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► **To cite this version:**

Matthew R Foreman, François Goudail. On the equivalence of optimization metrics in Stokes polarimetry. *Optical Engineering*, 2019, 58 (08), pp.082410. 10.1117/1.OE.58.8.082410 . hal-04491958

**HAL Id: hal-04491958**

**<https://hal.science/hal-04491958>**

Submitted on 6 Mar 2024

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# On the equivalence of optimization metrics in Stokes polarimetry

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**Abstract.** Optimization of polarimeters has historically been achieved using an assortment of performance metrics. Selection of an optimization parameter is frequently made, however, on an ad hoc basis. In this article we rigorously demonstrate that optimization strategies in Stokes polarimetry based on three common metrics; namely the condition number of the instrument matrix, the determinant of the associated Gram matrix, or the equally weighted variance, are formally equivalent. In particular, using each metric we derive the same set of constraints on the measurement states, and show that these can be satisfied using spherical 2 designs. Our conclusions are equally applicable to optimization of the illumination states in Mueller matrix polarimetry.

**Keywords:** polarimetry, optimization, spherical design, instrument matrix, equally weighted variance.

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## 1 Introduction

Quantitative analysis of the state of polarization of light provides a powerful tool in modern science. Applications vary from microscopy, biomedical diagnosis and astrophysics<sup>1-3</sup> to crystallographic, material and single molecule studies.<sup>4,5</sup> Whilst the polarization state of light itself, can be used to transmit information, hence presenting new opportunities in optical data storage and communications,<sup>6-9</sup> changes in polarization induced by a material can alternatively be used for object detection<sup>10</sup> or to characterize sample properties, such as chirality or molecular orientation.<sup>11-13</sup>

Stokes polarimeters, which allow a complete characterization of the polarization state of input light as described by the associated  $4 \times 1$  Stokes vector  $\mathbf{S}$ , comprise of  $N (\geq 4)$  distinct measurements that can be multiplexed in, for example, time,<sup>14</sup> frequency<sup>15</sup> or space.<sup>16</sup> Fundamentally, each constituent measurement outputs an intensity  $I_j (j \in [1, N])$  which is proportional to the projection of the incident Stokes vector onto an analysis state  $\mathbf{W}_j$ , i.e.,  $I_j = \mathbf{W}_j^T \mathbf{S}$ . Central to

28 the description and design of Stokes polarimeters is hence the so-called instrument or measure-  
29 ment matrix  $\mathbb{W} = [\mathbf{W}_1, \mathbf{W}_2, \dots]^T$  formed from stacking the set of analysis vectors. In order to  
30 obtain an estimate of the Stokes vector from the set of projections  $I_j$ , the measurement matrix must  
31 be inverted. So as to limit noise propagation through this inversion process, optimization of the  
32 measurement matrix is hence frequently performed. Optimization in this vein has been performed  
33 using different metrics including the associated information content,<sup>17–19</sup> matrix determinant,<sup>20–22</sup>  
34 signal to noise ratio,<sup>23</sup> equally weighted variance<sup>24,25</sup> and condition number.<sup>21–23,25–29</sup>

35 Mueller matrix polarimeters, on the other hand, combine a Stokes polarimeter with use of  
36 multiple incident polarized states so as to measure the full Mueller matrix of an object. Variation  
37 of the probing polarization states (as can be described using an analogous illumination matrix)  
38 therefore introduces additional degrees of freedom, hence admitting further optimization.<sup>17,28–32</sup>  
39 Application specific optimization of polarimeters has also been reported, for example, in detection  
40 and imaging problems the polarization contrast is a more suitable metric.<sup>33,34</sup>

41 Recently, the equivalence of a number of optimization metrics, namely the equally weighted  
42 variance, the condition number of  $\mathbb{W}$  and the determinant of the associated Gram matrix, was  
43 discussed by Foreman et al.<sup>35</sup> Additionally, Foreman et al. proved that a Stokes polarimeter is  
44 optimal (as characterized by these metrics) when the set of analysis states defines a spherical 2  
45 design<sup>36</sup> on the unit Poincaré sphere. A re-examination of the equivalence between these metrics  
46 is, however, necessary due to an error in the proof presented in Ref. 35. The goal of this article is  
47 therefore to provide a rigorous proof that the conclusions of Ref. 35 hold. Our derivations also elicit  
48 greater insight into the optimization of Stokes polarimeters. We additionally note that our results  
49 are equally applicable to optimization of the probing states used in Mueller matrix polarimetry due  
50 to the similar matrix structure of the problem.<sup>31,37</sup>

51 **2 Optimal polarimetry with spherical 2-designs**

The instrument matrix,  $\mathbb{W}$ , of a polarimeter is an  $N \times 4$  matrix, the rows of which are the Stokes vectors of the  $N$  polarization states being analyzed, normalized such that the polarimeter is passive.

