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Zero-Sum Stochastic Games with Vanishing Stage Duration and Public Signals

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Abstract

We consider the behaviour of λ -discounted zero-sum games as the discount factor λ tends to 0 (that is, the players are more and more patient), in the context of games with stage duration. In stochastic games with stage duration h , players act at times $0, h, 2h, \dots$, and the payoff and leaving probabilities are proportional to h . When h tends to 0, such games approximate games in continuous time. The asymptotic behavior of the values (when both λ and h tend to 0) was already studied in the case of stochastic games with perfect observation of the state and in the state-blind case.

We consider the same question for the case of stochastic games with imperfect observation of the state. In such games, players are given at each stage a public signal that depends only on the current state. Our main result states that there exists a stochastic game with public signals, with no limit value (as the discount factor λ goes to 0) if stage duration is 1, but with a limit value when stage duration h and discount factor λ both tend to 0. Informally speaking, it means that the limit value in discrete time does not exist, but the limit value in continuous time (i.e., when h tends to 0) exists. Such a situation is impossible in the case of stochastic games with perfect observation of the state.

Keywords: Stochastic games, Zero-sum stochastic games, Stochastic games with public signals, Shapley operator, Varying stage duration, Viscosity solution, Continuous-time Markov games, Limit value

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Notation:

- \mathbb{N}^* is the set of all positive integers; $\mathbb{N}_0 := \mathbb{N}^* \cup \{0\}$;
- $\mathbb{R}_+ := \{x : x \in \mathbb{R} \text{ and } x \geq 0\}$;
- If C is a finite set, then $\Delta(C)$ is the set of probability measures on C ;
- If X is a finite set, and $f, g : X \rightarrow \mathbb{R}$ are two functions, then $\langle f(\cdot), g(\cdot) \rangle := \sum_{x \in X} f(x)g(x)$;
If $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in X$, then $\langle x_1, x_2 \rangle := \sum_{i=1}^n x_i y_i$;
- If X is a finite set, $\zeta \in \Delta(X)$, and μ is a $|X| \times |X|$ matrix, then for any $x \in X$

$$(\zeta * \mu)(x) := \sum_{x' \in X} \zeta(x') \cdot \mu_{x'x};$$

- If I, J are sets and $g : I \times J \rightarrow \mathbb{R}$ is a function, then

$$\text{Val}_{I \times J}[g(i, j)] := \sup_{x \in \Delta(I)} \inf_{y \in \Delta(J)} \left(\int_{I \times J} g(i, j) dx(i) \otimes dy(j) \right) = \inf_{y \in \Delta(J)} \sup_{x \in \Delta(I)} \left(\int_{I \times J} g(i, j) dx(i) \otimes dy(j) \right).$$

I.e. $\text{Val}_{I \times J}[g(i, j)]$ is the value of a one-shot zero-sum game with action spaces I, J and with payoff function g . Note that we assume that the sets I, J, g are such that the value exists.

1 Introduction

Zero-sum stochastic games with perfect observation of the state were first defined in [Sha53]. Such a game is played in discrete time as follows. At each stage, player 1 and player 2 perfectly observe the current state and previous players' actions, and choose their mixed actions. Depending on players' actions and the current state, player 1 receives some payoff, player 2 receives the opposite of this payoff, and the next state is chosen according to a probability law. For a fixed discount factor $\lambda \in (0, 1]$ and a given starting state ω , player 1 wants to maximize the λ -discounted total payoff $E \left(\lambda \sum_{i=1}^{\infty} (1 - \lambda)^{i-1} g_i \right)$, where g_i is i -th stage payoff; player 2 wants to minimize it. Under some weak conditions maxmin and minmax coincide, and the resulting quantity is called the λ -discounted value denoted by $v_\lambda(\omega)$.

A similar model of stochastic games, in which players cannot perfectly observe the current state, is called stochastic games with public signals ([MSZ15, §IV.1]). In such games, the players can observe only the initial probability distribution p on the states, the previous actions, and a public signal given to players at the start of each stage. One can define the value $v_\lambda(p)$ in the same way as above. A particular case of such games are state-blind stochastic games, in which there is a unique signal.

An interesting question is what happens if the players are more and more patient, i.e., what happens with v_λ when λ tends to 0. In many frameworks $v_\lambda(\cdot)$ converges uniformly when λ tends to 0. This was proved for different types of stochastic games in [Eve58], [MZ71], [Koh74], [BK76], [AMS95], [Ros00], [RS01], [BGV15], [Zil23]. However, v_λ may diverge when λ tends to 0, even in simple frameworks. [Zil16b] shows an example of a finite state-blind stochastic game without the limit value. [RZ20] develops the previous paper, considering another example of a game without a limit value, similar to the one in [Zil16b]. See [Vig13] for an example of a game with perfect observation of the state and without a limit value, but with compact action spaces, finite state space, continuous payoff and transition functions. See also [SV15] for a paper about a general idea to construct counterexamples in different frameworks.

Zero-sum stochastic games with perfect observation of the state and with stage duration were introduced in [Ney13]. Given a stochastic game Γ (with perfect observation of the state) with stage duration 1, Neyman considers a family Γ_h of stochastic games in which players act at time $0, h, 2h, \dots$, and the payoffs and leaving probabilities are normalized at each stage, i.e. they are proportional to h . This gives the value $v_{h,\lambda}$ depending both on the discount factor λ and the stage duration h . It is interesting to consider small h , since in this case game with stage duration h approximates a game played in continuous time with λ -discounted payoff $\int_0^{+\infty} \lambda e^{-\lambda t} g_t dt$, see [Ney13, §3], [SV16, §8], [Nov24, §6] for more details.

Neyman considered in [Ney13] the asymptotics of $v_{h,\lambda}$ when either h or λ tends to 0. Among other things, he proved that when h is vanishing (i.e., h tends to 0), the value of a finite game with stage duration h converges. A corollary of this result states that the limit value (when the discount factor λ tends to 0) of the game with stage duration 1 should coincide with the limit value of the game with vanishing stage duration (i.e., $\lim_{\lambda \rightarrow 0} \lim_{h \rightarrow 0} v_{h,\lambda}$), and one exists if and only if another exists. Afterwards, the article [SV16] obtained new results; it considers a more general case in which the state and action spaces may be compact, and stage durations h_n may depend on the stage number n . In our paper, we also assume that stage duration may depend on the stage number.

The study of **zero-sum stochastic games with stage duration and public signals** is the main goal of this article. It is not evident how to generalize the definition of games with stage duration to the case of public signals. So, we first consider a particular class of stochastic games with public signals, and for this class we give a natural definition of games with stage duration. This particular case is going to be sufficient for all of our main results. Afterwards, we give a definition of games with stage duration for a general framework of stochastic games with public signals.

A particular case of games with stage duration are state-blind stochastic games with stage duration, which were introduced and studied in [Nov24]. An example of a finite state-blind stochastic game from [Nov24] shows in particular that the limit value (when the discount factor λ tends to 0) of the game with stage duration 1 *does not* coincide with the limit value of the game with vanishing stage duration (i.e., $\lim_{\lambda \rightarrow 0} \lim_{h \rightarrow 0} v_{h,\lambda}$).

The above example alludes that there may exist finite games with public signals, in which the limit value of the game with vanishing stage duration exists, but the limit value of the game with stage duration 1 *does not*. Our main result, **Theorem 1** from §4.4 gives an example of such a game. Informally speaking, it means that the limit value of the game played in discrete time does not exist, but the limit value of the game played in continuous time (i.e., when $\sup h_i$ tends to 0) exists. Such a situation is impossible in the case of stochastic games with perfect observation of the state.

The game from Theorem 1 is similar to the ones considered in [Zil16b] and [RZ20], which implies that there is no limit value in the case when stage duration is 1. The existence of the limit value of the game with vanishing stage duration is proved mainly by using a result from [Nov24] relating the values of λ -discounted state-blind games with vanishing stage duration and viscosity solutions of a differential equation, as well as techniques from [SV15].

2 Organization of the paper

In §3, we introduce a particular case of stochastic games with public signals that is enough to construct an example from Theorem 1.

In §4, we introduce stochastic games with stage duration. In §4.1, we introduce and give some known results about stochastic games with stage duration in the case of perfect observation of the state. In §4.2, we do the same in the state-blind case. In §4.3, we give a new definition of stochastic games with stage duration and with public signals. In §4.4, we give our main result about such games (Theorem 1). In §5, we give a detailed proof of this result.

In §6, we give some final comments. In §6.1, we give a definition of stochastic games with stage duration and with public signals for a general model of stochastic games with public signals. In §6.2, we give an alternative definition, which is not very natural but makes games with stage duration much easier to study. In §6.3, we briefly discuss why our main result also applies to the model from [Sor17], in which players act in a continuous-time game, but are allowed to take actions only at times t_1, t_2, t_3, \dots . In §6.4, we discuss some open questions and conjectures.

3 The model of zero-sum stochastic games with public signals, in which the signal depends only on the current state

A *zero-sum stochastic games with public signals, in which the signal depends only on the current state* is a 7-tuple $(A, \Omega, f, \{I_\alpha\}_{\alpha \in A}, \{J_\alpha\}_{\alpha \in A}, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$, where:

- A is a finite non-empty set of signals;
- Ω is a finite non-empty set of states;
- $f : \Omega \rightarrow A$ is a partition of Ω ;
- I_α (with $\alpha \in A$) is a finite non-empty set of actions of player 1, if the current state is in $f^{-1}(\alpha)$;
- J_α (with $\alpha \in A$) is a finite non-empty set of actions of player 2, if the current state is in $f^{-1}(\alpha)$;
- $g_m : \bigcup_{\omega \in \Omega} (I_{f(\omega)} \times J_{f(\omega)} \times \omega) \rightarrow \mathbb{R}$ is the m -th stage payoff function of player 1;
- $P_m : \bigcup_{\alpha \in A} (I_\alpha \times J_\alpha) \rightarrow \{\text{row-stochastic matrices } |\Omega| \times |\Omega|\}$ is the transition probability function at the m -th stage.

Recall that a matrix $A = (a_{ij})$ is called row-stochastic if $a_{ij} \geq 0$ for all i, j , and $\sum_j a_{ij} = 1$ for any fixed i .

We denote by $P_m(i, j)(\omega_a, \omega_b)$ the (ω_a, ω_b) -th element of the matrix $P_m(i, j)$.

The game $(A, \Omega, f, \{I_\alpha\}_{\alpha \in A}, \{J_\alpha\}_{\alpha \in A}, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$ proceeds in stages as follows. Before the first stage, an initial state $\omega_1 \in \Omega$ is chosen according to some probability law p_0 , and players receive the signal $\alpha_1 = f(\omega_1)$. At each stage $n \in \mathbb{N}^*$:

1. The current state is $\omega_n \in \Omega$. Players do not observe it, but they observe the signal $\alpha_n \in A$ and the actions of each other at previous stage;
2. Players choose their mixed actions. Player 1 chooses $x_n \in \Delta(I_{\alpha_n})$ and player 2 chooses $y_n \in \Delta(J_{\alpha_n})$;
3. An action $i_n \in I_{\alpha_n}$ of player 1 (respectively $j_n \in J_{\alpha_n}$ of player 2) is chosen according to the probability measure $x_n \in \Delta(I_{\alpha_n})$ (respectively $y_n \in \Delta(J_{\alpha_n})$);
4. Player 1 obtains a payoff $g_n = g_n(i_n, j_n, \omega_n)$, while player 2 obtains payoff $-g_n$. The new state ω_{n+1} is chosen according to the probability law $P(i_n, j_n)(\omega_n, \cdot)$. The new signal is $\alpha_{n+1} = f(\omega_{n+1})$.

The above description of the game is known to the players.

A state ω is called absorbing if for any i, j we have $P(i, j)(\omega, \omega) = 1$

Remark 1 (Special cases of the above construction). We fix a stochastic game $G = (A, \Omega, f, \{I_\alpha\}_{\alpha \in A}, \{J_\alpha\}_{\alpha \in A}, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$.

1. If A is a singleton, then G is called a *state-blind stochastic game*;
2. If $\alpha_n = \omega_n$ for all $n \in \mathbb{N}^*$, then the game is called a stochastic game with perfect observation of the state. Such games are widely studied in the literature.

Remark 2 (Games with infinite state space). In the construction above, we assumed that the state space Ω is finite. However, this is not always the case in this paper. If Ω is not finite, then instead of transition probability functions $P_k : I \times J \rightarrow \{\text{row-stochastic matrices } |\Omega| \times |\Omega|\}$ we are going to consider $\tilde{P}_k : I \times J \rightarrow \Omega \times \Omega$ such that:

1. For each $i \in I, j \in J, \omega_1, \omega_2 \in \Omega$ we have $\tilde{P}_k(i, j)(\omega_1, \omega_2) \geq 0$;
2. For each fixed i, j, ω_1 there is only finite number of ω_2 such that $\tilde{P}_k(i, j)(\omega_1, \omega_2) > 0$;
3. For each fixed i, j, ω_1 we have $\sum_{\omega_2 \in \Omega} \tilde{P}_k(i, j)(\omega_1, \omega_2) = 1$.

$\tilde{P}_k(i, j)(\omega_1, \omega_2)$ is still the probability to change the state from ω_1 to ω_2 , if at the current stage the players' action profile is (i, j) . So, we will still call $\tilde{P}_k(i, j)$ the transition probability functions. Note that if the state at the current stage is ω_1 , there is only finite number of possible states at the next stage. In the general model of stochastic games, this is not always the case, but we will need only this particular case.

Remark 3. In the original construction given by Lloyd Shapley, the payoff functions g_m and the transition probability functions P_m do not depend on the stage number. However, we need the dependency on the stage number.

Remark 4. Usually $I_\alpha = I$ and $J_\alpha = J$ for all $\alpha \in A$. It is easy to see that such a model is equivalent to the model presented here. (Indeed, one can add some "duplicate" actions to the sets I_β with $|I_\beta| < \max_{\alpha \in A} |I_\alpha|$).

By Remark 4, in the remaining of this section we assume (without loss of generality) that $I_\alpha = I$ and $J_\alpha = J$ for all $\alpha \in A$. So, we are going to talk about the stochastic game $(A, \Omega, f, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$.

A *history of length* $t \in \mathbb{N}$ for the stochastic game $(A, \Omega, f, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$ is $(\alpha_1, i_1, j_1, \alpha_2, i_2, j_2, \dots, \alpha_{t-1}, i_{t-1}, j_{t-1}, \alpha_t)$. The set of all histories of length t is $H_t := A \times (I \times J \times A)^{t-1}$. A (*behavior*) *strategy* of player 1 (respectively player 2) is a function $\sigma : \bigcup_{t \geq 1} H_t \rightarrow \Delta(I)$ (respectively $\tau : \bigcup_{t \geq 1} H_t \rightarrow \Delta(J)$). Players' strategies induce probability distribution on the set

$A \times (I \times J \times A)^{\mathbb{N}^*}$. (Indeed, strategies induce a probability distribution on the set H_1 , then on the set H_2 , etc. By Kolmogorov extension theorem, this probability can be extended in a unique way to the set $A \times (I \times J \times A)^{\mathbb{N}^*}$). In particular, given an initial probability distribution $p \in \Delta(\Omega)$, strategies $\sigma : \bigcup_{t \geq 1} H_t \rightarrow \Delta(I), \tau : \bigcup_{t \geq 1} H_t \rightarrow \Delta(J)$, and induced by them probability distribution $P_{\sigma, \tau}^p$ on $A \times (I \times J \times A)^{\mathbb{N}^*}$, we can consider an expectation $E_{\sigma, \tau}^p$ of a random variable on $\bigcup_{t \geq 1} H_t$.

Now, we need to choose how to compute a total payoff function.

Definition 1 (Stochastic game $\Gamma(\{b_m\}_{m \in \mathbb{N}^*})$ with payoff $\sum_{m=1}^{\infty} b_m g_m$). Fix a sequence $\{b_m\}_{m \in \mathbb{N}^*}$ with $b_m \geq 0$ ($m \in \mathbb{N}^*$) and with $0 < \sum_{m=1}^{\infty} b_m < \infty$. Fix also a stochastic game $\Gamma = (A, \Omega, f, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$. The (normalized) total payoff of stochastic game $\Gamma(\{b_m\}_{m \in \mathbb{N}^*})$ with payoff $\sum_{m=1}^{\infty} b_m g_m$ depends on a strategy profile (σ, τ) and initial distribution $p \in \Delta(\Omega)$, and is equal to

$$E_{\sigma, \tau}^p \left(\frac{1}{\sum_{m=1}^{\infty} b_m} \sum_{i=1}^{\infty} b_i g_i \right).$$

Remark 5 (λ -discounted game Γ^λ). The λ -discounted game Γ^λ is obtained if we set in the above definition $b_m = (1 - \lambda)^{m-1}$ for all $m \in \mathbb{N}^*$.

Analogously to single-shot zero-sum games, we may define the value and the (ε -)optimal strategies of stochastic games.

