detectable length ℓ^*_γ which separates consistency and non-consistency of the test is obtained for $\gamma = 1/2 - 1/a$. Indeed, for a given length ℓ^* of epidemics in the simulated sample, the probability of rejection of H_0 increases with the ratio ℓ^*/ℓ^*_γ.

Consider now the general case where both ℓ^* and μ^* vary with n. It is convenient here to see the epidemics as a rectangular signal hidden in a noise. We can then choose a function $\tau(\mu^*, \ell^*)$ non-decreasing in each of its arguments and try to interpret the detectability condition in terms of τ. Particularly, we can choose some norm of the rectangular signal. The L^1 and L^2 norms correspond to $\tau_1(\mu^*, \ell^*) = \ell^*|\mu^*|$ and $\tau_2(\mu^*, \ell^*) = \sqrt{\ell^*}|\mu^*|$, respectively. The latter is the square root of the energy of the signal. For this one, the minimal value of τ_2 provided by (26) for the detection of the signal depends essentially on a non-increasing function of the relative length of the signal ℓ^*/n, namely,

$$\sqrt{\ell^*}|\mu^*| \asymp \max(L_1(n), 1)\Big(\frac{n}{\ell^*}\Big)^{1/a},$$

obtained with $\gamma = 1/2 - 1/a$. In [9] a similar problem of testing the presence of a constant change in mean within an interval of a given size was considered for the model with independent Gaussian noise.

Here, the following condition that separates the detectable signals with mean μ_n from the non-detectable ones was obtained by using the minimax approach,

$$\sqrt{l^*}|\mu_n| \asymp \sqrt{2}\sigma\sqrt{\log\frac{n}{l^*}}, \quad l^*/n \to 0, \quad n \to \infty,$$

where σ is the noise variance. To obtain a final result of this type for noise in RV_a remains an open question. One may remark, nevertheless, that it is possible to reach some logarithmic rate for the function of relative length of the signal with our ratio statistics if we can afford some excursion outside the

framework of regular variation of the X_k's. This can be done like in [25] by using the Hölderian functional central limit theorem with generalized weight function ρ, see [24], assuming some exponential moments for the noise. To this end, we replace $T_n^{(\gamma)}$ by $T_n^{(\rho)} := \max_{1\le\ell\le n} \rho(\ell/n)^{-1} \max_{0\le k\le n-\ell} |S_{k,\ell}|$. Here under H_0, $T_n^{(\rho)} = O_P(n^{1/2})$. A simple adaptation of the proof of Th. 4 leads to the condition $d_n := \ell^*\mu^* n^{-1/2}\rho(\ell^*/n)^{-1} \to \infty$ under an adequate assumption on the tail of X_1. In particular, if for some $\beta > 1/2$ and every positive λ, $\mathsf{E}\exp(\lambda|X_1|^{1/\beta}) < \infty$, then one can choose $\rho(h) = h^{1/2}|\log(ch)|^\beta$ for some constant $c > 0$ and this leads to the detection sufficient condition

$$\sqrt{\ell^*}|\mu^*| = d_n \log^\beta\left(c\frac{n}{\ell^*}\right), \quad \text{with } d_n \to \infty.$$

3 Tests with the maximal ratio statistics

3.1 Critical values

In this section, we consider model (1), where $X \in \mathrm{RV}_a$ with $a > 1$. The null hypothesis H_0 is rejected in favour of the alternative H_A when

$$\mathrm{MR}_{\gamma,n} \ge B_\alpha, \tag{27}$$

where the asymptotic critical value B_α has the preassigned asymptotic level of significance α. Due to Th. 1, $B_\alpha > 1$ is determined by the equation

$$4B_\alpha^a(1 + B_\alpha^a)^{-2} = \alpha \tag{28}$$

if $\gamma > \max\{0, 1/2 - 1/a\}$ and by the equation

$$\int_0^\infty F_\gamma(B_\alpha y)\,\mathrm{d}F_\gamma(y) = \frac{1}{2} - \frac{1}{2}\sqrt{1-\alpha} \tag{29}$$

if $a > 2$ and $0 \le \gamma < 1/2 - 1/a$, where the function F_γ is defined in (14). The distribution of F_γ was evaluated on a grid of size 1000 and by running Monte-Carlo simulations with 400 runs. The results are presented in Table 5 given in the Appendix.

If one takes $\gamma = 1/2 - 1/a$, then the choice of the critical value B_α depends on the either of conditions (15), (16).

From equation (28) we find

$$B_\alpha := \left[\alpha^{-1}(2 - \alpha + 2\sqrt{1-\alpha})\right]^{1/a}.$$

Hence, in the case $\gamma > \max\{0, 1/2 - 1/a\}$, B_α depends on γ only indirectly. If $1 < a \le 2$, then the test can be written as

$$\sup_{0<\gamma\le 1} \mathrm{MR}_{\gamma,n} \ge \left[\alpha^{-1}(2 - \alpha + 2\sqrt{1-\alpha})\right]^{1/a}.$$

And if $a > 2$, then the test can be written as

$$\sup_{1/2-1/a<\gamma\leq 1} \mathrm{MR}_{\gamma,n} \geq \left[\alpha^{-1}(2-\alpha+2\sqrt{1-\alpha})\right]^{1/a}.$$

To find approximate values of B_α satisfying (29), we proceed as follows. We consider i.i.d. standard normal random variables $Z_1, Z_2, \ldots, Z_N$ with $N = 4m$ and define $T_{Nj}^{(\gamma)}(Z)$, $1 \leq j \leq 4$ as in the abbreviations (11). Put also

$$Q_N(x) := P\left(M\left(T_{N1}^{(\gamma)}(Z), \ldots, T_{N4}^{(\gamma)}(Z)\right) > x\right), \quad x > 0.$$

Then the limit tail distribution of $\mathrm{MR}_{\gamma,n}$ is $\lim_{N\to\infty} Q_N(x)$, $x > 0$.

We note in passing that by absolute continuity of the distribution of the Z_i, $T_{Nj}^{(\gamma)}(Z) > 0$ a.s. for $j = 1, \ldots, 4$, so $M\left(T_{N1}^{(\gamma)}(Z), \ldots, T_{N4}^{(\gamma)}(Z)\right) \geq 1$ almost surely, whence $Q_N(x) = 1$ for $0 < x < 1$ and the same holds for its limit. By independence and identical distributions,

$$1 - Q_N(x) = \left(1 - P\left(\max\left\{\frac{T_{N1}^{(\gamma)}(Z)}{T_{N2}^{(\gamma)}(Z)}, \frac{T_{N2}^{(\gamma)}(Z)}{T_{N1}^{(\gamma)}(Z)}\right\} > x\right)\right)^2.$$

For $x \geq 1$, the events $\{T_{N1}^{(\gamma)}(Z)/T_{N2}^{(\gamma)}(Z) > x\}$ and $\{T_{N2}^{(\gamma)}(Z)/T_{N1}^{(\gamma)}(Z) > x\}$ are disjoint, hence

$$P\left(\max\left\{\frac{T_{N1}^{(\gamma)}(Z)}{T_{N2}^{(\gamma)}(Z)}, \frac{T_{N2}^{(\gamma)}(Z)}{T_{N1}^{(\gamma)}(Z)}\right\} > x\right) = P\left(\frac{T_{N1}^{(\gamma)}(Z)}{T_{N2}^{(\gamma)}(Z)} > x\right) + P\left(\frac{T_{N2}^{(\gamma)}(Z)}{T_{N1}^{(\gamma)}(Z)} > x\right).$$

By identical distribution of $\left(T_{N1}^{(\gamma)}(Z), T_{N2}^{(\gamma)}(Z)\right)$ and $\left(T_{N2}^{(\gamma)}(Z), T_{N1}^{(\gamma)}(Z)\right)$,

$$Q_N(x) = 1 - \left(1 - 2P\left(\frac{T_{N1}^{(\gamma)}(Z)}{T_{N2}^{(\gamma)}(Z)} > x\right)\right)^2, \quad x \geq 1. \tag{30}$$

Hence, we find approximate asymptotic critical values $B_\alpha = B_\alpha(\gamma)$ by solving

$$P\left(Z_N^{(\gamma)} > B_\alpha(\gamma)\right) = \frac{1}{2}\left(1 - \sqrt{1-\alpha}\right),$$

where $Z_N^{(\gamma)} := T_{N1}^{(\gamma)}(Z)/T_{N2}^{(\gamma)}(Z)$.

The distribution of $Z_N^{(\gamma)}$ was evaluated on a grid of size 10000 and by running a Monte-Carlo simulation with 3000 runs. The results are presented in Table 6 given in the Appendix.

3.2 Simulation study: independent random variables

To illustrate the finite sample behavior of test (27) with independent random variables, we conduct an experiment using Monte-Carlo simulations. We use the symmetrized Pareto and Log-Gamma distributions to generate X regularly varying with index a. Data are generated by the model $Y_{nk} = \mu_n^* \mathbb{1}_{\{k^*+1,\cdots,k^*+\ell^*\}}(k) + X_k$.

To investigate the effect of test (27) on various alternative conditions we consider samples of sizes $n = 200, 300, 600, 1000$ with various relative lengths of changed mean segment ℓ^*/n, adapting the parameter γ to match the various cases in Th. 1. The power of the test is estimated with 1000 repetitions. The nominal size was chosen as $\alpha = 0.05$, 0.1.

3.2.1 Performance of the test with varying parameters of tail index a, amplitude μ_n^* and relative length ℓ^*/n

First, to illustrate the effect of the tail index a on empirical power scores, we take only two values of the parameter of regular variation, $a = 2.5$ and $a = 5$, since clearly, bigger values of a cause higher empirical power of test (7). Figure 1 illustrates the empirical power in setting (13) for different values of a for the samples generated from Pareto distribution with $n = 300$, $\ell^*/n = j/30, j = 1, ..., 7, \mu_n^* = 0.2c, c = 1, ..., 10$ and $\alpha = 0.05$.

More precisely, for the values $\gamma = 0.09$ and $\gamma = 0.29$, Figure 1 depicts the variation of the empirical power as a function of one of the parameters amplitude μ_n^* and relative length ℓ^*/n for several fixed values of the other. The choice of the parameter γ in this case is made arbitrarily since under setting (13) the requirement for γ to be less than the threshold $1/2 - 1/a$ is fulfilled. From both types of figures it can be seen that the greater the index a, the amplitude of the change μ_n^* or the length of the changed segment ℓ^*, the bigger the empirical power of the test when fixing any two of these three parameters. The intuitive explanation is as follows (see also the comments after Th. 4):

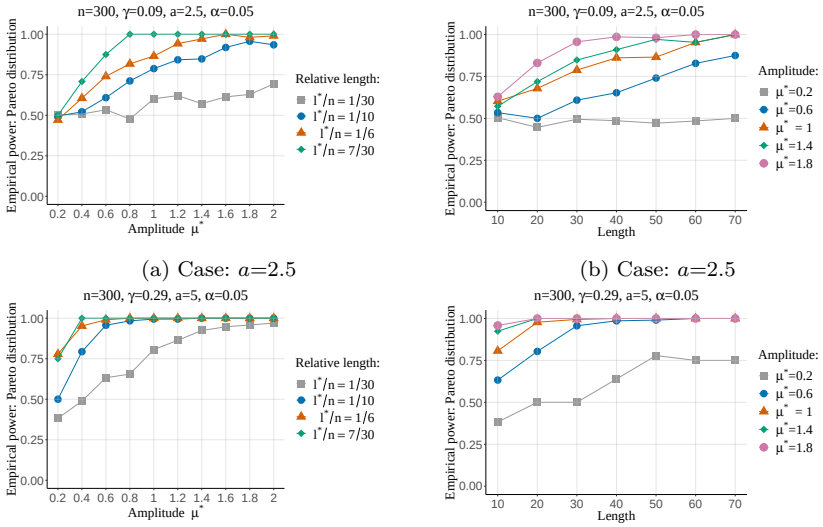
- tail parameter a: the lower the index a, the heavier tail the generated sample. Consequently, it is complicated for the test to distinguish epidemic periods when there are many large values in the sample, thus the power of the test may be naturally lower;
- the amplitude of the change μ_n^*: anew, the effect on the test of μ_n^* is the same as with parameter a: a lower mean value may indicate a lower power of the test (7);
- the length of the changed segment ℓ^*: it is also clear in this case that a shorter length of the changed segment may reduce the test performance.

Of course the three parameters can offset each other in favor of the test power. For example, although the amplitude μ_n^* is small and the length of the changed segment ℓ^* is also short, the test power improves significantly with a larger parameter a. Figure 5 in the Appendix represents the latter simulation,

but with a significance level $\alpha = 0.1$, from which it can be seen that all the findings are valid in this case as well, but test (7) performs condiderably better.

3.2.2 Performance of the test with varying parameter γ and sample size n

So far, only one part of the maximal ratio test with various parameters has been examined, i.e., under the setting (13). However, the performance of the test with particular realization can be controlled via the parameter γ allowing distinct limiting distributions.



