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# Intensity fluctuation-invariant maximum likelihood estimation of speckle order

FRANÇOIS GOUDAIL<sup>1,\*</sup> AND JAN DUPONT<sup>1</sup>

<sup>1</sup>Laboratoire Charles Fabry, Institut d'Optique Graduate School, CNRS, Université Paris-Saclay, 91127 Palaiseau cedex, France

\*Corresponding author: francois.goudail@institutoptique.fr

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**We present a speckle order estimation method invariant to mean intensity variations inside the sample. It is based on the acquisition of two statistically independent speckle images. Cramer Rao Lower bound (CRLB) and maximum likelihood (ML) estimator in the presence of pure speckle noise are derived, and shown to have performance very close to classical estimation methods that are not invariant to mean intensity fluctuations. This method is also shown to be robust to Poisson shot noise and additive noise, and is validated on an optical experiment with a standard camera. It is thus useful for estimating speckle order in real-world coherent images.** © 2018 Optical Society of America

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Coherent imaging systems produce speckle [1]. This phenomenon can be considered useful in some applications, since it conveys information about the observed scene [2–4], or considered as noise hiding the useful information [5–8]. In all cases, its properties have to be carefully estimated. The statistical distribution of speckle is accurately represented as a Gamma law, which is defined by two parameters: mean and order [1]. We focus in this paper on the estimation of the order. This subject has already been investigated in the case of samples with uniform mean, and the maximum likelihood (ML) estimator has been studied [9]. However, in optical images, samples are seldom spatially uniform, and the fluctuation of the mean within the sample from one element to the other induces a noise to which the standard ML estimator is very sensitive. We present in this Letter a speckle order estimation method that is invariant to mean fluctuations within the sample. It is based on the acquisition of two statistically independent speckle images. They can be obtained for example by acquiring partially polarized speckles through two orthogonally oriented polarizers. We determine the Cramer Rao Lower Bound (CRLB) of this method, which is surprisingly very close to the CRLB for classical estimation assuming uniform mean within the sample. The robustness of this method to photon shot noise and additive noise is also investigated. The validity and advantages of this approach are

finally demonstrated through optical experiment.

Let us consider a sample  $X = \{x_i, i \in [1, N]\}$  of  $N$  realizations of a Gamma random variable whose probability density function (pdf) is defined as follows:

$$P_X(x) = \frac{L^L x^{L-1}}{\Gamma(L)m^L} \exp\left(-\frac{Lx}{m}\right) \quad (1)$$

where  $m$  is the mean (expectation) and  $L$  the order. The variance of a Gamma random variable is equal to  $m^2/L$ . Thus as  $L$  increases, its variance decreases, and it can also be shown that its pdf tends to a Gaussian [1].

The objective is to estimate  $m$  and  $L$  in the maximum likelihood (ML) sense. The expression of the loglikelihood is:

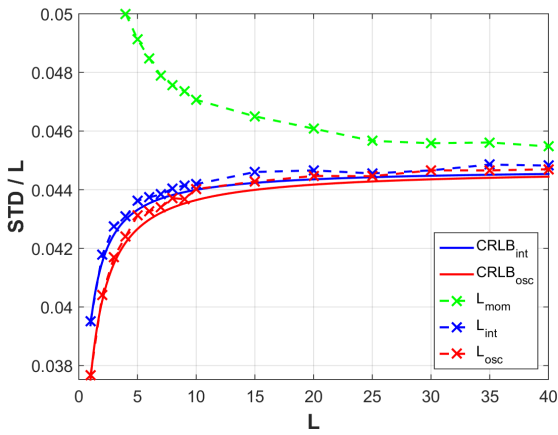
$$\begin{aligned} \ell(X) &= \sum_{i=1}^N \log P_X(x_i) = NL \log\left(\frac{L}{m}\right) + (L-1) \sum_{i=1}^N \log x_i \\ &\quad - N \log \Gamma(L) - \frac{L}{m} \sum_{i=1}^N x_i \end{aligned} \quad (2)$$

