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# Minimality in Finite-Dimensional ZW-Calculi

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**Abstract.** The ZW-calculus is a graphical language capable of representing 2-dimensional quantum systems (qubit) through its diagrams, and manipulating them through its equational theory. We extend the formalism to accommodate finite dimensional Hilbert spaces beyond qubit systems.

First we define a qudit version of the language, where all systems have the same arbitrary finite dimension  $d$ , and show that the provided equational theory is both complete – i.e. semantical equivalence is entirely captured by the equations – and minimal – i.e. none of the equations are consequences of the others. We then extend the graphical language further to accommodate all finite dimensional Hilbert spaces at the same time. We again show the completeness of the provided equational theory.

## 1 Introduction

Graphical languages for quantum computations are a product of the categorical quantum mechanics program [1, 13] devoted to studying the foundations of quantum mechanics through the prism of category theory. These graphical languages come in different flavours, depending on which generators are used to build the diagrams (graphical representations of the quantum operators), and critically, displaying different kinds of interactions between said generators. The ZX-calculus describes the interaction between two complementary bases [11], the ZW-calculus, the interaction between the two “spiders” derived from the “GHZ” and “W” states, the only two fully entangled tripartite states up to SLOCC-equivalence [12], and the ZH-calculus the interaction between the same GHZ-state inferred spider and a spider obtained by generalising the Hadamard gate [3].

The equations that describe these interactions form “equational theories”, that define syntactic equivalence classes of diagrams, that are also semantically equivalent. When the syntactic equivalence matches perfectly the semantical one (i.e. when two diagrams represent the same quantum operator iff they can be turned into one another), we say that the equational theory is *complete*. Complete equational theories have been found for the aforementioned graphical languages, betimes for restrictions of them [2, 4, 9, 10, 18, 20, 23, 24, 33].

As is customary for the computer science part of the community, the focus was largely set onto *qubit* systems, i.e. systems where the base quantum system is 2-dimensional, yet this is enough to get applications in optimisation [5, 25], quantum error correction [6, 22, 32], verification [16, 21], simulation [26–28]... However, physics allow for qudit systems (where the base quantum system is  $d$ -dimensional with  $d > 2$ ) and even infinite dimensional systems. Several attempts have hence been made to go beyond the qubit case [7, 19, 34], but it was only recently that a complete equational theory was found for  $d$ -dimensional (i.e. qudit) systems [30] and later for finite dimensional systems (so-called “qufinite”, i.e. for the category **FdHilb**) [36]. The results were obtained by generalising both the ZX and the ZW calculi and mixing them together. The W-node in particular allows for a neat intuitive (and unique) normal form for the diagrams. Satisfying the necessary conditions for every diagram to be normalisable then yields a complete equational theory. However, we believe that the ones obtained in [30, 36] are far from being *minimal*, due in particular to the presence the generators from both the ZX and the ZW calculi.

From a foundational perspective, it can be enlightening to know if an equation is a defining property of quantum systems (and hence necessary), or on the contrary if it is derivable from more fundamental properties (see e.g. [14, 15]). The redundancy in the equational theory may also cause issues when trying to transport the completeness result to other diagrammatic languages for qudit systems (every equation has to be proven in the new language, hence the fewer the better); or when trying to generalise further, e.g. to the **FdHilb** setting.

We argue here that the ZW-calculus is enough to get a natural normal form (akin to that of [30]) even in the qudit version, and provide two elegant equational theories, differing primarily in how we deal with the parameters in one of the generators. We show that the equational theories are *complete*, resorting to the normal form instead of transporting the completeness result from [30], for the reason described above. We also show that the first equational theory is minimal, meaning that none of the equations can be derived from the others, hence avoiding the aforementioned redundancy in the presentation.

We then adapt diagrams and the equational theory of the graphical language to accommodate all finite dimensional Hilbert spaces (**FdHilb**), in a way that requires no additional generator and only two new equations. Here again we prove the completeness of the equational theory, by leveraging that of the qudit setting.

The paper is split into two parts, Section 2 and Section 3, devoted respectively to the **Qudit**<sub>*d*</sub> version, and to the **FdHilb** version. In the **Qudit**<sub>*d*</sub> version, diagrams and their interpretation are introduced in Section 2.1 and the equational theory is introduced and discussed in Section 2.2. We then show its minimality in Section 2.3 and its completeness in Section 2.4. In the **FdHilb** version, diagrams and their interpretation are introduced in Section 3.1, and the equational theory is introduced and shown to be complete in Section 3.2. All missing proofs are provided in the appendix.

## The Dirac Notation

All the upcoming diagrams can be given an interpretation as a linear map, in the appropriate category. In quantum information, it is usual to express such linear maps using the so-called Dirac notation. The current section hence serves as a gentle introduction to this notation.

Let  $d \geq 2$ . In the  $d$ -dimensional Hilbert space  $\mathbb{C}^d$ , the canonical basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

is usually denoted  $\{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$ . All 1-qudit systems have states that live in  $\mathbb{C}^d$  and that can hence be represented by linear combinations of the elements of this basis:  $|\psi\rangle = a_0|0\rangle + a_1|1\rangle + \dots + a_{d-1}|d-1\rangle$  (notice that the “ket” notation  $|\cdot\rangle$  is used for states in general, not only basis elements).

To combine systems, we use the tensor product (Kronecker product):  $(\cdot \otimes \cdot)$  which is a fairly standard operation on linear maps. In particular, the overall state obtained by composing two 1-qudit systems in respective states  $|\psi\rangle$  and  $|\varphi\rangle$  is simply  $|\psi\rangle \otimes |\varphi\rangle$ . Notice that  $\{|i\rangle \otimes |j\rangle\}_{0 \leq i < d, 0 \leq j < d'}$  forms a basis of  $\mathbb{C}^d \otimes \mathbb{C}^{d'} \simeq \mathbb{C}^{d \times d'}$ . It is customary to write  $|\psi, \varphi\rangle$  to abbreviate  $|\psi\rangle \otimes |\varphi\rangle$ .

The “bra” notation  $\langle \cdot |$  is used to represent the dagger (the conjugate transpose) of a state, i.e.  $\langle \psi | = |\psi\rangle^\dagger = \overline{|\psi\rangle}^\top$ . The choice of the “bra-ket” notation is such that composing a bra with ket forms the bracket, the usual inner product in  $\mathbb{C}^d$ :

$$\langle \psi | \circ |\varphi\rangle = \langle \psi | \varphi \rangle$$

Linear combinations of kets and bras of the canonical basis can be used to represent any linear map of the correct dimensions, e.g. the 1-qudit identity:

$$id = \sum_{k=0}^{d-1} |k\rangle \langle k|$$

## 2 ZW-Calculus for Qudit Systems


In this section, we introduce a graphical language for quantum systems that all have the same fixed dimension  $d$  ( $d \geq 2$ ): qudit systems.


## 2.1 Diagrams of $ZW_d$ and their Interpretation


First, we need to introduce the mathematical objects at the heart of the graphical language – the diagrams – what they represent.

**The Diagrams** As is traditional for graphical languages for finite-dimensional quantum systems, we work with a  $\dagger$ -compact prop [29, 31, 37]. Categorically speaking, this is a symmetric, compact closed monoidal category generated by a single object, endowed with a contravariant endofunctor that behaves well with the symmetry and the compact structure. The following explains some of these concepts in more detail.

Let us denote  $ZW_d$  the  $\dagger$ -compact prop generated by:




▷ the Z-spider  :  $n \rightarrow m$ , for  $r \in \mathbb{C}$


▷ the W-node  :  $1 \rightarrow n$

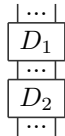


▷ the kets  :  $0 \rightarrow 1$ , for  $1 \leq k < d$

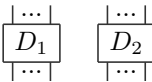


▷ the global scalars  $r : 0 \rightarrow 0$ , for  $r \in \mathbb{C}$

The symmetry and the compact structure of the prop are represented respectively by:

▷  :  $2 \rightarrow 2$  and  
 ▷  :  $0 \rightarrow 2$ ,  :  $2 \rightarrow 0$ ,

and the identity as  :  $1 \rightarrow 1$ . All these generators can be composed:

▷ sequentially:  :=   $\circ$  

▷ in parallel:  :=   $\otimes$  

The symmetry satisfies the following identities:

$$\text{crossing} = \text{parallel lines}$$

$$\text{box D with crossing} = \text{box D with cup and cap}$$

and the compact structure:

$$\text{cup and cap} = \text{cup}$$

$$\text{crossing} = \text{cup}$$

$$\text{cup and cap} = \text{vertical line} = \text{cup and cap}$$

This compact structure in particular allows us to define the "upside-down" version of the generators, for instance:

$$\text{W-node} := \text{W-node with cup and cap}$$

$$\text{ket} := \text{ket with cup}$$

The  $\dagger$  functor is defined inductively as:

$$\begin{array}{l}
\triangleright (D_2 \circ D_1)^\dagger = D_1^\dagger \circ D_2^\dagger \\
\triangleright (D_1 \otimes D_2)^\dagger = D_1^\dagger \otimes D_2^\dagger \\
\triangleright \left( \begin{array}{c} | \\ | \end{array} \right)^\dagger = \begin{array}{c} | \\ | \end{array} \\
\triangleright \left( \begin{array}{c} \diagup \\ \diagdown \end{array} \right)^\dagger = \begin{array}{c} \diagdown \\ \diagup \end{array} \\
\triangleright \left( \begin{array}{c} \cup \\ \cup \end{array} \right)^\dagger = \begin{array}{c} \cup \\ \cup \end{array} \\
\triangleright \left( \begin{array}{c} \cap \\ \cap \end{array} \right)^\dagger = \begin{array}{c} \cap \\ \cap \end{array}
\end{array}
\qquad
\begin{array}{l}
\triangleright \left( \begin{array}{c} \dots \\ n \\ \dots \\ r \\ \dots \\ m \end{array} \right)^\dagger = \begin{array}{c} \dots \\ m \\ \dots \\ r \\ \dots \\ n \end{array} \\
\triangleright \left( \begin{array}{c} \bullet \\ \dots \\ n \end{array} \right)^\dagger = \begin{array}{c} \dots \\ \bullet \end{array} \\
\triangleright \left( \begin{array}{c} \bullet \\ | \end{array} \right)^\dagger = \begin{array}{c} | \\ \bullet \end{array} \\
\triangleright (r)^\dagger = \bar{r}
\end{array}$$

Notice that thanks to the identities satisfied by the  $\dagger$ -compact prop, the  $\dagger$ -functor is involutive.

As will be made clearer in what follows, in  $\mathbf{ZW}_d$ ,  $d \geq 2$  represents the dimension of the "base" quantum system, called qudit. As this  $d$  will be fixed in the following, we may forget to specify it. For convenience, we define an empty white node as a parameter 1 Z-spider:

$$\begin{array}{c} \dots \\ \cup \\ \dots \end{array} := \begin{array}{c} \dots \\ \textcircled{1} \\ \dots \end{array}$$

and give the  $1 \rightarrow 0$  W-node a special symbol, akin to the dagger of the kets (as its interpretation, as we will see later, is merely  $\langle 0|$ ):

$$\begin{array}{c} | \\ \bullet \end{array} := \begin{array}{c} | \\ \blacktriangle \end{array}$$

**The Interpretation** The point of the diagrams of the  $\mathbf{ZW}_d$  is to represent quantum operators on multipartite  $d$ -dimensional systems. The way those are usually specified is thanks to the category  $\mathbf{Qudit}_d$ . This forms again a symmetric  $\dagger$ -compact prop, where the base object is  $1 := \mathbb{C}^d$ , and morphisms  $n \rightarrow m$  are linear maps  $\mathbb{C}^{d^n} \rightarrow \mathbb{C}^{d^m}$ . The symmetry and the compact structure correspond to their counterparts in  $\mathbf{ZW}_d$ , they will be stated out in the following, as part of the interpretation of  $\mathbf{ZW}_d$  diagrams. The  $\dagger$  functor is the usual  $\dagger$  of linear maps in  $\mathbb{C}$ .