(a) Case: a=2.5

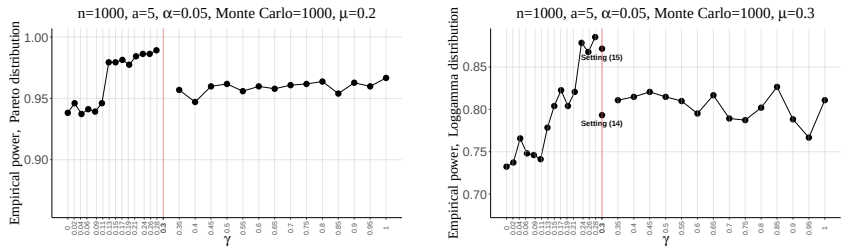
(b) Case: a=2.5

(c) Case: a=5

(d) Case: a=5

Fig. 1: Empirical power of the test under the setting (13) with $\alpha = 0.05$

Figure 2 depicts the empirical power of test (7) under various $\gamma = 0.01, 0.1, 0.2, 0.29, 0.3^*$ refer to (15), 0.3_* refer to (16), $0.31, 0.5, 0.7, 0.9$ values with tail index $a = 5$, $\alpha = 0.05$, sample size of 1000, changed mean segment of 0.2 for Pareto and 0.3 for Log-Gamma distributions, and 1000 Monte-Carlo simulations. In such a scenario, due to Th. 1, the threshold $\gamma = \max(1/2 - 1/a, 0) = \max(1/2 - 1/5, 0) = 0.3$ (red vertical line) divides the alternative into four settings: (12), (13), (15) and (16). Remarkably, the effect of γ is quite strong in the results of the empirical power of the test where it's the highest when γ is very close to $\frac{1}{2} - \frac{1}{a}$ from the left. Figure 6 in the Appendix illustrates the latter simulation with $\alpha = 0.1$ and the distinction between two limiting distribution is even more evident, when, again, the performance of the test is superior when γ is very close to $\frac{1}{2} - \frac{1}{a}$ under limiting distribution (13).



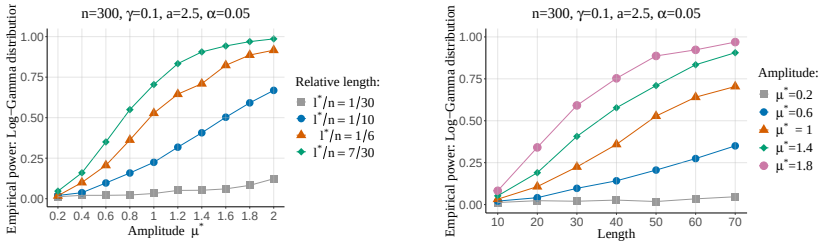
(a) Pareto distribution (b) Log-Gamma distribution

Fig. 2: Empirical power of the test under various γ values with $\alpha = 0.05$

Now, we investigate the role of the parameter γ in more depth by accounting other parameters. The detailed exploration under various settings in Th. 1 is illustrated in Figure 4 with samples of a size $n = 300$ from the Log-Gamma distribution with tail index $a = 2.5$ and $\alpha = 0.05$ under different relative lengths $\ell^*/n = j/30, j = 1, ..., 7$, and sizes of changed mean segments $\mu_n = 0.2c, c = 1, ..., 10$. In Figure 4a and 4b the depicted situation is under setting (12), in Figure 4c and 4d under setting (13), in Figure 4e and 4f under setting (15), Figure 4g and 4h under setting (16).

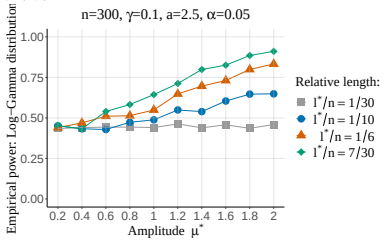
For all the settings in Th. 1, the empirical power of the test is much larger, when the length of changed mean segment is longer and the amplitude μ^* of the change is bigger. This result is straightforward. Nevertheless, in the same circumstances, the tests under setting (13) have higher empirical power than the tests under (12), (15) or (16) settings. Figure 7 in the Appendix illustrates the latter simulation with $\alpha = 0.1$.

Let us return to the tail parameter a. In reality, its true value is seldom known, so various estimators are used to evaluate a. We choosed here the well known Hill's estimator [15]. When analyzing both settings (12) and (13), the choice of γ is arbitrary. Moreover, settings (15) and (16) give the same limiting distribution as settings (12) and (13), respectively. Accordingly, the performance of the test with Hill estimator is analyzed only for the settings (15) and (16), when γ depends directly on the parameter a via $\gamma = \max(\frac{1}{2} - \frac{1}{a}, 0)$. Figure 3 portrays the empirical power of the test with evaluated parameter $\hat{a}$ from Hill estimator where samples of a size $n = 300$ are generated from Log-gamma distribution with true tail parameter $a = 2.5$. Results are estimated with $\alpha = 0.05$. Most of the time, Hill estimator is bigger than the true value of a by $\mathsf{E}\,\hat{a} = 4$ under the setting (15) (see Figure 3a and 3b). For this reason, the solution of (28), which directly depends on the parameter a, is smaller than the true critical value. For example, for $\alpha = 0.05$, the true critical value is equal to $B_{0.05} = 3.82$ and the evaluated critical value is approximately equal to $\mathsf{E}\,\hat{B}_{0.05} \approx 2.56$. Consequently, the performance of the test is slightly better when a is evaluated from Hill estimator. Figure 8 in the Appendix illustrates the latter simulation with $\alpha = 0.1$.

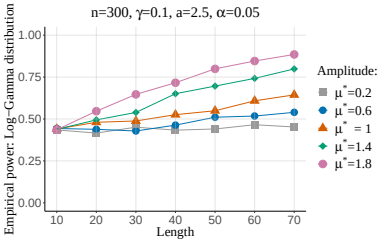


(a) Case: $\gamma = \max(1/2\text{-}1/a, 0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$

(b) Case: $\gamma = \max(1/2\text{-}1/a, 0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$



(c) Case: $\gamma = \max(1/2\text{-}1/a, 0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$

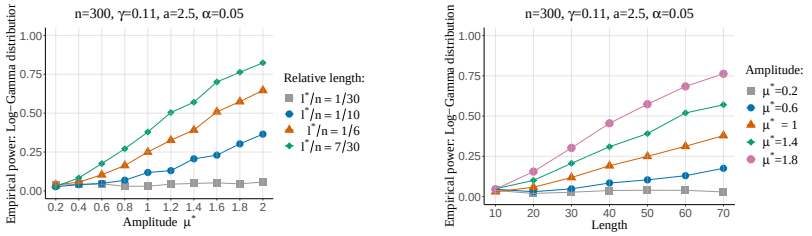


(d) Case: $\gamma = \max(1/2\text{-}1/a, 0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$

Fig. 3: Empirical power of the test under settings (15) (top row) and (16) (bottom row) when parameter a is estimated from Hill estimator with $\alpha = 0.05$

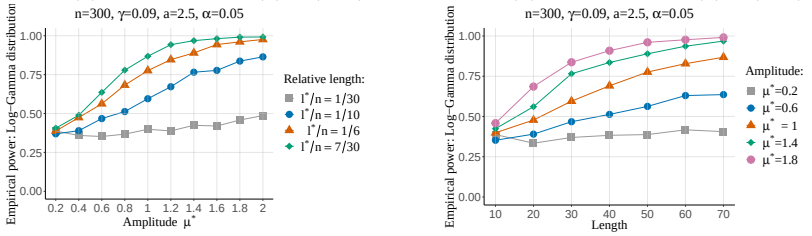
As mentioned earlier, the lower is the index a, the heavier is the tail of the generated sample. For this reason it is interesting to investigate type I errors. Figure 9 in the Appendix illustrates false positive probability of the test with Log-Gamma distribution under settings (12) and (13) with various values of th parameter a , $n = 300$, $\alpha = 0.05$.

Table 4 given in the Appendix summarizes the performance of test (7) under the Log-Gamma and Pareto distributions with regular variation index $a = 5$, sample size of $n = 200, 600, 1000$, amplitude of the change $\mu^* = 0.3$, with length ℓ^* of approximately $\frac{1}{7}, \frac{1}{5}, \frac{1}{4}$ of the sample size n. Increasing sample size, length and amplitude of the change also increase the empirical power. However, the choice of the parameter γ depends on the threshold $\frac{1}{2} - \frac{1}{a}$ where the empirical power is the highest when γ is very close to the threshold from the left, regardless of sample size or length of changed segment.



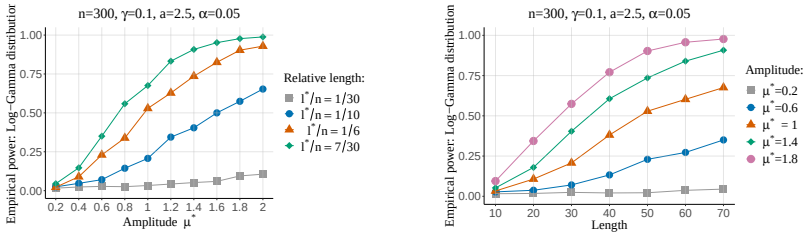
(a) Case: $\gamma > \max(1/2\text{-}1/a,0)$

(b) Case: $\gamma > \max(1/2\text{-}1/a,0)$



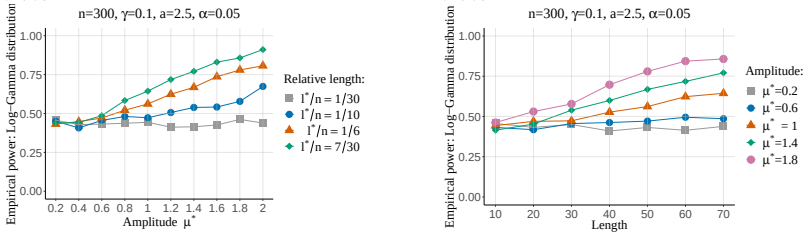
(c) Case: $0 < \gamma < \max(1/2\text{-}1/a,0)$

(d) Case: $0 < \gamma < \max(1/2\text{-}1/a,0)$



(e) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$

(f) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$



(g) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$

(h) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$

Fig. 4: Empirical power of the test under various settings with $\alpha = 0.05$. The top row corresponds to the situation under setting (12), second row corresponds to the situation under setting (13), third row corresponds to the situation under setting (15) and bottom row corresponds to the situation under setting (16).

3.3 Simulation study: dependent random variables

This section briefly investigates the performance of test (7) under some dependency settings used in Th. 2 and Th. 3 as the findings differ little from the above ones. To illustrate the finite sample behavior of test (7) we use the autoregressive and the moving average processes

$$\mathrm{AR}(p): \quad X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \epsilon_t^{(i)}; \qquad \mathrm{MA}(q): \quad X_t = \sum_{j=1}^{q} \psi_j \epsilon_{t-j}^{(i)} + \epsilon_t^{(i)},$$

where $\phi_1 = 0.5$, $\psi_1 = 0.5$ for $p, q = 1$ and $\phi_1 = 0.5$, $\phi_2 = 0.25$, $\psi_1 = 0.5$, $\psi_2 = -0.25$ for $p, q = 2$. The innovations $\epsilon_t^{(i)}$ are i.i.d. from symmetrized Pareto (denoted as $i = 1$) and symmetrized Log-gamma (denoted as $i = 2$) distributions with the parameter $a = 2.5$. For the change in mean, we consider $Y_{nt} = \mu_n^* \mathbb{1}_{\{t^*+1,\cdots,t^*+\ell^*\}}(t) + X_t$ with various values of μ_n^*.

As mentioned in Section 2, Th. 2 and Th. 3 generalizes the extension to the dependence case. The illustrations of the latter theorems are analyzed according to the parameters selected in the previous section. More precisely, Table 1 portrays the empirical power of the maximal ratio test under various alternatives corresponding to Th. 2 with $n = 300$, $\ell^*/n = j/30, j = 1, 5, 7, \mu_n^* = 0.2c, c = 1, ..., 10$ and $\alpha = 0.05$. Parameter γ is chosen to be equal to 0.09. AR(p) and MA(q) processes are generated with Pareto and Log-gamma innovations having a tail index $a = 2.5$ in both cases. First, the performance of the maximal ratio test seems superior for processes with Pareto innovations and when the order of the process is smaller for both AR(p) and MA(q). Moreover, the empirical power of an autoregressive process seems higher than with a moving average process. However, other findings are similar to the ones of independent case as the detectability of a change increases with the length ℓ^* and the amplitude μ^*.

Next, we examine the realization of maximal ratio test under conditions of Th. 3, which are shown in Table 2. Except for γ, all the parameters are the same as for the latter simulation. Our choice of γ, restricted by $\gamma > \max(0, 1/2 - 1/a)$, is here $\gamma = 0.11$. Note that $a = 2.5$. As in the independent case, the empirical power of the maximal ratio test is significantly smaller under the same conditions, but greatly improves with higher values of the length ℓ^* and the amplitude μ_n^* of the change. However, it does not reach the level specified in the setting (13). And again we come to the same conclusion that the empirical power is the highest when the alternative limiting distribution has the form (13). For comparison with i.i.d. case, see Table 3.