Annulling its gradient with respect to  $m$  and  $L$  yields that the ML estimate of the mean is  $\hat{m} = \sum_{i=1}^N x_i / N$  and that the ML estimate of the order is the solution of the following equation:

$$\log(L) - \psi(L) = \log(\hat{m}) - \frac{1}{N} \sum_{i=1}^N \log x_i \quad (3)$$

where  $\psi(z) = d \log \Gamma(z) / dz$  is the first derivative of the logarithm of the Gamma function (also called digamma function [10]). The solution of Eq. (3) will be denoted  $\hat{L}_{int}$ , where the subscript "int" stands for "intensity image", since in optics, Gamma distributed measurements are often obtained from intensity speckle. This equation has to be solved numerically by using a root-finding function. In this Letter, we have used the `fzero` function of Matlab, that uses a combination of bisection, secant, and inverse quadratic interpolation methods, to solve for all the considered non closed-form estimators, that is, Eqs. 3, 10 and 11.

The Cramer-Rao Lower Bound (CRLB) is a lower bound on the estimation variance reached by any unbiased estimator, and also represents the asymptotic variance of the ML estimator as  $N$  increases [11]. The CRLB for estimation of  $m$  and  $L$  are obtained by computing the Fisher matrix, that is the expectation of the matrix of second derivatives of the loglikelihood [11]. In our



**Fig. 1.** Plot of  $\sqrt{\text{CRLB}_{\text{int}}}$ ,  $\sqrt{\text{CRLB}_{\text{osc}}}$ , standard deviations of  $\hat{L}_{\text{mom}}$ ,  $\hat{L}_{\text{int}}$ , and  $\hat{L}_{\text{osc}}$ , divided by  $L$  and as a function of speckle order  $L$ .

case, it is easily shown from Eq. (2) that this matrix is diagonal. Taking the negative inverse of this matrix yields the following value of the CRLB for estimation of  $L$ :

$$\text{CRLB}_{\text{int}} = \frac{1}{N \left[ \psi'(L) - \frac{1}{L} \right]} \quad (4)$$

where  $\psi'(z)$  is the derivative of  $\psi(z)$  (also called trigamma function [10]). The Taylor expansion of the trigamma function is [10]:

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + o(1/z^3) \quad (5)$$

Using this expression, it is easily shown that the square root of the CRLB, which corresponds to a bound on standard deviation, is approximately an affine function of  $L$ :

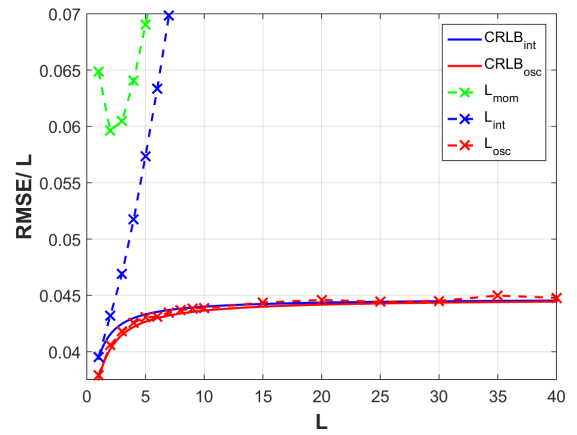
$$\sqrt{\text{CRLB}_{\text{int}}} \simeq \frac{1}{\sqrt{N}} \left( \sqrt{2}L - \frac{1}{3\sqrt{2}} \right) \quad (6)$$

This approximation is all the more valid as  $L$  is large.

We have represented in Fig. 1 the variation of  $\sqrt{\text{CRLB}_{\text{int}}}/L$  as a function of  $L$ , for a sample size  $N = 1000$  (blue solid curve). It is verified that it tends to a constant equal to  $\sqrt{2}/N = 0.047$  as  $L$  increases. We have also represented an estimate of the standard deviation of the ML estimator  $\hat{L}_{\text{int}}$  divided by  $L$  (blue dotted curve). This estimate has been obtained by Monte Carlo simulations with  $10^5$  realizations of a Gamma-distributed sample. It is verified that for this value of  $N$ , the ML estimator variance is very close to the CRLB for any value of  $L$ .

