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Market exit and minimax regret

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Abstract

This paper shows how minimax regret sheds new light on an old economic topic, market-exit games. It focuses on wars of attrition, namely overcrowded duopoly markets where the strategic variable is the exit time. The only symmetric Nash equilibrium of the game studied is a mixed-strategy equilibrium that leads to a null expected payoff, i.e. the payoff a firm gets when it immediately exits the market. This result is not convincing, both from a behavioral and from a strategic viewpoint. The minimax regret approach, that builds upon opposite regrets – exiting the market too late and exiting the market too early – is more convincing and ensures that both firms obtain a strictly positive expected payoff.

Highlights

- Minimax regret behavior sheds new light on the optimal exit-times in overcrowded markets
- Minimax regret behavior provides a new behavioral content to mixed strategies
- In overcrowded duopolies, if the firms’ aim consists in minimizing regrets, then they get better payoffs than in the mixed-strategy Nash equilibrium.

Keywords: war of attrition, minimax regret, mixed-strategy Nash equilibrium, maximin payoff, overcrowded market.

JEL Classification: C72, D4

Market exit and minimax regret

1. Introduction

We show in this paper how the concept of minimax regret sheds new light on exit times in overcrowded markets. We study an overcrowded duopoly market where both firms decide if and when they exit the market, knowing that the surviving firm gets a positive monopoly profit. Information is complete and both firms are identical (same profit function). The only symmetric Nash equilibrium (NE) of this war of attrition (Maynard Smith, 1974) is a full support mixed-strategy equilibrium where each firm gets a null equilibrium payoff, i.e. the payoff obtained by immediately leaving the market. Most authors bypass this unconvincing result by introducing some incomplete information, for example on the duopoly profits. By doing so, they get a pure-strategy symmetric NE with positive payoffs (see, for example, Fudenberg and Tirole (1986)), that is also more easy to test experimentally (Hörisch and Kirchkamp (2010), Oprea et al. (2013)).

In this paper, we stick to the complete information context, but we construct the mixed strategies in a new way. We focus on the minimax regret approach introduced to game theory by Linhart and Radner (1989) and developed by Halpern and Pass (2012), Hayashi (2008), Hayashi and Yoshimoto (2012) and Renou and Schlag (2010). The notion of minimax regret is well known in single-agent decision problems with a strong uncertainty, in that it goes back to Savage (1951) and Niehans (1948). An economic agent, if he is unable to anticipate a future state of the world (state of Nature) may opt for a strategy that minimizes his regret, which is the difference between the payoff obtained with his decision and the payoff obtained with the best decision in the realized state of the world. In a game, players are not confronted with Nature – at least not exclusively – but with other players. So they may suffer from a strong strategic uncertainty, in that it may be difficult, for many reasons, to anticipate how the other players will play. Players may be unsure about others' rationality and, even with rational players, they may be unsure of their way of playing, particularly when many strategies are rationalizable (Pearce, 1984). Wars of attrition are such contexts. In the game studied, leaving the market at any positive time t is a best response when the opponent exits the market early and before time t . Additionally, it is better to leave the market immediately (at time 0), when the opponent decides to never exit the market. So it is difficult to anticipate the opponent's behavior and it becomes more convincing to play a strategy that does not generate too much regret, rather than trying to best reply to an unknown strategy.

The minimax regret philosophy also adapts well to contexts that generate different types of regret. Such contexts are numerous. In a Bertrand duopoly, for example, by playing a price lower than the competitor's one, a firm may regret not having played a slightly higher price that would have allowed it to serve the entire market with a higher profit. But by playing a higher price than the competitor's, a firm simply regrets having lost its part of the market. In first-price sealed-bid auctions, a bidder may regret having submitted too low a bid, in that he was ready to pay the higher winning bid, but he may also regret having submitted too high a bid if he could have won the auction by bidding less (see Filiz-Ozbay and Ozbay (2007), Hayashi and Yoshimoto (2012)). And so on... The same fact is true in the market-exit game. In this "should I stay or should I go" problem, each firm may suffer from two types of regret: it is worth staying in the market, even while losing money, if the opponent exits the market fast (so a firm may regret leaving the market too early), but it is better to leave the market immediately if the opponent exits the market late (so a firm may regret leaving the market too late, in that the potential monopoly payoffs will not cover the excessive duopoly losses). The minimax regret criterion, and more particularly the mixed-strategy minimax regret concept, adapts well to contexts with multiple types of regret, in that it balances the possible regrets in order to minimize them.