			AR(1)			AR(2))			MA(1)			MA(2)		
			$\ell^*=10$	$\ell^*=50$	$\ell^*=70$	$\ell^*=10$	$\ell^*=50$	$\ell^*=70$	$\ell^*=10$	$\ell^*=50$	$\ell^*=70$	$\ell^*=10$	$\ell^*=50$	$\ell^*=70$
Log-Gamma innovations	$\alpha=0.05$	$\mu_n^*=0.2$	0.4792	0.4981	0.4673	0.3854	0.3971	0.3700	0.4233	0.4038	0.4246	0.3390	0.3481	0.3998
		$\mu_n^*=0.6$	0.4536	0.5096	0.5585	0.3835	0.4990	0.5496	0.3911	0.4865	0.5665	0.3788	0.4702	0.5764
		$\mu_n^*=1$	0.5009	0.6308	0.6587	0.3873	0.6490	0.7768	0.4242	0.6500	0.7579	0.3977	0.6615	0.7808
		$\mu_n^*=1.6$	0.4886	0.7452	0.8452	0.4034	0.8375	0.9187	0.4309	0.8394	0.9097	0.4025	0.8673	0.9444
		$\mu_n^*=2$	0.5019	0.8462	0.8889	0.4337	0.9327	0.9762	0.4640	0.9019	0.9504	0.4280	0.9404	0.9841
	$\alpha=0.1$	$\mu_n^*=0.2$	0.5748	0.5529	0.5734	0.4848	0.5087	0.4980	0.5095	0.5067	0.5466	0.4508	0.4606	0.4673
		$\mu_n^*=0.6$	0.5483	0.6038	0.6121	0.4877	0.5625	0.6458	0.4953	0.5663	0.6081	0.4754	0.5615	0.6339
		$\mu_n^*=1$	0.5521	0.6962	0.7401	0.4640	0.7365	0.8264	0.5284	0.7279	0.7986	0.4782	0.7625	0.8333
		$\mu_n^*=1.6$	0.5890	0.8212	0.8780	0.4943	0.9038	0.9593	0.5133	0.8750	0.9484	0.4924	0.9096	0.9563
		$\mu_n^*=2$	0.5653	0.8798	0.9425	0.5369	0.9423	0.9831	0.5322	0.9308	0.9692	0.5417	0.9673	0.9861
Pareto innovations	$\alpha=0.05$	$\mu_n^*=0.2$	0.5701	0.5760	0.5685	0.5246	0.5029	0.5129	0.5417	0.5337	0.5764	0.4688	0.5029	0.5169
		$\mu_n^*=0.6$	0.5597	0.6635	0.6756	0.5152	0.6644	0.7113	0.5559	0.6490	0.7113	0.4773	0.6721	0.7163
		$\mu_n^*=1$	0.5748	0.7490	0.7956	0.5095	0.8240	0.8829	0.5218	0.8038	0.8681	0.4981	0.8192	0.8909
		$\mu_n^*=1.6$	0.6392	0.8423	0.9256	0.5492	0.9221	0.9722	0.5464	0.9000	0.9702	0.5786	0.9404	0.9742
		$\mu_n^*=2$	0.6136	0.8942	0.9534	0.5814	0.9731	0.9871	0.6080	0.9625	0.9772	0.5824	0.9683	0.9921
	$\alpha=0.1$	$\mu_n^*=0.2$	0.6705	0.6606	0.6409	0.6127	0.6019	0.6349	0.5862	0.6144	0.6161	0.5900	0.5827	0.6022
		$\mu_n^*=0.6$	0.6326	0.7183	0.7411	0.5786	0.7029	0.7996	0.6259	0.7192	0.7907	0.5729	0.7317	0.8056
		$\mu_n^*=1$	0.6259	0.8000	0.8482	0.6042	0.8673	0.9246	0.6354	0.8385	0.8889	0.5407	0.8606	0.9236
		$\mu_n^*=1.6$	0.6761	0.9106	0.9335	0.6487	0.9529	0.9891	0.6420	0.9442	0.9802	0.6439	0.9587	0.9841
		$\mu_n^*=2$	0.6638	0.9337	0.9782	0.6610	0.9740	0.9921	0.6723	0.9663	0.9871	0.6581	0.9798	0.9940

Table 1: Empirical power of the test under Th. 2 conditions with $a = 2.5$, $\gamma = 0.09$ and $n = 300$

			AR(1)			AR(2))			MA(1)			MA(2)		
			$\ell^*=10$	$\ell^*=50$	$\ell^*=70$	$\ell^*=10$	$\ell^*=50$	$\ell^*=70$	$\ell^*=10$	$\ell^*=50$	$\ell^*=70$	$\ell^*=10$	$\ell^*=50$	$\ell^*=70$
Log-Gamma innovations	$\alpha=0.05$	$\mu_n^*=0.2$	0.0947	0.0779	0.0833	0.0275	0.0298	0.0377	0.0511	0.0510	0.0556	0.0256	0.0288	0.0337
		$\mu_n^*=0.6$	0.0947	0.1058	0.1200	0.0398	0.0673	0.0843	0.0407	0.0769	0.1131	0.0350	0.0635	0.1111
		$\mu_n^*=1$	0.0814	0.1615	0.1786	0.0398	0.1692	0.2351	0.0473	0.1779	0.2431	0.0379	0.1625	0.2609
		$\mu_n^*=1.6$	0.0890	0.2606	0.3730	0.0388	0.3404	0.4970	0.0473	0.3087	0.4732	0.0341	0.3644	0.5804
		$\mu_n^*=2$	0.0758	0.3510	0.5020	0.0559	0.4788	0.6776	0.0587	0.4279	0.6131	0.0521	0.5154	0.7312
	$\alpha=0.1$	$\mu_n^*=0.2$	0.1723	0.1846	0.1925	0.1146	0.1000	0.1141	0.1430	0.1337	0.1458	0.1013	0.1038	0.0982
		$\mu_n^*=0.6$	0.1771	0.2077	0.2470	0.0881	0.1779	0.2341	0.1420	0.2058	0.2381	0.1089	0.1990	0.2232
		$\mu_n^*=1$	0.1705	0.2673	0.3849	0.1023	0.3135	0.4415	0.1402	0.2933	0.3780	0.1117	0.3404	0.4712
		$\mu_n^*=1.6$	0.1884	0.4163	0.5685	0.1288	0.5471	0.7123	0.1383	0.5250	0.6399	0.1288	0.5827	0.7560
		$\mu_n^*=2$	0.2112	0.5279	0.6875	0.1307	0.6837	0.8413	0.1383	0.6769	0.7788	0.1326	0.7125	0.8690
Pareto innovations	$\alpha=0.05$	$\mu_n^*=0.2$	0.1676	0.1635	0.1647	0.1042	0.1192	0.1240	0.1297	0.1337	0.1300	0.1070	0.1029	0.1240
		$\mu_n^*=0.6$	0.1439	0.1952	0.2460	0.0985	0.1827	0.2688	0.1222	0.2106	0.2728	0.0909	0.1875	0.2510
		$\mu_n^*=1$	0.1525	0.3135	0.3988	0.0975	0.3500	0.4415	0.1468	0.3029	0.4593	0.0994	0.3519	0.4762
		$\mu_n^*=1.6$	0.1515	0.4510	0.5615	0.1117	0.5673	0.7034	0.1515	0.5519	0.6806	0.1212	0.5644	0.7460
		$\mu_n^*=2$	0.1913	0.5529	0.6766	0.1392	0.6856	0.8353	0.1591	0.6442	0.8105	0.1392	0.7115	0.8581
	$\alpha=0.1$	$\mu_n^*=0.2$	0.2661	0.2750	0.3165	0.2027	0.1923	0.2431	0.2188	0.2663	0.2450	0.2140	0.2067	0.1915
		$\mu_n^*=0.6$	0.3125	0.3317	0.3869	0.2093	0.3596	0.4018	0.2528	0.3260	0.4286	0.2083	0.3567	0.4196
		$\mu_n^*=1$	0.3153	0.4779	0.5268	0.2235	0.5154	0.6339	0.2188	0.4904	0.6379	0.2311	0.5356	0.6706
		$\mu_n^*=1.6$	0.3097	0.6404	0.7371	0.2604	0.7404	0.8720	0.2680	0.7308	0.8442	0.2453	0.7615	0.8671
		$\mu_n^*=2$	0.3030	0.7067	0.8433	0.2803	0.8375	0.9296	0.2898	0.7942	0.8968	0.2718	0.8356	0.9385

Table 2: Empirical power of the test under Th. 3 conditions with $a = 2.5$, $\gamma = 0.11$ and $n = 300$

			AR(1)	AR(2)	MA(1)	MA(2)	iid
Pareto innovations	$\gamma=0.29$	$\alpha=0.05$	0.78	0.8242	0.7883	0.8192	0.8742
		$\alpha=0.1$	0.8133	0.8525	0.845	0.8533	0.9117
	$\gamma=0.31$	$\alpha=0.05$	0.6233	0.6891	0.645	0.6758	0.7425
		$\alpha=0.1$	0.7292	0.7641	0.7433	0.7633	0.8458

Table 3: Empirical power of the test under Th. 2 and Th. 3 conditions with changed mean segment $\mu_n^* = 0.5$, $a = 5$, length $\ell^* = \sqrt{n}$, where $n = 300$

4 Asymptotic of the weighted maximal increment

Now we present theoretical results about the asymptotic behavior of $T_n^{(\gamma)}$ for $\gamma \in [0, 1]$ under H_0 for i.i.d. $X, X_1, \ldots$ Throughout this section, $T_n^{(\gamma)}$=$T_n^{(\gamma)}(X)$. If $X \in \mathrm{RV}_a$ and $\gamma > \max\{0, 1/2 - 1/a\}$, the limit distribution of $T_n^{(\gamma)}$ is

found in [21]. In the following theorem we recall the result and add the case $\gamma = 1/2 - 1/a$, $a > 2$ as well.

Theorem 5. *Assume that $X, X_1, X_2, \dots$ are i.i.d. random variables, $X \in \mathrm{RV}_a$ with $a > 1$, and $\mathsf{E}\, X = 0$.*

(a) If $\gamma > \max\{0, 1/2 - 1/a\}$, then

$$\lim_{n\to\infty} P\left(a_n^{-1} T_n^{(\gamma)} \le x\right) = \exp(-x^{-a}) =: \Phi_a(x), \quad x \ge 0. \tag{31}$$

(b) If $a > 2$, $\gamma = 1/2 - 1/a$ and

$$\lim_{t\to\infty} t^a P(|X| > t) = \infty, \tag{32}$$

then (31) holds as well.

Statement (a) is proved in [21]. To prove (b) one needs to do some modifications in the proof of (a) only. For the reader's convenience, we provide all the details. The following auxiliary results are the main ingredients of the proof.

Lemma 6. *Assume that $X \in \mathrm{RV}_a$, $a > 1$ and $\mathsf{E}\, X = 0$. Let $X_1, X_2, \dots$ be independent copies of X, $S_0 = 0$, $S_k = \sum_{i=1}^k X_i$, $k \ge 1$. Then for any $\gamma \ge 0$ and $h \ge 1$,*

$$\lim_{n\to\infty} P\left(a_n^{-1} \max_{1\le\ell\le h} \ell^{-\gamma} \max_{0\le k\le n-\ell} |S_{k+\ell} - S_k| \le x\right) = \Phi_a(x), \quad x \quad > 0.$$

Lemma 7. *Let (Y_i) be a sequence of i.i.d. random variables. Then, for any $b > 0$, $\gamma \ge 0$, $h \ge 1$ and $H \le n$,*

$$P\left(\max_{h\le\ell\le H} \ell^{-\gamma} \max_{0\le k\le n-\ell} \Big|\sum_{j=k+1}^{k+\ell} Y_j\Big| > b\right) \le 2 \sum_{j=J_1}^{J_0} Q_j,$$

$$Q_j := 2^j P\Big(\max_{1\le k\le 2n2^{-j}} \Big|\sum_{j=1}^{k} Y_j\Big| > b(n2^{-j})^\gamma\Big),$$

where $J_0 = \log_2(n/h), J_1 = \log_2(n/H) + 1$ and $\log_2 x$ denotes the dyadic logarithm.

Lemma 8. *Let $X \in \mathrm{RV}_a$, $a > 1$.*

(i) For any $0 < s < a$,

$$\mathsf{E}\,|X|^s \mathbb{1}_{\{|X| > y a_n\}} \le \frac{2a}{a-s} a_n^s n^{-1} y^{s-a}, \tag{33}$$

for n large enough, uniformly in $y \in [1, \infty)$.

(ii) For any $s > a$,

$$\mathsf{E}\,|X|^s \mathbb{1}_{\{|X|\le ya_n\}} \le \frac{2a}{s-a} a_n^s n^{-1} y^{s-a}, \tag{34}$$

for n large enough, uniformly in $y \in [1,\infty)$.

First two lemmas are proved in [21] (see, Lemma 2.2 and Lemma 3.3 therein), whereas the third one is proved in [28] (see Lemma 2.4 therein).