Definition 2. Fix a stochastic game $\Gamma = (A, \Omega, f, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$. The stochastic game $\Gamma(\{b_m\}_{m \in \mathbb{N}^*})$ with payoff $\sum_{m=1}^{\infty} b_m g_m$ is said to have a (normalized) value $v : \Delta(\Omega) \rightarrow \mathbb{R}$ if for all $p \in \Delta(\Omega)$ we have

$$v(\omega) = \sup_{\sigma} \inf_{\tau} E_{\sigma, \tau}^{\omega} \left(\frac{1}{\sum_{m=1}^{\infty} b_m} \sum_{i=1}^{\infty} b_i g_i \right) = \inf_{\tau} \sup_{\sigma} E_{\sigma, \tau}^{\omega} \left(\frac{1}{\sum_{m=1}^{\infty} b_m} \sum_{i=1}^{\infty} b_i g_i \right).$$

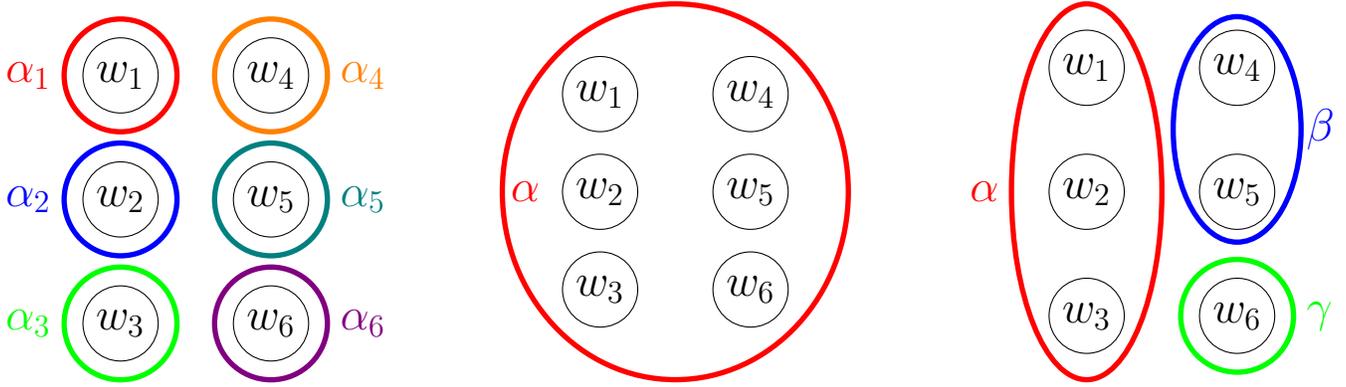
Sometimes instead of the transition probability function P , we consider the *kernel* $q : I \times J \rightarrow \{\text{matrices } |\Omega| \times |\Omega|\}$ defined by the expression

$$q(i, j)(\omega, \omega') = \begin{cases} P(i, j)(\omega, \omega') & \text{if } \omega \neq \omega'; \\ P(i, j)(\omega, \omega) - 1 & \text{if } \omega = \omega'. \end{cases}$$

In particular, we are sometimes going to define a stochastic game by using kernels instead of transition probability functions. Note that for any $\omega \in \Omega$ we have $\sum_{\omega' \in \Omega} q(i, j)(\omega, \omega') = 0$.

Now, let us discuss the above construction.

Figure 1 gives an example of a game with the same number of states, but with a different signalling structure on the states.



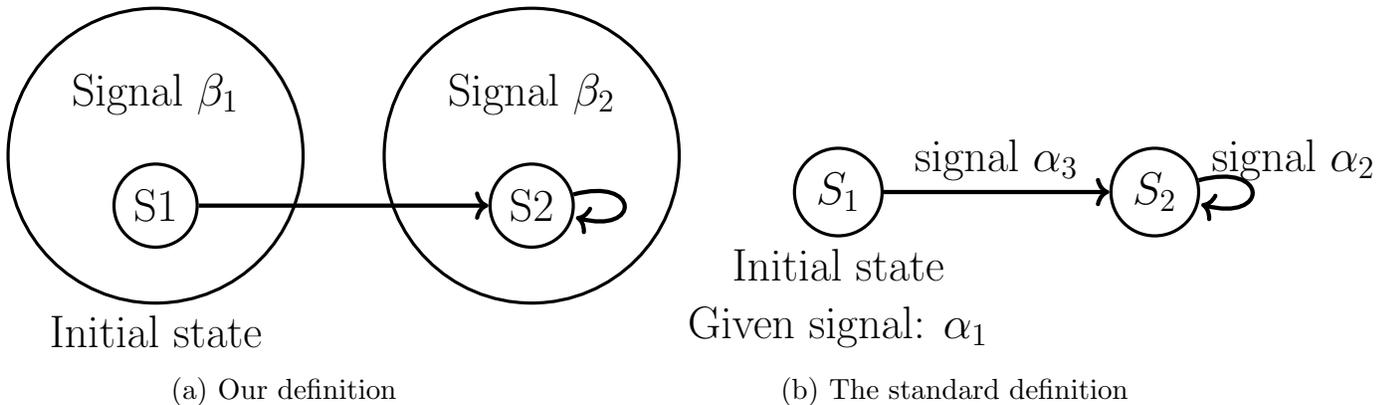
(a) The perfect observation of the state, i.e. there are 6 signals $\alpha_1, \dots, \alpha_6$; and $f(w_i) := \alpha_i$.

(b) The state-blind case. There is only one signal α , and $f(w_i) := \alpha$.

(c) Neither the perfect observation of the state nor the state-blind case. There are 3 signals, and
 $f(w_1) = f(w_2) = f(w_3) = \alpha$,
 $f(w_4) = f(w_5) = \beta$,
 $f(w_6) = \gamma$.

Figure 1: An example of different partition functions in a game with 6 states w_1, \dots, w_6 .

Remark 6 (Difference with the standard definition of stochastic games with public signals). Note that the above definition is not standard. Usually, there is no partition function f , and instead of function $P : I \times J \rightarrow \{\text{matrices } \Omega \times \Omega\}$ there is a function $P' : I \times J \times \Omega \rightarrow \Delta(\Omega \times A)$, and the next state and signal is chosen according to P' . (The first state and signal are chosen according to $p'_0 \in \Delta(\Omega \times A)$). Note that our definition is a particular case of the standard definition. To show this, consider a 0-players game with two states, S_1 and S_2 , where S_1 is the initial state. The probability to go from the state S_1 to S_2 is 1 and the probability to go from the state S_2 to S_1 is 1. In our definition, we need to specify a partition of the set of the states (See Figure 2a). In the standard definition, we need to specify a signal for each transition, and also an initial signal (See Figure 2b). If $\alpha_2 = \alpha_3$, then two definitions are equivalent (Take $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$). The case $\alpha_2 \neq \alpha_3$ cannot be reproduced with our definition.



Initial state

(a) Our definition

Initial state
Given signal: α_1

(b) The standard definition

Figure 2: Difference between the definition given by us and the standard definition.

4 Discounted zero-sum stochastic games with stage duration and public signals

The goal of this section is to define and state some results about games with stage duration and with public signals. We first recall two particular cases: the case of perfect observation of the state in §4.1 and state-blind case in §4.2.

In §4.3, we generalize the above notions for the setting of stochastic games with public signals, and in §4.4, we state our main result about such games.

Thus §4.1 and §4.2 consider already known notions and results, while §4.3 and §4.4 consider new notions and results.

Let us fix the notation that will be used during the entire section. Let T_∞ be a partition of \mathbb{R}_+ ; in other words, T_∞ is a strictly increasing sequence $\{t_n\}_{n \in \mathbb{N}^*}$ such that $t_1 = 0$ and $t_n \xrightarrow{n \rightarrow \infty} +\infty$. For each given partition T_∞ , denote $h_n = t_{n+1} - t_n$ for each $n \in \mathbb{N}^*$.

Throughout this entire subsection, we fix a stochastic game $(A, \Omega, f, I, J, g, q)$, where q is the kernel of the game. (Note that payoff function g and kernel q do not depend on the stage number).

4.1 Zero-sum stochastic games with stage duration in the case of perfect observation of the state

This subsection is based on [SV16, Ney13], and the text is mostly taken from [Nov24].

In a stochastic game with n -th stage duration h_n , state space Ω and strategy spaces I and J of player 1 and player 2 are independent of the partition T_∞ . Payoff function and kernel depend on T_∞ . For $n \in \mathbb{N}^*$, the n -th stage payoff function is $g_n = h_n g$, and the n -th stage kernel function is $q_n = h_n q$. The following definition summarizes this.

Definition 3. Given a stochastic game $(A, \Omega, f, I, J, g, q)$ with perfect observation of the state and a partition T_∞ of \mathbb{R}_+ , the *stochastic game with perfect observation of the state and with n -th stage duration h_n* is the stochastic game $(A, \Omega, f, I, J, \{h_n g\}_{n \in \mathbb{N}^*}, \{h_n q\}_{n \in \mathbb{N}^*})$ with perfect observation of the state.

In the rest of this subsection, we assume that all stochastic games are with perfect observation of the state, and we are going to omit the phrase “with perfect observation of the state”.

We are interested in behavior of games when the duration of each stage is vanishing, i.e., we want to know what happens when $\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0$.

Definition 4 (λ -discounted stochastic game with stage duration). Let α_h be an admissible family of discount factors with asymptotic discount rate λ .

λ -discounted stochastic game with n -th stage duration h_n is a stochastic game with payoff

$$\lambda \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i-1} (1 - \lambda h_j) \right) h_i g_i.$$

Denote by $v_{T_\infty, \lambda}$ the normalized value of such a game.

Remark 7 (Why such a definition of discounted stochastic games with stage duration?). We have chosen the total payoff in such a way that when all h_i are small, the value of the λ -discounted stochastic game with stage duration is close to the value of the analogous λ -discounted continuous-time game. See [Nov24, §3.3.3 and §6] for more details.

Now, we give some new results. We denote by $v_{1, \lambda}$ the value of the λ -discounted stochastic game in which each stage has duration 1. The following two propositions are about discounted games. The following proposition is [SV16, Proposition 7.4], it is a partial generalization of [Ney13, Theorem 1].

Proposition 1. For a fixed stochastic game $(A, \Omega, f, I, J, g, q)$, there is $K > 0$ such that for any $\lambda \in (0, +\infty)$ and partition T_∞ of \mathbb{R}_+ , we have

$$\left\| v_{T_\infty, \lambda} - v_{1, \frac{\lambda}{1+\lambda}} \right\| \leq K \sup_{i \in \mathbb{N}^*} h_i.$$

The following proposition is [SV16, Proposition 7.6].

Proposition 2. For a fixed stochastic game $(A, \Omega, f, I, J, g, q)$ there is $K > 0$ such that for any partition $T_\infty = \{t_n\}_{n \in \mathbb{N}^*}$ of \mathbb{R}_+ we have

$$\|v_{T_\infty, \lambda} - v_{1, \lambda}\| \leq K\lambda.$$

Recall that if X is a finite set, and $f, g : X \rightarrow \mathbb{R}$ are two functions, then $\langle f(\cdot), g(\cdot) \rangle := \sum_{x \in X} f(x)g(x)$.

Corollary 1. Fix a stochastic game $(A, \Omega, f, I, J, g, q)$. We have

1. The uniform limit $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}$ exists and is equal to $v_{1, \frac{\lambda}{1+\lambda}}$;
2. The uniform limit $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}$ exists and is a unique solution of the equation (in $v(\omega)$)

$$\lambda v(\omega) = \text{Val}_{I \times J} [\lambda g(i, j, \omega) + \langle q(i, j)(\omega, \cdot), v(\cdot) \rangle]; \quad (1)$$

3. For any partition T_∞ of \mathbb{R}_+ , the uniform limits $\lim_{\lambda \rightarrow 0} v_{T_\infty, \lambda}$ and $\lim_{\lambda \rightarrow 0} v_{1, \lambda}$ both exist or do not exist simultaneously. In the case of existence, for any two partitions T_∞^1, T_∞^2 of \mathbb{R}_+ we have $\lim_{\lambda \rightarrow 0} v_{T_\infty^1, \lambda} = \lim_{\lambda \rightarrow 0} v_{T_\infty^2, \lambda}$;
4. If the uniform limit $\lim_{\lambda \rightarrow 0} v_{1, \lambda}$ exists, then

$$\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} \left(\lim_{\lambda \rightarrow 0} v_{T_\infty, \lambda} \right) = \lim_{\lambda \rightarrow 0} \left(\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda} \right) = \lim_{\lambda \rightarrow 0} v_{1, \lambda}.$$

In particular, all the above limits exist. If the uniform limit $\lim_{\lambda \rightarrow 0} v_{1, \lambda}$ does not exist, then no of the above uniform limits exist.

Proof of Corollary 1. Assertions 1, 3, 4 follow directly from Propositions 1 and 2. Assertion 2 follows from assertion 1 by Proposition 4 below. \square

Afterwards, we are going to see that many of statements of the above corollary do not hold if there is no perfect observation of the state.

Remark 8. When defining stochastic games, we assumed that action sets I, J and a set of states Ω are finite. In this case it is proven in [BK76] that the limit value $\lim_{\lambda \rightarrow 0} v_{1, \lambda}$ always exists. However, we can also consider games in which I, J , or Ω are not finite. In this case, the existence of even the pointwise limit is not guaranteed. See [Zil16b] or [RZ20] for an example of a game with finite action spaces and compact state space without such a limit, and see [Vig13] or [Zil16b, §4.3] for an example of a game with compact action spaces and finite state space without such a limit. See also [SV15] for a construction which can build games without a limit value.

The above remark combined with Corollary 1(4) provides us with examples of games in which the pointwise limit $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}$ does not exist.

4.2 Zero-sum state-blind stochastic games with stage duration

In this subsection, we recall recent results from [Nov24].

Definition 5. Fix a zero-sum state-blind stochastic game $(A, \Omega, f, I, J, g, q)$, where q is the kernel. The *state-blind stochastic game with n -th stage duration h_n* is the state-blind stochastic game

$$(A, \Omega, f, I, J, \{h_m g\}_{m \in \mathbb{N}^*}, \{h_m q\}_{m \in \mathbb{N}^*}).$$

We can define λ -discounted total payoff as in §4.1.

Recall that if X is a finite set, $\zeta \in \Delta(X)$, and μ is a $|X| \times |X|$ matrix, then for any $x \in X$

$$(\zeta * \mu)(x) := \sum_{x' \in X} \zeta(x') \cdot \mu_{x'x}.$$

Definition 6. A function $u : \Delta(\Omega) \rightarrow \mathbb{R}$ is called a *viscosity solution* of the partial differential equation

$$\lambda v(p) = \mathbf{val}_{I \times J}[\lambda g(i, j, p) + \langle p * q(i, j), \nabla v(p) \rangle]$$

if:

1. for any \mathcal{C}^1 function $\psi : \Delta(\Omega) \rightarrow \mathbb{R}$ with $u - \psi$ having a strict local maximum at $\bar{w} \in \Omega$ we have $\lambda \psi(p) \leq \mathbf{val}_{I \times J}[\lambda g(i, j, p) + \langle p * q(i, j), \nabla \psi(p) \rangle]$;
2. for any \mathcal{C}^1 function $\psi : \Delta(\Omega) \rightarrow \mathbb{R}$ with $u - \psi$ having a strict local minimum at $\bar{w} \in \Omega$ we have $\lambda \psi(p) \geq \mathbf{val}_{I \times J}[\lambda g(i, j, p) + \langle p * q(i, j), \nabla \psi(p) \rangle]$.

The following proposition is [Nov24, Corollary 3], it is going to have an important role in the proof of the main result.

Proposition 3. If $(A, \Omega, f, I, J, g, q)$ is a state-blind stochastic game, then $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p)$ exists and is a unique viscosity solution of a partial differential equation (in $v(p)$)

$$\lambda v(p) = \mathbf{val}_{I \times J}[\lambda g(i, j, p) + \langle p * q(i, j), \nabla v(p) \rangle].$$

Remark 9. The state-blind case is more difficult than the case of perfect observation of the states. If we denote by T_h the Shapley operator (see §5.1.3) of the game in which each stage has duration h , then in the case of perfect observation of the state we have $T_h = hT_1 + (1-h)Id$. This makes studying of such games relatively easy. In the case of state-blind stochastic games, such an equality is not true anymore, and because of that many properties from Corollary 1 about stochastic games with perfect observation of the state do not hold anymore.

For instance, in [Nov24, Example 2] we have $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p) \neq v_{1, \frac{\lambda}{1+\lambda}}(p)$, so that Corollary 1(1) does not hold. Moreover, $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p) \neq v_{1, \lambda'}(p)$ for any $\lambda' \in (0, 1)$. Analogously, we have $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p) \neq \lim_{\lambda \rightarrow 0} v_{1, \lambda}(p)$, so that Corollary 1(4) does not hold.

4.3 Zero-sum stochastic games with stage duration and public signals

Definition 7. Fix a zero-sum stochastic game $(A, \Omega, f, I, J, g, q)$, in which the signal depends only on the current state, where q is the kernel. The *stochastic game with n -th stage duration h_n* is the stochastic game

$$(A, \Omega, f, I, J, \{h_m g\}_{m \in \mathbb{N}^*}, \{h_m q\}_{m \in \mathbb{N}^*}).$$

We can define λ -discounted total payoff as in §4.1.

Remark 10. In §6.1 later, we generalize this definition for a more general model of stochastic games with public signals.

4.4 Main result: an example of a game in which the pointwise limit $\lim_{\lambda \rightarrow 0} v_{1,\lambda}$ does not exist, but the uniform limit $\lim_{\lambda \rightarrow 0} \limsup_{i \in \mathbb{N}^*} h_i \rightarrow 0 v_{T_\infty, \lambda}$ exists

Remark 9 above alludes that there may be a stochastic game in which the limit value $\lim_{\lambda \rightarrow 0} v_{1,\lambda}$ does not exist, but the limit value of the game with vanishing stage duration ($\lim_{\lambda \rightarrow 0} \limsup_{\substack{i \in \mathbb{N}^* \\ h_1 + h_2 + \dots = +\infty}} h_i \rightarrow 0 v_{T_\infty, \lambda}$) exists. This is indeed the case, and we give an example of such a game here.