We show that the mixed-strategy minimax regret behavior is completely different from the mixed-strategy symmetric NE behavior and that it ensures, if played by both firms, a strictly positive expected profit for both firms.

Throughout the paper, for the purposes of applied economy, we study the market-exit game in both discrete and continuous time. In section 2 we introduce the game studied, recall the symmetric mixed-strategy NE in the continuous-time game, and establish this equilibrium in the discrete-time game. We also compare the mixed-strategy NE with the behavior observed in a classroom experiment. In section 3 we turn to the minimax regret behavior, both in the discrete and in the continuous setting. In section 4 we discuss the philosophy behind the mixed-strategy NE and the mixed-strategy minimax regret behavior. In section 5 we show that the mixed-strategy minimax regret behavior, contrary to the mixed-strategy NE, always leads to a strictly positive expected payoff when this behavior is adopted by both firms. Section 6 concludes on the meaning of mixed strategies. It also opens the discussion on bankruptcy, i.e. on the necessity to put limits on the maximal exit time.

2. Exit-time game and Nash equilibrium

The game goes as follows: two firms compete in a duopoly market, in discrete or continuous time. The market is overcrowded and each firm gets the negative duopoly profit D as long as both firms stay in the market. If one firm decides to exit the market, then the other firm gets the positive monopoly profit M forever. Time is discounted at rate r , with $r > 0$. In many countries, a firm cannot indefinitely earn negative profits, so the duopoly market cannot survive more than T periods (an amount of time T in the continuous setting). If the duopoly is still alive at time T , both firms are declared bankrupt and forced to leave the market.

Throughout the paper, we work with the reduced normal form of the game.

In the discrete-time game, each firm decides in which period t it will leave the duopoly market (if the opponent has not yet left). Leaving at time 0 means immediately leaving the market, in which case the firm earns a null payoff. Leaving in period $t > 0$, if the opponent stays in the market longer, leads to the negative payoff D in each period, from period 1 to period t , i.e., $D + \frac{D}{1+r} + \dots + \frac{D}{(1+r)^{t-1}} = \sum_{i=0}^{t-1} \frac{D}{(1+r)^i}$. If the firm leaves in period t and the opponent leaves in period u , with $t > u$, then the firm's payoff becomes: $\sum_{i=0}^{u-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^u} \frac{(1+r)}{r}$ if $u > 0$, and $\frac{M(1+r)}{r}$ if $u = 0$, because it gets the positive monopoly payoff forever, from period $u+1$ onwards.

For the model to be of interest, we suppose that there exists a positive period t^* , such that $\sum_{i=0}^{t^*-2} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{t^*-1}} \cdot \frac{1+r}{r} > 0$ and $\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{t^*}} \cdot \frac{1+r}{r} < 0$, so that it is worth staying in the duopoly market for one additional period if the opponent decides to leave in period t^*-1 , but it is better to exit the market at time 0 if the opponent leaves in period t^* or later.

Given that $\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{t^*}} \cdot \frac{1+r}{r} = \frac{D(1 - \frac{1}{(1+r)^{t^*}})(1+r)}{r} + \frac{M}{(1+r)^{t^*}} \frac{1+r}{r}$, t^* is the first period checking $t^* \geq \frac{\ln(\frac{M-D}{-D})}{\ln(1+r)}$. We assume $\frac{\ln(\frac{M-D}{-D})}{\ln(1+r)} > 1$, hence $-D < M/r$, which means that it is worth having one period of the duopoly loss when getting the monopoly profit thereafter forever.