We may hence interpret diagrams of the  $\mathbf{ZW}_d$ -calculus thanks to the functor  $\llbracket \cdot \rrbracket : \mathbf{ZW}_d \rightarrow \mathbf{Qudit}_d$  inductively defined as follows:

$$\begin{array}{l}
\llbracket D_2 \circ D_1 \rrbracket = \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket \\
\llbracket D_1 \otimes D_2 \rrbracket = \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket \\
\llbracket \begin{array}{c} | \\ | \end{array} \rrbracket = \sum_k |k\rangle\langle k| \\
\llbracket \begin{array}{c} \diagup \\ \diagdown \end{array} \rrbracket = \sum_{k,\ell} |\ell, k\rangle\langle k, \ell| \\
\llbracket \begin{array}{c} \cup \\ \cup \end{array} \rrbracket = \llbracket \begin{array}{c} \cup \\ \cup \end{array} \rrbracket^\dagger = \sum_k |k, k\rangle \\
\llbracket \begin{array}{c} \cap \\ \cap \end{array} \rrbracket = \llbracket \begin{array}{c} \cap \\ \cap \end{array} \rrbracket^\dagger = \sum_k |k, k\rangle
\end{array}
\qquad
\begin{array}{l}
\llbracket \begin{array}{c} \dots \\ r \\ \dots \\ m \end{array} \rrbracket = \sum_{k=0}^{d-1} r^k \sqrt{k!}^{n+m-2} |k^m\rangle\langle k^n| \\
\llbracket \begin{array}{c} \bullet \\ \dots \\ n \end{array} \rrbracket = \sum_{\substack{k \in \{0, \dots, d-1\} \\ i_1 + \dots + i_n = k}} \sqrt{\binom{k}{i_1, \dots, i_n}} |i_1, \dots, i_n\rangle\langle k| \\
\llbracket \begin{array}{c} \bullet \\ | \end{array} \rrbracket = \sqrt{k!} |k\rangle \\
\llbracket r \rrbracket = r
\end{array}$$

where  $\binom{k}{i_1, \dots, i_n} = \frac{k!}{i_1! \dots i_n!}$  is a multinomial coefficient. Notice that the interpretation of the  $1 \rightarrow 0$  W-node is simply:

$$\llbracket \begin{array}{c} | \\ \bullet \end{array} \rrbracket = \sum_{\substack{k \in \{0, \dots, d-1\} \\ 0=k}} \sqrt{\binom{k}{0}} |k\rangle = \langle 0|$$

The presence of  $\sqrt{\phantom{x}}$  on the coefficients is not particularly relevant, and is simply an artefact of us maintaining some symmetry between generators and their dagger. Indeed, as explained below, we have  $\llbracket \begin{array}{c} \bullet \\ \bullet \end{array} \rrbracket = k!$ , meaning that the coefficient  $k!$  needs to be split between both nodes, resulting in either an asymmetric presentation or a square root.

Notice also that the interpretation of the Z-spider differs from more usual generalisations of its qubit counterpart, because of the  $\sqrt{k^{n+m-2}}$  which depends on the degree of the spider. While it makes the interpretation of the diagrams slightly more complicated, it allows us – as will be stated later – to quite conveniently generalise equations from the qubit ZW-calculus, and hence have a simpler equational theory. It will be shown in the following (Corollary 1), that the above set of generators makes for a universal calculus, i.e. any linear map of  $\mathbf{Qudit}_d$  can be represented by a  $\mathbf{ZW}_d$ -diagram.

To gain intuition about the upcoming equations between diagrams, it can be useful to *semantically* decompose a diagram into sums of simpler ones<sup>1</sup>. To do so, it can be convenient to understand  $|k\rangle$  as a bunch of  $k$  indistinguishable particles.

The W-node spreads the  $k$  “particles” that enter it by a multinomial distribution:

$$\left[ \begin{array}{c} k \\ \text{W} \\ \dots \end{array} \right] = \sum_{i_1 + \dots + i_n = k} \binom{k}{i_1, \dots, i_n} \left[ \begin{array}{c} i_1 \dots i_n \\ \dots \end{array} \right]$$

The  $2 \rightarrow 1$  W-node takes two bunches of particles  $k$  and  $\ell$  and regroups them into one, and yields the null state if  $k + \ell$  exceeds the “capacity” (i.e. the dimension) of a single wire:

$$\left[ \begin{array}{c} k \quad \ell \\ \text{W} \\ \dots \end{array} \right] = \begin{cases} \left[ \begin{array}{c} k + \ell \\ \dots \end{array} \right] & \text{if } k + \ell < d \\ \bar{0} & \text{if } k + \ell \geq d \end{cases}$$

This will be proven graphically (Lemma 7) from the upcoming equational theory (Figure 1). When  $k + \ell < d$ , the fact that there is no additional scalar is due to the rescaling of the  $k$ -dots to represent  $\sqrt{k!} |k\rangle$ . This rescaling also makes the “copy” more natural: The Z-spider  $1 \rightarrow n$  copies any bunch of  $k$  particles entering it, yielding global scalar  $r^k$  in the process:

$$\left[ \begin{array}{c} k \\ \text{Z} \\ \dots \\ n \end{array} \right] = r^k \left[ \begin{array}{c} k \quad k \\ \dots \\ n \end{array} \right]$$

This will again be proven graphically (Lemma 8) from the equational theory.

The rescaling, however, forces:

$$\left[ \begin{array}{c} k \\ \text{b} \\ \ell \end{array} \right] = k! \quad \text{and} \quad \left[ \begin{array}{c} k \\ \text{b} \\ \ell \end{array} \right] = 0 \quad \text{if } k \neq \ell$$

Finally, it can be useful to decompose the identity as a linear combination of products of kets and bras:

$$\left[ \begin{array}{c} | \\ \dots \\ \langle \end{array} \right] = \sum_{k=0}^{d-1} \frac{1}{k!} \left[ \begin{array}{c} k \\ \dots \\ k \end{array} \right]$$

## 2.2 Equational Theory

With the above interpretation of the  $\mathbf{ZW}_d$ , different diagrams may yield the same linear map. All axioms of symmetric  $\dagger$ -compact props in particular preserve the interpretation. More generally, we may want to relate together *all* diagrams that have the same semantics. This is done through an equational theory, i.e. a set of equations that can be applied locally in a diagram without changing the semantics of the whole.

<sup>1</sup> Notice that here, such decompositions are merely semantical. The upcoming completeness is only interested in equivalence between single diagrams.

**Equations of the  $ZW_d$ -Calculus** On top of the axioms of symmetric  $\dagger$ -compact props, we assume some conventional equations about the topology of the generators, which should align with the symmetries of the symbols used to depict them. The Z-spider does not distinguish between any of its connections: it is “flexsymmetric” [8], meaning that we can interchange any of its legs without changing the semantics. Graphically, for any permutation of wires  $\sigma$ :

On the other hand, the binary  $W$ -node is only co-commutative, which, together with the upcoming Equation (a), means that all the outputs of the  $n$ -ary  $W$ -node can be exchanged, i.e. for any permutation of wire  $\sigma$ :

With all this in place, we can give the core of the equational theory, in Figure 1. When diagram  $D_1$  can be turned into diagram  $D_2$  using the rules of  $ZW_d$ , we write  $ZW_d \vdash D_1 = D_2$ .

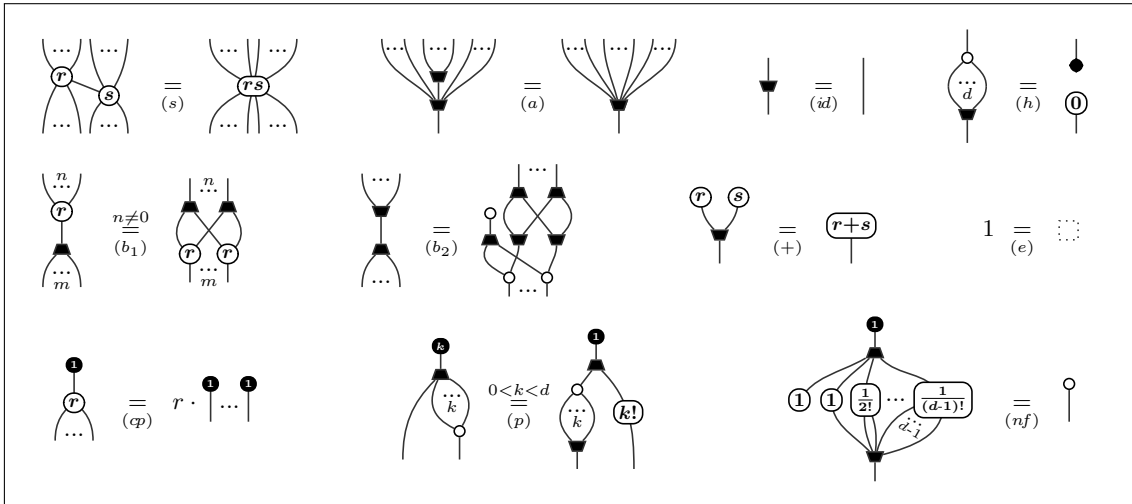


Fig. 1: Equational theory  $ZW_d$  for the qudit  $ZW$ -calculus.

*Remark 1.* In this framework, we can tensor global scalars together  $r \otimes s$ , which graphically could be confused with their product  $rs$ . This is actually unambiguous in the equational theory, as:

$$ZW_d \vdash r \otimes s \stackrel{(cp)}{=} \begin{array}{c} \bullet \\ \vdots \\ r \end{array} \begin{array}{c} \bullet \\ \vdots \\ s \end{array} \stackrel{(cp)}{=} \begin{array}{c} \bullet \\ \vdots \\ \begin{array}{c} r \\ \vdots \\ s \end{array} \end{array} \stackrel{(s)}{=} \begin{array}{c} \bullet \\ \vdots \\ rs \end{array} \stackrel{(cp)}{=} rs$$

Moreover, Equation (e) states that global scalar 1 is the empty diagram. Scalar multiplication is assumed to be automatically applied, and scalar 1 is assumed to be automatically removed in the following.