Proof of Th. 5 (b). By Lemma 6, we only need to prove that for each $\varepsilon > 0$,

$$\lim_{h\to\infty} \limsup_{n\to\infty} Q_n(h,\varepsilon) = 0, \tag{35}$$

where

$$Q_n(h,\varepsilon) := P\Big(\max_{h\le\ell\le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} \big|S_{k+\ell} - S_k\big| > \varepsilon a_n \Big).$$

Here and throughout the proof we take $\gamma = \gamma_a = 1/2 - 1/a$. To our aim, we consider the truncated random variables

$$X_i' = X_i \mathbb{1}\{|X_i| \le h^\gamma a_n\}, \quad \widetilde{X}_i = X_i' - \mathsf{E}\,(X_i'),\ i = 1,\dots,n,$$

and the corresponding partial sums

$$S_k' = \sum_{i=1}^k X_i' \ \text{ and } \ \widetilde{S}_k = \sum_{i=1}^k \widetilde{X}_i, \quad k \ge 1$$

with $S_0' = \widetilde{S}_0 = 0$. From Lemma 7, we conclude

$$\begin{aligned} Q_n(h,\varepsilon) &\le P\Big(\max_{1\le k\le n} |X_k| \ge h^\gamma a_n \Big) + P\Big(\max_{h\le\ell\le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} |S_{k+\ell}' - S_k'| > \varepsilon a_n \Big) \\ &\le P\Big(\max_{1\le k\le n} |X_k| \ge h^\gamma a_n \Big) + 2 \sum_{j=1}^{\log_2(n/h)} 2^j Q_{nj}(h,\varepsilon), \end{aligned}$$

where

$$Q_{nj}(h,\varepsilon) := P\left(\max_{1\le k\le 2n2^{-j}} |S_k'| > \varepsilon (n2^{-j})^\gamma a_n \right).$$

Since

$$\lim_{n\to\infty} P\big(\max_{1\le k\le n} |X_k| > h^\gamma a_n \big) = 1 - \exp(-h^{-\gamma a}) \xrightarrow[h\to\infty]{} 0,$$

the proof of (35) reduces to

$$\lim_{h\to\infty} \limsup_{n\to\infty} \sum_{j=1}^{\log_2(n/h)} 2^j Q_{nj}(h,\varepsilon) = 0. \tag{36}$$

Noticing that $\mathsf{E}\,X' = \mathsf{E}\,(X - X\mathbb{1}_{\{|X|>h^\gamma a_n\}}) = -\mathsf{E}\,(X\mathbb{1}_{\{|X|>h^\gamma a_n\}})$ because $\mathsf{E}\,X = 0$, we have by Lemma 8 (i)

$$|\mathsf{E}\,(S_k')| \le k|\mathsf{E}\,(X')| \le k\mathsf{E}\,\big(|X|\mathbb{1}_{\{|X|>h^\gamma a_n\}}\big) \le k\frac{2a}{a-1} a_n n^{-1} h^{\gamma(1-a)}.$$

This yields

$$\max_{1\le k\le 2n2^{-j}} |S'_k| \le \max_{1\le k\le 2n2^{-j}} |\widetilde{S}_k| + 2n2^{-j}\frac{2a}{a-1}a_n n^{-1}h^{\gamma(1-a)}.$$

Moreover, as $j \le \log_2(n/h)$, we have $2^{-j} \le 1 \le (n2^{-j})^\gamma$ since $\gamma \ge 0$. Hence for h large enough and uniformly in $j \le \log_2(n/h)$,

$$\max_{1\le k\le 2n2^{-j}} |S'_k| \le \max_{1\le k\le 2n2^{-j}} |\widetilde{S}_k| + \frac{\varepsilon}{2}(n2^{-j})^\gamma a_n.$$

Hence for h large enough and uniformly in j such that $2^j \le n/h$,

$$Q_{nj}(h,\varepsilon) \le \widetilde{Q}_{nj}(h,\varepsilon),$$

where

$$\widetilde{Q}_{nj}(h,\varepsilon) := P\left(\max_{1\le k\le 2n2^{-j}} |\widetilde{S}_k| > (\varepsilon/2)(n2^{-j})^\gamma a_n\right).$$

Fix $p > a$. Since $(|\widetilde{S}_k|, k \ge 1)$ is a submartingale, by Doob and Markov inequalities we obtain

$$\widetilde{Q}_{nj}(h,\varepsilon) \le \left(\frac{p}{p-1}\right)^p \left(\frac{2}{\varepsilon}\right)^p \left((n2^{-j})^\gamma a_n\right)^{-p} \mathsf{E}\left|\widetilde{S}_{2n2^{-j}}\right|^p.$$

By Rosenthal inequality,

$$\mathsf{E}\left|\widetilde{S}_{2n2^{-j}}\right|^p \le c_p\Big[(n2^{-j})^{p/2}(\mathsf{E}\,(|X|^2\mathbb{1}_{\{|X|\le a_n h^\gamma\}})^{p/2} + n2^{-j}\mathsf{E}\,|X|^p\mathbb{1}_{\{|X|\le a_n h^\gamma\}}\Big],$$

where the constant $c_p > 0$ depends on p only. Applying Lemma 8, we obtain

$$\mathsf{E}\left|\widetilde{S}_{2n2^{-j}}\right|^p \le c_p\Big[(n2^{-j})^{p/2}(\mathsf{E}\,X^2)^{p/2} + n2^{-j}\frac{2a}{p-a}a_n^p n^{-1}h^{\gamma(p-a)}\Big].$$

This leads to

$$\begin{aligned}\widetilde{Q}_{nj}(h,\varepsilon) &\le C_p\left(\frac{2}{\varepsilon}\right)^p [(n2^{-j})^\gamma a_n]^{-p}\Big[(n2^{-j})^{p/2}\sigma^p + n2^{-j}\frac{2a}{p-a}a_n^p n^{-1}h^{\gamma(p-a)}\Big]\\ &= C_p\left(\frac{2}{\varepsilon}\right)^p \Big[\sigma^p a_n^{-p}(n2^{-j})^{-\gamma p+p/2} + \frac{2a}{p-a}n^{-1}h^{\gamma(p-a)}(n2^{-j})^{-\gamma p+1}\Big],\end{aligned}$$

where $C_p := (1-1/p)^{-p}c_p$ and $\sigma^2 = \mathsf{E}\,X^2$. Hence, recalling that $\gamma = \gamma_a = 1/2 - 1/a$ and $p > a$, we obtain

$$\begin{aligned}&\sum_{j=1}^{\log_2(n/h)} 2^j Q_{nj}(h,\varepsilon) \le \sum_{j=1}^{\log_2(n/h)} 2^j \widetilde{Q}_{nj}(h,\varepsilon)\\ &\le C_p\left(\frac{2}{\varepsilon}\right)^p \left[\sigma^p n^{p/a}a_n^{-p}\sum_{j=1}^{\log_2(n/h)} 2^{-(p/a-1)j} + \frac{2a}{p-a}n^{-\gamma p}h^{\gamma(p-a)}\sum_{j=1}^{\log_2(n/h)} 2^{\gamma pj}\right]\\ &\le C_p\left(\frac{2}{\varepsilon}\right)^p \left[\frac{2^{p/a}}{2^{p/a}-2}\sigma^p a_n^{-p}n^{p/a} + \frac{2^{p/2+1}a}{(p-a)(2^{p/2}-2^{p/a})}h^{1-a/2}\right].\end{aligned}$$

We see, that when $a > 2$, (36) and hence (35) are valid provided that $\lim_{n\to\infty} a_n^{-1} n^{1/a} = 0$. This is guaranteed by condition (32). Indeed, as $a_n = \inf\{x \in \mathbb{R}; P(|X| > x) \le 1/n\}$, $P(|X| > 2a_n) \le n^{-1}$, so choosing $t = 2a_n$ in (18) gives $2^a a_n^a P(|X| > 2a_n) \to \infty$, hence $2^a a_n^a n^{-1} \to \infty$ or equivalently $n^{1/a} a_n^{-1} \to 0$. □

The following theorem proved in [25] is the key of the proof of (b) in Th. 1.

Theorem 9. *Let $a > 2$. Assume that $\mathsf{E}\, X = 0$ and*

$$\lim_{t\to\infty} t^a P(|X| > t) = 0. \tag{37}$$

Then for $0 \le \gamma < 1/2 - 1/a$,

$$\lim_{n\to\infty} P(n^{-1/2+\gamma}\sigma^{-1} T_n^{(\gamma)} \le x) = P(T^{(\gamma)} \le x), \quad x \ge 0,$$

where $\sigma^2 = \mathsf{E}\, X^2$ and

$$T^{(\gamma)} := \sup_{0\le s<t\le 1} \frac{|W(t) - W(s)|}{|t-s|^\gamma}, \tag{38}$$

W denoting a standard Brownian motion on $[0, 1]$.

Theorem 10. *Let $a > 2$ and $\gamma = 1/2 - 1/a$. Assume that $\mathsf{E}\, X = 0$. The sequence $(n^{-1/a} T_n^{(\gamma)})_{n\ge 1}$ is stochastically bounded if and only if*

$$A := \sup_{t>0} t^a P(|X| > t) < \infty. \tag{39}$$

Proof. We estimate for arbitrary $b > 0$ the probability

$$P_n(b) = P\left(n^{-1/a} T_n^{(\gamma)} > b\right).$$

First, we truncate the random variables X_j at the level $bn^{1/a}$ by setting

$$\widetilde{X}_i := X_i \mathbb{1}(|X_i| \le bn^{1/a}), \quad X_i' := \widetilde{X}_i - \mathsf{E}\, \widetilde{X}_i, \quad i = 1, \dots, n.$$

Recalling the abbreviations (11), this gives

$$P\left(n^{-1/a} T_n^{(\gamma)} > b\right) \le P\left(\max_{1\le i\le n} |X_i| > bn^{1/a}\right) + P\left(n^{-1/a} T_n^{(\gamma)}(\widetilde{X}) > b\right).$$

Next, $T_n^{(\gamma)}(\widetilde{X}) \le T_n^{(\gamma)}(X') + T_n^{(\gamma)}(\mathsf{E}\, \widetilde{X})$ and by stationarity, $T_n^{(\gamma)}(\mathsf{E}\, \widetilde{X}) = n^{1-\gamma} |\mathsf{E}\, \widetilde{X}|$.

Since $\mathsf{E}\, X = 0$, $|\mathsf{E}\, \widetilde{X}| = \left|\mathsf{E}\, X \mathbb{1}\{|X| > bn^{1/a}\}\right|$, so using (39) and the elementary equality $\mathsf{E}\, |X| \mathbb{1}\{|X| > c\} = cP(|X| > c) + \int_c^\infty P(|X| > t)\, \mathrm{d}t$, leads to $|\mathsf{E}\, \widetilde{X}| \le \frac{Aa}{a-1} b^{1-a} n^{1/a-1}$. Consequently,

$$n^{-1/a} T_n^{(\gamma)}(\widetilde{X}) \le n^{-1/a} T_n^{(\gamma)}(X') + \frac{Aa}{a-1} n^{1/a-1/2} b^{-a+1}.$$

By Lemma 6,

$$P(n^{-1/a}T_n^{(\gamma)}(X') > b) \le 2\sum_{j=1}^{\log n} 2^j P\Big(\max_{1\le k\le 2n2^{-j}} \Big|\sum_{i=1}^{k} X_i'\Big| > bn^{1/a}(n2^{-j})^{\gamma}\Big).$$

Proceeding as in the proof of Th. 5, see the estimation of $\widetilde{Q}_{nj}(h,\varepsilon)$, we obtain

$$P(n^{-1/a}T_n^{(\gamma)}(X') > b) \le c_p b^{-p}$$

for $b \ge 1$, and complete the proof of sufficiency.

As for the necessity of (39), we just notice that the stochastic boundedness of $n^{-1/a}T_n^{(\gamma)}$ yields $n^{1/a}\max_{1\le k\le n}|X_k| = O_P(1)$ which for i.i.d. X_k's implies the boundedness of L, giving (39). □

5 Proofs of the theorems on $\mathrm{MR}_{\gamma,n}$

5.1 Proof of Th. 1

First we prove (a). Since the random variables $T_{nj}, j = 1,\dots,4$ are independent and recalling that $n = 4m$, we have by Th. 5 a)

$$a_m^{-1}(T_{n1}, T_{n2}, T_{n3}, T_{n4}) \xrightarrow[m\to\infty]{\mathcal{D}} (T_1, T_2, T_3, T_4),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution and T_1, T_2, T_3, T_4 are independent random variables with Fréchet distribution:

$$P(T_j \le x) = \exp(-x^{-a}), \quad x > 0,$$

for any $j = 1,\dots,4$. By measurable mapping,

$$\mathrm{MR}_{\gamma,n} \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{MR}_{\gamma} = M(T_1,T_2,T_3,T_4) \overset{\text{a.s.}}{=} \max\left(\frac{T_1}{T_3}, \frac{T_3}{T_1}, \frac{T_2}{T_4}, \frac{T_4}{T_2}\right).$$

Since the random variables $\max(T_1/T_3, T_3/T_1)$ and $\max(T_2/T_4, T_4/T_2)$ are independent and identically distributed,

$$P(\mathrm{MR}_{\gamma} \le x) = P^2\left(\max\left(\frac{T_1}{T_3}, \frac{T_3}{T_1}\right) \le x\right).$$

Next, we observe that $P(\mathrm{MR}_{\gamma} \le x) = 0$ for $x < 1$, whereas for $x \ge 1$,

$$\left\{\max\left(\frac{T_1}{T_3}, \frac{T_3}{T_1}\right) \le x\right\} = \left\{\frac{1}{x} \le \frac{T_1}{T_3} \le x\right\}.$$

Hence for $x \ge 1$,

$$P(\mathrm{MR}_{\gamma} \le x) = P^2\left(\frac{1}{x} \le \frac{T_1}{T_3} \le x\right).$$

The distribution function of the ratio T_1/T_3 of two independent Fréchet random variables with the same parameter a is easy to derive:

$$\begin{aligned} P(T_1/T_3 \le x) = P(T_1 \le xT_3) &= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \mathbb{1}_{\{u \le xv\}} \, \mathrm{d}P_{T_1}(u) \right) \mathrm{d}P_{T_3}(v) \\ &= \int_{\mathbb{R}_+} \exp\left(-(xv)^{-a}\right) \, \mathrm{d}P_{T_3}(v) \\ &= \int_0^\infty \exp\left(-(xv)^{-a}\right) av^{-a-1} \exp\left(-v^{-a}\right) \, \mathrm{d}v \\ &= \int_0^1 t^{(x^{-a})} \, \mathrm{d}t = \frac{1}{1+x^{-a}}. \end{aligned}$$

Hence

$$P\left(\frac{1}{x} \le \frac{T_1}{T_3} \le x\right) = \frac{1}{1+x^{-a}} - \frac{1}{1+x^a} = \frac{x^a - 1}{x^a + 1}.$$

This yields

$$P(\mathrm{MR}_\gamma > x) = \begin{cases} 1 & \text{if } \ x < 1, \\ 4x^a(1+x^a)^{-2} & \text{if } \ x \ge 1, \end{cases}$$

and completes the proof of (a).