For example, if $T=6$, $-D=M=1$ and $r=0.25$, we get the normal-form game in matrix 1.

In this example, $t^*=4$. It is profitable to stay in the market if the opponent leaves before or in period 3, in that the firm achieves at least the positive payoff 0.12 (thanks to the monopoly profit from period 4 onwards), but it is better to exit the market at time 0 if the opponent

leaves in period $t^*=4$ or later (in which case the best payoff when leaving later – obtained with a monopoly profit from period 5 onwards – is -0.904).

		Firm 2						
		0	1	2	3	4	5	6
Firm 1	0	(0, 0)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)	(0, 5)
	1	(5, 0)	(-1, -1)	(-1, 3)	(-1, 3)	(-1, 3)	(-1, 3)	(-1, 3)
	2	(5, 0)	(3, -1)	(-1.8, -1.8)	(-1.8, 1.4)	(-1.8, 1.4)	(-1.8, 1.4)	(-1.8, 1.4)
	3	(5, 0)	(3, -1)	(1.4, -1.8)	(-2.44, -2.44)	(-2.44, 0.12)	(-2.44, 0.12)	(-2.44, 0.12)
	4	(5, 0)	(3, -1)	(1.4, -1.8)	(0.12, -2.44)	(-2.952, -2.952)	(-2.952, -0.904)	(-2.952, -0.904)
	5	(5, 0)	(3, -1)	(1.4, -1.8)	(0.12, -2.44)	(-0.904, -2.952)	(-3.3616, -3.3616)	(-3.3616, -1.7232)
	6	(5, 0)	(3, -1)	(1.4, -1.8)	(0.12, -2.44)	(-0.904, -2.952)	(-1.7232, -3.3616)	(-3.68928, -3.68928)

Matrix 1: market-exit game for $T=6$, $M= -D=1$ $r=0.25$

In the continuous-time game, each firm decides at which time t it exits the duopoly market (if the opponent has not yet left), with t from 0 to T . Leaving at time 0 leads to a null payoff. Leaving at time t , when the opponent stays longer in the market, leads to the payoff $\int_0^t D e^{-rs} ds$. And when the opponent leaves the market sooner, at time $u < t$, then the payoff becomes $\int_0^u D e^{-rs} ds + \int_u^\infty M e^{-rs} ds$. In the continuous-time game, it is profitable to stay in the market, relative to exiting immediately, if the opponent exits the market at time t^* or sooner – relatively later than t^* – for t^* checking: $\int_0^{t^*} D e^{-rs} ds + \int_{t^*}^\infty M e^{-rs} ds = 0$. So we get $t^* = \ln(-\frac{M-D}{D})/r$. $t^*=2.77$ in the numerical example.

In both the discrete-time game and the continuous-time game, there is only one symmetric NE and it is in mixed strategies.

Proposition 1 (close to Maynard Smith (1974))

In the continuous-time game, the unique mixed-strategy symmetric NE is given by:

- *The support of the equilibrium is $[0, T]$*
- *The cumulative probability distribution on $[0, T]$ is given by: $F(t) = 1 - e^{-\frac{Dr}{M}t}$*
- *T is a mass point played with probability $g(T) = e^{-\frac{Dr}{M}T}$*

Proof: see Appendix A

It derives from this proposition that the probabilities decrease from 0 to T . The density function $f(t) = -\frac{Dr}{M} e^{-\frac{Dr}{M}t}$ decreases at rate Dr/M and has a mass point on T .