Accordingly the instrument matrix has the parametric form

$$\mathbb{W} = \frac{1}{2} \begin{bmatrix} 1 & \mathbf{w}_1^T \\ 1 & \mathbf{w}_2^T \\ \vdots & \vdots \\ 1 & \mathbf{w}_N^T \end{bmatrix} \triangleq \frac{1}{2} \begin{bmatrix} \mathbf{r} & \mathbb{Q} \end{bmatrix}, \quad (1)$$

52 where  $\mathbf{r}$  is an  $N \times 1$  vector of ones and  $\mathbb{Q}$  is the matrix formed from the  $3 \times 1$  vectors  $\mathbf{w}_j$  ( $j \in [1, N]$ )  
 53 of unit norm. Note that throughout this work bold notation is used to signify column vectors whilst  
 54 blackboard font denotes matrices. Please also note that we have assumed an "ideal" instrument  
 55 matrix, in the sense that the transmittance and degree of polarization of all the rows are equal  
 56 to one. The generalization of the obtained results to arbitrary instrumentation matrices will be  
 57 discussed in Section 4.

In Stokes polarimetry, one performs  $N$  intensity measurements  $I_j, j \in [1, N]$  by projecting the input Stokes vector  $\mathbf{S}$  onto each of the  $N$  analyzers described by the  $N$  rows of the matrix  $\mathbb{W}$ . If these measurements are stacked in an  $N$ -dimensional vector  $\mathbf{I} = [I_1, I_2, \dots, I_N]^T$ , and if we assume that the measurements are perturbed by white additive noise, we obtain

$$\mathbf{I} = \mathbb{W}\mathbf{S} + \mathbf{B}, \quad (2)$$

where  $\mathbf{B}$  is an  $N \times 1$  random vector with covariance matrix  $\sigma^2 \mathbb{I}_N$  and  $\mathbb{I}_n$  denotes the  $n \times n$  identity

matrix. The maximum-likelihood estimate of  $\mathbf{S}$  is obtained by

$$\hat{\mathbf{S}} = \mathbb{W}^+ \mathbf{I} \quad (3)$$

where

$$\mathbb{W}^+ = (\mathbb{W}^T \mathbb{W})^{-1} \mathbb{W}^T \quad (4)$$

denotes the pseudo-inverse matrix. The estimate  $\hat{\mathbf{S}}$  is a random vector of mean  $\mathbf{S}$  (i.e., the estimator is unbiased), and of covariance matrix<sup>17,23,24</sup>

$$\mathbb{K}_{\mathbf{S}} = \sigma^2 (\mathbb{W}^T \mathbb{W})^{-1}. \quad (5)$$

The estimation variances of each element of the Stokes vector estimate are the diagonal elements of this matrix. A natural goal of polarimeter optimization is to find the measurement matrix  $\mathbb{W}$  that minimizes the sum of these variances, that is, the trace of  $\mathbb{K}_{\mathbf{S}}$ . The corresponding performance metric is called the equally weighted variance (EWV), i.e.,

$$\text{EWV} = \sigma^2 \text{tr}[\mathbb{G}^{-1}] \quad (6)$$

where

$$\mathbb{G} = \mathbb{W}^T \mathbb{W} \quad (7)$$

58 denotes the Gram matrix associated with  $\mathbb{W}$ .

To optimize the EWV, we first express the Gram matrix  $\mathbb{G}$  in block format, viz.

$$\mathbb{G} = \frac{1}{4} \begin{bmatrix} N & \mathbf{r}^T \mathbf{Q} \\ \mathbf{Q}^T \mathbf{r} & \mathbf{Q}^T \mathbf{Q} \end{bmatrix} \triangleq \begin{bmatrix} A & \mathbf{B}^T \\ \mathbf{C} & \mathbb{D} \end{bmatrix}. \quad (8)$$

The inverse of the Gram matrix can then be expressed in the form<sup>38</sup>

$$\mathbb{G}^{-1} = \begin{bmatrix} A^{-1} + A^{-1} \mathbf{B}^T \mathbb{M}^{-1} \mathbf{C} A^{-1} & -A^{-1} \mathbf{B}^T \mathbb{M}^{-1} \\ -\mathbb{M}^{-1} \mathbf{C} A^{-1} & \mathbb{M}^{-1} \end{bmatrix}, \quad (9)$$

where the matrix

$$\mathbb{M} = (\mathbb{D} - \mathbf{C} A^{-1} \mathbf{B}^T) \quad (10)$$

is the Schur complement of the upper left block of  $\mathbb{G}$ . This implies that the trace we seek can be written as

$$\text{tr}[\mathbb{G}^{-1}] = A^{-1} + A^{-1} \mathbf{B}^T \mathbb{M}^{-1} \mathbf{C} A^{-1} + \text{tr}[\mathbb{M}^{-1}]. \quad (11)$$

