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► **To cite this version:**

Davide Di Giusto, Laurence Bergougnoux, Cristian Marchioli, Élisabeth Guazzelli. Influence of small inertia on Jeffery orbits. *Journal of Fluid Mechanics*, 2024, 979, pp.A42. 10.1017/jfm.2023.1007 . hal-04492675

HAL Id: hal-04492675

<https://hal.science/hal-04492675v1>

Submitted on 6 Mar 2024

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Influence of small inertia on Jeffery orbits

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(Received xx; revised xx; accepted xx)

We experimentally investigate the rotational dynamics of neutrally-buoyant axisymmetric particles in a simple shear flow. A custom-built shearing cell and a multi-view shape-reconstruction method are used to obtain direct measurements of the orientation and period of rotation of particles having oblate and prolate shapes (such as spheroids and cylinders) of varying aspect ratios. By systematically changing the viscosity of the fluid, we examine the effect of inertia (which may be originated from either phase) on the dynamical behaviour of these suspended particles up to particle Reynolds number of approximately one. While no significant effect on the period of rotation is found in this small-inertia regime, a systematic drift among several rotations toward limiting stable orbits is observed. Prolate particles are seen to drift towards the tumbling orbit in the plane of shear, whereas oblate particles are driven either to the tumbling or to the vorticity-aligned spinning orbits, depending on their initial orientation. These results are compared to recent small-inertia asymptotic theories, assessing their range of validity, as well as to numerical simulations in the small-inertia regime for both prolate and oblate particles.

Key words: Particle/fluid flow, Suspensions, Slender-body theory

MSC Codes (*Optional*) 76-05, 76T99

1. Introduction

Suspensions of axisymmetrical (or quasi-axisymmetrical) particles are ubiquitous in natural and industrial applications, including plankton dynamics in the ocean (Marchioli *et al.* 2019; Guasto *et al.* 2012), micro-plastic sea-water contamination (Ross *et al.* 2021), red-blood cells dynamics (Goldsmith 1996), paper production processes (Lundell *et al.* 2011), drag reduction in pipelines and naval applications (Hoyt 1972), particle-reinforced composites (Bao *et al.* 1991). A description of these systems is particularly challenging, as their rheology strongly depends on the relative orientation of the particles within the flow. Nevertheless, as long as

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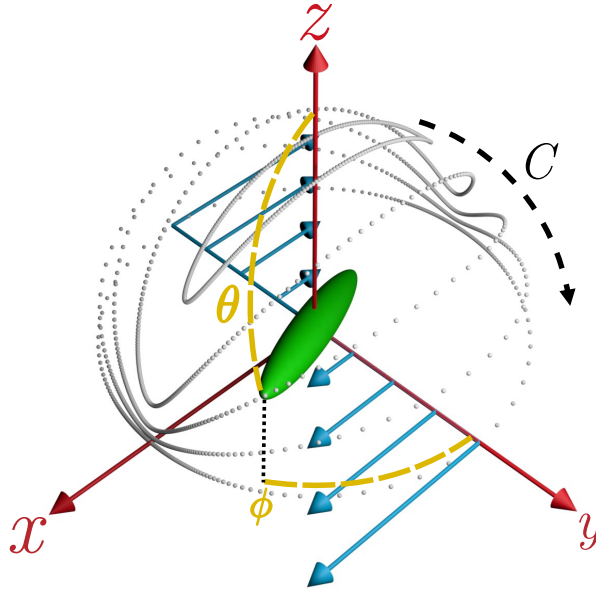


Figure 1: Representation of five different Jeffery orbits for an spheroid of aspect ratio $r = 10$. The azimuthal and polar angles, ϕ and θ , are displayed as yellow dashed lines. The trajectories followed by one of the spheroid extremities are obtained by integrating equation 1.1 in time and displayed as dotted lines on the surface of a sphere of radius equal to the particle half-length ℓ . The orbit constant C increases across the represented orbits following the dashed arrow, from $C = 0$ in the z axis-aligned log-rolling orbit to $C = \infty$ in the tumbling orbit in the x, z plane. See Supplementary Materials for animations.

32 these particles are shorter than the smallest relevant flow scale, the theory of Jeffery (1922)
 33 represents the most common approach to their modelling (Paschkewitz *et al.* 2004; Voth &
 34 Soldati 2017; Gustavsson *et al.* 2017).

35 In the absence of inertial and Brownian forces, Jeffery (1922) found that the axis of
 36 revolution of an axisymmetric rigid particle suspended in a simple shear flow rotates along
 37 one of an infinite one-parameter family of closed periodic orbits, known as Jeffery orbits,
 38 depicted in figure 1. The time change in orientation is given by the equation for the unit
 39 vector \mathbf{n} parallel to the axis of revolution:

$$40 \quad \dot{\mathbf{n}} = \boldsymbol{\Omega} \cdot \mathbf{n} + \frac{r^2 - 1}{r^2 + 1} [\mathbf{E} \cdot \mathbf{n} - \mathbf{n}(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})], \quad (1.1)$$

41 where $r = \ell/a$ is the particle aspect ratio with ℓ the particle half-length and a its radius.
 42 The orientation changes fully with the rate of rotation of the flow, $\boldsymbol{\Omega}$, and only by a fraction
 43 $(r^2 - 1)/(r^2 + 1)$ of the rate of strain of the flow, \mathbf{E} . Note that the second term within
 44 square parentheses on the right hand side of equation (1.1) is just meant to keep $|\mathbf{n}| = 1$.
 45 Considering the uniform shearing motion defined by $(\dot{\gamma}y, 0, 0)$ where $\dot{\gamma}$ is the flow shear
 46 rate, the orientation of the particle axis of revolution is described by the azimuthal and polar

47 angle, ϕ and θ , shown in figure 1. The Jeffery orbits are of the form (with $t = 0$ when $\phi = 0$):

$$48 \quad \tan \phi = \frac{1}{r} \tan \left[\frac{\dot{\gamma} t}{r + (1/r)} \right], \quad (1.2)$$

$$49 \quad \tan \theta = \frac{Cr}{(\cos^2 \phi + r^2 \sin^2 \phi)^{1/2}}, \quad (1.3)$$

50 where the constant of integration C is known as the orbit constant. The period of rotation
 51 is $T_J = 2\pi(r + 1/r)/\dot{\gamma}$. The rotational motions consist of infinitely many possible spherical
 52 ellipses, limited by a tumbling orbit in the (x, y) plane on the equator of the sphere ($C = \infty$)
 53 and a spinning orbit aligned with the vorticity z -axis on the pole of the sphere ($C = 0$).

54 Since their formulation, many experimental studies have reported Jeffery orbits for
 55 spheroids as well as for other axisymmetrical particles such as fibres having $r \gg 1$
 56 and disks having $r \ll 1$ (see e.g. Taylor 1923; Binder 1939; Forgacs & Mason 1959;
 57 Goldsmith & Mason 1962a; Karnis *et al.* 1966; Moses *et al.* 2001; Einarsson *et al.* 2016).
 58 Interestingly, fibres were observed to rotate with a period comparable to that of spheroids with
 59 a lower particle aspect ratio (Trevelyan & Mason 1951). Subsequent experiments with disks
 60 (Goldsmith & Mason 1962b) and high-aspect ratio fibres (Anczurowski & Mason 1968)
 61 highlighted the shape equivalence existing between spheroidal and cylindrical particles,
 62 which could be estimated precisely, moving from a simple linear proportionality (Burgers
 63 1938; Trevelyan & Mason 1951) to a semi-empirical asymptotic theory (Cox 1971), and
 64 finally to a data-driven formula (Harris & Pittman 1975). Note that also fore-aft symmetrical
 65 particles that possess a discrete rotation symmetry and certain mirror symmetries but do
 66 not have a continuous rotation symmetry obey Jeffery's theory, just with different shape
 67 parameters, as shown in the seminal paper by Bretherton (1962) and more recently by Fries
 68 *et al.* (2017).

69 In contrast to the case of a suspension of spheres (Einstein 1906, 1911), the first
 70 modification to the viscosity of a suspension of spheroids is indeterminate in the limit
 71 of the derivation of Jeffery (1922), as the particles exist in a dynamical state depending only
 72 on their initial orientation and without steady-state preferential orientation. To solve this
 73 indeterminacy, Jeffery himself was the first to suggest that spheroids would eventually align
 74 with the local vorticity, driven by the terms neglected in his calculations, namely flow and
 75 particle inertia. Taylor (1923) experimentally confirmed this conjecture, whereas different
 76 conclusions were later reached by Saffman (1956), who concluded that small particle inertia
 77 breaks the Jeffery orbits for nearly spherical particles, showing that log-rolling is stable
 78 for nearly spherical prolate particles and unstable for nearly spherical oblate particles. In
 79 the following years, few experiments explored the influence of inertia on Jeffery orbits.
 80 Preliminary efforts characterised the phenomenon in terms of a slow variation of orbit
 81 constant C among consecutive rotations for negligible particle Reynolds number (Goldsmith
 82 & Mason 1962b; Stover & Cohen 1990). Karnis *et al.* (1963) found that fibres and disks
 83 suspended in circular tubes at particle Reynolds numbers $Re_p \sim 10^{-4}$ would eventually
 84 rotate in the tumbling ($C = \infty$) and spinning ($C = 0$) orbits respectively. Yet, to the best
 85 of our knowledge, no experimental study has produced a rigorous characterisation of the
 86 influence of inertia on the Jeffery orbits.

87 Instead, weak inertial effects have been thoroughly addressed in a number of theoretical
 88 works, from Subramanian & Koch (2005, 2006) and Einarsson *et al.* (2015a,b) to Dabade
 89 *et al.* (2016) and Marath & Subramanian (2017, 2018). In particular, Subramanian & Koch
 90 (2005) examined the inertial effects on fibre motion in simple shear flow, focusing on the
 91 slender-body limit (infinite aspect ratio). In this limit, they were able to derive the fibre orbit
 92 equations up to $O(Re)$ where Re is the Reynolds number based on the fibre length. Their

93 findings were later reproduced both by Einarsson *et al.* (2015a,b) who exploited asymptotic
 94 perturbation theory to extend to small shear Reynolds numbers the range of validity of the
 95 equation originally proposed by Saffman for spheroidal particles with arbitrary aspect ratio
 96 and later by Dabade *et al.* (2016) who also analysed inertial effects in terms of modifications
 97 to the Jeffery orbit constant. A common conclusion reached by these independent studies
 98 is that the most striking effect of fluid and/or particle inertia is to lift the degeneracy of
 99 the infinitely many stable Jeffery orbits. This leads to a drift in the trajectory followed by
 100 the particles, which are driven towards a limiting stable orbit through consecutive rotations.
 101 Prolate spheroids are only pushed towards the tumbling orbit ($C = \infty$), whereas the situation
 102 is more complex for oblate spheroids which are attracted to either the sole spinning orbit
 103 ($C = 0$) or both the vorticity-aligned ($C = 0$) and the tumbling ($C = \infty$) orbits, depending
 104 on whether their aspect ratio is larger or smaller than a critical value of approximately 0.14
 105 (Einarsson *et al.* 2015b; Dabade *et al.* 2016). Interestingly, Marath & Subramanian (2017)
 106 also suggested a second-order effect of inertia on the period of rotation, while Rosén *et al.*
 107 (2015) discussed the influence of flow confinement over the stability of the tumbling and
 108 spinning orbits.

109 The problem has been also tackled numerically. Several studies based on the Lattice-
 110 Boltzmann method have considered prolate and oblate spheroids in the near-sphere limit (Qi
 111 & Luo 2003; Huang *et al.* 2012; Mao & Alexeev 2014). Simulations offer the advantage of
 112 easily separating between fluid and particle inertia, characterised by the particle Reynolds
 113 number $Re_p = \rho_f \ell^2 \dot{\gamma} / \mu$ and the Stokes number $St = \rho_p \ell^2 \dot{\gamma} / \mu = Re_p \rho_p / \rho_f$, respectively,
 114 where ρ_p is the density of the particle, ρ_f that of the fluid, and μ the dynamical viscosity.
 115 By exploring a wide parameter space ($0 < Re_p < 300$ and $0 < St < 1200$) for spheroids
 116 with moderate aspect ratio $2 \leq r \leq 6$, these authors found a good agreement with the
 117 theoretical findings of Dabade *et al.* (2016) and Einarsson *et al.* (2015a) at $Re_p \sim 10$,
 118 but also the emerging of other stable rotation states (spinning, inclined spinning, inclined
 119 tumbling, or kayaking) for different combinations of increasing particle and flow inertia
 120 (Rosén *et al.* 2015). Interestingly, these simulations and previous experiments by Zettner &
 121 Yoda (2001) reported an increase of the period of rotation with flow inertia, proportional to
 122 $(Re_{p,cr} - Re_p)^{-0.5}$, where $Re_{p,cr}$ is a critical particle Reynolds number above which the
 123 considered particles were observed to stop rotating.