Example 1. Consider the following two-players zero-sum stochastic game G_1 with two public signals. The game has the states $+, ++, +^*, -, --, -^*$. The states $+^*$ and $-^*$ are absorbing. The states $+^*, +, ++$ have payoff 1, and the states $-^*, --, -$ have payoff -1 , for any actions of the players. There are two public signals PLUS and MINUS, and we have $f(++) = f(+) = f(+^*) = \text{PLUS}$ and $f(--) = f(-) = f(-^*) = \text{MINUS}$. Assume that the initial probability is $p = (p_1, p_2, p_3, p_4, p_5, p_6)$, where

$$p_i = \begin{cases} \text{Probability that the starting state is } +, & \text{if } i = 1; \\ \text{Probability that the starting state is } ++, & \text{if } i = 2; \\ \text{Probability that the starting state is } +^*, & \text{if } i = 3; \\ \text{Probability that the starting state is } -, & \text{if } i = 4; \\ \text{Probability that the starting state is } --, & \text{if } i = 5; \\ \text{Probability that the starting state is } -^*, & \text{if } i = 6. \end{cases}$$

See Figure 3.

In states $--, -, \text{ and } -^*$, the actions of player 1 are T, B, Q , and the actions of player 2 are L, R . In states $++, +, \text{ and } +^*$, the actions of player 1 are T, M, B , and the actions of player 2 are L, M, R, Q . The transition matrices for non-absorbing states are given by Tables 1–4:

	L	R
T	$\frac{1}{2}\{--\} + \frac{1}{2}\{-\}$	$--$
B	$--$	$\frac{1}{2}\{--\} + \frac{1}{2}\{-\}$
Q	$-^*$	$-^*$

Table 1: State $--$

	L	R
T	$-$	$--$
B	$--$	$-$
Q	$++$	$++$

Table 2: State $-$

	L	M	R	Q
T	$\frac{1}{3}\{++\} + \frac{2}{3}\{+\}$	$++$	$++$	$+^*$
M	$++$	$\frac{1}{3}\{++\} + \frac{2}{3}\{+\}$	$++$	$+^*$
B	$++$	$++$	$\frac{1}{3}\{++\} + \frac{2}{3}\{+\}$	$+^*$

Table 3: State $++$

	L	M	R	Q
T	$+$	$++$	$++$	$--$
M	$++$	$+$	$++$	$--$
B	$++$	$++$	$+$	$--$

Table 4: State $+$

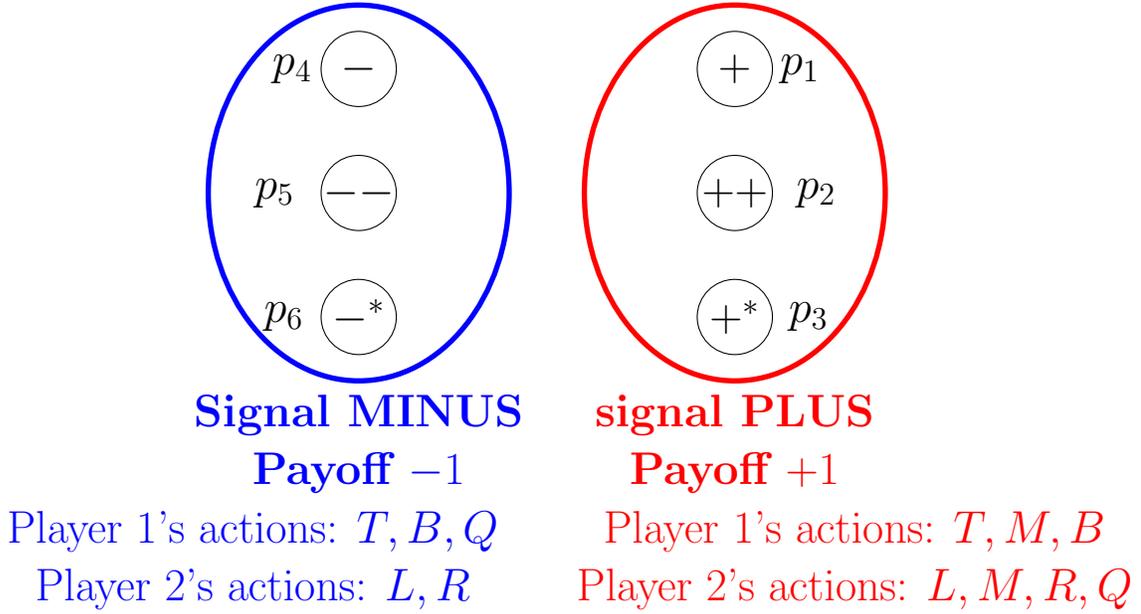


Figure 3: Signalling structure on the states. The letter next to a state is the probability that this is the initial state.

We denote by G_1^λ the game G_1 with λ -discounted payoff.

Theorem 1. For the game G_1^λ from Example 1:

1. The pointwise limit $\lim_{\lambda \rightarrow 0} v_{1,\lambda}(p)$ does not exist;
2. The uniform limit $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p)$ exists, and we have

$$\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p) = \begin{cases} \frac{7}{11}p_1 + \frac{7}{11}p_2 + p_3 - \frac{5}{11}p_4 - \frac{5}{11}p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} \geq \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and } \\ & \frac{p_5}{p_4+p_5} \geq \frac{2}{3} \text{ or } p_4 = p_5 = 0; \\ -\frac{5}{11}p_1 + p_2 + p_3 - \frac{5}{11}p_4 - \frac{5}{11}p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} < \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and } \\ & \frac{p_5}{p_4+p_5} \geq \frac{2}{3} \text{ or } p_4 = p_5 = 0; \\ \frac{7}{11}p_1 + \frac{7}{11}p_2 + p_3 - \frac{7}{11}p_4 + p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} \geq \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and } \\ & \frac{p_5}{p_4+p_5} < \frac{2}{3} \text{ or } p_4 = p_5 = 0; \\ -\frac{5}{11}p_1 + p_2 + p_3 - \frac{7}{11}p_4 + p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} < \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and } \\ & \frac{p_5}{p_4+p_5} < \frac{2}{3} \text{ or } p_4 = p_5 = 0. \end{cases}$$

5 The proof of Theorem 1

In this section, we present the proof of Theorem 1. In §5.1, we give some preliminaires. In §5.2, we show that the game from Example 1 has the same dynamics as a game from [RZ20, §3.4], for which

it is proven in [RZ20] that the pointwise limit $\lim_{\lambda \rightarrow 0} v_{1,\lambda}$ does not exist. In §5.3, we use Corollary 1 to prove that the uniform limit $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}$ exists and to find it.

5.1 Preliminaires

5.1.1 General model of stochastic games with public signals

A zero-sum stochastic game with public signals is a 6-tuple $(A, \Omega, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m^1\}_{m \in \mathbb{N}^*})$, where:

- A is a finite non-empty set of signals;
- Ω is a finite non-empty set of states;
- I is a finite non-empty set of actions of player 1;
- J is a finite non-empty set of actions of player 2;
- $g_m : I \times J \times \Omega \rightarrow \mathbb{R}$ is the m -th stage payoff function of player 1;
- $P_m^1 : I \times J \times \Omega \rightarrow \Delta(\Omega \times A)$ is the transition probability function in the m -th stage.

The game $(A, \Omega, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m^1\}_{m \in \mathbb{N}^*})$ proceeds in stages as follows. Before the first stage, an initial state $\omega_1 \in \Omega$ and an initial signal α_1 is chosen according to some probability law $p_0 \in \Delta(\Omega \times A)$. At each stage $n \in \mathbb{N}^*$:

1. The current state is $\omega_n \in \Omega$. Players do not observe it, but they observe the signal $\alpha_n \in A$ and the actions of each other at the previous stage;
2. Players choose their mixed actions. Player 1 chooses $x_n \in \Delta(I_{\alpha_n})$ and player 2 chooses $y_n \in \Delta(J_{\alpha_n})$;
3. An action $i_n \in I_{\alpha_n}$ of player 1 (respectively $j_n \in J_{\alpha_n}$ of player 2) is chosen according to the probability measure $x_n \in \Delta(I_{\alpha_n})$ (respectively $y_n \in \Delta(J_{\alpha_n})$);
4. Player 1 obtains a payoff $g_n = g_n(i_n, j_n, \omega_n)$, while player 2 obtains payoff $-g_n$. The new state ω_{n+1} and new signal is chosen according to the probability law $P_n^1(i_n, j_n, \omega_n)$.

The above description of the game is known to the players.

We can define strategies, λ -discounted and repeated finitely times games, total payoffs, values, in the same way as in §3.

We say that a state $\omega \in \Omega$ is controlled by player 1 if for any $i \in I, j_1, j_2 \in J, n \in \mathbb{N}^*$ we have $g_n(i, j_1, \omega) = g_n(i, j_2, \omega)$ and $P_n(i, j_1, \omega) = P_n(i, j_2, \omega)$. Analogously, a state ω is controlled by player 2 if for any $i_1, i_2 \in I, j \in J, n \in \mathbb{N}^*$ we have $g_n(i_1, j, \omega) = g_n(i_2, j, \omega)$ and $P_n(i_1, j, \omega) = P_n(i_2, j, \omega)$.

5.1.2 Any stochastic game with public signals is equivalent to a stochastic game with perfect observation of the state

Let (A, Ω, I, J, g, P) be a finite stochastic game with public signals (we use the general definition from §5.1.1).

The function $g^\gamma : I \times J \times \Delta(\Omega) \rightarrow \mathbb{R}$ is defined by $g^\gamma(i, j, p) = \sum_{\omega \in \Omega} p(\omega) g(i, j, \omega)$.

Now we define $P^\gamma : I \times J \times \Omega \rightarrow \Delta(A \times \Omega)$. For each $\omega \in \Omega, p \in \Delta(\Omega), i \in I, j \in J$ denote

$$P(i, j, p)(\omega, \alpha) = \sum_{\omega' \in \Omega} p(\omega') \cdot P(i, j, \omega')(\omega, \alpha).$$

If players have belief $p \in \Delta(\Omega)$ about the current state, then after playing $(i, j) \in I \times J$, the probability to receive the signal α is

$$P^\alpha(i, j, p) = \sum_{\omega \in \Omega} P(i, j, p)(\omega, \alpha).$$

If players have belief $p \in \Delta(\Omega)$ about the current state, then after playing $(i, j) \in I \times J$ and

receiving signal $\alpha \in A$ their posterior belief that the current state is ω is equal to

$$P(i, j, \alpha)(p, \omega) := \begin{cases} \frac{P(i, j, p)(\omega, \alpha)}{P^\alpha(i, j, p)} & , \text{ if } P^\alpha(i, j, p) \neq 0; \\ 0 & , \text{ otherwise.} \end{cases}$$

The function $P^\gamma : I \times J \rightarrow \Delta(\Omega) \times \Delta(\Omega)$ is defined by

$$P^\gamma(i, j)(p, p') = \begin{cases} P^\alpha(i, j, p) & , \text{ if } p'(\omega) = P_n(i, j, \alpha)(p, \omega) \text{ for all } \omega \in \Omega; \\ 0 & , \text{ otherwise.} \end{cases}$$

5.1.3 Shapley operator

Proposition 4 (Shapley operator for stochastic games with public signals with λ -discounted total payoff $\lambda \sum_{i=1}^{\infty} (1 - \lambda)^{i-1} g_i$). Consider the stochastic game (A, Ω, I, J, g, P) with public signals, where P is the probability distribution function. The value $v(p)$ of such a game exists and is a unique fixed point of the operator

$$T : \{\text{Continuous functions on } \Delta(\Omega)\} \rightarrow \{\text{Continuous functions on } \Delta(\Omega)\},$$

$$v(p) \mapsto \text{Val}_{I \times J} \left[\lambda g^\gamma(i, j, p) + (1 - \lambda) \sum_{p' \in \Delta(\Omega)} P^\gamma(i, j)(p, p') \cdot v(p') \right].$$

Remark 11. By §5.1.2 the sum in the proposition is finite, so the operator T is well-defined.

This proposition follows from §5.1.2 and from [MSZ15, Theorem IV.3.2].

5.2 The pointwise limit $\lim_{\lambda \rightarrow 0} v_{1, \lambda}$ does not exist

When considering stochastic games with signals in this and in the next subsection, we use the general definition of stochastic games with public signals from §5.1.1.

We consider a two-signals game \tilde{G}_1 from [RZ20], which does not have a value and whose dynamics is analogous to the dynamics of the game which is considered in Theorem 1.

The game \tilde{G}_1 has the states $+, ++, +^*, -, --, -^*$. The states $+, ++, +^*$ have payoff $+1$, and the states $-, --, -^*$ have payoff -1 , for any actions of the players. The states $+^*$ and $-^*$ are absorbing. The signal set is $\{\alpha, \beta\}$, and the action set for both players is $\{C, Q\}$. Before the first stage, the players receive the signal α (respectively β) if the current state is $+, ++, \text{ or } +^*$ (respectively $-, --, \text{ or } -^*$). The states $-, --$ are controlled by player 1, and the states $+, ++$ are controlled by player 2. The transition probabilities are described by Figure 4.

The arrow from state s_1 to the state s_2 with label (X, p, γ) tells that if player that controls state s_1 chooses action X , then with probability p he goes to state s_2 and receives signal γ .

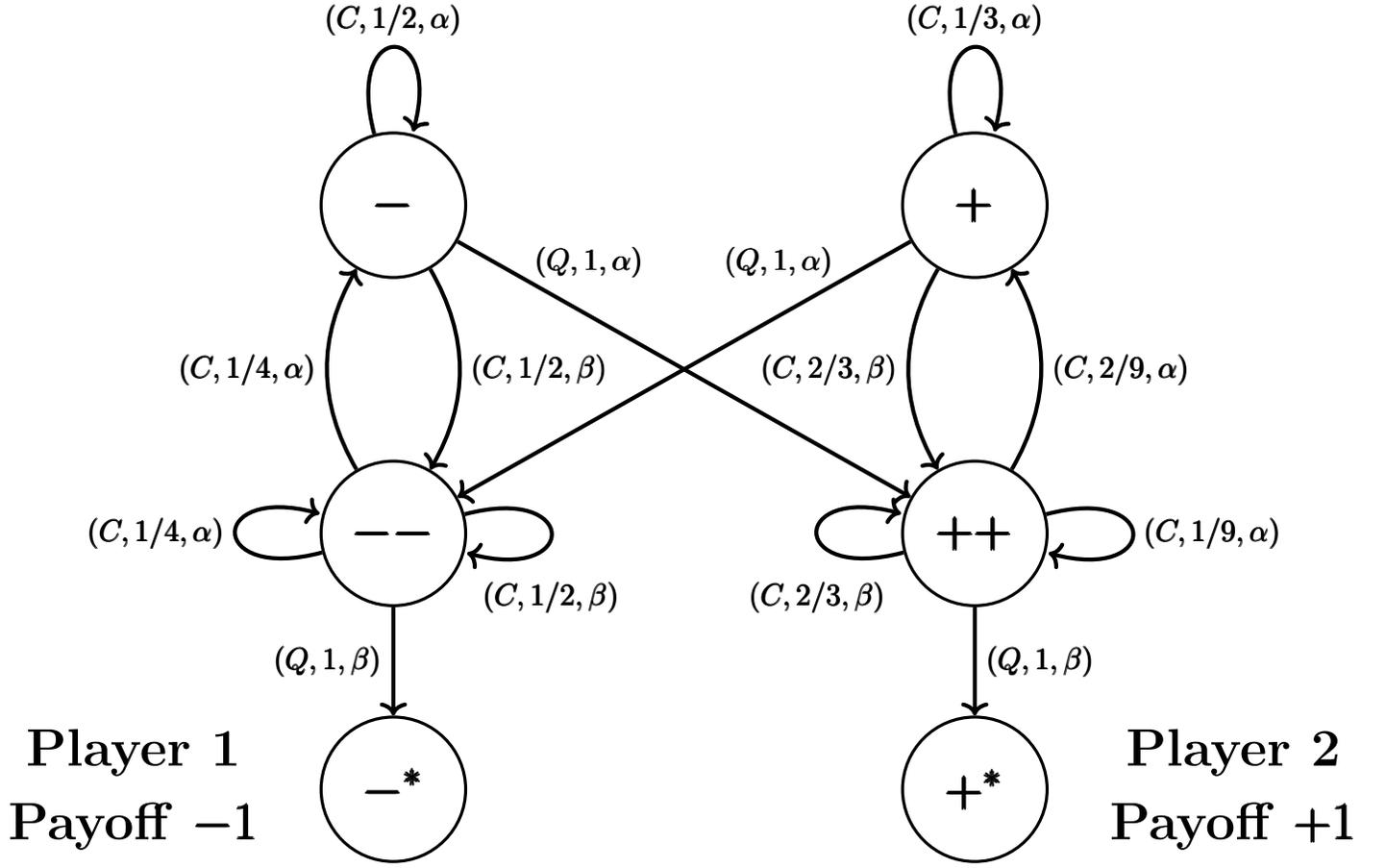


Figure 4: Transitions in the game \tilde{G}_1 .

Note that in this game, players know if they are in the left or in the right side of the game. If the current state is $-$ or $--$, and player 1 plays C , then players obtain public signal α with probability $1/2$ and public signal β with probability $1/2$. If the current state is $+$ or $++$, and player 2 plays C , then players obtain public signal α with probability $1/3$ and public signal β with probability $2/3$.

Denote by \tilde{v}_λ the value of λ -discounted game \tilde{G}_1^λ with starting state $--$.

Proposition 5. For each fixed $\lambda \in (0, 1]$, the value \tilde{v}_λ of the described above game \tilde{G}_1^λ coincides with the value $v_{1,\lambda}$ of the game G_1^λ from Theorem 1 with starting state $--$.