In the discrete-time game we get:

Proposition 2

We note p_t , the probability of exiting the market in period t , t from 0 to T . In the discrete-time game, the only mixed-strategy symmetric NE is given by:

$$p_0 = -\frac{rD}{M+r(M-D)}$$

$$p_t = \left(\frac{M(1+r)}{M+r(M-D)}\right)^t p_0 \text{ for } t \text{ from } 1 \text{ to } T-1$$

$$p_T = \left(\frac{M(1+r)}{M+r(M-D)}\right)^T$$

Proof: see Appendix B

Again, we get decreasing probabilities from 0 to $T-1$ (given that $\frac{M(1+r)}{M+r(M-D)} < 1$) and an extra probability p_T , larger than p_{T-1} . The probabilities decrease at rate $(p_{t+1} - p_t)/p_t = \frac{rD}{M+r(M-D)}$ from p_0 to p_{T-1} , so the equilibrium distribution behaves similarly in the discrete-time game and in the continuous-time game. There is no discontinuity between the discrete and the continuous model, which is not always the case in a war of attrition.

These probabilities, in the numerical example, are given and illustrated in figure 1.

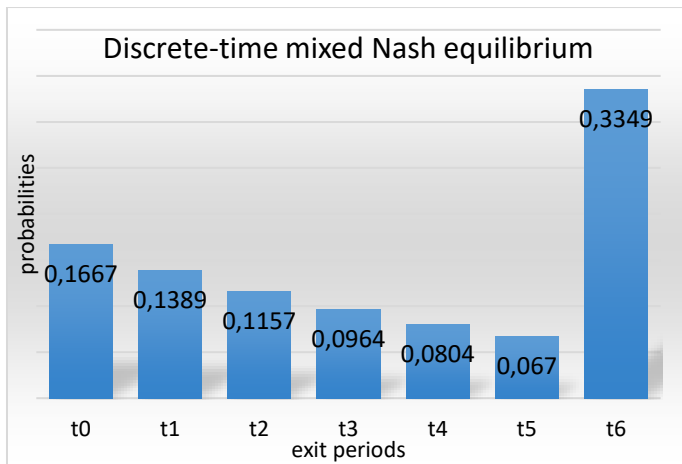


Figure 1: mixed-strategy Nash equilibrium for $T=6$, $M=-D=1$, $r=0.25$

In both the discrete-time game and the continuous-time game, the equilibrium payoff is 0 by construction.

This NE is not appealing from an economic and from a strategic viewpoint. Firstly, why should a firm play in a mixed-strategy way, risking a negative payoff, just to earn an expected payoff equal to 0, whereas it could be sure to not lose any money by immediately exiting the market? The mixed-strategy NE does no better than the maximin-payoff strategy which simply consists in exiting the market at time 0 (see Appendix C). Secondly, in the asymmetric

pure-strategy Nash equilibria, one firm gets 0 and the other firm gets the positive monopoly payoff forever; so the mean payoff is positive and it is reasonable to expect to find a way of playing for both firms that leads to a positive payoff.

The game in figure 1 was played in a classroom experiment. The experiment was run during a third-year class in game theory¹ at the faculty of economics and management of the University of Strasbourg in the academic year 2020-2021. In total, 283 students participated in the experiment and they were free to decline and quit if they did not want to play. The question put to the students was: “At which time (0,1,2,3,4,5,6) do you exit the market if your opponent has not left yet?” Their answers are given in figure 2.