Substituting Eq. (8) into Eq. (10), the Schur complement takes the form:

$$\mathbb{M} = \frac{1}{4} \left( \mathbf{Q}^T \mathbf{Q} - \frac{\mathbf{q} \mathbf{q}^T}{N} \right) \quad (12)$$

where  $\mathbf{q} = \mathbb{Q}^T \mathbf{r}$  is an  $N$ -dimensional vector. Upon using the identity<sup>38</sup>

$$(\mathbb{Z} + \mathbf{x}\mathbf{y}^T)^{-1} = \mathbb{Z}^{-1} - \frac{\mathbb{Z}^{-1}\mathbf{x}\mathbf{y}^T\mathbb{Z}^{-1}}{1 + \mathbf{y}^T\mathbb{Z}^{-1}\mathbf{x}} \quad (13)$$

with  $\mathbf{x} = -\mathbf{y} = \mathbf{q}/\sqrt{N}$  and  $\mathbb{Z} = \mathbb{Q}^T\mathbb{Q}$ , we find

$$\mathbb{M}^{-1} = 4(\mathbb{Q}^T\mathbb{Q})^{-1} + 4\frac{(\mathbb{Q}^T\mathbb{Q})^{-1}\mathbf{q}\mathbf{q}^T(\mathbb{Q}^T\mathbb{Q})^{-1}}{N - \mathbf{q}^T(\mathbb{Q}^T\mathbb{Q})^{-1}\mathbf{q}}. \quad (14)$$

Direct substitution from Eqs. (8) and (14) into Eq. (11) yields

$$\begin{aligned} \text{tr}[\mathbb{G}^{-1}] = 4 \left\{ \frac{1}{N} + \text{tr} [(\mathbb{Q}^T\mathbb{Q})^{-1}] \right. \\ \left. + \frac{\mathbf{q}^T [N(\mathbb{Q}^T\mathbb{Q})^{-2} + (\mathbb{Q}^T\mathbb{Q})^{-1}] \mathbf{q}}{N[N - \mathbf{q}^T(\mathbb{Q}^T\mathbb{Q})^{-1}\mathbf{q}]} \right\}, \quad (15) \end{aligned}$$

59 where we have also used the cyclic property of the trace operation and the identity  $\text{tr}[\mathbb{X}\mathbf{q}^T\mathbf{q}] =$   
60  $\mathbf{q}^T\mathbb{X}\mathbf{q}$  for arbitrary  $\mathbb{X}$ .<sup>38</sup>

Noting that  $N > 0$  and that  $\mathbb{Q}^T\mathbb{Q}$  is positive definite, it follows immediately that the first two terms in Eq. (15) are positive. We show in Appendix A that the third term is also positive. Consequently, the trace in Eq. (15) is minimal when its three terms are minimal. The first term is constant, and the third is minimal when it is null, that is, when  $\mathbf{q} = \mathbb{Q}^T \mathbf{r} = \mathbf{0}$  or equivalently

$$\sum_{n=1}^N \mathbf{w}_n = \mathbf{0}. \quad (16)$$

61 Importantly, Eq. (16) expresses a polynomial constraint that must be satisfied by an optimal mea-  
62 surement matrix, and is equivalent to that given in Eq. (4) of Ref. 35.

When Eq. (16) holds, minimizing  $\text{tr}[\mathbb{G}^{-1}]$  is equivalent to minimizing  $\text{tr}[(\mathbb{Q}^T\mathbb{Q})^{-1}]$ . This optimization has to be done under the constraint that the trace of the matrix  $\mathbb{Q}^T\mathbb{Q}$  is constant as follows from the normalization of  $\mathbf{w}_j$ . Indeed, since each row of the matrix  $\mathbb{Q}$  is a unit-norm vector, we have

$$\text{tr}[\mathbb{Q}^T\mathbb{Q}] = \text{tr}[\mathbb{Q}\mathbb{Q}^T] = N. \quad (17)$$