Proof. Assume that the initial probability distribution is $p = (p_1, p_2, p_3, p_4, p_5, p_6)$, where

$$p_i = \begin{cases} \text{Probability that the starting state is } ++, & \text{if } i = 1; \\ \text{Probability that the starting state is } +, & \text{if } i = 2; \\ \text{Probability that the starting state is } +*, & \text{if } i = 3; \\ \text{Probability that the starting state is } --, & \text{if } i = 4; \\ \text{Probability that the starting state is } -, & \text{if } i = 5; \\ \text{Probability that the starting state is } -*, & \text{if } i = 6. \end{cases}$$

Note that before the first stage, the players know if the current state belongs to the set $\{+, ++, +*\}$ or not. So, it is sufficient to consider two cases:

Case 1: initial probability is $p = (0, 0, 0, p_4, p_5, 0)$ with $p_4 + p_5 = 1$;

Case 2: initial probability is $p = (p_1, p_2, 0, 0, 0, 0)$ with $p_1 + p_2 = 1$.

Case 1: initial probability is $p = (0, 0, 0, p_4, p_5, 0)$.

By Proposition 4 the value \tilde{v}_λ of the game \tilde{G}_1^λ is a unique solution of the equation

$$\tilde{v}_\lambda(p) = -\lambda + (1 - \lambda) \max \left\{ \underbrace{\frac{1}{2}\tilde{v}_\lambda(p')}_{\text{Player 1 plays } C, \text{ the signal is } \alpha} + \underbrace{\frac{1}{2}\tilde{v}_\lambda(p'')}_{\text{Player 1 plays } C, \text{ the signal is } \beta}, \underbrace{p_5\tilde{v}_\lambda(p''')}_{\text{Player 1 plays } Q, \text{ the signal is } \alpha} + \underbrace{p_4(-1)}_{\text{Player 1 plays } Q, \text{ the signal is } \beta} \right\},$$

where

$$p' = \left(0, 0, 0, \frac{p_4}{2}, p_5 + \frac{p_4}{2}, 0\right); \quad p'' = (0, 0, 0, 1, 0, 0); \quad p''' = (1, 0, 0, 0, 0, 0).$$

Now, we consider the game G_1^λ from Theorem 1. Suppose that in the one-shot game from Proposition 4, player 2's optimal strategy is a mixed action $\frac{1}{2}L + \frac{1}{2}R$. In this case by Proposition 4 the value $v_{1,\lambda}(p)$ is a unique solution of the equation

$$v_{1,\lambda}(p) = -\lambda + (1 - \lambda) \max \left\{ \underbrace{\frac{1}{2}v_{1,\lambda}(p')}_{\text{Player 1 plays } T, \text{ Player 2 plays } L} + \underbrace{\frac{1}{2}v_{1,\lambda}(p'')}_{\text{Player 1 plays } T, \text{ Player 2 plays } R} \text{ OR } \underbrace{p_5v_{1,\lambda}(p''')}_{\text{Player 1 plays } Q}, \underbrace{p_4(-1)}_{\text{Player 1 plays } Q} \right\}. \quad (2)$$

OR
 Player 1 plays B , Player 2 plays R OR Player 1 plays B , Player 2 plays L

Thus if in the one-shot game from Proposition 4 player 2 plays the mixed action $\frac{1}{2}L + \frac{1}{2}R$, then we proved that $\tilde{v}_\lambda = v_{1,\lambda}$. So, it remains to prove that $\frac{1}{2}L + \frac{1}{2}R$ is an optimal strategy of player 2 in this one-shot game.

Note that if $\bar{p} = (0, 0, 0, \bar{p}_4, \bar{p}_5, 0)$ and if $\bar{p}_5 > p_5$, then from the structure of the game it follows that $v_{1,\lambda}(\bar{p}) \geq v_{1,\lambda}(p)$. Now, suppose that a player 2's optimal strategy in the above one shot game is $kL + (1 - k)R$, where $k \in [0, 1], k \neq 1/2$. Without the loss of generality assume that $k > 1/2$. In that case by playing a pure action T or a pure action Q , player 1 can obtain at least the same quantity as in (2).

Case 2: initial probability is $p = (p_1, p_2, 0, 0, 0, 0)$.

By Proposition 4 the value \tilde{v}_λ of the game \tilde{G}_1^λ is a unique solution of the equation

$$\tilde{v}_\lambda(p) = \lambda + (1 - \lambda) \min \left\{ \underbrace{\frac{1}{3}\tilde{v}_\lambda(p')}_{\text{Player 2 plays } C, \text{ the signal is } \alpha} + \underbrace{\frac{2}{3}\tilde{v}_\lambda(p'')}_{\text{Player 2 plays } C, \text{ the signal is } \beta}, \underbrace{p_2\tilde{v}_\lambda(p''')}_{\text{Player 2 plays } Q, \text{ the signal is } \alpha} + \underbrace{p_1(+1)}_{\text{Player 2 plays } Q, \text{ the signal is } \beta} \right\},$$

where

$$p' = \left(\frac{p_1}{3}, p_2 + \frac{2p_1}{3}, 0, 0, 0, 0\right); \quad p'' = (1, 0, 0, 0, 0, 0); \quad p''' = (0, 0, 0, 1, 0, 0).$$

Now, we consider the game G_1^λ from Theorem 1. Suppose that in the one-shot game from Proposition 4, player 1's optimal strategy is a mixed action $\frac{1}{3}T + \frac{1}{3}M + \frac{1}{3}B$. In this case by Proposition 4 the value $v_{1,\lambda}(p)$ is a unique solution of the equation

$$v_{1,\lambda}(p) = \lambda + (1 - \lambda) \min \left\{ \underbrace{\frac{1}{3}v_{1,\lambda}(p')}_{\text{Players' action profile is } (T,L) \text{ OR } (M,M) \text{ OR } (B,R)} + \underbrace{\frac{2}{3}v_{1,\lambda}(p'')}_{\text{Players' action profile is not } (T,L) \text{ OR } (M,M) \text{ OR } (B,R)}, \underbrace{p_2v_{1,\lambda}(p''')}_{\text{Player 2 plays } Q} + p_1(+1) \right\}.$$

Thus if in the one-shot game from Proposition 4 player 1 plays the mixed action $\frac{1}{3}T + \frac{1}{3}M + \frac{1}{3}B$, then we proved that $\tilde{v}_\lambda = v_{1,\lambda}$. One can prove that $\frac{1}{3}T + \frac{1}{3}M + \frac{1}{3}B$ is an optimal strategy of player 1 in this one-shot game in the same way as in case 1. \square

The following known result proves the first part of Theorem 1.

Proposition 6. [RZ20, §3.4]. $\lim_{\lambda \rightarrow 0} \tilde{v}_\lambda$ does not exist.

Remark 12. Considered by us game \tilde{G}_1 from [RZ20, §3.4] is very similar to a game from the paper [Zil16b, §2], which is the first counterexample of a finite stochastic game with public signals for which the limit value (when the discount fact λ tends to 0) does not exist.

5.3 The uniform limit $\lim_{\lambda \rightarrow 0} \lim_{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0} v_{T_\infty, \lambda}$ exists

The proof of this part is quite long, so we are going to cut it into 5 parts:

- In §5.3.1, we consider a dissection of the game into two state-blind “half-games”, each of which we are going to study.
- In §5.3.2, we consider a heuristic calculation of the value of each “half-game” with vanishing stage duration.
- In §5.3.3, we consider a formal calculation of the value of each “half-game” with vanishing stage duration, by checking that each heuristically obtained value from §5.3.2 is a classical solution of the partial differential equation from Proposition 3.
- In §5.3.4, we finish the proof by calculating the value of the initial game with vanishing stage duration through the values of two “half-games” with vanishing stage duration.
- In §5.3.5, we give some comments about the proof.

5.3.1 Start of the proof: Dissection of the game into two “half-games”

Here, we consider the game from Theorem 1 with n -th stage duration h_n , denoted by G_{T_∞} . The transition matrices at stage n for non-absorbing states are given by Tables 5–8:

	L	R
T	$(1 - \frac{1}{2}h_n) \{--\} + \frac{h_n}{2} \{-\}$	--
B	--	$(1 - \frac{1}{2}h_n) \{--\} + \frac{h_n}{2} \{-\}$
Q	$(1 - h_n) \{--\} + h_n \{-^*\}$	$(1 - h_n) \{--\} + h_n \{-^*\}$

Table 5: State --

	L	R
T	-	$(1 - h_n) \{-\} + h_n \{--\}$
B	$(1 - h_n) \{-\} + h_n \{--\}$	-
Q	$(1 - h_n) \{-\} + h_n \{++\}$	$(1 - h_n) \{-\} + h_n \{++\}$

Table 6: State -

	L	M	R	Q
T	$(1 - \frac{2h_n}{3}) \{++\} + \frac{2h_n}{3} \{+\}$	++	++	$(1 - h_n) \{++\} + h_n \{+^*\}$
M	++	$(1 - \frac{2h_n}{3}) \{++\} + \frac{2h_n}{3} \{+\}$	++	$(1 - h_n) \{++\} + h_n \{+^*\}$
B	++	++	$(1 - \frac{2h_n}{3}) \{++\} + \frac{2h_n}{3} \{+\}$	$(1 - h_n) \{++\} + h_n \{+^*\}$

Table 7: State ++

	L	M	R	Q
T	$+$	$(1 - h_n)\{+\} + h_n\{++\}$	$(1 - h_n)\{+\} + h_n\{++\}$	$(1 - h_n)\{+\} + h_n\{--\}$
M	$(1 - h_n)\{+\} + h_n\{++\}$	$+$	$(1 - h_n)\{+\} + h_n\{++\}$	$(1 - h_n)\{+\} + h_n\{--\}$
B	$(1 - h_n)\{+\} + h_n\{++\}$	$(1 - h_n)\{+\} + h_n\{++\}$	$+$	$(1 - h_n)\{+\} + h_n\{--\}$

Table 8: State $+$

Now, for each partition of T_∞ , we are going to consider two families of auxiliary two-players state-blind games, $G_{T_\infty}^-(k_-)$ and $G_{T_\infty}^+(k_+)$.

Definition 8. The game $G_{T_\infty}^-(k_-)$ (where $k_- \in [-1, +1]$) is a game with states $-$, $--$, $-^*$, k_-^* . The states $-$, $--$, $-^*$ have n -th stage payoff $-h_n$, while the state k_-^* has n -th stage payoff k_-h_n , for any actions of the players. The states $-^*$ and k_-^* are absorbing. In states $--$, $-$, the actions of player 1 are T , B , Q , and the actions of player 2 are L , R . The transition matrices for non-absorbing states are given by Tables 5–6, but in Table 6 one needs to replace $++$ by k_-^* .

Definition 9. The game $G_{T_\infty}^+(k_+)$ (where $k_+ \in [-1, +1]$) is a game with states $+$, $++$, $+^*$, k_+^* . The states $+$, $++$, $+^*$ have n -th stage payoff $+h_n$, while the state k_+^* has n -th stage payoff k_+h_n , for any actions of the players. The states $-^*$ and k_+^* are absorbing. The actions of player 1 are T , M , B , and the actions of player 2 are L , M , R , Q . The transition matrices for non-absorbing states are given by Tables 7–8, but in Table 8 one needs to replace $--$ by k_+^* .

Informally speaking, G_{T_∞} is a combination of two "half-games" $G_{T_\infty}^-(k_-)$ and $G_{T_\infty}^+(k_+)$, see Figure 5.

We denote by $G_{T_\infty, \lambda}$ the game G_{T_∞} with a λ -discounted payoff. We denote by $v_{T_\infty}^-(k, p)$ (respectively, $v_{T_\infty}^+(k, p)$) the value of the game $G_{T_\infty, \lambda}^-(k)$ (respectively, $G_{T_\infty, \lambda}^+(k)$) with a λ -discounted payoff, if starting state is $--$ (respectively, $++$) with probability $p \in [0, 1]$ and $-$ (respectively, $+$) with probability $1 - p$.

Lemma 1. For any $p \in [0, 1]$ and any $k_-, k_+ \in [-1, 1]$, we have $v_{T_\infty, \lambda}^-(k_-, p) = (k_- + 1)v_{T_\infty, \lambda}^-(0, p) + k_-$ and $v_{T_\infty, \lambda}^+(k_+, p) = (1 - k_+)v_{T_\infty, \lambda}^+(0, p) + k_+$.

Proof. Note that the structure of the game $G_{T_\infty, \lambda}^-(0)$ and the fact that $k_- \geq -1$ implies that any optimal in the game $G_{T_\infty, \lambda}^-(0)$ strategy is also optimal in the game $G_{T_\infty, \lambda}^-(k_-)$. Thus there exists $\alpha \in [0, 1]$ such that for any $k_- \in [-1, 1]$ we have $v_{T_\infty, \lambda}^-(k_-, p) = \alpha k_- + (-1)(1 - \alpha)$. By taking $k_- = 0$, we obtain

$$v_{T_\infty, \lambda}^-(0, p) = -1 + \alpha \iff \alpha = v_{T_\infty, \lambda}^-(0, p) + 1.$$

Analogously, any optimal in the game $G_{T_\infty, \lambda}^+(0)$ strategy is also optimal in the game $G_{T_\infty, \lambda}^+(k_+)$ as long as $k_+ \leq 1$. Thus there exists $\beta \in [0, 1]$ such that for any $k_+ \in [-1, 1]$ we have $v_{T_\infty, \lambda}^+(k_+, p) = \beta k_+ + 1(1 - \beta)$. By taking $k_+ = 0$, we obtain

$$v_{T_\infty, \lambda}^+(0, p) = 1 - \beta \iff \beta = 1 - v_{T_\infty, \lambda}^+(0, p).$$

From this we easily obtain the assertion of the proposition. \square

Thus if we want to find uniform limits $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^-(k_-, p)$ and $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^+(k_+, p)$ with $k_-, k_+ \in [0, 1]$, it is sufficient to find $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^-(0, p)$ and $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^+(0, p)$. In the next lemma, we are going to find these two functions.

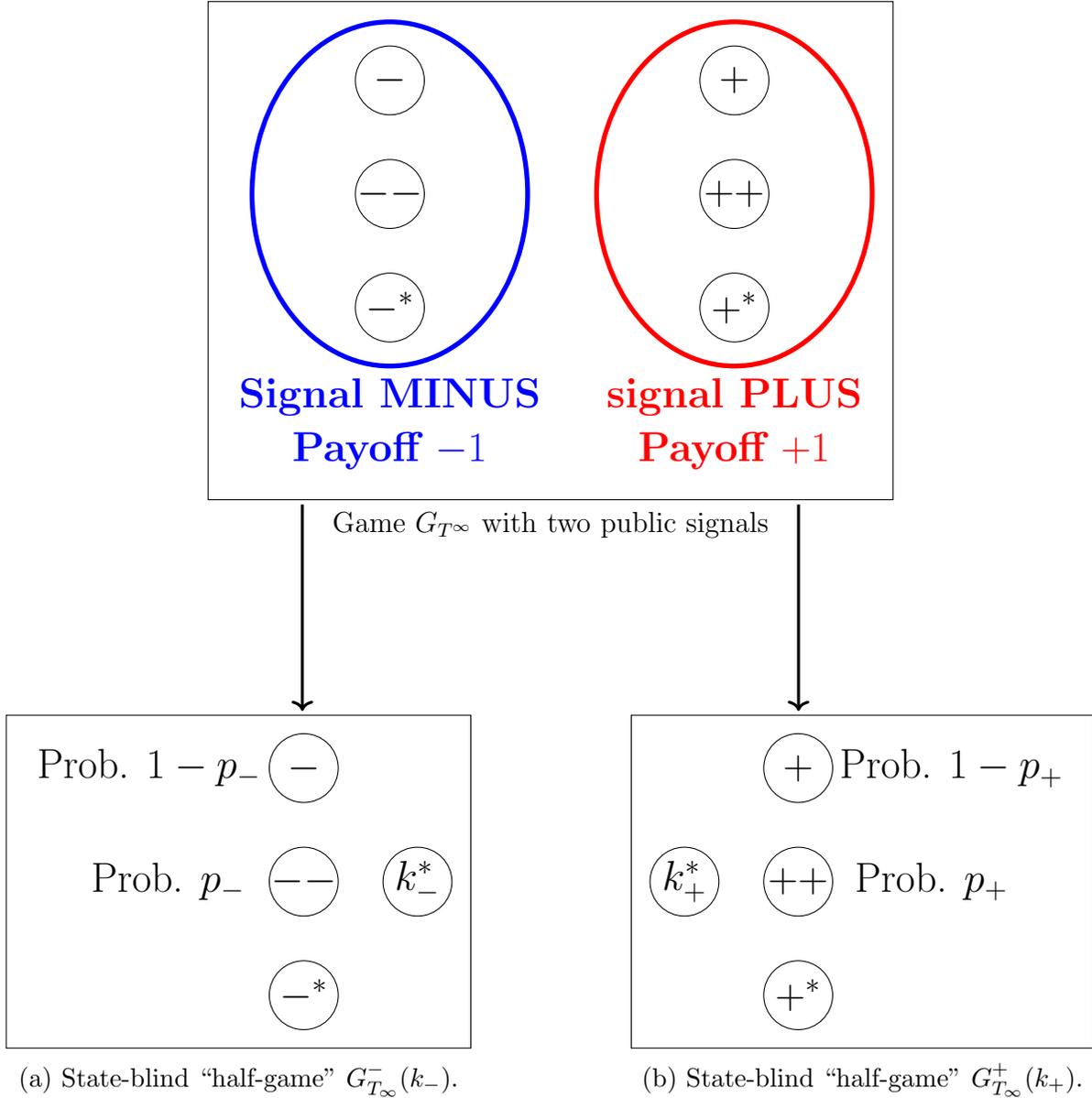


Figure 5: Game G_{T^∞} with two public signals, and received out of it state-blind “half-games” $G_{T^\infty}^-(k_-)$ and $G_{T^\infty}^+(k_+)$.