124 In this manuscript, we present the results of an experimental investigation on the rotational
 125 dynamics of neutrally-buoyant spheroidal and cylindrical particles (both prolate and oblate)
 126 subjected to simple shearing flows in the small-inertia regime ($Re_p = St \lesssim 1$). Since particles
 127 and fluid have the same density, we cannot distinguish between specific particle inertia effects
 128 and fluid inertia effects. As we will show, however, our findings are valid regardless of the
 129 specific phase that is responsible for the observed inertial bias. The experimental methods
 130 are described in § 2. The particles and fluids used are described in § 2.1. The custom-built
 131 shearing cell is presented in § 2.2. The multi-view shape-reconstruction method used to
 132 obtain direct measurements of the particle orientation is introduced in § 2.3 with details of
 133 the data processing in § 2.4 and analysis in § 2.5. The experimental results are presented in
 134 § 3. Typical evolution of the orbits for both oblate and prolate particles are shown in § 3.1.
 135 The period of rotation is examined in § 3.2 where a thorough analysis of the equivalence
 136 in shape between spheroids and cylinders is proposed. The drift through successive orbits
 137 owing to the effect of inertia is analysed in § 3.3 and compared to the asymptotic theories:
 138 first in terms of the time-variation of the particle orientation vector, \mathbf{n} , (Einarsson *et al.*
 139 2015b) and then as the discrete variation of orbit constant, ΔC , against C (Dabade *et al.*
 140 2016). Concluding remarks are drawn in § 4.

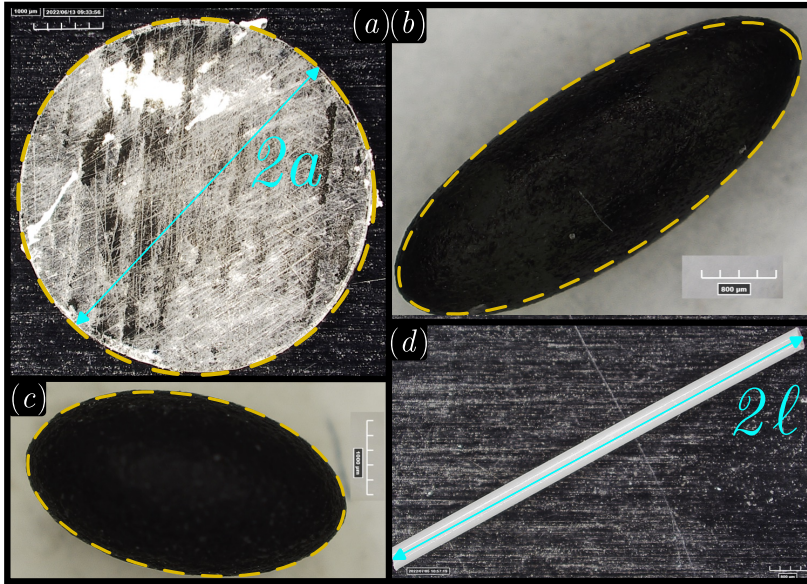


Figure 2: Some typical particles used in the experiments: (a) top view of the disk ‘CYL005’ cut by laser cutter; (b) side view of the spheroidal particle ‘ELL3’; (c) side view of the spheroidal particle ‘ELL06’; (d) side view of the fibre ‘CYL20’ manually cut from fishing line. The yellow dashed curves in panels (a), (b) and (c) represents the circularity of the given ideal shapes.

141 2. Experimental methods

142

2.1. Particles and fluids

143 Some typical cylindrical and spheroidal particles considered in this study are displayed in
 144 figure 2. Their shape is determined by taking multiple pictures with a Hirox RH-2000 digital
 145 microscope with a resolution of 221 pixels per mm. Then, 10 measurements are manually
 146 made using the software ImageJ to produce a statistical characterisation of their length, 2ℓ ,
 147 and diameter, $2a$, and consequently of their aspect ratio, r , as reported in table 1 where the
 148 characteristics of all the particles used are listed. Particles are separated in three different
 149 batches according to their production method. The first batch (batch I) comprises two low
 150 aspect-ratio fibres (CYL2, CYL10) and all the (oblate and prolate) spheroids. These particles
 151 are produced out of UV-sensitive resin using a stereolithography-3D printer with a resolution
 152 of $25\ \mu\text{m}$ and have an estimated density of $1200\ \text{kg}/\text{m}^3$. The second batch (batch II) includes
 153 two fibres cut from a fishing line and having an estimated density of $1160\ \text{kg}/\text{m}^3$. Finally, the
 154 last batch (batch III) contains all the disks obtained by laser cutting of rigid Plexiglas sheets
 155 having a density of $1180\ \text{kg}/\text{m}^3$. Cylinders and spheroids coming from batches I and III are
 156 further polished using sandpaper after production: for the former, this is done manually to
 157 preserve their sharp edges; for the latter, this is undertaken inside a custom sand-paper box
 158 shaken at 1000 rpm by a (Hauschild DAC 150.1 FVZ) speed mixer. Since the typical values
 159 of the Young modulus E of the three materials is of the order of few gigaPascals, the ratio
 160 between viscous and elastic forces $\mathbb{B} = 8\pi\mu\dot{\gamma}\ell^4/(Ea^4\pi/4)$ is vanishingly small ($\mathbb{B} \sim 10^{-5}$)
 161 and the particles do not deform within the flow (Du Roure *et al.* 2019).

162 The fluids used in the experiments are prepared by mixing pure water and citric acid until
 163 the solution meets the estimated density of the selected particle. The density of the fluid
 164 ρ_f is sampled by means of a highly accurate (Antoon Par) densimeter with an estimated
 165 uncertainty of $4\ \text{kg}/\text{m}^3$. Experiments are performed at imposed shear rate $\dot{\gamma}$ and the fluid

Name	Shape	r	ℓ (mm)	a (mm)	κ	Batch
ELL02	Oblate spheroid	0.20 ± 0.01	0.58 ± 0.02	2.91 ± 0.01	0.22 ± 0.01	I
ELL06	Oblate spheroid	0.56 ± 0.01	1.27 ± 0.01	2.28 ± 0.01	0.17 ± 0.01	I
ELL2	Prolate spheroid	1.72 ± 0.02	1.78 ± 0.01	1.04 ± 0.01	0.13 ± 0.01	I
ELL3	Prolate spheroid	2.67 ± 0.06	2.53 ± 0.01	0.95 ± 0.02	0.19 ± 0.01	I
ELL5	Prolate spheroid	5.1 ± 0.1	2.64 ± 0.01	0.52 ± 0.01	0.20 ± 0.01	I
ELL9	Prolate spheroid	9.0 ± 0.1	2.62 ± 0.01	0.29 ± 0.01	0.19 ± 0.01	I
ELL13	Prolate spheroid	13 ± 1	4.7 ± 0.2	0.36 ± 0.02	0.35 ± 0.02	I
CYL005	Disk	0.05 ± 0.01	0.14 ± 0.01	2.58 ± 0.06	0.19 ± 0.09	III
CYL009	Disk	0.10 ± 0.01	0.15 ± 0.02	1.51 ± 0.01	0.11 ± 0.01	III
CYL01	Disk	0.11 ± 0.01	0.31 ± 0.01	2.90 ± 0.04	0.21 ± 0.03	III
CYL02	Disk	0.20 ± 0.01	0.31 ± 0.01	1.56 ± 0.02	0.12 ± 0.01	III
CYL06	Disk	0.56 ± 0.01	1.03 ± 0.01	1.85 ± 0.02	0.14 ± 0.01	III
CYL2	Fibre	1.33 ± 0.02	1.78 ± 0.02	1.34 ± 0.01	0.13 ± 0.01	I
CYL10	Fibre	9.0 ± 0.2	2.59 ± 0.01	0.29 ± 0.01	0.19 ± 0.01	I
CYL15	Fibre	15.4 ± 0.2	3.48 ± 0.01	0.23 ± 0.01	0.26 ± 0.01	II
CYL20	Fibre	20.5 ± 0.7	4.54 ± 0.01	0.22 ± 0.01	0.34 ± 0.01	II

Table 1: Characteristics of all the particles used in the experiments. Columns from left to right: code name, shape, mean aspect ratio r , half-length ℓ , radius a , confinement ratio κ , and identification of the production method.

166 density is constrained to match that of the particle ($\rho_p = \rho_f$), yielding $Re_p = St$, with
167 Re_p and St defined in § 1. This implies that inertia is mainly controlled by tuning the fluid
168 viscosity μ . This is achieved by adding a certain percentage of Ucon oilTM to the solution,
169 which increases its viscosity from that of pure water up to $\sim 1 Pa \cdot s$ proportionally to its
170 concentration in our experiments, while also slightly decreasing its density.

171

2.2. Shearing cell

172 The shearing cell apparatus is displayed in panel (a) of figure 3. It is located in a room at
173 a controlled temperature of $23 \pm 1^\circ C$ and consists of a small tank (500 mm long, 40 mm
174 wide, and 90 mm deep) with 10 mm thick transparent walls on the long sides which has
175 been adapted from the previous work of Metzger & Butler (2012). Two metallic cylinders of
176 diameter 20 mm hang from the lid of the tank and are equipped with polylactic acid (PLA)
177 cylindrical supports that increase their diameter to $L_y = 27$ mm. One is free to rotate and
178 the other is coupled to a transmission shaft through a rolling bearing. A transparent belt is
179 kept under tension between the two cylinders similarly to the photographic roll used in film
180 cameras. It is cut into a rectangular sheet from a 0.1 mm thick flexible MylarTM film and
181 has its extremities kept together by metal staplers. Laser-cut holes in its upper edge enable a
182 gear wheel glued to the rotating cylinder to grip the belt. A rotating motor powered by a DC
183 power supply is connected to the transmission shaft through a drive belt to rotate the gear
184 wheel and drive the transparent belt inside the cell in an infinite loop at constant velocity.
185 The belt is 70 mm tall, extending from nearly the bottom of the shear cell to approximately
186 15 mm above the free surface.

187 The fluid within the two parallel sides of the belt is submitted to a confined linear shear
188 where the neutral zero-velocity line lies between the two moving walls. The absolute system
189 of reference is defined as x being the flow direction, y the gradient direction, and z the
190 vorticity direction, parallel to the gravity. In our experiments, the origin is always set at the
191 initial position of the particle placed by hand at the centre of the shear flow. The confinement
192 ratio of the particles is defined as the ratio between the particle length and the distance
193 between the two sides of the transparent belt as $\kappa = 2\ell/L_y$ (Zettner & Yoda 2001). Mean

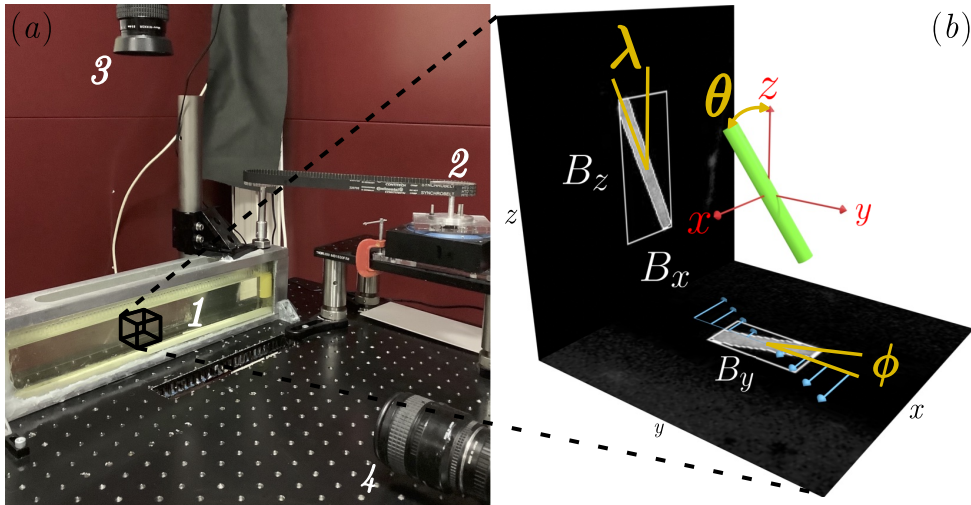


Figure 3: (a) Picture of the experimental setup, where the tank filled with fluid between the two sides of the transparent plastic belt is visible (1), together with the rotating motor and the motion transmission system (2), and the two cameras, looking at the (x, y) flow-gradient plane (3) and at the (x, z) flow-vorticity plane (4). (b) Sketch of the output given by the dual-camera video-recording system. The reference frame is defined at the centre of the particle. The azimuthal and polar angles, ϕ and θ , as well as the projected angle, λ , and the three components of the Axes-Aligned Bounding Box, \mathbf{B} , are also represented. The two recorded frames are displayed on the corresponding flow-gradient and flow-vorticity planes, to appreciate the contrast between the particle projections and the background as well as the detected particle contours and Axes-Aligned Bounding Boxes.

μ (Pa s)	$\dot{\gamma}$ (s^{-1})
0.054 ± 0.001	3.16 ± 0.03
0.400 ± 0.001	3.64 ± 0.05

Table 2: Shear rate measurements for two low and high values of viscosity. Mean values and uncertainties are calculated over 500 velocity fields obtained by Particle Image Velocimetry. The effective shear rates for the other fluids used are obtained by linear interpolation once the fluid characteristics have been finely tuned to match the density of the particles. It is possible to appreciate the small increase of the shear rate $\dot{\gamma}$ with increasing viscosity μ .

194 values of the confinement ratio are reported in table 1. These values are similar to that
 195 considered in the simulations by Rosén *et al.* (2015), namely $\kappa = 0.2$. Based on the results
 196 of these simulations, a limited confinement effect on some statistical observables may be
 197 expected. This is discussed in more detail in § 3.3 and in §4.

