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# A link between contractive and dissipative Hamiltonian systems

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**Abstract:** This paper should be intended as a preliminary work in which we describe a connection between contractive systems and dissipative Hamiltonian dynamics. In particular, we show that it is always possible to obtain from the Jacobian of the autonomous system, given the contraction metric, the structural matrices of a Hamiltonian dynamics. We show how to obtain in closed-form such matrices for (asymptotically) stable linear systems, by exploiting a modification of the Jordan block form. We then conclude the paper by analytically performing the analysis on an example of nonlinear contractive system.

*Keywords:* Contractive systems, dissipative Hamiltonian dynamics, variational dynamics, modified Jordan form, structural matrices

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## 1. CONTRACTIVE SYSTEMS AND DISSIPATIVE HAMILTONIAN DYNAMICS

### 1.1 Variational and prolonged systems

In Crouch and van der Schaft [1987] the authors show how an input-output-state system, defined on the smooth manifold  $\mathcal{X}$ , can be extended/prolonged (or ‘lifted’ as mentioned in Forni and Sepulchre [2013]) to a system on the  $2n$ -dimensional tangent bundle  $\mathcal{T}\mathcal{X}$  of the manifold  $\mathcal{X}$ , with  $2m$  inputs and  $2p$  outputs.

For this paper, we specialize our discussion on autonomous (closed) system

$$\Sigma : \begin{cases} \dot{x} = f(x) & x(0) = x_0 \end{cases} \quad (1)$$

where  $x \in \mathcal{X} \subseteq \mathbb{R}^n$ , with  $f$  continuously differentiable for all  $x \in \mathcal{X}$ , where  $\mathcal{X}$  is enclosed into a compact set.

Given an admissible state trajectory  $t \mapsto x(t)$  for  $\Sigma$ , we define the *variational system* along such a trajectory as the time-varying system

$$D\Sigma : \begin{cases} \dot{\xi} = \frac{\partial f}{\partial x}(x)\xi, & \xi(0) = \xi_0 \end{cases} \quad (2)$$

with state  $\xi \in \mathbb{R}^n$ .

The *prolonged system* of  $\Sigma$  hence corresponds to the augmented  $2n$ -dimensional system on the tangent bundle  $\mathcal{T}\mathcal{X}$  composed of the a ‘parallel’ configuration of  $\Sigma$  and  $D\Sigma$ , i.e., the prolonged dynamics is given by the dynamics

$$\mathcal{T}\Sigma : \begin{cases} \dot{x} = f(x), & x(0) = x_0 \\ \dot{\xi} = \frac{\partial f}{\partial x}(x)\xi, & \xi(0) = \xi_0. \end{cases} \quad (3)$$

In the standard notation  $\xi = \delta x$ , as employed, e.g., in van der Schaft [2013]. In the case of autonomous systems, as treated in Forni and Sepulchre [2013], the variational state  $\xi = \delta x$  refers to the tangent vector

to the parameterized curve connecting any two system trajectories. The infinitesimal variations  $\delta x(t)$  on the state  $x(t)$  can also be interpreted as being a generic ‘virtual displacement’ along any possible direction on the tangent space  $\mathcal{T}_x\mathcal{X}$ . In other words, considering the infinitesimal variations of a family of state trajectories of  $\Sigma$ , i.e.,  $x(t, \varepsilon)$  parameterized by  $\varepsilon \in (-\delta, \delta)$ , for some positive real  $\delta$ , with  $x(t, 0) = x(t)$ , the virtual displacement can be defined as

$$\delta x(t) = \frac{\partial x}{\partial \varepsilon}(t, 0),$$

satisfy the dynamical equations of  $D\Sigma$ , see van der Schaft [2013], Crouch and van der Schaft [1987] for further details.

However, if we specialize such a direction to be the one of the ‘state velocity’, i.e., we obtain the dynamics of  $\xi = \dot{x}$ . We thus describe the time evolution of the vectorfield  $f(x)$  and we can properly determine the initial conditions of the variational system  $D\Sigma$ , i.e.,  $\xi_0 = \xi(0) = f(x_0)$ .

