



HAL
open science

CONJUGATE VARIABLES APPROACH TO MIXED q -ARAKI WOODS ALGEBRAS: FACTORIALITY AND NON-INJECTIVITY

Manish Kumar

► **To cite this version:**

Manish Kumar. CONJUGATE VARIABLES APPROACH TO MIXED q -ARAKI WOODS ALGEBRAS: FACTORIALITY AND NON-INJECTIVITY. *Journal of Mathematical Physics*, 2023, 64 (9), pp.19. <hal-04496487>

HAL Id: hal-04496487

<https://hal.science/hal-04496487v1>

Submitted on 8 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

CONJUGATE VARIABLES APPROACH TO MIXED q -ARAKI-WOODS ALGEBRAS: FACTORIALITY AND NON-INJECTIVITY

MANISH KUMAR

ABSTRACT. We establish factoriality and non-injectivity in full generality for the mixed q -Araki-Woods von Neumann algebra associated to a separable real Hilbert space $\mathbf{H}_{\mathbb{R}}$ with $\dim \mathbf{H}_{\mathbb{R}} \geq 2$, a strongly continuous one parameter group of orthogonal transformations on $\mathbf{H}_{\mathbb{R}}$, a direct sum decomposition $\mathbf{H}_{\mathbb{R}} = \oplus_i \mathbf{H}_{\mathbb{R}}^{(i)}$, and a real symmetric matrix (q_{ij}) with $\sup_{i,j} |q_{ij}| < 1$. This is achieved by first proving the existence of conjugate variables for a finite number of generators of the algebras (when $\dim \mathbf{H}_{\mathbb{R}} < \infty$), following the lines of Miyagawa-Speicher and Kumar-Skalski-Wasilewski. The conjugate variables belong to the factors in question and satisfy certain Lipschitz conditions.

1. INTRODUCTION

We study the factoriality and non-injectivity question of mixed q -Araki-Woods von Neumann algebras. These are non-tracial counterpart of mixed q -Gaussian algebras introduced by Speicher [Spe] and Bożejko-Speicher [BS₂] which followed their definition of (non-mixed) q -Gaussian algebras in [BS₁]. Mixed q -Gaussian algebras are associated with two data $(\mathbf{H}_{\mathbb{R}}, (q_{ij}))$ where $\mathbf{H}_{\mathbb{R}}$ is a real separable Hilbert space and $-1 < q_{ij} = q_{ji} < 1$ are real numbers for $1 \leq i, j \leq \dim \mathbf{H}_{\mathbb{R}}$ such that $\sup_{i,j} |q_{ij}| < 1$. The von Neumann algebra is then generated by a family of operators $\{l_i + l_i^*\}_{1 \leq i \leq \dim \mathbf{H}_{\mathbb{R}}}$ acting on certain deformed Fock space where l_i 's obey the mixed q_{ij} commutation relations:

$$l_i l_j^* - q_{ij} l_j^* l_i = \delta_{ij}.$$

The non-mixed scenario corresponds to the case where $q_{ij} = q$ for all i, j . The structure of these algebras have attracted deep analysis over the decades; in particular factoriality of (mixed) q -Gaussian algebras is work of many hands. Ricard [Ric] established the factoriality of q -Gaussian algebras in full generality (for $\dim \mathbf{H}_{\mathbb{R}} \geq 2$) while several partial results were obtained earlier in [BKS, Kro₁, Sni]. Following the ideas of [Ric], Skalski and Wang in [SW] solved the factoriality question of mixed q -Gaussian algebras in full generality ($\dim \mathbf{H}_{\mathbb{R}} \geq 2$) while again some partial results preceded this in [Kro₂, NZ]. The citations listed here are by no means complete for the study of various other structural properties of the Gaussian algebras.

Both q -Gaussian and mixed q -Gaussian von Neumann algebras are tracial and are natural deformations of Voiculescu's free Gaussian functor associated to real Hilbert spaces. Another natural generalization of the free Gaussian functor was introduced by Shlyakhtenko [Shl] in the non-tracial framework which can also be regarded as analogues of injective factors coming from CAR relations. The associated algebras depend on one parameter groups of orthogonal transformations U_t on real Hilbert spaces $\mathbf{H}_{\mathbb{R}}$. They are non-injective factors of type III as soon as $\dim \mathbf{H}_{\mathbb{R}} \geq 2$ and U_t is non-trivial, and they are called free Araki-Woods factors.

Hiai in [Hia] combined the functor of [BS₁] and [Shl] to produce the so called q -Araki-Woods von Neumann algebras associated to $(\mathbf{H}_{\mathbb{R}}, U_t, q)$. These algebras have equally received a lot of attention, particularly because they provide a concrete approach to the study of certain type III factors. On one hand factoriality and non-injectivity was determined completely for the free Araki-Woods algebras (i.e. when $q = 0$) in [Shl], whereas the same for q -deformation counterparts for $q \neq 0$ appears to be more involved, with only partial results obtained over the years in

1991 *Mathematics Subject Classification*. Primary: 46L36; Secondary 46L10, 46L53, 46L65.

Key words and phrases. mixed q -Araki-Woods von Neumann algebra, factoriality, non-injectivity, conjugate variables.

[Hia, Ne₁, BM, SW, BMRW, KSW]. Remark that the factoriality was established by Bikram-Mukherjee-Ricard-Wang in [BMRW] for $\dim \mathbf{H}_{\mathbb{R}} \geq 3$, and for $\dim \mathbf{H}_{\mathbb{R}} \geq 2$ with U_t ergodic and q large. The complete answer to the factoriality and non-injectivity question of such algebras have been settled recently by Skalski, Wasilewski and the author in [KSW]. Unlike the free case scenario, various structural properties of q -deformed algebras like fullness, absence of Cartan subalgebras, solidity etc are still far from full understanding.

In [BKM₁], Bikram, Kumar R. and Mukherjee further extended Hiai's construction to now include the *mixed* q -Gaussian functor of [BS₂] as well as the functor of [Shl]. The concerned von Neumann algebras are called *mixed q -Araki-Woods von Neumann algebras*. Now the algebras are associated with data consisting of $(U_t, \mathbf{H}_{\mathbb{R}}^{(i)}, (q_{ij})_{i,j \in N}, T)$ where N is a countable set, $\mathbf{H}_{\mathbb{R}}^{(i)}, i \in N$ is a real separable Hilbert space, parameters $q_{ij} \in (-1, 1)$ satisfy $\sup_{i,j \in N} |q_{ij}| < 1$, U_t is a strongly continuous one parameter group of orthogonal operators on $\mathbf{H}_{\mathbb{R}} := \bigoplus_{i \in N} \mathbf{H}_{\mathbb{R}}^{(i)}$ such that each $\mathbf{H}_{\mathbb{R}}^{(i)}$ is invariant under U_t , and $T \in \mathbf{B}(\mathbf{H}_{\mathbb{R}} \otimes \mathbf{H}_{\mathbb{R}})$ is a Yang-Baxter operator which interpolates $\mathbf{H}_{\mathbb{R}}^{(i)} \otimes \mathbf{H}_{\mathbb{R}}^{(j)}$ and $\mathbf{H}_{\mathbb{R}}^{(j)} \otimes \mathbf{H}_{\mathbb{R}}^{(i)}$ with the twist q_{ij} . The mixed q -Araki-Woods algebra $\Gamma_T(\mathbf{H}_{\mathbb{R}}, U_t)$ is then generated by a set of operators $\{a(\xi) + a^*(\xi); \xi \in \mathbf{H}_{\mathbb{R}}^{(i)}, i \in N\}$ acting on a deformed Fock-space, satisfying the following commutation relations:

$$a(\xi)a^*(\eta) - q_{ij}a^*(\eta)a(\xi) = \langle \xi, \eta \rangle_U$$

for $\xi \in \mathbf{H}_{\mathbb{R}}^{(i)}$ and $\eta \in \mathbf{H}_{\mathbb{R}}^{(j)}$ where $\langle \cdot, \cdot \rangle_U$ is a twisted scalar product on the complexification of $\mathbf{H}_{\mathbb{R}}$ with the twist arising out of the representation U_t (see the formal construction in Section 2).

Previous investigations of mixed q -Araki-Woods algebras in [BKM₁] established factoriality and the type classification of $\Gamma_T(\mathbf{H}_{\mathbb{R}}, U_t)$ under different conditions i.e. whenever U_t is weakly mixing or U_t admits a fixed non-zero vector or when U_t is almost-periodic on an infinite dimensional space $\mathbf{H}_{\mathbb{R}}$ with the set of eigenvalues of A having a non-zero limit point where A is the analytic generator of U_t . They also prove non-injectivity of the algebras in certain cases i.e. when the weakly mixing part of U_t is non-zero. Several other nice properties are illustrated in [BKM₁] mostly generalizing those of their sister algebras of non-mixed situation.

However the case when the dimension of the underlying Hilbert space $\mathbf{H}_{\mathbb{R}}$ is finite could not be fully understood. In this paper, we finally settle the factoriality and non-injectivity question of mixed q -Araki-Woods algebras in full generality. We summarize our main results in the following theorem which is a combination of Theorem 4.2 and Theorem 4.3:

Theorem 1.1. *Let $(\mathbf{H}_{\mathbb{R}} = \bigoplus_{i \in N} \mathbf{H}_{\mathbb{R}}^{(i)}, U_t, (q_{ij})_{i,j \in N}, T)$ be given with $\dim \mathbf{H}_{\mathbb{R}} \geq 2$, $q = \max_{i,j} |q_{ij}| < 1$ and N countable. Then the von Neumann algebra $\Gamma_T(\mathbf{H}_{\mathbb{R}}, U_t)$ is a non-injective factor of type*

$$\begin{cases} \text{III}_1 & \text{if } G = \mathbb{R}_*^{\times}, \\ \text{III}_{\lambda} & \text{if } G = \lambda^{\mathbb{Z}}, 0 < \lambda < 1, \\ \text{II}_1 & \text{if } G = \{1\}, \end{cases}$$

where $G < \mathbb{R}_*^{\times}$ is the closed subgroup generated by the eigenvalues of the generator A of $(U_t)_{t \in \mathbb{R}}$.

Our approach towards the proof utilises the concepts of conjugate variables following the lines of arguments by Miyagawa-Speicher [MS], Kumar-Skalski-Wasilewski [KSW] and Nelson [Ne₂]. The key idea of previous attempts towards factoriality in [BKM₁] is the study of mixing properties of certain subalgebras which allows one to control the relative commutant of the centralizer, whereas our methods take completely different route.

We begin with the construction and basic properties of mixed q -Araki-Woods von Neumann algebras in Section 2. We recollect the notions of dual/conjugate variables in Section 3 and then exhibit in Theorem 3.4 the existence of a system of conjugate variables for a set of generators for the von Neumann algebra $\Gamma_T(\mathbf{H}_{\mathbb{R}}, U_t)$ whenever $\dim \mathbf{H}_{\mathbb{R}} < \infty$. The conjugate variables are shown to lie in the ambient von Neumann algebras (Corollary 3.5).

Moreover we prove in Corollary 3.10 that the conjugate variables are Lipschitz (see Section 3 for the definition of Lipschitz condition). Remark that the Lipschitz condition was not addressed in [KSW] even for $q_{ij} = q$ case. In this paper we do not precisely indicate an application of the Lipschitz conjugate variables, nevertheless we believe that the Lipschitz condition may provide

pleasant applications similar to the tracial case and we are working to achieve some in our upcoming work. To mention one striking application from the tracial settings, in conjugation with abstract results of [DI], the Lipschitz condition allowed [MS] to furnish a different proof for the absence of Cartan subalgebras in q -Gaussian factors with finite symbols.

Finally in Section 4, the existence of conjugate variables allows us to invoke the abstract results of Nelson [Ne₂] to prove factoriality and non-injectivity of $\Gamma_T(\mathbf{H}_{\mathbb{R}}, U_t)$ for $\mathbf{H}_{\mathbb{R}}$ finite dimensional. For a general (separable) Hilbert space $\mathbf{H}_{\mathbb{R}}$ with U_t being almost periodic (remaining cases are dealt in [BKM₁]), an inductive limit argument is used to deduce factoriality while some additional arguments are added to deduce non-injectivity.

After the completion of this article, the author learnt that parts of the results about non-injectivity of $\Gamma_T(\mathbf{H}_{\mathbb{R}}, U_t)$ have been obtained independently by [BKM₂]. I thank Rahul Kumar for making me aware of their results from [BKM₂] and for sending the draft which was not available online. More precisely they prove non-injectivity of $\Gamma_T(\mathbf{H}_{\mathbb{R}}, U_t)$ under certain conditions on the dimension of a fixed spectral projection of the analytic generator A of U_t (see [BKM₂, Theorem 4.3, Corollary 4.4]). However their investigations which still leave a large number of cases open are inspired from the ideas of [No₁] proving the equivalent conditions of non-semi-discreteness, and they do not take into account the point of view of conjugate and dual variables as we have done.

2. CONSTRUCTION OF MIXED q -ARAKI-WOODS VON NEUMANN ALGEBRAS

We briefly describe the construction of mixed q -Araki-Woods von Neumann algebras and refer the readers to [BKM₁] for a detailed treatment. As a convention, all Hilbert spaces are separable and inner products are linear in the second variable. For any real or complex Hilbert space H , $\mathbf{B}(H)$ denotes the algebra of bounded operators on H .

Let $\mathbf{H}_{\mathbb{R}}$ be a real Hilbert space, and let U_t be a strongly continuous orthogonal representations of \mathbb{R} on $\mathbf{H}_{\mathbb{R}}$. Consider the complexification $\mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathbf{H}_{\mathbb{R}}$, whose inner product is denoted by $\langle \cdot, \cdot \rangle_{\mathbf{H}_{\mathbb{C}}}$. Identify $\mathbf{H}_{\mathbb{R}}$ inside $\mathbf{H}_{\mathbb{C}}$ by $\mathbf{H}_{\mathbb{R}} \otimes 1$, so that $\mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{R}} + i\mathbf{H}_{\mathbb{R}}$. Given $\xi = \xi_1 + i\xi_2 \in \mathbf{H}_{\mathbb{C}}$ for some $\xi_1, \xi_2 \in \mathbf{H}_{\mathbb{R}}$, write the canonical involution $\bar{\xi}$ as

$$\bar{\xi} = \xi_1 - i\xi_2.$$

We extend U_t to a strongly continuous one parameter group of unitaries on the complexification $\mathbf{H}_{\mathbb{C}}$, which is still denoted by U_t . Let A denote the positive, self-adjoint, and non-singular analytic (unbounded) operator on $\mathbf{H}_{\mathbb{C}}$ obtained via Stone's theorem, which satisfies $U_t = A^{it}$ for all $t \in \mathbb{R}$. Note that $\overline{A\xi} = A^{-1}\bar{\xi}$ for any ξ in the domain of A . In particular, $\sigma(A) \cap (0, \infty)$ is symmetric around 1 i.e. for $\lambda \neq 0$, $\lambda \in \sigma(A)$ if and only if $1/\lambda \in \sigma(A)$, where $\sigma(A)$ is the spectrum of A .