We thus have to solve the following constrained optimization problem, set in Lagrange form

$$\Psi(\mathbb{Q}) = \text{tr}[(\mathbb{Q}^T\mathbb{Q})^{-1}] - \mu (\text{tr}[\mathbb{Q}^T\mathbb{Q}] - N) \quad (18)$$

where  $\mu$  is a Lagrange multiplier. The Lagrange function can also be expressed as

$$\Psi(\boldsymbol{\beta}) = \sum_{j=1}^3 \frac{1}{\beta_j} + \mu \left( \sum_{j=1}^3 \beta_j - N \right) \quad (19)$$

where  $\beta_j$ ,  $j \in [1, 3]$ , are the positive eigenvalues of the matrix  $\mathbb{Q}^T\mathbb{Q}$ . Equating the gradient of Eq. (19) with respect to  $\boldsymbol{\beta}$  to zero and enforcing the constraint ( $\partial\Psi/\partial\mu = 0$ ) yields  $\beta_j = 1/\sqrt{\mu} = N/3$  for all  $j \in [1, 3]$ , such that

$$\mathbb{Q}^T\mathbb{Q} = \sum_{j=1}^m \mathbf{w}_j\mathbf{w}_j^T = \frac{N}{3}\mathbb{I}_3. \quad (20)$$

Eq. (20) is the second set of polynomial constraints derived in Ref. 35. The form of the Gram



matrix  $\mathbb{G}$  which hence minimizes the EWV of the instrument matrix is thus

$$\mathbb{G} = \mathbb{W}^T \mathbb{W} = \frac{N}{12} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (21)$$

According to Eq. (5), the corresponding covariance of the Stokes vector estimate is hence:

$$\mathbb{K}_{\mathbf{S}} = \frac{4}{N} \sigma^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \quad (22)$$

63 This result is important since it specifies, in a very simple closed-form, the fundamental limit  
 64 of the estimation variance that can be reached by a Stokes polarimeter with a given number of  
 65 measurement vectors in the presence of additive noise. For example, we note that the minimum  
 66 achievable variance on an estimate of the intensity (i.e. the first element of the Stokes vector) is  
 67 three times better than that on the other Stokes parameters. Moreover, the covariance matrix is  
 68 diagonal, which means that the fluctuations of each element of the Stokes vector estimator are  
 69 statistically independent. This property is important when performing theoretical computations  
 70 involving Stokes vector estimators. Incidentally, we note that the minimum value of the equally  
 71 weighted variance is  $\text{EWV} = 40\sigma^2/N$ .

72 Finally, we can show that the conditions expressed by Eqs. (16) and (20) are satisfied when

73 the set of measurement states on the Poincaré sphere, defined by  $\{\mathbf{w}_j\}$ ,  $j \in [1, N]$ , constitute a  
74 spherical 2 design (see Appendix B) as reported in Ref. 35. A spherical  $t$ -design is defined as a  
75 collection of  $N$  points  $\{\mathbf{w}_j\}$  on the surface of the unit sphere (in our case in  $\mathcal{R}^3$ ) for which the  
76 normalized integral of any polynomial function,  $f(\mathbf{w})$ , of degree  $t$  or less is equal to the average  
77 taken over the  $N$  points. The Platonic solids, i.e., the regular tetrahedron ( $N = 4$ ), the octahedron  
78 ( $N = 6$ ), the cube ( $N = 8$ ), the icosahedron ( $N = 12$ ), and the dodecahedron ( $N = 20$ ), are  
79 spherical 2 designs, as well as many other measurement frames illustrated in.<sup>35</sup>

### 80 3 Equivalence of optimization metrics

81 We will now demonstrate that the optimization of two other popular metrics, namely the condition  
82 number and the determinant of the Gram matrix, lead to exactly the same measurement frames as  
83 the EWV, so that these three criteria are strictly equivalent.

#### 84 3.1 Condition number

The condition number  $\kappa$  of the instrument matrix is defined by  $\kappa = \|\mathbb{W}\|\|\mathbb{W}^+\|$  where  $\mathbb{W}^+$  is the  
pseudo-inverse matrix and the Hilbert-Schmidt matrix norm<sup>38</sup> of any matrix  $\mathbb{P}$  is defined as:

$$\|\mathbb{P}\| = (\text{tr}[\mathbb{P}^T \mathbb{P}])^{1/2} = (\text{tr}[\mathbb{P} \mathbb{P}^T])^{1/2} \quad (23)$$

Our choice of normalization of the measurement states  $\mathbf{W}_j = [1, \mathbf{w}_j]^T / 2$  implies that

$$\|\mathbb{W}\|^2 = \text{tr}[\mathbb{W}^T \mathbb{W}] = \frac{N}{2}. \quad (24)$$