For comparison purpose, we have also plotted an estimate of the standard deviation of the moment estimator of  $L$ , divided by  $L$ , using the same type of Monte Carlo simulation. The moment estimator is defined as  $\hat{L}_{\text{mom}} = \text{MEAN}(X)^2/\text{VAR}(X)$ , where  $\text{MEAN}(X)$  is the empirical mean and  $\text{VAR}(X)$  the empirical variance of the sample. It is verified in Fig. 1 (green dotted curve) that this estimator has a larger standard deviation than the ML estimator, and reaches a larger asymptotic value as  $L$  increases.

The problem with the ML and moment estimators is that if the value of  $m$  fluctuates within the sample from one element to the other, so that  $m_i$  depends on the sample index  $i$ , this induces a noise source to which these estimators is very sensitive. We have



**Fig. 2.** Plot of  $\sqrt{\text{CRLB}_{\text{int}}}$ ,  $\sqrt{\text{CRLB}_{\text{osc}}}$ , RMSE of  $\hat{L}_{\text{mom}}$ ,  $\hat{L}_{\text{int}}$ , and  $\hat{L}_{\text{osc}}$ , divided by  $L$  and as a function of speckle order  $L$ , in the presence of mean fluctuations within the sample, with  $\sigma_m = 0.1$ .

performed a Monte Carlo simulation assuming that the mean of the  $i^{\text{th}}$  element of the sample is  $m_i = m_0(1 + n_i)$ , where  $n_i$  is a Gaussian random variable of zero mean and standard deviation  $\sigma_m = 0.1$ . We have represented in Fig. 2 the estimated root mean square error (RMSE) of  $\hat{L}_{\text{mom}}$  (green dotted curve) and  $\hat{L}_{\text{int}}$  (blue dotted curve). Mean value fluctuations mainly introduce a bias in the estimation. The bias of an estimator  $\hat{L}$  is defined as  $\text{BIAS} = | \langle \hat{L} \rangle - L |$ , where  $\langle \cdot \rangle$  denotes ensemble averaging, and the  $\text{RMSE} = (\text{BIAS}^2 + \text{STD}^2)^{1/2}$  takes into account both the bias and the standard deviation (STD). It is seen in Fig. 2 that the RMSE of both estimators dramatically increase as  $L$  increases, which is normal since speckle variance is inversely proportional to  $L$ : as  $L$  increases, the relative importance of mean variations over speckle fluctuations increases.

In order to solve this problem, one needs an estimator that is independent of the mean  $m_i$ . One solution is to use two independent speckle realizations. In practice, this can be obtained by moving a diffuser in front of the scene, or, if the speckle is partially polarized, by analyzing it with two orthogonal polarizers [1]. In this case, we have two samples:  $X$ , of fluctuating mean parametrized as  $m_i^X = F_i(1 + u)/2$ , and  $Y$ , of mean  $m_i^Y = F_i(1 - u)/2$ , where  $F_i = m_i^X + m_i^Y$ . The parameter  $u = (m_i^X - m_i^Y)/(m_i^X + m_i^Y)$  represents the relative difference of mean intensities between the two channels (it is independent of the  $F_i$ ). In a polarimetric configuration, it would represent the Orthogonal State Contrast (OSC) of the speckle. The OSC is equal to the degree of polarization when pure depolarizers are considered, which is a reasonable hypothesis in the presence of natural textures [12].

In order to construct a sample statistics that is independent of  $F_i$ , we will consider the empirical OSC defined as

$$R = \{ \rho_i, i \in [1, N] \} \quad \text{with} \quad \rho_i = \frac{x_i - y_i}{x_i + y_i} \quad (7)$$

The pdf of this sample has been shown to be equal to [5]:

$$P_R(\rho) = \frac{\Gamma(2L)}{2^{2L-1}[\Gamma(L)]^2} (1 - u^2)^L \frac{(1 - \rho^2)^{L-1}}{(1 - u\rho)^{2L}} \quad (8)$$

It is verified that this pdf only depends on  $u$  and  $L$ , and not on the values  $F_i$ : it is thus independent of the fluctuations of the mean.

Of course, we will assume that  $u$  does not change within the sample. For example, in the polarimetric measurement scheme mentioned above, it would mean that the polarization properties are uniform within the sample.