Following Shlyakhtenko [Shl], consider the deformed inner product $\langle \cdot, \cdot \rangle_U$ on $\mathbf{H}_{\mathbb{C}}$ defined by

$$\langle \xi, \zeta \rangle_U = \langle \xi, 2(1 + A^{-1})^{-1}\zeta \rangle \quad \forall \xi, \zeta \in \mathbf{H}_{\mathbb{C}}.$$

We denote the completion of $\mathbf{H}_{\mathbb{C}}$ with respect to $\langle \cdot, \cdot \rangle_U$ by \mathbf{H}_U . The space $(\mathbf{H}_{\mathbb{R}}, \|\cdot\|_{\mathbf{H}_{\mathbb{R}}})$ is naturally identified as a subspace of \mathbf{H}_U such that $\mathbf{H}_{\mathbb{R}} + i\mathbf{H}_{\mathbb{R}}$ is dense in \mathbf{H}_U and $\mathbf{H}_{\mathbb{R}} \cap i\mathbf{H}_{\mathbb{R}} = \{0\}$. Let N denote a countable set $\{1, \dots, r\}$ for $r \in \mathbb{N}$ or $r = \infty$. Fix a decomposition of $\mathbf{H}_{\mathbb{R}}$

$$\mathbf{H}_{\mathbb{R}} = \bigoplus_{i \in N} \mathbf{H}_{\mathbb{R}}^{(i)}$$

into a direct sum of invariant subspaces of U_t . Fix $-1 < q_{ij} = q_{ji} < 1$ for $i, j \in N$ such that $\sup_{i, j \in N} |q_{ij}| < 1$. We will often write $q(i, j)$ instead of q_{ij} . For each $i, j \in N$ define the operator $T_{ij} : \mathbf{H}_{\mathbb{R}}^{(i)} \otimes \mathbf{H}_{\mathbb{R}}^{(j)} \rightarrow \mathbf{H}_{\mathbb{R}}^{(j)} \otimes \mathbf{H}_{\mathbb{R}}^{(i)}$ to be the bounded linear extension of the assignment

$$(2.1) \quad \xi \otimes \eta \mapsto q_{ij}\eta \otimes \xi$$

for $\xi \in \mathbf{H}_{\mathbb{R}}^{(i)}, \eta \in \mathbf{H}_{\mathbb{R}}^{(j)}$. Set $T = \bigoplus_{i, j \in N} T_{ij} \in \mathbf{B}(\mathbf{H}_{\mathbb{R}} \otimes \mathbf{H}_{\mathbb{R}})$. Naturally T extends linearly to $\mathbf{H}_{\mathbb{C}} \otimes \mathbf{H}_{\mathbb{C}}$ and then to a bounded operator on $\mathbf{H}_U \otimes \mathbf{H}_U$ in such a way that $T(\mathbf{H}_U^{(i)} \otimes \mathbf{H}_U^{(j)}) \subseteq \mathbf{H}_U^{(j)} \otimes \mathbf{H}_U^{(i)}$.

Moreover T satisfies the following properties:

$$\begin{aligned} T &= T^*, \\ \|T\| &< 1, \text{ and} \\ (I \otimes T)(T \otimes I)(I \otimes T) &= (T \otimes I)(I \otimes T)(T \otimes I) \end{aligned}$$

where I denotes the identity on \mathbf{H}_U . The last of the above conditions is called Yang-Baxter equation. For each $k \in \mathbb{N}$, define $T_k \in \mathbf{B}(\mathbf{H}_U^{\otimes k+1})$ by

$$(2.2) \quad T_k = \underbrace{I \otimes \cdots \otimes I}_{k-1} \otimes T$$

and by amplification also on $\mathbf{H}_U^{\otimes n}$ for all $n \geq k+1$ (which again is denoted by T_k by abuse of notation). For $n \in \mathbb{N}$, let S_n denote the symmetric group of n symbols and let τ_k be the transposition between k and $k+1$ for $1 \leq k \leq n-1$. Let $\Phi : S_n \rightarrow \mathbf{B}(\mathbf{H}_U^{\otimes n})$ be the unique (quasi-multiplicative) map satisfying $\Phi(1) = I$ and $\Phi(\tau_k) = T_k$, and define $P^{(n)} \in \mathbf{B}(\mathbf{H}_U^{\otimes n})$ by

$$P^{(n)} = \sum_{\sigma \in S_n} \Phi(\sigma).$$

Then $P^{(n)}$ is a strictly positive operator on $\mathbf{H}_U^{\otimes n}$ ([BS₂]); Consider the inner product on $\mathbf{H}_U^{\otimes n}$ defined by

$$\langle \xi, \eta \rangle_T = \langle \xi, P^{(n)} \eta \rangle_{\mathbf{H}_U^{\otimes n}}$$

and denote by $\mathbf{H}_U^{\otimes n T}$ the completion of $\mathbf{H}_U^{\otimes n}$ with respect to $\langle \cdot, \cdot \rangle_T$. Define the T -deformed Fock space $\mathcal{F}_T(\mathbf{H}_U)$ to be the direct sum $\bigoplus_{n \geq 0} \mathbf{H}_U^{\otimes n T}$, where \mathbf{H}_U^0 is understood to be the one-dimensional space $\mathbb{C}\Omega$ for a distinguished unit vector Ω . We now consider the free left annihilation operator $l(\xi)$ for any $\xi \in \mathbf{H}_U$ given on a dense subspace of $\mathcal{F}_T(\mathbf{H}_U)$ by the assignment

$$(2.3) \quad l(\xi)\Omega = 0, \quad l(\xi)\xi_1 \otimes \cdots \otimes \xi_n = \langle \xi, \xi_1 \rangle \xi_2 \otimes \cdots \otimes \xi_n$$

for $\xi_i \in \mathbf{H}_U, 1 \leq i \leq n, n \geq 1$. The following lemma is crucial for our purposes (see [MS, Lemma 2.2] and [Boz, Theorem 6] for proof):

Lemma 2.1. *The free annihilation operator $l(\xi)$ extends to a bounded linear operator on $\mathcal{F}_T(\mathbf{H}_U)$ with*

$$\|l(\xi)\| \leq \frac{\|\xi\|_U}{\sqrt{w(q)}}, \quad w(q) = (1 - q^2)^{-1} \prod_{k=1}^{\infty} \frac{(1 - q^k)}{(1 + q^k)}$$

where $q = \max_{i,j \in N} |q_{ij}| < 1$.

We next consider the T -deformed creation and annihilation operators on $\mathcal{F}_T(\mathbf{H}_U)$, denoted $a^*(\xi)$ and $a(\xi)$ respectively for $\xi \in \mathbf{H}$, given by the assignments

$$(2.4) \quad \begin{aligned} a^*(\xi)\Omega &= \xi, \quad a^*(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n, \quad \text{and} \\ a(\xi)\Omega &= 0, \quad a(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = l(\xi)(1 + T_1 + T_1 T_2 + \cdots + T_1 \cdots T_{n-1})\xi_1 \otimes \cdots \otimes \xi_n \end{aligned}$$

for $\xi_1 \otimes \cdots \otimes \xi_n \in \mathbf{H}_U^{\otimes n}, n \geq 1$, where $l(\xi)$ is the free annihilation operator as in (2.3) and each $T_k \in \mathbf{B}(\mathbf{H}_U^{\otimes n})$ is the operator as in (2.2) for $1 \leq k \leq n$. The operators $a^*(\xi)$ and $a(\xi)$ extend to bounded linear maps on $\mathcal{F}_T(\mathbf{H}_U)$ and they are adjoint to each other satisfying $\|a(\xi)\| = \|a^*(\xi)\| \leq \|\xi\|_U (1 - q)^{-1/2}$ (see [BS₂]). One easily verifies that if $\xi_k \in \mathbf{H}_U^{(i_k)}$ for $i_k \in N$ and $1 \leq k \leq n$, then

$$(2.5) \quad a(\xi)(\xi_n \otimes \cdots \otimes \xi_1) = \sum_{k=1}^n \langle \xi, \xi_k \rangle_U \prod_{t=1}^{n-k} q(i_k, i_{t+k}) \xi_n \otimes \cdots \otimes \widehat{\xi_k} \otimes \cdots \otimes \xi_1$$

where $\widehat{\xi_k}$ means the absence of the vector ξ_k . We adopt the following notations in order to minimize the length of calculations:

- $\xi_1 \cdots \xi_n = \xi_1 \otimes \cdots \otimes \xi_n$ for any $\xi_i \in \mathbf{H}_U, 1 \leq i \leq n$.
- $C_q = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}$ for $0 < q < 1$.
- $[n]_q = 1 + q + \cdots + q^{n-1}$ for $n \geq 1$ and $[0]_q = 1$.

- $[n]_q! = \prod_{k=1}^n [k]_q$ for $n \geq 1$ and $[0]_q! = 1$.

We now mention an explicit description for the inner product of two simple tensors. We could not trace anywhere a formula written in the following form, so we include the proof for completeness (also see [BKM₁, Lemma 2.1]).

Lemma 2.2. For $\xi_k \in \mathbb{H}_U^{(i_k)}$ and $\eta_k \in \mathbb{H}_U^{(j_k)}$, $i_k, j_k \in N$, $1 \leq k \leq n$, we have

$$\begin{aligned}
(2.6) \quad \langle \xi_n \cdots \xi_1, \eta_n \cdots \eta_1 \rangle_T &= \sum_{\sigma \in S_n} \prod_{\substack{1 \leq u < v \leq n \\ \sigma(u) > \sigma(v)}} q(i_u, i_v) \prod_{t=1}^n \langle \xi_t, \eta_{\sigma(t)} \rangle_U \\
&= \sum_{\sigma \in S_n} \prod_{\substack{1 \leq u < v \leq n \\ \sigma(u) > \sigma(v)}} q(j_{\sigma(u)}, j_{\sigma(v)}) \prod_{t=1}^n \langle \xi_t, \eta_{\sigma(t)} \rangle_U \\
&= \sum_{\sigma \in S_n} \prod_{\substack{1 \leq u < v \leq n \\ \sigma(u) > \sigma(v)}} q(j_u, j_v) \prod_{t=1}^n \langle \xi_{\sigma(t)}, \eta_t \rangle_U \\
&= \sum_{\sigma \in S_n} \prod_{\substack{1 \leq u < v \leq n \\ \sigma(u) > \sigma(v)}} q(i_{\sigma(v)}, i_{\sigma(u)}) \prod_{t=1}^n \langle \xi_{\sigma(t)}, \eta_t \rangle_U.
\end{aligned}$$

Proof. Note first that the last four expressions are indeed the same; for example to check that the first two expressions on the right side are same, observe that each term in the first sum of these is non-zero only if $\langle \xi_t, \eta_{\sigma(t)} \rangle_U \neq 0$ for all $1 \leq t \leq n$ which is possible only if $i_t = j_{\sigma(t)}$, in which case we have $q(i_u, i_v) = q(j_{\sigma(u)}, j_{\sigma(v)})$ for $1 \leq v, u \leq n$. Similarly, the first sum and the last sum on the right are equal by noting that

$$(2.7) \quad \prod_{\substack{1 \leq u < v \leq n \\ \sigma(u) > \sigma(v)}} q(i_u, i_v) = \prod_{\substack{1 \leq r < s \leq n \\ \sigma^{-1}(r) > \sigma^{-1}(s)}} q(i_{\sigma^{-1}(r)}, i_{\sigma^{-1}(s)})$$

for any $\sigma \in S_n$. We shall prove equality of the left hand side with the first on the right. The proof is by induction on n . For $n = 1$ the equality is obvious. So assume them to be true for some $n \geq 1$; then calculate

$$\begin{aligned}
&\langle \xi_{n+1} \cdots \xi_1, \eta_{n+1} \cdots \eta_1 \rangle_T = \langle \xi_{n+1} \cdots \xi_1, a^*(\eta_{n+1})\eta_n \cdots \eta_1 \rangle_T \\
&= \langle a(\eta_{n+1})(\xi_{n+1} \cdots \xi_1), \eta_n \cdots \eta_1 \rangle_T \\
&= \left\langle \sum_{k=1}^{n+1} \langle \eta_{n+1}, \xi_k \rangle_U \prod_{t=1}^{n+1-k} q(i_k, i_{k+t}) \xi_{n+1} \cdots \widehat{\xi}_k \cdots \xi_1, \eta_n \cdots \eta_1 \right\rangle_T \quad (\text{by eq. (2.5)}) \\
&= \sum_{k=1}^{n+1} \langle \xi_k, \eta_{n+1} \rangle_U \prod_{t=1}^{n+1-k} q(i_k, i_{k+t}) \left\langle \xi_{n+1} \cdots \widehat{\xi}_k \cdots \xi_1, \eta_n \cdots \eta_1 \right\rangle_T \\
&= \sum_{k=1}^{n+1} \prod_{t=1}^{n+1-k} q(i_k, i_{k+t}) \sum_{\sigma \in S_n} \prod_{\substack{1 \leq u < v \leq n \\ \sigma(u) > \sigma(v)}} q(i_u, i_v) \langle \xi_k, \eta_{n+1} \rangle_U \left\langle \xi_{n+1} \cdots \widehat{\xi}_k \cdots \xi_1, \eta_{\sigma(n)} \cdots \eta_{\sigma(1)} \right\rangle_{\mathbb{H}_U^{\otimes n}} \\
&= \sum_{k=1}^{n+1} \prod_{t=1}^{n+1-k} q(i_k, i_{k+t}) \sum_{\substack{\sigma \in S_{n+1} \\ \sigma(k)=n+1}} \prod_{\substack{1 \leq u < v \leq n \\ \sigma^{-1}(u) > \sigma^{-1}(v)}} q(i_{\sigma^{-1}(u)}, i_{\sigma^{-1}(v)}) \langle \xi_{n+1} \cdots \xi_1, \eta_{\sigma(n+1)} \cdots \eta_{\sigma(1)} \rangle_{\mathbb{H}_U^{\otimes n+1}}
\end{aligned}$$

where in the last equality we have used (2.7), and for any fixed $k \in \{1, \dots, n\}$ we have identified any permutation $\sigma \in S_n$ with a permutation, say $\sigma^{(k)}$, in S_{n+1} by declaring $\sigma^{(k)}(k) = n+1$ and

keeping the order preserved, that is,

$$\sigma^{(k)}(t) = \begin{cases} \sigma(t) & \text{if } 1 \leq t < k, \\ n+1 & \text{if } t = k, \\ \sigma(t-1) + 1 & \text{if } k < t \leq n+1. \end{cases}$$