Moreover, using the expression of the pseudo-inverse in Eq. (4), it is easily shown that

$$\|\mathbb{W}^+\|^2 = \text{tr}[(\mathbb{W}^+)^T \mathbb{W}^+] = \text{tr}[(\mathbb{W}^T \mathbb{W})^{-1}] = \frac{\text{E}\mathbb{W}\mathbb{V}}{\sigma^2}. \quad (25)$$

Consequently, one can write

$$\kappa = \frac{\sqrt{N}}{\sqrt{2}\sigma} \sqrt{\text{E}\mathbb{W}\mathbb{V}}. \quad (26)$$

85 For any measurement matrix verifying Eq. (1), the condition number is thus simply proportional to  
 86 the square root of the E $\mathbb{W}\mathbb{V}$ . It is thus obvious that minimizing the condition number is equivalent  
 87 to minimizing the E $\mathbb{W}\mathbb{V}$ .

### 88 3.2 Determinant of the Gram matrix

89 The first works on Stokes polarimeter optimization considered devices with a minimal number  
 90 ( $N = 4$ ) of measurement vectors.<sup>26</sup> Optimization of such systems used the determinant of the  
 91 matrix  $\mathbb{W}$  (which for this value of  $N$  is square and non singular) as a performance metric. In this  
 92 case the optimal structure found dictated that the measurement vectors defined a regular tetrahe-  
 93 dron on the Poincaré sphere, a result that we also found above by optimizing the E $\mathbb{W}\mathbb{V}$ . We show  
 94 in this section that this result comes from the strict equivalence of these two optimization metrics.  
 95 This equivalence can be generalized to any value of  $N$  if one considers the optimization of the  
 96 determinant of the Gram matrix  $\mathbb{G}$ , since for  $N > 4$  the matrix  $\mathbb{W}$  itself is rectangular and its  
 97 determinant is thus not defined. Notice that this equivalence was mentioned in Ref. 35, but there  
 98 was an erroneous step in the logic presented in that work (see appendix C for more detail).

99 We intend here to show that maximization of the determinant  $|\mathbb{G}|$  yields the same polynomial

100 constraints embodied in Eqs. (16) and (20). Considering the block form of the Gram matrix in  
 101 Eq. (8), its determinant can be written as<sup>38</sup>

$$\begin{aligned}
 |\mathbb{G}| &= |A - \mathbf{B}^T \mathbb{D}^{-1} \mathbf{C}| |\mathbb{D}| \\
 &= \frac{1}{256} [N - \mathbf{r}^T \mathbb{Q} (\mathbb{Q}^T \mathbb{Q})^{-1} \mathbb{Q}^T \mathbf{r}] |\mathbb{Q}^T \mathbb{Q}|.
 \end{aligned} \tag{27}$$

102 Maximizing this expression means maximizing the two factors appearing in the product. The first  
 103 factor is maximized if the positive subtractive term is zero, that is to say when the vector  $\mathbb{Q}^T \mathbf{r} = \mathbf{0}$ ,  
 104 corresponding to the first polynomial constraint expressed in Eq. (16).

For the second factor, we note that  $|\mathbb{Q}^T \mathbb{Q}| = \prod_{j=1}^3 \beta_j$  where  $\beta_j, j \in [1, 3]$ , are the eigenvalues of the matrix  $\mathbb{Q}^T \mathbb{Q}$  which are positive since  $\mathbb{Q}^T \mathbb{Q}$  is positive definite. Moreover, according to Eq. (17), the matrix  $\mathbb{Q}^T \mathbb{Q}$  has constant trace. Maximization of  $|\mathbb{Q}^T \mathbb{Q}|$  is thus once again a constrained optimization problem which can be solved using the method of Lagrange multipliers. We will consider maximization of  $\ln |\mathbb{Q}^T \mathbb{Q}|$  since the logarithmic function is monotonically increasing. The Lagrange function then becomes

$$\Psi(\boldsymbol{\beta}) = \sum_{j=1}^3 \ln \beta_j - \mu \left( \sum_{j=1}^3 \beta_j - N \right). \tag{28}$$

105 Following the standard optimization procedure we find, similarly to Section 2, that  $\beta_j = 1/\mu =$   
 106  $N/3$  for all  $j \in [1, 2, 3]$ . As shown in Section 2, the second polynomial constraint expressed in  
 107 Eq. (20) then follows. Therefore, we have ultimately shown that minimization of the EWW (and  
 108 thus also of the condition number of the instrument matrix) of a polarimeter yields the same set of  
 109 optimality constraints as maximizing the determinant of the associated Gram matrix.