As highlighted in Spirito et al. [2024], this choice of the variational system direction recalls the definition of Forward Contraction introduced in [Forni and Sepulchre 2013, sec. III.C].

### 1.2 Contractive dynamics

The contraction theory approach was first introduced in books such as Demidovich [1967] and Willems [1970] (see also Pavlov et al. [2004]), and extended in Lohmiller and Slotine [1998], Forni and Sepulchre [2013], Andrieu et al. [2016]. It is well-known and accepted by the community, that the contraction approach plays a crucial role in studying the stability and/or the attractiveness of invariant manifolds for nonlinear autonomous systems, acting as a counterpart of Lyapunov stability theory. In particular, one can determine the exponential convergence property by studying the properties of the variational dynamics. More specifically, this attractiveness property is equivalent

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to the existence of a positive definite quadratic form (called Lyapunov-Finsler metric in Forni and Sepulchre [2013]) which is decreasing along the flow of the autonomous system under consideration. A weaker notion of a contractive system also called a partial-contractive system in Lohmiller and Slotine [1998], is that of a semi-contractive system.

*Definition 1.1.* (Semi-contractive systems). *System (1) is said to be semi-contractive if there exists a  $C^1$  function  $P : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$ , two strictly positive real numbers  $\underline{p}$  and  $\bar{p}$  such that  $P$  has a time derivative computed along the flow of the vectorfield  $f$ , i.e., column-wise we have  $\dot{P}(x) = \frac{\partial P(x)}{\partial x} f(x)$ , and it satisfies*

$$\underline{p}I \leq P(x) \leq \bar{p}I,$$

$$\dot{P}(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f^\top}{\partial x}(x) P(x) \leq 0, \quad \forall x \in \mathcal{X}. \quad (4)$$

A system is said to be *contractive* if (4) holds with a strict sign, i.e.,

$$\dot{P}(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f^\top}{\partial x}(x) P(x) < 0, \quad \forall x \in \mathcal{X}. \quad (5)$$

A strictly related and independent concept is the one of incremental stability, that is the exponential attractiveness between any two trajectories of a system dynamics. In particular, it has been shown in [Andrieu et al. 2016, Prop. 1] that if the vectorfield  $f(x)$  is globally Lipschitz with bounded second derivative, the two concepts of contractive and incrementally stable systems are *equivalent*. Additionally, they are both equivalent, see [Andrieu et al. 2016, Prop. 1], to the global exponential attractiveness of the manifold  $\mathcal{E} = \{(x, \xi) : \xi = 0\}$  for the prolonged systems  $\mathcal{T}\Sigma$ . Furthermore, thanks to the direction choice  $\xi = \dot{x}$ , we can conclude that the manifold  $\mathcal{E}$  is the set of equilibrium points of the dynamics, which consequently implies that the system has a single asymptotically stable equilibrium in the contraction region.

### 1.3 Dissipative Hamiltonian system

In classical mechanics, the notion of dissipative Hamiltonian systems is well-known and it forms the basis for the port-Hamiltonian systems first introduced in Maschke and van der Schaft [1993], the latter being the open systems version of the former (also referred to as closed systems).

*Definition 1.2.* (Hamiltonian system). *A nonlinear system (1), with  $x \in \mathcal{X} \subset \mathbb{R}^n$ , is said to be Hamiltonian if its vectorfield has the following structure*

$$\dot{x} = (J(x) - R(x)) \nabla H(x) \quad (6)$$

where  $J : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$  is skew symmetric,  $R : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$  is a symmetric matrix, and  $H : \mathcal{X} \rightarrow \mathbb{R}$  is called the Hamiltonian function.