All rules up to (cp) are fairly standard generalisations of rules of the qubit  $ZW$ -calculus (with (b<sub>2</sub>) being inspired from [30] to avoid using a *fermionic swap*, as explained below). The non-conventional  $\sqrt{k}^{n+m-2}$  coefficients in the interpretation of the Z-spider seem to be necessary for

Equations (s), (b<sub>1</sub>) and (+) to all work. Equation (p) relates  $|k\rangle$  for  $k > 1$  to  $|1\rangle$ . On the left hand side of the equation, the Z-spider forces the  $k$  rightmost outputs from the W-node to carry the same state, hence applying  $|k\rangle$  on top of the W-node results in only two terms: one where the  $k$  particles all go to the left, and one where they are spread on the  $k$  inputs of the Z-spider, resulting in a mere  $|1\rangle$  on the right wire. This can be achieved by applying a  $|1\rangle$  on the input of a W-node, which results in  $|0, 1\rangle + |1, 0\rangle$  and by “multiplying” the number of particles on the left by  $k$ . The Z-spider parameter  $k!$  is here to deal with the unnormalisation of one of the two terms:

$$\left[ \begin{array}{c} \bullet \\ \vdots \\ k \\ \circ \end{array} \right] = \left[ \begin{array}{c} \bullet \\ \circ \end{array} \right] + k! \left[ \begin{array}{c} \bullet \\ \circ \end{array} \right] = \left[ \begin{array}{c} \bullet \\ \vdots \\ k \\ \circ \end{array} \right]$$

Finally, Equation (nf) simply states what the normal form (to be defined later) of the unary Z-spider should be.

**Proposition 1.** *All equations in  $ZW_d$  are sound, i.e.:*

$$ZW_d \vdash D_1 = D_2 \implies \llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$$

*Proof.* This is a straightforward verification for most of the equations. They can all be proven using the aforementioned identities in the semantics of the diagrams, especially the decomposition of the identity. Equation (b<sub>2</sub>) requires the Vandermonde identity:

$$\sum_{k_1 + \dots + k_p = m} \binom{n_1}{k_1} \dots \binom{n_p}{k_p} = \binom{n_1 + \dots + n_p}{m}.$$

**Related Work** The first and only result to date of a complete equational theory for a graphical language describing qudit systems comes from the “ZXW-calculus” [30]. There, the authors start from a qudit version of the ZX-Calculus and most probably end up requiring a W-node in the definition of a normal form, and hence in the equational theory leading to completeness. We argue here that we can get a complete equational theory, purely inside the ZW-calculus. Doing so, we end up with fewer generators, and as a consequence, fewer, more intuitive equations in the equational theory.

Focussing on ZW-calculus is not a new idea. The first ever completeness proof for qubit graphical languages was in the (qubit) ZW-calculus, introduced in [12] and tweaked and made complete in [18, 20]. The ZW-calculus noticeably has very nice combinatorial properties different from those of its counterparts, which in particular allows for a very natural notion of normal form. It is hence not surprising that some attempts were made to get a complete equational theory of qudit systems purely in ZW. There have then been tentative generalisations for qudit systems, in particular in [19] where  $q$ -arithmetic is used, and in [35] where the W-node is generalised in a different way (and that we encounter in [30]).

The system we are actually the closest with, is **QPath** [17]. Our W-node is merely the “triangle” node of **QPath** that we truncated to a finite dimension, and we generalised their “line weight” to an  $n$ -ary Z-spider. The degree-2 Z-spider furthermore has exactly the same interpretation as the line weight. While in **QPath** the triangle nodes satisfy a bialgebra, this is not the case when truncating to finite dimension. Here we could either resort to define a “fermionic swap” that would replace the usual swap in the bialgebra (as in [19] and [35]), or give a context in which the bialgebra works (as is done in [30]). While such a “fermionic swap” exists in our setting, it does not have all the nice properties of the qubit fermionic swap, that in particular allow us to see it as a quasi-proper swap. Instead we went with the latter solution, which as it turns out works in our setting, despite the W-node having a different interpretation from that of [30], and we end up with Equation (b<sub>2</sub>).

A final important distinction with [35] and [30] is that we decided that the Z-spider would have a single parameter, instead of a  $(d - 1)$ -sized list of them that would describe all but the 0-th weights in the description of the Z-spider as a linear combination of the basis states. This makes the overall presentation leaner, with the downside of requiring Equation (nf) that fully describes the

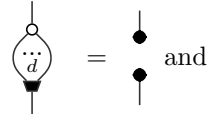


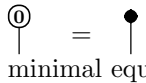
normal form of the unary Z-spider. In Appendix B, we give a presentation of the qudit ZW-calculus with lists as parameters. This simplifies Equation (nf) at the cost of more complex bookkeeping and operations on Z-spider parameters.

### 2.3 Minimality

Minimality of an equational theory states that every single equation is necessary: none can be derived from the others. Said otherwise, as soon as we remove one of the equations, some equalities (that were previously provable) become unprovable. Minimality is a fundamental property, as it allows us to pinpoint properties that are necessary to our model, and a contrario those that are consequences of the necessary ones. Notice however that there is usually not a single minimal equational theory, as it often happens that one equation can be replaced by an equivalent one.

In trying to prove minimality, it often happens that two equations fail to be proven necessary individually, but that the pair (i.e. at least one of the two) can be proven necessary. Such cases underline some sort of proximity between the two equations, and the obstacle it poses to minimality can sometimes be circumvented (somewhat artificially) by merging them into a single, potentially

slightly less intuitive, equation. This happened once here: we merged equations  and

, that we initially had as axioms, into Equation (h). This finally provides us with a minimal equational theory for qudit ZW-calculus.

To prove that an equation is necessary, we define a non-standard interpretation which is preserved by all the equations (including the axioms of †-compact props), except the equation of interest. When such an interpretation is exhibited, we can safely conclude that the equation is necessary, as if it were a consequence of the others, it would also preserve this interpretation. Interpretations like these sometimes simply take the form of a quantity that turns out to be invariant for all equations except the one that is considered.

In the following, we show that the equational theory  $ZW_d$  from Figure 1 is minimal, i.e. that none of the equations can be derived from the others. It is to be noted that most of the equations in Figure 1 are schemas, that is they are parametrised, and the equation is assumed for all possible values of the parameters. Our minimality result is “weak” in the sense that for each equation schema, we show that *at least* one of the occurrences cannot be derived from the other equations, but we do not pinpoint for which parameters the equation is necessary or not. Nevertheless:

**Theorem 1.** *All equations in  $ZW_d$  are necessary, hence  $ZW_d$  is minimal.*

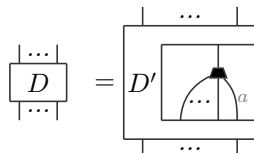
Several arguments in the proof require to distinguish between Z-spiders that have a non-trivial influence on the semantics of the diagrams, from those that essentially have no effect on the semantics. This is captured by the following definition:

**Definition 1 (Effective Z-spider).** *Let  $z$  be a Z-spider in diagram  $D$ . For  $x \in \mathbb{C}$ , we denote by  $D[z * = x]$  the diagram obtained by multiplying the parameter of  $z$  by  $x$  in  $D$ . We say that  $z$  is effective if:*

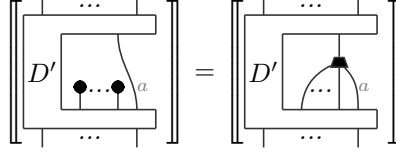
$$\forall x \in \mathbb{C} \setminus \{1\}, \forall \lambda \in \mathbb{C}, \llbracket D[z * = x] \rrbracket \neq \lambda \llbracket D \rrbracket$$

There is a similar notion that we need for determining whether a W-node could essentially just be an identity from its input to one of its outputs, the other outputs merely bearing a  $|0\rangle$ :

**Definition 2 (Trivial W-node and sole effective output).** *Let  $D$  be a diagram with a W-node. Let's call a one of its outputs:*



We say that edge  $a$  is the sole effective output of the  $W$ -node if:



If a  $W$ -node has a sole effective output, we say that it is trivial.

We can now prove Theorem 1:

*Proof.* We consider each of these equations individually:

- (s) When applying the transformation that turns all  $Z$ -spider parameters and global scalars to their real part ( $r \mapsto \text{Re}(r)$ ), Equation (s) is the only one that is not preserved.
- (a) It is the only equation permitting to create non-trivial  $W$ -nodes with arity  $> \frac{d^2}{2}$  from a diagram that only has  $W$ -nodes with arity  $\leq \frac{d^2}{2}$ .
- (id) It is the only equation that can create nodes connected to boundaries from a node-free diagram.
- (cp) It is the only equation that creates a non-1 scalar
- (e) It is the only equation that can create generators out of empty diagrams.
- (p) It is the only equation able to introduce generator  $\begin{array}{c} \bullet \\ | \end{array}$  for  $k > 1$ .
- (nf) Take the interpretation that maps  $\begin{array}{c} \bullet \\ | \end{array}$  to  $\begin{array}{c} \bullet \\ | \end{array}$ . All rules hold (up to colinearity) except (nf).
- (+) Take the interpretation that maps all the  $Z$ -spider parameters (and the global scalars) to their absolute value ( $r \mapsto |r|$ ). This interpretation preserves all equations except (+).
- (b<sub>1</sub>) Consider diagrams as graphs, and define an “effective  $Z$ -path” in the diagram as a path 1) that goes from a boundary to another boundary, and 2) that only goes through effective  $Z$ -spiders or through trivial  $W$ -nodes, through their sole effective output. All equations except (b<sub>1</sub>) preserve the existence of effective  $Z$ -paths.
- (b<sub>2</sub>) Consider diagrams as graphs, and define a “ $W$ -path” in the diagram as a path 1) that goes from a boundary to another boundary, 2) which cannot use two outputs of a  $W$ -node (if it goes through a  $W$ -node, it has to use the input edge) and 3) that does not go through a  $Z$ -spider. All equations, except (b<sub>2</sub>), preserve the existence of a  $W$ -path. (b<sub>2</sub>) is the only equation that can bring the number of  $W$ -paths from non-zero to zero (which is done by adding a  $Z$ -spider on the path).
- (h) To each wire in a diagram  $D$ , we associate a number  $0 \leq k < d$  (or more graphically we annotate each wire by some number  $k$ ). The procedure to do so is as follows:

1. annotate all wires with  $d - 1$

2. rewrite the annotations using the following rules, until a fixed point is reached:

$$\begin{array}{l}
 \bullet \quad a \begin{array}{c} \bullet \\ | \end{array} \xrightarrow{a \neq 0} 0 \begin{array}{c} \bullet \\ | \end{array} \quad \text{and} \quad a \begin{array}{c} \bullet \\ | \end{array} \xrightarrow{k \leq a} k \begin{array}{c} \bullet \\ | \end{array} \\
 \\
 \bullet \quad \begin{array}{c} a_0 \dots a_k \\ \text{---} \\ \text{---} \\ \text{---} \\ a_{k+1} \dots a_q \end{array} \rightarrow \begin{array}{c} a \dots a \\ \text{---} \\ \text{---} \\ \text{---} \\ a \dots a \end{array} \quad \text{with } a := \min(a_i) \\
 \\
 \bullet \quad \begin{array}{c} a \\ \text{---} \\ \text{---} \\ \text{---} \\ b_1 \dots b_n \end{array} \rightarrow \begin{array}{c} \min(a, \sum b_i) \\ \text{---} \\ \text{---} \\ \text{---} \\ \min(a, b_1) \dots \min(a, b_n) \end{array}
 \end{array}$$

This simple procedure obviously terminates, as a step is only applied if at least one of the annotations is decreased. By considering inputs and outputs of  $D$  (which are the only wires that can be guaranteed to remain during rewrites with  $ZW_d$ ), we can check that Equation (h) is the only one that can modify the outcome of the procedure.

**Some Useful Derivable Equalities** The equational theory being minimal, there are several useful and intuitive identities that were left out, as they can be proven to be derivable. The following lemmas will in particular be useful in the upcoming proof of completeness:



We say of any diagram in the image of  $\mathcal{N}$  that it is in normal form.

