Homogenization of a soft elastic or perfectly viscoplastic material reinforced by fibers

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Noname manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract We study the effective properties of a linear elastic or perfectly viscoplastic composite consisting of a soft matrix reinforced by possibly very stiff fibers. The effective material is characterized by the emergence of a discrepancy between the displacement in the fibers and the overall averaged displacement, giving rise to a concentration strain in the matrix. This phenomenon is attended by the appearance a combination of stretching, bending and torsional energies stored in the fibers. This work completes and corrects results previously obtained by the author et al., who failed to notice the torsional contribution.

Keywords Homogenization · Linear elasticity · Fibered structure · Anisotropy · Steady-state creep

Mathematics Subject Classification (2020) 35B27 · 74B05 · 74E10 · 74G70 · 74Q05 · 74Q15 · 74Q99

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1 Introduction

In this paper, we determine the effective behavior of a two-phase linear elastic composite made up of a very soft matrix reinforced by possibly very stiff fibers. Our approach consists in first studying asymptotically a problem of the type

$$\begin{aligned} &\inf_{\boldsymbol{u}_\varepsilon \in W_0^{1,p}(\Omega;\mathbb{R}^3)} F_\varepsilon(\boldsymbol{u}_\varepsilon) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx, \quad p\in(1,+\infty), \quad \boldsymbol{f}\in L^p(\Omega;\mathbb{R}^3) \\ &F_\varepsilon(\boldsymbol{u}_\varepsilon) = \varepsilon^p \int_{\Omega\setminus T_\varepsilon} f(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))dx + l_\varepsilon \int_{T_\varepsilon} g(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))dx, \\ &\boldsymbol{e}(\boldsymbol{u}_\varepsilon) = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon + \boldsymbol{\nabla}^T\boldsymbol{u}_\varepsilon), \end{aligned} \tag{1}$$

and subsequently particularizing the setting of linear elasticity ($p=2$, f, g quadratic). This perspective yields a simpler formulation of the limit problem and a clearer mathematical insight. Moreover, the case $p\neq 2$ has an interesting application in material sciences (see Remark 1 (v)). The functions f and g in (1) are assumed to satisfy a growth condition of order p, T_ε is an ε-periodic distribution of fibers (parallel to $\boldsymbol{e}_3$) of arbitrarily shaped cross-section of size of order ε, and $\Omega := \Omega'\times(0,L)$ a bounded Lipschitz domain of $\mathbb{R}^3$. We are aiming at deriving the limit problem associated with (1) as $\varepsilon\to 0$, for every possible choice of the stiffness parameter l_ε. All results obtained for Problem (1) are new.

Since the seminal work of Tod Arbogast, Jim Douglas Jr. and Ulrich Hornung [2], high contrast homogenization problems of this type have attracted a lot attention in a variety of physical contexts (see the many references in [19]). Problem (1) has been investigated in [7, Example 1] in the case of a connected stiff phase. The case of a vanishing volume fraction of very stiff fibers embedded in a matrix of stiffness of order 1 was addressed in [5]. A common feature between [5] and the present work lies in the study of the asymptotic behavior of the displacement in the fibers. In this regard, whenever possible, we will refer to the proofs developed in [5].

We show that the behavior of the solution $\boldsymbol{u}_\varepsilon$ to (1) is conditioned by the order of magnitude of the coefficient l_ε in terms of the parameters k and $\varkappa$ defined by

$$k := \lim_{\varepsilon\to 0} k_\varepsilon \in (0,+\infty]; \quad \varkappa := \lim_{\varepsilon\to 0} \varepsilon^p k_\varepsilon \in [0,+\infty]; \quad k_\varepsilon := l_\varepsilon |S|. \tag{2}$$

If $\varkappa > 0$, we prove that the displacement in the fibers is locally approximated to by a helicoidal vector field of resultant $\delta\boldsymbol{e}_3$ and resulting momentum $\boldsymbol{v}$ on their principal axes, $\boldsymbol{v}$ being orthogonal to these axes. This means, in simpler words, that the fibers locally undergo a displacement $\boldsymbol{v}$ of their principal axes and a rotation of angle δ around these axes. The field $\boldsymbol{v}$ approximates as $\varepsilon\to 0$ to the local average $\boldsymbol{v}_\varepsilon$ of the solution $\boldsymbol{u}_\varepsilon$ to (1) over the cross-sections of the fibers, and δ to their mean angular rotation δ_ε around $\boldsymbol{e}_3$. We establish that

$\boldsymbol{u}_\varepsilon$ and $\frac{v_{\varepsilon 3}}{\varepsilon}$ converge to $\boldsymbol{u}$ and w, respectively, where $(\boldsymbol{u}, \boldsymbol{v}, w, \delta)$ is the unique solution to

$$\begin{cases} \inf\limits_{(\boldsymbol{u},\boldsymbol{v},w,\delta)\in L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2} \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) - \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx, \\ \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) = \displaystyle\int_\Omega c^f(\boldsymbol{u}-\boldsymbol{v},0) + \varkappa g^{hom}(\boldsymbol{\tau})dx, \quad \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix}, \end{cases} \tag{3}$$

the functions c^f, g^{hom} and the domain $\mathcal{D}_2$ being defined by (9), (10) and (17), respectively. The second term of Φ_2 describes a combination of bending, extensional and torsional energies.

If $\varkappa = 0$, we find that the effective angle of rotation of the fibers is much larger, characterized by a function θ approximating to $\theta_\varepsilon = \frac{\delta_\varepsilon}{\varepsilon}$. We obtain a limit problem of the form

$$\begin{cases} \inf\limits_{(\boldsymbol{u},\boldsymbol{v},\theta)\in L^2(\Omega;\mathbb{R}^3)\times\mathcal{D}_1} \Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) - \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx, \\ \Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) = \displaystyle\int_\Omega c^f(\boldsymbol{v}-\boldsymbol{u},\theta)dx + k g^{hom}(\boldsymbol{\tau})dx, \quad \boldsymbol{\tau} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v_3}{\partial x_3} \\ \frac{\partial \theta}{\partial x_3} \end{pmatrix}, \end{cases} \tag{4}$$

where $\mathcal{D}_1$ is defined by (16). The second term of Φ_1 represents a combination of extensional and torsional energies. This contribution corrects the results stated in [7, Example 2], where the presence of torsional stored energy in the effective material was disregarded. The limit field $\boldsymbol{u}$ is a solution, unique if $\varkappa > 0$, to $\inf_{\boldsymbol{u}\in L^p} F(\boldsymbol{u}) - \int_\Omega \boldsymbol{f}.\boldsymbol{u}dx$, where $F(\boldsymbol{u}) = \inf_{(\boldsymbol{v},\theta)\in\mathcal{D}_1} \Phi_1(\boldsymbol{u},\boldsymbol{v},\theta)$ if $\varkappa = 0$ (resp. $\inf_{(\boldsymbol{v},w,\delta)\in\mathcal{D}_2} \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta)$ if $\varkappa > 0$). The functional F is the Γ-limit of the sequence (F_ε) in the weak topology of L^p (see [12]).

When $\varkappa = 0$, the solution to (1) may fail to be bounded in L^p, depending on $\boldsymbol{f}$. Loosely speaking, the composite is insufficiently reinforced to resist transversal body forces, and collapses. Its cohesion can be recovered by suitably reinforcing it with extra stiff components. The equations ruling such multiphase composites being straightforwardly deduced from those governing two-phase ones (see Remark 1 (ii)), it is convenient to present our main result in the context of two-phase composites under the assumption that the solution $\boldsymbol{u}_\varepsilon$ to (1) is bounded in $L^p(\Omega;\mathbb{R}^3)$ if $\varkappa = 0$. However, the reader should keep in mind that the relevant physical context for the case $\varkappa = 0$ lies in multiphase composites.

We next particularize the case of linear elasticity: namely, we assume that $p = 2$ and that f, g are given by

$$f(\boldsymbol{S}) = \frac{1}{2}\mathbf{A}\boldsymbol{S}:\boldsymbol{S}; \qquad g(\boldsymbol{S}) = \frac{1}{2}\mathbf{B}\boldsymbol{S}:\boldsymbol{S} \qquad \forall \boldsymbol{S}\in\mathbb{S}, \tag{5}$$

in terms of fourth order tensors $\mathbf{A}, \mathbf{B}$ satisfying

$$\begin{array}{l} \mathsf{A}_{ijkh} = \mathsf{A}_{jikh} = \mathsf{A}_{khij},\ \mathsf{B}_{ijkh} = \mathsf{B}_{jikh} = \mathsf{B}_{khij}\ \forall (i,j,k,h) \in \{1,2,3\}^4, \\ \mathbf{A}\boldsymbol{S} : \boldsymbol{S} \geq c|\boldsymbol{S}|^2, \qquad \mathbf{B}\boldsymbol{S} : \boldsymbol{S} \geq c|\boldsymbol{S}|^2 \quad \forall \boldsymbol{S} \in \mathbb{S}^3,\ (c>0). \end{array} \tag{6}$$

Problem (1) is then equivalent to

$$\begin{cases} -\operatorname{\mathbf{div}} \boldsymbol{\sigma}_\varepsilon = \boldsymbol{f} \quad \text{in} \quad \Omega, \quad \boldsymbol{\sigma}_\varepsilon = \mathbf{C}_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon), \\ \boldsymbol{e}(\boldsymbol{u}_\varepsilon) = \dfrac{1}{2}(\boldsymbol{\nabla} \boldsymbol{u}_\varepsilon + \boldsymbol{\nabla}^T \boldsymbol{u}_\varepsilon), \quad \mathbf{C}_\varepsilon = \varepsilon^2 \mathbf{A} \mathbb{1}_{\Omega \setminus T_\varepsilon} + l_\varepsilon \mathbf{B} \mathbb{1}_{T_\varepsilon}, \\ \boldsymbol{u}_\varepsilon \in H_0^1(\Omega; \mathbb{R}^3), \end{cases} \tag{7}$$

whose solution represents the displacement field at equilibrium of a clamped elastic fibered composite submitted to a density $\boldsymbol{f}$ of applied body forces.

In the case $\varkappa > 0$, Problem (7) has been studied in [7, Example 3] in the setting of linear isotropic elasticity, i. e. when $\mathbf{C}_\varepsilon \boldsymbol{S}$ takes the form $\lambda_\varepsilon (\operatorname{tr} \boldsymbol{S}) \boldsymbol{I} + 2\mu_\varepsilon \boldsymbol{S}$, and in [10] in the anisotropic case for fibers of circular cross-sections. The vibration problem associated with (7) has been investigated in [3] in the isotropic setting for every values of k, $\varkappa$. Here, we solve the case $\varkappa = 0$ in the anisotropic case, extend the case $\varkappa > 0$ to arbitrary cross-sections, express (w, δ) in terms of $\boldsymbol{v}$, and synthesize the cases $\varkappa \in \{0, +\infty\}$.

If $\varkappa = 0$, the homogenized Problem (4) obtained above takes the form

$$\inf_{(\boldsymbol{u},\boldsymbol{v},\theta) \in L^2(\Omega;\mathbb{R}^3) \times \mathcal{D}_1} \Phi_1(\boldsymbol{u}, \boldsymbol{v}, \theta) - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} dx,$$

$$\Phi_1(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \frac{1}{2} \begin{pmatrix} \boldsymbol{v} - \boldsymbol{u} \\ \theta \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} \boldsymbol{v} - \boldsymbol{u} \\ \theta \end{pmatrix} + \frac{k}{2} \begin{pmatrix} \frac{\partial v_3}{\partial x_3} \\ \frac{\partial \theta}{\partial x_3} \end{pmatrix} \cdot \boldsymbol{G}_{\{3,4\}\{3,4\}} \begin{pmatrix} \frac{\partial v_3}{\partial x_3} \\ \frac{\partial \theta}{\partial x_3} \end{pmatrix} dx.$$

where $\boldsymbol{C}$ and $\boldsymbol{G}$ are the 4×4 positive definite symmetric matrices associated to the quadratic forms $2c^f$ and $2g^{hom}$, respectively, and $\boldsymbol{G}_{\{3,4\}\{3,4\}}$ denotes the 2×2 submatrix of $\boldsymbol{G}$ deduced by removing its first and second rows and columns. This problem is equivalent to a coupled system of equations of the type

$$\begin{cases} \boldsymbol{C}_{\{1,2,3\}\{1,2,3,4\}} \begin{pmatrix} \boldsymbol{u} - \boldsymbol{v} \\ \theta \end{pmatrix} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{C}_{\{1,2\}\{1,2,3,4\}} \begin{pmatrix} \boldsymbol{v} \\ \theta \end{pmatrix} = 0 & \text{in } \Omega, \\ \boldsymbol{C}_{\{3,4\}\{1,2,3,4\}} \begin{pmatrix} \boldsymbol{v} \\ \theta \end{pmatrix} - k \boldsymbol{G}_{\{3,4\}\{3,4\}} \begin{pmatrix} \frac{\partial^2 v_3}{\partial x_3^2} \\ \frac{\partial^2 \theta}{\partial x_3^2} \end{pmatrix} = 0 & \text{in } \Omega, \\ (\boldsymbol{u}, \boldsymbol{v}, \theta) \in L^2(\Omega; \mathbb{R}^3) \times \mathcal{D}_1. \end{cases}$$

If $\varkappa > 0$, we show that the auxiliary variables w, δ can be expressed in terms of $\boldsymbol{v}$. Namely,

$$\begin{pmatrix} w \\ \delta \end{pmatrix} = \boldsymbol{E} \frac{\partial}{\partial x_3} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where $\boldsymbol{E}$ is defined by (24). Accordingly, the limit problem takes the form

$$\begin{cases} \inf\limits_{(\boldsymbol{u},\boldsymbol{v})\in\mathcal{D}} \Phi(\boldsymbol{u},\boldsymbol{v}) - \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx \\ \Phi(\boldsymbol{u},\boldsymbol{v}) = \displaystyle\int_\Omega \frac{1}{2}(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{C}_{\{1,2,3\}\{1,2,3\}}(\boldsymbol{v}-\boldsymbol{u}) + \frac{\varkappa}{2}\frac{\partial^2\boldsymbol{v}'}{\partial x_3^2}\cdot\boldsymbol{H}\frac{\partial^2\boldsymbol{v}'}{\partial x_3^2}dx, \end{cases}$$

where $\boldsymbol{v}' = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2$ and $\boldsymbol{H}$ is the symmetric positive definite 2×2 matrix given by (27). Equivalently, the couple $(\boldsymbol{u},\boldsymbol{v})$ is the unique solution to

$$\begin{cases} \boldsymbol{C}_{\{1,2,3\}\{1,2,3\}}(\boldsymbol{u}-\boldsymbol{v}) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{C}_{\{1,2\}\{1,2,3\}}(\boldsymbol{v}-\boldsymbol{u}) + \varkappa\boldsymbol{H}\dfrac{\partial^4\boldsymbol{v}'}{\partial x_3^4} = 0 & \text{in } \Omega, \\ (\boldsymbol{u},\boldsymbol{v})\in\mathcal{D}. \end{cases}$$

Our main result is presented in Section 3 and proved in Section 6. The linear elasticity problem (7) is discussed in Section 4. Section 5 is devoted to preliminary results and a priori estimates. The Euler-Lagrange equations of the limit problem are derived, in the elasticity case, in Section 7.