Lemma 2. We have

$$\begin{aligned}
 1) \quad \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T^\infty, \lambda}^-(0, p) = w_\lambda^-(p) &:= \begin{cases} -\frac{p+\lambda}{1+\lambda} & , \text{ if } p < \frac{4\lambda+2}{4\lambda+3}; \\ -1 + \frac{(4\lambda)^{4\lambda/3}}{(1+\lambda)(3+4\lambda)^{1+(4\lambda/3)}} (3p-2)^{-4\lambda/3} & , \text{ if } p \geq \frac{4\lambda+2}{4\lambda+3}. \end{cases} \\
 2) \quad \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T^\infty, \lambda}^+(0, p) = w_\lambda^+(p) &:= \begin{cases} \frac{p+\lambda}{1+\lambda} & , \text{ if } p < \frac{9\lambda+6}{9\lambda+8}; \\ 1 - \frac{2(18\lambda)^{9\lambda/8}}{(1+\lambda)(8+9\lambda)^{1+(9\lambda/8)}} (8p-6)^{-9\lambda/8} & , \text{ if } p \geq \frac{9\lambda+6}{9\lambda+8}. \end{cases}
 \end{aligned}$$

Both limits are uniform.

This lemma is very important in the proof of Theorem 1. We will give a formal proof later, because this proof only verifies that $w_\lambda^-(p)$ and $w_\lambda^+(p)$ satisfy some partial differential equations, but does it not show how one can come up with these functions. So, first we are going to give an informal proof.

5.3.2 Heuristic calculation of the value of each “half-game” with vanishing stage duration

With the help of games similar to the game \tilde{G}_1 from the previous subsection, we will derive $w_\lambda^-(p)$ and $w_\lambda^+(p)$.

Heuristic (not a strict) proof of Lemma 2.

We consider two families (each parametrized by $h \in (0, 1]$) of three-signals games, \tilde{G}_h^- and \tilde{G}_h^+ respectively, whose dynamics is analogous to the dynamics of games $G_{T^\infty}^-(0)$ and $G_{T^\infty}^+(0)$ respectively.

The game \tilde{G}_h^- (respectively \tilde{G}_h^+) has action space $\{C, Q\}$, state space $\{-, --, -*, 0^*\}$ (respectively $\{+, ++, +*, 0^*\}$), and the signal set is $\{\alpha, \beta, \delta\}$. The states $-$, $--$, $-*$ have payoff $-h$, the states $+$, $++$, $+*$ have payoff $+h$, and the state 0^* has payoff 0, for any actions of the players. The states $-*$, $+*$, 0^* are absorbing. The states $-$, $--$ are controlled by player 1, and the states $+$, $++$ are controlled by player 2. The transition probabilities are described by Figure 6.

The arrow from state s_1 to the state s_2 with label (X, p, γ) tells that if player that controls state s_1 plays action X , then with probability p he goes to state s_2 and receives signal γ .

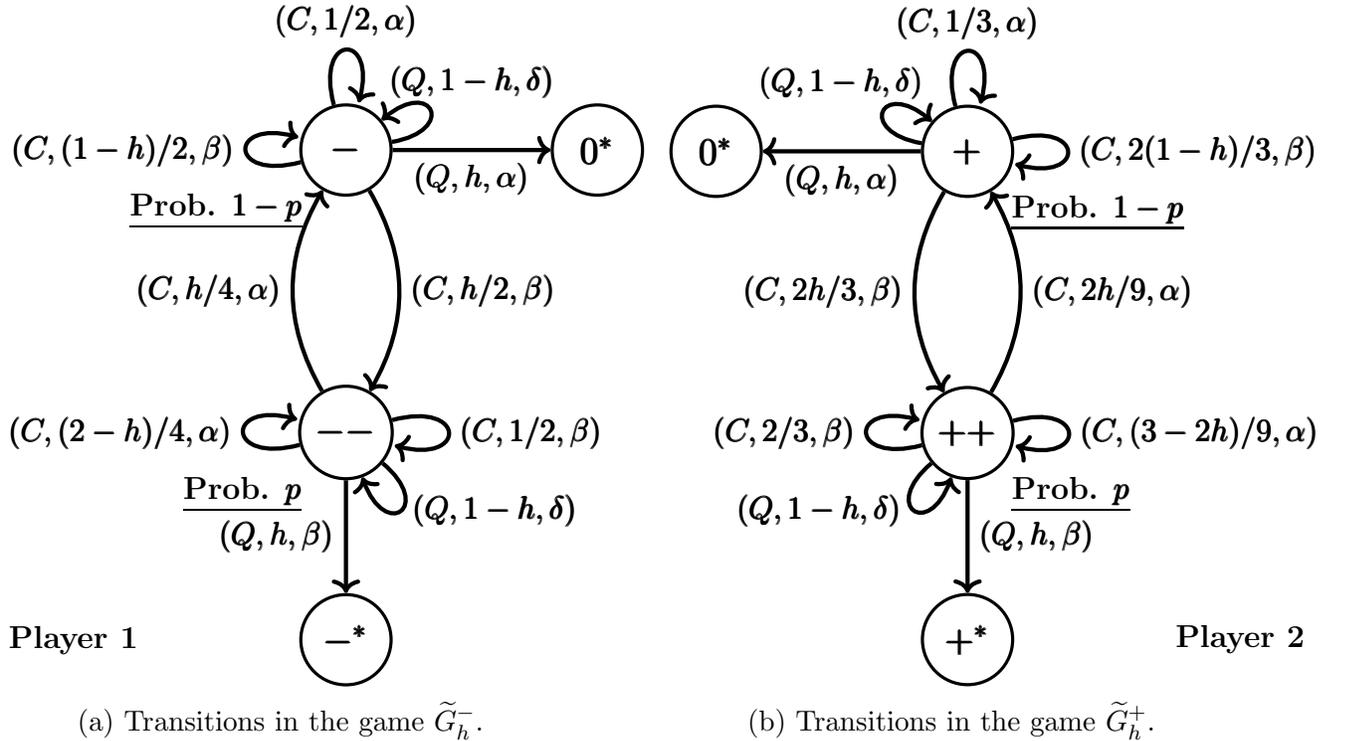


Figure 6: Transitions in games \tilde{G}_h^- and \tilde{G}_h^+ .

Informally speaking, the game \tilde{G}_1^- is the left half of the game \tilde{G}_1 (considered in §5.2), while \tilde{G}_1^+ is the right half of the game \tilde{G}_1 .

We denote by $\tilde{G}_{h,\lambda}^-$ (respectively $\tilde{G}_{h,\lambda}^+$) the game \tilde{G}_h^- (respectively \tilde{G}_h^+) with a λ -discounted payoff, and denote by $\tilde{v}_-^{h,\lambda}(p)$ (respectively $\tilde{v}_+^{h,\lambda}(p)$) the value of the game $\tilde{G}_{h,\lambda}^-$ (respectively $\tilde{G}_{h,\lambda}^+$).

Proposition 7. For each fixed $\lambda \in (0, 1]$ and $h \in [0, 1]$, the value $\tilde{v}_-^{h,\lambda}$ of the game $\tilde{G}_{h,\lambda}^-$ coincides with the value $v_{h,\lambda}^-$ of the game $G_{T^\infty,\lambda}^-(0)$, in which each stage has duration h .

Proposition 8. For each fixed $\lambda \in (0, 1]$ and $h \in [0, 1]$, the value $\tilde{v}_+^{h,\lambda}$ of the game $\tilde{G}_{h,\lambda}^+$ coincides with the value $v_{h,\lambda}^+$ of the game $G_{T^\infty,\lambda}^+(0)$, in which each stage has duration h .

The proof of these propositions are very similar to the proof of Proposition 5.

Remark 13. Note that the game \tilde{G}_h^- (respectively \tilde{G}_h^+) is *not* a game \tilde{G}_1^- (respectively \tilde{G}_1^+), in which the duration of each stage is h . See Figure 7.

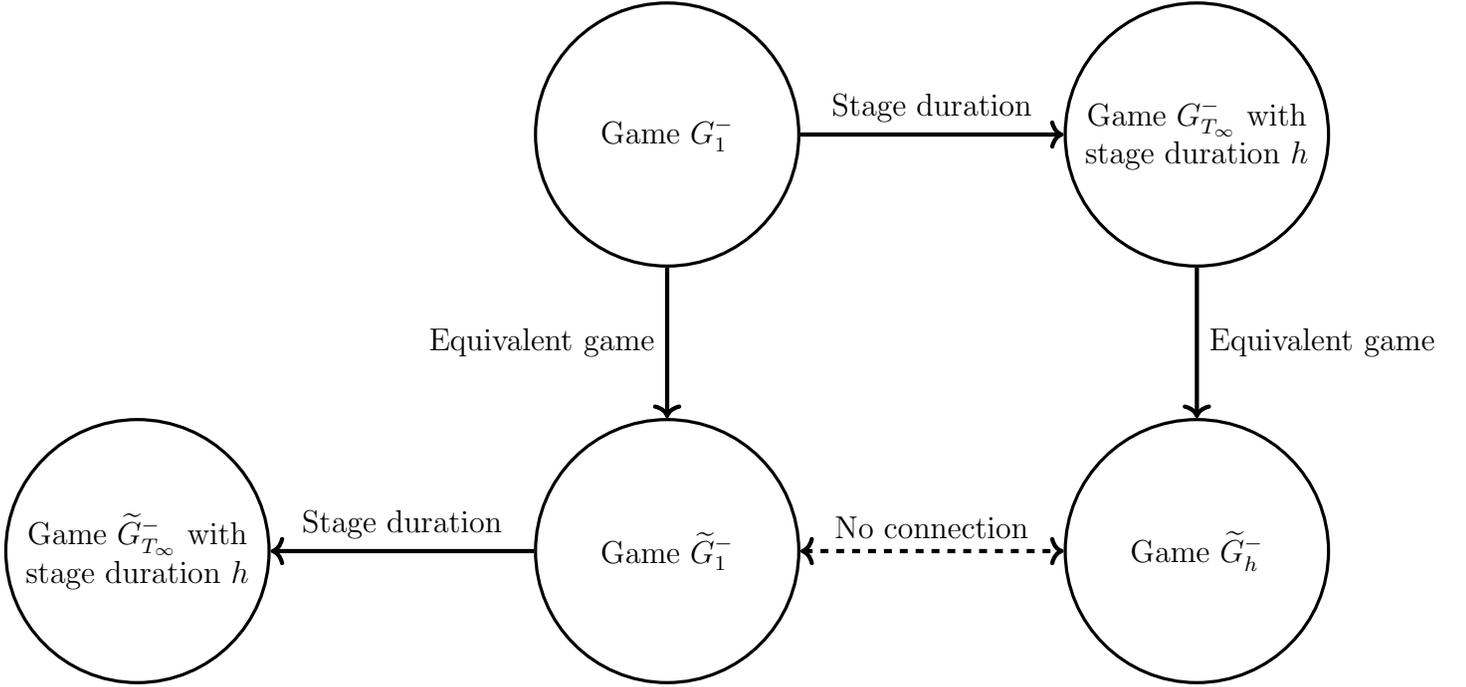


Figure 7: Connection between the game $G_{T_\infty}^-$ and the game \tilde{G}_1^- . The analogous is true for games $G_{T_\infty}^+$ and \tilde{G}_1^+ .

Note that in the game \tilde{G}_h^- , if player 1 takes action C , then both players obtain public signal α with probability $1/2$ and public signal β with probability $1/2$, while in the game \tilde{G}_h^+ , if player 2 takes action C , then both players obtain public signal α with probability $1/3$ and public signal β with probability $2/3$.

The value $\tilde{v}_-^{h,\lambda}$ of the λ -discounted game $\tilde{G}_{h,\lambda}^-$. We assume that the starting state is $--$ with probability $p \in [0, 1]$ and $-$ with probability $1 - p$.

By Proposition 4 we have

$$\tilde{v}_-^{h,\lambda}(p) = -\lambda h + (1 - \lambda h) \max \left\{ \underbrace{-hp}_{\text{Player 1 plays } Q, \text{ the signal is } \alpha \text{ or } \beta} + \underbrace{(1 - h)\tilde{v}_-^{h,\lambda}(p)}_{\text{Player 1 plays } Q, \text{ the signal is } \delta}; \underbrace{\frac{1}{2}\tilde{v}_-^{h,\lambda}\left(p - \frac{hp}{2}\right)}_{\text{Player 1 plays } C, \text{ the signal is } \alpha} + \underbrace{\frac{1}{2}\tilde{v}_-^{h,\lambda}(p + h - hp)}_{\text{Player 1 plays } C, \text{ the signal is } \beta} \right\}. \quad (3)$$

We assume that there exists $p^* \in [0, 1]$ such that for $p \leq p^*$ player 1 prefers to play Q , and for $p > p^*$ player 1 prefers to play C . In that case, we have for $p \leq p^*$

$$-\lambda h + (1 - \lambda h) \left(-hp + (1 - h)\tilde{v}_-^{h,\lambda}(p) \right) = \tilde{v}_-^{h,\lambda}(p) \iff \tilde{v}_-^{h,\lambda}(p) = \frac{(h\lambda - 1)p - \lambda}{1 + (1 - h)\lambda}.$$

p^* is an approximate solution of the equation

$$-\lambda h + (1 - \lambda h) \left(-hp + (1 - h) \frac{(h\lambda - 1)p - \lambda}{1 + (1 - h)\lambda} \right) = \\ -\lambda h + (1 - \lambda h) \left(\frac{(h\lambda - 1) \left(p - \frac{hp}{2} \right) - \lambda}{2(1 + (1 - h)\lambda)} + \frac{(h\lambda - 1)(p + h - hp) - \lambda}{2(1 + (1 - h)\lambda)} \right),$$

from which it is easy to find p^* and see that

$$p^* = \frac{4\lambda + 2 - 2\lambda h}{4\lambda + 3 - 7\lambda h} \xrightarrow{h \rightarrow 0} \frac{4\lambda + 2}{4\lambda + 3}.$$

So we take $p^* = \frac{4\lambda + 2}{4\lambda + 3}$. For $p \geq p^*$, $\tilde{v}_-^{h,\lambda}(p)$ is a solution of the equation (in $f(p)$)

$$f(p) = -\lambda h + (1 - \lambda h) \left(\frac{1}{2} f \left(p - \frac{hp}{2} \right) + \frac{1}{2} f(p + h - hp) \right).$$

Assuming that $\tilde{v}_-^{h,\lambda}(p)$ is differentiable if $p > p^*$, we have

$$\tilde{v}_-^{h,\lambda}(p) = -\lambda h + (1 - \lambda h) \left(\frac{1}{2} \left(\tilde{v}_-^{h,\lambda}(p) - \frac{1}{2} hp \left(\tilde{v}_-^{h,\lambda} \right)'(p) \right) + \frac{1}{2} \left(\tilde{v}_-^{h,\lambda}(p) + (h - hp) \left(\tilde{v}_-^{h,\lambda} \right)'(p) \right) \right) + o(h).$$

Thus we have for small h

$$\begin{cases} \lambda \tilde{v}_-^{h,\lambda}(p) \approx -\lambda - \frac{1}{4} p \left(\tilde{v}_-^{h,\lambda} \right)'(p) + \frac{1}{2} (1 - p) \left(\tilde{v}_-^{h,\lambda} \right)'(p), & \text{if } p \in (p^*, 1); \\ \tilde{v}_-^{h,\lambda}(p^*) = \frac{-p^* - \lambda}{1 + \lambda}. \end{cases}$$

By solving the differential equation, we obtain for small h

$$\tilde{v}_-^{h,\lambda}(p) = -1 + C(3p - 2)^{-4\lambda/3},$$

where $C \in \mathbb{R}$. Taking into account the border condition, we obtain for small h

$$\tilde{v}_-^{h,\lambda}(p) = -1 + \frac{(4\lambda)^{4\lambda/3}}{(1 + \lambda)(3 + 4\lambda)^{1 + (4\lambda/3)}} (3p - 2)^{-4\lambda/3}.$$

Thus we have for small h

$$\tilde{v}_-^{h,\lambda}(p) \approx \begin{cases} -\frac{p + \lambda}{1 + \lambda} & , \text{if } p < \frac{4\lambda + 2}{4\lambda + 3}; \\ -1 + \frac{(4\lambda)^{4\lambda/3}}{(1 + \lambda)(3 + 4\lambda)^{1 + (4\lambda/3)}} (3p - 2)^{-4\lambda/3} & , \text{if } p \geq \frac{4\lambda + 2}{4\lambda + 3}. \end{cases} \quad (4)$$

The value $\tilde{v}_+^{h,\lambda}$ of the λ -discounted game $\tilde{G}_{h,\lambda}^+$. We assume that the starting state is ++ with probability $p \in [0, 1]$ and + with probability $1 - p$.

By Proposition 4 we have

$$\tilde{v}_+^{h,\lambda}(p) = -\lambda h + (1 - \lambda h) \min \left\{ \underbrace{hp}_{\substack{\text{Player 2 plays } Q, \\ \text{the signal is } \alpha \text{ or } \beta}} + \underbrace{(1 - h)\tilde{v}_+^{h,\lambda}(p)}_{\substack{\text{Player 2 plays } Q, \\ \text{the signal is } \delta}}; \underbrace{\frac{1}{3}\tilde{v}_+^{h,\lambda} \left(p - \frac{2hp}{3} \right)}_{\substack{\text{Player 2 plays } C, \\ \text{the signal is } \alpha}} + \underbrace{\frac{2}{3}\tilde{v}_+^{h,\lambda}(p + h - hp)}_{\substack{\text{Player 2 plays } C, \\ \text{the signal is } \beta}} \right\}. \quad (5)$$

Now, by doing calculations analogous to the ones given above, we obtain that

$$\tilde{v}_+^{h,\lambda}(p) \approx \begin{cases} \frac{p + \lambda}{1 + \lambda} & , \text{if } p < \frac{9\lambda + 6}{9\lambda + 8}; \\ 1 - \frac{2(18\lambda)^{9\lambda/8}}{(1 + \lambda)(9\lambda + 8)^{1 + (9\lambda/8)}} (8p - 6)^{-9\lambda/8} & , \text{if } p \geq \frac{9\lambda + 6}{9\lambda + 8}. \end{cases} \quad (6)$$

□

5.3.3 Strict calculation of the value of each “half-game” with vanishing stage duration

Now, we give a formal proof.