## 110 4 Discussion

111 The main conclusion of the previous sections is the following: among all the measurement matri-  
 112 ces verifying Eq. (1), the ones that maximize the condition number, the EWV and the determinant  
 113 are exactly the same. It can be said that this results unifies many previous works on polarimeter op-  
 114 timization: for example, the early work of Azzam<sup>26</sup> (based on determinant), Ambirajan and Look<sup>22</sup>  
 115 (based on condition number and determinant), Sabatke et al.<sup>24</sup> (based on EWV and determinant),  
 116 and Tyo<sup>43</sup> (based on condition number), among many others.

Modeling of  $\mathbb{W}$  with Eq. (1) implies that the transmittance and the degree of polarization of all rows of  $\mathbb{W}$  are equal to one. This hypothesis is frequently done in polarimetry, but it is interesting to consider the case where it is not fulfilled. In the general case, each row of the measurement matrix may have a different transmission  $t_i, i \in [1, N]$ , and a different degree of polarization  $P_i, i \in [1, N]$ . In this case, the measurement matrix can be expressed in the following form:

$$\mathbb{W} = \frac{1}{2} \begin{bmatrix} t_1 & t_1 P_1 \mathbf{w}_1^T \\ t_2 & t_2 P_2 \mathbf{w}_2^T \\ \vdots & \vdots \\ t_N & t_N P_N \mathbf{w}_N^T \end{bmatrix} \triangleq \frac{1}{2} \begin{bmatrix} \mathbb{T} \mathbf{r} & \mathbb{T} \mathbb{P} \mathbb{Q} \end{bmatrix}, \quad (29)$$

117 where  $\mathbb{T} = \text{diag}(t_1, \dots, t_N)$  and  $\mathbb{P} = \text{diag}(P_1, \dots, P_n)$ . In this general form, it is easy to demon-  
 118 strate that

$$\|\mathbb{W}\|^2 = \frac{1}{4} \sum_{i=1}^N (1 + P_i^2) t_i^2 \quad (30)$$

119 Consequently, one can generalize Eq. (26) to:

$$\kappa = \frac{\sqrt{\sum_{i=1}^N (1 + P_i^2) t_i^2}}{2\sigma} \sqrt{\text{EWV}} \quad (31)$$

120 This entails a generalization of the result obtained in Section 3: for arbitrary, but fixed values  
 121 of transmissions and degree of polarization of the analysis vectors, optimization of the vectors'  
 122 positions on the normalized Poincaré sphere (i.e., of the vectors  $\mathbf{w}_n$ ) yields the same result if the  
 123 condition number or the EWV are used as criteria. If, in addition, the transmissions and degree of  
 124 polarizations are identical for all the N measurements, it is easy to show that the resulting optimal  
 125 structures are spherical 2 designs. If they are not identical, the optimal structures are no longer the  
 126 spherical designs. Determining the optimal structures in this case is an interesting perspective to  
 127 the present work.

Another important practical question is which one of the three considered metrics is the most appropriate for evaluating the performance of any polarimeter, be it optimal or not. Indeed, on this point of view, the metrics are not equivalent. This is most easily seen by noticing that the three metrics only depend on the measurement matrix through the eigenvalues of  $\mathbb{G}$ , denoted  $\mu_i, i \in [1, 4]$ , and have the following form:

$$|\mathbb{G}| = \prod_{i=1}^4 \mu_i \quad (32)$$

$$\kappa = \left( \sum_{i=1}^4 \mu_i \right)^{1/2} \left( \sum_{i=1}^4 \frac{1}{\mu_i} \right)^{1/2} \quad (33)$$

$$\text{EWV} = \sum_{i=1}^4 \frac{1}{\mu_i} \quad (34)$$

128 It is seen that although they look similar, they are different. In particular, two different sets of  
129 eigenvalues may lead to the same value of  $\kappa$ , but different values of  $|\mathbb{G}|$  and EWV, and vice versa.  
130 The question of how to choose the best metric could seem arbitrary, but we think there is a strong  
131 objective argument in favor of EWV. Indeed, the EWV corresponds to an estimation variance,  
132 which has a clear and useful statistical meaning. For example, it makes it handier to compare  
133 two different polarimeter structures: saying that polarimeter A has a EWV twice than polarimeter  
134 B just means that the variance of the estimated Stokes vector is twice. In sharp contrast, a ratio  
135 between two determinants or two condition numbers is much less easy to interpret in terms of  
136 estimation error.