2 Notations

In this paper, $\{\boldsymbol{e}_1,\boldsymbol{e}_2,...,\boldsymbol{e}_N\}$ stands for the canonical basis of $\mathbb{R}^N$. Points in $\mathbb{R}^3$ or $\mathbb{Z}^3$ and real-valued functions are represented by symbols beginning with a lightface minuscule (example $x, i, \det\boldsymbol{A}$...) and vectors and vector-valued functions by symbols beginning by a boldface minuscule (examples: $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{f}$, $\mathbf{div}\boldsymbol{\sigma}_\varepsilon$,...). Matrices and matrix-valued functions are represented by symbols beginning by a boldface majuscule with the following exceptions: $\boldsymbol{\nabla u}$ (displacement gradient), $\boldsymbol{e}(\boldsymbol{u})$ (linearized strain tensor). Fourth order tensors are represented by sans-serif boldface majuscules (examples: $\mathbf{A}$, $\mathbf{B}$). We use upright symbols for operators (examples: $\mathbf{v}_\varepsilon$, θ_ε,...). We denote by u_i or $(\boldsymbol{u})_i$ the components of a vector $\boldsymbol{u}$ and by A_{ij} or $(\boldsymbol{A})_{ij}$ those of a matrix $\boldsymbol{A}$, by A_{ijkl} or $(\mathbf{A})_{ijkl}$ those of a fourth order tensor $\mathbf{A}$ (that is $\boldsymbol{u} = \sum_{i=1}^3 u_i\boldsymbol{e}_i = \sum_{i=1}^3 (\boldsymbol{u})_i\boldsymbol{e}_i$; $\boldsymbol{A} = \sum_{i,j=1}^3 A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \sum_{i,j=1}^3 (\boldsymbol{A})_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$). For any two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ in $\mathbb{R}^3$, the symmetric product $\boldsymbol{a}\odot\boldsymbol{b}$ is defined by $\boldsymbol{a}\odot\boldsymbol{b} := \frac{1}{2}(\boldsymbol{a}\otimes\boldsymbol{b} + \boldsymbol{b}\otimes\boldsymbol{a})$. We do not employ the repeated index convention for summation. We denote by $\boldsymbol{A}:\boldsymbol{B} = \sum_{i,j=1}^3 A_{ij}B_{ij}$ the inner product of two matrices, by ε_{ijk} the three-dimensional alternator, by $\boldsymbol{u}\wedge\boldsymbol{v} = \sum_{i,j,k=1}^3 \varepsilon_{ijk}u_jv_k\boldsymbol{e}_i$ the exterior product in $\mathbb{R}^3$, by $\mathbb{S}^m$ ($m\in\mathbb{N}^*$) the set of all real symmetric matrices of order m. The symbol $\boldsymbol{I}$ represents the 3×3 identity matrix. For every $m\times n$ matrix $\boldsymbol{A}$, $\alpha\subset\{1,...,m\}$, $\beta\subset\{1,...,n\}$, we denote by $\boldsymbol{A}_{\alpha,\beta}$ the submatrix of $\boldsymbol{A}$ whose entries lie in the rows of $\boldsymbol{A}$ indexed by α and in the columns of $\boldsymbol{A}$ indexed by β (example: $\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}_{\{1,2\}\{2,3\}} = \begin{pmatrix}2&3\\5&6\end{pmatrix}$) We denote by $[s]$ the integer part

of a real number s. The letter D denotes the open disc of $\mathbb{R}^2$ of center 0 and radius 1. The letter C stands for different positive constants whose precise values may vary.
For every $x \in \mathbb{R}^3$, we write $x = (x', x_3)$ for $x' = (x_1, x_2)$. For any two weakly differentiable fields $\boldsymbol{\psi} : \mathbb{R}^2 \to \mathbb{R}^3$ and $\boldsymbol{u} : \mathbb{R}^3 \to \mathbb{R}^3$, we set

$$\begin{aligned}
&\boldsymbol{e}_y(\boldsymbol{\psi}) := \sum_{\alpha,\beta=1}^{2} \frac{1}{2}\left(\frac{\partial \psi_\alpha}{\partial y_\beta} + \frac{\partial \psi_\beta}{\partial y_\alpha}\right)\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta + \sum_{\alpha=1}^{2} \frac{\partial \psi_3}{\partial y_\alpha}\boldsymbol{e}_\alpha \odot \boldsymbol{e}_3,\\
&\boldsymbol{e}_{x'}(\boldsymbol{u}) := \sum_{\alpha,\beta=1}^{2} \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta + \sum_{\alpha=1}^{2} \frac{\partial u_3}{\partial x_\alpha}\boldsymbol{e}_\alpha \odot \boldsymbol{e}_3,\\
&\boldsymbol{\psi}' := \psi_1\boldsymbol{e}_1 + \psi_2\boldsymbol{e}_2, \quad \boldsymbol{u}' := u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2.
\end{aligned} \tag{8}$$

For every $(\boldsymbol{a}, \zeta) \in \mathbb{R}^3 \times \mathbb{R}$ and every convex, p-positively homogeneous function $f : \mathbb{S}^3 \to [0, +\infty)$, we define

$$c^f(\boldsymbol{a}, \zeta) := \inf\left\{ \int_Y f\left(\boldsymbol{e}_y(\boldsymbol{\psi})\right) dy \;\middle|\; \begin{array}{l} \boldsymbol{\psi} \in W^{1,p}_\sharp(Y;\mathbb{R}^3),\ \displaystyle\int_Y \boldsymbol{\psi} dy = 0, \\ \boldsymbol{\psi}(y) = \boldsymbol{a} + \zeta \boldsymbol{e}_3 \wedge \boldsymbol{y} \ \text{ in } \ S \end{array} \right\}, \tag{9}$$

where $Y := [-1/2, 1/2[^2$ and $W^{1,p}_\sharp(Y;\mathbb{R}^3)$ denotes the set of Y-periodic members of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^3)$.
For every convex, p-positively homogeneous function $g : \mathbb{S}^3 \to [0, +\infty)$, we set

$$\begin{aligned}
&g^{hom}(\boldsymbol{\tau}) := \inf_{\boldsymbol{q} \in W^{1,p}(S;\mathbb{R}^3)} \fint_S g\left(\boldsymbol{e}_y(\boldsymbol{q}) + \boldsymbol{M}(\boldsymbol{\tau})\right) dy, \qquad \forall \boldsymbol{\tau} \in \mathbb{R}^4,\\
&\boldsymbol{M}(\boldsymbol{\tau}) := \begin{pmatrix} 0 & 0 & -y_2\frac{\tau_4}{2} \\ 0 & 0 & y_1\frac{\tau_4}{2} \\ -y_2\frac{\tau_4}{2} & y_1\frac{\tau_4}{2} & \tau_3 - \sum_{\alpha=1}^{2}\tau_\alpha y_\alpha \end{pmatrix}.
\end{aligned} \tag{10}$$

3 Main result

Let $\Omega := \Omega' \times (0, L)$ be a bounded Lipschitz cylindrical domain of $\mathbb{R}^3$, S a bounded Lipschitz domain of $\mathbb{R}^2$ satisfying

$$D \subset S, \qquad \overline{S} \subset \mathring{Y}, \qquad Y := \left[-\frac{1}{2}, \frac{1}{2}\right)^2, \qquad \fint_S \boldsymbol{y} dy = 0,$$

T_ε the ε-periodic distribution of fibers defined by (see fig. 1)

$$\begin{aligned}
&T_\varepsilon := S_\varepsilon \times (0, L) = \bigcup_{i \in I_\varepsilon} T^i_\varepsilon, \qquad T^i_\varepsilon := S^i_\varepsilon \times (0, L), \qquad S^i_\varepsilon := \varepsilon(i + S),\\
&S_\varepsilon := \bigcup_{i \in I_\varepsilon} S^i_\varepsilon, \qquad I_\varepsilon := \{i \in \mathbb{Z}^2, \quad Y^i_\varepsilon \subset \Omega'\}, \qquad Y^i_\varepsilon := \varepsilon i + \varepsilon Y,
\end{aligned} \tag{11}$$

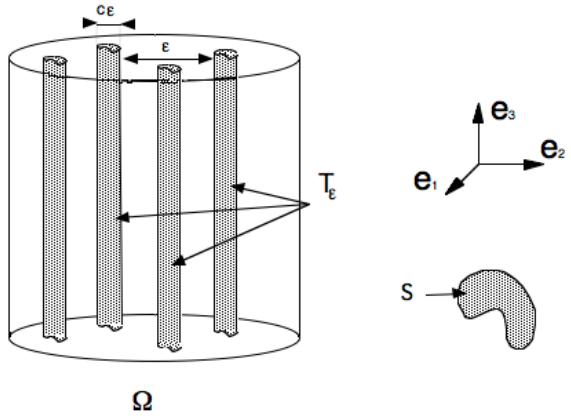

Fig. 1 The fibered structure

and $f, g : \mathbb{S}^3 \to \mathbb{R}$ two strictly convex, p-positively homogeneous functions satisfying

$$c|\boldsymbol{S}|^p \leq f(\boldsymbol{S}) \leq C|\boldsymbol{S}|^p, \qquad c|\boldsymbol{S}|^p \leq g(\boldsymbol{S}) \leq C|\boldsymbol{S}|^p \quad \forall \boldsymbol{S} \in \mathbb{S}^3, \tag{12}$$

for some $p \in (1, +\infty)$ and some positive constants c, C. Under these assumptions, we show that the asymptotic behavior as $\varepsilon \to 0$ of the solution $\boldsymbol{u}_\varepsilon$ of (1) depends on the parameters k and $\varkappa$ defined by (2). We assume that (see Remark 1 (i), (ii))

$$\boldsymbol{u}_\varepsilon \text{ is bounded in } L^p(\Omega; \mathbb{R}^3) \text{ if } \varkappa = 0, \tag{13}$$

and

$$k > 0. \tag{14}$$

Case $\varkappa = 0$. We prove that, up to a subsequence, $\boldsymbol{u}_\varepsilon$ converges to $\boldsymbol{u}$ and $(\boldsymbol{v}_\varepsilon, \theta_\varepsilon)$ given by

$$\begin{aligned}
\boldsymbol{v}_\varepsilon(x) &:= \sum_{i \in I_\varepsilon} \left(\fint_{S_\varepsilon^i} \boldsymbol{u}_\varepsilon(s, x_3) d\mathcal{H}^2(s) \right) \mathbb{1}_{Y_\varepsilon^i}(x'), \\
\theta_\varepsilon(x) &:= \sum_{i \in I_\varepsilon} \frac{1}{\fint_S |y|^2 dy} \fint_{S_\varepsilon^i} (-y_{\varepsilon 2} u_{\varepsilon 1} + y_{\varepsilon 1} u_{\varepsilon 2})(s, x_3) d\mathcal{H}^2(s) \mathbb{1}_{Y_\varepsilon^i}(x'), \\
\boldsymbol{y}_\varepsilon(x') &:= \sum_{i \in I_\varepsilon} ((x_1 - \varepsilon i_1)\boldsymbol{e}_1 + (x_2 - \varepsilon i_2)\boldsymbol{e}_2) \, \mathbb{1}_{Y_\varepsilon^i}(x'),
\end{aligned} \tag{15}$$

converges to $(\boldsymbol{v},\theta)$, where $(\boldsymbol{u},\boldsymbol{v},\theta)$ is a solution to

$$
\begin{aligned}
&(\mathcal{P}_1^{hom}): \inf_{(\boldsymbol{u},\boldsymbol{v},\theta)\in L^2(\Omega;\mathbb{R}^3)\times\mathcal{D}_1} \Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx,\\
&\Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) = \int_\Omega c^f(\boldsymbol{v}-\boldsymbol{u},\theta)dx + kg^{hom}(\boldsymbol{\tau})dx, \quad \boldsymbol{\tau} = \begin{pmatrix} 0\\ 0\\ \frac{\partial v_3}{\partial x_3}\\ \frac{\partial\theta}{\partial x_3}\end{pmatrix},\\
&\mathcal{D}_1 := \left\{(\boldsymbol{v},\theta)\in L^p(\Omega;\mathbb{R}^4) \;\middle|\; \begin{array}{l} v_3,\theta\in L^p(\Omega';W_0^{1,p}(0,L)),\\ v_3=\theta=0 \;\text{ if }\; k=+\infty\end{array}\right\}.
\end{aligned}
\tag{16}
$$

Case $\varkappa > 0$. We show that the convergences of the case $\varkappa = 0$ hold for the entire sequences. Besides, the sequence $\left(\frac{1}{\varepsilon}v_{\varepsilon 3}, \frac{1}{\varepsilon}\theta_\varepsilon\right)$ converges to (w,δ), where $(\boldsymbol{u},\boldsymbol{v},w,\delta)$ is the unique solution to

$$
\begin{aligned}
&\inf_{(\boldsymbol{u},\boldsymbol{v},w,\delta)\in L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2} \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx,\\
&\Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) = \int_\Omega c^f(\boldsymbol{v}-\boldsymbol{u},0) + \varkappa g^{hom}(\boldsymbol{\tau})dx, \quad \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2}\\ \frac{\partial^2 v_2}{\partial x_3^2}\\ \frac{\partial w}{\partial x_3}\\ \frac{\partial\delta}{\partial x_3}\end{pmatrix},\\
&\mathcal{D}_2 := \left\{(\boldsymbol{v},w,\delta)\in L^p(\Omega;\mathbb{R}^5) \;\middle|\; \begin{array}{l} v_\alpha\in L^p(\Omega';W_0^{2,p}(0,L)),\forall\alpha\in\{1,2\},\\ v_3=0,\\ (w,\delta)\in L^p(\Omega';W_0^{1,p}(0,L))^2\end{array}\right\}.
\end{aligned}
\tag{17}
$$

Theorem 1 *Let $\boldsymbol{u}_\varepsilon$ be the solution to (1). Assume (2), (11), (13) and (14). Then, up to a subsequence, $(\boldsymbol{u}_\varepsilon)$ weakly converges in $L^p(\Omega;\mathbb{R}^3)$ to $\boldsymbol{u}$, $(\boldsymbol{v}_\varepsilon,\theta_\varepsilon)$ defined by (15) weakly converges in $L^p(\Omega;\mathbb{R}^4)$ to $(\boldsymbol{v},\theta)$, and, if $\varkappa>0$, $(\frac{1}{\varepsilon}v_{\varepsilon 3}, \frac{1}{\varepsilon}\theta_\varepsilon)$ weakly converges in $(L^p(\Omega))^2$ to (w,δ), where*

(i) if $\varkappa = 0$, $(\boldsymbol{u},\boldsymbol{v},\theta)$ is a solution to (16),

(ii) if $\varkappa>0$, $v_3=\theta=0$ and $(\boldsymbol{u},\boldsymbol{v},w,\delta)$ is the unique solution to (17).

Moreover, c^f and g^{hom} defined by (9) and (10) satisfy

$$
\begin{aligned}
&c(|\boldsymbol{a}|^p+|\zeta|^p)\le c^f(\boldsymbol{a},\zeta)\le C(|\boldsymbol{a}|^p+|\zeta|^p) && \forall(\boldsymbol{a},\zeta)\in\mathbb{R}^4,\\
&c|\boldsymbol{\tau}|^p\le g^{hom}(\boldsymbol{\tau})\le C|\boldsymbol{\tau}|^p && \forall\boldsymbol{\tau}\in\mathbb{R}^4,
\end{aligned}
\tag{18}
$$

for some $C\ge c>0$.