The goal is now to determine estimates of  $u$  and  $L$ . The loglikelihood of sample  $R$  has the following expression:

$$\begin{aligned} \ell(R) = & N[\log \Gamma(2L) - 2\log \Gamma(L)] - (2L - 1)N \log 2 \\ & + (L - 1) \sum_i \log(1 - \rho_i^2) - L \sum_i \log \left[ \frac{(1 - u\rho_i)^2}{1 - u^2} \right] \end{aligned} \quad (9)$$

Annuling the partial derivative of  $\ell(R)$  with respect to  $u$  yields that the ML estimate of  $u$ , denoted  $\hat{u}$ , is solution of the equation

$$\frac{1}{N} \sum_{i=1}^N \frac{\rho_i}{1 - u\rho_i} = \frac{u}{1 - u^2} \quad (10)$$

Annuling the partial derivative of  $\ell(R)$  with respect to  $L$ , and replacing  $u$  with its ML estimate  $\hat{u}$ , yields the ML estimate of  $L$ , denoted  $\hat{L}_{\text{osc}}$ , as the solution of the following equation in  $L$ :

$$2[\psi(2L) - \psi(L)] = \frac{1}{N} \sum_{i=1}^N \log \left[ \frac{(1 - \hat{u}\rho_i)^2}{1 - \rho_i^2} \right] - \log \left( \frac{1 - \hat{u}^2}{4} \right) \quad (11)$$

To determine the CRLB, one has to compute the Fisher matrix. Using the expression of the loglikelihood in Eq. (9), it can be shown after cumbersome but elementary calculations that  $\langle \partial^2 \ell / \partial u \partial L \rangle = 0$ . The Fisher matrix is thus diagonal. Using the expression of the loglikelihood in Eq. (9), it is easily shown that the CRLB on  $L$  is equal to:

$$\text{CRLB}_{\text{osc}} = - \frac{1}{\langle \frac{\partial^2 \ell}{\partial L^2} \rangle} = \frac{1}{2N[\psi'(L) - 2\psi'(2L)]} \quad (12)$$

Using the Taylor series of  $\psi'(z)$  in Eq. (5), one obtains:

$$\sqrt{\text{CRLB}_{\text{osc}}} \simeq \frac{1}{\sqrt{N}} \left( \sqrt{2L} - \frac{1}{2\sqrt{2}} \right) \quad (13)$$

By comparing with Eq. (6), it is seen that the CRLB of  $L$  estimated from OSC is very close to the CRLB of  $L$  estimated from intensity measurement. It is even slightly lower for all values of  $L$ , but tends to be equal as  $L$  increases.

It is also interesting to notice that  $\text{CRLB}_{\text{osc}}$  does not depend on the value of  $u$ . This may seem surprising, since for example, if  $u$  is close to 1, the means  $m_i^Y$  are very low, thus less accurately estimated and it would be expected that the CRLB is higher. This is due to the fact that only speckle noise is taken into account in our model. However, in practical measurements, other types of noise sources are present, such as Poisson shot noise and additive noise. Immunity to these noise sources will be better if  $u$  is close to zero, that is, if the means in channels  $X$  and  $Y$  are well balanced. We will study the robustness of the method to extraneous sources of noise at the end of this Letter. For the moment, we can say that the CRLB in Eq. (12) represents a fundamental limit on the estimation variance, since it only depends on speckle's intrinsic variability, that cannot be avoided contrary to the other, extraneous sources of noise.