This functor  $\mathcal{N}$  creates a diagram whose interpretation is the starting state:

**Proposition 2.**  $\forall |\psi\rangle \in \mathbf{Qudit}_d[0, n], \llbracket \mathcal{N}(|\psi\rangle) \rrbracket = |\psi\rangle.$

As a simple consequence of this proposition, any qudit operator can be represented by a diagram of  $\mathbf{ZW}$ :

**Corollary 1 (Universality).**

$$\forall f \in \mathbf{Qudit}_d[n, m], \exists D_f \in \mathbf{ZW}_d[n, m], \llbracket D_f \rrbracket = f$$

As we defined the normal form as the image of a map from the semantics, any diagram can only be associated to a unique normal form.

**Compositions of Normal Forms** Our goal now is to show that any diagram can be put in normal form. To do so, we show that all generators can be put in normal form, and that all compositions of diagrams in normal form can be put in normal form.

We start by showing the latter for the tensor product:

**Proposition 3.** *The spatial composition of diagrams in normal form can be put in normal form, i.e.  $\mathbf{ZW}_d \vdash \mathcal{N}(v_1) \otimes \mathcal{N}(v_2) = \mathcal{N}(v_1 \otimes v_2).$*

When turning arbitrary operators into states, the sequential composition turns into the application of cups  $\cup$  onto pairs of outputs of the state, as:

$$\begin{array}{c} \vdots \\ \boxed{D_1} \\ \vdots \\ \boxed{D_2} \\ \vdots \end{array} = \begin{array}{c} \boxed{\lceil D_2 \rceil} \quad \boxed{\lceil D_1 \rceil} \quad \vdots \\ \cup \quad \cup \quad \cup \\ \vdots \end{array} \underset{3}{=} \begin{array}{c} \boxed{\lceil D_2 \rceil \otimes \lceil D_1 \rceil} \quad \vdots \\ \cup \quad \cup \quad \cup \\ \vdots \end{array}$$

**Proposition 4.** *The diagram obtained by applying a cup  $\cup$  to two outputs of a diagram in normal form can be put in normal form.*

**Normal Forms of the Generators** Then we move on to showing that all the generators can be put in normal form.

**Proposition 5.** *The Z-spider can be put in normal form.*

**Proposition 6.** *The diagram obtained by applying  $\nabla$  to two outputs of a normal form can be put in normal form.*

**Corollary 2.** *The W-node can be put in normal form.*

**Proposition 7.** *Generators  $\overset{\bullet}{\lceil} \rceil$  can be put in normal form.*

**Theorem 2 (Completeness for Qudit Systems).** *The language is complete: for any two diagrams  $D_1$  and  $D_2$  of the  $\mathbf{ZW}_d$ -calculus:*

$$\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \iff \mathbf{ZW}_d \vdash D_1 = D_2$$

*Proof.* By Propositions 5 and 7, and Corollary 2, any generator of the language can be put in normal form. Thanks to Propositions 3 and 4, compositions of diagrams in normal form can be put in normal form. As a consequence, any diagram can be put in normal form. By uniqueness of this normal form, if two diagrams share the same semantics, they can be rewritten into the same diagram. This proves completeness.

### 3 Finite Dimensional Hilbert Spaces

In the previous setting, all systems are required to be  $d$ -dimensional for some fixed  $d$ . Here we relax that constraint, which allows us to represent morphisms of **FdHilb**

**FdHilb** [1] is the strict symmetric monoidal  $\dagger$ -compact category of finite dimensional Hilbert spaces. Its objects are tensor products of finite dimensional Hilbert spaces  $\mathbb{C}^d (d \in \mathbb{N} \setminus \{0\})$ , and its morphisms are linear maps between them. The symmetry and the compact structure are naturally extended from that of **Qudit** $_d$ .

In this new setting, we will be able to represent *all* morphisms of **FdHilb**, at the cost of annotating the wires of the diagrams to keep track of their dimensions. Instead of the dimension itself, we rather annotate the wire with its dimension  $-1$ , i.e. with the largest  $k$  such that  $|k\rangle$  is allowed on the wire. We call such  $k$  the *capacity* of the wire. This makes the bookkeeping a little bit less tedious.

#### 3.1 Diagrams and Interpretation

We also require the following constraints for the capacities around each generator:

- ▷ All capacities around a Z-spider are the same
- ▷ The input capacity of the W-node must be larger than each of its outputs

The first constraint follows from the fact that Z-spiders in ZW can be seen as a generalisation of graph edges (more precisely they can be seen as hyperedges). Hence the whole hyperedge should have a single capacity. The second constraint simply comes from the fact that a larger capacity on the outputs of a W-node will never be used, so we might as well prevent it. When considering  $1 \rightarrow 1$  W-nodes, which represent projections, this restriction allows us to see at a glance which side has the largest dimension.

The first restriction further allows us to put the capacity annotation on the Z-spider rather than on all its legs, making annotating diagrams less cumbersome.

We now work with a  $\dagger$ -compact symmetric monoidal category, but which is not a prop anymore. Our base objects are  $\mathbb{C}^d$  for  $d \in \mathbb{N} \setminus \{0\}$ . Every pair of objects can be composed with  $\otimes$  to form a third object, with  $\otimes$  being associative, and with the tensor unit  $I$  being  $I := \mathbb{C}^1$ . We work with a strict monoidal category, so we consider  $I \otimes \mathbb{C}^d = \mathbb{C}^d = \mathbb{C}^d \otimes I$ . To simplify notations, we represent objects  $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \dots \otimes \mathbb{C}^{d_n}$  by a list of capacities  $\langle d_1 - 1, d_2 - 1, \dots, d_n - 1 \rangle$  (the tensor product simply becomes the concatenation of lists). The tensor unit is represented by  $\langle \rangle$ .

In this new setting, the generators are generalised as follows:

$$\triangleright \text{Z-spiders } \begin{array}{c} \langle \dots \rangle \\ \text{---} \\ \langle a \mid \langle r \rangle \mid \dots \rangle \\ \text{---} \\ \langle \dots \rangle \\ \text{---} \\ \langle \dots \rangle \\ \text{---} \\ \langle \dots \rangle \\ \text{---} \\ \langle \dots \rangle \\ \text{---} \\ \langle \dots \rangle \\ \text{---} \\ \langle \dots \rangle \\ \text{---} \\ \langle \dots \rangle \end{array} : \langle a, \dots, a \rangle \rightarrow \langle a, \dots, a \rangle \text{ with } r \in \mathbb{C}$$

$$\triangleright \text{W-nodes } \begin{array}{c} \langle a \rangle \\ \text{---} \\ \langle \dots \rangle \\ \text{---} \\ \langle b_1 \rangle \text{---} \dots \text{---} \langle b_n \rangle \end{array} : \langle a \rangle \rightarrow \langle b_1, \dots, b_n \rangle$$

$$\text{with } a \geq \max_{1 \leq i \leq n} (b_i)$$

$$\triangleright \text{kets } \begin{array}{c} \langle \dots \rangle \\ \text{---} \\ \langle a \mid \langle k \rangle \mid \dots \rangle \\ \text{---} \\ \langle \dots \rangle \end{array} : \langle \rangle \rightarrow \langle a \rangle \text{ with } k \leq a$$

$$\triangleright \text{global scalars } r : \langle \rangle \rightarrow \langle \rangle \text{ with } r \in \mathbb{C}$$

with the symmetry and the compact structure being generalised to

$$\triangleright \begin{array}{c} \langle a \rangle \text{---} \langle b \rangle \\ \text{---} \\ \langle b \rangle \text{---} \langle a \rangle \end{array} : \langle a, b \rangle \rightarrow \langle b, a \rangle \text{ and}$$

$$\triangleright \left( \begin{array}{c} \langle a \rangle \\ \text{---} \\ \langle a \rangle \end{array} : \langle \rangle \rightarrow \langle a, a \rangle, \begin{array}{c} \langle a \rangle \\ \text{---} \\ \langle a \rangle \end{array} : \langle a, a \rangle \rightarrow \langle \rangle \right),$$

and the identity as  $\begin{array}{c} | \\ \text{---} \\ a \end{array} : \langle a \rangle \rightarrow \langle a \rangle$ .

Diagrams can still be composed together both sequentially and in parallel. The composition of categories prevents us from composing diagrams with unmatched objects (e.g. two Z-spiders with different capacities in sequence), but the conditions on the capacities of the W-nodes still have to be checked for a diagram to be valid. Such valid diagrams are called **ZW<sub>f</sub>**-diagrams, and are graphical representations of the morphisms of the †-compact symmetric monoidal category **FdHilb**.

The compact structure still allows us to define upside-down versions of the W-node and the kets. Again, we give the  $1 \rightarrow 0$  W-node a special symbol:

$$\begin{array}{c} | \\ \bullet \\ \text{---} \\ a \end{array} := \begin{array}{c} | \\ \blacktriangle \\ \text{---} \\ a \end{array}$$

The interpretation of these diagrams is now a monoidal functor  $\llbracket \cdot \rrbracket : \mathbf{ZW}_f \rightarrow \mathbf{FdHilb}$  inductively defined as:

$$\begin{aligned} \llbracket D_2 \circ D_1 \rrbracket &= \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket & \llbracket \begin{array}{c} a \\ \text{---} \\ \text{---} \\ b \end{array} \rrbracket &= \sum_{k=0}^a \sum_{\ell=0}^b |\ell, k\rangle \langle k, \ell| \\ \llbracket D_1 \otimes D_2 \rrbracket &= \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket & \llbracket \begin{array}{c} \dots \\ \text{---} \\ \text{---} \\ \dots \\ a \\ \text{---} \\ \text{---} \\ \dots \\ m \end{array} \rrbracket &= \sum_{k=0}^a r^k \sqrt{k!}^{n+m-2} |k^m\rangle \langle k^n| \\ \llbracket \begin{array}{c} | \\ \text{---} \\ a \end{array} \rrbracket &= \sum_{k=0}^a |k\rangle \langle k| & \llbracket \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ a \end{array} \rrbracket &= \sum_{k=0}^a |k, k\rangle \\ \llbracket \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ a \end{array} \rrbracket &= \llbracket \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ a \end{array} \rrbracket^\dagger & \llbracket \begin{array}{c} \bullet \\ \text{---} \\ a \end{array} \rrbracket &= \begin{cases} \sqrt{k!} |k\rangle & \text{if } 0 < k \leq a \\ \vec{0} & \text{otherwise} \end{cases} \\ \llbracket r \rrbracket &= r & & \end{aligned}$$

$$\llbracket \begin{array}{c} a \\ \text{---} \\ \text{---} \\ \dots \\ \text{---} \\ b_1 \dots b_n \end{array} \rrbracket = \sum_{\substack{0 \leq k_i \leq b_i \\ k_1 + \dots + k_n \leq a}} \sqrt{\binom{k_1 + \dots + k_n}{k_1, \dots, k_n}} |k_1, \dots, k_n\rangle \langle k_1 + \dots + k_n|$$

Notice that we use the same notation for the interpretation of **ZW<sub>d</sub>**-diagrams, and for the interpretation of the **ZW<sub>f</sub>**-diagrams. Which interpretation we are referring to should be clear from the context.