198 The shearing flow is examined by shedding light onto three different (x, y) planes at
 199 different depths and using Particle Image Velocimetry (PIV). These sections are chosen
 200 at least 10 mm below the free surface and above the small unconfined fluid layer laying
 201 at the bottom of the cell to characterise a confined region of ~ 30 mm depth where to
 202 operate the experiments. The viscosity of the fluid μ is characterised by accurate rheological
 203 measurements with an error of 0.001 Pa s on the measurements fully taking into account the
 204 real uncertainties. PIV measurements of the shear rate are performed only for two low and

205 high values of viscosity, reported in table 2. Then, the effective shear rate for each experiment
 206 is estimated by linear interpolation between these two measurements, once the fluid properties
 207 have been fixed and its density matched to that of the given particle. Secondary circulations
 208 are naturally present in such confined flow and can be characterised by finely tuning the
 209 parameters of the PIV. They appear to be insignificant when compared to the mean shear.
 210 Also, no significant variation of the shear rate is observed across the depth of the region of
 211 interest in the shear cell. To summarise, we observe a linear shear rate in a three-dimensional
 212 region the sizes of which are determined by the width of the camera field in the flow direction
 213 ($L_x = 140$ mm), by the distance between the two inner sides of the transparent plastic belt in
 214 the gradient direction ($L_y = 27$ mm) and by the depth of the different PIV measurements in
 215 the vorticity direction ($L_z = 30$ mm).

216

2.3. Measurements

217 A multi-view video recording system is deployed in the experiments, as shown in panel (b)
 218 of figure 3. Jeffery orbits are three-dimensional, but given the axisymmetric nature of the
 219 particles, the number of variables that specify the orientation reduces to two. Therefore,
 220 measurements of the particle spin were not performed. The objective of this multi-view
 221 system is to provide a three dimensional reconstruction of the orientation of each particle
 222 given by its direction vector \mathbf{n} . Whereas three or more particle projections would permit
 223 a direct resolution (Eberly 1999), we are limited to two complementary images of the
 224 axisymmetric particle which are post-processed to reconstruct the orientation as explained
 225 in § 2.4.

226 The cameras are two Allied Prosilica GX1910 cameras, with a resolution of 1920×1080
 227 pixels. Both cameras are equipped with a Nikon Micro-Nikkor 55 mm f2.8 objective, imaging
 228 the shear cell from a distance of approximately half a meter with a resolution of 20 pixels
 229 per mm. As they have their axes perpendicular, these cameras provide two complementary
 230 images: one of the flow-vorticity plane and the other of the flow-gradient plane, as seen in
 231 figure 3(a). Considering that the depth of field is enhanced to an estimated value of 3 cm, we
 232 verified that distortion and diffraction phenomena are negligible by measuring a check-board
 233 patterned object in the flow volume of interest inside the shear cell. The two cameras are
 234 controlled by an in-house developed Matlab script.

235 Experiments are performed as follows. After having filled the cell with the density matched
 236 fluid and started shearing, a single particle is manually positioned at the centre of the camera
 237 fields, as close as possible to the neutral plane in the middle of the shearing flow. The particle
 238 is pre-sheared for a short time to avoid any influence of the positioning operation on its
 239 dynamics. Note that, when repeated, this positioning method results in randomly-varying
 240 initial orientations despite all the care that can be taken. Then, the recording of the two
 241 videos is started and the cameras are synchronised by a manually activated light signal. The
 242 recording is stopped before the particle leaves the camera fields. Experiments are typically
 243 repeated between 5 and 10 times for a given particle at each particle Reynolds number, as
 244 reported in table 3. Typically, at least two complete periods of rotation must be observed in
 245 order to validate the run. This requirement happened to be rather challenging for the most
 246 slender particles.

247

2.4. Image processing

248 Each recorded video is processed by an in-house developed Python script based on the Canny
 249 method (Canny 1986), implemented using the OpenCV library (Bradski 2000). The script
 250 performs a simple detection routine based on the contrast between the background and the
 251 object to be detected. After having reduced the camera field to a small square section of size

Particle	$Re_p, n_{runs}, \overline{\Delta t_{run}}/T$							
ELL02	0.040, 5, 6.06	0.05, 5, 2.77						
ELL06	0.010, 5, 8.40	0.02, 5, 6.90	0.03, 5, 4.62	0.44, 11, 6.46	1.04, 6, 9.71			
ELL2	0.003, 5, 5.49 0.120, 10, 8.65	0.006, 5, 6.39 0.132, 10, 3.22	0.023, 5, 3.09 0.329, 10, 5.75	0.038, 5, 16.17 0.375, 4, 11.07	0.042, 5, 4.06	0.073, 10, 3.37	0.085, 8, 13.28	
ELL3	0.544, 7, 7.20	1.300, 6, 5.31						
ELL5	0.008, 5, 1.68	0.014, 5, 2.63	0.051, 5, 2.69	0.092, 5, 2.04	0.161, 10, 1.74	0.291, 10, 1.67		
ELL9	0.007, 5, 2.63	0.010, 5, 2.07	0.048, 10, 2.16	0.088, 10, 1.70	0.158, 6, 1.73	0.286, 5, 2.03	0.642, 9, 3.63	
ELL13	0.180, 5, 2.00							
CYL005	0.032, 5, 2.15	0.100, 10, 2.65	0.236, 10, 2.20	0.247, 10, 2.72	0.347, 11, 2.53	0.778, 10, 2.44		
CYL009	0.011, 5, 2.54	0.088, 10, 5.12	0.195, 11, 2.84	0.344, 10, 3.40				
CYL01	0.042, 5, 3.57	0.074, 5, 3.22	0.336, 10, 4.99	1.318, 10, 3.42				
CYL02	0.005, 5, 3.73	0.009, 5, 6.16						
CYL06	0.039, 10, 8.27	0.246, 10, 12.85						
CYL2	0.002, 5, 5.51	0.005, 10, 6.48	0.033, 10, 4.23	0.059, 10, 6.98	0.187, 10, 4.31	0.336, 5, 7.57		
CYL9	0.007, 5, 2.51 0.180, 13, 3.24 0.790, 10, 3.53	0.013, 5, 2.28 0.230, 5, 1.39 0.986, 14, 2.85	0.047, 5, 2.03 0.250, 10, 2.52	0.080, 10, 5.33 0.260, 11, 2.82	0.085, 5, 2.12 0.410, 10, 1.73	0.100, 10, 3.98 0.480, 10, 3.78	0.150, 10, 4.39 0.510, 15, 3.03	
CYL15	0.059, 5, 2.08							
CYL20	0.100, 5, 1.89							

Table 3: For each particle type used in the experiments (first column), the particle Reynolds number, Re_p , the number of runs, n_{runs} , and the mean duration of the shearing normalized by the (experimentally measured) mean period of rotation, $\overline{\Delta t_{run}}/T$, are provided.

252 $4 \max(\ell, a) \times 4 \max(\ell, a)$ around the particle, a Gaussian blur and manual threshold filters
253 are applied to reduce the noise and improve the contrast by smoothing out possible defects
254 of the transparent plastic belt. At this point, there is a strong intensity gradient between the
255 particle and the background. Identifying this gradient using the Canny edge filter provides
256 the closed contour of the given particle. A least-squared optimisation method yields the
257 orientation angle (fitEllipse function of the OpenCv library), while the extent of the contour
258 in the aligned directions leads to the estimation of its minimum two-dimensional Axis-
259 Aligned Bounding Box (boundingRect function of the OpenCv library). By Axis-Aligned
260 Bounding Box (AABB hereinafter), we mean the smallest rectangle (parallelepiped when
261 generalising to three dimensions) that is tangent to the particle projection (particle in three
262 dimensions). The three measurements are stored and the script analyses the videos frame by
263 frame. The accuracy of the proposed particle detection method has been evaluated against
264 8100 virtual images of randomly oriented disks and fibres, with a resolution of 7 pixels over

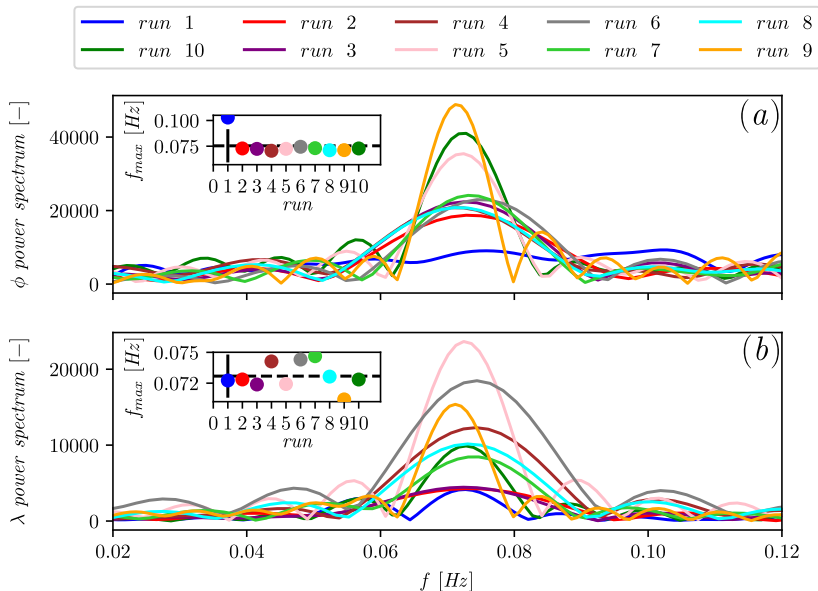


Figure 4: Power spectra of the two projected angles (a) ϕ and (b) λ , for 10 different experimental runs for the fibre CYL10 of aspect ratio $r = 9$ at $Re_p = 0.08$. The insets in each panel show the maximum peak frequency versus the considered runs.

265 the smallest particle dimension. We found that the Euclidean norm error between theoretical
 266 and measured AABBs is below 4 pixels, while the absolute error on the measured angle
 267 rarely reached a value above 2° .

268 Two lists of positions, orientations, and AABB components of the particle projections
 269 onto the flow-gradient and in the flow-vorticity planes in time are the result of the post-
 270 processing of each experiment made by the Python script. Then, by a suitable re-scaling,
 271 it is possible to combine the information collected by the two cameras and produce three-
 272 dimensional measurements of the translation and rotation of the particles. The procedure
 273 is non-trivial for the estimation of the orientation of each particle because, as displayed in
 274 figure 3, while the particle projection in the (x, y) plane provides the azimuthal angle ϕ ,
 275 the projection in the (x, z) plane determines the angle λ , which only corresponds to θ in
 276 the flow-aligned positions. Therefore, given the possibility of producing long lists of three-
 277 dimensional AABBs by knowing the particle orientation vector, we choose a data-driven
 278 method to regress the orientation of the particle by a two-camera view system. As detailed in
 279 Appendix A, we deploy a simple Deep Learning model, implemented using Tensorflow and
 280 trained over synthetic data, to perform a three-variable regression and estimate the particle
 281 orientation vector \mathbf{n} from the experimentally measured three-dimensional AABBs.

282

2.5. Data analysis

283 The data processing of § 2.4 provides the orientation of the axisymmetric particle. This is
 284 the key information needed when constructing the Jeffery orbits, discussed in § 3.1, and
 285 obtaining the time evolution of the three components of \mathbf{n} , discussed in § 3.3. There are two
 286 other important quantities: The period of rotation and the orbit constant, described below
 287 and used in § 3.2 and § 3.3, respectively.

288 Since experimental runs can last up to several particle rotations, the measured projected
 289 angles, ϕ and λ , can be interpreted as time series and characterised by Fourier analysis.
 290 The Fourier transform of the angular signals can be computed. As a typical example, the

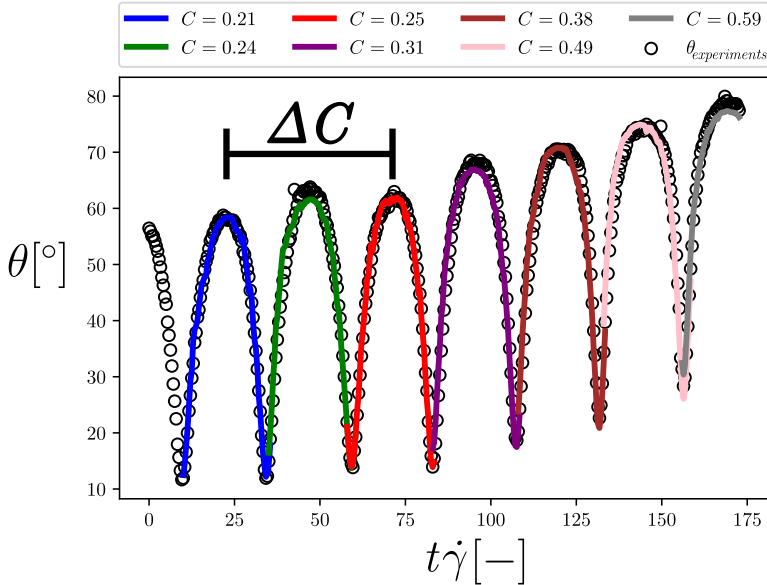


Figure 5: Time-evolution of the reconstructed angle $\theta = \arccos n_3$ (black circles). The fitted forms of equation (1.3) are also plotted as solid lines. The different colours correspond to the estimated values of the orbit constant C reported in the legend. The quantity ΔC represents the variation of the orbit constant in each of the measurements.