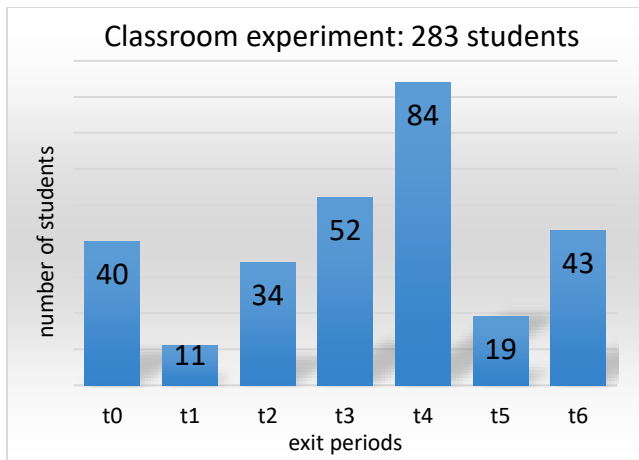


Figure 2: Students' answers in the market-exit game

Clearly, the students' behavior distribution is strongly different from the mixed-strategy NE distribution. The peak is on t^* and nearly half of the students (48.06%) choose t^* or t^*-1 , whereas in the NE a firm leaves the market at t^* or t^*-1 with a probability of only 17.68%. Moreover, the probabilities increase over time (from period 1 to period 4) whereas they decrease in the NE.

3. Exit-time game and minimax regret behavior

In a normal-form game with N players i , pure strategy sets S_i and utility functions u_i , player i 's regret from playing the pure strategy \tilde{s}_i when the opponents play the pure strategies s_{-i} is $r_i(\tilde{s}_i, s_{-i}) = \max_{s_i \in S_i} u_i(s_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i})$. The maximal regret that s_i leads to is $R_i(s_i) = \max_{s_{-i} \in S_{-i}} r_i(s_i, s_{-i})$. Player i 's minimax regret is $\min_{s_i \in S_i} R_i(s_i)$ (see Linhart and Radner (1989), Halpern and Pass (2012) and Renou and Schlag (2010) for more details).

¹ The students played the game before knowing the concepts of dominance and Nash equilibrium.

We first illustrate the minimax-regret concept in the discrete numerical example before switching to the general game. To do so, we compute, for each strategy, the regret it leads to in the context of each opponent's strategy, this regret being the difference between the best-reply payoff and the payoff obtained with the chosen strategy. For example, if firm 1 leaves in period 2 whereas firm 2 leaves in period 3, firm 1 gets two times the negative duopoly profit (-1.8), whereas it can get 0.12 (three times the negative duopoly profit plus the endless positive monopoly payoff from period 4 onwards) with the best decision, which consists in exiting the market in period 4, 5 or 6. So firm 1's regret when firm 2 leaves in period 3 and firm 1 leaves in period 2 is $0.12 - (-1.8) = 1.92$. Matrix 2 is firm 1's matrix of regrets:

		Firm 2						
		0	1	2	3	4	5	6
Firm 1	0	5	3	1.4	0.12	0	0	0
	1	0	4	2.4	1.12	1	1	1
	2	0	0	3.2	1.92	1.8	1.8	1.8
	3	0	0	0	2.56	2.44	2.44	2.44
	4	0	0	0	0	2.952	2.952	2.952
	5	0	0	0	0	0.904	3.3616	3.3616
	6	0	0	0	0	0.904	1.7232	3.68928

Matrix 2: regret matrix for firm 1 in the market-exit game for $T=6$, $M=-D=1$, $r=0.25$

For each strategy, we compute the maximal regret (in bold in matrix 2). For example, when firm 1 chooses to exit the market in period 2, its maximal regret is 3.2 and it is obtained when firm 2 also chooses to leave the market in period 2. If so, firm 1 earns two times the negative duopoly payoff by leaving in period 2 (-1.8) whereas it could obtain two times the duopoly payoff, plus the endless monopoly payoff from period 3 onwards (1.4), by leaving at least one period later (hence the regret is $1.4 - (-1.8) = 3.2$).

So, in pure strategies, the strategy that minimizes the maximal regret consists in exiting the market in period 3, i.e. t^*-1 (regret=2.56). This result can be generalized as follows.