### 3.2 Complete Equational Theory

We once again equip the language with an equational theory **ZW<sub>f</sub>**, defined in Figure 2. This equational theory only slightly differs from the one for qudit systems in Figure 1. It is interesting to notice that 1) the associativity of the W-node is broken down into two equations (a) and (a'), whose choice depends on the capacities involved, 2) the W-bialgebra equation (b<sub>2</sub>) does not need a context anymore, but instead side conditions on the capacities, 3) the equation that allows for a normal form is greatly simplified, and 4) we now have a version of the copy of a white node through a W-node (xp).

The category **Qudit<sub>d</sub>** is a full subcategory of **FdHilb**, and as such there is an obvious inclusion functor **Qudit<sub>d</sub>**  $\xrightarrow{i_d}$  **FdHilb**. This inclusion transports to the ZW-calculi: we can turn any **ZW<sub>d</sub>**-diagram into a **ZW<sub>f</sub>**-diagram through  $\iota_d$  in such a way that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{ZW}_d & \xleftarrow{\iota_d} & \mathbf{ZW}_f \\ \llbracket \cdot \rrbracket \downarrow & & \downarrow \llbracket \cdot \rrbracket \\ \mathbf{Qudit}_d & \xleftarrow{i_d} & \mathbf{FdHilb} \end{array}$$

The functor  $\iota_d$  simply takes a **ZW<sub>d</sub>**-diagram and annotates all its wires with  $d - 1$ .

We show that the present equational theory is complete. To do so, we need to adapt the notion of normal form from qudit systems (and we again use the map/state duality to focus on states rather than arbitrary morphisms):

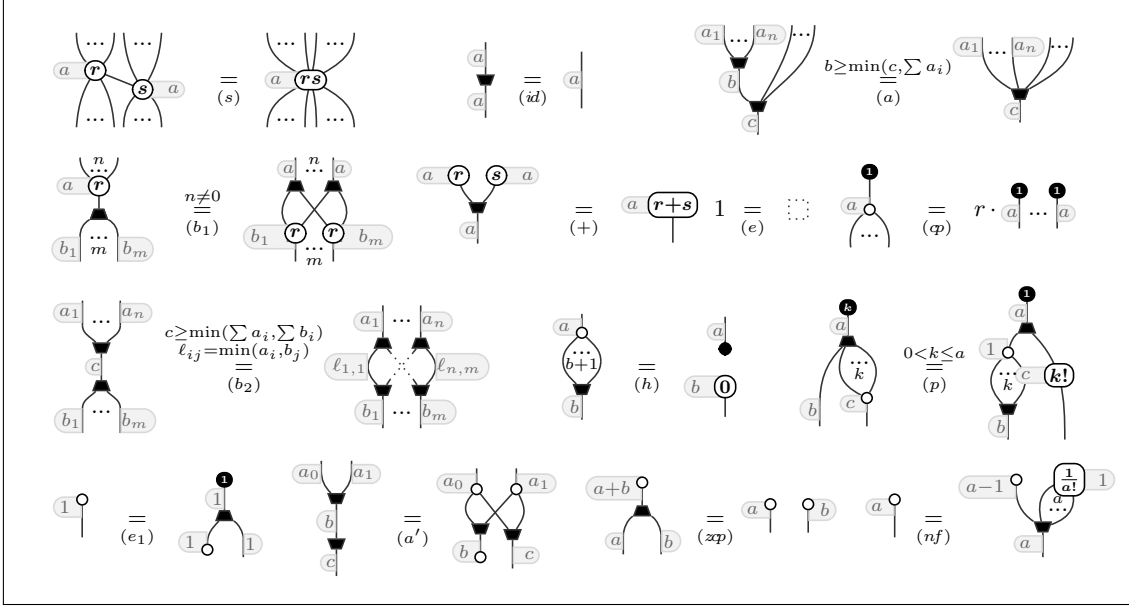
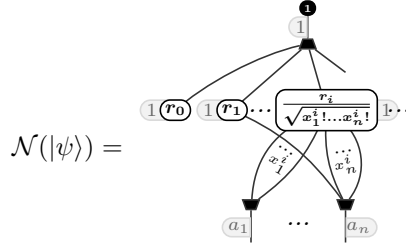


Fig. 2: Equational theory  $\mathbf{ZW}_f$  for the finite-dimensional  $\mathbf{ZW}$ -calculus.

**Definition 4.** We define  $\mathcal{N} : \mathbf{FdHilb} \rightarrow \mathbf{ZW}_f$  as the functor such that, for any  $n$ -ary state  $|\psi\rangle = r_0 |0\dots 0\rangle + r_1 |0\dots 01\rangle + \dots + r_i |x^i_1 \dots x^i_n\rangle + \dots \in \mathbf{FdHilb}[\langle \cdot \rangle, \langle a_1, \dots, a_n \rangle]$ :



We say of any diagram in the image of  $\mathcal{N}$  that it is in normal form. Notice that  $x^i_j \leq a_j$  for all  $\langle i \rangle \leq i < a_1 \dots a_n$  and  $1 \leq j \leq n$ .

We can once again show that  $\mathcal{N}$  builds a diagram that represents  $|\psi\rangle$ :

**Lemma 10.**

$$\forall |\psi\rangle \in \mathbf{FdHilb}[0, \langle a_1, \dots, a_n \rangle], \llbracket \mathcal{N}(|\psi\rangle) \rrbracket = |\psi\rangle$$

We hence get universality of the language as a direct consequence:

**Corollary 3.**

$$\forall f \in \mathbf{FdHilb}[\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle], \\ \exists D_f \in \mathbf{ZW}_f[\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle], \llbracket D_f \rrbracket = f$$

Using the normal form, we can then leverage the completeness from the qudit  $\mathbf{ZW}$ -calculus to get the similar result in the current setting:

**Theorem 3 (Completeness for Finite Dimensional Systems).** The language is complete: for any two diagrams  $D_1$  and  $D_2$  of the  $\mathbf{ZW}_f$ -calculus:

$$\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \iff \mathbf{ZW}_f \vdash D_1 = D_2$$

*Proof.* The proof, in its entirety in Appendix C, uses the completeness of  $\mathbf{ZW}_d$  for qudit systems. We once again use the map / state duality to restrict the proof to states. Then, we proceed as follows:

1. We show that all equations of  $ZW_d$  are derivable from  $ZW_f$ , i.e. that  $ZW_d \vdash D_1 = D_2 \implies ZW_f \vdash \iota_d(D_1) = \iota_d(D_2)$  for all  $d \geq 2$
2. We show that any  $ZW_f$ -diagram  $D$ , with maximum capacity  $\leq d - 1$  can be turned into

$$ZW_f \vdash \begin{array}{c} \boxed{D} \\ \hline a_1 \quad \dots \quad a_n \end{array} = \begin{array}{c} \boxed{\iota_d(D_d)} \\ \hline \begin{array}{c} d \quad \dots \quad d \\ \uparrow \quad \dots \quad \uparrow \\ a_1 \quad \dots \quad a_n \end{array} \end{array}$$

for some  $ZW_d$ -diagram  $D_d$

3. Using rules of  $ZW_d$ , we can turn  $D_d$  in (qudit) normal form
4. We show that  $\iota_d(D)$  with  $D$  in  $ZW_d$  normal form can be turned in  $ZW_f$  normal form. Consequently, any  $ZW_f$ -diagram can be put in  $ZW_f$  normal form
5. We conclude thanks to the uniqueness of the  $ZW_f$  normal form.

**Related Work** Another complete presentation of a graphical language for **FdHilb** was recently announced [36]. This one builds upon the aforementioned ZXW-calculus, and introduces a new generator that takes two systems, of dimensions  $a$  and  $b$ , and builds a system of dimension  $a \times b$ . Our approach builds upon  $ZW_d$ , the qudit version of the ZW-calculus from Section 2 and hence starts with fewer generators and equations. As a consequence, the graphical language for **FdHilb** we end up with has fewer equations as well.

Moreover, we did not require a new generator, and simply promoted the qudit W-node to work with any mix of dimensions in a natural manner, which was enough to provide us with universality. This way we deal with dimensions is less constraining, as the generator used in [36] only allows to multiply dimensions together. For instance, a (non-trivial)  $\langle 2 \rangle \rightarrow \langle 3 \rangle$  diagram would require at least going through a 6-dimensional system, while in our setting one can directly go from a 2-dimensional system to a 3-dimensional one.

## 4 Conclusion

In this paper, we explored the potential for a minimal yet complete diagrammatic language for quantum mechanics beyond qubit systems. This starts with a well-chosen generalisation of the generators of the ZW-calculus, allowing us to have few and (mostly) intuitive equations. For both qudit systems and finite dimensional systems, we showed that the diagrams are universal, and that the equational theories are complete for their respective interpretation. In the case of qudit systems, we further showed that the equational theory was minimal. As we used it to build the equational theory for finite dimensional systems, we conjecture the latter is also minimal, or very close to being so. It turns out that transporting the minimality theorem from the first setting to the second one is challenging, as some minimality arguments break when adding or modifying equations. Proving minimality in the second setting hence requires a fair amount of work, and is hence left as an open question.

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**Lemma 13.**

$$\text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & & \uparrow \\ \bullet & & \bullet \end{array} \end{array}$$

**Lemma 17.** If  $0 \leq k < d$ , we have:

$$\text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{c} \circ \\ \uparrow \\ \bullet \end{array} \end{array}$$

**Lemma 14.** If  $k \neq \ell$ :

$$\text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{c} \circ \\ \uparrow \\ \bullet \end{array} \end{array} = k! \quad \text{and} \quad \text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{c} \circ \\ \uparrow \\ \bullet \end{array} \end{array} = 0$$

**Lemma 18.** The bialgebra between  $W$ -nodes can be used in the following context, if  $0 \leq k_i < d$  for all  $i$ :

**Lemma 15.**

$$\text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} = r \cdot \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{c} \circ \\ \uparrow \\ \bullet \end{array} \end{array}$$

$$\text{ZW}_d \vdash \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} = \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array}$$

**Lemma 16.** If  $0 \leq k < d$ :

$$\text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \begin{array}{c} \circ \\ \uparrow \\ \bullet \end{array} \end{array}$$

Now, since the lemmas' ordering does not follow their dependencies any more, we provide in Figure 3 a small (automatically generated) graph of all dependencies between them, so as to convince the reader that there is no circular proof. Equations from the equational theories are not displayed in the graph, they are assumed to be available at any point.