Proof of Lemma 2. We prove assertion 1). First, note that $w_\lambda^-(p)$ is continuously differentiable. If $p \neq \frac{4\lambda+2}{4\lambda+3}$, then it is clear. To prove the differentiability at $p = \frac{4\lambda+2}{4\lambda+3}$, one may verify that

$$\left(-\frac{p+\lambda}{1+\lambda}\right)' \left(\frac{4\lambda+2}{4\lambda+3}\right) = \left(-1 + \frac{(4\lambda)^{4\lambda/3}}{(1+\lambda)(3+4\lambda)^{1+(4\lambda/3)}} (3p-2)^{-4\lambda/3}\right)' \left(\frac{4\lambda+2}{4\lambda+3}\right) = -\frac{1}{1+\lambda}.$$

Denote $I = \{T, B, Q\}, J = \{L, R\}$. Denote also by $\bar{p} = (p_1, p_2, p_3, p_4)$ a probability distribution on the states of the game $G_{T_\infty, \lambda}^-(0)$. Here

$$p_i = \begin{cases} \text{Probability that the starting state is } -, & \text{if } i = 1; \\ \text{Probability that the starting state is } --, & \text{if } i = 2; \\ \text{Probability that the starting state is } -*, & \text{if } i = 3; \\ \text{Probability that the starting state is } 0^*, & \text{if } i = 4. \end{cases}$$

Denote by $\bar{v}_{T_\infty, \lambda}(\bar{p})$ the value of the λ -discounted game $G_{T_\infty, \lambda}^-(0)$ with initial probability distribution \bar{p} . It is clear that

$$\bar{v}_{T_\infty, \lambda}(\bar{p}) = (p_1 + p_2)v_{T_\infty, \lambda}^-\left(0, \frac{p_2}{p_1 + p_2}\right) - p_3. \quad (7)$$

Since $G_{T_\infty, \lambda}^-(0)$ is a state-blind game, by Corollary 3 we know that $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} \bar{v}_{T_\infty, \lambda}(\bar{p})$ is a unique viscosity solution of a partial differential equation

$$\lambda v(\bar{p}) = -\lambda(p_1 + p_2 + p_3) + \text{val}_{I \times J} [\langle \bar{p} * q(i, j), \nabla v(\bar{p}) \rangle]; \quad (8)$$

We are going to check that

$$\bar{w}_\lambda(\bar{p}) := (p_1 + p_2)w_\lambda^-\left(0, \frac{p_2}{p_1 + p_2}\right) - p_3$$

is a classical solution of (8). First, we need to compute the kernel q for the game $G_{T_\infty, \lambda}^-(0)$. By using Tables 5–6 (with $h_n = 1$ and with replacing state $++$ by state 0^*), we obtain

$$\begin{aligned} q(i, j)(--, --) &= \begin{cases} 0 & , \text{if } (i, j) = (T, R) \text{ or } (B, L); \\ -\frac{1}{2} & , \text{if } (i, j) = (T, L) \text{ or } (B, R); \\ -1 & , \text{if } (i, j) = (Q, L) \text{ or } (Q, R). \end{cases} & q(i, j)(--, -) &= \begin{cases} \frac{1}{2} & , \text{if } (i, j) = (T, L) \text{ or } (B, R); \\ 0 & , \text{otherwise.} \end{cases} \\ q(i, j)(--, -*) &= \begin{cases} 1 & , \text{if } (i, j) = (Q, L) \text{ or } (Q, R); \\ 0 & , \text{otherwise.} \end{cases} & q(i, j)(--, 0^*) &= 0 \quad \forall (i, j); \\ q(i, j)(-, -) &= \begin{cases} 0 & , \text{if } (i, j) = (T, L) \text{ or } (B, R); \\ -1 & , \text{otherwise.} \end{cases} & q(i, j)(-, --) &= \begin{cases} 1 & , \text{if } (i, j) = (T, R) \text{ or } (B, L); \\ 0 & , \text{otherwise.} \end{cases} \\ q(i, j)(-, -*) &= 0 \quad \forall (i, j); & q(i, j)(-, 0^*) &= \begin{cases} 1 & , \text{if } (i, j) = (Q, L) \text{ or } (Q, R); \\ 0 & , \text{otherwise.} \end{cases} \\ q(i, j)(-*, s) &= 0 \quad \forall (i, j), s; & q(i, j)(0^*, s) &= 0 \quad \forall (i, j), s. \end{aligned}$$

Hence

$$\bar{p} * q(i, j) = \begin{cases} (\frac{1}{2}p_2, -\frac{1}{2}p_2, 0, 0) & , \text{ if } (i, j) = (T, L) \text{ or } (B, R); \\ (-p_1, p_1, 0, 0) & , \text{ if } (i, j) = (T, R) \text{ or } (B, L); \\ (-p_1, -p_2, p_2, p_1) & , \text{ if } (i, j) = (Q, L) \text{ or } (Q, R). \end{cases}$$

Now, we need to find the value of the following one-shot matrix game

	L	R
T	$\frac{p_2}{2} \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right)$	$-p_1 \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right)$
B	$-p_1 \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right)$	$\frac{p_2}{2} \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right)$
Q	$-p_1 \frac{\partial \bar{w}_\lambda}{\partial p_1} - p_2 \frac{\partial \bar{w}_\lambda}{\partial p_2} - p_2$	$-p_1 \frac{\partial \bar{w}_\lambda}{\partial p_1} - p_2 \frac{\partial \bar{w}_\lambda}{\partial p_2} - p_2$

(9)

Note that either Q or $\frac{1}{2}T + \frac{1}{2}B$ is an optimal strategy of player 1 in this game. Indeed, if

$$\frac{p_2}{2} \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right) = -p_1 \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right),$$

then it is clear. Otherwise, if in an optimal strategy player 1 plays Q with a probability $p_Q \in [0, 1]$, then player 1 plays T with the same probability as he plays B . Since the payoff depends on p_Q linearly, its maximum is achieved either at $p_Q = 0$ or $p_Q = 1$ (or both).

Now, we consider two cases:

Case 1: $\frac{p_2}{p_1+p_2} < \frac{4\lambda+2}{4\lambda+3}$;

Case 2: $\frac{p_2}{p_1+p_2} \geq \frac{4\lambda+2}{4\lambda+3}$.

Case 1: $\frac{p_2}{p_1+p_2} < \frac{4\lambda+2}{4\lambda+3}$. By (7) we have

$$\frac{\partial \bar{w}_\lambda}{\partial p_1} = -\frac{\lambda}{1+\lambda}; \quad \frac{\partial \bar{w}_\lambda}{\partial p_2} = -1; \quad \frac{\partial \bar{w}_\lambda}{\partial p_3} = -1; \quad \frac{\partial \bar{w}_\lambda}{\partial p_4} = 0.$$

Hence the game (9) may be rewritten as follows.

	L	R
T	$\frac{p_2}{2(1+\lambda)}$	$\frac{-p_1}{1+\lambda}$
B	$\frac{-p_1}{1+\lambda}$	$\frac{p_2}{2(1+\lambda)}$
Q	$\frac{\lambda p_1}{1+\lambda}$	$\frac{\lambda p_1}{1+\lambda}$

We have

$$\begin{aligned} \left(\frac{1}{2} \cdot \frac{p_2}{2(1+\lambda)} + \frac{1}{2} \cdot \frac{-p_1}{1+\lambda} \right) - \frac{\lambda p_1}{1+\lambda} &= \frac{p_2 - (4\lambda+2)p_1}{4(1+\lambda)} = \frac{(4\lambda+3)p_2 - (4\lambda+2)(p_1+p_2)}{4(1+\lambda)} \\ &= \frac{(4\lambda+3)(p_1+p_2) \left(\frac{p_2}{p_1+p_2} - \frac{4\lambda+2}{4\lambda+3} \right)}{4(1+\lambda)} \leq 0. \end{aligned}$$

Hence Q is an optimal strategy of player 1. Now, it remains to verify that partial differential equation (8) holds. We have

$$\begin{aligned} &\lambda \bar{w}_\lambda(\bar{p}) + \lambda(p_1 + p_2 + p_3) - \text{val}_{I \times J} [(\bar{p} * q(i, j), \nabla \bar{w}_\lambda(\bar{p}))] \\ &= \left(-\lambda(p_1 + p_2) \left(\frac{\frac{p_2}{p_1+p_2} + \lambda}{1+\lambda} \right) - \lambda p_3 \right) + \lambda(p_1 + p_2 + p_3) - \frac{\lambda p_1}{1+\lambda} \\ &= \frac{-\lambda p_2 - \lambda^2(p_1 + p_2)}{1+\lambda} + \frac{\lambda(p_1 + p_2)(1+\lambda)}{1+\lambda} - \frac{\lambda p_1}{1+\lambda} = 0. \end{aligned}$$

Case 2: $\frac{p_2}{p_1+p_2} \geq \frac{4\lambda+2}{4\lambda+3}$. In this case we have

$$\begin{aligned}\frac{\partial \bar{w}_\lambda}{\partial p_1} &= -1 + \frac{1}{(4\lambda+3)(\lambda+1)} \left(\frac{4\lambda(p_1+p_2)}{(4\lambda+3)(p_2-2p_1)} \right)^{4\lambda/3} + \frac{p_2(p_1+p_2)^{4\lambda/3}}{\lambda+1} \left(\frac{4\lambda}{(4\lambda+3)(p_2-2p_1)} \right)^{1+(4\lambda/3)}; \\ \frac{\partial \bar{w}_\lambda}{\partial p_1} &= -1 + \frac{1}{(4\lambda+3)(\lambda+1)} \left(\frac{4\lambda(p_1+p_2)}{(4\lambda+3)(p_2-2p_1)} \right)^{4\lambda/3} - \frac{p_1(p_1+p_2)^{4\lambda/3}}{\lambda+1} \left(\frac{4\lambda}{(4\lambda+3)(p_2-2p_1)} \right)^{1+(4\lambda/3)}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} &= \frac{1}{1+\lambda} \left(\frac{4\lambda(p_1+p_2)}{(4\lambda+3)(p_2-2p_1)} \right)^{1+(4\lambda/3)}; \\ -p_1 \frac{\partial \bar{w}_\lambda}{\partial p_1} - p_2 \frac{\partial \bar{w}_\lambda}{\partial p_2} - p_2 &= p_1 - \frac{p_1+p_2}{(1+\lambda)(3+4\lambda)} \left(\frac{4\lambda(p_1+p_2)}{(4\lambda+3)(p_2-2p_1)} \right)^{4\lambda/3}.\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}& \left(\frac{1}{2} \cdot \frac{p_2}{2} \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right) + \frac{1}{2} \cdot (-p_1) \left(\frac{\partial \bar{w}_\lambda}{\partial p_1} - \frac{\partial \bar{w}_\lambda}{\partial p_2} \right) \right) + \left(p_1 \frac{\partial \bar{w}_\lambda}{\partial p_1} + p_2 \frac{\partial \bar{w}_\lambda}{\partial p_2} + p_2 \right) \\ &= (p_1+p_2) \left(-\frac{p_1}{p_1+p_2} + \frac{1}{4\lambda+3} \left(\frac{4\lambda}{(4\lambda+3) \left(1 - \frac{3p_1}{p_1+p_2}\right)} \right)^{4\lambda/3} \right).\end{aligned}$$

Since $\frac{p_2}{p_1+p_2} \geq \frac{4\lambda+2}{4\lambda+3}$, it follows that $\frac{p_1}{p_1+p_2} \leq \frac{1}{4\lambda+3} < \frac{1}{3}$. Consider a function $f : [0, 1/3] \rightarrow \mathbb{R}$, defined by

$$f_\lambda(p) = -p + \frac{1}{4\lambda+3} \left(\frac{4\lambda}{(4\lambda+3)(1-3p)} \right)^{4\lambda/3}.$$

We have

$$f'_\lambda(p) = -1 + \left(\frac{4\lambda}{(4\lambda+3)(1-3p)} \right)^{1+(4\lambda/3)}, \text{ and } f'_\lambda(p) = 0 \iff p = \frac{1}{4\lambda+3}.$$

Note that $f'_\lambda(p)$ is a strictly increasing function, with $f'_\lambda(0) < 0$ and $f'_\lambda(p) \rightarrow +\infty$ as $p \rightarrow 1/3$. Thus for any $p \in [0, 1/3]$ we have

$$f_\lambda(p) \geq f_\lambda\left(\frac{1}{4\lambda+3}\right) = 0.$$

Hence $\frac{1}{2}T + \frac{1}{2}B$ is an optimal strategy of player 1. Now, it remains to verify that partial differential equation (8) holds. We have

$$\begin{aligned}& \lambda \bar{w}_\lambda(\bar{p}) + \lambda(p_1+p_2+p_3) - \text{val}_{I \times J} [\langle \bar{p} * q(i, j), \nabla \bar{w}_\lambda(\bar{p}) \rangle] \\ &= \left(-\lambda(p_1+p_2) \left(-1 + \frac{(4\lambda)^{4\lambda/3}}{(1+\lambda)(3+4\lambda)^{1+(4\lambda/3)}} \left(\frac{3p_2}{p_1+p_2} - 2 \right)^{-4\lambda/3} \right) - \lambda p_3 \right) + \lambda(p_1+p_2+p_3) \\ &- \left(\frac{p_2}{4} - \frac{p_1}{2} \right) \frac{1}{1+\lambda} \left(\frac{4\lambda(p_1+p_2)}{(4\lambda+3)(p_2-2p_1)} \right)^{1+(4\lambda/3)} = 0.\end{aligned}$$

This is the end of the proof of assertion 1.

The proof of assertion 2 is analogous. So, we are only going to give a general plan.

Denote $I = \{T, M, B\}$, $J = \{L, M, R, Q\}$. Denote also by $\bar{p} = (p_1, p_2, p_3, p_4)$ a probability distribution on the states of the game $G_{T_\infty, \lambda}^+(0)$. Here

$$p_i = \begin{cases} \text{Probability that the starting state is } +, & \text{if } i = 1; \\ \text{Probability that the starting state is } ++, & \text{if } i = 2; \\ \text{Probability that the starting state is } +^*, & \text{if } i = 3; \\ \text{Probability that the starting state is } 0^*, & \text{if } i = 4. \end{cases}$$

Consider the equation (in $v(\bar{p})$)

$$\lambda v(\bar{p}) = -\lambda(p_1 + p_2 + p_3) + \text{val}_{I \times J}[A], \quad (10)$$

where A is the one-shot matrix game

	L	M	R	Q
T	$\frac{2p_2}{3} \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$-p_1 \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$-p_1 \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$-p_1 \frac{\partial v}{\partial p_1} - p_2 \frac{\partial v}{\partial p_2} + p_2$
M	$-p_1 \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$\frac{2p_2}{3} \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$-p_1 \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$-p_1 \frac{\partial v}{\partial p_1} - p_2 \frac{\partial v}{\partial p_2} + p_2$
B	$-p_1 \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$-p_1 \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$\frac{2p_2}{3} \left(\frac{\partial v}{\partial p_1} - \frac{\partial v}{\partial p_2} \right)$	$-p_1 \frac{\partial v}{\partial p_1} - p_2 \frac{\partial v}{\partial p_2} + p_2$

Now, it remains to consider two cases ($\frac{p_2}{p_1+p_2} < \frac{9\lambda+6}{9\lambda+8}$ and $\frac{p_2}{p_1+p_2} \geq \frac{9\lambda+6}{9\lambda+8}$) to check that

$$\bar{w}_\lambda(\bar{p}) := (p_1 + p_2)w_\lambda^+ \left(0, \frac{p_2}{p_1 + p_2} \right) + p_3$$

is a classical solution of (10). □

Finally, we are ready to prove Theorem 1.

5.3.4 The end of the proof: combining two “half-games” into a single one

Proof of Theorem 1. Recall that we denote $p = (p_1, p_2, p_3, p_4, p_5, p_6)$, where

$$p_i = \begin{cases} \text{Probability that the starting state is } +, & \text{if } i = 1; \\ \text{Probability that the starting state is } ++, & \text{if } i = 2; \\ \text{Probability that the starting state is } +^*, & \text{if } i = 3; \\ \text{Probability that the starting state is } -, & \text{if } i = 4; \\ \text{Probability that the starting state is } --, & \text{if } i = 5; \\ \text{Probability that the starting state is } -^*, & \text{if } i = 6. \end{cases}$$

Denote also $p_+ = (0, 1, 0, 0, 0, 0)$ and $p_- = (0, 0, 0, 0, 1, 0)$.