Remark 1 (i) In the case $\varkappa=0$, by (16) we have

$$
\Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) = \Phi_1\left(\boldsymbol{u}+\boldsymbol{\psi}',\boldsymbol{v}-\boldsymbol{\psi}',\theta\right) \qquad \forall\boldsymbol{\psi}\in L^p(\Omega;\mathbb{R}^3),
$$

hence, if $\boldsymbol{f}'\neq 0$, the infimum (16) is equal to $-\infty$ and the minimization problem (16) has no solution. Consequently, by Theorem 1, the assertion (13)

fails to hold. Moreover, $\lim_{\varepsilon\to 0}|\boldsymbol{u}_\varepsilon|_{L^p}=+\infty$: otherwise (13) would hold for a subsequence and, by Theorem 1, Problem (16) would have a solution. On the other hand, if $\boldsymbol{f}'=0$, Problem (16) has infinitely many solutions, but even then, we only can prove that $|u_{\varepsilon 3}|_{L^p}\leq C$. Indeed, when $\varkappa=0$, F_ε is not coercive in L^p, more precisely does not satisfy $\int_\Omega|\boldsymbol{w}|^p dx\leq CF_\varepsilon(\boldsymbol{w})$ for some $C>0$. Coercivity of F_ε and boundedness of $\boldsymbol{u}_\varepsilon$ are achieved when considering multiphase composites reinforced, besides the fibers, by suitable extra stiff components such as fibers distributed in two other independent directions, fibers satisfying $\varkappa>0$, stiff plates, a connected stiff phase, or any combination of these... (see [7, Examples 1 and 2], [4, Remarks 3.17, 3.18], [3, Section 5]). Accordingly, the main application of the case $\varkappa=0$ of Theorem 1 lies in multiphase composites, the effective behavior of which being easily deduced, as commented below, from Theorem 1 (i).

(ii) *Multiphase media.* When a multiphase medium comprising several families of periodically distributed stiff components is considered, the effective energy is deduced by adding up the effective energies stored in each family of stiff component (second term of Φ_α) and taking into account, in the definition of c^f, the effective displacement in each stiff component. For instance, when two ε-periodic disconnected families $T_\varepsilon^{(1)}$, $T_\varepsilon^{(2)}$ of fibers parallel to $\boldsymbol{e}_3$ embedded in a soft matrix and satisfying $\varkappa^{(1)}=0$, $\varkappa^{(2)}>0$ and $⨍_{S^{(1)}}\boldsymbol{y}dy=0$ are considered, the boundedness of $\boldsymbol{u}_\varepsilon$ in L^p is ensured and the effective problem takes the form

$$\inf_{(\boldsymbol{u},\boldsymbol{\varrho})\in L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_1^{(1)}\times\mathcal{D}_2^{(2)}}\Phi(\boldsymbol{u},\boldsymbol{\varrho})-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}dx,$$

$$\Phi(\boldsymbol{u},\boldsymbol{\varrho})=\int_\Omega c^f(\boldsymbol{u},\boldsymbol{\varrho})+k^{(1)}g^{hom,(1)}(\boldsymbol{\tau}^{(1)})+\varkappa^{(2)}g^{hom,(2)}(\boldsymbol{\tau}^{(2)})dx,$$

$$\boldsymbol{\tau}^{(1)}=\begin{pmatrix}0\\0\\\frac{\partial v_3^{(1)}}{\partial x_3}\\\frac{\partial\theta^{(1)}}{\partial x_3}\end{pmatrix},\quad\boldsymbol{\tau}^{(2)}=\begin{pmatrix}\frac{\partial^2 v_1^{(2)}}{\partial x_3^2}\\\frac{\partial^2 v_2^{(2)}}{\partial x_3^2}\\\frac{\partial w^{(2)}}{\partial x_3}\\\frac{\partial\delta^{(2)}}{\partial x_3}\end{pmatrix},\quad\boldsymbol{\varrho}=((\boldsymbol{v}^{(1)},\theta^{(1)}),(\boldsymbol{v}^{(2)},w^{(2)},\delta^{(2)})),$$

where

$$c^f(\boldsymbol{u},\boldsymbol{\varrho})=\inf\left\{\int_Y f\left(\boldsymbol{e}_y(\boldsymbol{\psi})\right)dy\;\middle|\;\begin{array}{l}\boldsymbol{\psi}\in W^{1,p}_\sharp(Y;\mathbb{R}^3),\ \int_Y\boldsymbol{\psi}dy=\boldsymbol{u},\\\boldsymbol{\psi}(y)=\boldsymbol{v}^{(1)}+\theta^{(1)}\boldsymbol{e}_3\wedge\boldsymbol{y}\ \text{ in }\ S^{(1)},\\\boldsymbol{\psi}(y)=\boldsymbol{v}^{(2)}\ \text{ in }\ S^{(2)}\end{array}\right\}.$$

(iii) *Boundary conditions.* When mixed boundary conditions are considered, the analysis and the limit problem are the same, up to slight differences. However, Dirichlet boundary conditions should be enforced on a subset of $\partial\Omega$ containing one of the extremity of the fibers to ensure the weak relative compactness of the solution to (1) if $\varkappa>0$ (resp. of its third component if $\varkappa=0$).

(iv) *Metal Matrix Composites.* When $p \neq 2$, Problem (1) is relevant to perfectly viscoplastic materials such as metals at high temperatures. In this context, the solution $\boldsymbol{u}$ to (1) describes the velocity field in a hot metal matrix fibered composite with high viscosity contrast undergoing a steady-state creep. Steady-state creep denominates the quasi-static motion experienced by metals, ceramics or alloys at high temperatures under the influence of applied body forces. During this process, the stress tensor $\boldsymbol{\sigma}$ and the rate of strain $\boldsymbol{e}(\boldsymbol{u})$ are linked by a phenomenological constitutive relation of the type $\boldsymbol{\sigma} = l\frac{\partial g}{\partial \boldsymbol{e}}(\boldsymbol{e}(\boldsymbol{u}))$ discovered by Norton [18], where g is a convex function with p-growth at infinity and l a viscosity parameter. Solids governed by this law, called the Norton-Hoff model of viscoplasticity, are said to be perfectly viscoplastic. The inverse relation, which takes the form $\boldsymbol{e}(\boldsymbol{u}) = h(\boldsymbol{\sigma})$ for some function h satisfying a growth condition of order $q = \frac{1}{p-1}$, is called a q-power law. In general, q is about 5 ($p \sim 1,2$). In a steady state creep, the acceleration is negligible and the linear momentum equation $\mathbf{div}\boldsymbol{\sigma} + \boldsymbol{f} = 0$ is equivalent to a minimization problem of the type $\inf_{\mathcal{D}} \left\{\int_\Omega lg(\boldsymbol{e}(\boldsymbol{u})) - \boldsymbol{f}\cdot\boldsymbol{u}dx\right\}$ analogous to (1). Metal matrix composites have numerous industrial applications. References on these subjects can be found in the Materials Science and Engineering literature [11, 13, 15].

4 Linear elasticity case

In this section, we particularize Theorem 1 when $p = 2$ and f, g are given by (5). Then, c^f and g^{hom} are quadratic form on $\mathbb{R}^4$ associated with 4×4 symmetric matrices $\frac{1}{2}\boldsymbol{C}$ and $\frac{1}{2}\boldsymbol{G}$ which, by (18), are definite positive. Denoting by $\boldsymbol{q}(\boldsymbol{\tau})$ the unique solution in $H^1(S;\mathbb{R}^3)/\mathbb{R}^3$ to the minimization problem (5,10), the components of $\boldsymbol{G}$ are given by (see Section 7.3)

$$G_{ij} = -\int_S \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_i)) : \boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j))dy + \int_S \mathbf{B}\boldsymbol{M}(\boldsymbol{e}_i) : \boldsymbol{M}(\boldsymbol{e}_j)dy. \tag{19}$$

Denoting by $\boldsymbol{\psi}(\boldsymbol{a},\zeta)$ the unique solution in $H^1_\sharp(Y;\mathbb{R}^3)$ to the minimization problem (5,9), and setting $(\boldsymbol{a},\zeta) = \sum_{i=1}^3 a_i\boldsymbol{e}_i + \zeta\boldsymbol{e}_4$, the components of $\boldsymbol{C}$ are given by

$$C_{ij} = \int_Y \mathbf{A}\boldsymbol{e}_y(\boldsymbol{\psi}(\boldsymbol{e}_i)) : \boldsymbol{e}_y(\boldsymbol{\psi}(\boldsymbol{e}_j))dy.$$

Case $\varkappa = 0$

The limit problem (16) takes the form

$$\begin{aligned}
&\inf_{(\boldsymbol{u},\boldsymbol{v},\theta)\in L^2(\Omega;\mathbb{R}^3)\times\mathcal{D}_1} \Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx,\\
&\Phi_1(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \frac{1}{2}\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \theta\end{pmatrix}\cdot\boldsymbol{C}\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \theta\end{pmatrix} + \frac{k}{2}\begin{pmatrix}\frac{\partial v_3}{\partial x_3}\\ \frac{\partial\theta}{\partial x_3}\end{pmatrix}\cdot\boldsymbol{G}_{\{3,4\}\{3,4\}}\begin{pmatrix}\frac{\partial v_3}{\partial x_3}\\ \frac{\partial\theta}{\partial x_3}\end{pmatrix}dx,
\end{aligned} \tag{20}$$

equivalent to the system of linear elliptic equations

$$\begin{cases} \boldsymbol{C}_{\{1,2,3\}\{1,2,3,4\}} \begin{pmatrix} \boldsymbol{u}-\boldsymbol{v} \\ \theta \end{pmatrix} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{C}_{\{1,2\}\{1,2,3,4\}} \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ \theta \end{pmatrix} = 0 & \text{in } \Omega, \\ \boldsymbol{C}_{\{3,4\}\{1,2,3,4\}} \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ \theta \end{pmatrix} - k\boldsymbol{G}_{\{3,4\}\{3,4\}} \begin{pmatrix} \frac{\partial^2 v_3}{\partial x_3^2} \\ \frac{\partial^2 \theta}{\partial x_3^2} \end{pmatrix} = 0 & \text{in } \Omega, \\ (\boldsymbol{u},\boldsymbol{v},\theta) \in L^2(\Omega;\mathbb{R}^3) \times \mathcal{D}_1. \end{cases} \tag{21}$$

Case $\varkappa > 0$

The limit problem (17) takes the form

$$\begin{cases} \inf\limits_{(\boldsymbol{u},\boldsymbol{v},w,\delta)\in L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2} \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) - \displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx, \\ \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) = \displaystyle\int_\Omega \frac{1}{2} \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ 0 \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ 0 \end{pmatrix} + \frac{\varkappa}{2} \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix} \cdot \boldsymbol{G} \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix} dx, \end{cases} \tag{22}$$

equivalent to

$$\begin{cases} \boldsymbol{C}_{\{1,2,3\}\{1,2,3\}}(\boldsymbol{u}-\boldsymbol{v}) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{C}_{\{1,2\}\{1,2,3\}}(\boldsymbol{v}-\boldsymbol{u}) + \varkappa \left(\boldsymbol{G}_{\{1,2\}\{1,2\}} \dfrac{\partial^4 \boldsymbol{v}'}{\partial x_3^4} + \boldsymbol{G}_{\{1,2\}\{3,4\}} \dfrac{\partial^3}{\partial x_3^3} \begin{pmatrix} w \\ \delta \end{pmatrix} \right) = 0 & \text{in } \Omega, \\ \boldsymbol{G}_{\{3,4\}\{1,2\}} \dfrac{\partial^3 \boldsymbol{v}'}{\partial x_3^3} + \boldsymbol{G}_{\{3,4\}\{3,4\}} \dfrac{\partial^2}{\partial x_3^2} \begin{pmatrix} w \\ \delta \end{pmatrix} = 0 & \text{in } \Omega, \\ (\boldsymbol{u},\boldsymbol{v},w,\delta) \in L^2(\Omega;\mathbb{R}^3) \times \mathcal{D}_2. \end{cases} \tag{23}$$

The variables w, δ can be eliminated: we find that

$$\begin{pmatrix} w \\ \delta \end{pmatrix} = \boldsymbol{E} \frac{\partial \boldsymbol{v}'}{\partial x_3}, \qquad \boldsymbol{E} = -\boldsymbol{G}^{-1}_{\{3,4\}\{3,4\}} \boldsymbol{G}_{\{3,4\}\{1,2\}}. \tag{24}$$

We deduce that $(\boldsymbol{u},\boldsymbol{v})$ is the unique solution to

$$\begin{aligned} &\inf_{(\boldsymbol{u},\boldsymbol{v})\in\mathcal{D}} \Phi(\boldsymbol{u},\boldsymbol{v}) - \int_\Omega \boldsymbol{f}.\boldsymbol{u}dx \\ &\Phi(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \frac{1}{2}(\boldsymbol{v}-\boldsymbol{u}) \cdot \boldsymbol{C}_{\{1,2,3\}\{1,2,3\}}(\boldsymbol{v}-\boldsymbol{u}) + \frac{\varkappa}{2} \frac{\partial^2 \boldsymbol{v}'}{\partial x_3^2} \cdot \boldsymbol{H} \frac{\partial^2 \boldsymbol{v}'}{\partial x_3^2} dx, \\ &\mathcal{D} = \left\{ (\boldsymbol{u},\boldsymbol{v}) \in L^2(\Omega;\mathbb{R}^3) \times L^2(\Omega';H_0^2(0,L;\mathbb{R}^3)),\ v_3 = 0 \right\}, \end{aligned} \tag{25}$$

or equivalently to

$$\begin{cases} \boldsymbol{C}_{\{1,2,3\}\{1,2,3\}}(\boldsymbol{u}-\boldsymbol{v}) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{C}_{\{1,2\}\{1,2,3\}}(\boldsymbol{v}-\boldsymbol{u}) + \varkappa \boldsymbol{H}\dfrac{\partial^4 \boldsymbol{v}'}{\partial x_3^4} = 0 & \text{in } \Omega, \\ (\boldsymbol{u},\boldsymbol{v}) \in \mathcal{D}, \end{cases} \tag{26}$$

being $\boldsymbol{H}$ the symmetric positive definite 2×2 matrix defined by

$$\boldsymbol{H} = \boldsymbol{G}_{\{1,2\}\{1,2\}} + \boldsymbol{G}_{\{1,2\}\{3,4\}}\boldsymbol{E}. \tag{27}$$

Corollary 1 *Let $\boldsymbol{u}_\varepsilon$ be the solution to (7). Assume $p=2$, (2), (13) and (14). Then the convergences stated in Theorem 1 hold, where*

(i) if $\varkappa = 0$, $(\boldsymbol{u},\boldsymbol{v},\theta)$ is a solution to the problem (20) equivalent to (21),

(ii) if $\varkappa > 0$, $(\boldsymbol{u},\boldsymbol{v},w,\delta)$ is the unique solution to the problem (22) equivalent to (23). Equivalently, (w,δ) is given by (24) and $(\boldsymbol{u},\boldsymbol{v})$ is the unique solution to the problem (25) equivalent to (26).

Remark 2 When the material constituting the fibers is isotropic, the matrix $\boldsymbol{G}$ is diagonal hence $\boldsymbol{G}_{\{3,4\}\{1,2\}} = 0$ so that, by (24), $w=\delta=0$. It is not diagonal, in general, otherwise (see [6, Remark 1] for explicit examples).