*Proof (Proof of Lemma 1).*

$$\text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(s)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(id)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(id)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array}$$

*Proof (Proof of Lemma 2).*

$$\text{ZW}_d \vdash \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(s)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(s)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array}$$

*Proof (Proof of Lemma 3).*

$$\text{ZW}_d \vdash \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(cp)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(id)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(cp)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array} \stackrel{(id)}{=} \begin{array}{c} \dots \\ \downarrow \\ \begin{array}{ccc} \circ & & \circ \\ \uparrow & \uparrow & \uparrow \\ \bullet & & \bullet \end{array} \end{array}$$

*Proof (Proof of Lemma 4).*

$$\text{ZW}_d \vdash \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(0)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(h)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array} \stackrel{(a)}{=} \begin{array}{c} \bullet \\ \downarrow \\ \circ \end{array}$$

*Proof (Proof of Lemma 5).*

$$\text{ZW}_d \vdash \begin{array}{c} \begin{array}{cc} \circ & \circ \\ \uparrow & \uparrow \\ \bullet & \bullet \end{array} \\ \downarrow \\ \dots \end{array} \stackrel{(s)}{=} \begin{array}{c} \begin{array}{cc} \circ & \circ \\ \uparrow & \uparrow \\ \bullet & \bullet \end{array} \\ \downarrow \\ \dots \end{array} \stackrel{(+)}{=} \begin{array}{c} \begin{array}{cc} \circ & \circ \\ \uparrow & \uparrow \\ \bullet & \bullet \end{array} \\ \downarrow \\ \dots \end{array} \stackrel{(s)}{=} \begin{array}{c} \begin{array}{cc} \circ & \circ \\ \uparrow & \uparrow \\ \bullet & \bullet \end{array} \\ \downarrow \\ \dots \end{array}$$









*Proof (Proof of Proposition 2).* First notice that:

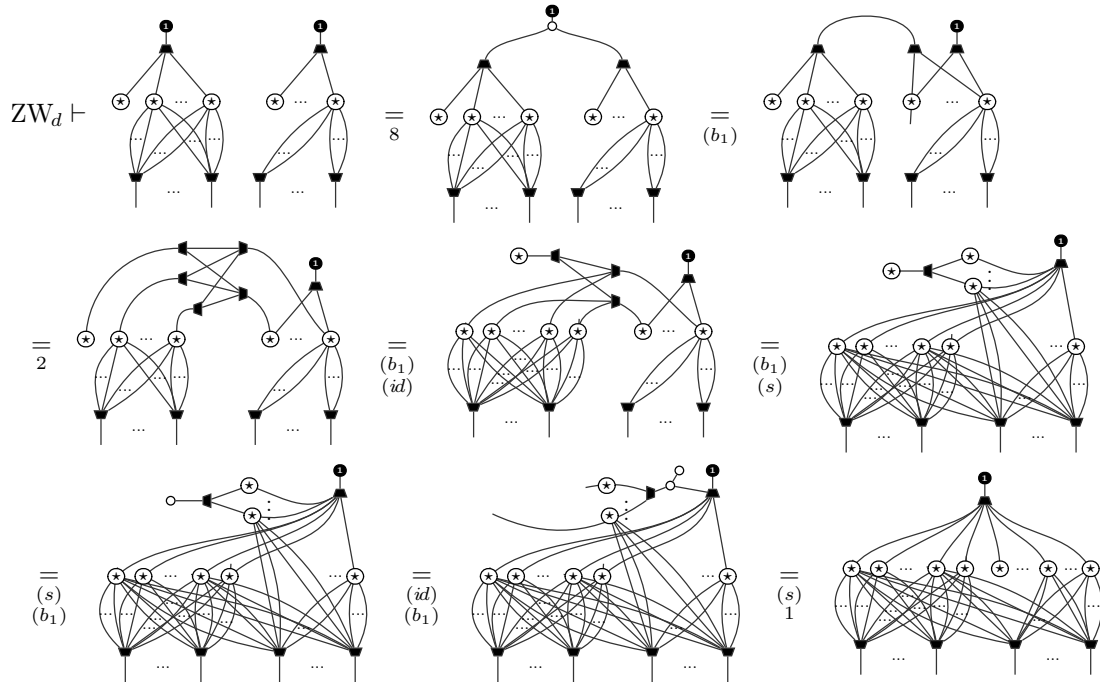
$$\begin{aligned} \left[ \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \dots \quad \dots \end{array} \right] &= |1, 0, \dots, 0\rangle + |0, 1, \dots, 0\rangle + \dots + |0, 0, \dots, 1\rangle \\ &= \left[ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] + \left[ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] + \dots + \left[ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] \end{aligned}$$

In a given term, each  $|0\rangle$  will merely cancel the Z-spider connected to it, and bring no contribution to the resulting state. Every contribution is brought by the  $|1\rangle$ . The number of parallel edges dictates to which basis state the contribution will go to. Consider for instance the  $i$ th term:

$$\begin{aligned} \left[ \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \end{array} \right] &= \left[ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \end{array} \right] = \left[ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \end{array} \right] \\ &= \frac{r_i}{\sqrt{x_1^i! \dots x_n^i!}} \left[ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \end{array} \right] = \frac{r_i}{\sqrt{x_1^i! \dots x_n^i!}} \left[ \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \end{array} \right] = r_i |x_1^i, \dots, x_n^i\rangle \end{aligned}$$

We do indeed recover the  $i$ th term in  $|\psi\rangle$ . We have something similar for all terms, so in the end  $[\mathcal{N}(|\psi\rangle)] = |\psi\rangle$ .

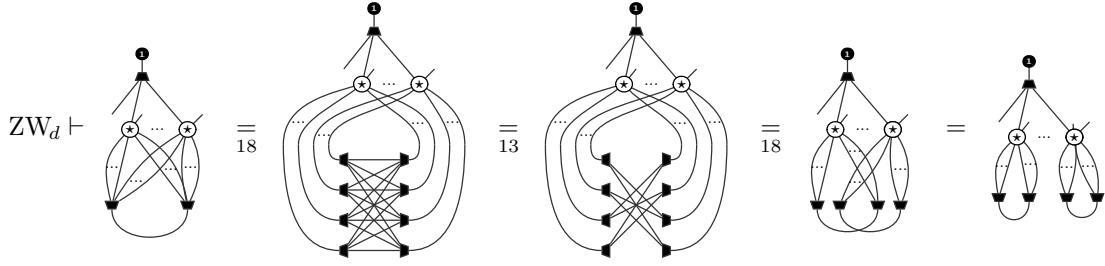
*Proof (Proof of Proposition 3).* Putting two diagrams in normal form side by side, we get:



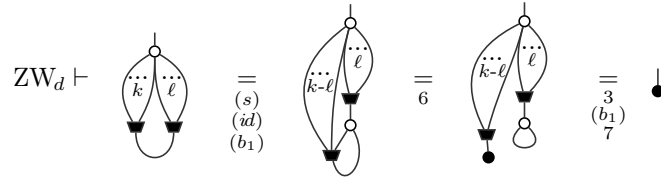
The diagram obtained at the end of this process is indeed in normal form.



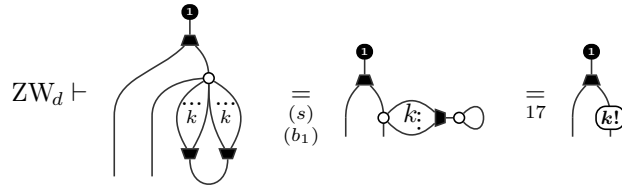
*Proof (Proof of Proposition 4).* First, we can show that the cup can “distribute” to each white node in the normal form:



We can then treat each white node independently. If a white node had  $k$  connections with the first output, and  $\ell$  with the second one, with  $k \neq \ell$ , then it can simply be removed as:

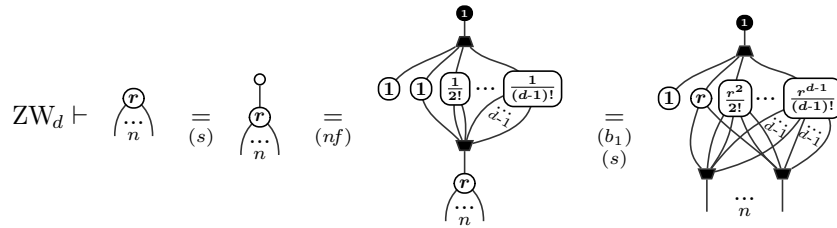


This  $\bullet$  then copies through the white node (with  $(b_1)$ ) and gets absorbed by the top and bottom W-nodes of the normal form (by *id*). If  $k = \ell$ , however the “cup gadget” disappears and the white node gets a new parameter, as:



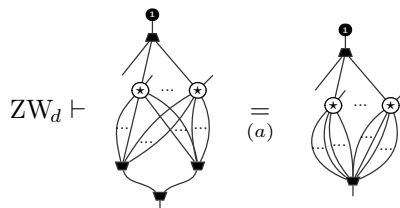
After the last two simplifications, it is possible that two white nodes end up with exactly the same connections. It is then possible to merge them (performing the sum of their parameters), using Lemma 5. After doing all these simplifications, the diagram is again in normal form.

*Proof (Proof of Proposition 5).* First, if  $n > 0$ :



If  $n = 0$ , then  $\textcircled{r}$  can be obtained by compositions:  $\textcircled{r} = \textcircled{r} \circ$ . It can hence be turned in normal form by Propositions 3 and 4.

*Proof (Proof of Proposition 6).* By simply using (a), we get:





Notice that in particular:

$$(r, r^2, \dots, r^{d-1}) \star (s, s^2, \dots, s^{d-1}) = ((r+s), (r+s)^2, \dots, (r+s)^{d-1}).$$

These two operations redefine the effects of Equations (s) and (+) on the parameters of the Z-spiders. The rest of the equational theory remains the same, except (nf) can be replaced by (e<sub>1</sub>). The obtained equational theory is given in Figure 4. Equation (e<sub>1</sub>) gives the normal form of a unary Z-spider, but with the simplest non-trivial support. It is then possible to derive Equation (nf) by decomposing arbitrary lists of parameters into convolutions of this simple one.

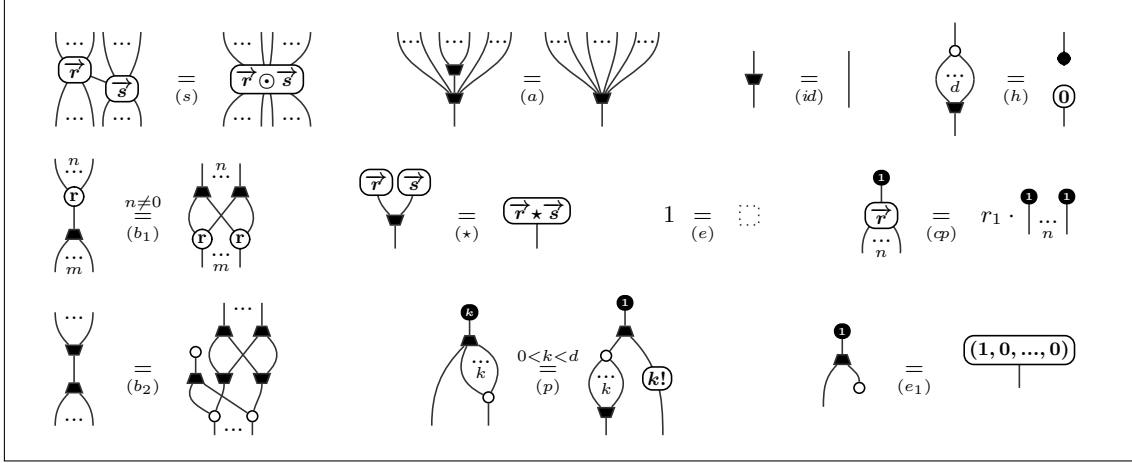


Fig. 4: Alternative equational theory  $ZW'_d$  for the qudit  $ZW$ -calculus.

To prove that the equational theory is complete, it suffices to prove that  $ZW_d$  is derivable with  $ZW'_d$ . Since all equations of  $ZW_d$  but (nf) are either exactly in  $ZW'_d$  or are special cases of equations of  $ZW'_d$ , it suffices to show that  $ZW'_d \vdash (nf)$ . Notice also that all lemmas and propositions derivable in  $ZW_d$  remain derivable in  $ZW'_d$  if they do not use (nf).