Proposition 3

When $t^ \leq T$, in the discrete-time game, the pure minimax regret strategy consists in leaving the market in period t^*-1 if $-\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i} > \frac{M}{(1+r)^{t^*-1}} \frac{(1+r)}{r}$, in period t^* if not. In the continuous-time game, the pure minimax regret strategy consists in leaving the market at time t^* . This regret is equal to $\frac{-DM}{r(M-D)}$ in the continuous-time game.*

When $t^ > T$, the pure minimax regret strategy consists in leaving the market in period T , both in the discrete-time game and in the continuous-time game.*

Proof: see Appendix D

In the numerical example, $-\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i} = 2.952$ and $\frac{M}{(1+r)^{t^*-1}} \frac{(1+r)}{r} = 2.56$, so the pure minimax regret strategy consists in leaving in period $t^*-1 = 3$. The minimax regret is at the junction between the two types of regret: the regret a firm experiences when leaving too early (so it regrets the monopoly payoff $\frac{M}{(1+r)^{t^*-1}} \frac{(1+r)}{r}$ it could get by staying in the market for an additional period) and the regret due to not leaving at time 0 (it could avoid the duopoly losses $-\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i}$).

If the numerical example were studied in continuous time, the pure minimax regret strategy would consist in leaving at time $t^* = 2.77$ and the minimax regret would be equal to 2.

Yet $\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i}$, $-\frac{M}{(1+r)^{t^*-1}} \frac{(1+r)}{r}$ and $\frac{DM}{r(M-D)}$ may be large negative numbers, namely in comparison with the largest payoffs in the game ($M(1+r)/r$ in the discrete-time game and M/r in the continuous-time game). It is possible to lower the minimax regret by switching to mixed strategies. The idea is to construct a mixed strategy that minimizes the regret regardless of the exit time chosen by the opponent. Thus far we follow Renou and Schlag (2010). We call p_t firm 1's probability of exiting in period t . The probabilities are chosen in order to minimize the maximal regret, y , regardless of the exit time chosen by firm 2. In the discrete numerical example, this amounts to solving the optimization program:

$$\begin{aligned}
& \min_{y, p_0, p_1, p_2, p_3, p_4, p_5, p_6} y \\
& \text{u.c. } 5p_0 \leq y \\
& 3p_0 + 4p_1 \leq y \\
& 1.4p_0 + 2.4p_1 + 3.2p_2 \leq y \\
& 0.12p_0 + 1.12p_1 + 1.92p_2 + 2.56p_3 \leq y \\
& p_1 + 1.8p_2 + 2.44p_3 + 2.952p_4 + 0.904p_5 + 0.904p_6 \leq y \\
& p_1 + 1.8p_2 + 2.44p_3 + 2.952p_4 + 3.3616p_5 + 1.7232p_6 \leq y \\
& p_1 + 1.8p_2 + 2.44p_3 + 2.952p_4 + 3.3616p_5 + 3.68928p_6 \leq y \\
& p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1 \\
& 0 \leq p_t \quad t \text{ from 0 to 6}
\end{aligned}$$

This program has a unique solution,

$$\left(y = \frac{205}{144}, p_0 = \frac{41}{144}, p_1 = \frac{41}{288}, p_2 = \frac{41}{192}, p_3 = \frac{41}{128}, p_4 = \frac{5}{128}, p_5 = 0, p_6 = 0 \right)$$

The minimax regret distribution is represented in figure 3.