291 power spectra of ten experimental runs are plotted against the frequency in figure 4
 292 for the fibre CYL10. This figure shows that the signal power is coherently resonating around
 293 a characteristic frequency, calculated as the frequency f_{max} of the peak of each power
 294 spectrum. The period of rotation of the given particle, T , is then estimated as the inverse
 295 of this frequency. Measurements are then collected over several runs for a sound statistical
 296 characterisation, choosing between the angles ϕ and λ according to a minimum squared error
 297 criterion. Averages are computed and uncertainties are estimated as standard deviations over
 298 all the experimental runs at a given Re_p .

299 The components of the reconstructed particle orientation vector \mathbf{n} can provide the azimuthal
 300 and polar angles, defined as:

$$301 \quad n_1 = \sin(\phi) \sin(\theta), \quad (2.1)$$

$$302 \quad n_2 = \cos(\phi) \sin(\theta), \quad (2.2)$$

$$303 \quad n_3 = \cos(\theta). \quad (2.3)$$

304 Equation (1.3) can then be fitted over the reconstructed values of ϕ and θ to estimate the orbit
 305 constant C of the given Jeffery orbits, as displayed in figure 5. This is done by a non-linear
 306 least squares minimisation (curvefit function of the Scipy Python library) over an observation
 307 window manually centred around each flow-aligned position and with total width comparable
 308 to half the period of rotation, producing two separate orbit constant measurements for each
 309 Jeffery orbit. Orbit constant variations ΔC are then calculated as discrete differences over
 310 one period of rotation. This choice is motivated by a classical approach to the analysis of
 311 periodic dynamical systems (Glendinning 1994), according to which the flow of the orbit
 312 constant C is obtained by its discrete variations between flow-aligned positions over one
 313 period T . Recalling the orbit coordinates τ and C introduced by Leal & Hinch (1971), this
 314 is equivalent to dropping the temporal dependency of the system (τ) and focusing only on
 315 the evolution of its phase (C), similarly to what done in Dabade *et al.* (2016). Therefore, the

316 intensity of the orbit variation ΔC has a sign indicating the direction of the orbit constant
 317 motion, i.e. positive when towards the tumbling orbit ($C = \infty$) and negative when towards
 318 the spinning orbit ($C = 0$), and a value the magnitude of which quantifies the stability of a
 319 given Jeffery orbit (more unstable orbits will experience stronger variations).

320 3. Experimental results

321 3.1. Jeffery orbits

322 Typical experimental Jeffery orbits for a fibre of aspect ratio $r = 9$, for an oblate spheroid
 323 of aspect ratio $r = 0.6$, and for the disk of aspect ratio $r = 0.1$ are shown in figure 6 (a),(b)
 324 and (c),(d) and (e),(f) respectively. The coloured dots represent the intersection of the axis
 325 given by the orientation vector \mathbf{n} with the half sphere of radius ℓ for the prolate particles
 326 and a for the oblate particles, respectively. The location of the intersection was captured
 327 during three different experimental runs for each panel and reconstructed as detailed in § 2.4.
 328 The corresponding Jeffery orbits at zero inertia are also displayed as solid black lines. They
 329 were obtained by integration of equation (1.1) using the first flow-aligned orientation of each
 330 experimental run as initial condition.

331 At low inertia, i.e. for the fibre of panel (a) at $Re_p = 0.08$, the spheroid of panel (c) at
 332 $Re_p = 0.02$, and the disk of panel (e) at $Re_p = 0.05$, there is no significant change between
 333 the successive rotations for runs with different initial conditions, i.e. different orbit constants.
 334 The experimental orbits are in good agreement with the theoretical Jeffery orbits, represented
 335 by the black spherical ellipses.

336 As inertia becomes finite, i.e. for the fibre of panel (b) at $Re_p = 1$, the spheroid of panel
 337 (d) at $Re_p = 0.43$, and the disk of panel (f) at $Re_p = 1.32$, the picture slightly changes. The
 338 experimental orbits are still spherical ellipses, but they depart from the zero-inertia orbits as
 339 their shapes evolve between successive rotations.

340 For the fibre of panel (b), the orbits are expanding towards the tumbling orbit ($C = \infty$) on
 341 the equator of the sphere, i.e. in the (x, y) flow-gradient plane. It is worth mentioning that
 342 the orbits are not equally unstable as the fibre experiences a stronger drift when describing
 343 intermediate orbits (run 13) than when close to the rolling (run 14) and tumbling (run 3)
 344 cycles.

345 The oblate particles are also drifting through consecutive orbits, but their behaviour is
 346 more complex as two limiting stable orbits exist. As shown in panel (d), the spheroid can
 347 either drift to a tumbling orbit ($C = \infty$) in the plane of shear (run 10) or to a spinning orbit
 348 ($C = 0$), i.e. aligning its orientation vector with the direction of vorticity, z (runs 5 and 8).
 349 It is interesting to note that the two orbits closer to the pole of the half sphere (runs 5 and
 350 8) are attracted toward the spinning orbit ($C = 0$) while the other orbit, which is starting
 351 with a much larger orbit constant (run 10), is drawn toward the tumbling orbit ($C = \infty$) on
 352 the equator of the half sphere. A similar description holds for the disk of panel (f), which
 353 exhibits both consecutive rotations in the tumbling orbit ($C = \infty$, run 2) and a systematic
 354 drift towards the spinning orbit ($C = 0$, runs 1 and 9).

355 A comment is in order regarding the separation between consecutive rotations, which
 356 appears larger for the more slender particles, i.e. the fibre with $r = 9$ of panel (b) and the
 357 disk with $r = 0.1$ of panel (f), than in the case of the spheroid with $r = 0.6$ of panel (d).

358 3.2. Period of rotation

359 The dimensionless period of rotation, $T\dot{\gamma}/2\pi$, of the axisymmetric particles is displayed
 360 against particle aspect ratio, r , in figure 7. The main panel (a) shows all the results obtained
 361 for both prolate and oblate particles, while the two smaller panels distinguish between (b)

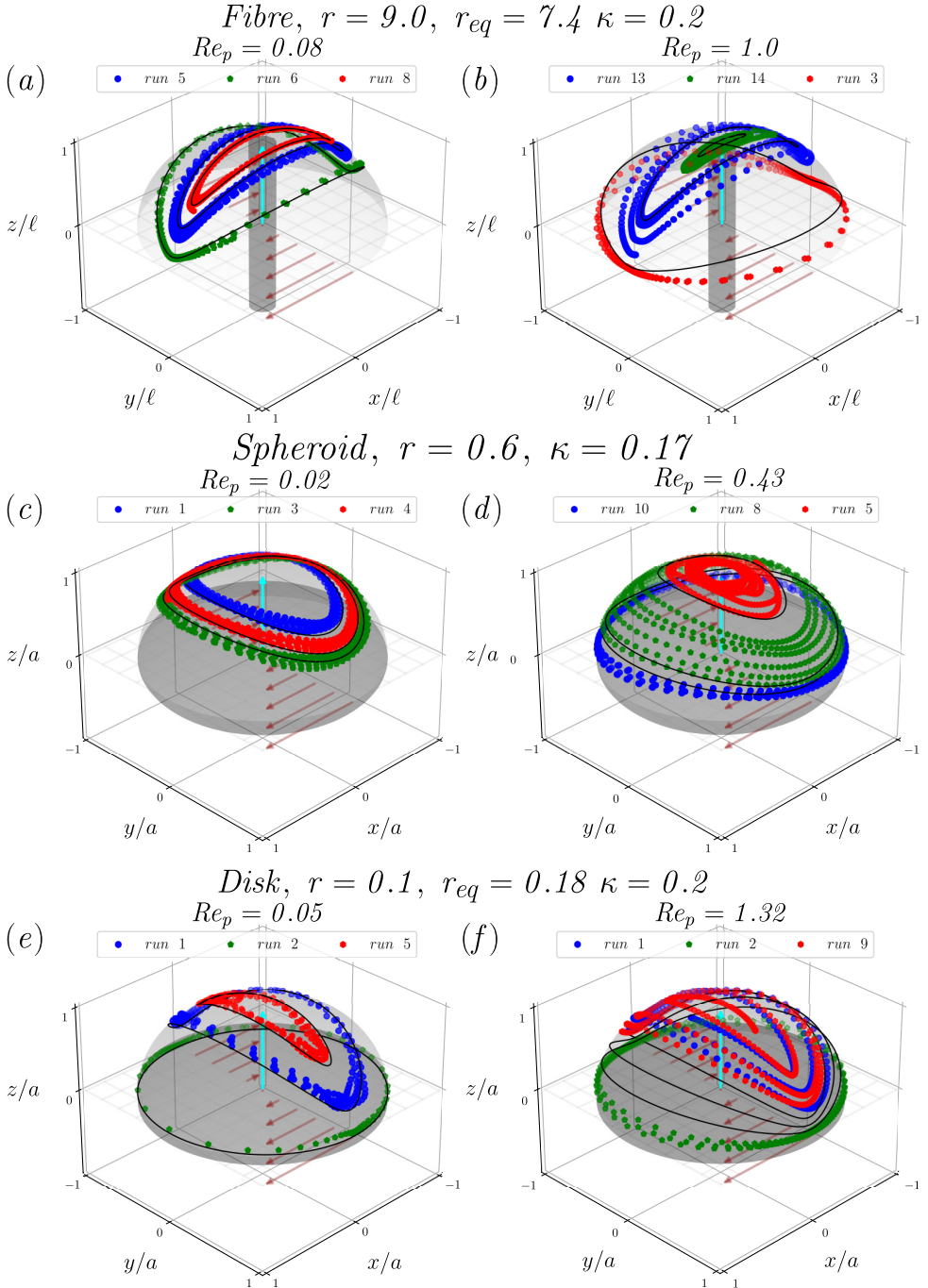


Figure 6: Experimental Jeffery orbits at two Reynolds numbers for the fibre CYL10 (top-row panels), the spheroid ELL06 (middle-row panels) and the disk CYL01 (bottom-row panels): (a) Fibre, $r = 9.0$, $Re_p = 0.08$; (b) Fibre, $r = 9.0$, $Re_p = 1.0$; (c) Spheroid, $r = 0.6$, $Re_p = 0.02$; (d) Spheroid, $r = 0.6$, $Re_p = 0.43$; (e) Disk, $r = 0.1$, $Re_p = 0.05$; (f) Disk, $r = 0.1$, $Re_p = 1.32$. The particles considered in this figure are shown in the vorticity-aligned position with their orientation vector \mathbf{n} highlighted in cyan. The coloured dots represent the intersections of the axis given by the orientation vector \mathbf{n} with the half sphere of radius ℓ for the prolate particles and a for the oblate particles, respectively. The corresponding Jeffery orbits are also displayed as solid black lines and were obtained by integrating equation (1.1) from an initial condition given by the first flow-aligned orientation of each experiment. See Supplementary Materials for animations.

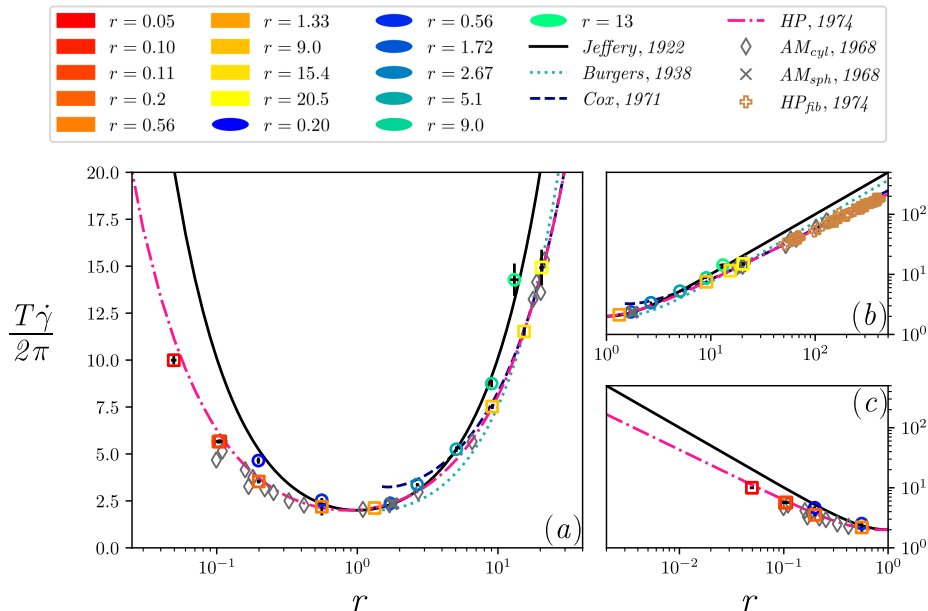


Figure 7: Period of rotation, T , of the axisymmetrical particles against the particle aspect ratio r . The period is made dimensionless using the shear rate $\dot{\gamma}$ and normalised by a factor 2π . Panels: (a) both prolate and oblate, (b) prolate, and (c) oblate particles. The experimental values are displayed as coloured rectangles (cylindrical particles) and circles (spheroidal particles). Each point is the average over all the available experiments for all particle Reynolds numbers ($Re_p \lesssim 1$). The theories of Jeffery (1922) and Burgers (1938), the semi-empirical correlation of Cox (1971), and the empirical expression of Harris & Pittman (1975) are displayed as a solid black line, a dotted cyan line, a dashed blue line and a dash-dotted pink line, respectively. The experiments of Anczurowski & Mason (1968) are displayed as empty grey diamonds (cylinders) and one solid x (spheroid), while the data of Harris & Pittman (1975) correspond to empty brown plus symbols. See Supplementary Materials for the directory of the figure including the data and the Jupyter notebook.