Similarly, one proves (b). Since the random variables $T_{nj}, j = 1, \ldots, 4$, are independent, we have by Th. 9

$$m^{-1/2+\gamma}\sigma^{-1}(T_{n1}, T_{n2}, T_{n3}, T_{n4}) \xrightarrow[m\to\infty]{\mathcal{D}} \left(U_1^{(\gamma)}, U_2^{(\gamma)}, U_3^{(\gamma)}, U_4^{(\gamma)}\right),$$

where $U_1^{(\gamma)}, U_2^{(\gamma)}, U_3^{(\gamma)}, U_4^{(\gamma)}$ are independent random variables with the same distribution as $T^{(\gamma)}$ defined by (38). By measurable mapping,

$$\mathrm{MR}_{\gamma,n} \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{MR}_\gamma \overset{\text{a.s.}}{=} \max\left(\frac{U_1^{(\gamma)}}{U_3^{(\gamma)}}, \frac{U_3^{(\gamma)}}{U_1^{(\gamma)}}, \frac{U_2^{(\gamma)}}{U_4^{(\gamma)}}, \frac{U_4^{(\gamma)}}{U_2^{(\gamma)}}\right).$$

Since $\max\left(U_1^{(\gamma)}/U_3^{(\gamma)}, U_3^{(\gamma)}/U_1^{(\gamma)}\right)$ and $\max\left(U_2^{(\gamma)}/U_4^{(\gamma)}, U_4^{(\gamma)}/U_2^{(\gamma)}\right)$ are independent and identically distributed random variables,

$$P(\mathrm{MR}_\gamma \le x) = P^2\left(\max\left(\frac{U_1^{(\gamma)}}{U_3^{(\gamma)}}, \frac{U_3^{(\gamma)}}{U_1^{(\gamma)}}\right) \le x\right) = P^2\left(\frac{1}{x} \le \frac{U_1^{(\gamma)}}{U_3^{(\gamma)}} \le x\right).$$

Observing that

$$P\left(\frac{U_1^{(\gamma)}}{U_3^{(\gamma)}} \le x\right) = \int_0^\infty F_\gamma(xy) \, \mathrm{d}F_\gamma(y),$$

we complete the proof of (b) following the same steps as for (30).

The statement (c) combines both proofs of cases (a) and (b) together with Th. 5 and Th. 9.

Finally, (d) follows from Th. 10, noting that with A defined by (39),

$$P(n^{-1/a}T_n^{(\gamma)} \le \delta) \le P(n^{-1/a}\max_{1\le k\le n}|X_k| \le \delta) \le \Big(1-\frac{A}{\delta^a n}\Big)^n \to \exp\{-A\delta^{-a}\}$$

as $n\to\infty$, so $n^{-1/a}T_n^{(\gamma)}$ is $O_P(1)$ and bounded away from 0 in probability.

5.2 Proof of Th. 2

Let $\mathcal{H}^o_{4,\gamma}$ denote the product Hölder space endowed with $\|\ \|_{4,\gamma}$, where

$$\mathcal{H}^o_{4,\gamma} := \prod_{j=1}^4 \mathcal{H}^o_\gamma\left[\frac{j-1}{4},\frac{j}{4}\right], \quad \|(f_1,f_2,f_3,f_4)\|_{4,\gamma} := \max_{1\le j\le 4}\|f_j\|_\gamma.$$

Clearly, $\mathcal{H}^o_{4,\gamma}$ is a separable Banach space. For $f\in\mathcal{H}^o_\gamma[0,1]$ and $j=1,\dots,4$, denote by R_jf the restriction of f to the segment $[(j-1)/4, j/4]$, put $R_j^0 f := R_jf - f((j-1)/4)$. We remark in passing that $\omega_\gamma(R_jf,\delta)=\omega_\gamma(R_j^0f,\delta)$. Next, we define $\Phi(f) := (R_1^0f, R_2^0f, R_3^0f, R_4^0f)$. The linear map $\Phi:\mathcal{H}^o_\gamma[0,1]\to\mathcal{H}^o_{4,\gamma}$ is continuous since $\|\Phi(f)\|_{4,\gamma}\le\|f\|_\gamma$. To complete these functional preliminaries, we introduce the map $\Psi_\gamma:\mathcal{H}^o_{4,\gamma}\to\mathbb{R}^4_+$,

$$(f_1,f_2,f_3,f_4)\longmapsto\left(\omega_\gamma\Big(f_1,\frac14\Big),\omega_\gamma\Big(f_2,\frac14\Big),\omega_\gamma\Big(f_3,\frac14\Big),\omega_\gamma\Big(f_4,\frac14\Big)\right).$$

As each ω_γ involved in this formula is absolutely scalable and subadditive, these properties are inherited by Ψ_γ. It follows that Ψ_γ is Lipschitzian, hence continuous on $\mathcal{H}^o_{4,\gamma}$.

Recalling that $n=4m$, we express $T_{nj}^{(\gamma)}$ in terms of ξ_n as

$$T_{4m,j}^{(\gamma)} = (4m)^{-\gamma}\max_{1\le\ell\le m}\max_{(j-1)m\le k\le jm-\ell}\frac{\big|\xi_n\left(\frac{k+\ell}{4m}\right)-\xi_n\left(\frac{k}{4m}\right)\big|}{\left(\frac{\ell}{4m}\right)^\gamma}.$$

Since for a polygonal line function g on the segment $[b,c]$, $\omega_\gamma(g,c-b)$ is reached at two vertices, see Lemma 3 in [19], this can be recast as

$$T_{4m,j}^{(\gamma)} = 4^{-\gamma}m^{-\gamma}\omega_\gamma\left(R_j\xi_{4m};\frac14\right) = 4^{-\gamma}m^{-\gamma}\omega_\gamma\left(R_j^0\xi_{4m};\frac14\right), \quad j=1,\dots,4.$$

It follows that for $n=4m$,

$$n^{-1/2+\gamma}\sigma^{-1}\Big(T_{n1}^{(\gamma)},\dots,T_{n4}^{(\gamma)}\Big) = (\Psi_\gamma\circ\Phi)\big(n^{-1/2}\sigma^{-1}\xi_n\big).$$

As $(X_k)_{k\ge1}$ satisfies the HFCLT(γ), this gives, by measurable mapping,

$$n^{-\frac12+\gamma}\sigma^{-1}\Big(T_{n1}^{(\gamma)},\dots,T_{n4}^{(\gamma)}\Big)\xrightarrow[n\to\infty]{\mathcal{D}}\left(\omega_\gamma\Big(R_1^0W,\frac14\Big),\dots,\omega_\gamma\Big(R_4^0W,\frac14\Big)\right). \quad (40)$$

In view of the independence of the increments of W, the four components of the above limiting random vector are independent random variables. To find the marginal distribution we observe that

$$\begin{aligned}\omega_\gamma\Big(R_j^0 W, \frac{1}{4}\Big) &= \sup_{\substack{0<t-s\le\frac{1}{4}\\ s,t\in[\frac{j-1}{4},\frac{j}{4}]}} \frac{|W(t)-W(s)|}{(t-s)^\gamma}\\ &= \sup_{\substack{0<v-u\le 1\\ u,v\in[0,1]}} \frac{\big|W\big(\frac{j-1}{4}+\frac{v}{4}\big)-W\big(\frac{j-1}{4}+\frac{u}{4}\big)\big|}{\big(\frac{v-u}{4}\big)^\gamma}\\ &= 4^\gamma \omega_\gamma(V_j,1),\end{aligned}$$

where

$$V_j(u) := W\Big(\frac{j-1}{4}+\frac{u}{4}\Big) - W\Big(\frac{j-1}{4}\Big), \quad u\in[0,1].$$

V_j is a centered Gaussian process on $[0,1]$ with $V_j(0)=0$, independent increments and $\mathsf{E}\, V_j(u)^2 = u/4$. Hence $W_j := 2V_j$ is a standard Brownian motion on $[0,1]$. Noticing that $\omega_\gamma\Big(R_j^0 W, \frac{1}{4}\Big) = \frac{4^\gamma}{2}\omega_\gamma(W_j,1)$, we can rewrite (40) as

$$\frac{2}{4^\gamma} n^{-1/2+\gamma}\sigma^{-1}\Big(T_{n1}^{(\gamma)},\dots,T_{n4}^{(\gamma)}\Big) \xrightarrow[n=4m\to\infty]{\mathcal{D}} \Big(\omega_\gamma\Big(W_1,\frac{1}{4}\Big),\dots,\omega_\gamma\Big(W_4,\frac{1}{4}\Big)\Big),$$

where $W_1,\dots,W_4$ are independent standard Brownian motions on $[0,1]$. Since $n=4m$, the normalization above is $\frac{2}{4^\gamma} n^{-1/2+\gamma}\sigma^{-1} = m^{-1/2+\gamma}\sigma^{-1}$. Now, putting

$$U_j^{(\gamma)} := \omega_\gamma\Big(W_j,\frac{1}{4}\Big), \quad j=1,\dots,4,$$

the proof is completed exactly as the proof of Th. 1 (b).

In the i.i.d. case, the proof was simpler due to the independence of the components for each n which enabled us to apply directly the HFCLT componentwise. In the above proof we use only asymptotic independence which is encoded in the independence of the increments of the Brownian motion.

The constant σ in the definition of HFCLT(γ) is not necessarily equal to $\mathsf{E}\, X_1^2$. For instance, in the strong mixing case or in the associated case, see e.g. [14], $\sigma^2 = \mathsf{E}\, X_1^2 + \sum_{i=2}^\infty \mathsf{E}\, X_1 X_i$.

In [12], Giraudo obtains an extension of the HFCLT with a limit $\sqrt{\eta}\,W$ where the random variable η is independent of W. The above proof can be adapted to this case via Fubini arguments (keeping the same η for the four subsegments of $[0,1]$). Clearly, η disappears in the computation of MR_γ.

5.3 Proof of Th. 3

We notice that for $a>1$, (19) implies (18), so the series in (17) converges almost surely and $X_t \in \mathrm{RV}_a$.

Proof of Th.3 i). Here the sequence of observations is $(X_i)_{i\geq 1}$ where

$$X_i = X_i^{(q)} := \sum_{j=1}^{q} \psi_j \epsilon_{i-j}.$$

By [20, Th. 3.2],

$$a_m^{-1} T_{n,j}^{(\gamma)}(X^{(q)}) \xrightarrow[n=4m\to\infty]{\mathcal{D}} c_{\gamma,q} T_j, \quad j = 1, \ldots, 4$$

where T_j has the Fréchet distribution with parameter a and

$$c_{\gamma,q} := \max_{1\leq \ell \leq q} \ell^{-\gamma} \max_{1\leq k\leq q-\ell+1} \left| \sum_{i=k}^{k+\ell-1} \psi_i \right|.$$

Unfortunately, the random variables $T_{n,j}^{(\gamma)}(X^{(q)})$, $j = 1, \ldots, 4$ are dependent, so we cannot deduce from the above weak convergences the convergence of the vector $a_m^{-1}\big(T_{n,1}^{(\gamma)}(X^{(q)}), \ldots, T_{n,4}^{(\gamma)}(X^{(q)})\big)$. To avoid this drawback, we remove of each of the blocks the q last variables X_i^q, replacing $T_{n,j}^{(\gamma)}(X^{(q)})$ by the corresponding $T_{n,j,q}^{(\gamma)}(X^{(q)})$, defined by (11), which are independent. Let us check this. In the spirit of (11), we denote by $\mathrm{mf}(u, v, Z)$ the set of random variables of the form $f(Z_{u+1}, \ldots, Z_v)$ where f is real valued, defined on some Borel subset of $\mathbb{R}^{v-u+1}$ and Borel measurable. Then for $i \geq 1$, $X_i^{(q)} \in \mathrm{mf}(i - q - 1, i - 1, \epsilon)$, whence

$$\begin{aligned} T_{n,j,q}^{(\gamma)}(X^{(q)}) \in \mathrm{mf}\big((j-1)m, jm-q, X^{(q)}\big) &\subset \mathrm{mf}\big((j-1)m-q, jm-q-1, \epsilon\big) \\ &\subset \mathrm{mf}\big((j-1)m-q, jm-q, \epsilon\big). \end{aligned}$$

So the four random variables $T_{n,j,q}^{(\gamma)}(X^{(q)})$ are independent as measurable functions of disjoint subsequences of the independent sequence $(\epsilon_t)_{t\in\mathbb{Z}}$. A similar result could be obtained by removing only the $q-1$ last variables X_i^q in each block. Our preference for the q last variables leads to some writing simplification below.