Note that the extra crossings that occur in the permutation $\sigma^{(k)}$ as compared to σ are exactly those between the pairs $(k, n+1)$ and $(t, \sigma^{(k)}(t))$ for $t > k$; in such case we get from (2.7) (and by setting $\tau = \sigma^{(k)}$) that

$$\begin{aligned} \prod_{\substack{1 \leq u < v \leq n+1 \\ \tau(u) > \tau(v)}} q(i_u, i_v) &= \prod_{\substack{1 \leq u < v \leq n+1 \\ \tau^{-1}(u) > \tau^{-1}(v)}} q(i_{\tau^{-1}(v)}, i_{\tau^{-1}(u)}) \\ &= \prod_{\substack{1 \leq u < v \leq n \\ \tau^{-1}(u) > \tau^{-1}(v)}} q(i_{\tau^{-1}(v)}, i_{\tau^{-1}(u)}) \cdot \prod_{t=k+1}^{n+1} q(i_k, i_t). \end{aligned}$$

Finally we obtain

$$\begin{aligned} \langle \xi_{n+1} \cdots \xi_1, \eta_{n+1} \cdots \eta_1 \rangle_T &= \sum_{k=1}^{n+1} \sum_{\substack{\sigma \in \mathcal{S}_{n+1} \\ \sigma(k) = n+1}} \prod_{\substack{1 \leq u < v \leq n+1 \\ \sigma(u) > \sigma(v)}} q(i_u, i_v) \langle \xi_{n+1} \cdots \xi_1, \eta_{\sigma(n+1)} \cdots \eta_{\sigma(1)} \rangle_{\mathbb{H}_U^{\otimes n+1}} \\ &= \sum_{\sigma \in \mathcal{S}_{n+1}} \prod_{\substack{1 \leq u < v \leq n+1 \\ \sigma(u) > \sigma(v)}} q(i_u, i_v) \langle \xi_{n+1} \cdots \xi_1, \eta_{\sigma(n+1)} \cdots \eta_{\sigma(1)} \rangle_{\mathbb{H}_U^{\otimes n+1}} \end{aligned}$$

which is exactly what was to be shown. \square

For each $\xi \in \mathbb{H}_{\mathbb{R}}$, consider the operator $s(\xi) = a(\xi) + a^*(\xi) \in \mathbf{B}(\mathcal{F}_T(\mathbb{H}))$ and denote by $\Gamma_T(\mathbb{H}_{\mathbb{R}}, U_t)$ the von Neumann algebra generated by $\{s(\xi); \xi \in \mathbb{H}_{\mathbb{R}}\}$, which we call as *mixed q-Araki-Woods von Neumann algebra*. We will often denote this von Neumann algebra by \mathbf{M}_T (or simply by \mathbf{M} if T is clear from the context). The vacuum vector Ω is a cyclic and separating vector for \mathbf{M}_T ; so \mathbf{M}_T is in standard form as an algebra acting on $\mathcal{F}_T(\mathbb{H}_U)$. Let φ denote the canonical normal faithful state $\langle \cdot, \Omega \rangle$ on $\Gamma_T(\mathbb{H}_{\mathbb{R}}, U_t)$. Denote by $L^2(\mathbf{M}_T)$ (or by $L^2(\mathbf{M}_T, \varphi)$) the GNS Hilbert space of \mathbf{M}_T with respect to the state φ . As mentioned above $L^2(\mathbf{M}_T)$ is naturally identified with $\mathcal{F}_T(\mathbb{H}_U)$. Denote by $\mathcal{F}_T(\mathbb{H}_U)_{\text{alg}}$ the subspace spanned by finite simple tensors. Then $\mathcal{F}_T(\mathbb{H}_U)_{\text{alg}}$ is contained in $\mathbf{M}_T\Omega$ (see [BKM₁]); hence for all $\xi \in \mathcal{F}_T(\mathbb{H}_U)_{\text{alg}}$, there is a unique operator $W(\xi) \in \mathbf{M}_T$ such that $W(\xi)\Omega = \xi$. We refer the readers to [BKM₁] for modular theory and other basic facts about the algebras \mathbf{M}_T .

3. DUAL AND CONJUGATE VARIABLES FOR MIXED q -ARAKI-WOODS ALGEBRAS

In this section we will consider only the case of finite-dimensional $\mathbb{H}_{\mathbb{R}}$ and an orthogonal representation U_t acting on $\mathbb{H}_{\mathbb{R}}$. Assume then that N is a finite set and $\mathbb{H}_{\mathbb{R}} = \bigoplus_{i \in N} \mathbb{H}_{\mathbb{R}}^{(i)}$ where each $\mathbb{H}_{\mathbb{R}}^{(i)}$ is a U_t -invariant finite dimensional real Hilbert space. Fix $-1 < q(i, j) = q(j, i) < 1$ for $i, j \in N$, let $q = \max_{i, j \in N} |q(i, j)| < 1$ and consider the operator T on $\mathbf{B}(\mathbb{H}_U \otimes \mathbb{H}_U)$ as defined in (2.1). Assume that $\dim \mathbb{H}_{\mathbb{R}} = d$. For this whole Section, we fix an orthonormal basis $\{e_1, \dots, e_d\}$ of \mathbb{H}_U constructed in such a way that each $e_k \in \mathbb{H}_{\mathbb{C}}^{(j_k)}$ for some $j_k \in N$ (the basis can be chosen so as to contain eigenvectors of the analytic generator A of U_t ; see [Shl, Hia]). For each $1 \leq k \leq d$, set $A_k = W(e_k)$. Note that the set $\{A_k\}_{1 \leq k \leq d}$ is algebraically free such that the unital $*$ -subalgebra $\mathbb{C}[A_1, \dots, A_d]$ generated by the set is strongly dense in $\mathbf{M}_T = \Gamma_T(\mathbb{H}_{\mathbb{R}}, U_t)$.

We say that a tuple (D_1, \dots, D_d) of unbounded operators on $\mathcal{F}_T(\mathbb{H}_U)$ with $\mathcal{F}_T(\mathbb{H}_U)_{\text{alg}}$ contained in their domains and Ω contained in the domain of their adjoints is a (*normalized*) *dual system* for (A_1, \dots, A_d) if for all $k, \ell \in \{1, \dots, d\}$

$$[D_k, A_\ell] = \langle \bar{e}_\ell, e_k \rangle_U P_{\mathbb{C}\Omega} = \varphi(A_\ell A_k) P_{\mathbb{C}\Omega} \quad \text{and} \quad D_k \Omega = 0,$$

where $P_{\mathbb{C}\Omega}$ is the projection onto the one-dimensional subspace $\mathbb{C}\Omega$, and as before \bar{e}_ℓ is the complex conjugate of e_ℓ . The existence of dual system ensures that each D_k is a closable operator on $\mathcal{F}_T(\mathbf{H}_U)$.

Next recall that if $\mathbf{B} \subseteq \mathbf{M}_T$ is a $*$ -subalgebra, then a *derivation* is a \mathbb{C} -linear map $\delta : \mathbf{B} \rightarrow \mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}}$ which satisfies the Leibniz rule: $\delta(ab) = a\delta(b) + b\delta(a)$ for all $a, b \in \mathbf{B}$. Here \mathbf{M}_T^{op} denotes the opposite algebra equipped with the natural state φ^{op} . Then the *quasi-free difference quotients* ∂_k are defined as unique derivations from $\mathbb{C}[A_1, \dots, A_d]$ into $\mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}}$ such that

$$\partial_k(A_\ell) := \varphi(A_\ell A_k) \mathbf{1} \otimes \mathbf{1}$$

for all $k, \ell \in \{1, \dots, d\}$, where $\mathbf{1}$ is the identity of \mathbf{M}_T . The *conjugate variable* will be a vector $\xi_k \in L^2(\mathbf{M}_T)$ such that

$$\langle \xi_k, x \mathbf{1} \rangle_{L^2(\mathbf{M}_T)} = \langle \mathbf{1} \otimes \mathbf{1}, \partial_k(x) \mathbf{1} \otimes \mathbf{1} \rangle_{L^2(\mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}})}$$

for all $x \in \mathbb{C}[A_1, \dots, A_d]$. Remark that the existence of the conjugate variable is equivalent to $\mathbf{1} \otimes \mathbf{1} \in \text{Dom}(\partial_k^*)$ when ∂_k is considered an (unbounded) operator from $L^2(\mathbf{M}_T)$ to $L^2(\mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}})$; in this case it is given precisely by $\xi_k = \partial_k^*(\mathbf{1} \otimes \mathbf{1})$. We also recall that the existence of dual variables implies existence of conjugate variables:

Proposition 3.1 (See [MS, Theorem 2.5], [KSW, Proposition 3.1]). *Suppose that (D_1, \dots, D_d) is a normalized dual system for (A_1, \dots, A_d) . Then $(D_1^* \Omega, \dots, D_d^* \Omega)$ are conjugate variables for (A_1, \dots, A_d) .*

Existence of Dual variables. We now show the existence of a system of dual variables for the tuple (A_1, \dots, A_d) of operators corresponding to a fixed orthonormal basis $\{e_1, \dots, e_d\}$ of \mathbf{H}_U . We follow the lines of investigation as in [MS, KSW]. In order to keep this article self contained, we repeat here some of the notions considered in [MS]. Firstly we recall below the rules of drawing partitions of the set $\{0, 1, \dots, n\}$ consisting of singletons and pairs:

- (1) Consider $n + 1$ vertices $n > n - 1 > \dots > 1 > 0$.
- (2) 0 must be connected to some $k \in \{1, \dots, n\}$ with height 1.
- (3) $\ell \in \{1, \dots, k - 1\}$ must be coupled with one of $\{k + 1, \dots, n\}$ with height $\ell + 1$.
- (4) Vertices which are not coupled with $\{1, \dots, k - 1\}$ should be singletons and are drawn with straight lines to the top.

Define $B(n + 1)$ as the set of partitions that satisfy the above rules. For $\pi \in B(n + 1)$, denote by $p(\pi)$ the set of pairings in π , and by $s(\pi)$ the set of singletons in π . Also for any $k \in \{0, \dots, n\}$, write $\pi(k) = \ell$ if $(k, \ell) \in p(\pi)$ and $\pi(k) = k$ if $k \in s(\pi)$.

We set some notations. Write $[d] = \{1, \dots, d\}$ and let $[d]^*$ denote the set of words in $[d]$. Write

$$e_w = e_{\alpha_1} \cdots e_{\alpha_n} = e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n} \quad \text{and} \quad e_\emptyset = e_{()} = \Omega,$$

for any word $w = \alpha_1 \cdots \alpha_n \in [d]^*$ of length $n \geq 1$ and empty word $()$. The set $\{e_w; w \in [d]^*\}$ is linearly independent and spans $\mathcal{F}_T(\mathbf{H}_U)_{\text{alg}}$. Observe that for any $\alpha \in [d]$ and for any word $\alpha_n \cdots \alpha_1 \in [d]^*$ of length $n \geq 1$, the expression $A_\alpha = W(e_\alpha) = a^*(e_\alpha) + a(\bar{e}_\alpha)$ yields

$$\begin{aligned} (3.1) \quad A_\alpha(e_{\alpha_n \cdots \alpha_1}) &= e_\alpha e_{\alpha_n \cdots \alpha_1} + \sum_{k=1}^n \langle \bar{e}_\alpha, e_{\alpha_k} \rangle_U \prod_{k < t \leq n} q(i_k, i_t) e_{\alpha_n \cdots \hat{\alpha}_k \cdots \alpha_1} \\ &= e_\alpha e_{\alpha_n \cdots \alpha_1} + \sum_{k=1}^n \langle \bar{e}_\alpha, e_{\alpha_k} \rangle_U \prod_{t=1}^{n-k} q(i_k, i_{n-t+1}) e_{\alpha_n \cdots \hat{\alpha}_k \cdots \alpha_1} \end{aligned}$$

where $i_k \in N$ is such that $e_{\alpha_k} \in \mathbf{H}_\mathbb{C}^{(i_k)}$ for $1 \leq k \leq n$.

Taking cue from the approach of [MS] and [KSW], we now establish the existence of dual/conjugate variables in the non-tracial context of mixed q -Araki-Woods von Neumann algebras.

Proposition 3.2. *The algebraic formula for the tuple (D_1, \dots, D_d) of the dual variables of (A_1, \dots, A_d) is given as follows; for fixed $\alpha \in [d]$ and $w = \alpha_n \cdots \alpha_1 \in [d]^*$ such that $e_\alpha \in \mathbf{H}_\mathbb{C}^{(i_\alpha)}$*

and $e_{\alpha_k} \in \mathbf{H}_{\mathbb{C}}^{(i_k)}$, for some $i_0, i_k \in N$, $1 \leq k \leq n$:

$$(3.2) \quad D_{\alpha} \Omega = 0, \quad D_{\alpha}(e_{\alpha_n \dots \alpha_1}) = \sum_{\pi \in B(n+1)} (-1)^{\pi(0)-1} Q(\pi) E(\pi) e_{s(\pi)},$$

where $s(\pi)$ is the set of singletons of π , and

$$Q(\pi) := \prod_{0 \leq v < u < \pi(0)} q(i_v, i_u) \cdot \prod_{\substack{0 < v < u < \pi(0) \\ \pi(v) > \pi(u)}} q(i_{\pi(v)}, i_{\pi(u)}) \cdot \prod_{\substack{\pi(0) < v < u \leq n \\ v \in s(\pi), u \notin s(\pi)}} q(i_v, i_u)$$

$$E(\pi) := \prod_{\substack{(k, \ell) \in p(\pi) \\ k > \ell}} \langle \bar{e}_{\alpha_k}, e_{\alpha_{\ell}} \rangle_U$$

$$e_{s(\pi)} := e_{\alpha_{\ell_s} \dots \alpha_{\ell_1}} = e_{\alpha_{\ell_s}} \otimes \dots \otimes e_{\alpha_{\ell_1}} \quad \text{for } s(\pi) = \{\ell_s > \dots > \ell_1\}$$

with the prescription that $\alpha_0 = \alpha$.