By Lemma 1 we have

$$\begin{cases} v_{T_\infty, \lambda}(p_+) = v_{T_\infty, \lambda}^+(v_{T_\infty, \lambda}(p_-), 1) = (1 - v_{T_\infty, \lambda}(p_-))v_{T_\infty, \lambda}^+(0, 1) + v_{T_\infty, \lambda}(p_-); \\ v_{T_\infty, \lambda}(p_-) = v_{T_\infty, \lambda}^-(v_{T_\infty, \lambda}(p_+), 1) = (1 + v_{T_\infty, \lambda}(p_+))v_{T_\infty, \lambda}^-(0, 1) + v_{T_\infty, \lambda}(p_+). \end{cases}$$

By solving this system (with variables $v_{T_\infty, \lambda}(p_+)$, $v_{T_\infty, \lambda}(p_-)$), we obtain

$$\begin{cases} v_{T_\infty, \lambda}(p_+) = -\frac{v_{T_\infty, \lambda}^-(0, 1) + v_{T_\infty, \lambda}^+(0, 1) - v_{T_\infty, \lambda}^-(0, 1)v_{T_\infty, \lambda}^+(0, 1)}{v_{T_\infty, \lambda}^-(0, 1) - v_{T_\infty, \lambda}^+(0, 1) - v_{T_\infty, \lambda}^-(0, 1)v_{T_\infty, \lambda}^+(0, 1)}, \\ v_{T_\infty, \lambda}(p_-) = -\frac{v_{T_\infty, \lambda}^-(0, 1) + v_{T_\infty, \lambda}^+(0, 1) + v_{T_\infty, \lambda}^-(0, 1)v_{T_\infty, \lambda}^+(0, 1)}{v_{T_\infty, \lambda}^-(0, 1) - v_{T_\infty, \lambda}^+(0, 1) - v_{T_\infty, \lambda}^-(0, 1)v_{T_\infty, \lambda}^+(0, 1)}. \end{cases} \quad (11)$$

By Lemma 2 we have

$$\begin{aligned} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} (v_{T_\infty, \lambda}^-(0, 1)) &= -1 + \frac{(4\lambda)^{4\lambda/3}}{(1+\lambda)(3+4\lambda)^{1+(4\lambda/3)}} (3 \cdot 1 - 2)^{-4\lambda/3}; \\ \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} (v_{T_\infty, \lambda}^-(0, 1)) &= 1 - \frac{2(18\lambda)^{9\lambda/8}}{(1+\lambda)(8+9\lambda)^{1+(9\lambda/8)}} (8 \cdot 1 - 6)^{-9\lambda/8}. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} (v_{T_\infty, \lambda}^-(0, 1)) &= -1 + \lim_{\lambda \rightarrow 0} \frac{(4\lambda)^{4\lambda/3}}{(1+\lambda)(3+4\lambda)^{1+(4\lambda/3)}} = -\frac{2}{3}; \\ \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} (v_{T_\infty, \lambda}^+(0, 1)) &= 1 - \lim_{\lambda \rightarrow 0} \frac{2(18\lambda)^{9\lambda/8} \cdot 2^{-9\lambda/8}}{(1+\lambda)(8+9\lambda)^{1+(9\lambda/8)}} = \frac{3}{4}. \end{aligned} \tag{12}$$

By combining (11) and (12), we obtain

$$\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p_+) = 7/11 \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p_-) = -5/11.$$

If $p \in [0, 1]$ and starting state is $p(+) = (1-p, p, 0, 0, 0, 0)$, then by Lemmas 1 and 2 we have

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p(+)) &= \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^+\left(-\frac{5}{11}, p\right) = \frac{16}{11} \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^+(0, p) - \frac{5}{11} \\ &= \begin{cases} \frac{16}{11} \cdot \frac{3}{4} - \frac{5}{11} = \frac{7}{11} & , \text{if } p \geq 3/4; \\ \frac{16}{11}p - \frac{5}{11} & , \text{if } p < 3/4. \end{cases} \end{aligned}$$

If $p \in [0, 1]$ and the starting state is $p(-) = (0, 0, 0, 1-p, p, 0)$, then by Lemmas 1 and 2 we have

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p(-)) &= \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^-\left(\frac{7}{11}, p\right) = \frac{18}{11} \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^-(0, p) + \frac{7}{11} \\ &= \begin{cases} -\frac{18}{11} \cdot \frac{2}{3} + \frac{7}{11} = -\frac{5}{11} & , \text{if } p \geq 2/3; \\ -\frac{18}{11}p + \frac{7}{11} & , \text{if } p < 2/3. \end{cases} \end{aligned}$$

Finally, we consider the most general starting probability distribution on the states $p = (p_1, p_2, p_3, p_4, p_5, p_6)$. Assuming that $p_1 + p_2 > 0$ and $p_4 + p_5 > 0$, denote

$$\bar{p}(+) = \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}, 0, 0, 0, 0 \right) \quad \text{and} \quad \bar{p}(-) = \left(0, 0, 0, \frac{p_4}{p_4 + p_5}, \frac{p_5}{p_4 + p_5}, 0 \right).$$

We have

$$\begin{aligned}
& \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p) \\
&= p_3 - p_6 + (p_1 + p_2) \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(\bar{p}(+)) + (p_4 + p_5) \lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(\bar{p}(-)) \\
&= \begin{cases} p_3 - p_6 + \frac{7}{11}(p_1 + p_2) - \frac{5}{11}(p_4 + p_5) & , \text{ if } \frac{p_2}{p_1 + p_2} \geq \frac{3}{4} \text{ and } \frac{p_5}{p_4 + p_5} \geq \frac{2}{3}; \\ p_3 - p_6 + (p_1 + p_2) \left(\frac{16}{11} \cdot \frac{p_2}{p_1 + p_2} - \frac{5}{11} \right) - \frac{5}{11}(p_4 + p_5) & , \text{ if } \frac{p_2}{p_1 + p_2} < \frac{3}{4} \text{ and } \frac{p_5}{p_4 + p_5} \geq \frac{2}{3}; \\ p_3 - p_6 + \frac{7}{11}(p_1 + p_2) - (p_4 + p_5) \left(-\frac{18}{11} \cdot \frac{p_5}{p_5 + p_6} + \frac{7}{11} \right) & , \text{ if } \frac{p_2}{p_1 + p_2} \geq \frac{3}{4} \text{ and } \frac{p_5}{p_4 + p_5} < \frac{2}{3}; \\ p_3 - p_6 + (p_1 + p_2) \left(\frac{16}{11} \cdot \frac{p_2}{p_1 + p_2} - \frac{5}{11} \right) - (p_4 + p_5) \left(-\frac{18}{11} \cdot \frac{p_5}{p_5 + p_6} + \frac{7}{11} \right) & , \text{ if } \frac{p_2}{p_1 + p_2} < \frac{3}{4} \text{ and } \frac{p_5}{p_4 + p_5} < \frac{2}{3}. \end{cases}
\end{aligned}$$

Thus we obtained the expressions from Theorem 1. Doing the same for the cases when $p_1 = p_2 = 0$ or/and $p_4 = p_5 = 0$, one can see that the expressions from Theorem 1 also hold for these cases. \square

5.3.5 A discussion of the proof

Here, we give several remarks discussing the above proof.

Remark 14 (The main idea of the proof). In order to prove that the limit value (when first $\sup h_i \rightarrow 0$ and later $\lambda \rightarrow 0$) exists, we considered a “decomposition” of the game G_1 from Theorem 1 into two “half-games”, $G_{T_\infty}^-(0)$ and $G_{T_\infty}^+(0)$. Afterwards, in Lemma 2 we found the limit value for each of these two “half-games”. Afterwards, in the above proof of Theorem 1 we found the limit value of the game G_1 by combining two “half-games”. The same idea was used in [SV15]. (However, there it was used to prove the inexistence of the limit value).

Remark 15. In this (somewhat informal) remark, we try to understand for what reason the game G_1^λ from Theorem 1 has no limit value in discrete time (if each stage duration is $h = 1$), but has a limit value in continuous time (if stage durations h_i all tend to 0).

First, we consider the discrete case (each stage duration is $h = 1$). Consider the game \tilde{G}_1^λ from §5.2 with starting state $--$. (Recall that games G_1^λ and \tilde{G}_1^λ are equivalent, see Proposition 5). It is easy to see that by playing only C , after some time player 1’s belief that the current state is $-$ may be $1 - (1/2)^n$, where $n \in \mathbb{N}_0$. Of course, the larger is n , the larger is the average time that player 1 needs to wait to achieve the belief $1 - (1/2)^n$. However, the closer is the discount factor λ to 0, the longer player 1 may afford himself to wait. Thus, if λ is close to 0, then he can with very high chance go to the state $++$, and do it in such a way that his payoff before going to the state $++$ is almost 0. See Figure 8. Now, consider the game \tilde{G}_1^λ with starting state $++$. By playing only C , after some time player 2’s belief that the current state is $+$ may be $1 - (2/3)^n$, where $n \in \mathbb{N}_0$. And as before, if λ is close to 0, then player 2 can with very high chance go to the state $--$, and do it in such a way that his payoff before going to the state $--$ is almost 0. Thus, both player 1 and player 2 can (if discount factor λ is small) move the state from unfavorable position to favorable position, obtaining almost no payoff while being in the unfavorable position. (In particular, the value when starting in the state $--$ is equal to the value when starting in the state $++$). Because of this quick change from one player to another one, there is an oscillation when λ tends to 0. See [Zil16b] and [RZ20] for more details. See also [SV15].

Now, we consider the continuous case ($\sup h_i \rightarrow 0$). Consider the game $G_{T_\infty, \lambda}$, where starting state is $--$ with probability p and $-$ with probability $1 - p$. The goal of player 1 is to quickly make his belief \tilde{p} that the current state is $--$ smaller, and afterwards start playing Q . Looking at (3) from

§5.3.2, we see that after player 1 plays C , player 1's a posteriori belief \tilde{p} is going to be either $p - \frac{hp}{2}$, if the public signal is α , or $p + h - hp$, if the public signal is β . Since the probability to obtain the signal α and the probability to obtain the signal β are both equal to $1/2$, the expectation of the a posteriori belief \tilde{p} equals

$$\frac{1}{2}(h - hp) + \frac{1}{2} \cdot \frac{-hp}{2} = \frac{h}{4}(2 - 3p).$$

See Figure 9. Thus if $p \leq 2/3$, then player 1 immediately starts playing Q until he gets to the state $++$. If $p > 2/3$, then player 1 may play C for some time in order to make \tilde{p} smaller. As λ becomes smaller, he can wait longer and longer, thus allowing \tilde{p} to get closer to $2/3$. See Figure 10.

Analogously, if in the game $G_{T_\infty, \lambda}$ the starting state is $++$ with probability p and $+$ with probability $1 - p$, then by looking at (5) from §5.3.2, we see that after player 2 plays C , player 2's a posteriori belief \tilde{p} is going to be either $p - \frac{2hp}{3}$, if the public signal is α , or $p + h - hp$, if the public signal is β . Since the probability to obtain the signal α equals $1/3$ and the probability to obtain the signal β equals $2/3$, the expectation of the a posteriori belief \tilde{p} equals

$$\frac{2}{3}(h - hp) + \frac{1}{3} \cdot \frac{-2hp}{3} = \frac{h}{9}(6 - 8p).$$

See Figure 11. Thus if $p \leq 3/4$, then player 1 immediately starts playing Q until he gets to the state $--$. If $p > 3/4$, then player 1 may play C for some time in order to make \tilde{p} smaller. As λ becomes smaller, he can wait longer and longer, thus allowing \tilde{p} get closer to $3/4$.

Thus, neither player 1 nor player 2 can (if discount factor λ is small) move the state from unfavorable position to favorable position for sure, because with high probability they are going to go to an absorbing state, $-^*$ for player 1 and $+^*$ for player 2. (In particular, the value when starting in the state $--$ is *not* equal to the value when starting in the state $++$). Thus, there is no oscillation when λ tends to 0.

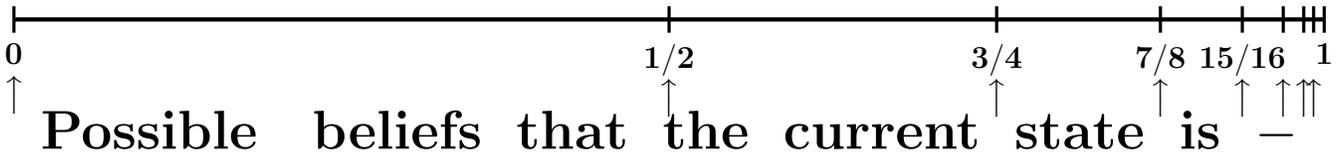


Figure 8: Discrete case: Possible beliefs that the current state is $-$. As λ becomes smaller, player 1 can wait longer and longer to achieve higher probabilities.

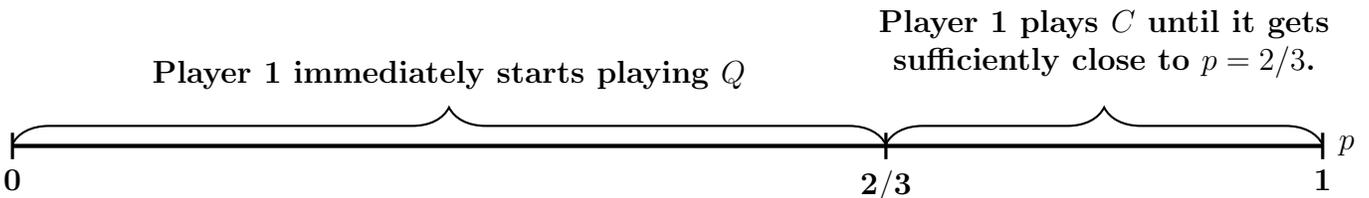
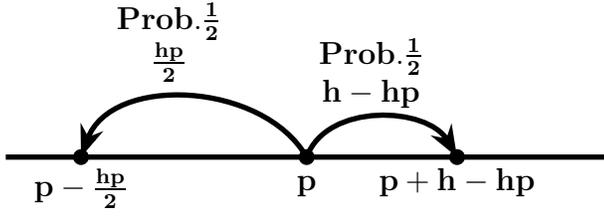
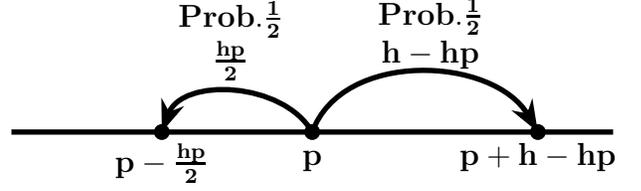


Figure 10: Continuous case with small λ : With probability $p < 2/3$ that the current state is $--$, player 1 should immediately start playing Q . Otherwise, his belief \tilde{p} will start to increase until it becomes $\tilde{p} = 2/3$, which is bad for player 1. With probability $p \geq 2/3$ that the current state is $--$, player 1 can very quickly decrease his belief \tilde{p} until it becomes $\tilde{p} \approx 2/3$, which is good for him.

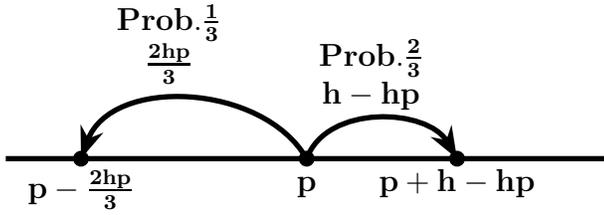


(a) $p > 2/3$ and player 1 plays C . In this case $\tilde{p} = \frac{1}{2}(h - hp) + \frac{1}{2} \cdot \frac{-hp}{2} = \frac{h}{4}(2 - 3p) < 0$, thus if λ is small, then player 1 prefers to play C until \tilde{p} is close to $2/3$.

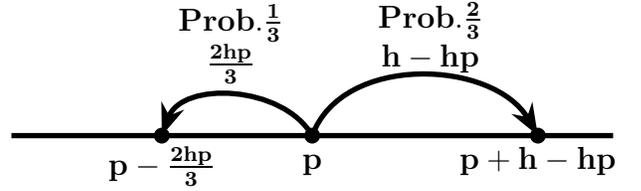


(b) $p < 2/3$ and player 1 plays C . In this case $\tilde{p} = \frac{1}{2}(h - hp) + \frac{1}{2} \cdot \frac{-hp}{2} = \frac{h}{4}(2 - 3p) > 0$, thus player 1 prefers to play Q until the state changes.

Figure 9: Continuous case with player 1 playing C : What happens with the probability p that the current state is $--$.



(a) $p > 3/4$ and player 2 plays C . In this case $\tilde{p} = \frac{2}{3}(h - hp) + \frac{1}{3} \cdot \frac{-2hp}{3} = \frac{h}{9}(6 - 8p) < 0$, thus if λ is small, then player 2 prefers to play C until \tilde{p} is close to $3/4$.



(b) $p < 3/4$ and player 2 plays C . In this case $\tilde{p} = \frac{2}{3}(h - hp) + \frac{1}{3} \cdot \frac{-2hp}{3} = \frac{h}{9}(6 - 8p) < 0$, thus player 2 prefers to play Q until the state changes.

Figure 11: Continuous case with player 1 playing C : What happens with the probability p that the current state is $++$.

6 Final comments

6.1 Definition of games with stage duration for a general model of games with public signals

First, we need to give another general construction of games with public signals, which is equivalent to the one given in §5.1.1.

A zero-sum stochastic game with public signals is a 7-tuple $(A, \Omega, \{f_m\}_{m \in \mathbb{N}^*}, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m^2\}_{m \in \mathbb{N}^*})$, where:

- A is a finite non-empty set of signals;
- Ω is a finite non-empty set of states;
- I is a finite non-empty set of actions of player 1;
- J is a finite non-empty set of actions of player 2;
- $g_m : I \times J \times \Omega \rightarrow \mathbb{R}$ is the m -th stage payoff function of player 1;
- $P_m^2 : I \times J \times \Omega \rightarrow \Delta(\Omega)$ is the transition probability function at the m -th stage;
- $f_m : I \times J \times \Omega \times \Omega \rightarrow \Delta(A)$ is the signalling function at the m -th stage.