Now, exactly as in the proof of Th. 1, by independence of components,

$$a_{m-q}^{-1} \left(T_{n,1,q}^{(\gamma)}(X^{(q)}), \ldots, T_{n,4,q}^{(\gamma)}(X^{(q)}) \right) \xrightarrow[n=4m\to\infty]{\mathcal{D}} c_{\gamma,q}(T_1, T_2, T_3, T_4), \quad (41)$$

where $T_1, \ldots, T_4$ are i.i.d. with Fréchet distribution with parameter a. As $\mathbb{R}_+^4 \setminus D$, the discontinuities set of M, has null measure for the distribution of $(T_1, \ldots, T_4)$, this gives by measurable mapping and homogeneity of M,

$$\begin{aligned} M\Big(T_{n,1,q}^{(\gamma)}(X^{(q)}), \ldots, T_{n,4,q}^{(\gamma)}(X^{(q)})\Big) &= M\Big(a_{m-q}^{-1}\big(T_{n,1,q}^{(\gamma)}(X^{(q)}), \ldots, T_{n,4,q}^{(\gamma)}(X^{(q)})\big)\Big) \\ &\xrightarrow[n=4m\to\infty]{\mathcal{D}} M(T_1, T_2, T_3, T_4). \end{aligned}$$

Now, to prove Th.3 i), it is enough to prove the convergence in probability to zero of $M\left(T^{(\gamma)}_{n,1,q}(X^{(q)}),\dots,T^{(\gamma)}_{n,4,q}(X^{(q)})\right)-M\left(T^{(\gamma)}_{n,1}(X^{(q)}),\dots,T^{(\gamma)}_{n,4}(X^{(q)})\right)$ or by homogeneity that of

$$M\left(a^{-1}_{m-q}T^{(\gamma)}_{n,1,q}(X^{(q)}),\dots,a^{-1}_{m-q}T^{(\gamma)}_{n,4,q}(X^{(q)})\right) - M\left(a^{-1}_{m}T^{(\gamma)}_{n,1}(X^{(q)}),\dots,a^{-1}_{m}T^{(\gamma)}_{n,4}(X^{(q)})\right).$$

This last convergence in turn is implied by

$$a^{-1}_{m}T^{(\gamma)}_{n,j}(X^{(q)}) - a^{-1}_{m-q}T^{(\gamma)}_{n,j,q}(X^{(q)}) \xrightarrow[n=4m\to\infty]{\mathrm{P}} 0, \quad j=1,\dots,4. \tag{42}$$

To check (42), we will prove the L^1 convergence, which reduces the problem in checking that $\mathsf{E}\,|\Delta_n|$ tends to zero, where

$$\Delta_n := a^{-1}_{n}T^{(\gamma)}_{n}(X^{(q)}) - a^{-1}_{n-q}T^{(\gamma)}_{n-q}(X^{(q)}).$$

Noticing that $T^{(\gamma)}_{n}(X^{(q)}) \geq T^{(\gamma)}_{n-q}(X^{(q)}) \geq 0$ and that (a^{-1}_n) is non-increasing in n, we get

$$\begin{aligned}\mathsf{E}\,|\Delta_n| \leq{}& \mathsf{E}\left(a^{-1}_{n}T^{(\gamma)}_{n}(X^{(q)})\right) - \frac{a_{n-q}}{a_n}\mathsf{E}\left(a^{-1}_{n-q}T^{(\gamma)}_{n-q}(X^{(q)})\right)\\ &+\left(1-\frac{a_{n-q}}{a_n}\right)\mathsf{E}\left(a^{-1}_{n-q}T^{(\gamma)}_{n-q}(X^{(q)})\right).\end{aligned}$$

As (a_n) is a regularly varying sequence, it is easily seen that a_{n-q}/a_n tends to 1, so the problem is reduced in proving the convergence of moment

$$\mathsf{E}\left(a^{-1}_{n}T^{(\gamma)}_{n}(X^{(q)})\right) \xrightarrow[n\to\infty]{} \mathsf{E}\,(c_{\gamma,q}T), \quad \text{with } P(T\leq x)=\exp(-x^{-a}),\ x>0.$$

Since we already have the convergence in distribution by [20, Th. 3.2], we need only to check the uniform integrability. To this aim we will prove that for $1<r<a$,

$$\sup_{n\geq 1}\mathsf{E}\left(a^{-1}_{n}T^{(\gamma)}_{n}(X^{(q)})\right)^{r} < \infty. \tag{43}$$

Now, we observe that

$$\begin{aligned}T^{(\gamma)}_{n}(X^{(q)}) &= \max_{1\leq\ell\leq n}\ell^{-\gamma}\max_{0\leq k\leq n-\ell}\left|\sum_{i=k+1}^{k+\ell}\sum_{j=1}^{q}\psi_j\epsilon_{i-j}\right|\\ &= \max_{1\leq\ell\leq n}\ell^{-\gamma}\max_{0\leq k\leq n-\ell}\left|\sum_{j=1}^{q}\psi_j\sum_{i=k+1}^{k+\ell}\epsilon_{i-j}\right|\\ &\leq \sum_{j=1}^{q}|\psi_j|\max_{1\leq\ell\leq n}\ell^{-\gamma}\max_{0\leq k\leq n-\ell}\left|\sum_{i=k+1}^{k+\ell}\epsilon_{i-j}\right|.\end{aligned}$$

By convexity of $t \mapsto t^r$, puting $K := \sum_{j=1}^{q} |\psi_j|$, we get

$$T_n^{(\gamma)}(X^{(q)})^r \le K^{r-1} \sum_{j=1}^{q} |\psi_j| \left(\max_{1\le \ell \le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} \left| \sum_{i=k+1}^{k+\ell} \epsilon_{i-j} \right| \right)^r .$$

By stationarity of $(\epsilon_t)_{t\in\mathbb{Z}}$, $\max_{1\le\ell\le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} \left|\sum_{i=k+1}^{k+\ell} \epsilon_{i-j}\right|$ has the same r-moment as $\max_{1\le\ell\le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} \left|\sum_{i=k+1}^{k+\ell} \epsilon_i\right|$. This implies that

$$\begin{aligned} \mathsf{E} \left(a_n^{-1} T_n^{(\gamma)}(X^{(q)}) \right)^r &\le K^r \mathsf{E} \left(a_n^{-1} \max_{1\le \ell \le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} \left| \sum_{i=k+1}^{k+\ell} \epsilon_i \right| \right)^r \\ &= K^r \mathsf{E} \left(a_n^{-1} T_n^{(\gamma)}(\epsilon) \right)^r . \end{aligned}$$

By (4.5) in [28, Th. 4.2], $\mathsf{E} \left(a_n^{-1} T_n^{(\gamma)}(\epsilon) \right)^r$ tends to $\mathsf{E}\, T_1^r$, so (43) follows and the proof of i) is complete. □

Proof of Th.3 ii). Now, the observations X_i have the representation

$$X_i = \sum_{j=1}^{q} \psi_j \epsilon_{i-j} + \sum_{j=q+1}^{\infty} \psi_j \epsilon_{i-j} = X_i^{(q)} + \sum_{j=q+1}^{\infty} \psi_j \epsilon_{i-j}.$$

We introduce the following random vectors of $\mathbb{R}_+^4$.

$$\begin{aligned} V_m &:= \left(T_{m,1}^{(\gamma)}(X), \ldots, T_{m,4}^{(\gamma)}(X) \right), \; V_m^{(q)} := \left(T_{m,1}^{(\gamma)}(X^{(q)}), \ldots, T_{m,4}^{(\gamma)}(X^{(q)}) \right), \\ V &:= (T_1, \ldots, T_4). \end{aligned}$$

If we prove the weak convergence of $a_m^{-1} V_m$ to cV for some constant c, then the convergence (12), i.e., the weak convergence of $M(V_m)$ to $M(V)$, follows by measurable mapping and homogeneity of M. For the choice of c, it is natural to try with

$$c = c_{\gamma,\infty} := \lim_{q\to\infty} c_{\gamma,q} = \sup_{\ell\ge 1} \ell^{-\gamma} \sup_{k\ge 1} \left| \sum_{i=k}^{k+\ell-1} \psi_i \right|, \tag{44}$$

which is finite in view of (19).

To establish the weak convergence of $a_m^{-1} V_m$ to $c_{\gamma,\infty} V$, it suffices to prove that

$$\lim_{m\to\infty} \mathsf{E}\, f(a_m^{-1} V_m) = \mathsf{E}\, f(c_{\gamma,\infty} V) \tag{45}$$

for each bounded function $f : \mathbb{R}_+^4 \to \mathbb{R}$ such that $|f(x) - f(y)| \le \|x - y\|$, $x, y \in \mathbb{R}_+^4$, where say $\|(x_1, \ldots, x_4)\| := \sum_{j=1}^{4} |x_j|$. To this end we note that

$$\begin{aligned} &\left| \mathsf{E}\, f(a_m^{-1} V_m) - \mathsf{E}\, f(c_{\gamma,\infty} V) \right| \le \\ &\quad \mathsf{E}\, \|a_m^{-1}(V_m - V_m^{(q)})\| + \left| \mathsf{E}\, f(a_m^{-1} V_m^{(q)}) - \mathsf{E}\, f(c_{\gamma,q} V) \right| + |c_{\gamma,q} - c_{\gamma,\infty}| \mathsf{E}\, \|V\|. \end{aligned}$$

From (41) and (42), $a_m^{-1}V_m^{(q)}$ converges in distribution to $c_{\gamma,q}V$, hence

$$\limsup_{n\to\infty}\left|\mathsf{E}\, f(a_m^{-1}V_m)-\mathsf{E}\, f(c_{\gamma,\infty}V)\right| \le \sup_{m\ge 1}\mathsf{E}\,\|a_m^{-1}(V_m-V_m^{(q)})\| + |c_{\gamma,q}-c_{\gamma,\infty}|\mathsf{E}\,\|V\|. \tag{46}$$

Next, setting $V_{m,l}:=T_{m,l}^{(\gamma)}(X)$, $V_{m,l}^{(q)}:=T_{m,l}^{(\gamma)}(X^{(q)})$ for $l=1,\dots,4$,

$$|V_{m,l}-V_{m,l}^{(q)}| \le \sum_{j=q+1}^{\infty}|\psi_j|\max_{1\le\ell\le m}\ell^{-\gamma}\max_{(l-1)m\le k\le lm-\ell}\left|\sum_{i=k+1}^{k+\ell}\epsilon_{i-j}\right|.$$

Like in the proof of i), this implies by stationarity of $(\epsilon_t)_{t\in\mathbb{Z}}$,

$$\mathsf{E}\,|V_{m,l}-V_{m,l}^{(q)}| \le \left(\sum_{j=q+1}^{\infty}|\psi_j|\right)\mathsf{E}\,\left(a_m^{-1}T_m^{(\gamma)}(\epsilon)\right).$$

By (4.5) in [28, Th. 4.2], $\lim_{m\to\infty}\mathsf{E}\,\left(a_m^{-1}T_m^{(\gamma)}(\epsilon)\right)=\mathsf{E}\,T_1$, whence $C:=\sup_{m\ge1}\mathsf{E}\,\left(a_m^{-1}T_m^{(\gamma)}(\epsilon)\right)$, is finite and therefore

$$\sup_{m\ge1}\mathsf{E}\,\|a_m^{-1}(V_m-V_m^{(q)})\| \le 4C\sum_{j=q+1}^{\infty}|\psi_j|.$$

Thanks to this estimate and (44), letting q tend to infinity in (46) gives

$$\limsup_{n\to\infty}\left|\mathsf{E}\, f(a_m^{-1}V_m)-\mathsf{E}\, f(c_{\gamma,\infty}V)\right|=0,$$

which proves (45) and completes the proof of Th. 3. □

Remark 11. As a by-product of this proof, we obtain a slight improvement of [20, Th. 3.6], where the convergence of $a_n^{-1}T_n^{(\gamma)}(X)$ to T_1 is proved under a stronger assumption than (19), namely $\sum_{j=3}^{\infty}|\psi_j|\kappa_j<\infty$, where $\kappa_j=(j\log\log j)^{1/2}$ if $\operatorname{var}\epsilon_0<\infty$ or $j^{1/r}$ for some $r<a$, if $a<2$ or $a=2$, and $\operatorname{var}\epsilon_0=\infty$.