In particular, the skew-symmetric matrix  $J(x)$  describes the underlying Dirac structure (i.e., the energy transfer between different physical domains), the symmetric matrix  $R(x)$  refers to the presence of dissipative elements in the system, and the Hamiltonian function  $H(x)$  is associated to the system energy. We say that a nonlinear Hamiltonian system (6) is (strictly) *dissipative* wherever its associated energy/storage function, or Hamiltonian,  $H$

is lower bounded and satisfies equation ((8)) (7). This definition can be easily verified by checking the (positive) semi-positive definiteness of the symmetric matrix  $R$ , via the energy balance leading to the following inequality, respectively

$$\dot{H} = \nabla H^\top (J - R) \nabla H = -\nabla H^\top R \nabla H \leq 0. \quad (7)$$

$$\dot{H} = \nabla H^\top (J - R) \nabla H = -\nabla H^\top R \nabla H < 0. \quad (8)$$

Since we generally assume that  $H$  is lower-bounded, one can use it as a Lyapunov function and prove the stability of the system's origin/equilibrium point, possibly exploiting Lasalle's invariance principle. In particular, the asymptotic convergence property has only been shown for linear dissipative Hamiltonian systems, as reported in the following Proposition.

*Proposition 1.* *Given constant matrices  $J$  and  $R$ , and Hamiltonian function  $H = \frac{1}{2} x^\top Q x$  with  $Q = Q^\top \geq 0$ . Denote by  $k$  the dimension of the kernel of  $R \geq 0$ , and assume it is spanned by  $\{r_1, \dots, r_k\}$ . Then the dissipative Hamiltonian system (6) is asymptotically stable if and only if the matrix*

$$(sI - QJ) [r_1 | r_2 | \dots | r_k]$$

has rank  $k$ , for every  $s = j\omega$ ,  $\omega \in \mathbb{R}$ .

This proposition has been shown in [Prajna et al. 2002, Proposition 6], and its proof is thus omitted.

In the remainder of the paper, our main result is presented in the next section, i.e., Section 2, in which we describe the link between contractive systems and dissipative Hamiltonian dynamics. We then show, in Section 3, a method to bring a stable linear system into a dissipative Hamiltonian dynamics by exploiting a modification of the Jordan block form whose details are given in the document appendix. We then perform, as a nonlinear example, the described analysis on a contractive dynamics, in Section 4, providing the system metric and obtaining the relative Hamiltonian structural matrices. However, although concluding analytically that the system is then contractive is too complex to deal with at the present stage, the presented nonlinear dynamics can be used as a possible reference point or benchmark for future works. We then summarize the paper content in Section 5. The notation paragraph can be found at the beginning of the appendix.

## 2. MAIN RESULT

The main result of this paper, expressed by the following Lemma along with the successive Theorem, describes a strong link between contracting systems and Hamiltonian dynamics.

*Lemma 1.* *Consider an autonomous system (1) and assume there exists matrices  $J(x) = -J(x)^\top$ ,  $R(x) = R^\top(x)$ , and  $Q(x) = Q(x)^\top > 0$  upper bounded (or invertible) for all  $x \in \mathcal{X}$ , such that*

$$\frac{\partial f(x)}{\partial x} = [J(x) - R(x)] Q(x) \quad (9)$$

then the system (1) is contractive in  $\mathcal{X}$  with metric  $Q(x)$  if

$$\dot{Q}(x) - 2Q(x)R(x)Q(x) < 0, \quad \forall x \in \mathcal{X}. \quad (10)$$

*Proof.* Because we have (9), where  $Q(x)$  is assumed to be positive definite for all  $x$ , we can take as Lyapunov function the quadratic form

$$\mathcal{H} = \frac{1}{2}\xi^\top Q(x)\xi + c$$

for some real value  $c$ , and its time derivative reads as

$$\dot{\mathcal{H}} = \frac{1}{2}\xi^\top \dot{Q}(x)\xi + \xi^\top Q(x)[J(x) - R(x)]Q(x)\xi < 0, \forall x$$

by assumption. Hence, we proved that the system is contractive with metric  $Q(x)$ .  $\square$

Although assuming (9) might seem too restrictive for a general system, it is not the case. In particular, as we exploited in the following theorem, the matrix  $Q(x)$  possibly plays the role of the contractive metric  $P(x)$  introduced in Sec. 1.2, and the latter satisfies the LMI (5) for contractive systems.

Moreover, it is easy to show that any contracting system has the form (9) and can be brought into it once the metric  $P(x)$  is known, as shown in the following theorem.