362 prolate and (c) oblate shapes and focus on the asymptotic limits. The data are obtained by
 363 averaging over all the available experiments for all particle Reynolds number ($Re_p \lesssim 1$),
 364 meaning that we choose in these plots not to take into account any possible influence of
 365 inertia on the period of rotation. Such influence will be addressed later at the end of this
 366 section. The data are displayed as empty rectangles for cylindrical particles and circles for
 367 spheroidal particles and explore a moderate range of aspect ratios ($0.05 \lesssim r \lesssim 21$). They
 368 complement the previous experimental results of Anczurowski & Mason (1968) for disks
 369 and fibres (empty grey diamonds) as well as one prolate spheroid (solid grey x), and those
 370 of Harris & Pittman (1975) for fibres with higher aspect ratio (empty brown pluses), also
 371 reported in the figure.

372 The present measurements for the period of spheroidal particles span over two decades of
 373 r around $r = 1$, extending over the experimental dataset of Anczurowski & Mason (1968).
 374 Their agreement with the theory of Jeffery (1922), $T_J \dot{\gamma} = 2\pi(r + 1/r)$, displayed as a solid
 375 black curve, is excellent. We remark here that the expression just given can be written as
 376 $2\pi/T_J = (\dot{\gamma}/2)\sqrt{1 - \Lambda^2}$ if the Bretherton constant, or shape parameter, $\Lambda = (r^2 - 1)/(r^2 + 1)$
 377 is used. In contrast, the measured period of cylindrical particles systematically lies below
 378 the Jeffery curve, meaning that the period of a cylinder is always smaller than that of the
 379 corresponding spheroid at the same r . This difference is minimal around $r = 1$ and increases

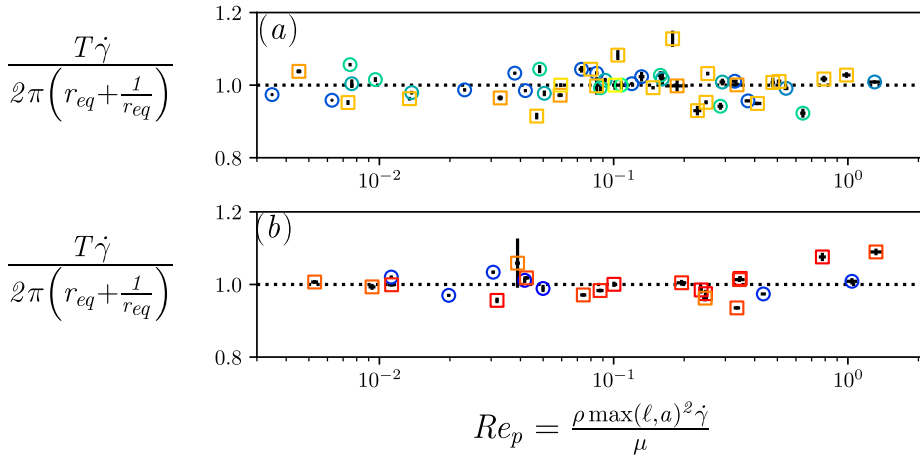


Figure 8: Period of rotation, T , of the axisymmetrical particles against the particle Reynolds number, Re_p for: (a) fibres and prolate spheroids and (b) disks and oblate spheroids. The period is now normalised by the Jeffery period, $2\pi(r_{eq} + 1/r_{eq})/\dot{\gamma}$. The dotted black line corresponds thus to the Jeffery period within this normalisation. The experiments are displayed as coloured rectangles (cylindrical particles) and circles (spheroidal particles) with the same colour code for the aspect ratios as in figure 7. See Supplementary Materials for the directory of the figure including the data and the Jupyter notebook.

380 with increasing slenderness or flatness but not in the same manner. Interestingly, a shorter
 381 period of rotation is measured for the disks than for the fibres, as it is clearly evidenced by
 382 comparing the prolate CYL20 ($r = 20.5$) and the oblate CYL005 ($r = 0.05$). These data are
 383 in good agreement with those available in the literature (Anczurowski & Mason 1968; Harris
 384 & Pittman 1975).

385 Since the pioneering work of Trevelyan & Mason (1951), it has been suggested that an
 386 equivalent aspect ratio, r_{eq} , can be found for cylindrical particles to recover the Jeffery period.
 387 In particular, r_{eq} can be computed from $T\dot{\gamma} = 2\pi(r_{eq} + 1/r_{eq})$ using the measured period of
 388 rotation for T . Different expressions have been proposed for r_{eq} . In an earlier work for the
 389 case of fibres, Burgers (1938) showed that the disturbance caused by a cylinder of axis ratio
 390 r will be reproduced by an spheroid of $r_{eq} = 0.74r$, an expression which would eventually
 391 diverge at high r as well as underestimate the period of rotation in the near-sphere limit. Our
 392 measurements for $10 \lesssim r \lesssim 20$ show good agreement with this proposed equivalence. Later
 393 on, by fitting his asymptotic theory to the measurements of Anczurowski & Mason (1968),
 394 Cox (1971) provided an expression for slender cylinders as $r_{eq} = 1.24r/\sqrt{\log r}$. Our data
 395 are in good agreement with this formula for $r > 5$. Then, Harris & Pittman (1975) proposed
 396 an unweighted least square log-log fit over their measurements for slender rods leading to
 397 $r_{eq} = 1.14r/r^{0.156}$. They found that Cox's semi-empirical prediction was an overestimate in
 398 the asymptotic limit and their power-law function of r showed an excellent agreement with all
 399 the considered experimental measurements available at that time. This empirical correlation
 400 provides an excellent match with our present data for both prolate and oblate cylinders. It is
 401 even accurately predicting the additional measurements with the thinnest disk (CYL005, red
 402 rectangle in figure 7).

403 To conclude this section, we move to the influence of inertia on the period of rotation.
 404 Figure 8 shows the period of rotation, T , normalised by the Jeffery period, $2\pi(r_{eq} + 1/r_{eq})/\dot{\gamma}$,
 405 against the particle Reynolds number, Re_p . The data are now averaged over experimental runs
 406 at the same Re_p for each particle. While r_{eq} is equal to the aspect ratio r for the spheroidal

407 particles, its value for the cylindrical particles is determined by solving $T\dot{\gamma} = 2\pi(r_{eq} + 1/r_{eq})$
 408 using their measured period of rotation, T , at each Re_p . For both prolate and oblate particles,
 409 there is a good collapse of the data around unity, corresponding to the Jeffery period. The
 410 data are scattered within $\pm 20\%$ but do not indicate any systematic trend with increasing
 411 inertia. We can conclude that inertia does not affect significantly the period of rotation, T , for
 412 $Re_p \lesssim 1$, at least within the range of aspect ratios considered in our study. For much longer
 413 (very slender) fibres, some alignment might be observed in the presence of small inertia, as
 414 the findings of Subramanian & Koch (2005) seem to suggest, the same being possible for
 415 very thin disks, as suggested by the results of Rosén *et al.* (2015).

416

3.3. Drift

417 Having previously looked at the general behaviours of the Jeffery orbits in figure 6, we now
 418 examine in detail the influence of inertia on the time evolution of each component of the
 419 orientation vector \mathbf{n} . Figures 9, 10 and 11 display the three components of \mathbf{n} in the flow (n_1),
 420 gradient (n_2), and vorticity (n_3) directions against the dimensionless time, $t\dot{\gamma}$. For clarity of
 421 presentation, we have chosen to focus the discussion on a subset of three runs for a typical
 422 fibre and a disk at a small but finite Re_p and at a larger Re_p as well as for an oblate spheroid
 423 at a sole moderate Re_p . We also compare our results with the asymptotic theory of Einarsson
 424 *et al.* (2015a) as detailed in § B.1. It is important to stress that, while the theory considers an
 425 unbounded system, there is some degree of confinement in the experiments ($\kappa \approx 0.2$) which
 426 may affect the stability of the orbits (Rosén *et al.* 2015). Confinement effects are further
 427 discussed in § 4.

428 Let us first examine the influence of inertia on the dynamics of the fibre with aspect
 429 ratio $r = 9$ in figure 9. The general tendency is that the fibre drifts out of the vorticity axis
 430 towards the tumbling orbit in the plane of shear as evidenced by the systematic decrease of
 431 n_3 with successive oscillations. This effect is stronger for intermediate particle orientations,
 432 see runs 5 and 8 of panel (a) and runs 13 of panel (b), whereas it may appear somehow
 433 irregular for vorticity aligned orbits, see runs 2 of panel (a) and run 14 of panel (b). At small
 434 $Re_p = 0.15$, there is a good agreement between the theory and the experiments, see figure
 435 9 (a), suggesting that the range of application of the asymptotic theory of Einarsson *et al.*
 436 (2015a) can be extended to finite values of $Re_p \sim O(10^{-1})$ in the case of the fibres. The
 437 agreement is inevitably lost at larger values, e.g. $Re_p \sim O(1)$. At such Re_p , a faster drift
 438 toward the tumbling orbit is systematically obtained as compared to the experiments, as can
 439 be seen from figure 9 (b) in particular for the case of run 13.

440 The discussion is now repeated for the disk with aspect ratio $r = 0.05$ in figure 10. Most of
 441 the runs show a tendency for the disk to drift towards the vorticity axis, i.e. to move towards
 442 the spinning orbit where the particle lays flat in the flow-gradient plane. However, a few runs
 443 (with carefully-tuned initial conditions) show disks that are just tumbling with their axis in the
 444 plane of shear. This confirms that, unlike prolate particles (which are always driven towards
 445 the sole limiting tumbling orbit), oblate particles can drift towards two different orbits, the
 446 spinning or the tumbling orbits, depending upon their initial orientations. Whereas the theory
 447 of Einarsson *et al.* (2015a) successfully predicts the existence of two limiting orbits, it gives a
 448 much stronger overestimation of the drift through consecutive orbits for disks than for fibres,
 449 even for the smallest values of Re_p studied. This seems to imply that the range of application
 450 of the asymptotic theory is more limited in the oblate case than in the prolate case. This is
 451 clear in figure 10 (a) for $Re_p = 0.24$, but becomes even more evident in figure 10 (b) for
 452 $Re_p = 0.80$. At this larger Re_p , the theory predicts a rapid shift towards aligned positions
 453 while the experiments still show rotational behaviours.

454 Finally, we consider the oblate spheroid ELL06 with $r = 0.6$ at moderate $Re_p = 0.43$ in
 455 figure 11. At first glance, no remarkable difference emerges with regards to the slender disk

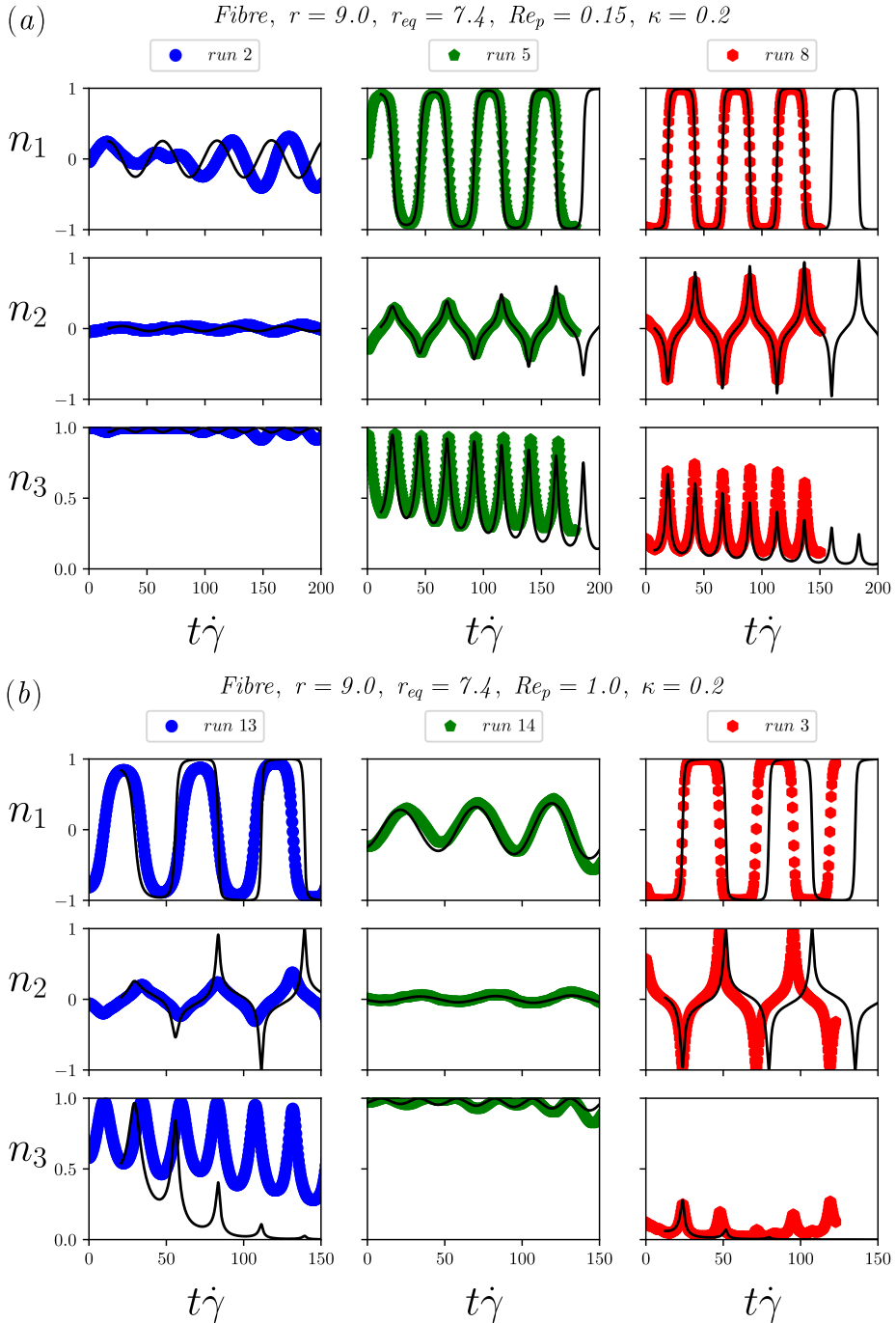


Figure 9: Evolution of the components of the orientation vector \mathbf{n} , displayed as vertically aligned panels for 3 typical runs against the dimensionless time $t\dot{\gamma}$, for the fibre CYL10 with aspect ratio $r = 9$ and confinement ratio $\kappa = 0.19$: (a) $Re_p = 0.15$; (b) $Re_p = 1.0$. Comparison with the theory of Einarsson *et al.* (2015a), presented in § B.1 is also given as black dashed lines. See Supplementary Materials for the directory of the figure including the data and the Jupyter notebook.