Remark 3.3. We have divided the terms of $Q(\pi)$ into three products. The first product (resp. the second product) corresponds to the crossings in the partition occurring on the line of the vertex ‘ u ’ (resp. ‘ $\pi(u)$ ’) for $0 < u < \pi(0)$, with the lines of another pair. While the third product corresponds to the crossings on the lines of the singletons. Note that $E(\pi)$ is non-zero only if $i_k = i_{\pi(k)}$ for any pair $(k, \pi(k)) \in p(\pi)$ (because $\xi \in \mathbf{H}_{\mathbb{C}}^{(i)}$ if and only if $\bar{\xi} \in \mathbf{H}_{\mathbb{C}}^{(i)}$ and $\mathbf{H}_{\mathbb{C}}^{(i)} \perp \mathbf{H}_{\mathbb{C}}^{(j)}$ if $i \neq j$). Thus in such cases, in the expression for $Q(\pi)$, the term $q(i_v, i_u)$ can be replaced by $q(i_{\pi(v)}, i_{\pi(u)})$ for any indices u, v which are part of the pairings of π and then we use symmetry of the matrix $(q(i, j))$ to get the following:

$$(3.3) \quad \begin{aligned} E(\pi)Q(\pi) &= E(\pi) \prod_{0 \leq v < u < \pi(0)} q(i_{\pi(v)}, i_{\pi(u)}) \cdot \prod_{\substack{0 < v < u < \pi(0) \\ \pi(v) > \pi(u)}} q(i_{\pi(v)}, i_{\pi(u)}) \cdot \prod_{\substack{\pi(0) < v < u \leq n \\ v \in s(\pi), u \notin s(\pi)}} q(i_v, i_u) \\ &= E(\pi) \prod_{0 \leq v < u < \pi(0)} q(i_v, i_u) \cdot \prod_{\substack{0 < v < u < \pi(0) \\ \pi(v) > \pi(u)}} q(i_v, i_u) \cdot \prod_{\substack{\pi(0) < v < u \leq n \\ v \in s(\pi), u \notin s(\pi)}} q(i_v, i_u). \end{aligned}$$

Proof. Consider the (unbounded) operator D_{α} on $\mathcal{F}_T(\mathbf{H}_U)$ defined as in (3.2) on vectors which span the subspace $\mathcal{F}_T(\mathbf{H}_U)_{\text{alg}}$. We will show that $[D_{\alpha}, A_{\beta}] = \langle \bar{e}_{\beta}, e_{\alpha} \rangle_U P_{\mathbb{C}\Omega}$ on $\mathcal{F}_T(\mathbf{H}_U)_{\text{alg}}$ for all $\beta \in [d]$. It is straightforward to check that

$$[D_{\alpha}, A_{\beta}] \Omega = D_{\alpha} A_{\beta} \Omega = D_{\alpha} e_{\beta} = \langle \bar{e}_{\beta}, e_{\alpha} \rangle_U \Omega.$$

Then for $\alpha_{n+1} \in [d]$, use (3.1) to compute

$$\begin{aligned} A_{\alpha_{n+1}} D_{\alpha}(e_{\alpha_n \dots \alpha_1}) &= \sum_{\sigma \in B(n+1)} (-1)^{\sigma(0)-1} Q(\sigma) E(\sigma) A_{\alpha_{n+1}} e_{s(\sigma)} \\ &= \sum_{\sigma \in B(n+1)} (-1)^{\sigma(0)-1} Q(\sigma) E(\sigma) e_{\alpha_{n+1}} e_{s(\sigma)} \\ &\quad + \sum_{\sigma \in B(n+1)} \sum_{v=1}^{|\sigma|} (-1)^{\sigma(0)-1} Q(\sigma) E(\sigma) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_{s(\sigma)_v}} \rangle_U \prod_{\substack{s(\sigma)_v < t \leq n \\ t \in s(\sigma)}} q(i_{s(\sigma)_v}, i_t) e_{s(\sigma) \setminus s(\sigma)_v} \end{aligned}$$

where $e_{s(\sigma) \setminus s(\sigma)_v}$ means that we omit the term $e_{\alpha_{s(\sigma)_v}}$ from $e_{s(\sigma)}$. Next we compute

$$\begin{aligned} &D_{\alpha} A_{\alpha_{n+1}}(e_{\alpha_n \dots \alpha_1}) \\ &= D_{\alpha} \left(e_{\alpha_{n+1} \dots \alpha_1} + \sum_{u=1}^n \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_u} \rangle_U \prod_{u < t \leq n} q(i_u, i_t) e_{\alpha_n \dots \hat{\alpha}_u \dots \alpha_1} \right) \\ &= D_{\alpha} e_{\alpha_{n+1} \dots \alpha_1} \\ &\quad + \sum_{u=1}^n \sum_{\pi \in B(n)} (-1)^{\pi(0)-1} Q(\pi, u) E(\pi, u) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_u} \rangle_U \prod_{u < t \leq n} q(i_u, i_t) e_{s(\pi, u)} \end{aligned}$$

where $e_{s(\pi,u)}$, $Q(\pi, u)$ and $E(\pi, u)$ respectively denote $e_{s(\pi)}$, $Q(\pi)$ and $E(\pi)$ only with the emphasis that $\pi \in B(n)$ acts on the $n-1$ letters $\alpha_n, \dots, \widehat{\alpha}_u, \dots, \alpha_1$. As in the proof of [MS, Proposition 4.5] we identify all terms in the last sum of $A_{\alpha_{n+1}} D_\alpha(e_{\alpha_n \dots \alpha_1})$ with some terms in the last sum of $D_\alpha A_{\alpha_{n+1}}(e_{\alpha_n \dots \alpha_1})$ as follows: take a pair (σ, v) with $\sigma \in B(n+1)$ and $v \in \{1, \dots, |s(\sigma)|\}$, then get a pair (π, u) where $u = s(\sigma)_v$ and $\pi \in B(n)$ is a partition of the set $\{0, \dots, n\} \setminus \{u\}$ obtained from σ in an order preserving fashion by removing the singleton $s(\sigma)_v$. Then observe that $\sigma(0) = \pi(0)$ and the pairings in $p(\sigma)$ and $p(\pi)$ remain the same while the singleton sets satisfy $s(\sigma) = s(\pi) \cup \{s(\sigma)_v\}$. This means that $e_{s(\pi,u)} = e_{s(\sigma) \setminus s(\sigma)_v}$ and $E(\pi, u) = E(\sigma)$ so that

$$E(\pi, u) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_u} \rangle_U e_{s(\pi,u)} = E(\sigma) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_{s(\sigma)_v}} \rangle_U e_{s(\sigma) \setminus s(\sigma)_v}.$$

Moreover the extra terms which appear in the product of $Q(\sigma)$ as compared to that of $Q(\pi, u)$ are the ones due to the crossing points on the line of the singleton $s(\sigma)_v$. Such crossings occur with those pairings in σ which have one of the vertices on the left of $u = s(\sigma)_v$. Therefore we get

$$Q(\sigma) = Q(\pi, u) \cdot \prod_{\substack{s(\sigma)_v < t \leq n \\ t \notin s(\sigma)}} q(i_{s(\sigma)_v}, i_t) = Q(\pi, u) \cdot \frac{\prod_{s(\sigma)_v < t \leq n} q(i_{s(\sigma)_v}, i_t)}{\prod_{\substack{s(\sigma)_v < t \leq n \\ t \in s(\sigma)}} q(i_{s(\sigma)_v}, i_t)}$$

where $\frac{0}{0}$ is understood to be 0; this implies by using $u = s(\sigma)_v$ that

$$Q(\sigma) \cdot \prod_{\substack{s(\sigma)_v < t \leq n \\ t \in s(\sigma)}} q(i_{s(\sigma)_v}, i_t) = Q(\pi, u) \cdot \prod_{u < t \leq n} q(i_u, i_t).$$

This shows that the term corresponding to σ and v in the second sum of $A_{\alpha_{n+1}} D_\alpha(e_{\alpha_n \dots \alpha_1})$ appears as the term corresponding to π and u in the second sum of $D_\alpha A_{\alpha_{n+1}}(e_{\alpha_n \dots \alpha_1})$, and at the same time note that this correspondence exhausts all the pairs (π, u) with $\pi \in B(n)$ and $u > \pi(0)$. Thus after the subtracting we get the following:

$$\begin{aligned} & -[D_\alpha, A_{\alpha_{n+1}}] e_{\alpha_n \dots \alpha_1} + D_\alpha e_{\alpha_{n+1} \dots \alpha_1} \\ &= \sum_{\sigma \in B(n+1)} (-1)^{\sigma(0)-1} Q(\sigma) E(\sigma) e_{\alpha_{n+1}} e_{s(\sigma)} \\ (3.4) \quad & - \sum_{u=1}^n \sum_{\substack{\pi \in B(n) \\ \pi(0) \geq u}} (-1)^{\pi(0)-1} Q(\pi, u) E(\pi, u) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_u} \rangle_U \prod_{u < t \leq n} q(i_u, i_t) e_{s(\pi,u)}. \end{aligned}$$

Now we try to understand the terms in the last sum above. Recall that each pair (π, u) with $\pi \in B(n)$ and $u \in \{1, \dots, n\}$ acts on the $n-1$ letters $\alpha_n, \dots, \widehat{\alpha}_u, \dots, \alpha_1$. As in the second step of the proof of [MS, Proposition 4.5], to each such pair (π, u) with $u \leq \pi(0)$, we associate a new partition $\pi' \in B(n+2)$ as follows: construct the partition $\pi' \in B(n+2)$ of $\{1, \dots, n+1\}$ out of (π, u) by inserting a new point at u and pairing it with $n+1$ with the requirement that $\pi' \setminus (n+1, u) = \pi$. We then have

$$\pi'(0) = \pi(0) + 1.$$

Observe that the set of singletons remains the same in both π and π' while the set of pairings satisfies $p(\pi') = p(\pi) \cup \{(n+1, u)\}$; hence we get

$$e_{s(\pi')} = e_{s(\pi,u)} \quad \text{and} \quad E(\pi') = E(\pi, u) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_u} \rangle_U.$$

Further note that there are some extra crossing points for the partition π' as compared to (π, u) due to the coupling $(n+1, u)$; such crossings occur on the line of t of the couplings $(t, \pi'(t))$ for $0 \leq t < u$, on the lines of both t and $\pi(t)$ of the couplings $(t, \pi'(t))$ for $\pi'(0) > t > u$, and with the singletons of π' . Therefore we obtain (whenever $E(\pi') \neq 0$) that

$$\begin{aligned} Q(\pi') &= Q(\pi, u) \cdot \prod_{0 \leq t < u} q(i_t, i_u) \cdot \prod_{u < t < \pi'(0)} q(i_t, i_u) \cdot \prod_{u < t < \pi'(0)} q(i_{\pi'(u)}, i_{\pi'(t)}) \cdot \prod_{t \in s(\pi')} q(i_t, i_{n+1}) \\ &= Q(\pi, u) \prod_{0 \leq t < u} q(i_{\pi'(t)}, i_u) \cdot \prod_{u < t < \pi'(0)} q(i_{\pi'(t)}, i_u) \cdot \prod_{u < t < \pi'(0)} q(i_u, i_{\pi'(t)}) \cdot \prod_{t \in s(\pi')} q(i_t, i_u) \end{aligned}$$

where to get the last expression, we have used the symmetry of the matrix $(q(i, j))$ and the same argument as in Remark 3.3 that whenever $E(\pi') \neq 0$, then $i_t = i_{\pi'(t)}$ for any coupling $(t, \pi'(t))$ so that we can replace any $q(i_t, i_r)$ by $q(i_{\pi(t)}, i_r)$, $r \in N$. On the other hand, if $E(\pi') = 0$ then $E(\pi, u) = 0$, so the corresponding term does not contribute in the sum. A careful observation of the last product above (and again the symmetry of the matrix $(q(i, j))$) then yields

$$(3.5) \quad Q(\pi') = Q(\pi, u) \cdot \prod_{u < t \leq n} q(i_u, i_t).$$

Therefore the expression in (3.4) can finally be rewritten in the following way:

$$\begin{aligned} & -[D_\alpha, A_{\alpha_{n+1}}](e_{\alpha_n \dots \alpha_1}) + D_\alpha e_{\alpha_{n+1}} \dots e_{\alpha_1} \\ &= \sum_{\sigma \in B(n+1)} (-1)^{\sigma(0)-1} Q(\sigma) E(\sigma) e_{\alpha_{n+1}} e_{s(\sigma)} \\ & \quad + \sum_{\substack{\pi' \in B(n+2) \\ \pi'(n+1) \text{ is not a singleton}}} (-1)^{\pi'(0)-1} Q(\pi') E(\pi') e_{s(\pi')} \\ &= \sum_{\pi' \in B(n+2)} (-1)^{\pi'(0)-1} Q(\pi') E(\pi') e_{s(\pi')} \end{aligned}$$

where the last equality follows because each partition $\sigma \in B(n+1)$ gets identified exactly to a partition $\sigma' \in B(n+2)$ in an order preserving fashion such that $n+1$ is a singleton in σ' ; in such case $\sigma(0) = \sigma'(0)$, $Q(\sigma) = Q(\sigma')$, $E(\sigma) = E(\sigma')$ and $s(\sigma') = \{n+1\} \cup s(\sigma)$ so that $e_{s(\sigma')} = e_{\alpha_{n+1}} e_{s(\sigma)}$. We have thus shown that $[D_\alpha, A_{\alpha_{n+1}}]e_{\alpha_n \dots \alpha_1} = 0$. This completes the proof. \square

We now give the main result of this Section in the following theorem. Note that since $\pi(n+1)$ has at least one singleton whenever n is even, $\langle \Omega, D_\alpha e_w \rangle = 0$ for any word w of even length and $\alpha \in [d]$. Also if $n = 2m - 1$ for $m \geq 1$, then in the summation of D_α , only those $\pi \in B(2m)$ contribute a non-zero term in the inner product $\langle \Omega, D_\alpha e_w \rangle$ which has no singleton; this occurs only if $\pi(0) = m$. In this case, we have

$$(3.6) \quad \langle \Omega, D_\alpha e_w \rangle = \sum_{\substack{\pi \in B(2m) \\ \pi(0)=m}} (-1)^{m-1} Q(\pi, \omega) E(\pi, \omega)$$

where $Q(\pi, \omega)$ and $E(\pi, \omega)$ are written respectively for $Q(\pi)$ and $E(\pi)$ in order to make the dependency on ω explicit.