*Theorem 2.1.* Assume (1) is contracting in  $\mathcal{X}$  with metric  $P(x)$ , then its variational dynamics (2) is a strict dissipative Hamiltonian with structural matrices

$$J(x) = \text{skew} \left[ \frac{\partial f(x)}{\partial x} P^{-1}(x) \right] \quad (11)$$

$$R(x) = -\text{sym} \left[ \frac{\partial f(x)}{\partial x} P^{-1}(x) \right], \quad (12)$$

Hamiltonian function  $H(x, \xi) = \frac{1}{2}\xi^\top P(x)\xi$ , and  $\nabla H(x) = P(x)\xi$ .

*Proof.* By definition of a contractive system, there exists a positive definite  $P(x)$  metric satisfying the LMI condition (5). In particular, we can always write

$$\frac{\partial f(x)}{\partial x} P^{-1}(x) = J(x) - R(x)$$

where  $J(x)$  and  $R(x)$  are the antisymmetric and the symmetric part of the resulting matrix on the left-hand side, i.e., (11) and (12). Thus, it is easy to see that (8) corresponds to (5). Indeed, from (5), we can write

$$\begin{aligned} \dot{P} + P \frac{\partial f(x)}{\partial x} P^{-1} P + P P^{-1} \frac{\partial f(x)}{\partial x}^\top P &< 0 \\ \dot{P}(x) + P(x) \left( \frac{\partial f(x)}{\partial x} P^{-1}(x) + P^{-1}(x) \frac{\partial f(x)}{\partial x}^\top \right) P(x) &< 0 \\ \dot{P}(x) - 2P(x)R(x)P(x) &< 0 \end{aligned}$$

by definition of  $R(x)$ , and this is exactly (8).  $\square$  Moreover, if we additionally pre- and post-multiply by  $P^{-1}(x)$  last inequality of the proof, we obtain the ‘simpler’ condition for contraction

$$\frac{dP^{-1}(x)}{dt} + 2R(x) > 0 \quad (13)$$

only involving the inverse of  $P(x)$  and the symmetric part of the system structure. In some sense, from this new point of view, the contractive property is not simply related to the positive definiteness of  $R(x)$ , but it is rather a property associated with the matrices pair  $(P^{-1}(x), R(x))$ , whenever  $P$  is not constant. Indeed, otherwise, such a contraction property only depends on the positive definiteness of  $R(x)$ .

*Remark 1.* It is worth noticing that the dissipative Hamiltonian system is a weaker notion of contractive system, especially when the metric  $P$  is constant (we talk about convergent systems rather than contractive systems) or when the Hamiltonian  $H$  is a quadratic form.

*Remark 2.* On the other hand, semi-convergent systems, i.e., inequality (5) does not hold with a strict sign, are a weaker notion of dissipative Hamiltonian system since they allow the gradient of  $H$  and thus  $H$  to depend on an external variable. Indeed, classically the gradient of  $H$  in a dissipative Hamiltonian dynamics (6) only depends on the system state, but in principle, the Hamiltonian of the system can be time-varying, especially when considering perturbed systems.

### 3. EXPLICIT DISSIPATIVE HAMILTONIAN DYNAMICS STRUCTURE OF LINEAR STABLE SYSTEMS

It is well known that linear asymptotically stable autonomous systems

$$\dot{x} = Ax \quad (14)$$

are contractive systems. Furthermore, one can find a constant contractive metric  $P$  by solving the standard Lyapunov inequality for some positive real  $q > 0$

$$PA + A^\top P \leq -2qI \quad (15)$$

and it is easy to show that any asymptotically stable autonomous linear system is indeed a strict linear dissipative Hamiltonian system

$$\dot{x} = (J - R)Qx \quad (16)$$

with Hamiltonian function

$$H = \frac{1}{2}x^\top Qx = \frac{1}{2}x^\top Px.$$

The structure matrices  $J$  and  $R$  are then given by

$$\begin{aligned} J &= \text{skew}(AP^{-1}) \\ R &= -\text{sym}(AP^{-1}) \end{aligned}$$

with  $P$  solution of (15). However, we can provide a much more constructive way to obtain  $J$ ,  $R$  and  $H$  for linear systems even when the system is stable but not asymptotically stable, by exploiting the eigenstructure of the state matrix  $A$ , as detailed in the following Lemma.