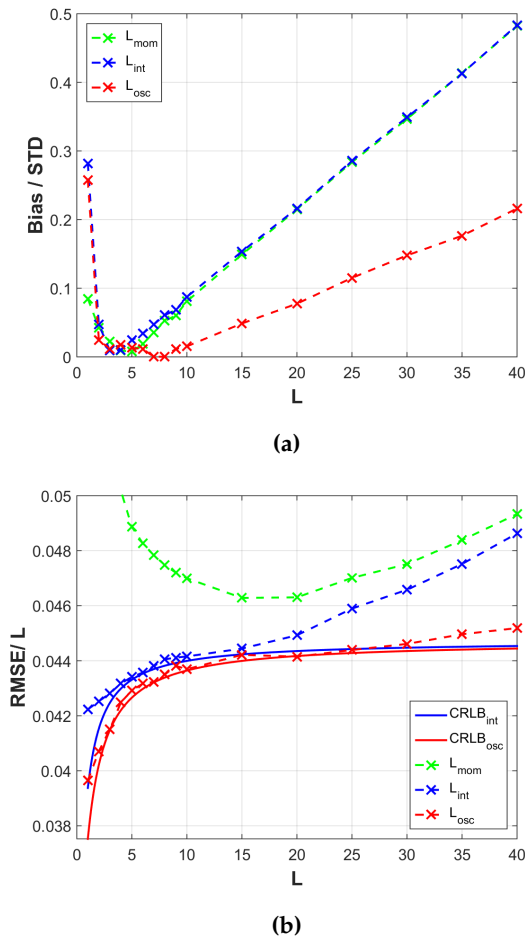
We have represented in Fig. 1 the variation of  $\sqrt{\text{CRLB}_{\text{osc}}}/L$  as a function of  $L$ , for a sample size  $N = 1000$  (red solid curve). It is indeed very close, and even a little smaller, than  $\sqrt{\text{CRLB}_{\text{int}}}/L$ . We have also represented the standard deviation of  $\hat{L}_{\text{osc}}$  divided by  $L$  (red dotted curve). It has been estimated by Monte Carlo

simulations with  $10^5$  realizations. It is seen that for this value of  $N$ , it reaches the CRLB for any value of  $L$ . It performs as well as  $\hat{L}_{\text{int}}$ , and even a little better for small values of  $L$ . We have also plotted in Fig. 2 the standard deviation of  $\hat{L}_{\text{osc}}$  in the presence of fluctuations of the mean, estimated by Monte Carlo simulation (red dotted curve). It is verified that its performance remains the same as in the presence of uniform mean, and it thus performs much better than  $\hat{L}_{\text{int}}$  and  $\hat{L}_{\text{mom}}$ . This method is thus very interesting to estimate speckle order in the presence of fluctuations.

However, in practical imaging situations, there are other sources of noise. First, the photo-detection process generates Poisson shot noise. Moreover, dark current fluctuations, readout noise, and fluctuations of background illumination from ambient sources give rise to fluctuations which level is independent of coherent illumination. They can thus be said "additive" and we will assume them Gaussian with total variance  $\sigma^2$ . Deriving the CRLB in the presence of Poisson and additive noise is complex and beyond the scope of this Letter. We will only illustrate here the influence of these noise sources on a specific example with realistic noise levels that can be encountered when imaging with standard digital cameras.

Let us consider a camera with maximal number of photoelectrons in one pixel (full well capacity) equal to 20,000 and perturbed by additive Gaussian noise of standard deviation 20 photoelectrons. The parameter  $u$  is set to 0, and the channel means are supposed constant within the sample. They are assumed to be equal to  $m^X = m^Y = 4000$  photoelectrons in order not to saturate the full well capacity for low values of  $L$  (i.e., when speckle variance is largest). In the presence of Poisson or additive noise, the ML estimator  $\hat{L}_{\text{osc}}$  defined in Eq. (11) cannot be used as is, since the measured values equal or less than zero would cause numerical errors. Thus all the sample values less than zero are set to one, which causes a bias in the estimation. This bias is larger for small values of  $L$ , where speckle realizations close to zero are more probable. Moreover, when  $L$  is large, the ratio of the variance of speckle (which is  $m^2/L$ ) to that of photon noise (which is  $m$ ) decreases, increasing the relative weight of Poisson noise and thus the inadequacy of the ML estimator. This introduces a bias for large values of  $L$ . We have represented in Fig. 3.a the bias of the ML estimator  $\hat{L}_{\text{osc}}$  divided by its standard deviation. It is verified that it is larger for small and large values of  $L$ , and that for these values, it becomes non-negligible with respect to the standard deviation. We have represented in Fig. 3.b the RMSE of the estimators  $\hat{L}_{\text{int}}$ ,  $\hat{L}_{\text{mom}}$ , and  $\hat{L}_{\text{osc}}$  divided by  $L$ . It is observed that the RMSE of all estimators are larger than when only speckle noise is present, but the difference is slight. It is also noticed that intensity-based estimators are more impacted by detection noise than  $\hat{L}_{\text{osc}}$ .