We denote the elementary lists  $\vec{e}_k := (\overbrace{0, \dots, 0}^{k-1}, 1, 0, \dots)$ . These can be shown to be decomposable as follows:

**Lemma 19.**

$$ZW'_d \vdash \begin{array}{c} \vec{e}_k \\ | \\ \circlearrowleft \end{array} = \begin{array}{c} \vec{e}_1 \quad \vec{e}_1 \quad \dots \quad \vec{e}_1 \\ | \quad | \quad \dots \quad | \\ 1 \quad e^{\frac{2i\pi}{k}} \quad \dots \quad e^{\frac{2i(k-1)\pi}{k}} \\ | \quad | \quad \dots \quad | \\ \circlearrowleft \quad \circlearrowleft \quad \dots \quad \circlearrowleft \\ | \\ \circlearrowleft \\ | \\ x_k \end{array}$$

where  $x_k = (k!)^{\frac{1}{k}}$  if  $k$  is odd and  $x_k = (k!)^{\frac{1}{k}} e^{\frac{i\pi}{k}}$  if  $k$  is even.

*Proof.* Proving this equality is more a matter of proving a non-trivial equation involving our convolution ( $\star$ ) than a diagrammatic one. The operation that is performed is essentially:

$$r_0 \vec{e}_1 \star \dots \star r_{k-1} \vec{e}_1 = \left( 1! \sum_i r_i, 2! \sum_{i_1, i_2} r_{i_1} r_{i_2}, 3! \sum_{i_1, i_2, i_3} r_{i_1} r_{i_2} r_{i_3}, \dots, k! r_0 \dots r_{k-1}, 0, \dots, 0 \right)$$

where we are looking for  $r_i$ s such that the above is equal to  $\vec{e}_k$ . Taking the  $r_i$ s to be the  $k$ th roots of unity makes all  $k-1$  first terms equal to 0, and the  $k$ th equal to  $(-1)^{k+1} k!$ . The  $(k!)^{\frac{1}{k}}$  and  $(k!)^{\frac{1}{k}} e^{\frac{i\pi}{k}}$  then simply correct for the  $(-1)^{k+1} k!$  that is left.





**Lemma 26.**

$$\text{ZW}_f \vdash \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \\ \dots \\ a \quad a \end{array} = \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \quad a \\ \dots \\ a \quad a \end{array}$$

*Proof.*

$$\begin{array}{c} \text{ZW}_f \vdash \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \\ \dots \\ a \quad a \end{array} \end{array} \stackrel{22}{=} \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \quad a \\ \dots \\ a \quad a \end{array} \stackrel{25}{=} \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \quad a \\ \dots \\ a \quad a \end{array} \stackrel{(b_2)}{=} \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \quad a \\ \dots \\ a \quad a \end{array} \\ \stackrel{(b_1) \quad (a) \quad (s)}{=} \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \quad a \\ \dots \\ a \quad a \end{array} \stackrel{(b_1) \quad (s)}{=} \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \quad a \\ \dots \\ a \quad a \end{array} \stackrel{(b_1) \quad (a)}{=} \begin{array}{c} a \quad a \\ \dots \\ \downarrow \\ a \quad a \\ \dots \\ a \quad a \end{array}$$

**Lemma 27.**

$$\text{ZW}_f \vdash \begin{array}{c} \textcircled{0} \\ | \\ a \end{array} = \begin{array}{c} \bullet \\ | \\ a \end{array}$$

*Proof.*

$$\text{ZW}_f \vdash \begin{array}{c} \textcircled{0} \\ | \\ a \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \bullet \\ | \\ \textcircled{0} \\ | \\ a \end{array} \stackrel{(h)}{=} \begin{array}{c} \bullet \\ | \\ \textcircled{0} \\ | \\ a \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \bullet \\ | \\ \textcircled{0} \\ | \\ a \end{array} \stackrel{(a)}{=} \begin{array}{c} \bullet \\ | \\ a \end{array}$$

**Lemma 28.**

$$\text{ZW}_f \vdash \begin{array}{c} \bullet \\ | \\ \textcircled{r} \\ | \\ a \end{array} = \begin{array}{c} \bullet \\ | \\ \dots \\ | \\ a \end{array}$$

*Proof.* The case where there are no outputs is dealt with

$$\text{ZW}_f \vdash \begin{array}{c} \bullet \\ | \\ \textcircled{r} \\ | \\ a \end{array} \stackrel{27}{=} \begin{array}{c} \textcircled{0} \\ | \\ \textcircled{r} \\ | \\ a \end{array} \stackrel{(s)}{=} a \textcircled{0} \stackrel{(s)}{=} \begin{array}{c} \textcircled{0} \\ | \\ \textcircled{0} \\ | \\ a \end{array} \stackrel{27}{=} \begin{array}{c} \bullet \\ | \\ a \end{array} \stackrel{(b_2)}{=} 1$$

The general case is a direct application of (b<sub>1</sub>) with m = 0.

**Lemma 29.**

$$\text{ZW}_f \vdash \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ a \end{array} = \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ a \end{array}$$

*Proof.*

$$\text{ZW}_f \vdash \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ a \end{array} \stackrel{(id) \quad 28}{=} \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ a \end{array} \stackrel{(p)}{=} \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ a \end{array} \stackrel{(id) \quad 28}{=} \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ a \end{array} \stackrel{(cp)}{=} \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ a \end{array}$$

**Lemma 30.** If  $b \geq a \geq k \geq b_i$ :

$$\text{ZW}_f \vdash \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ b \end{array} = \begin{array}{c} \bullet \\ | \\ b \end{array} \quad \text{and} \quad \text{ZW}_f \vdash \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ b_1 \quad b_n \end{array} = \begin{array}{c} \bullet \\ | \\ \textcircled{a} \\ | \\ b_1 \quad b_n \end{array}$$

*Proof.*

$$\begin{aligned}
 \text{ZW}_f \vdash \begin{array}{c} \bullet^a \\ \downarrow \\ b \end{array} & \stackrel{29}{=} \begin{array}{c} 1 \quad \dots \quad k \quad \dots \quad 1 \\ \downarrow \\ a \\ \downarrow \\ b \end{array} \stackrel{(a')}{=} \begin{array}{c} 1 \quad \dots \quad k \quad \dots \quad 1 \\ \downarrow \\ b \end{array} \stackrel{29}{=} \begin{array}{c} \bullet^k \\ \downarrow \\ b \end{array} \\
 \text{ZW}_f \vdash \begin{array}{c} \bullet^a \\ \downarrow \\ \begin{array}{c} \dots \\ b_1 \quad \dots \quad b_n \end{array} \end{array} & = \begin{array}{c} \bullet^k \\ \downarrow \\ a \\ \downarrow \\ \begin{array}{c} \dots \\ b_1 \quad \dots \quad b_n \end{array} \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \bullet^k \\ \downarrow \\ \begin{array}{c} \dots \\ b_1 \quad \dots \quad b_n \end{array} \end{array} \stackrel{(id)}{=} \begin{array}{c} \bullet^k \\ \downarrow \\ \begin{array}{c} \dots \\ b_1 \quad \dots \quad b_n \end{array} \end{array}
 \end{aligned}$$

**Lemma 31.** *Let  $a \geq b$ . Then:*

$$\text{ZW}_f \vdash \begin{array}{c} \circ^a \\ \uparrow \\ b \end{array} = \begin{array}{c} \circ \\ \downarrow \\ b \end{array}$$

*Proof.* We work by induction on  $a \geq b$ . If  $a = b$ , the result is a direct application of Equation (id). Else:

$$\begin{aligned}
 \text{ZW}_f \vdash \begin{array}{c} \circ^a \\ \uparrow \\ b \end{array} & \stackrel{(nf)}{=} \begin{array}{c} \circ^{a-1} \quad \begin{array}{c} \circ \\ \downarrow \\ \begin{array}{c} \dots \\ a \end{array} \end{array} \\ \uparrow \\ b \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \circ^{a-1} \quad \begin{array}{c} \circ \\ \downarrow \\ \begin{array}{c} \dots \\ a \end{array} \end{array} \\ \uparrow \\ b \end{array} \stackrel{ind.}{=} \begin{array}{c} \begin{array}{c} \circ \\ \downarrow \\ \begin{array}{c} \dots \\ a \end{array} \end{array} \\ \uparrow \\ b \end{array} \stackrel{(s)}{=} \begin{array}{c} \begin{array}{c} \circ \\ \downarrow \\ \begin{array}{c} \dots \\ a \end{array} \end{array} \\ \uparrow \\ b \end{array} \stackrel{(h)}{=} \begin{array}{c} \begin{array}{c} \circ \\ \downarrow \\ \begin{array}{c} \dots \\ a \end{array} \end{array} \\ \uparrow \\ b \end{array} \stackrel{(id)}{=} \begin{array}{c} \circ \\ \downarrow \\ b \end{array}
 \end{aligned}$$

**Definition 5.** *With  $d \geq a$ , we define the “ $a$ -restricted Z-spider as:*

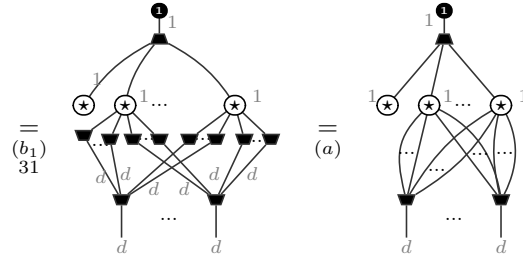
$$\begin{array}{c} \begin{array}{c} \dots \\ n \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ a \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ m \end{array} \end{array} := \begin{array}{c} \begin{array}{c} \dots \\ n \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ a \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ m \end{array} \end{array} \begin{array}{c} \circ^a \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \end{array}$$

**Lemma 32.**

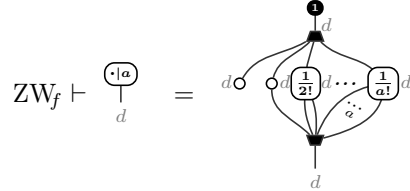
$$\text{ZW}_f \vdash \begin{array}{c} \bullet^d \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \end{array} = \begin{array}{c} \bullet^1 \\ \downarrow \\ \begin{array}{c} \dots \\ 1 \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \end{array}$$

*Proof.*

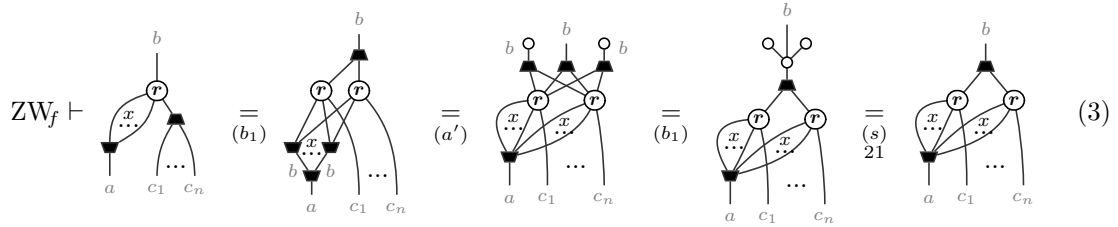
$$\begin{aligned}
 \text{ZW}_f \vdash \begin{array}{c} \bullet^d \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \end{array} & \stackrel{30}{=} \begin{array}{c} \bullet^1 \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \bullet^1 \\ \downarrow \\ \begin{array}{c} \dots \\ 1 \end{array} \\ \downarrow \\ \begin{array}{c} \dots \\ d \end{array} \end{array}
 \end{aligned}$$