*Lemma 2.* Given a stable linear autonomous system (14), and the modified Jordan form  $\mathbb{J} = \mathbb{T}^{-1}A\mathbb{T}$ , where  $\mathbb{T} = T\mathbb{D}$  as in (B.3). Then the system can always be put in dissipative Hamiltonian form (16) with structural matrices

$$\begin{aligned} J &= \text{skew}(\mathbb{T}\mathbb{J}\mathbb{T}^\top) \\ R &= -\text{sym}(\mathbb{T}\mathbb{J}\mathbb{T}^\top) \end{aligned}$$

and Hamiltonian function

$$H = \frac{1}{2}x^\top \mathbb{T}^{-\top} \mathbb{T}^{-1} x.$$

*Proof.* The proof readily comes from the definition of the modified Jordan block  $\mathbb{J}$  defined in the appendix, which is guaranteed to exist because the system is stable, so all the eigenvalues of  $A$  on the imaginary axis are simple. Then, we have

$$\dot{x} = Ax = \mathbb{T}\mathbb{J}\mathbb{T}^{-1}x = (\mathbb{T}\mathbb{J}\mathbb{T}^\top)\mathbb{T}^{-\top}\mathbb{T}^{-1}x$$

and by defining  $Q = \mathbb{T}^{-\top}\mathbb{T}^{-1}$  which is always full rank and positive definite, we can simply take the antisymmetric

and symmetric part of  $\mathbb{T}\mathbb{J}\mathbb{T}^\top$  to define  $J$  and  $R$ , respectively. By construction of the transformation  $\mathbb{T}$  we have indeed that  $R \geq 0$ .  $\square$

We thus see, that the transformation to get the modified Jordan form for the state matrix  $A$  ‘naturally’ defines a metric,  $\mathbb{T}^{-\top}\mathbb{T}^{-1}$ , through which one can determine the (semi-) contractivity property of the system.

Note that, to the best of the authors’ knowledge, the only available result that shows how to obtain the port-Hamiltonian system matrices is Beattie et al. [2019], whose approach is based on the solution of Algebraic Riccati inequality, of which (15) is a particular case. However, with the proposed approach, we describe a method to obtain the Hamiltonian system matrices, by relating them directly to the eigenstructure of the system state matrix.

For nonlinear systems, the problem is more involved and no result is available in the literature. A possible approach is currently under investigation and will possibly be the subject of future work.

#### 4. A NONLINEAR EXAMPLE

Let us consider the nonlinear dynamics introduced in Kawano and Ohtsuka [2015][ex.8], i.e.,

$$\begin{aligned} \dot{x}_1 &= -x_2(-1 + 4x_1 + 4x_2 - 4x_2^2) \\ \dot{x}_2 &= -2(x_1 + x_2 - x_2^2) \end{aligned} \quad (17)$$

for which it has been shown that the origin is globally asymptotically stable. The related Jacobian matrix is given by

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -4x_2 & 1 - 8x_2 + 12x_2^2 - 4x_1 \\ -2 & -2 + 4x_2 \end{bmatrix},$$

and for this system, we introduce the metric

$$P(x) = \begin{bmatrix} 1 & \frac{1}{2} - 2x_2 \\ \frac{1}{2} - 2x_2 & \frac{1}{4} + (2x_2 - \frac{1}{2})^2 \end{bmatrix}$$

whose principle minors are  $1 > 0$  and its determinant  $\det(P) = 1/4 > 0$ , thus showing it is everywhere positive definite. Thus, from the Jacobian expression and the introduced metric, we can deduce that the Hamiltonian matrices of the system’s variational dynamics  $J(x)$  and  $R(x)$ , respectively in (11) and (12), are given by

$$J(x) = \begin{bmatrix} 0 & 8x_2^2 - 8x_2 - 8x_1 + 2 \\ -(8x_2^2 - 8x_2 - 8x_1 + 2) & 0 \end{bmatrix}$$

and

$$R(x) = \begin{bmatrix} \star & 8x_2^2 - 16x_2 - 8x_1 + 2 \\ 8x_2^2 - 16x_2 - 8x_1 + 2 & -4 \end{bmatrix}$$

where

$$\star = 8x_1 + 16x_2 - 32x_1x_2 - 56x_2^2 + 32x_2^3 - 2.$$

One can easily see that  $R$  is not sign definite. However, to show contraction one can verify that  $\dot{P} - 2PRP < 0$ .