5 Preliminary results and a priori estimates

5.1 Two-scale convergence

A sequence $(\varphi_\varepsilon)\subset L^p(\Omega)$ two-scale converges to $\varphi_0\in L^p(\Omega\times Y)$ if (see [1,16, 17])

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon(x)\cdot\eta_0\left(x,\frac{y_\varepsilon(x')}{\varepsilon}\right)dx = \int_{\Omega\times Y}\varphi_0(x,y)\cdot\eta_0(x,y)dxdy, \quad \forall\,\eta_0\in C(\overline{\Omega}\times S) \quad (\text{notation: } \varphi_\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \varphi_0). \tag{28}$$

Any bounded sequence in $L^p(\Omega)$ admits a two-scale convergent subsequence. A sequence (η_ε) in $L^{p'}(\Omega)$ satisfying

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon\cdot\eta_\varepsilon dxdt = \int_{\Omega\times Y}\varphi_0\cdot\eta_0 dxdy \tag{29}$$

for some $\eta_0\in L^{p'}(\Omega\times Y)$ and for every two-scale convergent sequence (φ_ε) is said to be admissible. Sequences of the form $\eta_\varepsilon=\eta_0\left(x,\frac{y_\varepsilon(x')}{\varepsilon}\right)$ with $\eta_0\in L^\infty(\Omega,C_\sharp(Y;\mathbb{R}^3))\cup L^\infty_\sharp(Y,C(\overline{\Omega};\mathbb{R}^3))$ are admissible [1] and satisfy, for every sequence (φ_ε) in $L^p(\Omega)$ [3, Section 5]

$$\varphi_\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \varphi_0 \quad\Longrightarrow\quad \varphi_\varepsilon\eta_\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \varphi_0\eta_0.$$

In particular,

$$\varphi_\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \varphi_0 \qquad \Longrightarrow \qquad \begin{cases} \varphi_\varepsilon \mathbb{1}_{T_\varepsilon} \rightharpoonup\!\!\!\!\rightharpoonup \varphi_0 \mathbb{1}_S \\ \varphi_\varepsilon \mathbb{1}_{\Omega\setminus T_\varepsilon} \rightharpoonup\!\!\!\!\rightharpoonup \varphi_0 \mathbb{1}_{Y\setminus S}. \end{cases} \tag{30}$$

If $(\boldsymbol{\varphi}_\varepsilon) \subset L^p(\Omega;\mathbb{R}^k)$ two-scale converges to $\boldsymbol{\varphi}_0 \in L^p\left(\Omega\times Y;\mathbb{R}^k\right)$, then for every convex function $j:\mathbb{R}^k\to\mathbb{R}$, we have [5, Lemma 4.1]

$$\liminf_{\varepsilon\to 0}\int_\Omega j(\boldsymbol{\varphi}_\varepsilon)dx \geq \int_{\Omega\times Y} j(\boldsymbol{\varphi}_0)dxdy \geq \int_\Omega j\left(\int_Y \boldsymbol{\varphi}_0(x,y)dy\right)dx. \tag{31}$$

If $(\boldsymbol{\varphi}_\varepsilon)\subset W^{1,p}(\Omega;\mathbb{R}^3)$ is such that

$$\boldsymbol{\varphi}_\varepsilon \rightharpoonup \boldsymbol{\varphi} \text{ weakly in } L^p(\Omega;\mathbb{R}^3), \qquad \boldsymbol{\varphi}_\varepsilon \rightharpoonup\!\!\!\!\rightharpoonup \boldsymbol{\varphi}_0, \qquad \varepsilon \boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon) \rightharpoonup\!\!\!\!\rightharpoonup \boldsymbol{\Xi}_0,$$

then [7, Section 3]

$$\begin{aligned} &\int_Y \boldsymbol{\varphi}_0(.,y)dy = \boldsymbol{\varphi}, \qquad \boldsymbol{\varphi}_0 \in L^p(\Omega;W^{1,p}_\sharp(Y;\mathbb{R}^3)), \\ &\boldsymbol{\Xi}_0 = \boldsymbol{e}_y(\boldsymbol{\varphi}_0), \quad \varepsilon\boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\mathbb{1}_{\Omega\setminus T_\varepsilon} \rightharpoonup\!\!\!\!\rightharpoonup \boldsymbol{e}_y(\boldsymbol{\varphi}_0)\mathbb{1}_{Y\setminus S}. \end{aligned} \tag{32}$$

5.2 Existence and uniqueness of the solution to the limit problem (17) ($\varkappa > 0$).

In the following lemma, we establish the well posed character of the infimum problems defining c^f and g^{hom} and of the limit problem associated with (1) when $\varkappa > 0$.

Lemma 1 *(i) The infimum problems (9), (10) are achieved. The solution to (9) is unique. Problem (10) has a unique solution satisfying* $\mathbf{r}(\boldsymbol{q}) = 0$ *(see (34)).*
(ii) The mappings c^f *and* g^{hom} *satisfy the growth conditions 18.*
(iii) Problem (17) ($\varkappa > 0$*) has a unique solution.*

Proof (i) *Problem (9).* By [14, Theorem 4 p. 75], for any bounded Lipschitz domain U of $\mathbb{R}^2$ and any subspace V of $W^{1,p}(U;\mathbb{R}^2)$ such that $V\cap\mathcal{R} = \{0\}$, where $\mathcal{R}$ is the space of infinitesimal rigid displacements of $\mathbb{R}^2$ (that is the kernel of $\boldsymbol{\varphi}\to\boldsymbol{e}(\boldsymbol{\varphi})$), the following Korn inequality holds

$$\int_U |\boldsymbol{\varphi}|^p dx + \int_U |\boldsymbol{\nabla}\boldsymbol{\varphi}|^p dx \leq C\int_U |\boldsymbol{e}(\boldsymbol{\varphi})|^p dx \quad \forall \boldsymbol{\varphi}\in V. \tag{33}$$

Applying this to $V = \left\{\boldsymbol{\varphi}\in W^{1,p}_\sharp(Y;\mathbb{R}^2),\ \int_Y \boldsymbol{\varphi}dy = 0\right\}$, we infer

$$\int_Y |\boldsymbol{\varphi}|^p + |\boldsymbol{\nabla}\boldsymbol{\varphi}|^p dy \leq C\int_Y |\boldsymbol{e}(\boldsymbol{\varphi})|^p dy \quad \forall \boldsymbol{\varphi}\in W^{1,p}_\sharp(Y;\mathbb{R}^2),\ \int_Y \boldsymbol{\varphi}dy = 0.$$

By the Poincaré-Wirtinger inequality in $W^{1,p}(Y)$, we have

$$\int_Y |\eta|^p\, dy \le C \int_Y |\nabla \eta|^p dy, \quad \forall \eta \in W^{1,p}(Y),\ \int_Y \eta dy = 0.$$

We deduced from the above inequalities, applied to $\boldsymbol{\varphi} = \boldsymbol{\psi}'$ and $\eta = \psi_3$, that

$$\int_Y |\boldsymbol{\psi}|^p + |\nabla \boldsymbol{\psi}|^p dy \le C \int_Y |\boldsymbol{e}_y(\boldsymbol{\psi})|^p dy \qquad \forall \boldsymbol{\psi} \in W^{1,p}_\sharp(Y;\mathbb{R}^3),\ \int_Y \boldsymbol{\psi} dy = 0.$$

This, along with (12), implies that any minimizing sequence $(\boldsymbol{\psi}_n)$ of (9) is bounded in $W^{1,p}_\sharp(Y;\mathbb{R}^3)$ and weakly converges, up to a subsequence, to some $\boldsymbol{\psi}_\infty$ which, by the weak lower semicontinuity of $\boldsymbol{\psi} \to \int_Y f\left(\boldsymbol{e}_y(\boldsymbol{\psi})\right) dy$ and the weak closedness of the set on which the infimum is taken, is a solution to (9). By the strict convexity of f and the convexity of the set on which the infimum is taken, this solution is unique.

Problem (10). We set

$$\begin{aligned} \mathbf{r}(\boldsymbol{q}) &= \mathbf{r}'(\boldsymbol{q}') + \left(\fint_S q_3 dy \right) \boldsymbol{e}_3, \\ \mathbf{r}'(\boldsymbol{q}')(y) &= \fint_S \boldsymbol{q}' dy + \left(\frac{1}{\fint_S |\boldsymbol{y}|^2 dy} \fint_S \left(-y_2 q_1(y) + y_1 q_2(y)\right) dy \right) \boldsymbol{e}_3 \wedge \boldsymbol{y}. \end{aligned} \tag{34}$$

One can check that

$$\left\{ \boldsymbol{q}' - \mathbf{r}'(\boldsymbol{q}'),\ \boldsymbol{q}' \in W^{1,p}(S;\mathbb{R}^2) \right\} \cap \mathcal{R} = \{0\} \ \text{ and } \ \boldsymbol{e}(\mathbf{r}'(\boldsymbol{q}')) = 0,$$

hence, by (33),

$$\int_S |\boldsymbol{q}' - \mathbf{r}'(\boldsymbol{q}')|^p + |\nabla(\boldsymbol{q}' - \mathbf{r}'(\boldsymbol{q}'))|^p dy \le C \int_S |\boldsymbol{e}(\boldsymbol{q}')|^p dy, \quad \forall \boldsymbol{q}' \in W^{1,p}(S;\mathbb{R}^2).$$

Using the Poincaré-Wirtinger inequality $\int_S \left| q_3 - \fint_S q_3 dy \right| dy \le C \int_S |\nabla q_3|^p dy$ in $W^{1,p}(S)$, we deduce

$$\int_S |\boldsymbol{q} - \mathbf{r}(\boldsymbol{q})|^p + |\nabla(\boldsymbol{q} - \mathbf{r}(\boldsymbol{q}))|^p dy \le C \int_S |\boldsymbol{e}_y(\boldsymbol{q})|^p dy \qquad \forall \boldsymbol{q} \in W^{1,p}(S;\mathbb{R}^3).$$

Noticing that $\boldsymbol{e}_y(\boldsymbol{q} - \mathbf{r}(\boldsymbol{q})) = \boldsymbol{e}_y(\boldsymbol{q})$, we can choose a minimizing sequence $(\boldsymbol{q}_n)$ of Problem (10) such that $\mathbf{r}(\boldsymbol{q}_n) = 0$. The above inequality, along with (12), implies that $(\boldsymbol{q}_n)$ is bounded in $W^{1,p}(S;\mathbb{R}^3)$ and weakly converges, up to a subsequence, to some $\boldsymbol{q}_\infty$ which, by the weak lower semicontinuity of $\boldsymbol{q} \to \fint_S g\left(\boldsymbol{e}_y(\boldsymbol{q}) + \boldsymbol{M}(\boldsymbol{\tau})\right) dy$, is a solution to (10). The uniqueness of the solution to (10) in the set $\{\boldsymbol{q} \in W^{1,p}(S;\mathbb{R}^3),\ \mathbf{r}(\boldsymbol{q}) = 0\}$ results from the strict convexity of g.

(ii) The upper bounds result from (12). To prove the lower bounds, since c^f and g^{hom} are positively homogeneous of degree p, we only have to check that $c^f(\boldsymbol{a}, \zeta) > 0$ for every $(\boldsymbol{a}, \zeta) \neq 0$ and $g^{hom}(\boldsymbol{\tau}) > 0$ for every $\boldsymbol{\tau} \neq 0$: -

If $c^f(\boldsymbol{a},\zeta)=0$, the minimizer $\boldsymbol{\psi}$ of (9) satisfies $\boldsymbol{e}_y(\boldsymbol{\psi})=0$ in Y, thus $\boldsymbol{\psi}(y)=\boldsymbol{c}+\eta\boldsymbol{e}_3\wedge\boldsymbol{y}$ in Y for some $(\boldsymbol{c},\eta)\in\mathbb{R}^4$. As $\boldsymbol{\psi}(y)=\boldsymbol{a}+\zeta\boldsymbol{e}_3\wedge\boldsymbol{y}$ in S, $(\boldsymbol{c},\eta)=(\boldsymbol{a},\zeta)$, hence $\int_Y\boldsymbol{\psi}dy=\boldsymbol{a}=0$ and the periodicity of $\boldsymbol{\psi}$ implies $\zeta=0$. - If $g^{hom}(\boldsymbol{\tau})=0$ and $\boldsymbol{q}$ is the minimizer of (10), then $\boldsymbol{e}_y(\boldsymbol{q})+\boldsymbol{M}(\boldsymbol{\tau})=0$ a.e. in S. In particular $\frac{1}{2}\frac{\partial q_3}{\partial y_1}-y_2\frac{\tau_4}{2}=\frac{1}{2}\frac{\partial q_3}{\partial y_2}+y_1\frac{\tau_4}{2}=\tau_3-\sum_{\alpha=1}^2\tau_\alpha y_\alpha=0$ a.e. $\boldsymbol{y}\in S$. The latter equality implies $\tau_1=\tau_2=\tau_3=0$ ant the former, by the Schwarz theorem, yields $\tau_4=0$.

(iii) Given $(\boldsymbol{u},\boldsymbol{v},w,\delta)\in L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2$, we notice that, by Poincaré inequalities in $W_0^{1,p}(0,L)$ and $W_0^{2,p}(0,L)$, the norm of $(\boldsymbol{v},w,\delta)$ in $\mathcal{D}_2$, understood as its norm in $L^p(\Omega';W_0^{2,p}(0,L;\mathbb{R}^3))\times L^p(\Omega';W_0^{2,p}(0,L))^2$, is equivalent to the L^p norm of $\boldsymbol{\tau}$ defined by (17) which, by the second line of (18) and (17), is bounded from above by $C\Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta)$. Noticing that by the first line of (18), $\int_\Omega|\boldsymbol{u}-\boldsymbol{v}|^p dx\le\int_\Omega c^f(\boldsymbol{u}-\boldsymbol{v},0)dx\le C\Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta)$, we deduce that

$$|(\boldsymbol{u},\boldsymbol{v},w,\delta)|^p_{L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2}\le C\Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta),\quad\forall(\boldsymbol{u},\boldsymbol{v},w,\delta)\in L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2.$$

Hence, any minimizing sequence $(\boldsymbol{u}_n,\boldsymbol{v}_n,w_n,\delta_n)$ of (17) is bounded in $L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2$ (identifying $\mathcal{D}_2$ with $L^p(\Omega';W_0^{2,p}(0,L;\mathbb{R}^2))\times L^p(\Omega';W_0^{2,p}(0,L)^2)$ and weakly converges, up to a subsequence, to some $(\boldsymbol{u},\boldsymbol{v},w,\delta)$. The functional Φ_2 is convex and bounded on the unit ball of the Banach space $L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_2$, hence strongly continuous, thus weak lower semicontinuous, therefore $(\boldsymbol{u},\boldsymbol{v},w,\delta)$ is a solution to (17). By the strict convexity of Φ_2, resulting from that of c^f and g^{hom}, this solution is unique.

5.3 Auxiliary results for the upper bound

We state below a variant of [9, Theorem 1] proved in [8, Lemma 4.3]:

Lemma 2 *Let $\mathcal{H}\subset L^1(\Omega)$ such that for every finite family $(g_i)_{i\in I}$ of elements of $\mathcal{H}$ and every family $(\alpha_i)_{i\in I}$ of non-negative functions of $C^1(\overline{\Omega})$ satisfying $\sum_{i\in I}\alpha_i=1$, there exists $g\in\mathcal{H}$ such that $g\le\sum_{i\in I}\alpha_i g_i$. Then*

$$\inf_{g\in\mathcal{H}}\int_\Omega g(x)dx=\int_\Omega ess\inf_{g\in\mathcal{H}}g(x)dx. \tag{35}$$

The following two corollaries take a crucial part in the proof of the upperbound inequalities (Section 6.2).