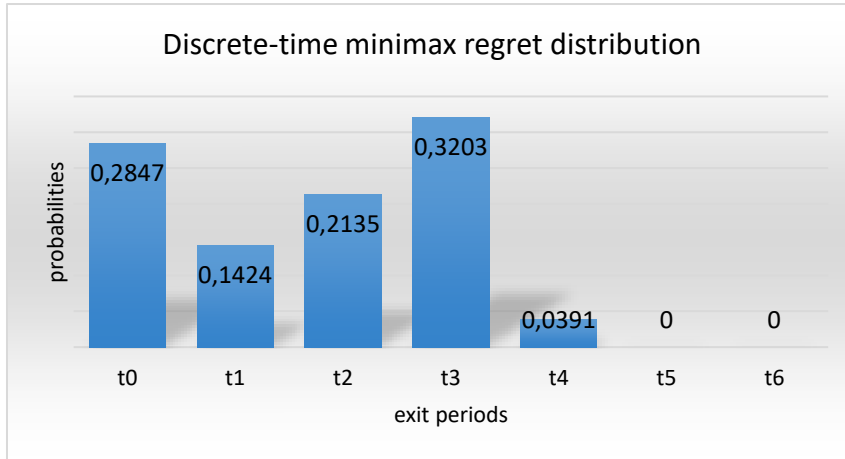


Figure 3: minimax regret distribution for $T=6$, $M=-D=1$, $r=0.25$

We observe that the probabilities increase from period 1 to period 3 (t^*-1) with the constant rate $(p_{t+1}-p_t)/p_t=0.5$, that leaving in period $4=t^*$ is played with a smaller probability, and that leaving immediately (time 0) is played with a probability that is different from the other probabilities. The minimax regret is $y=1.4236$ which means that a firm, regardless of the exit time chosen by the opponent, will never get less than the best-reply payoff minus 1.4236. Given that at the optimum all inequations equalize in y (see Appendix E), a firm, by playing this strategy, always gets the best possible payoff minus 1.4236. So, if firm 2 exits in period 0, 1, 2 or 3, firm 1 gets 3.5764, 1.5764, -0.0236 or -1.3036, respectively, and if firm 2 exits in periods 4, 5 or 6, firm 1 gets -1.4236. Hence, if the opponent also plays the minimax regret strategy, a firm's mean payoff is $0.7645 > 0$.

This distribution, in contrast with the mixed-strategy NE, better fits with the students' behavior in that t^*-1 and t^* , with the minimax regret strategy, have a large probability (36%) of being played, and in that the probabilities increase from p_1 to p_3 . Moreover, few students exit in period 5. Yet the distribution only partially expresses the students' behavior in that more than 15% of the students exit in period 6.²

We now give the minimax regret distribution in the general case:

Proposition 4

In the discrete-time game, the mixed minimax regret strategy consists in leaving the market with positive probability p_t in any period t from 0 to $\min(t^, T)$. The probabilities check the following equations:*

² The fact that the numerical example assigns a low probability to p_4 (much lower than that of p_3) should not be taken too seriously. p_{t^*} , depending on the numerical values of D , M and r , is sometimes larger and sometimes lower than p_{t^*-1} . This can be checked by switching to the numerical example $T=6$, $D=-1$, $M=1$ and $r=0.2$, where t^* is also equal to 4, but p_4 is larger than p_3 .

$$p_0 = -\left(\frac{D}{M}\right) \cdot \frac{(1+r)^{\min(t^*, T)-1}}{\left(\frac{M+r(M-D)}{M}\right)^{\min(t^*, T)}}$$

$$p_1 = \frac{r(M-D)}{M} p_0$$

$$p_t = \left(\frac{M+r(M-D)}{M}\right)^{t-1} \frac{r(M-D)}{M} p_0 \text{ for } t \text{ from } 2 \text{ to } \min(t^*-1, T-1)$$

$$p_{\min(t^*, T)} = \frac{p_0 \left(\frac{M+r(M-D)}{M}\right)^{\min(t^*, T)-1} \left((M-D) \left(\frac{1}{1+r}\right)^{\min(t^*, T)-1} + D \right)}{-D \left(1 - \left(\frac{1}{1+r}\right)^{\min(t^*, T)}\right)}$$

$$p_t = 0 \text{ for } t \text{ from } t^*+1 \text{ to } T \text{ when } t^* < T$$