The method used to obtain the metric  $P$ , to show that it is a contractive metric and an equivalent approach to show the stability of the involved system is currently under consideration and will be the subject of future work on the topic.

#### 5. CONCLUSIONS

In this paper, we present a link between contractive systems and dissipative Hamiltonian dynamics. In particular,

given a contractive autonomous system, we showed that its variational dynamics can always be considered to be a strictly dissipative Hamiltonian system. In particular, we establish a direct relationship between the two notions. For linear (asymptotically) stable autonomous systems we show a method to explicitly obtain the dissipative Hamiltonian realization of the system state matrix by exploiting a particular change of coordinates. We then perform the analysis on a nonlinear contractive dynamics.

#### Appendix A. NOTATION

We denote with  $\mathbb{R}$  the set of real numbers. Given a matrix  $P \in \mathbb{R}^{n \times n}$ , we denote by  $\sigma(P)$  its spectrum, while  $\lambda_i(P)$  its the  $i$ -th eigenvalue. Given a matrix  $M \in \mathbb{R}^{n \times n}$ , we denote by  $\text{skew}(M)$  and  $\text{sym}(M)$  its antisymmetric and its symmetric parts, i.e.,  $\text{skew}(M) = (M - M^\top)/2$  and  $\text{sym}(M) = (M + M^\top)/2$ .

##### A.1 Some additional nomenclature

Given any  $n \in \mathbb{N}$ , we define the following matrices

$$\mathbb{J}_n := I_n + N_n = \begin{bmatrix} 1 & 1 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 1 & 1 \end{bmatrix}, \quad N_n = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \quad (A.1)$$

where  $I_n$  is the identity matrix of dimension  $n$  and  $N_n$  is a nilpotent matrix with ones on the upper diagonal and zeros elsewhere. Given a number  $\mu \in \mathbb{R}$  and a natural number  $n \in \mathbb{N}$ , we denote

$$D_n(\mu) := \begin{cases} \text{diag}(1, \mu, \dots, \mu^{n-1}), & \text{if } \mu \neq 0, \\ I_n, & \text{if } \mu = 0. \end{cases} \quad (A.2)$$

Then, with the previous definition, the following identity

$$N_n D_n(\mu) = \mu D_n(\mu) N_n \quad (A.3)$$

holds for any  $\mu \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Similarly, for a matrix  $A \in \mathbb{R}^{r \times r}$  we can define

$$D_n(A) := \begin{cases} \text{diag}(I_r, A, \dots, A^{n-1}), & \text{if } A \neq 0, \\ I_{rn}, & \text{if } A = 0, \end{cases} \quad (A.4)$$

and in this case, the following identity can be verified

$$D_n(A)^{-1} (N_n \otimes I_r) D_n(A) = N_n \otimes A. \quad (A.5)$$

From [Meyer 2000, Ex. 7.2.5], we have that given a matrix  $A \in \mathbb{R}^{n \times n}$  of the form

$$A = \begin{bmatrix} b & a & & \\ c & b & a & \\ & \ddots & \ddots & \ddots \\ & & c & b & a \\ & & & c & b \end{bmatrix} = bI_n + aN_n + cN_n^\top, \quad (A.6)$$

with  $a \neq 0$  and  $c \neq 0$ , its eigenvalues  $\lambda_i$  are given by the explicit formula

$$\lambda_i = b + 2a\sqrt{\frac{c}{a}} \cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \dots, n\}.$$

and thus for the case  $c = a$  we have

$$\lambda_i = b + 2a \cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \dots, n\}. \quad (A.7)$$