Corollary 2 *For every $(\boldsymbol{\phi},\zeta)\in\mathcal{D}(\Omega;\mathbb{R}^3)\times\mathcal{D}(\Omega)$, the following holds (see (9))*

$$\begin{aligned}\int_\Omega c^f(\boldsymbol{\phi},\zeta)dx&=\inf\left\{\int_{\Omega\times(Y\setminus S)}f(\boldsymbol{e}_y(\boldsymbol{\eta}(x,y)))dxdy,\quad\boldsymbol{\eta}\in\mathcal{D}_{\boldsymbol{\phi},\zeta}\right\},\\ \mathcal{D}_{\boldsymbol{\phi},\zeta}&=\left\{\boldsymbol{\eta}\in C^1_c(\Omega;C^\infty_\sharp(Y;\mathbb{R}^3))\;\middle|\;\begin{array}{l}\displaystyle\int_Y\boldsymbol{\eta}(x,.)dy=0,\\ \boldsymbol{\eta}=\boldsymbol{\phi}+\zeta\boldsymbol{e}_3\wedge\boldsymbol{y}\quad in\quad\Omega\times S\end{array}\right\}.\end{aligned} \tag{36}$$

Proof We consider the following subset of $L^1(\Omega)$

$$\mathcal{H} := \left\{ g(x) = \int_{Y\setminus S} f(\boldsymbol{e}_y(\boldsymbol{\eta}(x,y)))dy,\ \boldsymbol{\eta} \in \mathcal{D}_{\boldsymbol{\phi},\zeta} \right\}.$$

Let $(g_i)_{i\in I}$ and $(\alpha_i)_{i\in I}$ be like in Lemma 2, and $(\boldsymbol{\eta}_i)_{i\in I} \subset \mathcal{D}_{\boldsymbol{\phi},\zeta}$ such that $g_i(x) = \int_Y f(\boldsymbol{e}_y(\boldsymbol{\eta}_i(x,y)))dy$ for every $i \in I$. Then the function

$$g(x) = \int_{Y\setminus S} f\left(\boldsymbol{e}_y\left(\sum_{i\in I} \alpha_i(x)\boldsymbol{\eta}_i(x,y)\right)\right) dy$$

belongs to $\mathcal{H}$ and, by the convexity of f, satisfies $g \le \sum_{i\in I} \alpha_i g_i$. We deduce from Lemma 2 that

$$\begin{aligned}
&\inf\left\{ \int_{\Omega\times(Y\setminus S)} f(\boldsymbol{e}_y(\boldsymbol{\eta}(x,y)))dxdy, \quad \boldsymbol{\eta} \in \mathcal{D}_{\boldsymbol{\phi},\zeta} \right\} \\
&= \inf_{g\in\mathcal{H}} \int_\Omega g(x)dx = \int_\Omega \operatorname{ess}\inf_{g\in\mathcal{H}} g(x)dx \\
&= \int_\Omega \operatorname{ess\,inf} \left\{ \int_{Y\setminus S} f(\boldsymbol{e}_y(\boldsymbol{\eta}(x,y)))dy, \left| \begin{array}{l} \boldsymbol{\eta} \in C^1_c(\Omega; C^\infty_\sharp(Y;\mathbb{R}^3)),\ \displaystyle\int_Y \boldsymbol{\eta}(x,.)dy = 0, \\ \boldsymbol{\eta} = \boldsymbol{\phi}(x) + \zeta(x)\boldsymbol{e}_3 \wedge \boldsymbol{y} \ \text{ in } \ \Omega\times S \end{array} \right. \right\} dx \\
&= \int_\Omega \inf \left\{ \int_{Y\setminus S} f(\boldsymbol{e}_y(\boldsymbol{\eta}(y)))dy, \left| \begin{array}{l} \boldsymbol{\eta} \in C^\infty_\sharp(Y;\mathbb{R}^3), \quad \displaystyle\int_Y \boldsymbol{\eta}dy = 0, \\ \boldsymbol{\eta}(y) = \boldsymbol{\phi}(x) + \zeta(x)\boldsymbol{e}_3 \wedge \boldsymbol{y} \ \text{ in } \ \Omega\times S \end{array} \right. \right\} dx \\
&= \int_\Omega c^f(\boldsymbol{\phi}(x), \zeta(x))dx,
\end{aligned}$$

where the last equality follows from the strong continuity on $W^{1,p}_\sharp(Y;\mathbb{R}^3)$ of $\boldsymbol{\varphi} \to \int_Y f(\boldsymbol{e}_y(\boldsymbol{\varphi}))dy$ and a density argument. The assertion (36) is proved. □

In the same manner, we have:

Corollary 3 *For every $\boldsymbol{\tau} \in \mathcal{D}(\Omega;\mathbb{R}^4)$, there holds (see (10))*

$$\int_\Omega g^{hom}(\boldsymbol{\tau})dx = \inf_{\boldsymbol{q}\in C^1_c(\Omega;C^\infty(\overline{S};\mathbb{R}^3))} \int_\Omega \fint_S g\left(\boldsymbol{e}_y(\boldsymbol{q}) + \boldsymbol{M}(\boldsymbol{\tau})\right) dxdy.$$

5.4 Apriori estimates, convergences and identification relations

In the following proposition, we analyze the asymptotic behavior of a sequence $(\boldsymbol{u}_\varepsilon)$ satisfying (13) and

$$\sup_{\varepsilon>0} F_\varepsilon(\boldsymbol{u}_\varepsilon) < +\infty, \tag{37}$$

of the sequence of its symetrized gradient, and of the auxiliary sequences $(\boldsymbol{v}_\varepsilon)$, (θ_ε), $\left(\frac{v_{\varepsilon 3}}{\varepsilon}\right)$, $\left(\frac{\theta_\varepsilon}{\varepsilon}\right)$ defined by (15). In what follows, the operators $\boldsymbol{u}_\varepsilon \to \boldsymbol{v}_\varepsilon$ and $\boldsymbol{u}_\varepsilon \to \theta_\varepsilon$ are denoted by v_ε and θ_ε: more precisely, for every $\boldsymbol{\varphi} \in L^p(\Omega;\mathbb{R}^3)$, we set

$$\begin{aligned}
&\mathrm{v}_\varepsilon(\boldsymbol{\varphi})(x) := \sum_{i\in I_\varepsilon} \left(\fint_{S_\varepsilon^i} \boldsymbol{\varphi}(s,x_3) d\mathcal{H}^2(s) \right) \mathbb{1}_{Y_\varepsilon^i}(x'),\\
&\theta_\varepsilon(\boldsymbol{\varphi})(x) := \sum_{i\in I_\varepsilon} \frac{1}{\fint_S |y|^2 dy} \int_{S_\varepsilon^i} (-y_{\varepsilon 2}\varphi_{\varepsilon 1} + y_{\varepsilon 1}\varphi_{\varepsilon 2})(s,x_3) d\mathcal{H}^2(s) \mathbb{1}_{Y_\varepsilon^i}(x').
\end{aligned} \tag{38}$$

Proposition 1 *Let $\boldsymbol{u}_\varepsilon$ be a sequence in $W_0^{1,p}(\Omega;\mathbb{R}^3)$ satisfying (13) and (37). Then, up to a subsequence, the following convergences hold:*

$$\begin{aligned}
&\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \text{ weakly in } L^p(\Omega;\mathbb{R}^3),\\
&\mathrm{v}_\varepsilon(\boldsymbol{u}_\varepsilon) \rightharpoonup \boldsymbol{v} \text{ weakly in } L^p(\Omega;\mathbb{R}^3), \quad \theta_\varepsilon(\boldsymbol{u}_\varepsilon) \rightharpoonup \theta \text{ weakly in } L^p(\Omega),\\
&\boldsymbol{u}_\varepsilon \mathbb{1}_{T_\varepsilon} \rightharpoonup\!\!\!\rightharpoonup (\boldsymbol{v} + \theta \boldsymbol{e}_3 \wedge \boldsymbol{y})\, \mathbb{1}_S(y),\\
&\boldsymbol{u}_\varepsilon \mathbb{1}_{T_\varepsilon} \rightharpoonup |S| \boldsymbol{v} \quad \text{weakly in } L^p(\Omega;\mathbb{R}^3),\\
&\theta, v_3 \in L^p(\Omega'; W_0^{1,p}(0,L))),
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\boldsymbol{u}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup \boldsymbol{u}_0, \qquad \varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) \rightharpoonup\!\!\!\rightharpoonup \boldsymbol{e}_y(\boldsymbol{u}_0), \quad \varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon)\mathbb{1}_{\Omega\setminus T_\varepsilon} \rightharpoonup\!\!\!\rightharpoonup \boldsymbol{e}_y(\boldsymbol{u}_0)\mathbb{1}_{Y\setminus S},\\
&\boldsymbol{u}_0 \in L^p(\Omega; W_\sharp^{1,p}(Y)), \qquad \int_Y \boldsymbol{u}_0(.,y)dy = \boldsymbol{u},
\end{aligned} \tag{40}$$

and

$$\boldsymbol{e}(\boldsymbol{u}_\varepsilon)\mathbb{1}_{T_\varepsilon} \rightharpoonup\!\!\!\rightharpoonup (\boldsymbol{e}_y(\boldsymbol{q}) + \boldsymbol{M}(\boldsymbol{\tau}))\, \mathbb{1}_S(y), \qquad \boldsymbol{\tau} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v_3}{\partial x_3} \\ \frac{\partial \theta}{\partial x_3} \end{pmatrix}, \tag{41}$$

for some $\boldsymbol{q} \in L^p(\Omega; W^{1,p}(S;\mathbb{R}^3))$, where $\boldsymbol{M}(\boldsymbol{\tau})$ is defined by (10).
If in addition $\varkappa > 0$, then, up to a subsequence,

$$\begin{aligned}
&\frac{u_{\varepsilon 3}}{\varepsilon}\mathbb{1}_{T_\varepsilon} \rightharpoonup |S| w \quad \text{weakly in } L^p(\Omega),\\
&\left(\frac{\mathrm{v}_{\varepsilon 3}(\boldsymbol{u}_\varepsilon)}{\varepsilon}, \frac{\theta_\varepsilon(\boldsymbol{u}_\varepsilon)}{\varepsilon}\right) \rightharpoonup (w,\delta) \quad \text{weakly in } L^p(\Omega)^2,\\
&\boldsymbol{v} \in L^p(\Omega'; W_0^{2,p}(0,L;\mathbb{R}^3)); \quad \delta, w \in L^p(\Omega'; W_0^{1,p}(0,L)),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{u_{\varepsilon 3}}{\varepsilon}\mathbb{1}_{T_\varepsilon} \rightharpoonup\!\!\!\rightharpoonup \left(w - \frac{\partial v_1}{\partial x_3} y_1 - \frac{\partial v_2}{\partial x_3} y_2\right) \mathbb{1}_S(y),\\
&\frac{1}{\varepsilon}\boldsymbol{e}(\boldsymbol{u}_\varepsilon)\mathbb{1}_{T_\varepsilon} \rightharpoonup\!\!\!\rightharpoonup (\boldsymbol{e}_y(\boldsymbol{l}) + \boldsymbol{M}(\boldsymbol{\tau}))\, \mathbb{1}_S(y), \qquad \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix},
\end{aligned}$$

for some $\boldsymbol{l} \in L^p(\Omega; W^{1,p}(S;\mathbb{R}^3))$. *Besides (see* (16), (17)*),*

$$\begin{aligned} &(\boldsymbol{v},\theta) \in \mathcal{D}_1 && \text{if } \varkappa = 0,\\ &(\boldsymbol{v},w,\delta) \in \mathcal{D}_2 && \text{if } \varkappa > 0. \end{aligned} \tag{42}$$

Proof Let $\boldsymbol{u}_\varepsilon$ be a sequence in $W_0^{1,p}(\Omega;\mathbb{R}^3)$ satisfying (13) and (37).

If $\varkappa = 0$, by (13) and (38) the sequences $(\boldsymbol{u}_\varepsilon)$, $(\mathbf{v}_\varepsilon(\boldsymbol{u}_\varepsilon))$ and $(\theta_\varepsilon(\boldsymbol{u}_\varepsilon))$ are bounded in $L^p(\Omega)$ (see (44)).

If $\varkappa > 0$, by applying the inequality (see [5, (4.41)])

$$\int_{T_\varepsilon} |\varphi_1|^p + |\varphi_2|^p + \left|\frac{\varphi_3}{\varepsilon}\right|^p dx \le \frac{C}{\varepsilon^p}\int_{T_\varepsilon} |\boldsymbol{e}(\boldsymbol{\varphi})|^p\, dx, \quad \forall \boldsymbol{\varphi} \in W_0^{1,p}(\Omega;\mathbb{R}^3),$$

to $\boldsymbol{\varphi} = \boldsymbol{u}_\varepsilon$, taking (1) and (2) into account, we deduce

$$\int_{T_\varepsilon} |\boldsymbol{u}_\varepsilon|^p dx \le C F_\varepsilon(\boldsymbol{u}_\varepsilon) \qquad \text{if} \quad \varkappa > 0, \tag{43}$$

thus $(\boldsymbol{u}_\varepsilon \mathbb{1}_{T_\varepsilon})$ is bounded in $L^p(\Omega;\mathbb{R}^3)$. By (38) and Jensen's inequality,

$$\begin{aligned} \int_\Omega |\mathbf{v}_\varepsilon(\boldsymbol{u}_\varepsilon)|^p\, dx &= \sum_{i\in I_\varepsilon} \int_0^L dx_3 \int_{Y_\varepsilon^i} \left| \fint_{S_\varepsilon^i} \boldsymbol{u}_\varepsilon(s,x_3) d\mathcal{H}^2(s)\right|^p dx' \\ &\le \sum_{i\in I_\varepsilon} \int_0^L dx_3 \int_{Y_\varepsilon^i} \fint_{S_\varepsilon^i} |\boldsymbol{u}_\varepsilon|^p(s,x_3) d\mathcal{H}^2(s) dx' \\ &= \frac{1}{|S|}\int_{T_\varepsilon} |\boldsymbol{u}_\varepsilon|^p dx, \end{aligned} \tag{44}$$

hence $(\mathbf{v}_\varepsilon(\boldsymbol{u}_\varepsilon))$ is bounded in L^p. The same holds for $(\theta_\varepsilon(\boldsymbol{u}_\varepsilon))$. By making suitable changes of variables in the Poincaré-Wirtinger inequality

$$\int_Y \left|\varphi - \fint_S \varphi\, ds'\right|^p dx' \le C\int_Y |\nabla\varphi|^p dx' \qquad \forall \varphi \in W^{1,p}(Y),$$

we infer

$$\int_{Y_\varepsilon^i} \left|\varphi - \fint_{S_\varepsilon^i} \varphi\, ds'\right|^p dx' \le C\varepsilon^p \int_{Y_\varepsilon^i} |\nabla\varphi|^p dx' \qquad \forall \varphi \in W^{1,p}(\Omega),\ \forall i \in I_\varepsilon.$$