We can conclude from these simulations that the proposed method should be sufficiently robust to Poisson and additive noise to be used efficiently in practice. In order to verify it, we have performed an optical imaging experiment where noise levels are comparable to those considered in the simulations. The coherent light source is a HeNe laser ( $\lambda = 633$  nm) and the illumination beam is collimated and fully polarized along the vertical direction. The illuminated sample is a white paper sheet, and the scattered field is projected with a rotating polarizer along the parallel (orthogonal) polarization directions to obtain images  $X$  ( $Y$ ). The OSC parameter  $u$  of the scattered field is quite low (around 0.13), thus the mean intensities are well balanced between the two polarimetric channels. The projected field is

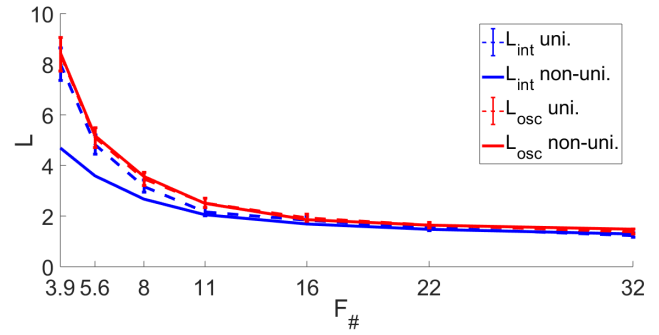


**Fig. 3.** Bias divided by STD (a) and RMSE divided by  $L$  (b) of  $\hat{L}_{mom}$ ,  $\hat{L}_{int}$ , and  $\hat{L}_{osc}$ , as a function of speckle order  $L$ , in the presence of Poisson noise ( $u = 0$ ,  $m^X = m^Y = 4000$ ,  $\max_i(X_i) = 20,000$ ) and additive Gaussian noise with standard deviation equal to 20.

imaged with a lens which  $f$ -number  $F_{\#}$  can vary between 3.9 and 32 thanks to an adjustable pupil diaphragm. The used camera is a CCD matrix with a  $9.9 \mu\text{m}$  square pitch and detection noise level of 20 photoelectrons. As  $F_{\#}$  increases, the correlation area of the speckle field increases and the order of the speckle integrated on each pixel decreases [1].

We have considered two areas in the image: a  $51 \times 51$  pixel region A where illumination intensity is uniform, and a  $5 \times 520$  pixel region B where the intensity varies along a Gaussian profile due to the illumination beam, with a lowest value equal to 30% of its maximum value. In Fig. 4, we have represented the estimation results of  $\hat{L}_{int}$  and  $\hat{L}_{osc}$  on uniform region A (dotted curves) and on non-uniform region B (solid curves). The error bars plotted on the uniform illumination curves correspond to  $\pm 3$  standard deviations, computed with Eq. (6) and Eq. (13). As predicted, the two estimators  $\hat{L}_{int}$  and  $\hat{L}_{osc}$  give equivalent results when illumination is uniform, since the corresponding curves fit with each other within the estimation precision given by error bars. When illumination is non-uniform, it is seen that only  $\hat{L}_{osc}$  is accurate, since its curve is exactly superposed with that obtained in the case of continuous illumination. On the

contrary,  $\hat{L}_{int}$  (blue solid curve) yields estimations that depart largely from the continuous illumination case, especially for larger values of  $L$  (smaller values of  $F_{\#}$ ).



**Fig. 4.** Experimental estimation of  $\hat{L}_{int}$ , and  $\hat{L}_{osc}$ , for a uniform (dashed line) and non-uniform (continuous line) illumination, plotted as a function of the focal number  $F_{\#}$ .

We have studied and validated a speckle order estimation method that is applicable on samples with non uniform mean. It is based on the acquisition of two statistically independent speckle images. This method performs as well as the classical methods when sample mean is uniform, and much better when it is non uniform. The CRLB of this method thus constitutes a closed-form expression of the fundamental limit of estimation variance of speckle order, since it only depends on speckle intrinsic variability. Moreover, this method has been shown robust to Poisson shot noise and additive noise, and validated on an optical experiment with a standard camera. It is thus useful for estimating speckle order in real-world coherent images. A particularly interesting perspective to this work is derive CLRB and ML estimators in the presence of Poisson and additive noise in order to more precisely assess the domain of validity and the limits of this method.

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