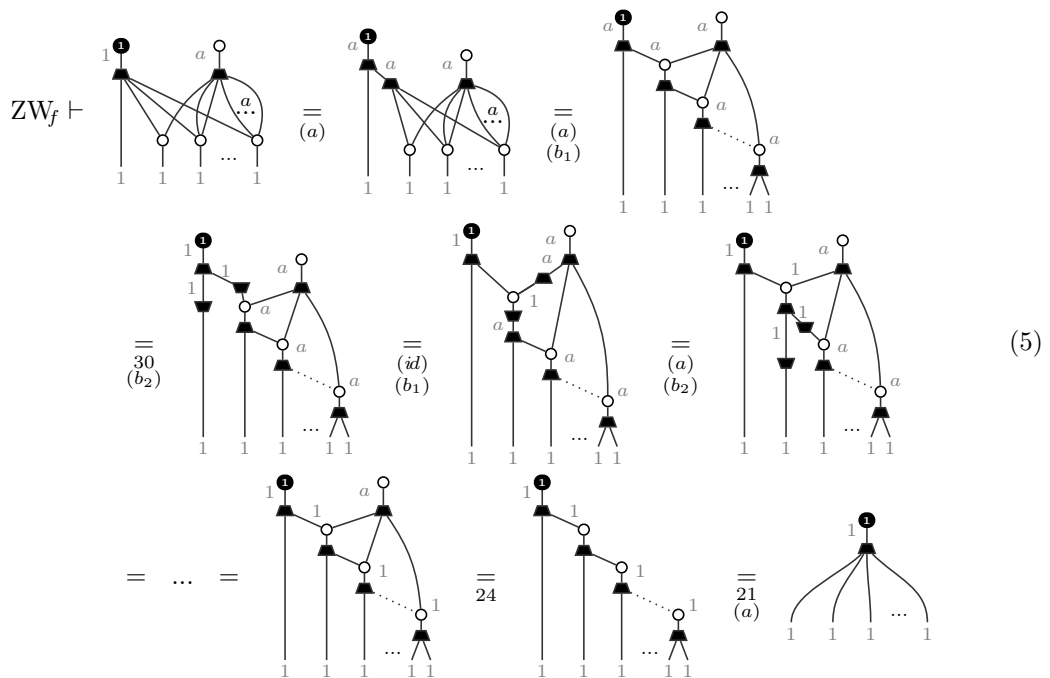
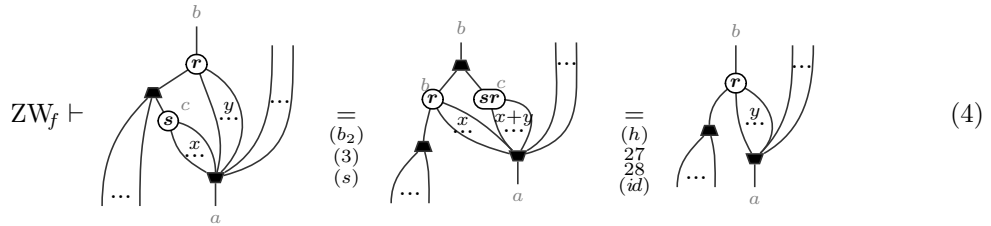
**Lemma 33.**



*Proof.* We need a few intermediary derivations first:



If  $x + y > a$ :





We then reason by induction on  $a$ . If  $a = 0$ , we have:

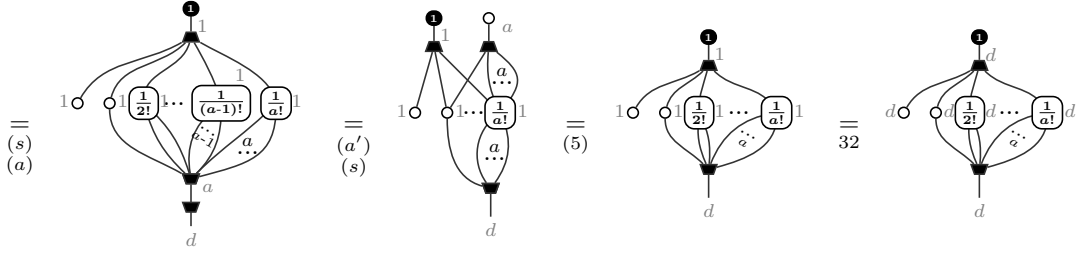
$$\begin{aligned}
 \text{ZW}_f \vdash \frac{\circlearrowleft \cdot 0}{d} &= \frac{\circlearrowleft 0}{d} \\
 &\stackrel{(id)}{=} \frac{\circlearrowleft 0}{d} \\
 &\stackrel{(h)}{=} \frac{\bullet 0}{d} \\
 &\stackrel{(28)}{=} \frac{\bullet 1}{d} \\
 &\stackrel{(q)}{=} \frac{\bullet 1}{d} \\
 &\stackrel{(id)}{=} \frac{\bullet 1}{d} \\
 &\stackrel{(a)}{=} \frac{\bullet 1}{d}
 \end{aligned}$$

The case  $a = 1$  is obtained as:

$$\begin{aligned}
 \text{ZW}_f \vdash \frac{\circlearrowleft \cdot 1}{d} &= \frac{\circlearrowleft 1}{d} \\
 &\stackrel{(\epsilon_1)}{=} \frac{\bullet 1}{d}
 \end{aligned}$$

Otherwise, if we suppose the result is proven for 1 and for  $a - 1$ , then we can prove:

$$\begin{aligned}
 \text{ZW}_f \vdash \frac{\circlearrowleft \cdot a}{d} &= \frac{\circlearrowleft a}{d} \\
 &\stackrel{(nf)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{ind.}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(s)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{ind. (5)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(30)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(q)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(b_1)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(30 (b_2))}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(3)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(4)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(s)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(s)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(21)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(s)}{=} \frac{\circlearrowleft a}{d} \\
 &\stackrel{(b_1)}{=} \frac{\circlearrowleft a}{d}
 \end{aligned}$$



**Proposition 9.** All equations of  $ZW_d$  are derivable from  $ZW_f$ , i.e.

$$\forall d \geq 2, ZW_d \vdash D_1 = D_2 \implies ZW_f \vdash \iota_d(D_1) = \iota_d(D_2)$$

*Proof.* First, notice that the result is obviously true for all axioms of compact-closed props. Then, it is enough to show the result for the equations in Figure 1, as all equations provable with  $ZW_d$  derive from them. Equations  $(s)$ ,  $(id)$ ,  $(a)$ ,  $(b_1)$ ,  $(+)$ ,  $(cp)$ ,  $(h)$  and  $(p)$  are directly translated to an equation of  $ZW_f$  through  $\iota_d$ . Then,  $(b_2)$  is exactly Lemma 26, and  $(nf)$  is Lemma 33 with  $a \leftarrow d$ .

**Proposition 10.** Let  $D \in \mathbf{ZW}_f$  be a diagram (state) with its largest capacity being  $d - 1$ . Then, there exists  $D_d \in \mathbf{ZW}_d$  such that:

$$ZW_f \vdash \boxed{D} = \boxed{\iota_d(D_d)}$$

*Proof.* Let us write  $\delta := d - 1$  for the maximum capacity in  $D$ . Using Lemma 22, we can “force” capacity  $\delta$  on all wires:

$$ZW_f \vdash a \Big| \stackrel{22}{=} \begin{array}{c} a \\ \downarrow \\ \delta \\ \downarrow \\ a \end{array}$$

Every original generator in  $D$  is then surrounded by W-nodes (connected to them by their output). These can be turned into purely qudit systems as follows:

$$ZW_f \vdash \begin{array}{c} \delta \quad \delta \\ \downarrow \quad \downarrow \\ \text{---} \text{---} \\ \downarrow \quad \downarrow \\ a \text{---} \text{---} \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(s)}{=} \begin{array}{c} \delta \quad \delta \\ \downarrow \quad \downarrow \\ \text{---} \text{---} \\ \downarrow \quad \downarrow \\ a \text{---} \text{---} \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(b_1)}{=} \begin{array}{c} \delta \\ \downarrow \\ \text{---} \text{---} \\ \downarrow \\ \delta \end{array} \stackrel{(a)}{=} \begin{array}{c} \delta \\ \downarrow \\ \text{---} \text{---} \\ \downarrow \\ \delta \end{array}$$

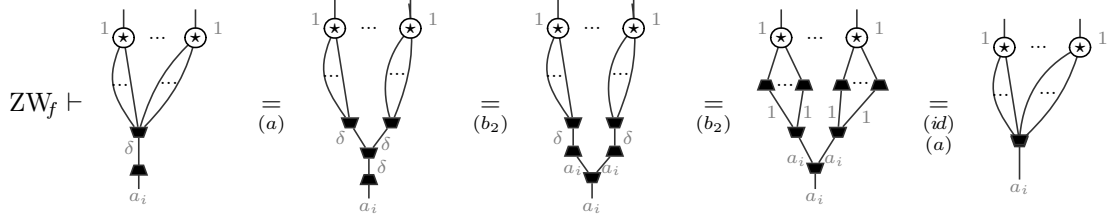
$$ZW_f \vdash \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \quad \downarrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(a)}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \quad \downarrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(b_2)}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} \stackrel{(a)}{=} \begin{array}{c} \delta \\ \downarrow \\ a \\ \downarrow \quad \downarrow \\ b \quad c \\ \downarrow \quad \downarrow \\ \delta \quad \delta \end{array} = \begin{array}{c} \delta \\ \downarrow \\ \text{---} \text{---} \\ \downarrow \\ \delta \end{array}$$

Finally, using  $(s)$  with 33, all “ $a$ -restricted” Z-spiders  $\delta \begin{pmatrix} n \\ \text{---} \\ r | a \\ \text{---} \\ m \end{pmatrix}$  can be turned into diagrams that only use capacity  $\delta$ . Doing so for all generators, we consume all the W-nodes created at the beginning of the proof through Lemma 22, except the ones that are connected to the outputs of the diagram.

We hence end up in the form  $\boxed{\iota_d(D_d)}$ .

**Proposition 11.** Let  $D_d$  be a  $\mathbf{ZW}_d$ -state in (qudit) normal form. Then  $\boxed{\iota_d(D_d)}$  can be put in (qufinite) normal form.

*Proof.* First, we may use Lemma 32 to turn all “internal” capacities in  $\iota_d(D_d)$  into 1.  $\iota_d(D_d)$  is hence technically in qufinite normal form. It remains to remove the W-nodes  $\langle \delta \rangle \rightarrow \langle a_i \rangle$  at the bottom of the diagram. This can be done as follows, considering each output individually:



Finally, we may remove Z-spiders connected  $k$  times to output  $a_i$  when  $k > a_i$  using Equation (h) followed by Lemma 28 and Equation (id).

*Proof (Proof of Theorem 3).* We can now show that any  $\mathbf{ZW}_f$ -state  $D$  can be put in normal form. First, use Proposition 10 to turn all but the outputs of the diagram into a qudit diagram. Using Proposition 9 and Theorem 2, we can turn the qudit-part of that diagram into qudit normal form. Finally, using Proposition 11, the whole diagram can be put in qufinite normal form.

If two  $\mathbf{ZW}_f$ -states  $D_1$  and  $D_2$  are semantically equivalent ( $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ ), then they can both be turned into normal form. By uniqueness of the normal form, the two diagrams are equal, i.e.  $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \implies \mathbf{ZW}_f \vdash D_1 = D_2$ .