137 Another strong advantage of the EWV is that it can be used for polarimeter optimization in the  
138 presence of non-additive noises sources. The EWV has for example been used to derive the optimal  
139 measurement frames in the presence of Poisson shot noise.<sup>39,40</sup> In this case, the expression of the  
140 covariance matrix of the Stokes estimate was different from Eq.(5). Consequently, the EWV was  
141 different from Eq. (6), and was thus no longer proportional of the square of the condition number,  
142 which had no statistical meaning. Furthermore, when measurements are simultaneously affected  
143 by several types of statistically independent noise sources, the value of the EWV is simply the sum  
144 of the EWVs corresponding to each noise source. This property has for example been recently  
145 employed to characterize the actual performance of micro-grid based polarimetric cameras in the  
146 presence of both additive detection noise and Poisson shot noise.<sup>41</sup>

147 In conclusion, the message of the present work could be summarized as follows. When op-  
148 timizing the estimation performance of a polarimeter in the presence of additive Gaussian noise,  
149 condition number, determinant, and EWV are three strictly equivalent metrics. When evaluating  
150 and comparing the performance of different polarimeters, or when optimizing polarimeters in the

151 presence of non-additive, non-Gaussian noise sources, the EWV has strong advantages compared  
152 with the two other metrics.

## 153 **5 Conclusions**

154 We have shown that optimization of the EWV, of the condition number, or of the determinant  
155 of the Gram matrix of a Stokes polarimeter leads to the same optimal measurement structures,  
156 namely, spherical 2 designs. These structures yield a very simple closed-form expression for the  
157 covariance matrix of the Stokes vector estimator, and thus of the variances of each element of the  
158 Stokes vector. These expressions constitute the fundamental limit of the estimation variance that  
159 can be reached by a Stokes polarimeter in the presence of additive noise.

160 As a conclusion, we would like to stress that although the three considered metrics are equiv-  
161 alent for polarimeter optimization in the presence of additive noise, the EWV has the simplest  
162 physical interpretation since it corresponds to an estimation variance, which has a clear and useful  
163 statistical meaning. As a consequence, in contrast to the two other metrics, the EWV can be used  
164 for polarimeter optimization in the presence of noise sources with non-additive, non-Gaussian, or  
165 mixed statistics.

166 A very interesting perspective is the determination of the optimal measurement matrices in the  
167 presence of non-additive noise sources. As said above, this problem has already been addressed  
168 by optimizing the EWV obtained after application of the pseudo-inverse estimator.<sup>39,40</sup> Although  
169 this procedure give satisfying results in practice,<sup>42</sup> it is not strictly optimal. Indeed, in the presence  
170 of non additive and non Gaussian noise, by virtue of the Cramér-Rao lower bound, the appropriate  
171 criterion is the trace of the inverse Fisher information matrix.<sup>17,18</sup> The value of this criterion  
172 corresponds to the EWV of an efficient estimator, which the pseudo-inverse estimator is not in



173 general. An exciting perspective is to work at determining the measurement matrices optimizing  
 174 this Fisher criterion, and at analyzing the differences between the obtained optimal structures and  
 175 the spherical 2 designs.

176 **Appendix A: Positivity of the third term of Eq. (15)**

We demonstrate in this appendix that the third term of the expression of  $\text{tr}[\mathbb{G}^{-1}]$  in Eq. (15) is positive definite. Since the matrix  $\mathbb{Q}^T\mathbb{Q}$  is by definition a positive matrix, the numerator of this term is also positive. We therefore need only analyse the denominator. Considering then the singular value decomposition  $\mathbb{Q} = \mathbb{U}\mathbb{F}\mathbb{V}^T$ , where  $\mathbb{U}$  and  $\mathbb{V}$  are unitary matrices and  $\mathbb{F}$  is diagonal, it is easily seen that

$$\mathbb{Q}(\mathbb{Q}^T\mathbb{Q})^{-1}\mathbb{Q}^T = \mathbb{U}\mathbb{F}\mathbb{U}^T \quad (35)$$

where  $\mathbb{F} = \mathbb{D}(\mathbb{D}^T\mathbb{D})^{-1}\mathbb{D}^T$  is a diagonal  $N \times N$  matrix. The first three diagonal elements of  $\mathbb{F}$  are unity, whereas the other elements are zero. We thus have

$$\mathbf{q}^T(\mathbb{Q}^T\mathbb{Q})^{-1}\mathbf{q} = \mathbf{v}^T\mathbb{F}\mathbf{v} = \sum_{i=1}^3 v_i^2 \quad (36)$$

where  $\mathbf{v} = \mathbb{U}^T\mathbf{r}$  is an  $N$ -dimensional vector. Moreover

$$\sum_{i=1}^3 v_i^2 \leq \sum_{i=1}^N v_i^2 = \|\mathbf{v}\|^2 = \|\mathbf{r}\|^2 = N \quad (37)$$

since  $\mathbb{U}$  is a unitary matrix. Hence

$$\mathbf{q}^T(\mathbb{Q}^T\mathbb{Q})^{-1}\mathbf{q} \leq N, \quad (38)$$

177 which means that the third term of Eq. (15) is positive.