Theorem 3.4. *For each $\alpha \in [d]$ we have $\Omega \in \text{Dom } D_\alpha^*$. Thus $(D_1^* \Omega, \dots, D_d^* \Omega)$ forms a set of conjugate variables for (A_1, \dots, A_d) .*

Proof. Fix $\alpha \in [d]$ and let $e_\alpha \in \mathbb{H}_\mathbb{C}^{(i_\alpha)}$ for some $i_\alpha \in N$. As in [MS, Theorem 4.6] and [KSW, Proposition 1.4] we will prove that the functional $\langle \Omega, D_\alpha(\cdot) \rangle_T$ on $\mathcal{F}_T(\mathbb{H}_U)_{\text{alg}}$ extends to a bounded map on $\mathcal{F}_T(\mathbb{H}_U)$. Fix then an element $\sum_{w \in [d]^*} \alpha_w e_w$ in $\mathcal{F}_T(\mathbb{H}_U)_{\text{alg}}$ where α_w is 0 except for finitely many, and then calculate using (3.6):

$$(3.7) \quad \langle \Omega, D_\alpha \left(\sum_{w \in [d]^*} \alpha_w e_w \right) \rangle_T = \sum_{m=1}^{\infty} \sum_{\substack{\pi \in B(2m) \\ \pi(0)=m}} \sum_{|w|=2m-1} \alpha_w (-1)^{m-1} Q(\pi, w) E(\pi, w).$$

Fix $m \geq 1$ and rewrite each word w of length $2m - 1$ as $w_1 \beta w'_1$ with w_1, w'_1 being words of length $m - 1$ and $\beta \in [d]$; then identify each $\pi \in B(2m)$, $\pi(0) = m$ with a permutation $\rho := \rho_\pi \in S_{m-1}$ requiring that the couplings satisfy $(t, m + \rho(t)) \in p(\pi)$ for $1 \leq t \leq m - 1$. Under such identification write $Q(\rho, w_1, w'_1, \beta)$ and $E(\rho, w_1, w'_1, \beta)$ respectively for $Q(\pi, w)$ and $E(\pi, w)$; which can further be rewritten as follows: if $w_1 = \alpha_{m-1} \dots \alpha_1$, $w'_1 = \alpha'_{m-1} \dots \alpha'_1$ and $e_{\alpha_t} \in \mathbb{H}_\mathbb{C}^{(i_t)}$, $e_{\alpha'_t} \in \mathbb{H}_\mathbb{C}^{(i'_t)}$,

$e_\beta \in \mathbf{H}_\mathbb{C}^{(j)}$ for some $j, i_t, i'_t \in N$, $1 \leq t \leq m-1$, then

$$\begin{aligned} E(\rho, w_1, w'_1, \beta) &= \langle \bar{e}_\beta, e_\alpha \rangle_U \prod_{t=1}^{m-1} \langle \bar{e}_{\alpha_{\rho(t)}}, e_{\alpha'_t} \rangle_U = \langle \bar{e}_\beta, e_\alpha \rangle_U \prod_{t=1}^{m-1} \langle \bar{e}_{\alpha_t}, e_{\alpha'_{\rho^{-1}(t)}} \rangle_U \\ Q(\rho, w_1, w'_1, \beta) &= \prod_{u=1}^{m-1} q(i_\alpha, i'_u) \cdot \prod_{1 \leq v < u \leq m-1} q(i'_v, i'_u) \cdot \prod_{\substack{1 \leq v < u \leq m-1 \\ \rho(v) > \rho(u)}} q(i_{\rho(v)}, i_{\rho(u)}). \end{aligned}$$

Moreover, whenever $E(\pi, w) \neq 0$, then we use the same arguments as in Remark 3.3 to get the following expression:

$$Q(\rho, w_1, w'_1, \beta) = \prod_{t=1}^{m-1} q(i_\alpha, i_{\rho(t)}) \cdot \prod_{1 \leq v < u \leq m-1} q(i_{\rho(v)}, i_{\rho(u)}) \cdot \prod_{\substack{1 \leq v < u \leq m-1 \\ \rho(v) > \rho(u)}} q(i_{\rho(v)}, i_{\rho(u)})$$

which can further be written as follows by rearranging the indices of the first two products and then by using the symmetry of the matrix $(q(i, j))$:

$$Q(\rho, w_1, w'_1, \beta) = \prod_{t=1}^{m-1} q(i_\alpha, i_t) \cdot \prod_{1 \leq v < u \leq m-1} q(i_v, i_u) \cdot \prod_{\substack{1 \leq v < u \leq m-1 \\ \rho(v) > \rho(u)}} q(i_{\rho(v)}, i_{\rho(u)}).$$

Therefore the sum in (3.7) can be rewritten as follows:

$$\begin{aligned} &\langle \Omega, D_\alpha \left(\sum_{w \in [d]^*} \alpha_w e_w \right) \rangle_T \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} \sum_{\beta \in [d]} \sum_{\rho \in S_{m-1}} \sum_{|w|=m-1} \sum_{|w'|=m-1} \alpha_{w\beta w'} Q(\rho, w, w', \beta) E(\rho, w, w', \beta) \end{aligned}$$

We set the following notations: For each word $w = \alpha_n \cdots \alpha_1$ with $e_{\alpha_t} = \mathbf{H}_\mathbb{C}^{(i_t)}$ for $i_t \in N$, $1 \leq t \leq n$, write $i_t = i_t^{(w)}$ in order to show dependency on w . Also write $\bar{e}_w = \bar{e}_{\alpha_n} \otimes \cdots \otimes \bar{e}_{\alpha_1}$ and $\rho(w) = \alpha_{\rho(n)} \cdots \alpha_{\rho(1)}$ for any permutation $\rho \in S_n$. In this notation, we note that $E(\rho, w_1, w'_1, \beta) = \langle \bar{e}_\beta, e_\alpha \rangle_U \langle e_{\rho(w)}, e_{w'} \rangle_{\mathbf{H}_U^{\otimes n}}$. We now calculate

$$\begin{aligned} &\sum_{\beta \in [d]} \sum_{\rho \in S_{m-1}} \sum_{|w|=m-1} \sum_{|w'|=m-1} \alpha_{w\beta w'} Q(\rho, w, w', \beta) E(\rho, w, w', \beta) \\ &= \sum_{\beta \in [d]} \sum_{|w|=m-1} \sum_{|w'|=m-1} \prod_{1 \leq v < u \leq m-1} q(i_v^{(w)}, i_u^{(w)}) \prod_{t=1}^{m-1} q(i_\alpha, i_t^{(w)}) \alpha_{w\beta w'} \langle \bar{e}_\beta, e_\alpha \rangle_U \\ &\quad \cdot \sum_{\rho \in S_{m-1}} \prod_{\substack{1 \leq t < r \leq m-1 \\ \rho(t) > \rho(r)}} q(i_{\rho(r)}^{(w)}, i_{\rho(t)}^{(w)}) \prod_{t=1}^{m-1} \langle \bar{e}_{w_{\rho(t)}}, e_{w'_t} \rangle_U \\ &= \sum_{\beta \in [d]} \sum_{|w|=m-1} \sum_{|w'|=m-1} \prod_{1 \leq v < u \leq m-1} q(i_v^{(w)}, i_u^{(w)}) \prod_{t=1}^{m-1} q(i_\alpha, i_t^{(w)}) \alpha_{w\beta w'} \langle \bar{e}_\beta, e_\alpha \rangle_U \langle \bar{e}_w, e_{w'} \rangle_T \\ &= \sum_{\beta \in [d]} \sum_{|w|=m-1} \prod_{1 \leq v < u \leq m-1} q(i_v^{(w)}, i_u^{(w)}) \prod_{t=1}^{m-1} q(i_\alpha, i_t^{(w)}) \langle \bar{e}_\beta, e_\alpha \rangle_U \left\langle \bar{e}_w, \sum_{|w'|=m-1} \alpha_{w\beta w'} e_{w'} \right\rangle_T. \end{aligned}$$

Now denote by $L_{w\beta}$ the composition of the m relevant free annihilation operators i.e. $L_{w\beta} = l(e_\beta)l(e_{\alpha_1}) \cdots l(e_{\alpha_n})$ for any word $w = \alpha_n \cdots \alpha_1$, where $l(e_\gamma)$ is as defined in (2.3) for any $\gamma \in [d]$. Recall our assumption that the set $\{e_\gamma\}_{\gamma \in [d]}$ is an orthonormal basis for \mathbf{H}_U so that $L_{w\beta}(e_{w''\gamma w'}) = 0$ for words w', w'' of length $m-1$ and $\gamma \in [d]$ such that $w''\gamma \neq w\beta$. Then for fixed w, β, α as

above we obtain

$$\begin{aligned}
\sum_{|w'|=m-1} \alpha_{w\beta w'} e_{w'} &= \sum_{\gamma \in [d]} \sum_{|w''|=m-1} \sum_{|w'|=m-1} \alpha_{w''\gamma w'} L_{w\beta}(e_{w''\gamma w'}) \\
&= L_{w\beta} \left(\sum_{\gamma \in [d]} \sum_{|w''|=m-1} \sum_{|w'|=m-1} \alpha_{w''\gamma w'} e_{w''\gamma w'} \right) \\
&= L_{w\beta} \left(\sum_{|w''|=2m-1} \alpha_{w''} e_{w''} \right)
\end{aligned}$$

so that

$$\langle e_w, \sum_{|w'|=m-1} \alpha_{w\beta w'} e_{w'} \rangle_T = \langle e_w, L_{w\beta} \left(\sum_{w'' \in [d]^*} \alpha_{w''} e_{w''} \right) \rangle_T.$$

which follows by noting that the operator $L_{w\beta}$ maps $e_{w'}$ with $|w'| = 2m' - 1$ to the subspace spanned by $\{e_v\}_{|v|=m-1}$ and hence $\langle e_w, L_{w\beta}(e_{w'}) \rangle_T = 0$ if $m \neq m'$ and $|w| = m - 1, |w'| = 2m' - 1$. Thus we get

$$\begin{aligned}
&\sum_{\beta \in [d]} \sum_{\rho \in S_{m-1}} \sum_{|w|=m-1} \sum_{|w'|=m-1} \alpha_{w\beta w'} Q(\rho, w, w', \beta) E(\rho, w, w', \beta) \\
&= \sum_{\beta \in [d]} \sum_{|w|=m-1} \prod_{1 \leq v < u \leq m-1} q(i_v^{(w)}, i_u^{(w)}) \prod_{t=1}^{m-1} q(i_\alpha, i_t^{(w)}) \langle \bar{e}_\beta, e_\alpha \rangle_U \left\langle \bar{e}_w, L_{w\beta} \left(\sum_{w'' \in [d]^*} \alpha_{w''} e_{w''} \right) \right\rangle_T
\end{aligned}$$

If we set

$$(3.8) \quad \xi_{m,\alpha} = \sum_{\beta \in [d]} \sum_{|w|=m-1} \prod_{1 \leq v < u \leq m-1} q(i_v^{(w)}, i_u^{(w)}) \prod_{t=1}^{m-1} q(i_\alpha, i_t^{(w)}) \langle e_\alpha, \bar{e}_\beta \rangle_U L_{w\beta}^*(\bar{e}_w)$$

then we have

$$\langle \Omega, D_\alpha \left(\sum_{w \in [d]^*} \alpha_w e_w \right) \rangle_T = \left\langle \sum_{m=1}^{\infty} (-1)^m \xi_{m,\alpha}, \sum_{w \in [d]^*} \alpha_w e_w \right\rangle_T$$

so long as the sum $\sum_{m=1}^{\infty} (-1)^m \xi_{m,\alpha}$ is a well-defined vector. We show that it indeed is and converges in norm:

$$\begin{aligned}
\|\xi_{m,\alpha}\|_T &\leq \sum_{\beta \in [d]} \sum_{|w|=m-1} q^{\frac{m(m-1)}{2}} \|e_\alpha\|_U \|\bar{e}_\beta\|_U \|L_{w\beta}^*\| \|\bar{e}_w\|_T \\
(3.9) \quad &\leq d B \sum_{|w|=m-1} q^{\frac{m(m-1)}{2}} C^m \|\bar{e}_w\|_T \\
&= d^m B^m C^m \sqrt{[m-1]_q!} q^{\frac{m(m-1)}{2}}
\end{aligned}$$

where $d = |[d]|$, $B = \max_{\gamma \in [d]} \|\bar{e}_\gamma\|$, $C = \max_{\gamma \in [d]} \|l(e_\gamma)\| < \infty$ (by Lemma 2.1) and $q = \max_{i,j} |q(i,j)| < 1$; and the last equality follows by noting from Lemma 2.6 that

$$(3.10) \quad \|\bar{e}_w\|_T^2 \leq \sum_{\sigma \in S_{m-1}} q^{\text{inv}(\sigma)} B^{2(m-1)} = [m-1]_q! B^{2(m-1)}$$

where $\text{inv}(\sigma)$ denotes the number of inversions in the permutation σ , so that

$$\sum_{|w|=m-1} \|\bar{e}_w\|_T \leq d^{m-1} B^{m-1} \sqrt{[m-1]_q!}.$$

Therefore we finally arrive at the claim:

$$\sum_{m=1}^{\infty} \|\xi_{m,\alpha}\| \leq \sum_{m=1}^{\infty} q^{\frac{m(m-1)}{2}} d^m B^m C^m \sqrt{[m-1]_q!} < \infty$$

by using ratio test. We have thus shown that

$$(3.11) \quad \xi_\alpha := \sum_{m=1}^{\infty} (-1)^m \xi_{m,\alpha} \in \mathcal{F}_T(\mathbf{H}_U)$$

and $\langle \Omega, D_\alpha(\cdot) \rangle_T = \langle \xi_\alpha, \cdot \rangle_T$ on $\mathcal{F}_T(\mathbf{H}_U)_{\text{alg}}$; hence the functional extends to a bounded linear map on $\mathcal{F}_T(\mathbf{H}_U)$. This proves that $\Omega \in \text{Dom } D_\alpha^*$ and $\xi_\alpha = D_\alpha^*(\Omega)$. \square

In our next result, we use Bożejko-Haagerup type inequality (see [No₁, Lemma 2]): For any $\xi \in \mathbf{H}_U^{\otimes n}$, $n \geq 1$, we have

$$(3.12) \quad \|\xi\|_T \leq \|W(\xi)\| \leq C_q^{3/2} (n+1) \|\xi\|_T$$

where as before $q = \sup_{i,j} |q(i,j)|$ and $C_q = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$. With the notations as in the proof of Theorem 3.4, we establish the following:

Corollary 3.5. *The conjugate system (ξ_1, \dots, ξ_d) corresponding to (D_1, \dots, D_d) is given by $\xi_\alpha = D_\alpha^* \Omega = \sum_{m=1}^{\infty} (-1)^{m-1} \xi_{m,\alpha}$ for $1 \leq \alpha \leq d := \dim \mathbf{H}_{\mathbb{R}}$ where for $m \geq 1$,*

$$\xi_{m,\alpha} = \sum_{\beta \in [d]} \sum_{|w|=m-1} \prod_{1 \leq v < u \leq m-1} q(i_v^{(w)}, i_u^{(w)}) \prod_{t=1}^{m-1} q(i_\alpha, i_t^{(w)}) \langle e_\alpha, \bar{e}_\beta \rangle_U L_{w\beta}^* (\bar{e}_w).$$

Moreover the sum $\sum_{m=1}^{\infty} (-1)^{m-1} W(\xi_{m,\alpha})$ converges in operator norm i.e. $\xi_\alpha \in \mathbf{M}_T \Omega$.