By substituting $\boldsymbol{u}_\varepsilon$ for φ, summing with respect to i over I_ε (see (11)) and integrating with respect to x_3 over $(0,L)$, we deduce from (38) and Korn's inequality in $W_0^{1,p}(\Omega;\mathbb{R}^3)$ that

$$\begin{aligned} \int_{Y_\varepsilon\times(0,L)} |\boldsymbol{u}_\varepsilon - \mathbf{v}_\varepsilon(\boldsymbol{u}_\varepsilon)|^p\, dx &\le C\varepsilon^p \int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^p dx \le C\varepsilon^p \int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^p dx \\ &\le C F_\varepsilon(\boldsymbol{u}_\varepsilon), \end{aligned} \tag{45}$$

where $Y_\varepsilon = \bigcup_{i\in I_\varepsilon} Y_\varepsilon^i$. However this, along with (43) and (44), only shows that $(\boldsymbol{u}_\varepsilon \mathbb{1}_{Y_\varepsilon\times(0,L)})$ is bounded in $L^p(\Omega;\mathbb{R}^3)$. To circumvent this difficulty, we extend $\boldsymbol{u}_\varepsilon$ by setting $\boldsymbol{u}_\varepsilon = 0$ in $\mathbb{R}^2 \setminus \Omega' \times (0,L)$ and substitute $\widetilde{Y} = \left[-\frac{3}{2}, \frac{3}{2}\right)^2$ for Y in the above argument. Noticing that $\Omega \subset \bigcup_{i\in I_\varepsilon} \widetilde{Y}_\varepsilon^i \times (0,L)$, $|\widetilde{Y}_\varepsilon^i| = 9|Y_\varepsilon^i|$, and $\sum_{i\in I_\varepsilon} \int_{\widetilde{Y}_\varepsilon^i\times(0,L)} |\boldsymbol{\nabla u}_\varepsilon|^p dx \le 9 \int_\Omega |\boldsymbol{\nabla u}_\varepsilon|^p dx$, we infer

$$\begin{aligned}\int_\Omega |\boldsymbol{u}_\varepsilon|^p dx &\le \sum_{i\in I_\varepsilon} \int_{\widetilde{Y}_\varepsilon^i\times(0,L)} |\boldsymbol{u}_\varepsilon|^p \, dx' \\ &\le C \sum_{i\in I_\varepsilon} \int_{\widetilde{Y}_\varepsilon^i} \left| \fint_{S_\varepsilon^i} \boldsymbol{u}_\varepsilon \, ds' \right|^p dx' + C\varepsilon^p \sum_{i\in I_\varepsilon} \int_{\widetilde{Y}_\varepsilon^i\times(0,L)} |\boldsymbol{\nabla u}_\varepsilon|^p dx \\ &\le C \int_\Omega |\mathbf{v}_\varepsilon(\boldsymbol{u}_\varepsilon)|^p \, dx + C F_\varepsilon(\boldsymbol{u}_\varepsilon).\end{aligned}$$

Taking (43) and (44) into account, we infer

$$\int_\Omega |\boldsymbol{u}_\varepsilon|^p dx \le C F_\varepsilon(\boldsymbol{u}_\varepsilon) \qquad \text{if} \quad \varkappa > 0. \tag{46}$$

Therefore, by (37), $(\boldsymbol{u}_\varepsilon)$ is bounded in $L^p(\Omega;\mathbb{R}^3)$ if $\varkappa > 0$.

Revisiting their proofs, we then find that all assertions stated in [5, Propositions 4.5, 4.6, 4.7] hold, up to the modifications described below in Remark 3. We deduce that all assertions stated in Proposition 1 hold, (40) being a consequence of (32).

Remark 3. Fibers of cross-sections of size $r_\varepsilon \ll \varepsilon$ were considered in [5]. Here, we have $r_\varepsilon = \varepsilon$. The notation $\varphi_\varepsilon \overset{m_\varepsilon}{\rightharpoonup} \varphi_0$ employed in [5] simply means $\varphi_\varepsilon \mathbb{1}_{T_\varepsilon} \rightharpoonup\!\!\!\!\rightharpoonup \varphi_0 \mathbb{1}_S(y)$. In [5] the fields $\mathbf{v}_\varepsilon(\boldsymbol{u}_\varepsilon)$ and $\theta_\varepsilon(\boldsymbol{u}_\varepsilon)$ are denoted by $\overline{\mathfrak{v}}_\varepsilon^S(\boldsymbol{u}_\varepsilon)$ and $(\mathrm{diam}S)\overline{\theta}_\varepsilon^S(\boldsymbol{u}_\varepsilon)$. The first line of [5, (4.58)] is irrelevant to our context and must be replaced by the weak convergence in $L^p(\Omega;\mathbb{R}^3)$ of $(\boldsymbol{u}_\varepsilon)$ to $\boldsymbol{u}$.

6 Proof of Theorem 1

In the spirit of the Γ-convergence method [12], we establish a lower bound in Section 6.1, an upper bound in Section 6.2, and conclude the proof in Section 6.3.

6.1 Lower bound inequality

Proposition 2 *Let $(\boldsymbol{u}_\varepsilon)$ be a sequence in $W_0^{1,p}(\Omega;\mathbb{R}^3)$ satisfying (13), (37) and, up to a subsequence, the convergences stated in Proposition 1. Then*

$$\begin{aligned}&\liminf_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) \ge \Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) && \text{if} \quad \varkappa = 0, \\ &\liminf_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) \ge \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) && \text{if} \quad \varkappa > 0.\end{aligned} \tag{47}$$

Proof. Assuming (13) and (37), we extract a subsequence of $\boldsymbol{u}_{\varepsilon_k}$ of $\boldsymbol{u}_\varepsilon$ such that $\liminf_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) = \lim_{k\to+\infty} F_{\varepsilon_k}(\boldsymbol{u}_{\varepsilon_k})$, and then a further subsequence (denoted $\boldsymbol{u}_\varepsilon$ for notational simplicity) such that all convergences stated in Proposition 1 hold. Setting

$$F_\varepsilon(\boldsymbol{u}_\varepsilon) = I_{\varepsilon 1} + I_{\varepsilon 3}; \quad I_{\varepsilon 1} := \int_{\Omega\setminus T_\varepsilon} \varepsilon^p f(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))dx; \quad I_{\varepsilon 3} := k_\varepsilon \int g(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))dm_\varepsilon,$$

we obtain (same proof as that of [5, (6.5)])

$$\begin{aligned} &\liminf_{\varepsilon\to 0} I_{\varepsilon 3} \geq k\int_\Omega g^{hom}(\boldsymbol{\tau})dx, \quad \boldsymbol{\tau} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v_3}{\partial x_3} \\ \frac{\partial \theta}{\partial x_3} \end{pmatrix} \quad \text{if } \varkappa = 0; \\ &\liminf_{\varepsilon\to 0} I_{\varepsilon 3} \geq \varkappa\int_\Omega g^{hom}(\boldsymbol{\tau})dx, \quad \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix} \quad \text{if } \varkappa > 0. \end{aligned} \tag{48}$$

By (30) and the two-scale convergences, stated in (39) and (40), of the sequences $(\boldsymbol{u}_\varepsilon \mathbb{1}_{T_\varepsilon})$ and $(\boldsymbol{u}_\varepsilon)$ to $\boldsymbol{v}(x) + \theta(x)\boldsymbol{e}_3 \wedge \boldsymbol{y}$ and $\boldsymbol{u}_0$, respectively, we have

$$\boldsymbol{u}_0(x,y) = \boldsymbol{v}(x) + \theta(x)\boldsymbol{e}_3 \wedge \boldsymbol{y} \qquad \text{a.e. in } \Omega \times S. \tag{49}$$

By the p-positive homogeneousness of f, $I_{\varepsilon 1} = \int_\Omega f(\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon \mathbb{1}_{\Omega\setminus T_\varepsilon}))dx$ hence, by the two-scale convergence $(\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon \mathbb{1}_{\Omega\setminus T_\varepsilon}))$ to $\boldsymbol{e}_y(\boldsymbol{u}_0)$ stated in (40) and the lower bound (31) applied to $j = f$, the following holds

$$\liminf_{\varepsilon\to 0} I_{\varepsilon 1} \geq \int_{\Omega\times Y\setminus S} f(\boldsymbol{e}_y(\boldsymbol{u}_0))dxdy = \int_{\Omega\times Y\setminus S} f(\boldsymbol{e}_y(\boldsymbol{u}_0 - \boldsymbol{u}(x)))dxdy.$$

On the other hand, by (40), (49), and the definition (9) of c^f, we have

$$\begin{aligned} &\int_{\Omega\times Y\setminus S} f(\boldsymbol{e}_y(\boldsymbol{u}_0 - \boldsymbol{u}(x)))dxdy \\ &\geq \int_\Omega \inf\left\{ \int_Y f\left(\boldsymbol{e}_y(\boldsymbol{\psi})\right)dy \;\middle|\; \begin{array}{l} \boldsymbol{\psi} \in W^{1,p}_\sharp(Y;\mathbb{R}^3), \ \int_Y \boldsymbol{\psi}dy = 0, \\ \boldsymbol{\psi} = \boldsymbol{v}(x) - \boldsymbol{u}(x) + \theta(x)\boldsymbol{e}_3 \wedge \boldsymbol{y} \ \text{ in } \ S \end{array} \right\} dx \\ &= \int_\Omega c^f(\boldsymbol{v} - \boldsymbol{u}, \theta)dx. \end{aligned}$$

Collecting (42), (48) and the above inequalities, we infer

$$\begin{aligned} &\liminf_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \Phi_1(\boldsymbol{u}, \boldsymbol{v}, \theta) && \text{if } \varkappa = 0, \\ &\liminf_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) \geq \Phi_2(\boldsymbol{u}, \boldsymbol{v}, w, \delta) && \text{if } \varkappa > 0. \end{aligned}$$

Proposition 2 is proved □

6.2 Upper bound inequality

Proposition 3 *Let* $(\boldsymbol{\varphi},\boldsymbol{\psi},\zeta,a,b)\in\mathcal{D}(\Omega;\mathbb{R}^3)^2\times(\mathcal{D}(\Omega))^3$ *such that*

$$\begin{aligned}&\psi_3=\zeta=0 \qquad \text{if } (k,\varkappa)=(+\infty,0),\\ &(\boldsymbol{\psi},\zeta,a,b)=0 \quad \text{if } \varkappa=+\infty,\end{aligned} \tag{50}$$

and let $\alpha>0$. *Then, there exists a sequence* $(\boldsymbol{\varphi}_\varepsilon)$ *in* $C^1_c(\Omega;\mathbb{R}^3)$ *satisfying*

$$\begin{aligned}&\boldsymbol{\varphi}_\varepsilon\rightharpoonup\boldsymbol{\varphi} \quad \text{weakly in } L^p(\Omega;\mathbb{R}^3),\\ &\mathbf{v}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\rightharpoonup\boldsymbol{\psi} \quad \text{weakly in } L^p(\Omega;\mathbb{R}^3), \quad \theta_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\rightharpoonup\zeta \quad \text{weakly in } L^p(\Omega),\\ &\frac{1}{\varepsilon}\left(\mathrm{v}_{\varepsilon 3}(\boldsymbol{\varphi}_\varepsilon),\theta_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\right)\rightharpoonup(a,b) \quad \text{weakly in } L^p(\Omega)^2 \quad \text{if} \quad 0<\varkappa<+\infty,\end{aligned} \tag{51}$$

and

$$\begin{aligned}&\limsup_{\varepsilon\to0}F_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\le\Phi_1(\boldsymbol{\varphi},\boldsymbol{\psi},\zeta)+\alpha \qquad \text{if} \quad \varkappa=0,\\ &\limsup_{\varepsilon\to0}F_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\le\Phi_2(\boldsymbol{\varphi},\boldsymbol{\psi},a,b)+\alpha \qquad \text{if} \quad \varkappa>0.\end{aligned} \tag{52}$$

Proof. We fix $\alpha>0$, $(\boldsymbol{\varphi},\boldsymbol{\psi},\zeta,a,b)\in\mathcal{D}(\Omega;\mathbb{R}^3)^2\times(\mathcal{D}(\Omega))^3$ satisfying (50), and $\boldsymbol{\varphi}_0\in\mathcal{D}_{\boldsymbol{\psi}-\boldsymbol{\varphi},\zeta}$ (see (36)) such that

$$\int_{\Omega\times(Y\setminus S)}f(\boldsymbol{e}_y(\boldsymbol{\varphi}_0))dxdy\le\int_\Omega c^f(\boldsymbol{\psi}-\boldsymbol{\varphi},\zeta)dx+\frac{\alpha}{2}. \tag{53}$$

Setting

$$T_\varepsilon^{\varepsilon^2}:=\{x\in\Omega,\operatorname{dist}(x,T_\varepsilon)<\varepsilon^2\},$$

we fix $(\xi_\varepsilon)\subset C^\infty(\overline{\Omega})$ satisfying

$$0\le\xi_\varepsilon\le1,\quad \xi_\varepsilon=1 \text{ in } T_\varepsilon,\quad \xi_\varepsilon=0 \text{ in } \Omega\setminus T_\varepsilon^{\varepsilon^2},\quad |\boldsymbol{\nabla}\xi_\varepsilon|\le\frac{C}{\varepsilon^2},$$

and consider the sequence $(\boldsymbol{\varphi}_\varepsilon)\subset C^1_c(\Omega;\mathbb{R}^3)$ defined by

$$\boldsymbol{\varphi}_\varepsilon(x):=(1-\xi_\varepsilon)\,\boldsymbol{\varphi}_0\left(x,\frac{y_\varepsilon(x')}{\varepsilon}\right)+\xi_\varepsilon\boldsymbol{\chi}_\varepsilon, \tag{54}$$

where $\boldsymbol{\chi}_\varepsilon$ is obtained, depending on k, $\varkappa$, by substituting ε for r_ε in [5, (6.28), (6.29) or (6.31)] if $\varkappa<+\infty$ and by setting $\boldsymbol{\chi}_\varepsilon=0$ if $\varkappa=+\infty$. Namely,
• if $0<k<+\infty$, by Corollary 3 we can fix $\boldsymbol{q}\in C^1_c(\Omega;C^\infty(\overline{S};\mathbb{R}^3))$ such that

$$k\int_\Omega\fint_S g\left(\boldsymbol{e}_y(\boldsymbol{q})+\boldsymbol{M}(\boldsymbol{\tau})\right)dxdy\le k\int_\Omega g^{hom}(\boldsymbol{\tau})dx+\frac{\alpha}{2},\qquad \boldsymbol{\tau}=\begin{pmatrix}0\\0\\\frac{\partial\psi_3}{\partial x_3}\\\frac{\partial\zeta}{\partial x_3}\end{pmatrix}, \tag{55}$$

where $\boldsymbol{M}(\boldsymbol{\tau})$ is defined by (10). We set

$$\boldsymbol{\chi}_\varepsilon:=\overline{\boldsymbol{\psi}}_\varepsilon+\overline{\zeta}_\varepsilon\boldsymbol{e}_3\wedge\frac{\boldsymbol{y}_\varepsilon}{\varepsilon}-\varepsilon\left(\frac{\partial\overline{\psi}_{\varepsilon1}}{\partial x_3}\frac{y_{\varepsilon1}}{\varepsilon}+\frac{\partial\overline{\psi}_{\varepsilon2}}{\partial x_3}\frac{y_{\varepsilon2}}{\varepsilon}\right)\boldsymbol{e}_3+\varepsilon\boldsymbol{q}\left(x,\frac{y_\varepsilon(x')}{\varepsilon}\right), \tag{56}$$

where

$$\overline{h}_\varepsilon(x) := \sum_{i\in I_\varepsilon} \left(\fint_{S_\varepsilon^i} h(s_1,s_2,x_3)ds_1ds_2 \right) \mathbb{1}_{Y_\varepsilon^i}(x_1,x_2) \quad \forall h \in L^p(\Omega). \tag{57}$$