5.4 Proof of Th. 4

Proof of Th.4 i). Denote for, $1\le\ell\le n$, $0\le k\le n-\ell$,

$$I_{k,\ell}=\{k+1,\dots,k+\ell\},$$

and set $I^*=I_{k^*,\ell^*}$. For any subset $I\subset\{0,1,\dots\}$, we denote by $|I|$ the number of elements in I. Consider $I_m^{(j)}=\{(j-1)m+1,\dots,jm\}, j=1,\dots,4$. If $\ell^*<n/4$, then either $I^*\subset I_m^{(j)}$ for one of $j=1,\dots,4$ or $I^*\subset I_{,}^{(j)}\cup I_n^{(j+1)}$

for one of $j = 1, 2, 3$. If $I^* \subset I_m^{(j)}$, we have, denoting by i the other integer with same parity as j in $\{1, 2, 3, 4\}$,

$$\mathrm{MR}_{n,\gamma} \geq \frac{T_{nj}}{T_{ni}}.$$

By Th. 5, $[\max(a_n, n^{1/2-\gamma})]^{-1} T_{ni} = O_P(1)$, whereas

$$T_{nj} \geq \ell^{*-\gamma} \Big| \sum_{k \in I^*} (\mu_n + X_k) \Big| \geq \ell^{*-\gamma} \Big| \sum_{i \in I^*} \mu_{ni} \Big| - O_P(c_n)$$

and the result follows.

If $I^* \subset I_m^{(j)} \cup I_m^{(j+1)}$, then with $i \neq j$ and of same parity, $i' \neq j+1$ and of same parity as $j+1$,

$$\mathrm{MR}_{n,\gamma} \geq \max\left(\frac{T_{nj}}{T_{ni}}, \frac{T_{n,j+1}}{T_{ni'}}\right) \geq \frac{T_{n,j+1} + T_{nj}}{2\max(T_{ni}, T_{ni'})}.$$

Both T_{ni} and $T_{ni'}$ are of order $\max(a_n, n^{1/2-\gamma})$ in probability. Since

$$T_{n,j+1} + T_{nj} \geq \ell^{*-\gamma} \Big| \sum_{i \in I^*} \mu_{ni} \Big| - O_P(c_n),$$

the result follows. □

Proof of Th.4 ii). First, we recall that $D = (0, \infty)^4$. It is elementary to check that M is Lispchitzian on every compact subset of D and that in particular on $K_r := [r, 1/r]^4$, $0 < r < 1$,

$$|M(y_1, \ldots, y_4) - M(x_1, \ldots, x_4)| \leq \frac{2}{r^3} \max_{1 \leq i \leq 4} |y_i - x_i|. \tag{47}$$

In what follows, we use the abbreviations (11) for sequences of random variables as well as for sequences of real numbers.

We recall in particular that $T_n^{(\gamma)}(x)$ induces a norm on $\mathbb{R}^n$. As $T_n^{(\gamma)} = T_n^{(\gamma)}(Y) = T_n^{(\gamma)}(X + \mu)$, where μ represents $(\mu_n(k))_{1 \leq k \leq n}$, this implies

$$|T_n^{(\gamma)} - T_n^{(\gamma)}(X)| \leq T_n^{(\gamma)}(\mu). \tag{48}$$

As $\mu = \mu^* \mathbb{1}_{I^*}$, it is easily seen that

$$T_n^{(\gamma)}(\mu) = \ell^{*(1-\gamma)} |\mu^*|. \tag{49}$$

Let us fix an arbitrary positive ε. By tightness of the distribution of V, there exists a compact subset K of D, such that $P(V \in K) > 1 - \varepsilon$. Next, we observe that if $r' < r$, $K_r \subset K_{r'}$ and that D is the union of all the K_r, $0 < r < 1$. Moreover, for $r \neq r'$, the boundaries ∂K_r and $\partial K_{r'}$ are disjoint and the collection $\{\partial K_r; 0 < r < 1\}$ is uncountable. All this implies that we can replace K by a compact K_r containing K and such that $P(V \in \partial K_r) = 0$:

$$P(V \in K_r) > 1 - \varepsilon, \quad P(V \in \partial K_r) = 0. \tag{50}$$

In view of assumption (23), $P(V \in \partial K_r) = 0$ implies that $P(V_n \in K_r)$ tends to $P(V \in K_r)$, so there is some integer n_1 such that

$$P(V_n \in K_r) > 1 - 2\varepsilon, \quad n \geq n_1. \tag{51}$$

Now, by assumption (24), there is some integer n_2 such that

$$c_n^{-1} \ell^{*(1-\gamma)} |\mu^*| < \frac{r}{2}, \quad n \geq n_2. \tag{52}$$

Putting

$$V_n' := c_n^{-1} \left(T_{n1}^{(\gamma)}, \ldots, T_{n4}^{(\gamma)} \right),$$

we remark that by (48), (49) and the elementary fact that if $0 < s < r < 1$, $1/s - 1/r > r - s$, for $n \geq n_2$ the membership of V_n in K_r implies the membership of V_n' in $K_{r/2}$. Consequently,

$$P(V_n \in K_{r/2}, V_n' \in K_{r/2}) > 1 - 2\varepsilon, \quad n \geq n_0 := \max(n_1, n_2). \tag{53}$$

Accounting (47), (48), (49), (52) and (53) this implies that

$$P\Big(|\mathrm{MR}_{\gamma,n} - M(V_n)| \leq \frac{8}{r^2} \Big) > 1 - 2\varepsilon, \quad n \geq n_0,$$

which gives the result by arbitrariness of ε and assumption (23). □

Remark 12. As a by-product of this proof, we obtain the following consistency result for the test with the statistics $T_n^{(\gamma)}$ instead of $\mathrm{MR}_{\gamma,n}$. Indeed, from (48) and (49), it immediately follows that if for some normalizing sequence $(c_n)_{n\geq 1}, c_n^{-1} T_n^{(\gamma)}(X) = O_P(1)$, then $c_n^{-1} T_n^{(\gamma)}$ tends to infinity in probability if $c_n^{-1} \ell^{*(1-\gamma)} |\mu^*|$ tends to infinity while $c_n^{-1} T_n^{(\gamma)} = O_P(1)$ if $c_n^{-1} \ell^{*(1-\gamma)} |\mu^*|$ tends to zero. So roughly speaking, the boundary between the detectable and not detectable epidemics is the same for both test statistics $\mathrm{MR}_{\gamma,n}$ and $T_n^{(\gamma)}$.

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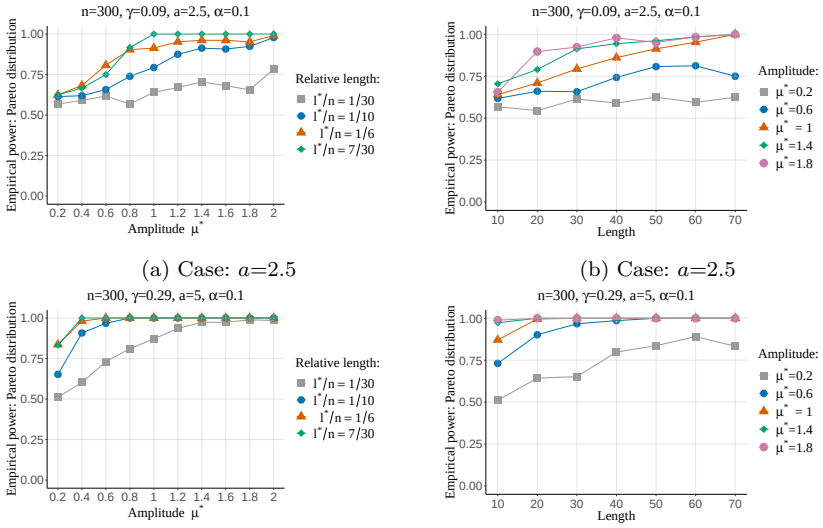
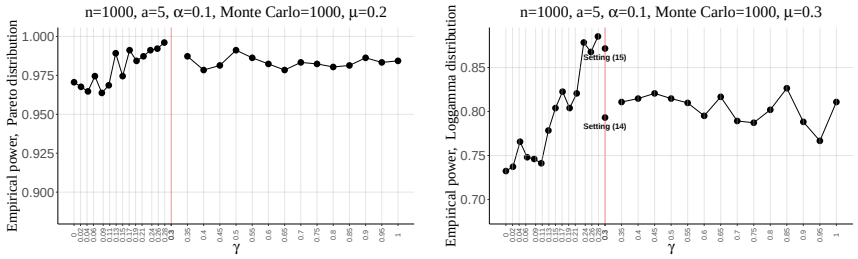
6 Appendix



(a) Case: a=2.5 (b) Case: a=2.5

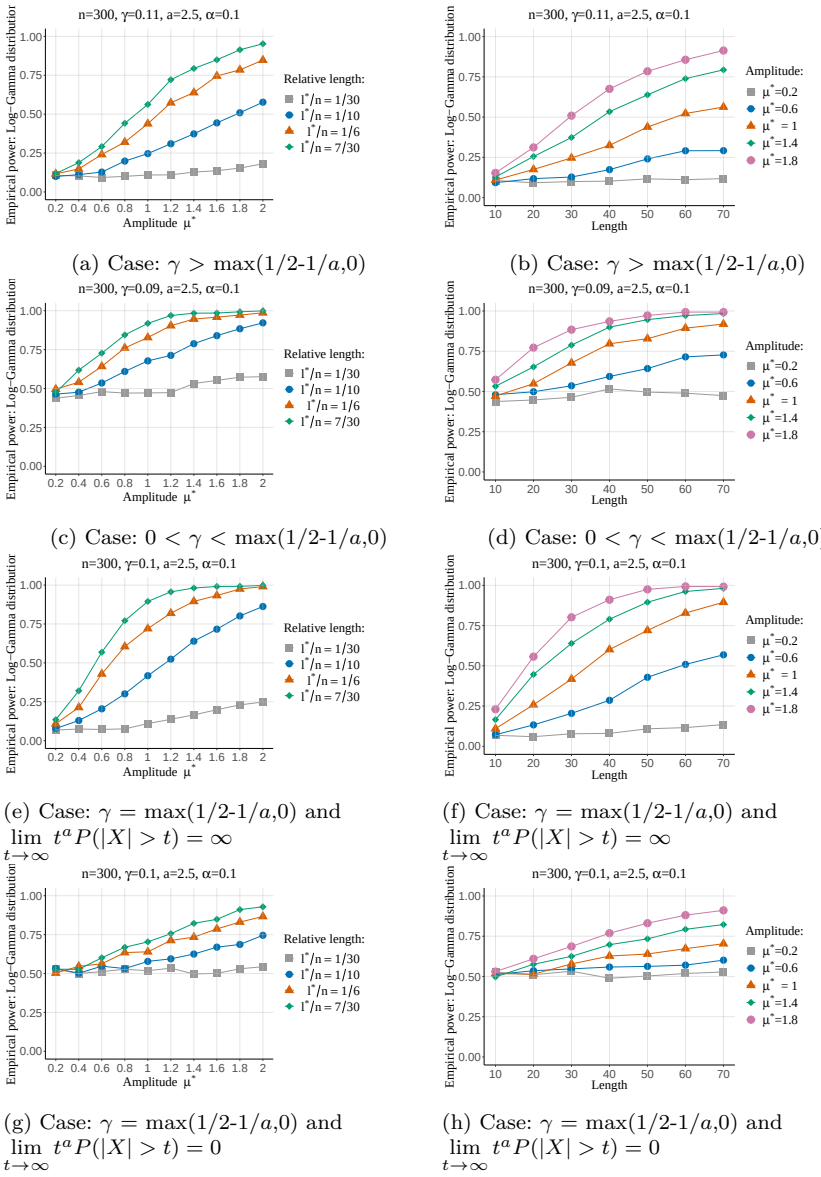
(c) Case: a=5 (d) Case: a=5

Fig. 5: Empirical power of the test under the setting (13) with $\alpha = 0.1$



(a) Pareto distribution (b) Log-Gamma distribution

Fig. 6: Empirical power of the test under various γ values with $\alpha = 0.1$



(a) Case: $\gamma > \max(1/2\text{-}1/a,0)$

(b) Case: $\gamma > \max(1/2\text{-}1/a,0)$

(c) Case: $0 < \gamma < \max(1/2\text{-}1/a,0)$

(d) Case: $0 < \gamma < \max(1/2\text{-}1/a,0)$

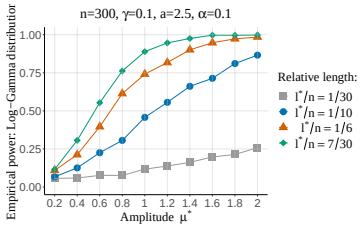
(e) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$

(f) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$

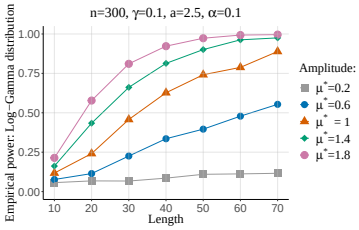
(g) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$

(h) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$

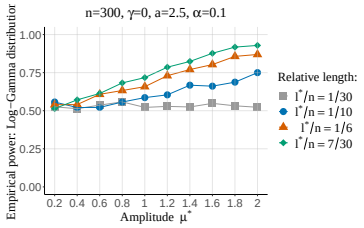
Fig. 7: Empirical power of the test under various settings with $\alpha = 0.1$. The top row corresponds to the situation under setting (12), second row corresponds to the situation under setting (13), third row corresponds to the situation under setting (15) and bottom row corresponds to the situation under setting (16).