Proof. We have already seen the expression for the conjugate variables ξ_α in (3.11) in Theorem 3.4. Note that the vector $L_{w\beta}^* (\bar{e}_w)$ belongs to the particle space $\mathbf{H}_U^{\otimes 2m-1}$ for any word w of length $m-1$ and $\beta \in [d]$. Arguing as in [MS], we invoke Bożejko's Haagerup type inequality as mentioned in (3.12) to infer that

$$\begin{aligned} \|W(\xi_{m,\alpha})\| &\leq \sum_{\beta \in [d]} \sum_{|w|=m-1} q^{\frac{m(m-1)}{2}} \|e_\alpha\|_U \|\bar{e}_\beta\|_U \|W(L_{w\beta}^* \bar{e}_w)\| \\ &\leq B \sum_{\beta \in [d]} \sum_{|w|=m-1} q^{\frac{m(m-1)}{2}} C_q^{3/2} (2m) \|L_{w\beta}^* (\bar{e}_w)\|_T \\ &\leq C_q^{3/2} d^m q^{\frac{m(m-1)}{2}} (2m) C^m B^m \sqrt{[m-1]_q!} \end{aligned}$$

where B, C are as written towards the end of proof of Theorem 3.4 and we have used estimates as in (3.9) and (3.10). Thus, by using the ratio test, it follows that

$$\sum_{m=1}^{\infty} \|W(\xi_{m,\alpha})\| \leq C_q^{3/2} \sum_{m=1}^{\infty} d^m q^{\frac{m(m-1)}{2}} (2m) C^m B^m \sqrt{[m-1]_q!} < \infty$$

concluding the proof. \square

Remark 3.6. (1) In the expression for the conjugate variables given in [MS, Corollary 4.7], certain products of free *right* annihilation operators appear, while in our result products of *left* annihilation operators appear. One can however verify that in the q -Gaussian case, the two situations indeed agree. This follows by noting that if $R_{\beta w}$ is a product of appropriate right free

annihilation operators then

$$\begin{aligned}
\sum_{|v|=2m-1} \sum_{|w|=m-1} \alpha_v \langle \bar{e}_w, R_{\beta w} e_v \rangle_q &= \sum_{|w'|=m-1} \sum_{|w|=m-1} \alpha_{w' \beta w} \langle e_w, e_{w'} \rangle_q \\
&= \sum_{|w|=m-1} \sum_{|w'|=m-1} \alpha_{w' \beta w} \langle e_{w'}, e_w \rangle_q \\
&= \sum_{|v|=2m-1} \sum_{|w'|=m-1} \alpha_v \langle \bar{e}_{w'}, L_{\beta w'} e_v \rangle_q
\end{aligned}$$

simply because in the q -Gaussian case, one chooses an orthonormal basis $\{e_1, \dots, e_d\}$ with $e_i \in \mathbf{H}_{\mathbb{R}}$ so that $\bar{e}_w = e_w$ and $\langle e_w, e_{w'} \rangle_T = \langle e_{w'}, e_w \rangle_T$ as they are always real for any two words w, w' (summation of the above form appears in the proof of Theorem 3.4). In particular, we can replace $L_{\beta w}$ by $R_{\beta w}$ in Corollary 3.5 which produces exactly the same formula as in [MS, Corollary 4.7]. Note that equality of above forms no longer holds in non-tracial settings, so left free annihilation operators seem to be more convenient choice.

(2) The orthonormal set $\{e_1, \dots, e_d\}$ can be chosen to be eigenvectors of the analytic generator A of U_t so that each element $W(e_k)$ is analytic in \mathbf{M}_T (w.r.t the modular automorphism group σ^φ of the canonical state φ); consequently the conjugate variables ξ_i 's are also analytic as they are norm-limits of vectors which are linear combinations of tensors of e_i 's.

(3) When $q_{ij} = q$ for all i, j , we get

$$\xi_\alpha = \sum_{m=1}^{\infty} (-1)^{m-1} \sum_{\beta \in [d]} \sum_{|w|=m-1} q^{\frac{m(m-1)}{2}} \langle e_\alpha, \bar{e}_\beta \rangle_U L_{w\beta}^* (\bar{e}_w),$$

which is slightly different than the formula in [KSW, Remark 1.5] because the initial assumption on the set $\{e_1, \dots, e_d\} \subseteq \mathbf{H}_{\mathbb{C}}$ in [KSW] is that it is orthonormal in the undeformed inner product of $\mathbf{H}_{\mathbb{C}}$ while our assumption is that the set is orthonormal in the deformed inner product of \mathbf{H}_U . In particular, if $q = 0$ then $\xi_{m,\alpha} = 0$ for all $m \geq 2$, so that

$$\xi_\alpha = \xi_{1,\alpha} = \sum_{\beta \in [d]} \langle e_\alpha, \bar{e}_\beta \rangle_U l(\beta)^* \Omega = \sum_{\beta \in [d]} \langle e_\alpha, \bar{e}_\beta \rangle_U e_\beta.$$

Lipschitz Conjugate Variables. Our aim now is to prove that the conjugate variables (ξ_1, \dots, ξ_d) as obtained above are *Lipschitz* i.e.

$$\xi_\ell \in \text{Dom } \bar{\partial}_k \text{ and } \bar{\partial}_k(\xi_\ell) \in \mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}}$$

for all $1 \leq k, \ell \leq d$, where ∂_k is the quasi-free difference quotients as defined in the beginning of this Section. For this purpose, we consider a different set of rules for partitions and counting their crossings exactly as in [MS, Section 5]. The partitions again consist of singletons and pairs, and we draw them in the following ways to locate crossings:

- (1) Consider $n + 1$ vertices $n > \dots > 1 > 0$.
- (2) The vertex 0 must be coupled with some $k \in \{1, \dots, n\}$ with height 1.
- (3) Each $\ell \in \{1, \dots, k-1\}$ is a singleton or is coupled with one of $\{k+1, \dots, n\}$ with height $\ell + 1$.
- (4) Vertices which are not coupled with one of $\{1, \dots, k-1\}$ should be singletons and are drawn with straight lines to the top.

Let $C(n+1)$ be the set of partitions defined by the rules above. For each $\pi \in C(n+1)$, denote by $s_\ell(\pi)$ and $s_r(\pi)$ respectively the set of singletons in the left area $n \geq l > \pi(0)$ and the set of singletons in the right area $\pi(0) > l \geq 1$. Let $p(\pi)$ denote the set of pairings.

We now give a formula for the action of ∂_k 's on the algebra $\mathbb{C}[A_1, \dots, A_d]$. The algebra $\mathbb{C}[A_1, \dots, A_d]$ is nothing but the space $\text{span}\{W(e_w); w \in [d]^*\}$. This follows by noting the following calculation (see eq. (2.5)):

$$(3.13) \quad W(\xi_{n+1} \cdots \xi_1) = W(\xi_{n+1})W(\xi_n \cdots \xi_1) - \sum_{k=1}^n \langle \bar{\xi}_{n+1}, \xi_k \rangle_U \prod_{t=1}^{n-k} q(i_k, i_{k+t}) W(\xi_n \cdots \widehat{\xi}_k \cdots \xi_1)$$

whenever $\xi_k \in \mathbf{H}_U^{(i_k)}$ for $i_k \in N$, $1 \leq k \leq n+1$. Therefore it suffices to understand the action of ∂_k 's on the operators $W(e_w)$ for each word w .

Proposition 3.7. *For each $\alpha \in [d]$ and $w = \alpha_n \cdots \alpha_1 \in [d]^*$, $n \geq 1$, we have*

$$\partial_\alpha W(e_w) = \sum_{\pi \in C(n+1)} (-1)^{|\rho(\pi)|-1} D(\pi) E(\pi) W(e_{s_\ell(\pi)}) \otimes W(e_{s_r(\pi)})^{\text{op}}$$

where

$$D(\pi) := \prod_{\substack{u,v \text{ belong to a pairing} \\ 0 \leq v < u < \pi(0)}} q(i_v, i_u) \cdot \prod_{\substack{u,v \text{ belong to a pairing} \\ 0 < v < u \leq \pi(0) \\ \pi(u) > \pi(v)}} q(i_{\pi(v)}, i_{\pi(u)}) \cdot \prod_{\substack{\pi(0) < v < u \leq n \\ v \in s_\ell(\pi), u \notin s_\ell(\pi)}} q(i_v, i_u)$$

$$E(\pi) := \prod_{\substack{(k,\ell) \in p(\pi) \\ k > \ell}} \langle \bar{e}_{\alpha_k}, e_{\alpha_\ell} \rangle_U$$

$$e_{s_\ell(\pi)} = e_{\alpha_{j_r}} \cdots e_{\alpha_{j_1}} \quad \text{for } s_\ell(\pi) = \{j_r > \dots > j_1\} \quad \text{and}$$

$$e_{s_r(\pi)} = e_{\alpha_{j'_s}} \cdots e_{\alpha_{j'_1}} \quad \text{for } s_r(\pi) = \{j'_s > \dots > j'_1\},$$

with the prescription that $\alpha_0 = \alpha$.

Remark 3.8. As in Proposition 3.2, we have divided the expression for $D(\pi, w)$ into three products. The terms in the first product (resp. the second product) corresponds to the crossings occurring on the line of the vertex ' u ' (resp. ' $\pi(u)$ ') of any pair $(u, \pi(u))$ for $0 < u < \pi(0)$ with the lines of vertices of another pair, while the third product corresponds to the crossings on the lines of a singleton in the left side of $\pi(0)$. We essentially do not care about the crossings on the line of the singletons on the right side of $\pi(0)$.

Proof. As in [MS, Proposition 5.1], the proof goes by induction on n . For $n = 1$, the expression is obvious because

$$\partial_\alpha (W(e_{\alpha_1})) = \partial_\alpha (A_{\alpha_1}) = \langle \bar{e}_{\alpha_1}, e_{\alpha_1} \rangle_U \mathbf{1} \otimes \mathbf{1}.$$

So assume this to be true for some $n \geq 1$. Use then (3.13) to calculate:

$$\begin{aligned} \partial_\alpha W(e_{\alpha_{n+1} \cdots \alpha_1}) &= \partial_\alpha \left(W(e_{\alpha_{n+1}}) W(e_{\alpha_n \cdots \alpha_1}) - \sum_{u=1}^n \prod_{u < t \leq n} q(i_u, i_t) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_u} \rangle_U W(e_{\alpha_n \cdots \hat{\alpha}_u \cdots \alpha_1}) \right) \\ &= \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_n} \rangle_U \mathbf{1} \otimes W(e_{\alpha_n \cdots \alpha_1})^{\text{op}} + (W(e_{\alpha_{n+1}}) \otimes \mathbf{1}) \partial_\alpha W(e_{\alpha_n \cdots \alpha_1}) \\ &\quad - \sum_{v=1}^n \prod_{v < t \leq n} q(i_v, i_t) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_v} \rangle_U \partial_\alpha W(e_{\alpha_n \cdots \hat{\alpha}_v \cdots \alpha_1}). \end{aligned}$$