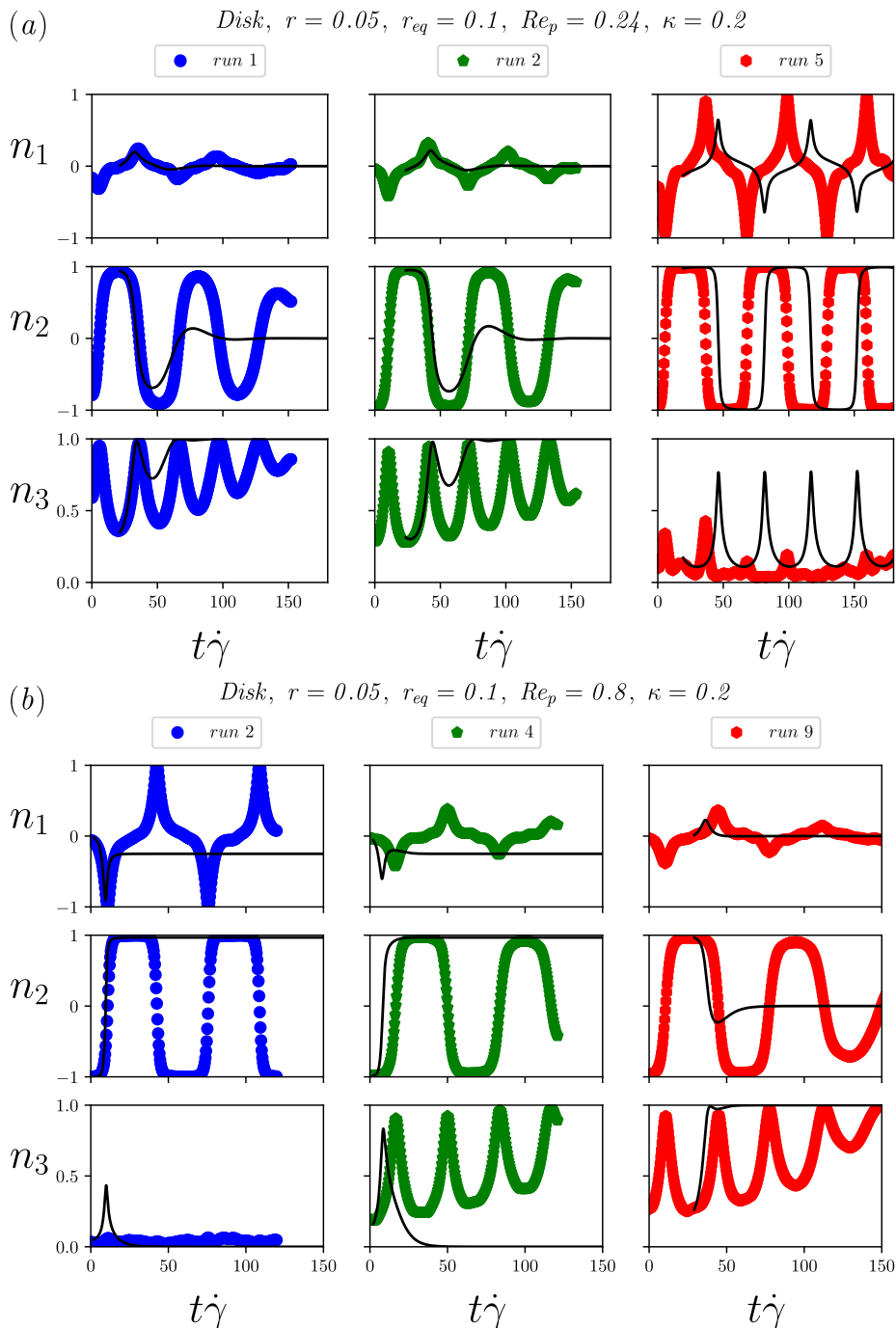


Figure 10: Evolution of the components of the orientation vector \mathbf{n} , displayed as vertically aligned panels for 3 typical runs against the dimensionless time $t\dot{\gamma}$, for the disk CYL005 with aspect ratio $r = 0.05$ and confinement ratio $\kappa = 0.19$: (a) $Re_p = 0.24$; (b) $Re_p = 0.8$. Comparison with the theory of Einarsson *et al.* (2015a), presented in § B.1 is also given as black dashed lines. See Supplementary Materials for the directory of the figure including the data and the Jupyter notebook.

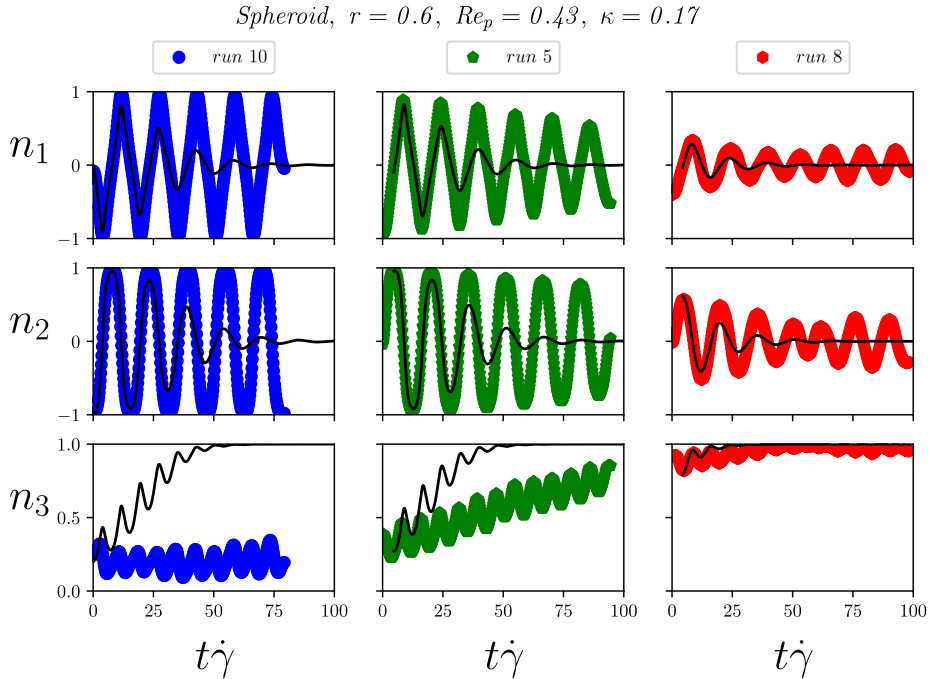


Figure 11: Evolution of the components of the orientation vector \mathbf{n} , displayed as vertically aligned panels for 3 typical runs against the dimensionless time $t\dot{\gamma}$, for the oblate spheroid ELL06 with aspect ratio $r = 0.6$ and confinement ratio $\kappa = 0.17$ at particle Reynolds number $Re_p = 0.43$. Comparison with the theory of Einarsson *et al.* (2015a), presented in § B.1 is also given as black dashed lines. See Supplementary Materials for the directory of the figure including the data and the Jupyter notebook.

456 CYL005, displayed in figure 10. This spheroid both tumbles in the plane of shear (run 10) or
 457 slowly drifts to the spinning orbit (runs 8 and 5). However, in addition to the already discussed
 458 discrepancy of the drift, there is a notable difference between the theory of Einarsson *et al.*
 459 (2015a) and the experiments regarding the dynamics of this nearly-spherical oblate spheroid.
 460 For this type of particle, the asymptotic theory only predicts the drift to an attracting spinning
 461 orbit at $Re_p \ll 1$. This is further discussed in connection to confinement effects in §4.

462 As previously mentioned in §1, the separation between the two limiting orbits for the
 463 oblate particles is predicted to depend on their initial orientations but also on their aspect
 464 ratio (Einarsson *et al.* 2015b; Dabade *et al.* 2016). To examine this difference in drift more
 465 closely and avoid the ambiguity of an arbitrary initial condition, we calculate the change in
 466 the orbit constant in a single Jeffery period as done in Dabade *et al.* (2016). As detailed in
 467 § 2.5, this implies the fitting of our experimental measurements to estimate the local values
 468 of the orbit constant C before taking discrete differences ΔC over each period of rotation.
 469 The results are shown in figure 12, where the normalised change in the orbit constant in a
 470 single period, $Re_p^{-1} \Delta C / (C^2 + 1)$, is plotted against the normalized orbit constant $C / (C + 1)$
 471 for all the experiments. Measurements referring to different values of Re_p are identified by
 472 different colours. The values $C / (C + 1) = 0$ and $C / (C + 1) = 1$ correspond to the spinning
 473 and tumbling modes, respectively. Following Dabade *et al.* (2016), to keep the drift finite in
 474 the near-sphere limit, the normalisation for ΔC also uses a factor depending on the particle
 475 eccentricity: $\xi_0 = \sqrt{1/(1 - 1/r^2)}$ for the prolate particles and $\xi_0 = \sqrt{1/(1 - r^2)}$ for the oblate
 476 particles.

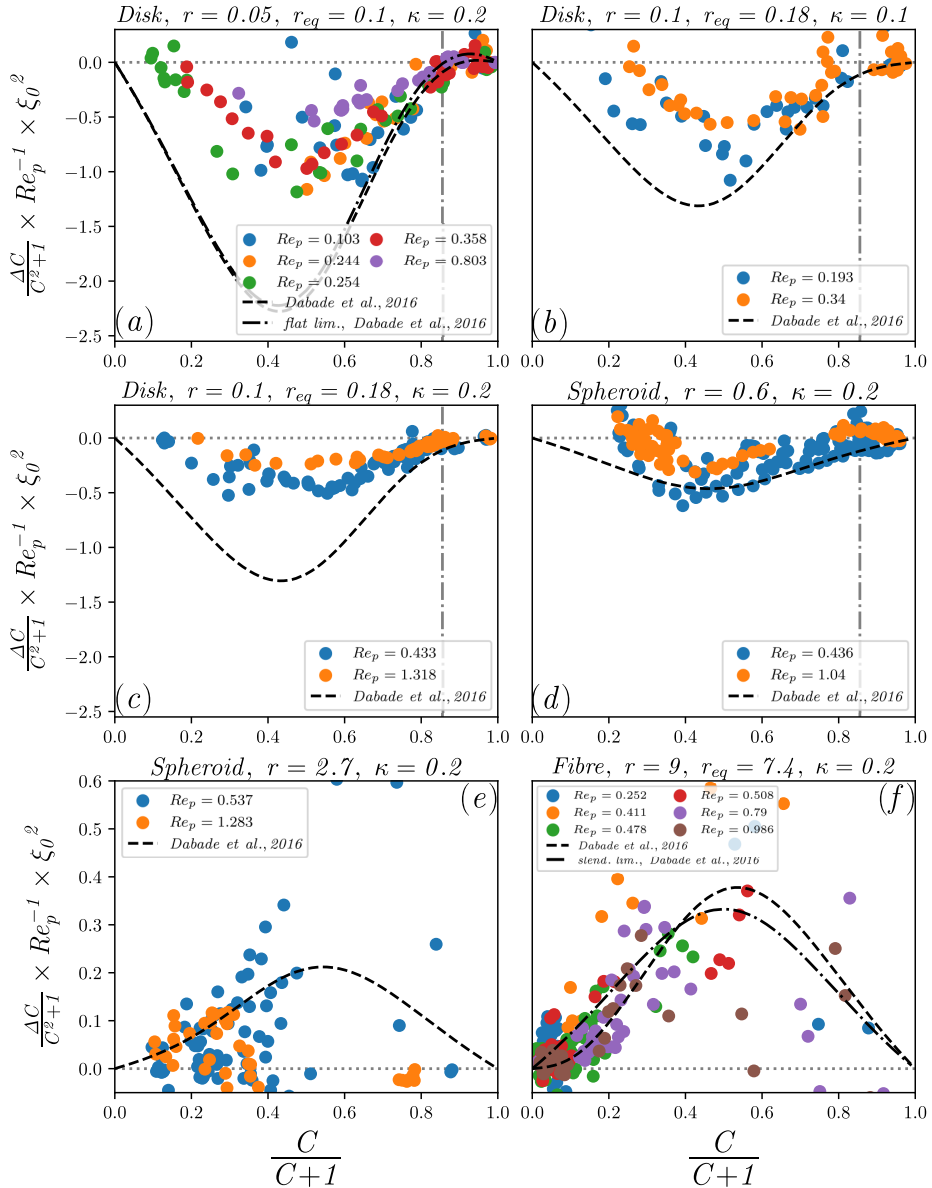


Figure 12: Discrete variation of the orbit constant over one period, ΔC , against the orbit constant C . Results refer to the following particle types (characterised by different colours for the varying Re_p): (a) disk CYL005 ($r = 0.05$ and $\kappa = 0.19$); (b) disk CYL009 ($r = 0.1$ and $\kappa = 0.11$); (c) disk CYL01, ($r = 0.11$ and $\kappa = 0.21$); (d) spheroid ELL06, $r = 0.56$ and $\kappa = 0.17$); (e) spheroid ELL3, ($r = 2.67$ and $\kappa = 0.19$); (f) fibre CYL10, ($r = 9.01$ and $\kappa = 0.19$). The drift is normalised to keep its value finite in the near-sphere limit as in Dabade *et al.* (2016), thus yielding a better comparison with their theory (black dashed line and black solid line in the slender limit). The predicted critical orbit constant $C^* = \sqrt{35}$, i.e. $C^*/(C^* + 1) \simeq 0.86$, separating the basins of attracting in the slender disk limit is also displayed as a dash-dotted grey line (Dabade *et al.* 2016). Data are collected over all the available experiments at a given Re_p (up to 10 runs), measuring the orbit constant and its variation as described in § 2.5. See Supplementary Materials for the directory of the figure including the data and the Jupyter notebook.