• If $k = +\infty$ and $\varkappa = 0$, we set

$$\boldsymbol{\chi}_\varepsilon := \overline{\boldsymbol{\psi}}_\varepsilon - \varepsilon \left(\frac{\partial \overline{\psi}_{\varepsilon 1}}{\partial x_3} \frac{y_{\varepsilon 1}}{\varepsilon} + \frac{\partial \overline{\psi}_{\varepsilon 2}}{\partial x_3} \frac{y_{\varepsilon 2}}{\varepsilon} \right) \boldsymbol{e}_3. \tag{58}$$

• If $0 < \varkappa < +\infty$, applying Corollary 3 we fix $\boldsymbol{q} \in C_c^1(\Omega; C^\infty(\overline{S};\mathbb{R}^3))$ satisfying the inequality deduced from (55) by replacing k by $\varkappa$ and setting $\boldsymbol{\tau} = \begin{pmatrix} \frac{\partial^2 \psi_1}{\partial x_3^2} \\ \frac{\partial^2 \psi_2}{\partial x_3^2} \\ \frac{\partial b}{\partial x_3} \\ \frac{\partial a}{\partial x_3} \end{pmatrix}$, and define

$$\boldsymbol{\chi}_\varepsilon(x) := \overline{\boldsymbol{\psi}}_\varepsilon(x) + \varepsilon \begin{pmatrix} -2\overline{a}_\varepsilon \frac{y_{\varepsilon 2}}{\varepsilon} \\ 2\overline{a}_\varepsilon \frac{y_{\varepsilon 1}}{\varepsilon} \\ \overline{b}_\varepsilon(x) - \frac{y_{\varepsilon 1}}{\varepsilon} \frac{\partial}{\partial x_3} \overline{\psi}_{\varepsilon 1} - \frac{y_{\varepsilon 2}}{\varepsilon} \frac{\partial}{\partial x_3} \overline{\psi}_{\varepsilon 2} \end{pmatrix} + \varepsilon^2 \boldsymbol{q}\left(x, \frac{y_\varepsilon}{\varepsilon}\right). \tag{59}$$

• If $\varkappa = +\infty$, we set $\boldsymbol{\chi}_\varepsilon = 0$. We split $F_\varepsilon(\boldsymbol{\varphi}_\varepsilon)$ into the sum of three terms:

$$F_\varepsilon(\boldsymbol{\varphi}_\varepsilon) = I_{\varepsilon 1} + I_{\varepsilon 2} + I_{\varepsilon 3}; \qquad I_{\varepsilon 1} := \int_{\Omega \setminus T_\varepsilon^{\varepsilon^2}} f(\varepsilon \boldsymbol{e}(\boldsymbol{\varphi}_0(x, y_\varepsilon(x')/\varepsilon)));$$
$$I_{\varepsilon 2} := \int_{T_\varepsilon^{\varepsilon^2} \setminus T_\varepsilon} f\left(\varepsilon \boldsymbol{e}\left(\boldsymbol{\varphi}_\varepsilon\right)\right) dx; \qquad I_{\varepsilon 3} := l_\varepsilon \int_{T_\varepsilon} g\left(\boldsymbol{e}\left(\boldsymbol{\chi}_\varepsilon\right)\right) dx.$$

For every $\boldsymbol{\Xi}_0 \in C_c(\Omega \times Y; \mathbb{S}^3)$, we have

$$\lim_{\varepsilon\to 0} \int_{\Omega \setminus T_\varepsilon^{\varepsilon^2}} \boldsymbol{\Xi}_0\left(x, \frac{y_\varepsilon(x')}{\varepsilon}\right) dx = \int_{\Omega \times (Y \setminus S)} \boldsymbol{\Xi}_0(x,y) dxdy,$$

hence, by the estimate $\varepsilon \boldsymbol{e}\left(\boldsymbol{\varphi}_0\left(x, \frac{y_\varepsilon(x')}{\varepsilon}\right)\right) = \boldsymbol{e}_y(\boldsymbol{\varphi}_0)\left(x, \frac{y_\varepsilon(x')}{\varepsilon}\right) + \mathrm{O}(\varepsilon)$,

$$\lim_{\varepsilon\to 0} I_{\varepsilon 1} = \int_{\Omega \times (Y \setminus S)} f(\boldsymbol{e}_y(\boldsymbol{\varphi}_0)) dxdy.$$

One can check that $|\boldsymbol{e}(\varepsilon \boldsymbol{\varphi}_\varepsilon)| \le C$ in $T_\varepsilon^{\varepsilon^2} \setminus T_\varepsilon$, therefore

$$|I_{\varepsilon 2}| \le C |T_\varepsilon^{\varepsilon^2} \setminus T_\varepsilon| \le C\varepsilon.$$

Substituting ε for r_ε in the proofs of [5, (6.23), (6.24)], we find

$$\liminf_{\varepsilon\to 0} I_{\varepsilon 3} \le k\int_\Omega g^{hom}(\boldsymbol{\tau})dx + \frac{\alpha}{2}, \quad \boldsymbol{\tau} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial \psi_3}{\partial x_3} \\ \frac{\partial \zeta}{\partial x_3} \end{pmatrix} \qquad \text{if } \varkappa = 0,$$

$$\liminf_{\varepsilon\to 0} I_{\varepsilon 3} \le \varkappa\int_\Omega g^{hom}(\boldsymbol{\tau})dx + \frac{\alpha}{2}, \quad \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial^2 \psi_1}{\partial x_3^2} \\ \frac{\partial^2 \psi_2}{\partial x_3^2} \\ \frac{\partial b}{\partial x_3} \\ \frac{\partial a}{\partial x_3} \end{pmatrix} \qquad \text{if } \varkappa > 0.$$

Combining the above inequalities, we obtain (52). Proposition 3 is proved. □

6.3 Conclusion of the proof of Theorem 1

Let $\boldsymbol{u}_\varepsilon$ be the solution to (1). Then, $F_\varepsilon(\boldsymbol{u}_\varepsilon) - \int_\Omega \boldsymbol{f}.\boldsymbol{u}_\varepsilon dx \le F_\varepsilon(0) = 0$, thus,

$$F_\varepsilon(\boldsymbol{u}_\varepsilon) \le C|\boldsymbol{u}_\varepsilon|_{L^p(\Omega;\mathbb{R}^3)}. \tag{60}$$

We deduce from (13) if $\varkappa = 0$, and from (46) and (60) if $\varkappa > 0$, that $(\boldsymbol{u}_\varepsilon)$ is bounded in L^p. This, along with (60), implies that $(\boldsymbol{u}_\varepsilon)$ satisfies (37), hence there exists a subsequence $(\boldsymbol{u}_{\varepsilon_k})_{k\in\mathbb{N}}$ satisfying the convergences stated in Propositions 1. Applying Proposition 2, we infer

$$\begin{aligned} &\liminf_{k\to+\infty} F_{\varepsilon_k}(\boldsymbol{u}_{\varepsilon_k}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_{\varepsilon_k}dx \ge \Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx \qquad \text{if } \varkappa = 0, \\ &\liminf_{k\to+\infty} F_{\varepsilon_k}(\boldsymbol{u}_{\varepsilon_k}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_{\varepsilon_k}dx \ge \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx \quad \text{if } \varkappa > 0. \end{aligned} \tag{61}$$

Let us fix $\alpha > 0$. The functionals Φ_i are convex and bounded on the unit ball of the Banach space $L^p(\Omega;\mathbb{R}^3)\times\mathcal{D}_i$ $(i\in\{1,2\})$, hence strongly continuous. By density, there exists $(\boldsymbol{\varphi},\boldsymbol{\psi},\zeta,a,b)\in\mathcal{D}(\Omega;\mathbb{R}^3)^2\times\mathcal{D}(\Omega)^3$ satisfying (50) and such that

$$\begin{aligned} &\Phi_1(\boldsymbol{\varphi},\boldsymbol{\psi},\zeta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx \le \inf(\mathcal{P}_1^{hom}) + \alpha \qquad \text{if} \quad \varkappa = 0, \\ &\Phi_2(\boldsymbol{\varphi},\boldsymbol{\psi},a,b) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx \le \inf(\mathcal{P}_2^{hom}) + \alpha \qquad \text{if} \quad \varkappa > 0. \end{aligned}$$

Applying Proposition 3, we fix $(\boldsymbol{\varphi}_\varepsilon)\subset C_c^1(\Omega;\mathbb{R}^3)$ verifying (51) and (52). Since $\boldsymbol{u}_\varepsilon$ is the solution to (1), we have

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon dx \le \limsup_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{\varphi}_\varepsilon) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}_\varepsilon dx,$$

thus, by (52),

$$\begin{aligned} &\limsup_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon dx \le \inf(\mathcal{P}_1^{hom}) + 2\alpha \qquad \text{if} \quad \varkappa = 0, \\ &\limsup_{\varepsilon\to 0} F_\varepsilon(\boldsymbol{u}_\varepsilon) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_\varepsilon dx \le \inf(\mathcal{P}_2^{hom}) + 2\alpha \qquad \text{if} \quad \varkappa > 0. \end{aligned} \tag{62}$$

We deduce from (61), (62) and the arbitrariness of α that $(\boldsymbol{u},\boldsymbol{v},\theta)$ is a solution to $\inf(\mathcal{P}_1^{hom})$ if $\varkappa = 0$, and $(\boldsymbol{u},\boldsymbol{v},w,\delta)$ is a solution to $\inf(\mathcal{P}_2^{hom})$ if $\varkappa > 0$. Theorem 1 is proved. □

7 Proof of Corollary 1

7.1 Case $\varkappa = 0$

If $(\boldsymbol{u},\boldsymbol{v},\theta)$ is a solution to (20), then for all $(\boldsymbol{\varphi},\boldsymbol{\psi},\zeta,t) \in \mathcal{D}(\Omega;\mathbb{R}^3)^2\times\mathcal{D}(\Omega)\times\mathbb{R}$,

$$\Phi_1(\boldsymbol{u},\boldsymbol{v},\theta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx \le \Phi_1(\boldsymbol{u}+t\boldsymbol{\varphi},\boldsymbol{v}+t\boldsymbol{\psi},\theta+t\zeta) - \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}+t\boldsymbol{\varphi})dx.$$

Dividing this inequation by $t \neq 0$ and sending t to ± 0, we infer

$$\int_\Omega \begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \theta\end{pmatrix}\cdot\boldsymbol{C}\begin{pmatrix}\boldsymbol{\psi}-\boldsymbol{\varphi}\\ \zeta\end{pmatrix} + k\begin{pmatrix}0\\0\\ \frac{\partial v_3}{\partial x_3}\\ \frac{\partial\theta}{\partial x_3}\end{pmatrix}\cdot\boldsymbol{G}\begin{pmatrix}0\\0\\ \frac{\partial\psi_3}{\partial x_3}\\ \frac{\partial\zeta}{\partial x_3}\end{pmatrix} dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx.$$

Choosing $\boldsymbol{\psi} = \zeta = 0$, we obtain

$$\int_\Omega \begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \theta\end{pmatrix}\cdot\boldsymbol{C}\begin{pmatrix}-\boldsymbol{\varphi}\\ 0\end{pmatrix} dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx,$$

and, letting $\boldsymbol{\varphi}$ vary in $\mathcal{D}(\Omega;\mathbb{R}^3)$, deduce

$$\boldsymbol{C}_{\{1,2,3\}\{1,2,3,4\}}\begin{pmatrix}\boldsymbol{u}-\boldsymbol{v}\\ \theta\end{pmatrix} = \boldsymbol{f} \quad \text{in } \Omega.$$

Selecting $\boldsymbol{\varphi} = \zeta = 0$, we get

$$\int_\Omega \begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \theta\end{pmatrix}\cdot\boldsymbol{C}\begin{pmatrix}\boldsymbol{\psi}\\ 0\end{pmatrix} + k\begin{pmatrix}0\\0\\ \frac{\partial v_3}{\partial x_3}\\ \frac{\partial\theta}{\partial x_3}\end{pmatrix}\cdot\boldsymbol{G}\begin{pmatrix}0\\0\\ \frac{\partial\psi_3}{\partial x_3}\\ 0\end{pmatrix} dx = 0,$$

and, integrating by parts and letting $\boldsymbol{\psi}$ vary, infer

$$\begin{aligned} &\boldsymbol{C}_{\{1,2\}\{1,2,3,4\}}\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \theta\end{pmatrix} = 0 && \text{in } \Omega, \\ &\boldsymbol{C}_{\{3\}\{1,2,3,4\}}\begin{pmatrix}\boldsymbol{v}-\boldsymbol{u}\\ \theta\end{pmatrix} - k\boldsymbol{G}_{\{3\}\{3,4\}}\begin{pmatrix}\frac{\partial^2 v_3}{\partial x_3^2}\\ \frac{\partial^2\theta}{\partial x_3^2}\end{pmatrix} = 0 && \text{in } \Omega. \end{aligned}$$

Choosing $\boldsymbol{\varphi} = \boldsymbol{\psi} = 0$ leads to

$$\int_\Omega \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ \theta \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \zeta \end{pmatrix} + k \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v_3}{\partial x_3} \\ \frac{\partial \theta}{\partial x_3} \end{pmatrix} \cdot \boldsymbol{G} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial \zeta}{\partial x_3} \end{pmatrix} dx = 0,$$

yielding, by the arbitrary nature of ζ,

$$\boldsymbol{C}_{\{4\}\{1,2,3,4\}} \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ \theta \end{pmatrix} - k\boldsymbol{G}_{\{4\}\{3,4\}} \begin{pmatrix} \frac{\partial^2 v_3}{\partial x_3^2} \\ \frac{\partial^2 \theta}{\partial x_3^2} \end{pmatrix} = 0 \qquad \text{in } \Omega,$$

thus $(\boldsymbol{u}, \boldsymbol{v}, \theta)$ is a solution to (21).