By using induction argument, we separately compute the last two terms in the above sum:

$$(3.14) \quad \begin{aligned} (W(e_{\alpha_{n+1}}) \otimes \mathbf{1}) \partial_\alpha W(e_{\alpha_n \cdots \alpha_1}) &= \sum_{\pi \in C(n+1)} (-1)^{|\rho(\pi)|-1} D(\pi) E(\pi) W(e_{\alpha_{n+1}}) W(e_{s_\ell(\pi)}) \otimes W(e_{s_r(\pi)})^{\text{op}} \\ &= \sum_{\pi \in C(n+1)} (-1)^{|\rho(\pi)|-1} D(\pi) E(\pi) W(e_{\alpha_{n+1}} e_{s_\ell(\pi)}) \otimes W(e_{s_r(\pi)})^{\text{op}} \\ &+ \sum_{\pi \in C(n+1)} (-1)^{|\rho(\pi)|-1} D(\pi) E(\pi) \sum_{u=1}^{n(\pi)} \prod_{t=u+1}^{n(\pi)} q(i_{j_u^\pi}, i_{j_t^\pi}) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_{j_u^\pi}} \rangle_U W(e_{\alpha_{j_n^\pi(\pi)} \cdots \hat{\alpha}_{j_u^\pi} \cdots \alpha_{j_1^\pi}}) \otimes W(e_{s_r(\pi)})^{\text{op}} \end{aligned}$$

where $n(\pi) = |s_\ell(\pi)|$ and $s_\ell(\pi) = \{j_{n(\pi)}^\pi > \dots > j_1^\pi\}$; then compute

$$(3.15) \quad \begin{aligned} & \sum_{v=1}^n \prod_{v < t \leq n} q(i_v, i_t) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_v} \rangle_U \partial_\alpha W(e_{\alpha_n \dots \widehat{\alpha}_v \dots \alpha_1}) \\ &= \sum_{\sigma \in C(n)} \sum_{v=1}^n \prod_{v < t \leq n} q(i_v, i_t) \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_v} \rangle_U (-1)^{|p(\sigma)|-1} D(\sigma, v) E(\sigma, v) W(e_{s_\ell(\sigma)}) \otimes W(e_{s_r(\sigma)})^{\text{op}} \end{aligned}$$

where $D(\sigma, v)$ and $E(\sigma, v)$ are written respectively for $D(\sigma)$ and $E(\sigma)$ only but just to emphasize that each $\sigma \in C(n)$ acts on the word $\alpha_n \dots \widehat{\alpha}_v \dots \alpha_1$. Note that each term of the last sum of (3.14) appears as one of the terms in the last sum of (3.15). To see this, we follow the similar argument as in Theorem 3.4. Let $\pi \in C(n+1)$ and $u \in \{1, \dots, n(\pi)\}$, where $n(\pi) = |s_\ell(\pi)|$. Set $v = s_\ell(\pi)_u = j_u^\pi$ and obtain the partition $\sigma \in C(n)$ of the set $\{0, \dots, n\} \setminus \{v\}$ in an order-preserving fashion by removing the singleton $s_\ell(\pi)_u = v$. Note that $p(\pi) = p(\sigma), s_\ell(\pi) = s_\ell(\sigma) \setminus \{v\}$ and $s_r(\pi) = s_r(\sigma)$. Hence $E(\pi) = E(\sigma)$. Also note that the extra crossings in π as compared to σ may be occurring only due to the straight line at $v = j_u^\pi$. Therefore we have

$$D(\pi) = D(\sigma, v) \cdot \prod_{\substack{t > v \\ t \notin s_\ell(\pi)}} q(i_v, i_t) = \frac{\prod_{v < t \leq n} q(i_v, i_t)}{\prod_{\substack{v < t \leq n \\ t \in s_\ell(\pi)}} q(i_v, i_t)} = \frac{\prod_{v < t \leq n} q(i_v, i_t)}{\prod_{u < r \leq n(\pi)} q(i_{j_u^\pi}, i_{j_r^\pi})}$$

where it is to be understood that $\frac{0}{0}$ is 0. This shows our claim that all terms of the last sum of (3.14) appear in the last sum of (3.15), and the same time note that the remaining terms in (3.15) correspond to those σ and v for which $v \leq \sigma(0)$. Again as in Theorem 3.4, we identify such pair (σ, v) i.e. $\sigma \in C(n)$ and $v \leq \sigma(0)$, with a partition $\sigma' \in C(n+2)$ which creates a new pair between v and $n+1$ with the requirement that $\sigma' \setminus (n+1, v) = \sigma$. Note that $\sigma'(0) = \sigma(0) + 1$, $s_\ell(\sigma') = s_\ell(\sigma), s_r(\sigma') = s_r(\sigma)$ and $|p(\sigma')| = |p(\sigma)| + 1$. Also note that the extra crossings in σ' occur with all the pairs (twice) and singletons (once) to the left of v or on the line of v ; so that whenever $E(\sigma) \neq 0$, then we have

$$D(\sigma') = D(\sigma) \cdot \prod_{v < t \leq n+1} q(i_v, i_t)$$

which follows by similar arguments as in (3.5) from Proposition 3.2. In particular, we get

$$E(\sigma') D(\sigma') = E(\sigma) D(\sigma) \cdot \prod_{v < t \leq n+1} q(i_v, i_t).$$

Finally by subtracting (3.14) from (3.15) yields

$$\begin{aligned} \partial_\alpha W(e_{\alpha_{n+1} \dots \alpha_1}) &= \langle \bar{e}_{\alpha_{n+1}}, e_\alpha \rangle_U \mathbb{1} \otimes W(e_{\alpha_n \dots \alpha_1})^{\text{op}} \\ &+ \sum_{\pi \in C(n+1)} (-1)^{|p(\pi)|-1} D(\pi) E(\pi) W(e_{\alpha_{n+1} e_{s_\ell(\pi)}}) \otimes W(e_{s_r(\pi)})^{\text{op}} \\ &+ \sum_{\substack{\sigma' \in C(n+2) \\ \sigma'(n+1) \text{ is not a singleton} \\ \sigma'(0) \neq n+1}} \langle \bar{e}_{\alpha_{n+1}}, e_{\alpha_v} \rangle_U (-1)^{|p(\sigma')|-1} D(\sigma') E(\sigma') W(e_{s_\ell(\sigma')}) \otimes W(e_{s_r(\sigma')})^{\text{op}} \\ &= \sum_{\sigma' \in C(n+2)} (-1)^{|p(\sigma')|-1} D(\sigma') E(\sigma') W(e_{s_\ell(\sigma')}) \otimes W(e_{s_r(\sigma')})^{\text{op}} \end{aligned}$$

where the last equality follows by noting that $\langle \bar{e}_{\alpha_{n+1}}, e_\alpha \rangle_U \mathbb{1} \otimes W(e_{\alpha_n \dots \alpha_1})^{\text{op}}$ corresponds to the partition in $C(n+2)$ which connects 0 to $n+1$ (in which case all other points are right singletons); while each $\pi \in C(n+1)$ corresponds to those partitions $\sigma' \in C(n+2)$ such that $n+1$ is a singleton, in which case we have $p(\sigma') = p(\pi), D(\sigma') = D(\pi), s_r(\sigma') = s_r(\pi)$ and $s_\ell(\sigma') = \{n+1\} \cup s_\ell(\pi)$ so that $e_{\alpha_{n+1} e_{s_\ell(\pi)}} = e_{s(\sigma')}$. This completes the proof by induction. \square

Remark 3.9. One can also obtain Theorem 3.4 using the formulas in Proposition 3.7 for the quasi-free difference quotients as the conjugate variables ξ_α 's are nothing but $\partial_\alpha^*(\mathbf{1} \otimes \mathbf{1})$ and it is an easy check that the expression $\langle \mathbf{1} \otimes \mathbf{1}, \partial_\alpha(\sum_{w \in [d]^*} W(e_w)) \rangle_{L^2(\mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}})}$ is exactly the one in eq. (3.7). Nevertheless Proposition 3.2 provides a crucial information for the operators D_α 's which can be helpful in other situations.

We keep the notations of Corollary 3.5 and Proposition 3.7 for the following result.

Corollary 3.10. *The system of conjugate variables (ξ_1, \dots, ξ_d) corresponding to the tuple (D_1, \dots, D_d) is Lipschitz.*

Proof. Fix $\alpha, \beta \in [d]$. First note that each $\xi_{m,\alpha} \in \mathcal{F}_T(\mathbf{H}_U)_{\text{alg}}$ so that $W(\xi_{m,\alpha}) \in \text{Dom } \partial_\beta$; hence by Proposition 3.7, $\partial_\beta W(\xi_{m,\alpha}) \in \mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}}$. We show that $\sum_{m=1}^\infty \|\partial_\beta W(\xi_{m,\alpha})\| < \infty$, from which it will follow that $\sum_{m=1}^\infty (-1)^{m-1} \partial_\beta W(\xi_{m,\alpha})$ converges both in norm and in $L^2(\mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}})$ proving that $W(\xi_\alpha) \in \text{Dom } \bar{\partial}_\beta$ and $\bar{\partial}_\beta W(\xi_\alpha) \in \mathbf{M}_T \otimes \mathbf{M}_T^{\text{op}}$. Assume that $e_\alpha \in \mathbf{H}_C^{(i_\alpha)}$ so that

$$\xi_{m,\alpha} = \sum_{\gamma \in [d]} \sum_{|w|=m-1} \prod_{1 \leq v < u \leq m-1} q(i_v^{(w)}, i_u^{(w)}) \prod_{t=1}^{m-1} q(i_\alpha, i_t^{(w)}) \langle e_\alpha, \bar{e}_\gamma \rangle_U L_{w\gamma}^*(\bar{e}_w).$$

By Proposition 3.7, note that for each word w' of length n

$$\|\partial_\beta W(e_{w'})\| \leq \sum_{\pi \in C(n+1)} |D(\pi)| |E(\pi)| \|W(e_{s_\ell(\pi)})\| \|W(e_{s_r(\pi)})\|.$$

But we have

$$\begin{aligned} |D(\pi)| &\leq q^{\frac{n(n+1)}{2}} \leq 1, \\ |E(\pi)| &\leq B^{|\rho(\pi)|} \leq B^n \quad \text{where } B = \max\{1, \|\bar{e}_\gamma\|_U\}, \\ \|W(e_{s_\ell(\pi)})\| &\leq C_q^{3/2} (|s_\ell(\pi)| + 1) \|e_{s_\ell(\pi)}\|_T \leq C_q^{3/2} (|s_\ell(\pi)| + 1) B^{|s_\ell(\pi)|} \sqrt{[|s_\ell(\pi)|]_q!} \\ &\leq C_q^{3/2} (n+1) B^n \sqrt{[n]_q!}, \\ \|W(e_{s_r(\pi)})\| &\leq C_q^{3/2} (n+1) B^n \sqrt{[n]_q!} \end{aligned}$$

where the last two inequalities are due to Bożejko-Haagerup type inequality and the estimate as in (3.10). Consequently we obtain

$$(3.16) \quad \begin{aligned} \|\partial_\beta W(e_{w'})\| &\leq |C(n+1)| B^{3n} C_q^3 (n+1)^2 [n]_q! \\ &\leq (n+1)! B^{3n} C_q^3 (n+1)^2 [n]_q!, \end{aligned}$$

where we have used the fact that the cardinality $|C(n+1)|$ of $C(n+1)$ is upper-bounded by $(n+1)!$ as we can think $C(n+1)$ as a subset of the symmetric group S_{n+1} . Note that the upper-bound of each $\|\partial_\beta W(e_{w'})\|$ depends only on the length of the word w' and is independent of the choice of the word itself. In particular, since for each fixed word w of length $m-1$ and $\gamma \in [d]$, $L_{w\gamma}^*(\bar{e}_w)$ belongs to the subspace spanned by $\{e_{w'}\}_{|w'|=2m-1}$, we write $L_{w\gamma}^*(\bar{e}_w) = \sum_{|w'|=2m-1} \lambda_{w'} e_{w'}$ for some scalars $\lambda_{w'} \in \mathbb{C}$ and then estimate

$$\begin{aligned} \|\partial_\beta W(L_{w\gamma}^* \bar{e}_w)\| &\leq \sum_{|w'|=2m-1} |\lambda_{w'}| \|\partial_\beta W(e_{w'})\| \\ &\leq \sum_{|w'|=2m-1} |\lambda_{w'}| (2m)! B^{6m-3} C_q^3 (2m)^2 [2m-1]_q!. \end{aligned}$$

Further observe that for all $|w'| = 2m-1$, we have $\lambda_{w'} = L_{w'} L_{\gamma w}^*(\bar{e}_w)$ so that

$$|\lambda_{w'}| \leq \|L_{w'}\| \|L_{w\gamma}^*\| \|\bar{e}_w\|_T \leq C^{2m-1} B^{m-1} \sqrt{[m-1]_q!},$$

where as in Theorem 3.4, $C = \max_{\gamma \in [d]} \|l(e_\gamma)\|$, B is as above and the estimate of (3.10) is used. It follows that

$$\|\partial_\beta W(L_{w\gamma}^* \bar{e}_w)\| \leq d^{2m-1} (2m)! C^{2m-1} B^{7m-4} C_q^3 (2m)^2 [2m-1]_q! \sqrt{[m-1]_q!}.$$

Gathering all the estimates above, we finally arrive at the following:

$$\begin{aligned} \|\partial_\beta W(\xi_{m,\alpha})\| &\leq \sum_{\gamma \in [d]} \sum_{|w|=m-1} q^{\frac{m(m-1)}{2}} \|e_\alpha\|_U \|\bar{e}_\gamma\|_U \|\partial_\beta W(L_{w\gamma}^* \bar{e}_w)\| \\ &\leq q^{\frac{m(m-1)}{2}} d^{3m-1} (2m)! C^{2m-1} B^{7m-4} C_q^3 (2m)^2 [2m-1]_q! \sqrt{[m-1]_q!} \end{aligned}$$

therefore we conclude by using the ratio test that

$$\sum_{m=1}^{\infty} \|\partial_\beta W(\xi_{m,\alpha})\| \leq \sum_{m=1}^{\infty} q^{\frac{m(m-1)}{2}} d^{3m-1} (2m)! C^{2m-1} B^{7m-4} C_q^3 (2m)^2 [2m-1]_q! \sqrt{[m-1]_q!} < \infty$$

thus completing the proof. \square

In terminologies of [Ne₂], the above results can be rephrased as follows. Let $\varphi = \langle \Omega, \cdot \Omega \rangle_T$ be the canonical normal faithful state on M_T and let σ^φ be the corresponding modular automorphism group. An element $x \in M_T$ is an *eigenoperator* (with respect to σ^φ) if $\sigma_t^\varphi(x) = \lambda^{it} x \forall t \in \mathbb{R}$ for some $\lambda > 0$. We choose the orthonormal set $\{e_1, \dots, e_d\}$ in H_U at the beginning of this Section in such a way that each $e_k \in H_{\mathbb{C}}$ is an eigenvector for the analytic generator A of U_t , so that $W(e_k)$ will be an eigenoperator for σ^φ because $\sigma_t(W(e_k)) = W(A^{-it} e_k) \forall t \in \mathbb{R}$, and that the set $\{W(e_1), \dots, W(e_d)\}$ is self-adjoint (see [Hia, Proof of Theorem 2.2] for such construction). Since the tuple $(W(e_1), \dots, W(e_d))$ admits a system (ξ_1, \dots, ξ_d) of conjugate variables as shown in Theorem 3.4, it has *finite free Fisher information* i.e.

$$\sum_{k=1}^d \|\xi_k\|_{L^2(M_T)}^2 < \infty,$$

in the terminology of [Ne₂, Definition 3.6]. We summarise our discussions in the following result:

Corollary 3.11 (see [KSW], Corollary 1.5). *Let $(H_{\mathbb{R}} = \oplus_{i \in N} H_{\mathbb{R}}^{(i)}, U_t, (q_{ij})_{i,j \in N})$ be given with $H_{\mathbb{R}}$ finite dimensional (and hence N a finite set), the von Neumann algebra M_T equipped with the canonical state φ is generated by a finite set $G = G^*$ of eigenoperators of the modular group of automorphisms σ_t^φ with finite free Fisher information.*

4. FACTORIALITY AND NON-INJECTIVITY

In this section, we prove that the mixed q -Araki-Woods von Neumann algebra M_T is a factor i.e. its center $M_T \cap M'_T = \mathbb{C}$, and that M_T is non-injective. Recall that a von Neumann algebra $M \subseteq \mathbf{B}(H)$ is *injective* if there exists an idempotent from $\mathbf{B}(H)$ onto M of norm 1. Moreover we also discuss the type classifications of the algebras M_T . Connes [Con] defined the *S-invariant* of a factor M with a separable predual to be the intersection of the spectra of the modular operators Δ_ϕ of all normal faithful states ϕ . The factor M is of type III if and only if $0 \in S(M)$. In this case, the type classification of M is given as follows:

$$S(M) = \begin{cases} \lambda^{\mathbb{Z}} \cup \{0\} & \text{if } M \text{ is of type III}_\lambda, 0 < \lambda < 1, \\ [0, \infty) & \text{if } M \text{ is of type III}_1, \\ \{0, 1\} & \text{if } M \text{ is of type III}_0. \end{cases}$$

Recall that the centralizer of M with respect to a normal faithful state ϕ is defined by $M^\phi = \{x \in M; \phi(xy) = \phi(yx) \forall y \in M\}$. If the centralizer M^ϕ is a factor, then $S(M)$ is just the spectrum of the modular operator Δ_ϕ corresponding to ϕ . We now state the following abstract results of Nelson [Ne₂]:

Theorem 4.1 ([Ne₂], Theorem A, Theorem B). *Let M be a von Neumann algebra with a faithful normal state ϕ . Suppose M is generated by a finite set $G = G^*$, $|G| \geq 2$ of eigenoperators of the modular group σ^ϕ with finite free Fisher information. Then $(M^\phi)' \cap M = \mathbb{C}$. In particular, M^ϕ is a II_1 factor and if $H < \mathbb{R}_*^\times$ is the closed subgroup generated by the eigenvalues of G then M is a factor of type*

$$\begin{cases} \text{III}_1 & \text{if } H = \mathbb{R}_*^\times \\ \text{III}_\lambda & \text{if } H = \lambda^{\mathbb{Z}}, 0 < \lambda < 1 \\ \text{II}_1 & \text{if } H = \{1\}. \end{cases}$$

Furthermore M^φ is full, and if M is a type III_λ factor, $0 < \lambda < 1$, then M is full.

Recall that a factor M is *full* if for any bounded sequence x_n in M with $\|\phi(\cdot x_n) - \phi(x_n \cdot)\| \rightarrow 0$ for all normal states ϕ on M , there is a bounded sequence $\lambda_n \in \mathbb{C}$ such that $x_n - \lambda_n \rightarrow 0$ $*$ -strongly. Remark that a full factor is necessarily non-injective.

We mention in the passing that if U_t is a strongly continuous orthogonal representations on a real Hilbert space $\mathbb{H}_\mathbb{R}$, then $\mathbb{H}_\mathbb{R}$ decomposes into a direct sum $\mathbb{K}_\mathbb{R} \oplus \mathbb{L}_\mathbb{R}$ of invariant subspaces of U_t , where $\mathbb{K}_\mathbb{R}$ is densely spanned by the eigenvectors of U_t . We call the part $(\mathbb{K}_\mathbb{R}, U_t|_{\mathbb{K}_\mathbb{R}})$ as the *almost periodic* and the part $(\mathbb{L}_\mathbb{R}, U_t|_{\mathbb{L}_\mathbb{R}})$ as *weakly mixing*. Note that if $\mathbb{L}_\mathbb{R}$ is non-zero, then it must be infinite-dimensional. Also the almost periodic part has a nice decomposition as representations on a direct sum of either one or two dimensional real spaces (see [Shl]). In particular if $\mathbb{H}_\mathbb{R} = \bigoplus_{i \in N} \mathbb{H}_\mathbb{R}^{(i)}$ is finite dimensional then each $\mathbb{H}_\mathbb{R}^{(i)}$ is almost periodic, hence decomposes as a direct sum of one or two dimensional invariant subspaces. We are now ready to give our main results. As before, we denote below the canonical normal faithful state $\langle \Omega, \cdot \Omega \rangle_T$ on M by φ .

Theorem 4.2. *Let $(\mathbb{H}_\mathbb{R} = \bigoplus_{i \in N} \mathbb{H}_\mathbb{R}^{(i)}, U_t, (q_{ij})_{i,j \in N})$ be given for some countable set N , with $\dim(\mathbb{H}_\mathbb{R}) \geq 2$. Then $M := \Gamma_T(\mathbb{H}_\mathbb{R}, U_t)$ is a factor. Moreover, if $G < \mathbb{R}_*^\times$ is the closed subgroup generated by the spectrum of A then M is a factor of type*

$$\begin{cases} \text{III}_1 & \text{if } G = \mathbb{R}_*^\times \\ \text{III}_\lambda & \text{if } G = \lambda^{\mathbb{Z}}, 0 < \lambda < 1 \\ \text{II}_1 & \text{if } G = \{1\}. \end{cases}$$

Proof. If $\dim \mathbb{H}_\mathbb{R} < \infty$ then the factoriality and type classification of M follow from Corollary 3.11 and Theorem 4.1. Therefore assume that $\dim \mathbb{H}_\mathbb{R} = \infty$. If the weakly mixing part of U_t is non-trivial, then M is a type III_1 factor by [BKM₁, Theorem 5.9].

So we may assume that U_t is almost periodic on an infinite dimensional space $\mathbb{H}_\mathbb{R}$. We use an inductive limit argument to prove factoriality and type classification; this may be apparent to experts but we provide details for completeness. As mentioned above, one can find an increasing sequence $\mathbb{H}_\mathbb{R}(n)$ of U_t -invariant finite-dimensional subspaces of $\mathbb{H}_\mathbb{R}$ such that each $\mathbb{H}_\mathbb{R}(n)$ is compatible with the direct sum i.e. $\mathbb{H}_\mathbb{R}(n) = \bigoplus_{i \in N} (\mathbb{H}_\mathbb{R}(n) \cap \mathbb{H}_\mathbb{R}^{(i)})$, and that their union spans a dense subspace of $\mathbb{H}_\mathbb{R}$. Denote the von Neumann algebra $\Gamma_T(\mathbb{H}_\mathbb{R}(n), U_t|_{\mathbb{H}_\mathbb{R}(n)})$ by M_n and the restriction of the state φ to M_n by φ_n . Through our assumption, we have implication that the unital $*$ -algebra $\bigcup_{n \geq 1} M_n$ is strongly dense in M . An appeal to [BKM₁, Proposition 6.1] entail that each M_n is naturally embedded in M as a subalgebra and that there is a φ -preserving conditional expectation E_n from M onto M_n , so that M_n is invariant under the modular automorphism group σ_t^φ of the state φ . Observe that $E_n(x) \rightarrow x$ strongly for all $x \in M$. One easily verifies that $M_n^{\varphi_n} \subseteq M^\varphi$.

We now show that M^φ is a factor. Indeed if $x \in (M^\varphi)' \cap M$, then $E_n(x) \in (M_n^{\varphi_n})' \cap M_n$ for all $n \geq 1$; but we know from Corollary 3.11 and Theorem 4.1 that $(M_n^{\varphi_n})' \cap M_n = \mathbb{C}$ (since $\dim \mathbb{H}_\mathbb{R}(n) < \infty$), therefore $E_n(x) = \varphi_n(E_n(x)) = \varphi(x)$ for all $n \geq 1$ which finally implies that $x = \varphi(x)$. We have thus shown that $(M^\varphi)' \cap M = \mathbb{C}$ so that M^φ is a factor; hence M is a factor. Since M^φ is a factor, the type classification of M follows from the spectral data of the modular operator Δ_φ of the state φ , which is related to the analytic generator A as given in [BKM₁, Theorem 3.13]. \square

We now show that the factor M_T is non-injective with the same data as in Theorem 4.2.

Theorem 4.3. *The factor $M = \Gamma_T(\mathbb{H}_\mathbb{R}, U_t)$ is not injective as soon as $\dim(\mathbb{H}_\mathbb{R}) \geq 2$.*

Proof. If U_t has non-trivial weakly mixing part, then M is non-injective by [BKM₁, Theorem 8.4]. So we may assume that U_t is almost periodic. As in the proof of [KSW, Theorem 3.5], we will find a non-injective subalgebra with a conditional expectation from the ambient factor M . So let $K_{\mathbb{R}}$ be a two-dimensional invariant subspace of U_t compatible with the direct sum decomposition i.e. $K_{\mathbb{R}} = \bigoplus_{i \in N} (K_{\mathbb{R}} \cap H_{\mathbb{R}}^{(i)})$. As in the proof of Theorem 4.2 above, there is a conditional expectation from M to the subalgebra $M_1 := \Gamma_T(K_{\mathbb{R}}, U_{t|_{K_{\mathbb{R}}}})$. Now if $U_{t|_{K_{\mathbb{R}}}}$ is trivial then $M_1 = M_1^{\varphi|_{M_1}}$ which in conjugation with Corollary 3.11 and Theorem 4.1 yields that M_1 is full and hence M_1 is non-injective. On the other hand if $U_{t|_{K_{\mathbb{R}}}}$ is ergodic then by Theorem 4.2, M_1 is a type III $_{\lambda}$ factor for some $0 < \lambda < 1$; the same combination of Corollary 3.11 and Theorem 4.1 then concludes that M_1 is full and hence non-injective. The proof is now complete. \square

Acknowledgments. I am grateful to Adam Skalski and Mateusz Wasilewski for carefully reading the initial draft and giving valuable comments. The author is partially supported by the National Science Center (NCN) grant no. 2020/39/I/ST1/01566.

REFERENCES

- [BKS] M. Bożejko, B. Kümmerer, and R. Speicher, q -Gaussian processes: noncommutative and classical aspects, *Comm. Math. Phys.* **185** (1997), no. 1, 129–154.
- [BS₁] M. Bożejko and R. Speicher, An example of a generalized Brownian motion, *Comm. Math. Phys.* **137** (1991), no. 3, 519–531.
- [BS₂] M. Bożejko and R. Speicher, Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces, *Math. Ann.* **300** (1994), 97–120.
- [Bož] M. Bożejko, Completely positive maps on Coxeter groups and the ultracontractivity of the q -Ornstein-Uhlenbeck semigroup, *Banach Center Publ.* **43** (1998), 87–93.
- [BKM₁] P. Bikram, R. Kumar R., and K. Mukherjee, Mixed q -deformed Araki-Woods von Neumann algebras, *J. Noncommut. Geom.*, to appear.
- [BKM₂] P. Bikram, R. Kumar R., and K. Mukherjee, On non-injectivity of mixed q -deformed Araki-Woods von Neumann algebras, *Kyoto J. Math.*, to appear.
- [BM] P. Bikram and K. Mukherjee, Generator masas in q -deformed Araki-Woods von Neumann algebras and factoriality, *J. Funct. Anal.* **273** (2017), no. 4, 1443–1478.
- [BMRW] P. Bikram, K. Mukherjee, É. Ricard and S. Wang, On the factoriality of q -deformed Araki-Woods von Neumann algebras, *Comm. Math. Phys.* **398** (2023), no. 2, 797–821.
- [Con] A. Connes, Une classification des facteurs de type III, *Ann. Scient. Ec. Norm. Sup.* **6** (1973), 133–252.
- [DI] Y. Dabrowski and A. Ioana, Unbounded derivations, free dilations, and indecomposability results for II₁ factors, *Trans. Amer. Math. Soc.* **368** (2016), no. 7, 4525–4560.
- [Hia] F. Hiai, q -deformed Araki-Woods algebras, in “Operator algebras and mathematical physics (Constanta, 2001),” Theta, Bucharest, 2003, pp. 169–202.
- [Kro₁] I. Królak, Factoriality of von Neumann algebras connected with general commutation relations – finite dimensional case, in “Quantum probability”, *Banach Center Publ.* **73** (2006), pp. 277–284.
- [Kro₂] I. Królak, Wick product for commutation relations connected with Yang-Baxter operators and new constructions of factors, *Comm. Math. Phys.* **210** (2000), no. 3, 685–701.
- [KSW] M. Kumar, A. Skalski and M. Wasilewski, Full solution of the factoriality question for q -Araki-Woods von Neumann algebras via conjugate variables, *Comm. Math. Phys.*, to appear, available at arXiv:2301.08619.
- [MS] A. Miyagawa and R. Speicher, A dual and conjugate system for q -Gaussians for all q , *Adv. Math.* **413** (2023), 108834.
- [Ne₁] B. Nelson, Free monotone transport without a trace, *Comm. Math. Phys.* **334** (2015), no. 3, 1245–1298.
- [Ne₂] B. Nelson, On finite free Fisher information for eigenvectors of a modular operator, *J. Funct. Anal.* **273** (2017), no. 7, 2292–2352.
- [NZ] B. Nelson and Q. Zeng, An application of free transport to mixed q -Gaussian algebras, *Proc. Amer. Math. Soc.* **144** (2016), no. 10, 4357–4366.
- [No₁] A. Nou, Non injectivity of the q -deformed von Neumann algebra, *Math. Ann.* **330** (2004), no. 1, 17–38.
- [Ric] É. Ricard, Factoriality of q -Gaussian von Neumann algebras, *Comm. Math. Phys.* **257** (2005), no. 3, 659–665.
- [Shl] D. Shlyakhtenko, Free quasi-free states, *Pacific J. Math.* **177** (2) (1997), 329–368.
- [Spe] R. Speicher, Generalized statistics of macroscopic fields, *Lett. Math. Phys.* **27** (1993), no. 2, 97–104.
- [SW] A. Skalski and S. Wang, Remarks on factoriality and q -deformations, *Proc. Amer. Math. Soc.* **146** (2018), no. 9, 3813–3823.
- [Sni] P. Śniady, Factoriality of Bożejko-Speicher von Neumann algebras, *Comm. Math. Phys.* **246** (2004), no. 3, 561–567.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-656 WARSZAWA,
POLAND

Email address: `mkumar@impan.pl`