## 178 **Appendix B: Satisfying Eqs. (16) and (20) with spherical $t$ designs**

Consider a finite set of points  $\{\mathbf{w}_j\}$  ( $j \in [1, N]$ ), which lie on the surface of the three dimensional unit sphere. The set of points  $\{\mathbf{w}_j\}$  are said to constitute a spherical  $t$  design if for *any* polynomial function  $f(\mathbf{w})$  of order  $t$  or lower

$$\sum_{j=1}^N f(\mathbf{w}_j) = N \int f(\mathbf{w}) d\sigma_{\mathbf{w}}, \quad (39)$$

179 where  $d\sigma_{\mathbf{w}}$  is the normalized surface area element of the unit sphere.

Proof that Eqs. (16) and (20) can be satisfied using spherical 2 designs follows by showing that we can generate the constraints through appropriate choice of polynomial functions  $f(\mathbf{w})$  of second order degree or less in Eq. (39). Considering first the case  $f(\mathbf{w}) = w_s$  ( $s \in [1, 3]$ ), substitution into Eq. (39) yields

$$\sum_{j=1}^N w_{sj} = N \int w_s d\sigma_{\mathbf{w}} \quad (40)$$

where  $w_{sj}$  is the value of the  $s$ th element of  $\mathbf{w}_j$ . We can express  $\mathbf{w}$  in terms of the usual spherical polar coordinates, i.e.,  $\mathbf{w} = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]^T$  such that  $4\pi d\sigma_{\mathbf{w}} = \sin \theta d\theta d\phi$ . It is then simple to show that  $\int w_s d\sigma_{\mathbf{w}} = 0$  for  $s \in [1, 3]$  such that Eq. (40) reduces to Eq. (16). Similarly,

using the polynomial function  $f(\mathbf{w}) = w_s w_t$  for  $\{s, t\} = \{1, 2, 3\}$  Eq. (39) becomes

$$\sum_{j=1}^m w_{sj} w_{tj} = N \int w_s w_t d\sigma_{\mathbf{w}}. \quad (41)$$

180 Evaluating the integral on the right hand side yields  $\delta_{st}/3$ , such that Eq. (41) reduces to Eq. (20),  
 181 therefore completing our proof. Although we have proven that Eqs. (16) and (20) can be satisfied  
 182 by a spherical 2 design, it is worthwhile to note that it automatically follows that they can also be  
 183 satisfied by a spherical design of higher order,  $t \geq 2$ , because a spherical  $t$  design is also a  $t - 1$   
 184 design.

### 185 **Appendix C: Previous derivation**

The constraints derived in Section 2 through direct minimization of the EWV were first derived by Foreman et. al exploiting a claimed equivalence between minimizing the trace of  $\mathbb{G}^{-1}$  and maximizing the determinant of  $\mathbb{G}$ . Specifically, using the definition of the matrix inverse and Jacobi's formula it was first shown that the condition number can be expressed in the form<sup>35</sup>

$$\kappa^2 = \frac{N}{2} \text{tr}[(\mathbb{W}^T \mathbb{W})^{-1}] = \frac{N}{2} \sum_{i=1}^4 \frac{\partial \ln |\mathbb{G}|}{\partial G_{ii}} \quad (42)$$

186 where  $G_{ii}$  are the diagonal elements of  $\mathbb{G}$ . Based on Eq. (42) Foreman et al. claim that the  
 187 equivalence of optimization metrics follows from the differential relation  $2d \ln \kappa = -d \ln |\mathbb{G}|$ .  
 188 Regrettably this relation does not follow from Eq. (42), nor in fact does it hold in general, as can  
 189 be seen by expressing both  $\ln[\text{tr}[\mathbb{G}^{-1}]] = 2 \ln \kappa + \text{const.}$  and  $\ln |\mathbb{G}|$  in terms of the eigenvalues of  
 190  $\mathbb{G}$ .

191 *Disclosures*

192 The authors declare they have no conflicts of interest.

193 *Acknowledgements*

194 The authors would like to thank Dr. A. Favaro for useful discussions. MRF also acknowledges  
195 financial support from the Royal Society through a Royal Society University Research Fellowship.

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