Given a matrix  $A$  of dimensions  $n \times n$  we suppose, without loss of generality, that its eigenvalues are ordered in decreasing order with respect to the real part, namely

$$\Re\{\lambda_1\} \geq \Re\{\lambda_2\} \geq \dots \geq \Re\{\lambda_n\}.$$

Let  $m \leq n$  be the total number of linearly independent (non-generalized) eigenvectors  $T_i^1 \neq 0$  associated with an eigenvalue  $\bar{\lambda}_i \in \sigma(A)$ ,  $i = 1, \dots, m$ , such that

$$AT_i^1 = \bar{\lambda}_i T_i^1 \quad \forall i = 1, \dots, m.$$

*Definition B.1.* (Jordan blocks dimension). For each  $i \in \{1, \dots, m\}$ , we define the values  $g_i \geq 1$  satisfying  $\sum_{i=1}^m g_i = n$ , such that there exist  $g_i - 1$  linearly independent generalized eigenvectors  $T_i^k \neq 0$ , for  $k = 2, \dots, g_i$ , associated to the corresponding eigenvalue  $\bar{\lambda}_i$  and satisfying

$$(A - \bar{\lambda}_i I)T_i^k = T_i^{k-1} \quad \forall k = 2, \dots, g_i.$$

The introduced notation allows us to determine in advance the number of distinct Jordan blocks  $m$  and their relative dimensions  $g_i$ , when the matrix  $A$  is transformed into its Jordan form  $J$ , see, e.g. Lancaster and Tismenetsky [1985],

$$T^{-1}AT = J, \quad \begin{cases} J := \text{blkdiag}(J_{\bar{\lambda}_1}, \dots, J_{\bar{\lambda}_m}), \\ J_{\bar{\lambda}_i} := \bar{\lambda}_i I_{g_i} + N_{g_i}, \quad i = 1, \dots, m. \end{cases} \quad (\text{B.1})$$

To obtain the Jordan form modification, see additional details in Spirito and Astolfi [2024a] and Spirito and Astolfi [2024b], we define the matrix

$$D := \text{blkdiag}(D_{g_1}(\bar{\lambda}_1), \dots, D_{g_m}(\bar{\lambda}_m)) \quad (\text{B.2})$$

and we determine the matrix  $\mathbb{J}$  as

$$\mathbb{J} := T^{-1}AT, \quad T := TD, \quad (\text{B.3})$$

in which  $T$  satisfies (B.1), and such a Jordan form  $\mathbb{J}$  as the following form

$$\mathbb{J} = \text{blkdiag}(\bar{\lambda}_1 \mathbb{J}_{g_1}, \dots, \bar{\lambda}_m \mathbb{J}_{g_m}), \quad (\text{B.4})$$

with the matrix  $\mathbb{J}_{g_i}$  defined as in (A.1), for each  $i = 1, \dots, m$ . Moreover, in the case of Jordan blocks associated complex eigenvalues, by using the definition (A.4), one can verify, for the real matrix representation of these Jordan blocks, the following identity

$$\mathbb{J}_{g_i} \otimes \bar{\Lambda}_i = (I_{g_i} + N_{g_i}) \otimes \bar{\Lambda}_i = D_{g_i}^{-1}(\bar{\Lambda}_i) J_{\bar{\Lambda}_i} D_{g_i}(\bar{\Lambda}_i)$$

which gives

$$\mathbb{J}_{g_i} \otimes \bar{\Lambda}_i = \begin{bmatrix} \bar{\Lambda}_i & \bar{\Lambda}_i & 0 & \dots & 0 \\ 0 & \bar{\Lambda}_i & \bar{\Lambda}_i & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \bar{\Lambda}_i & \bar{\Lambda}_i & \\ 0 & \dots & 0 & \bar{\Lambda}_i & \end{bmatrix} \quad (\text{B.5})$$

where  $\bar{\Lambda}_i$  is the real matrix representation of a pair of complex conjugate eigenvalues  $(\bar{\lambda}_i, \bar{\lambda}_{i+1})$ .

Other properties of the  $\mathbb{J}_n$  matrix can be found in Spirito and Astolfi [2024b] along with applications to solving generalized Lyapunov inequalities, while other applications of the modified normal form, related to stability properties, controller and observer design with prescribed performances (e.g., convergence time), can be found in Spirito and Astolfi [2024a] and Spirito and Astolfi [2024c].

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