477 Figure 12 displays results for the following cases: the thinnest disk with $r = 0.05$ in
 478 panel (a), two disks having similar r but two different confinement ratios in panels (b)
 479 ($r = 0.1, \kappa = 0.1$) and (c) ($r = 0.11, \kappa = 0.2$), a thicker oblate spheroid with $r = 0.56$ in
 480 panel (d). Also included are the results for two prolate particles: a prolate spheroid with
 481 $r = 2.7$ in panel (e) and a slender fibre with $r = 9$ in panel (f). These data are compared to
 482 the theoretical prediction of Dabade *et al.* (2016) considering both particle and fluid inertia
 483 ($St = Re_p$ in the present neutrally-buoyant case) as detailed in § B.2.

484 Despite the rather large scatter of the data, there are some clear trends and qualitative
 485 agreements with theoretical predictions. Prolate particles experience positive increments of
 486 the orbit constant, which drive them towards tumbling motion ($C = \infty$, namely $C/(C+1) =$
 487 1), while oblate particles experience negative variations as they are mostly attracted to the
 488 vorticity-aligned orbit ($C = 0$, namely $C/(C+1) = 0$). Moreover, as revealed from the scales
 489 of the ordinate axes, the drift intensity ΔC increases with the aspect ratio r , as more slender
 490 objects experience more unstable Jeffery orbits.

491 The oblate particles not only drift towards the spinning orbit but also to the tumbling orbit
 492 as evidenced by the clustering of points around zero and slightly above for orbit constants
 493 $C \gtrsim \sqrt{35}$, namely $C/(C+1) \gtrsim 0.86$, as computed by Dabade *et al.* (2016), see panels (a) to
 494 (c). This is in stark contrast with the strong incline taken by the fibre of aspect ratio $r = 9$ in
 495 panel (f) when leaving the spinning orbit ($C/(C+1) = 0$) which is unstable for this prolate
 496 particle.

497 An interesting behaviour is observed in panel (d) for the oblate spheroid with $r = 0.6$,
 498 a value well above the predicted critical aspect ratio of approximately 0.14 for which only
 499 drift toward the spinning motion is anticipated. As evidenced in figure 11, this particle is
 500 still experiencing tumbling and spinning orbits, in contradiction with theoretical predictions.
 501 The critical orbit constant seems also to have moved as positive increments of orbit constant
 502 are observed at $C \lesssim \sqrt{35}$, i.e. $C/(C+1) \lesssim 0.86$, in particular for the higher Re_p . Similar
 503 behaviour is observed for the disk with $r_{eq} = 0.18$ in panels (b) and (c), which correspond to
 504 different confinements. This may suggest a wider region of attraction for the tumbling mode.

505 The influence of confinement is clearly visible between panels (b) and (c), where are shown
 506 two disks at a comparable $r \approx 0.1$ but differing confinement ratios, $\kappa = 0.11$ and $\kappa = 0.21$.
 507 The smaller disk is in better agreement with the theory than the large disk having a twofold
 508 increase of confinement ratio, where the drift is less intense than that predicted by Dabade
 509 *et al.* (2016) for an unconfined viscous shear flow.

510 Finally, we address the scaling of the drift intensity with the particle Reynolds number.
 511 Clearly, experiments at higher particle Reynolds number ($Re_p \approx 1$) do not collapse with
 512 those at smaller Re_p . Despite the large scatters, this is particularly evidenced in panels
 513 (a),(c),(d) for oblate objects, where the intensity of the drift is weaker for the larger Re_p .
 514 This suggests a saturation effect above a certain inertial threshold.

515 4. Concluding remarks

516 We have examined the rotation of axisymmetrical particles suspended in a simple shear flow
 517 when inertia is progressively increased up to particle Reynolds number $Re_p \approx 1$. A custom-
 518 built shearing cell and a multi-view reconstruction method have been used to obtain direct
 519 measurements of the orientation and period of rotation of ideal bodies such as spheroids but
 520 also bodies of practical interest such as cylinders with different aspect ratios. This system is
 521 rather flexible and is amenable to study the alignment of small bodies with different shapes
 522 in simple shearing flows.

523 The first important result is that the axisymmetric particles still rotate with the Jeffery

524 period in this small-inertia regime. Our results also complement the data available in the
 525 literature (Anczurowski & Mason 1968; Harris & Pittman 1975) and provide a connection
 526 with those of Zettner & Yoda (2001) at higher Re_p , showing that the period of rotation
 527 of a cylinder is smaller than that of an spheroid with the same aspect ratio. An equivalent
 528 spheroidal aspect ratio can be inferred to recover the Jeffery period in the case of cylinders.
 529 Our results show that the empirical expression of Harris & Pittman (1975) still remains
 530 the best prediction for this equivalent aspect ratio, which we were able to validate over one
 531 further decade of slender oblate particles compared to available literature results.

532 The second major output is that we observe an irreversible drift across Jeffery orbits
 533 towards attracting limiting cycles. This drift is due to weak inertial effects, and its occurrence
 534 confirms, at least qualitatively, the asymptotic theories of Einarsson *et al.* (2015*b*) and Dabade
 535 *et al.* (2016). It is important to stress before proceeding to any comparison below that these
 536 theories are meant to be valid for $Re_p \ll 1$ as well as at infinite distance from walls. Clearly,
 537 some deviations between their predictions and experimental results begin to appear as one
 538 moves away from their range of validity as discussed below.

539 Our measurements indicate that prolate particles are uniquely driven towards a tumbling
 540 motion in the flow-gradient plane regardless of their initial orientation and aspect ratio.
 541 For small but finite Re_p , i.e. typically up to $O(10^{-1})$ in our experiments, there is a good
 542 quantitative agreement with the theories. For larger Re_p , i.e. of $O(1)$ in our experiments, a
 543 saturation of the phenomenon is observed and the asymptotic theories predict a faster drift
 544 toward the tumbling orbit. This is expected since these theories are strictly valid only for
 545 $Re_p \ll 1$ and cannot be expected to describe the dynamics at Re_p of order unity or larger,
 546 as also discussed in Rosén *et al.* (2015).

547 In stark contrast to prolate particles, which only possess a single attracting orbit, oblate
 548 particles are observed to drift towards two different orbits, the spinning orbit or the tumbling
 549 orbit, depending upon their initial orientation. Whereas the theories of Einarsson *et al.*
 550 (2015*b*) and Dabade *et al.* (2016) do predict the existence of two limiting orbits, their
 551 overestimation of the drift through consecutive orbits is more severe and even seen for small
 552 Re_p of $O(10^{-1})$ as also discussed in Rosén *et al.* (2015) when the theory is compared to DNS
 553 results. This may suggest that the predictive capabilities of the theories in the small-inertia
 554 regime are more limited for oblate particles and that their range of application at finite Re_p
 555 is narrower than for prolate particles. We remark here that part of the observed overestimation
 556 may be ascribed to unavoidable confinement effects (Rosén *et al.* 2015), which tend to lower
 557 the drift intensity and thus introduce a small bias in the experimental measurements. Again,
 558 it seems that these effects do affect more the oblate particles.

559 There is a notable discrepancy between the experiments and the predictions of Einarsson
 560 *et al.* (2015*b*) and Dabade *et al.* (2016) regarding the stability of the tumbling orbit for oblate
 561 particles. A bifurcation toward a single stable spinning orbit above a critical aspect ratio of
 562 approximately 0.14 is not observed, suggesting a wider basin of attraction for the tumbling
 563 mode in the experiments for small Re_p of $O(10^{-1})$ up to $O(1)$. The time-resolved simulations
 564 of Rosén *et al.* (2015) having a confinement of $\kappa = 0.2$ have shown a bifurcation between
 565 stable and unstable tumbling for a critical aspect ratio of approximately 0.13 at $Re_p = 1$,
 566 in fair agreement with the prediction 0.14 of the asymptotic theories. Rosén *et al.* (2015)
 567 even tracked the bifurcation for larger Re_p , showing that it can survive up to $Re_p = 5$, see
 568 the bifurcation diagram of their figure 4. Our experimental results for similar confinement
 569 ($\kappa = 0.2$) and even those with smaller confinement ($\kappa = 0.1$) still exhibit stable tumbling
 570 modes. While this finding is intriguing, it seems reasonable to assume that it cannot be
 571 attributed to confinement.

572 Overall, our results indicate that inertia plays a significant role in breaking the indetermi-
 573 nacy of the Jeffery orbits for the prolate particles, while we still observe the existence of two

574 limiting orbits for the oblate particles. This finding has deep consequences for the steady-state
 575 rheology of suspensions of axisymmetrical particles in a viscous dilute regime, i.e. when
 576 particle-particle interactions are negligible. Indeed, prolate particles will eventually drift
 577 toward the tumbling orbit, aligning asymptotically with the mean flow direction. Conversely,
 578 the effective viscosity of a suspension of oblate particles will always depend on their initial
 579 orientation and regardless of their aspect ratio, i.e. not only in the lower near-sphere limit
 580 ($0.14 < r < 1$) as predicted by Dabade *et al.* (2016). Of course, hydrodynamic and direct
 581 particle-particle interactions come into play and change the picture as soon as semi-dilute
 582 and concentrated suspensions are considered (Butler & Snook 2018).

583 A final comment is in order about the importance of including the effect of both the fluid
 584 and particle inertia in the numerical calculation of the rotational dynamics of axisymmetric
 585 particles suspended in complex flows, e.g. turbulent flows. In a turbulent flow, the modified
 586 rotation rate (see equation B 1) might affect the distribution of particle orientations (Sheikh
 587 *et al.* 2020). In homogeneous isotropic turbulence, axisymmetric prolate (*resp.* oblate)
 588 particles were seen to align with (*resp.* perpendicular to) the local vorticity vector, following
 589 the local flow stretching in a mostly spinning (*resp.* tumbling) rotational state (Ni *et al.* 2015;
 590 Byron *et al.* 2015). Accounting for the inertial torques would strengthen the influence of the
 591 local stretching, even if one might argue under which conditions the inertial contribution
 592 becomes relevant given the intermittent nature of the flow. One may indeed need to compare
 593 the drift time (of typical order of a few periods of rotation) with the typical time of the flow
 594 velocity fluctuations. The picture is even more complex in bounded turbulence, where the
 595 strong near-wall shear causes axisymmetric particles to follow the local flow stretching but
 596 aligns the vorticity vector in the spanwise direction (Zhao & Andersson 2016). In this case,
 597 inertial torques could especially modify the orientation statistics of oblate particles, which
 598 tend to align normal to the wall, with possible consequences for drag reduction (Wang *et al.*
 599 2021).

600 **Supplementary data.** Supplementary material and movies are available at
 601 <https://doi.org/10.1017/jfm.2023...>

602 **Acknowledgements.** We thank N. K. Marath and G. Subramanian for providing detailed information on
 603 their calculation for the drift of oblate spheroids.

604 **Funding.** D.D.G. acknowledges the Università Italo-Francese, Bando Vinci 2021, cap. 2, progetto C2-257
 605 ‘Fibre flessibili in flusso turbolento ad elevato numero di Reynolds’ for the generous funding.

606 **Declaration of interests.** The authors report no conflict of interest.

607 **Data availability statement.** The data that support the findings of this study are openly available in
 608 [repository name] at [http://doi.org/\[doi\]](http://doi.org/[doi]), reference number [reference number].