The game $(A, \Omega, \{f_m\}_{m \in \mathbb{N}^*}, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m^2\}_{m \in \mathbb{N}^*})$ proceeds in stages as follows. Before the first stage, an initial state $\omega_1 \in \Omega$ and an initial signal α_1 is chosen according to some probability law $p_0 \in \Delta(\Omega \times A)$. At each stage $n \in \mathbb{N}^*$:

1. The current state is $\omega_n \in \Omega$. Players do not observe it, but they observe the signal $\alpha_n \in A$ and

the actions of each other at the previous stage;

2. Players choose their mixed actions. Player 1 chooses $x_n \in \Delta(I_{\alpha_n})$ and player 2 chooses $y_n \in \Delta(J_{\alpha_n})$;
3. An action $i_n \in I_{\alpha_n}$ of player 1 (respectively $j_n \in J_{\alpha_n}$ of player 2) is chosen according to the probability measure $x_n \in \Delta(I_{\alpha_n})$ (respectively $y_n \in \Delta(J_{\alpha_n})$);
4. Player 1 obtains a payoff $g_n = g_n(i_n, j_n, \omega_n)$, while player 2 obtains payoff $-g_n$. The new state ω_{n+1} is chosen according to the probability law $P_n^2(i_n, j_n, \omega_n)$. The new signal is chosen according to the probability law $f_n(i_n, j_n, \omega_n, \omega_{n+1})$.

Remark 16. $P_n^2(i_n, j_n, \omega_n)(\omega_{n+1}) = 0$ means that after playing (i_n, j_n) at n -th stage, the probability to change the state from ω_n to ω_{n+1} is 0. Despite this, we still need to specify a distribution $f_n(i_n, j_n, \omega_n, \omega_{n+1}) \in \Delta(A)$. So, the construction of games with public signals presented here has more information about the game compared to the construction from §5.1.1, but this new information is useless.

Proposition 9 (Equivalence of the constructions). For each transition probability function P^1 from construction given in §5.1.1, there is a transition probability function P^2 and a signalling function f_m from construction presented here, such that the expected total payoff of both players coincide for any pair of strategies. And vice versa, for any transition probability function P^2 and a signalling function f_m from construction presented here, there is a transition probability function P^1 from construction given in §5.1.1 such that the expected total payoff of both players coincide for any pair of strategies.

Proof. Given P_n^1 from construction in §5.1.1, define for each $n \in \mathbb{N}^*$, $i_n \in I, j_n \in J, \alpha \in A, \omega_n, \omega_{n+1} \in \Omega$

$$P_n^2(i_n, j_n, \omega_n)(\omega_{n+1}) = \sum_{\alpha' \in A} P_n^1(i_n, j_n, \omega_n)(\omega_{n+1}, \alpha');$$

$$f_n(i_n, j_n, \omega_n, \omega_{n+1})(\alpha) = \begin{cases} P_n^1(i_n, j_n, \omega_n)(\omega_{n+1}, \alpha) / P_n^2(i_n, j_n, \omega_n)(\omega_{n+1}), & \text{if } P_n^2(i_n, j_n, \omega_n)(\omega_{n+1}) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Vice versa, given P_n^2 and f_n from construction given here, define for each $n \in \mathbb{N}^*$, $i_n \in I, j_n \in J, \alpha \in A, \omega_n, \omega_{n+1} \in \Omega$

$$P_n^1(i_n, j_n, \omega_n)(\omega_{n+1}, \alpha) = P_n^2(i_n, j_n, \omega_n)(\omega_{n+1}) \cdot f_n(i_n, j_n, \omega_n, \omega_{n+1})(\alpha). \quad \square$$

In §4.3, we gave the definition of games with stage duration and public signals. However, this definition was for a very specific class of games with public signals, in which given signal is determined and does not depend on the previous state. Now, we give a general definition.

Definition 10. Fix a zero-sum stochastic game $(A, \Omega, \{f_n\}_{n \in \mathbb{N}^*}, I, J, g, q)$ with public signals, where q is a kernel. The *stochastic game with n -th stage duration h_n* is the stochastic game

$$(A, \Omega, \{f_m\}_{m \in \mathbb{N}^*}, I, J, \{h_m g\}_{m \in \mathbb{N}^*}, \{h_m q\}_{m \in \mathbb{N}^*})$$

with public signals.

Remark 17. Even if $P_n^2(i_n, j_n, \omega_n)(\omega_n) = 0$ in a game with stage duration 1, we have $P_n^2(i_n, j_n, \omega_n)(\omega_n) > 0$ in a game with stage duration $h < 1$. So, even if the knowledge of $f_n(i_n, j_n, \omega_n, \omega_n) \in \Delta(A)$ may be useless when stage duration is 1, it is never useless and is always required when stage duration is $h < 1$.

6.2 Alternative definition of stochastic games with stage duration with public signals

In this section, we give a definition of stochastic games with stage duration, which is different from the definition which we gave in §6.1 (and different from the one given in §4.3).

Fix a zero-sum stochastic game $(A, \Omega, I, J, g, \{P_n\}_{n \in \mathbb{N}^*})$ with public signals, where P_n are transition probability functions. (This time we use construction of stochastic games from §5.1.1).

We assume that $A = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

Definition 11. The *stochastic game with n -th stage duration h_n* is the stochastic game

$$(\tilde{A}, \Omega, I, J, \{h_n g\}_{m \in \mathbb{N}^*}, \{\tilde{P}_m\}_{m \in \mathbb{N}^*}),$$

where $\tilde{A} = A \cup \{\alpha_{n+1}\}$ and for each $n \in \mathbb{N}^*$, $i_n \in I, j_n \in J, \alpha \in \tilde{A}, \omega_n, \omega_{n+1} \in \Omega$ we have

$$\tilde{P}_n(i, j, \omega_n)(\omega_{n+1}, \alpha) = \begin{cases} h_n P_n(i, j, \omega_n)(\omega_{n+1}, \alpha), & \text{if } \alpha \neq \alpha_{n+1}; \\ 1 - h_n, & \text{if } \omega_n = \omega_{n+1} \text{ and } \alpha = \alpha_{n+1}. \end{cases}$$

Remark 18. Informally speaking, in n -th stage, with probability h_n the next state is drawn according to the transition probability p_n , and players obtain the signal $\alpha \neq \alpha_{n+1}$, and with probability $1 - h_n$ the state stays the same, and the players obtain the signal α_{n+1} . In this case the players know for sure that the state has not changed.

Although this new definition does not seem to be very natural, it makes the stochastic games with stage duration with public signals equivalent to the stochastic games with perfect observation of the state and with stage duration.

Proposition 10. For any stochastic game $G = (A, \Omega, I, J, g, \{P_n\}_{n \in \mathbb{N}^*})$ there is a stochastic game \tilde{G} such that for any partition T_∞ of $[0, T)$ (with $t_{n+1} - t_n = h_n$), the stochastic game $G(T_\infty)$ with n -th stage duration h_n (considered as a stochastic game with state space $\Delta(\Omega)$) coincides with the stochastic game $\tilde{G}(T_\infty)$ with n -th stage duration h_n .

Proof. In §5.1.2 we showed that given a stochastic game G with public signals, we may consider an equivalent stochastic game $\Gamma(G)$ with perfect observation of the state. Given a stochastic game $\Gamma(G)$ with perfect observation of the state, we may consider the stochastic game $(\Gamma(G))(T_\infty)$ with perfect observation of the state and with n -th stage duration h_n . Analogously, given a stochastic game G with public signals, we may consider a stochastic game $G(T_\infty)$ with public signals and with n -th stage duration h_n . Given a stochastic game $G(T_\infty)$ with public signals, we may consider an equivalent stochastic game $\Gamma(G(T_\infty))$ with perfect observation of the state. Now note that the games $\Gamma(G(T_\infty))$ and $(\Gamma(G))(T_\infty)$ coincide. See Figure 12. \square

Remark 19. Thus with this alternative definition, all the proposition from §4.1 are true. In particular, in [Nov24, Example 2] and in the example from Theorem 1, we will not obtain any inconsistencies with stochastic games with perfect observation of the state and with stage duration.

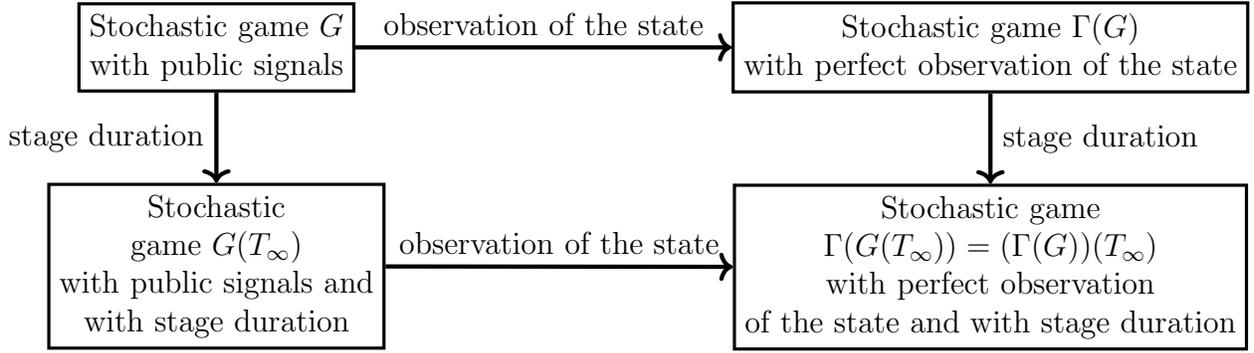


Figure 12

6.3 Theorem 1 for the discretization of continuous-time Markov game

Theorem 1 also holds in the model from [Sor17]. Let us introduce this model. As before, T_∞ is a partition of \mathbb{R}_+ ; in other words, T_∞ a strictly increasing sequence $\{t_n\}_{n \in \mathbb{N}^*}$ such that $t_1 = 0$ and $t_n \xrightarrow{n \rightarrow \infty} +\infty$. For each given partition T_∞ , denote $h_n = t_{n+1} - t_n$ for each $n \in \mathbb{N}^*$.

Loosely speaking, we want to consider continuous-time Markov game, in which players only allowed to act at times t_1, t_2, t_3, \dots . At time $t \in [t_i, t_{i+1})$ players should act according to their decision at time t_i . Let us give a formal definition.

Definition 12. Fix $\lambda \in (0, 1)$ and a stochastic game $(A, \Omega, f, I, J, g, q)$ with public signals, where q is a kernel. A *discretization of a continuous-time Markov game with public signals, in which n -th action is taken at time t_n* is the stochastic game $(A, \Omega, f, I, J, \{g_m\}_{m \in \mathbb{N}^*}, \{P_m\}_{m \in \mathbb{N}^*})$ with public signals, where

$$g_m(i, j, \omega) := \int_{t_m}^{t_{m+1}} \lambda e^{-\lambda t} g(i, j, \omega) dt \quad \text{and} \quad P_m(i, j)(\omega, \omega') := (\exp\{h_m q(i, j)\})(\omega, \omega')$$

is the m -th stage transition probability function.

The total payoff is

$$G_{T_\infty, \lambda}^{\text{cont}}(p) := E_{\sigma, \tau}^\omega \left(\int_0^{+\infty} \lambda e^{-\lambda t} g(i_t, j_t, \omega_t) dt \right).$$

Denote by $v_{T_\infty, \lambda}^{\text{cont}}(p)$ the value of a game with such a total payoff, and denote by $v_{1, \lambda}^{\text{cont}}(p)$ the value of a game with such a total payoff and in which $t_n = n - 1$ for all $n \in \mathbb{N}^*$.

Such games were studied in [Sor17], in which the following result was proved.

Proposition 11. [Sor17, Proposition 5.3]. If $(A, \Omega, f, I, J, g, q)$ is a discretization of a continuous-time state-blind Markov game, then the uniform limit $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}^{\text{cont}}(p)$ exists and is a unique viscosity

solution of the partial differential equation (in $v(p)$)

$$\lambda v(p) = \text{val}_{I \times J} [\lambda g(i, j, p) + \langle p * q(i, j), \nabla v(p) \rangle].$$

It is straightforward to see that the differential equation in Proposition 11 coincides with the differential equation in Proposition 3. The following theorem follows from Proposition 11 and the structure of the proof of Theorem 1.

Theorem 2. For the game G_1 from Example 1:

1. The pointwise limit $\lim_{\lambda \rightarrow 0} v_{1,\lambda}^{\text{cont}}(p)$ does not exist;
2. The uniform limit $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=+\infty}} v_{T_\infty,\lambda}^{\text{cont}}(p)$ exists, and we have

$$\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=+\infty}} v_{T_\infty,\lambda}^{\text{cont}}(p) = \begin{cases} \frac{7}{11}p_1 + \frac{7}{11}p_2 + p_3 - \frac{5}{11}p_4 - \frac{5}{11}p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} \geq \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and} \\ & \frac{p_5}{p_4+p_5} \geq \frac{2}{3} \text{ or } p_4 = p_5 = 0; \\ -\frac{5}{11}p_1 + p_2 + p_3 - \frac{5}{11}p_4 - \frac{5}{11}p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} < \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and} \\ & \frac{p_5}{p_4+p_5} \geq \frac{2}{3} \text{ or } p_4 = p_5 = 0; \\ \frac{7}{11}p_1 + \frac{7}{11}p_2 + p_3 - \frac{7}{11}p_4 + p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} \geq \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and} \\ & \frac{p_5}{p_4+p_5} < \frac{2}{3} \text{ or } p_4 = p_5 = 0; \\ -\frac{5}{11}p_1 + p_2 + p_3 - \frac{7}{11}p_4 + p_5 - p_6 & , \text{ if } \frac{p_2}{p_1+p_2} < \frac{3}{4} \text{ or } p_1 = p_2 = 0, \text{ and} \\ & \frac{p_5}{p_4+p_5} < \frac{2}{3} \text{ or } p_4 = p_5 = 0. \end{cases}$$

6.4 Conjectures and questions for the future

In this section we discuss some open problems.

The next conjecture considers the game from Theorem 1. We showed that the uniform limit $\lim_{\lambda \rightarrow 0} \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=+\infty}} v_{T_\infty,\lambda}(p)$ exists, while the pointwise limit $\lim_{\lambda \rightarrow 0} v_{1,\lambda}(p)$ does not exist. The next conjecture is a stronger variation of these two results.

Conjecture 1. For the game from Theorem 1, we have

1. For any fixed partition T_∞ , the pointwise limit $\lim_{\lambda \rightarrow 0} v_{T_\infty,\lambda}(p)$ does not exist;
2. We have $\left| \limsup_{\lambda \rightarrow 0} v_{T_\infty,\lambda}(p) - \liminf_{\lambda \rightarrow 0} v_{T_\infty,\lambda}(p) \right| \rightarrow 0$ as $\sup h_i$ tends to 0, uniformly in p .

Corollary 1(2) says that for any stochastic game with perfect observation of the state, the limit $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=T}} v_{T_\infty,k}(t, p)$ exists and is a unique solution of an equation. Proposition 3 says that for state-blind games, the same limit exists and is a unique viscosity solution of a partial differential equation. The natural question is to ask whether this is true for other signalling structures.

Question 1. Can we say that for any stochastic game with public signals (in the sense of definition from §6.1 or at least in the sense of definition from §4.3), the uniform limit $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=+\infty}} v_{T_\infty,\lambda}(p)$ exists

for any fixed $\lambda \in (0, 1)$? In the case of existence, can this uniform limit be found as a unique viscosity solution of some partial differential equation?

The following question asks how important is the order in which λ and $\sup h_i$ tend to 0.

Question 2 (Permutation of limits). Assume that the uniform limit $\lim_{\lambda \rightarrow 0} v_{T_\infty,\lambda}(p)$ exists for any fixed partition T_∞ , and the uniform limit $\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=+\infty}} v_{T_\infty,\lambda}(p)$ exists for any fixed λ . Can we say that double

limits

$$\lim_{\lambda \rightarrow 0} \left(\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=+\infty}} v_{T_\infty,\lambda}(p) \right) \quad \text{and} \quad \lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1+h_2+\dots=+\infty}} \left(\lim_{\lambda \rightarrow 0} v_{T_\infty,\lambda}(p) \right)$$

exist or do not exist simultaneously, and in the case of the existence they are equal to each other?

Remark 20. Note that:

1. Theorem 1 gives an example of a game in which one of the above double limits exists, whereas the other one does not. This is the reason why we require the existence of both $\lim_{\lambda \rightarrow 0} v_{T_\infty, \lambda}(p)$ and

$$\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p);$$

2. In the case of stochastic games with perfect observation of the state and with stage duration this conjecture is true, see Corollary 1(4).

For each fixed real positive number T and for each partition T_∞ with $t_n \xrightarrow{n \rightarrow \infty} T$, we may consider the stochastic game with public signals and with total payoff

$$\frac{1}{T} \left(\sum_{i=n}^{\infty} h_i g_i \right),$$

in which n -th stage kernel is still equal to hq . We denote by $v_{T_\infty, T}$ the value of the game with such a total payoff. Such games with stage duration were first studied in [SV16].

The next question motivated by Tauberian theorems for games in discrete time [Zil16a] and in continuous time [Khl16].

Question 3 (Tauberian theorem for games with vanishing stage duration). Can we say that the limits

$$\lim_{\lambda \rightarrow 0} \left(\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = +\infty}} v_{T_\infty, \lambda}(p) \right) \quad \text{and} \quad \lim_{T \rightarrow \infty} \left(\lim_{\substack{\sup_{i \in \mathbb{N}^*} h_i \rightarrow 0 \\ h_1 + h_2 + \dots = T}} v_{T_\infty, T}(p) \right)$$

exist or do not exist simultaneously, and in the case of existence they are equal to each other?

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