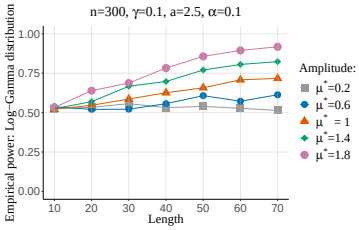
(a) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$



(b) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = \infty$



(c) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$



(d) Case: $\gamma = \max(1/2\text{-}1/a,0)$ and $\lim_{t\to\infty} t^a P(|X| > t) = 0$

Fig. 8: Empirical power of the test under settings (15) (top row) and (16) (bottom row) when parameter a is estimated from Hill estimator with $\alpha = 0.1$

			$n = 200$			$n = 600$			$n = 1000$		
			$\ell^* = 35$	$\ell^* = 40$	$\ell^* = 45$	$\ell^* = 80$	$\ell^* = 120$	$\ell^* = 140$	$\ell^* = 165$	$\ell^* = 200$	$\ell^* = 235$
Log-Gamma	$\alpha = 0.05$	$\gamma = 0.01$	0.4216	0.4377	0.4420	0.4850	0.6022	0.6581	0.6873	0.7520	0.8006
		$\gamma = 0.1$	0.3929	0.4061	0.4620	0.4683	0.5968	0.6413	0.6294	0.7520	0.7748
		$\gamma = 0.2$	0.5714	0.5741	0.5900	0.6514	0.7294	0.7945	0.7980	0.8314	0.8829
		$\gamma = 0.29$	0.6835	0.7026	0.7130	0.7403	0.8244	0.8567	0.8549	0.8824	0.9415
		$\gamma = 0.3_*$	0.6915	0.6848	0.6810	0.6717	0.7285	0.7678	0.7588	0.8157	0.8482
		$\gamma = 0.3^*$	0.5099	0.5336	0.5860	0.6074	0.7491	0.7826	0.7833	0.8578	0.9087
		$\gamma = 0.31$	0.4712	0.5000	0.5310	0.5405	0.6299	0.7194	0.7108	0.7804	0.8165
		$\gamma = 0.5$	0.4692	0.4881	0.5160	0.5079	0.6801	0.7045	0.6971	0.7824	0.8145
		$\gamma = 0.7$	0.4821	0.5168	0.4850	0.5396	0.6219	0.7036	0.7216	0.7696	0.8403
		$\gamma = 0.9$	0.4792	0.5138	0.5060	0.5308	0.6487	0.7016	0.6912	0.7804	0.8244
	$\alpha = 0.1$	$\gamma = 0.01$	0.5030	0.5524	0.5660	0.5898	0.7043	0.7460	0.7627	0.8010	0.8472
		$\gamma = 0.1$	0.4812	0.5296	0.5650	0.5511	0.6918	0.7421	0.7284	0.8088	0.8482
		$\gamma = 0.2$	0.6260	0.6344	0.6400	0.6567	0.7769	0.8043	0.8049	0.8696	0.8958
		$\gamma = 0.29$	0.7063	0.7273	0.7560	0.7535	0.8674	0.8587	0.8745	0.9206	0.9504
		$\gamma = 0.3_*$	0.7192	0.7194	0.6790	0.7148	0.7841	0.8221	0.8265	0.8549	0.8829
		$\gamma = 0.3^*$	0.6488	0.6680	0.7030	0.7201	0.8459	0.8814	0.8784	0.9265	0.9514
		$\gamma = 0.31$	0.5952	0.6255	0.6660	0.6540	0.7545	0.8053	0.7941	0.8657	0.9058
		$\gamma = 0.5$	0.6190	0.6383	0.6500	0.6717	0.7832	0.8113	0.8069	0.8853	0.8810
		$\gamma = 0.7$	0.6022	0.6156	0.6530	0.6576	0.7796	0.7945	0.8147	0.8490	0.8879
		$\gamma = 0.9$	0.5744	0.6235	0.6530	0.6391	0.7652	0.8103	0.7980	0.8480	0.8859
Pareto	$\alpha = 0.05$	$\gamma = 0.01$	0.8522	0.9042	0.9290	0.9472	0.9884	0.9970	0.9951	0.9990	1
		$\gamma = 0.1$	0.8383	0.8893	0.9210	0.9357	0.9848	0.9970	0.9912	1	1
		$\gamma = 0.2$	0.9345	0.9437	0.9610	0.9745	0.9973	1	1	1	1
		$\gamma = 0.29$	0.9514	0.9664	0.9780	0.9833	0.9982	1	1	1	1
		$\gamma = 0.31$	0.8899	0.9308	0.9250	0.9419	0.9964	0.9990	0.9951	0.9990	1
		$\gamma = 0.5$	0.8849	0.9249	0.9340	0.9595	0.9928	0.9990	0.9990	1	1
		$\gamma = 0.7$	0.8839	0.9160	0.9470	0.9621	0.9964	0.9960	0.9980	0.9990	1
		$\gamma = 0.9$	0.8849	0.9219	0.9390	0.9657	0.9919	0.9990	0.9980	0.9990	0.9990
	$\alpha = 0.1$	$\gamma = 0.01$	0.9167	0.9397	0.9550	0.9648	0.9964	0.9980	1	1	1
		$\gamma = 0.1$	0.8839	0.9338	0.9520	0.9551	0.9982	0.9990	0.9941	0.9990	1
		$\gamma = 0.2$	0.9395	0.9654	0.9720	0.9824	0.9982	1	0.9990	1	1
		$\gamma = 0.29$	0.9692	0.9792	0.9890	0.9930	0.9991	1	1	1	1
		$\gamma = 0.31$	0.9226	0.9595	0.9630	0.9762	0.9955	1	0.9990	1	1
		$\gamma = 0.5$	0.9325	0.9654	0.9700	0.9789	0.9982	1	1	1	1
		$\gamma = 0.7$	0.9375	0.9605	0.9730	0.9877	0.9946	0.9990	0.9990	0.9990	1
		$\gamma = 0.9$	0.9177	0.9555	0.9730	0.9692	0.9982	1	0.9971	1	1

Table 4: Constant change in mean, $\mu_n^* = 0.3$, $a = 5$, 0.3^* refer to (15) and 0.3_* to (16)

$\gamma\backslash\alpha$	0.01	0.025	0.05	0.075	0.1	0.15	0.2
0	2.9541	2.6914	2.3608	2.2338	2.1297	1.9124	1.8734
0.05	2.7869	2.6455	2.5091	2.4334	2.1182	2.0001	1.8852
0.1	3.0655	2.6363	2.4194	2.2677	2.1697	1.9778	1.7648
0.15	2.9574	2.3971	2.0329	2.0086	1.9711	1.8637	1.8004
0.2	2.5323	2.1146	2.0384	1.9896	1.9384	1.8224	1.7526
0.25	2.1312	2.0824	2.0030	1.8593	1.7569	1.7296	1.6374
0.3	2.0613	1.8489	1.7989	1.7331	1.7079	1.6168	1.5263
0.35	1.6921	1.6238	1.6077	1.5399	1.5282	1.4405	1.4162
0.4	1.6583	1.5223	1.4805	1.4561	1.4259	1.3669	1.3411
0.45	1.3999	1.3559	1.3212	1.3053	1.2856	1.2578	1.2219

Table 5: Asymptotic critical values for $0 \leq \gamma < 1/2 - 1/a$, $a > 2$.

$a\backslash\alpha$	0.01	0.025	0.05	0.075	0.1	0.15	0.2
4	4.4665	3.5454	2.9717	2.6764	2.4824	2.2277	2.0582
10	1.8197	1.6591	1.5460	1.4826	1.4386	1.3777	1.3347
50	1.1272	1.1066	1.0910	1.0819	1.0754	1.0662	1.0594
100	1.0617	1.0519	1.0445	1.0402	1.0370	1.0326	1.0293

Table 6: Asymptotic critical values for $\gamma > 1/2 - 1/a$, $a > 2$.

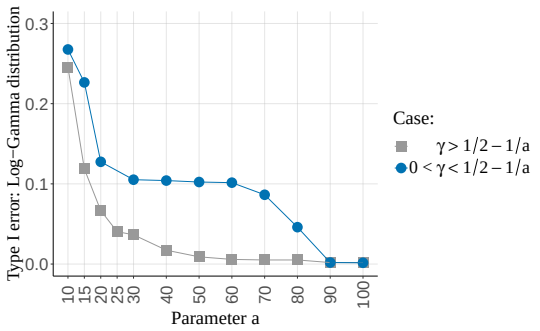


Fig. 9: Type I errors of the test under settings (12) and (13) with various parameter a values, $n = 300$, $\alpha = 0.05$

References

1. M. Basseville and N. Nikiforov. *The Detection of abrupt changes - Theory and applications*. Information and System sciences series. Prentice-Hall, 1993.
2. P. Billingsley. *Convergence of probability measures*. Wiley, New York, 1968.
3. N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*. Encyclopaedia of Mathematics and its Applications. Cambridge University Press, 1987.
4. B. E. Brodsky and B. S. Darkhovsky. *Non parametric methods in change point problems*. Kluwer Academic Publishers, Dordrecht, Boston, 1993.

5. J. Chen and A. K. Gupta. *Parametric statistical change point analysis*. Birkhäuser Verlag, Boston, 2000.
6. D. Commenges, J. Seal, and F. Pinatel. Inference about a change point in experimental neurophysiology. *Math. Biosc.*, 80:81–108, 1986.
7. M. Csörgő and L. Horváth. *Limit Theorems in Change-Point Analysis*. John Wiley & Sons, New York, 1997.
8. F. Enikeeva, A. Munk, and F. Werner. Bump detection in heterogeneous Gaussian regression. *Bernoulli*, 24(2):1266–1306, May 2018.
9. K. Frick, A. Munk, and H. Sieling. Multiscale change point inference. *J. Roy. Statist. Soc. Ser. B*, 76(3):495–580, 2014.
10. D. Giraudo. Hölderian weak invariance principle under a Hannan type condition. *Stochastic Processes and their Applications*, 126:290–311, 2016.
11. D. Giraudo. Hölderian weak invariance principle for stationary mixing sequences. *J. Theoret. Probab.*, 30(1):196–211, 2017.
12. D. Giraudo. Hölderian weak invariance principle under Maxwell and Woodroofe condition. *Brazilian Journal of Probability and Statistics*, 32(1):172–187, 2018.
13. E. Gombay. Testing for change-points with rank and sign statistics. *Statist. Probab. Lett.*, 20:49–56, 1994.
14. D. Hamadouche. Invariance principles in Hölder spaces. *Portugal. Math.*, 57:127–151, 2000.
15. B. M. Hill. A simple general approach to inference about the tail of a distribution. *Ann. Statist.*, 3(5):1163–1174, 1975.
16. M. Juodis, A. Račkauskas, and Ch. Suquet. Hölderian functional central limit theorems for linear processes. *ALEA Lat. Am. J. Probab. Math. Stat.*, 5:1–18, 2009.
17. J. Lamperti. On convergence of stochastic processes. *Trans. Amer. Math. Soc.*, 104:430–435, 1962.
18. B. Levin and J. Kline. CUSUM tests of homogeneity. *Statistics in Medicine*, 4:469–488, 1985.
19. J. Markevičiūtė, A. Račkauskas, and Ch. Suquet. Functional central limit theorems for sums of nearly nonstationary processes. *Lithuanian Math. J.*, 52(3):282–296, July 2012.
20. T. Mikosch and M. Moser. The limit distribution of the maximum increment of a random walk with dependent regularly varying jump sizes. *Probab. Theory Relat. Fields*, 156:249–272, 2013.
21. T. Mikosch and A. Račkauskas. The limit distribution of the maximum increment of a random walk with regularly varying jump size distribution. *Bernoulli*, 16(4):1016–1038, 2010.
22. T. Mikosch and G. Samorodnitsky. The supremum of a negative drift random walk with dependent heavy-tailed steps. *Ann. Appl. Probab.*, 10(3):1025–1064, 2000.
23. A. Račkauskas and Ch. Suquet. Necessary and sufficient condition for the Lamperti invariance principle. *Theory of Probability and Mathematical Statistics*, 68:115–124, 2003.
24. A. Račkauskas and Ch. Suquet. Necessary and sufficient condition for the Hölderian functional central limit theorem. *Journal of Theoretical Probability*, 17(1):221–243, 2004.
25. A. Račkauskas and Ch. Suquet. Hölder norm test statistics for epidemic change. *Journal of Statistical Planning and Inference*, 126(2):495–520, 2004.
26. A. Račkauskas and M. Wendler. Convergence of U-processes in Hölder spaces with application to robust detection of a changed segment. *Statistical Papers*, 61:1409–1435, 2020.
27. A. Ramanayake and A. K. Gupta. Tests for an epidemic change in a sequence of exponentially distributed random variables. *Biometrical Journal*, 45(8):946–958, 2003.
28. A. Račkauskas and Ch. Suquet. On Bernstein–Kantorovich invariance principle in Hölder spaces and weighted scan statistic. *ESAIM: PS*, 24:186–206, 2020.
29. Q. Yao. Tests for change-points with epidemic alternatives. *Biometrika*, 80:179–191, 1993.