7.2 Case $\varkappa > 0$

In the setting of linear elasticity, the effective problem (17) takes the form

$$\inf_{(\boldsymbol{u},\boldsymbol{v},w,\delta)\in L^2(\Omega;\mathbb{R}^3)\times\mathcal{D}_2} \Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) - \int_\Omega \boldsymbol{f}.\boldsymbol{u}dx, \tag{63}$$

where

$$\Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) := \int_\Omega \frac{1}{2} \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ 0 \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ 0 \end{pmatrix} + \frac{\varkappa}{2} \boldsymbol{\tau} \cdot \boldsymbol{G}\boldsymbol{\tau} dx, \ \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix},$$

$$\mathcal{D}_2 := \Big\{ (\boldsymbol{v}, w, \delta) \in L^2(\Omega'; H_0^2(0,L))^3 \times L^2(\Omega'; H_0^1(0,L))^2, \quad v_3 = 0, \\ \boldsymbol{v} = 0 \quad \text{and} \quad w = \delta = 0 \quad \text{if} \quad \varkappa = +\infty \Big\}. \tag{64}$$

If $(\boldsymbol{u}, \boldsymbol{v}, w, \delta)$ is a solution to (63), then for every $(\boldsymbol{\varphi}, \boldsymbol{\psi}, a, b) \in \mathcal{D}(\Omega;\mathbb{R}^3)^2 \times \mathcal{D}(\Omega)^2$ with $\psi_3 = 0$ and every $t \in \mathbb{R}$,

$$\Phi_2(\boldsymbol{u},\boldsymbol{v},w,\delta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx \le \Phi_2(\boldsymbol{u}+t\boldsymbol{\varphi}, \boldsymbol{v}+t\boldsymbol{\psi}, w+ta, \delta+tb) - \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}+t\boldsymbol{\varphi})dx.$$

Dividing this inequation by $t \neq 0$ and letting t tend to 0, we get

$$\int_\Omega \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ 0 \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} \boldsymbol{\psi}-\boldsymbol{\varphi} \\ 0 \end{pmatrix} + \varkappa \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix} \cdot \boldsymbol{G} \begin{pmatrix} \frac{\partial^2 \psi_1}{\partial x_3^2} \\ \frac{\partial^2 \psi_2}{\partial x_3^2} \\ \frac{\partial a}{\partial x_3} \\ \frac{\partial b}{\partial x_3} \end{pmatrix} dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx.$$

Choosing $\boldsymbol{\psi} = a = b = 0$, and letting $\boldsymbol{\varphi}$ vary in $\mathcal{D}(\Omega;\mathbb{R}^3)$, we infer

$$\sum_{k=1}^{3} C_{ik}(u_k - v_k) = f_i \quad \text{in } \Omega, \quad \forall i \in \{1,2,3\}.$$

Setting $\boldsymbol{\varphi} = a = b = 0$, we get

$$\int_\Omega \begin{pmatrix} \boldsymbol{v}-\boldsymbol{u} \\ 0 \end{pmatrix} \cdot \boldsymbol{C} \begin{pmatrix} \boldsymbol{\psi} \\ 0 \end{pmatrix} + \varkappa \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix} \cdot \boldsymbol{G} \begin{pmatrix} \frac{\partial^2 \psi_1}{\partial x_3^2} \\ \frac{\partial^2 \psi_2}{\partial x_3^2} \\ 0 \\ 0 \end{pmatrix} dx = 0,$$

then, integrating by parts and letting $\boldsymbol{\psi}$ vary, deduce

$$\sum_{k=1}^{3} C_{\alpha k}(v_k - u_k) + \varkappa \sum_{\beta=1}^{2} G_{\alpha\beta} \frac{\partial^4 v_\beta}{\partial x_3^4} + \varkappa G_{\alpha 3} \frac{\partial^3 w}{\partial x_3^3} + \varkappa G_{\alpha 4} \frac{\partial^3 \delta}{\partial x_3^3} = 0 \quad \text{in } \Omega,$$

for $\alpha \in \{1,2\}$. The choice $\boldsymbol{\varphi} = \boldsymbol{\psi} = b = 0$ leads to

$$\int_\Omega \varkappa \begin{pmatrix} \frac{\partial^2 v_1}{\partial x_3^2} \\ \frac{\partial^2 v_2}{\partial x_3^2} \\ \frac{\partial w}{\partial x_3} \\ \frac{\partial \delta}{\partial x_3} \end{pmatrix} \cdot \boldsymbol{G} \begin{pmatrix} 0 \\ 0 \\ \frac{\partial a}{\partial x_3} \\ 0 \end{pmatrix} dx = 0,$$

yielding

$$\sum_{\alpha=1}^{2} -\varkappa G_{3\alpha} \frac{\partial^3 v_\alpha}{\partial x_3^3} - \varkappa G_{33} \frac{\partial^2 w}{\partial x_3^2} - \varkappa G_{34} \frac{\partial^2 \delta}{\partial x_3^2} = 0 \qquad \text{in } \Omega,$$

while $\boldsymbol{\varphi} = \boldsymbol{\psi} = a = 0$ yields

$$\sum_{\alpha=1}^{2} -\varkappa G_{4\alpha} \frac{\partial^3 v_\alpha}{\partial x_3^3} - \varkappa G_{43} \frac{\partial^2 w}{\partial x_3^2} - \varkappa G_{44} \frac{\partial^2 \delta}{\partial x_3^2} = 0 \qquad \text{in } \Omega.$$

Collecting the above equations, we infer that $(\boldsymbol{u}, \boldsymbol{v}, w, \delta)$ is the unique solution to the system of equations

$$\begin{cases} \boldsymbol{C}_{\{1,2,3\}\{1,2,3\}}(\boldsymbol{u}-\boldsymbol{v}) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{C}_{\{1,2\}\{1,2,3\}}(\boldsymbol{v}-\boldsymbol{u}) + \varkappa \left(\boldsymbol{G}_{\{1,2\}\{1,2\}} \dfrac{\partial^4 \boldsymbol{v}'}{\partial x_3^4} + \boldsymbol{G}_{\{1,2\}\{3,4\}} \dfrac{\partial^3}{\partial x_3^3} \begin{pmatrix} w \\ \delta \end{pmatrix} \right) = 0 & \text{in } \Omega, \\ \boldsymbol{G}_{\{3,4\}\{1,2\}} \dfrac{\partial^3 \boldsymbol{v}'}{\partial x_3^3} + \boldsymbol{G}_{\{3,4\}\{3,4\}} \dfrac{\partial^2}{\partial x_3^2} \begin{pmatrix} w \\ \delta \end{pmatrix} = 0 & \text{in } \Omega, \\ (\boldsymbol{u}, \boldsymbol{v}, w, \delta) \in L^2(\Omega;\mathbb{R}^3) \times \mathcal{D}_2. \end{cases} \tag{65}$$

Since $\boldsymbol{G}$ is invertible, so is $\boldsymbol{G}_{\{3,4\}\{3,4\}}$. Setting $\boldsymbol{E} = -\boldsymbol{G}^{-1}_{\{3,4\}\{3,4\}}\boldsymbol{G}_{\{3,4\}\{1,2\}}$, the third line of (65) implies that $\frac{\partial^2}{\partial x_3^2}\begin{pmatrix} w \\ \delta \end{pmatrix} = \boldsymbol{E}\frac{\partial^3 \boldsymbol{v}'}{\partial x_3^3}$ in the sense of distributions in $\mathcal{D}'(\Omega;\mathbb{R}^2)$. By [6, Remark 1 (iv), (v)]), the matrices $\boldsymbol{G}_{\{3,4\}\{1,2\}}$ and $\boldsymbol{E}$ vanish in the isotropic case but not, in general, in the anisotropic case. We deduce that $\frac{\partial}{\partial x_3}\begin{pmatrix} w \\ \delta \end{pmatrix} = \boldsymbol{E}\frac{\partial^2 \boldsymbol{v}'}{\partial x_3^3} + \boldsymbol{k}(x_1, x_2)$, for some distribution $\boldsymbol{k} \in \mathcal{D}'(\Omega';\mathbb{R}^2)$. The last line of (65) then implies that $\boldsymbol{k} \in L^2(\Omega';\mathbb{R}^2)$. Integrating with respect to x_3 over $(0, L)$, taking into account the homogeneous Dirichlet boundary conditions satisfied by $w, \delta, \frac{\partial \boldsymbol{v}'}{\partial x_3}$ on $\Omega' \times \{0, L\}$, we infer that $\boldsymbol{k} = 0$, hence $\begin{pmatrix} w \\ \delta \end{pmatrix} = \boldsymbol{E}\frac{\partial \boldsymbol{v}'}{\partial x_3}$. The system (65) is thus equivalent to (24,26), where

$$\boldsymbol{H} = \boldsymbol{G}_{\{1,2\}\{1,2\}} + \boldsymbol{G}_{\{1,2\}\{3,4\}}\boldsymbol{E}; \qquad \boldsymbol{E} = -\boldsymbol{G}^{-1}_{\{3,4\}\{3,4\}}\boldsymbol{G}_{\{3,4\}\{1,2\}}.$$

One can check that $\boldsymbol{H}$ is symmetric. A straightforward computation shows that $\boldsymbol{\tau}\cdot \boldsymbol{G}\boldsymbol{\tau} = \frac{\partial^2 \boldsymbol{v}'}{\partial x_3^2}\cdot \boldsymbol{H}\frac{\partial^2 \boldsymbol{v}'}{\partial x_3^2}$, hence $\boldsymbol{H}$ is definite positive (because $\boldsymbol{G}$ is so). Coming back to (64), we deduce that the limit couple $(\boldsymbol{u}, \boldsymbol{v})$ is the unique solution to (25). The relative simplicity of the effective strain energy Φ in (25) covers the interesting torsional and extensional phenomena emerging in the fibers. The effective strain energy Φ_2 yields a better insight into the effective behavior of the fibers.

7.3 Justification of (19)

The components of the matrix $\boldsymbol{G}$ associated to the quadratic form $2g^{hom}$ on $\mathbb{R}^4$ defined by (5,10) are given by

$$G_{ij} = \fint_S \mathbf{B}\left(\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_i)) + \boldsymbol{M}(\boldsymbol{e}_i)\right) : \left(\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j)) + \boldsymbol{M}(\boldsymbol{e}_j)\right) dy, \tag{66}$$

where $\boldsymbol{q}(\boldsymbol{\tau})$ denotes the solution to the minimization problem (5,10) associated to the Euler-Lagrange equation

$$\begin{cases} -\operatorname{\mathbf{div}}_y \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{\tau})) = \operatorname{\mathbf{div}}_y \mathbf{B}\boldsymbol{M}(\boldsymbol{\tau}) & \text{in} \quad S, \\ \left(\mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{\tau}))\right)\cdot \boldsymbol{n} = -\left(\mathbf{B}\boldsymbol{M}(\boldsymbol{\tau})\right)\cdot \boldsymbol{n} & \text{on} \quad \partial S, \end{cases} \tag{67}$$

equivalent to

$$\fint_S \mathbf{B}\left(\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{\tau})) + \boldsymbol{M}(\tau)\right) : \boldsymbol{e}_y(\boldsymbol{l}) dy = 0 \qquad \forall \boldsymbol{l} \in H^1(S;\mathbb{R}^3). \tag{68}$$

By (66) and (68), we have

$$G_{ij} = \fint_S \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_i)) : \boldsymbol{M}(\boldsymbol{e}_j) dy + \fint_S \mathbf{B}\boldsymbol{M}(\boldsymbol{e}_i) : \boldsymbol{M}(\boldsymbol{e}_j) dy.$$

By (67), denoting by $\boldsymbol{n}'$ the outward normal to ∂S and setting $\boldsymbol{n} = \begin{pmatrix} \boldsymbol{n}' \\ 0 \end{pmatrix}$,

$$\begin{aligned}
&\fint_S \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_i)) : \boldsymbol{M}(\boldsymbol{e}_j)dy = \fint_S \boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_i)) : \mathbf{B}\boldsymbol{M}(\boldsymbol{e}_j)dy \\
&\quad = \frac{1}{|S|}\int_{\partial S} \boldsymbol{q}(\boldsymbol{e}_i) \otimes \boldsymbol{n} : \mathbf{B}\boldsymbol{M}(\boldsymbol{e}_j)d\mathcal{H}^1 - \fint_S \boldsymbol{q}(\boldsymbol{e}_i) \cdot \mathbf{div}_y(\mathbf{B}\boldsymbol{M}(\boldsymbol{e}_j))dy \\
&\quad = -\frac{1}{|S|}\int_{\partial S} \boldsymbol{q}(\boldsymbol{e}_i) \otimes \boldsymbol{n} : \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j))d\mathcal{H}^1 + \fint_S \boldsymbol{q}(\boldsymbol{e}_i) \cdot \mathbf{div}_y(\mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j)))dy
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\fint_S \boldsymbol{q}(\boldsymbol{e}_i) \cdot \mathbf{div}_y(\mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j)))dy \\
&\quad = \frac{1}{|S|}\int_{\partial S} \boldsymbol{q}(\boldsymbol{e}_i) \cdot (\mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j))\boldsymbol{n})d\mathcal{H}^1 - \fint_S \boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_i)) : \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j))dy \\
&\quad = \frac{1}{|S|}\int_{\partial S} \boldsymbol{q}(\boldsymbol{e}_i) \otimes \boldsymbol{n} : \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j))d\mathcal{H}^1 - \fint_S \mathbf{B}\boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_i)) : \boldsymbol{e}_y(\boldsymbol{q}(\boldsymbol{e}_j))dy.
\end{aligned}$$

The above equations imply (19).

Declarations

Competing interests. The author declares no competing interest.

References

1. Allaire, G.: Homogenization and two-scale convergence. SIAM J. Math. Anal. 23, 1482–1518 (1992) https://doi.org/10.1137/0523084
2. Arbogast, T., Douglas, J., Hornung, U.: Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Math. Anal., 21, 823–836 (1990) https://doi.org/10.1137/0521046
3. Bellieud, M. Torsion effects in elastic composites with high contrast. SIAM J. Math. Anal. 41 (6), 2514–2553 (2010) https://doi.org/10.1137/07069362X
4. Bellieud, M.: Homogenization of stratified elastic composites with high contrast. SIAM J. Math. Anal. 49-4, 2615-2665 (2017) https://doi.org/10.1137/15M1012050
5. Bellieud, M.: Homogenization of Norton-Hoff fibered composites with high viscosity contrast. SIAM J. Math. Anal. (2022). https://doi.org/10.1137/20M1345785
6. Bellieud, M.: Homogenization of an Anisotropic Elastic Material Reinforced by a Small Volume Fraction of Very Stiff Anisotropic Fibers. Non Local Effects. Bending Effects. Torsional Effects. J Elast (2023). https://doi.org/10.1007/s10659-023-09986-9
7. Bellieud, M., Bouchitté, G.: Homogenization of a soft elastic material reinforced by fibers. Asymptotic Analysis 32(2), 153–183 (2002)
8. Bouchitté, G., Dal Maso, G.: Integral representation and relaxation of convex local functionals on $BV(\Omega)$, Ann. Scuola Norm. Sup. Cl. Sci. IV 20, 483–533 (1993)
9. Bouchitte, G., Valadier, M.: Integral representation of convex functionals on a space of measures, J. Funct. Anal. 80, 398–420 (1988) https://doi.org/10.1016/0022-1236(88)90009-2

10. Charef, H., Sili, A.: The effective equilibrium law for a highly heterogeneous elastic periodic medium. Proc. Roy. Soc. Edinburgh Sect. A 143, no. 3, 507–561 (2013) https://doi.org/10.1017/S0308210511001053
11. Chawla, N, Chawla, K. K.: Metal Matrix Composites, Springer New York, NY (2016) https://doi.org/10.1007/978-1-4614-9548-2
12. Dal Maso, G.: An Introduction to Γ-Convergence. Birkhäuser (1993) https://doi.org/10.1007/978-1-4612-0327-8
13. Kassner, M. E.: Fundamentals of creep in Metals and Alloys, Butterworth-Heinemann, Elsevier, (2015) https://doi.org/10.1016/C2012-0-06071-1
14. Kondratiev, V.A., Oleinik, O.A.: Boundary value problems for a system in elasticity theory in unbounded domains. Korn's inequalities. (Russian) Uspekhi Mat. Nauk 43, no. 5(263), 55–98 (1988); translation in Russian Math. Surveys 43, no. 5, 65–119 (1988)
15. Lemaitre, J., Chaboche J., L.: Mechanics of Solid Materials. Cambridge University Press (1990)
16. Lukkassen, D., Nguetseng, G., Wall, P.:Two-scale convergence. Int. J. Pure Appl. Math. 2 (2002), no. 1, 35–86.
17. Nguetseng, G.: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. 20 608–623 (1989) https://doi.org/10.1137/0520043
18. Norton, F.H.: The Creep of Steel at High Temperature, Mc Graw-Hill, New York, USA (1929)
19. Smyshlyaev, V. P.: Propagation and localization of elastic waves in highly anisotropic periodic composites via two-scale homogenization. Mechanics of Materials. 41, 434–447 (2009) https://doi.org/10.1016/j.mechmat.2009.01.009