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614 **Appendix A. Three-dimensional particle orientation estimation**

615 A.1. *Three-dimensional Axis-Aligned Bounding Box (AABBB) calculation for cylinders* 616 *and spheroids*

617 In this section we briefly describe the geometrical relations that provide the AABBB **B** for
 618 cylindrical and spheroidal particles, having imposed their orientation vector **n**. The relation
 619 is straightforward for cylinders, whose projection corresponds to the sum of a line and two

620 capping circles (Quilez 2016). The eccentricity e of these two circles is defined as:

$$621 \quad e = a \cdot \sqrt{1 - \mathbf{n} \cdot \mathbf{n}} . \quad (\text{A } 1)$$

622 Given the centre points of the capping circles, \mathbf{p}_a and \mathbf{p}_b , \mathbf{B} is obtained by maximising the
623 only possible orientation extension:

$$624 \quad \mathbf{B} = \max ((\mathbf{p}_a + e), (\mathbf{p}_b + e)) - \min ((\mathbf{p}_a - e), (\mathbf{p}_b - e)) . \quad (\text{A } 2)$$

625 Performing the same calculation for spheroidal particles is slightly more complex, in view
626 of the lack of sharp edges. A generic spheroid that can only rotate but not translate may be
627 represented as a diagonal matrix. If the spheroid is initially aligned with the z axis such that
628 its orientation vector is $\mathbf{n} = (0, 0, 1)$, then the matrix reads as follows:

$$629 \quad \overline{\mathbf{O}} = \begin{bmatrix} a^{-2} & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & (a \cdot r)^{-2} \end{bmatrix} . \quad (\text{A } 3)$$

630 Neglecting the spin, any generic orientation attained by the spheroid can be determined by a
631 combination of the two polar angles ϕ and θ :

$$632 \quad \mathbf{n} = (\sin \phi \sin \theta, \cos \phi \sin \theta, \cos \theta) . \quad (\text{A } 4)$$

633 It is straightforward to build a rotation matrix that correlates the initial orientation of the
634 spheroid to the new one through equation (A 4). Therefore, the quadratic form of the spheroid
635 will also be rotated according to the new orientation as follows:

$$636 \quad \overline{\mathbf{Q}} = \overline{\mathbf{R}} \cdot \overline{\mathbf{O}} \cdot \overline{\mathbf{R}}^T . \quad (\text{A } 5)$$

637 Given the matrix $\overline{\mathbf{Q}}$, in Appendix A.3 we demonstrate that taking the square root of the
638 diagonal components of its inverse yields the halved sides of the AABB:

$$639 \quad \mathbf{B}_i = 2 \cdot \sqrt{\overline{\mathbf{Q}}_{i,i}^{-1}} . \quad (\text{A } 6)$$

640 The procedure just described is repeated over $360 \times 360 = 129600$ possible combinations
641 of values of ϕ and θ , in order to discretise any possible orientation in the first quadrant
642 of the three-dimensional Cartesian system. This results in a resolution of 0.25° for each
643 angle. Calculations are scripted in Python to generate extended data sets of AABBs and
644 their corresponding orientation vectors. The corresponding scripts are provided in the
645 Supplementary Materials.

646 A.2. Orientation reconstruction by multi-variable regression

647 The relation between the Axis-Aligned Bounding Box \mathbf{B} and the particle orientation vector
648 \mathbf{n} is non-linear and known in a closed form only for the inverse transformation. Therefore, to
649 solve the direct transformation, we choose a data-driven approach.

650 A Deep Learning model is trained using synthetic data generated geometrically as
651 described in the previous section. The model is implemented as a neural network in
652 TensorFlow (Abadi 2015), and performs a multi-variable regression, where the input is
653 an experimentally measured AABB and the output is the corresponding orientation vector \mathbf{n} .
654 Hidden dense layers are made of 256 fully connected weights, introducing non-linearity
655 through the *relu* activation function. Typically, one hidden layer would be sufficient for the
656 regression of the most slender shapes, however, three layers seem preferable to be able to
657 deal also with the nearly spherical objects, e.g. the spheroid ELL06. Normalization of the
658 Euclidean norm of the output orientation vector \mathbf{n} is finally ensured by a dedicated Lambda

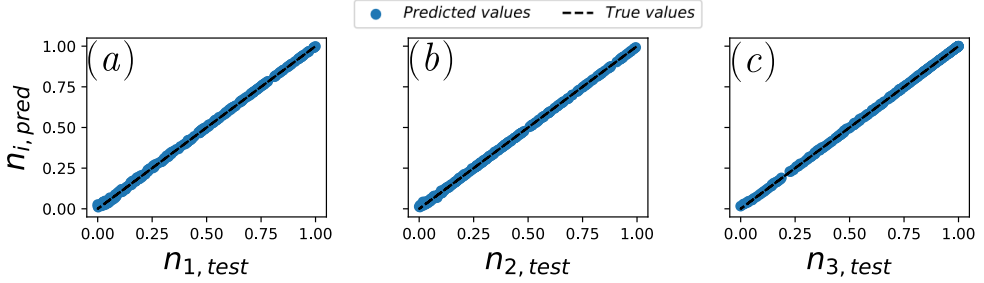


Figure 13: Predicted values against true values of the orientation vector \mathbf{n} obtained applying the Deep Learning model described in Appendix A.2 to a fibre with aspect ratio $r = 9.01$ (CYL10): (a) n_1 along x , (b) n_2 along y , (c) n_3 along z .

659 layer. The model is compiled and normally trained over at least 20 epochs by minimising a
 660 Custom Loss Function (CLF), which calculates the Euclidean norm of the difference between
 661 the true and predicted values of the orientation vector \mathbf{n} :

$$662 \quad CLF = \sqrt{\sum_{i=1}^3 (\mathbf{n}_{true,i} - \mathbf{n}_{pred,i})^2}. \quad (A 7)$$

663 In addition, the custom loss function strongly penalises any prediction yielding non-physical,
 664 lower-than-zero values for the orientation vector components. The chosen optimisation
 665 method is *Adam*, which is a randomized batched Gradient Descent method. A random
 666 20% splitting of the data set between training and testing samples is selected for the training.
 667 As displayed in figure 13, the regression for a fibre of aspect ratio $r = 9.01$ (CYL10) is highly
 668 reliable and yields precise estimations of the three components of the orientation vector \mathbf{n} .
 669 It is important to stress that the lack of availability of previously measured similar experiments
 670 prevents us from further testing the model. The Python script used to estimate the particle
 671 orientation vector \mathbf{n} from its AABBs is provided in the Supplementary Materials.

672 A.3. Relation between the Axes-Aligned Bounding Box of an spheroid and the principal 673 diagonal terms of its matrix of coefficients.

674 Given a generic n -dimensional hyper-spheroid \mathcal{E} :

$$675 \quad \mathcal{E} = \{x \in \mathbb{R}^n | (x - c)^T Q^{-1} (x - c) \leq 1\}, \quad (A 8)$$

676 where $c \in \mathbb{R}^n$ is the vector defining the centre of the spheroid and Q is a positive definite
 677 matrix of coefficients, and writing $g(x) = (x - c)^T Q^{-1} (x - c)$, the vector field orthogonal to
 678 the shell of the spheroid is:

$$679 \quad \nabla g(x) = 2Q^{-1} (x - c). \quad (A 9)$$

680 Considering the i^{th} axis and the corresponding projection matrix $P_i = e_i e_i^T$, the orthogonal
 681 vector field $\nabla g(x)$ and its projection $P_i \nabla g(x)$ will satisfy the condition:

$$682 \quad \nabla g(x) = P_i \nabla g(x), \quad (A 10)$$

683 at the two points where the spheroid \mathcal{E} touches the (smallest) bounding box, namely the
 684 smallest AABB. The above condition is equivalent to:

$$685 \quad (I_n - P_i) \underbrace{Q^{-1} (x - c)}_{\equiv y} = 0_n, \quad (A 11)$$

686 where the entry y_i will be the only non-zero value, i.e. $y = te_i$, or $x = c + tQe_i$. Intersecting
687 this line with the boundaries of the spheroid, t can be obtained as follows:

$$688 \quad t^2 = (e_i^T Q e_i)^{-1} = q_{ii}^{-1}, \quad (\text{A } 12)$$

689 where q_{ii} is the i^{th} entry of the inverse of the matrix Q . Therefore, the shell of the i^{th}
690 projection of the spheroid will touch the smallest AABB at the following two points:

$$691 \quad x_i = c_i \pm \frac{1}{\sqrt{q_{ii}}} e_i^T Q e_i = c_i \pm \frac{q_{ii}}{\sqrt{q_{ii}}} = c_i \pm \sqrt{q_{ii}}. \quad (\text{A } 13)$$

692 **Appendix B. Comparison with small-inertia theories**

693 **B.1. Model of Einarsson et al. (2015a)**

694 Einarsson *et al.* (2015a) characterised the influence of inertia on Jeffery orbits as additional
695 terms to equation (1.1), which then reads as:

$$696 \quad \dot{\mathbf{n}} = \mathbf{\Omega} \cdot \mathbf{n} + \frac{r^2 - 1}{r^2 + 1} [\mathbf{E} \cdot \mathbf{n} - \mathbf{n}(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})] + \beta_1 (\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}) \mathbb{P}(\mathbf{E} \cdot \mathbf{n}) + \\ 697 \quad + \beta_2 (\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n}) \mathbf{\Omega} \cdot \mathbf{n} + \beta_3 \mathbb{P}(\mathbf{\Omega} \cdot \mathbf{E} \cdot \mathbf{n}) + \beta_4 \mathbb{P}(\mathbf{E} \cdot \mathbf{E} \cdot \mathbf{n}), \quad (\text{B } 1)$$

698 where $\mathbb{P}(\mathbf{x}) = \mathbf{x} - (\mathbf{n} \cdot \mathbf{x}) \mathbf{n}$ is an operator that projects components in the \mathbf{n} -direction. The
699 first two terms on right-end side of equation (B 1) correspond to the original Jeffery equation
700 while the last four terms, each containing a scalar coefficient β_i , are the inertia-related terms.
701 The coefficients β_i are functions of the particle aspect ratio, provided graphically in figure 2
702 of Einarsson *et al.* (2015b) for $Re_p = St$ and reported here in figure 14 for the CYL10 case,
703 using the equivalent aspect ratio for cylinders. Equation (B 1) is integrated in time using the
704 odeint function of the Scipy Python library. The initial conditions are provided by the first
705 flow-aligned orientation measured in the CYL10 experimental run. The Python script used
706 to solve for equation (B 1) is provided in the Supplementary Materials.

707 **B.2. Model of Dabade et al. (2016)**

708 Dabade *et al.* (2016) characterize the inertia-driven drift from the Jeffery orbits through
709 consecutive rotations by means of discrete variations of the orbit constant ΔC in a single
710 Jeffery period. For particle inertia, this is given in their equation (5.19) in terms of the
711 eccentricity: $\xi_0 = \sqrt{1/(1 - 1/r^2)}$ for the prolate and $\xi_0 = \sqrt{1/(1 - r^2)}$ for the oblate particles.
712 The eccentricity parameter ξ_0 is calculated using the physical particle aspect ratio, r , for the
713 spheroids and the equivalent particle aspect ratio, r_{eq} , for the cylinders, obtained by averaging
714 over all the available experiments. Then, the integrals I_i, J_i are provided in Appendix C, while
715 the prolate F_i^p, G_i^p coefficients come from their equations (5.7-5.12). Following Dabade *et al.*
716 (2016) to obtain the oblate coefficients, one needs first to multiply the prolate coefficients by
717 the squared length-scale to obtain their dimensional form. Then the transformation prolate-to-
718 oblate must be applied ($\xi_0 < - > i (\xi_0^2 - 1)^{0.5}$), before returning to dimensionless variables
719 by dividing by the squared length-scale, as described below equation (5.12) of Dabade *et al.*
720 (2016). The fluid-inertia drift is still obtained from equation (5.19), where St is replaced by
721 Re_p and the F_i^f, G_i^f coefficients are given in equations (6.1-6.8). The Python script used
722 to calculate the discrete orbit variation ΔC as in Dabade *et al.* (2016) is provided in the
723 Supplementary Materials.

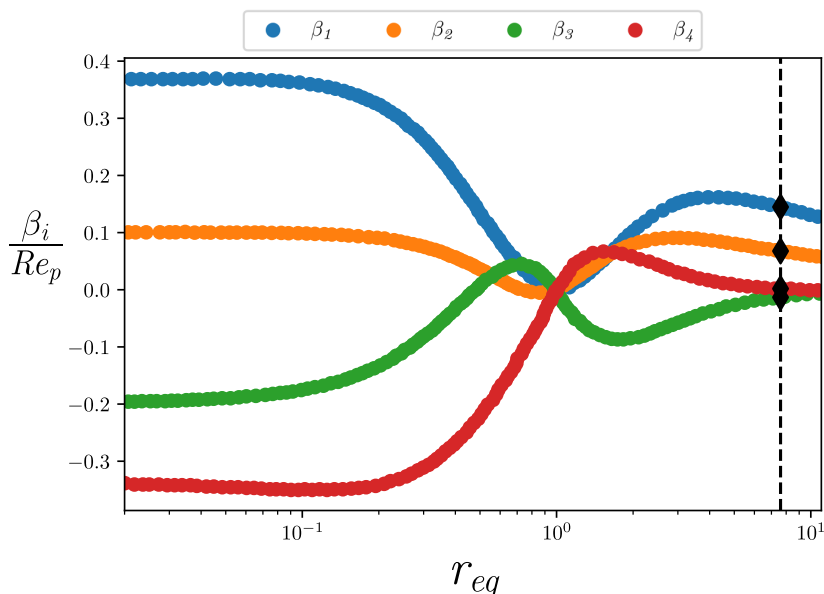


Figure 14: β_i coefficients of equation B 1 as given in Figure 2 of Einarsson *et al.* (2015b).
As an example, for the fibre CYL10, the coefficients β_i are chosen at the intersection between the dotted vertical line at $r_{eq} = 7.4$ with the respective curves.

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