$$\text{The minimax regret is equal to } \frac{p_0 M(1+r)}{r}$$

Proof: see Appendix E

We observe that p_0 and $p_{\min(t^*, T)}$ do not behave like the other probabilities, and that the probabilities from p_1 to $p_{\min(t^*, T)-1}$ increase with the constant increasing rate $\frac{r(M-D)}{M}$. If $t^* \leq T$, p_{t^*} can be small. As a matter of fact $D + (M-D) \left(\frac{1}{1+r}\right)^{t^*-1}$ is equal to $\frac{\left(D + \frac{D}{1+r} + \dots + \frac{D}{(1+r)^{t^*-2}} + \frac{M}{(1+r)^{t^*-1}} \cdot \frac{1+r}{r}\right)r}{r+1}$, the profit a firm earns by staying in the market longer than the opponent, who stays in up to period t^*-1 , multiplied by $r/(r+1)$. This profit becomes small when the interval between two periods becomes small. In contrast, if $t^* > T$, then $D + (M-D) \left(\frac{1}{1+r}\right)^{T-1}$ is generally not small so p_T can be large.

We now turn to the continuous-time game:

Proposition 5

$$t^* = \frac{\ln\left(\frac{M-D}{-D}\right)}{r}$$

In the continuous-time game, if $t^ \leq T$, the mixed-minimax regret strategy is given by:*

- *The support of the strategy is $[0, t^*]$*
- *The cumulative probability distribution on $[0, t^*]$ is given by: $F(t) = \left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}} \cdot e^{\frac{(M-D)rt}{M}}$*
- *0 is a mass point played with probability $g(0) = \left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}}$*
- *The minimax regret is equal to $\frac{g(0)M}{r}$.*

If $t^ > T$, the mixed-minimax regret strategy is given by:*

- The support of the strategy is $[0, T]$
- The cumulative probability distribution on $[0, T]$ is given by: $F(t) = g(0)e^{\frac{(M-D)rt}{M}}$
- 0 is a mass point played with probability $g(0) = (e^{rT} - 1)\left(\frac{-D}{M}\right)e^{-(M-D)rT/M} > 0$ and T is a mass point played with probability $g(T) = 1 + \frac{e^{rT}D}{M} - \frac{D}{M}$.
- $g(0)$ is increasing in T and equal to $\left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}}$ when $T=t^*$. Symmetrically, $g(T)$ is decreasing in T and equal to 0 when $T = t^*$.
- The minimax regret is equal to $\frac{g(0)M}{r}$

Proof: see Appendix F

If our discrete-time example (with $-D=M=1$ and $r=0.25$) were studied in a continuous setting, we would obtain $g(0)=0.25$ and a minimax regret equal to 1. We can observe that a regret of 1, which amounts to losing the monopoly payoff for one period, becomes acceptable given the potential best-payoffs (from 0 to 4).

In the continuous-time game, the probabilities increase at rate $r(M-D)/M$, like in the discrete-time game, from $0+$ to either t^* (when $t^* \leq T$) or T (if $t^* > T$).

If $t^* \leq T$, the discrete-time probability $p_0 = -\left(\frac{D}{M}\right) \cdot \frac{(1+r)^{t^*} - 1}{\left(\frac{M+r(M-D)}{M}\right)^{t^*}}$ becomes close to $g(0)$ in the continuous-time game as soon as $r \rightarrow 0$ in the discrete-time game.

When $T < t^*$, T becomes an additional mass point. $g(0)$ is increasing in T and equal to $\left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}}$ only for $T=t^*$, so the firm leaves the market at time 0 with a smaller probability. For example, in our numerical example, when $T=2 < t^*=2.77$, $-D=M=1$, $r=0.25$, $g(0)=0.239 < 0.25$ and $g(2)=0.351$. The minimax regret, equal to 0.955, is lower than 1, the minimax regret obtained for $T > t^*$.

At this level we have to explain why we choose to work with the reduced normal form and not with the extensive form of the duopoly exit game. This choice, which is of no importance as regards the Nash equilibrium, may have an impact as regards the concept of minimax regret. The problem is that the concept of minimax regret in extensive form games gives rise to some puzzling dilemmas that are, as far as we know, unsolved up to now. For example, it is not clear, in an extensive form game, whether, while calculating the regret linked to a decision, a player has to take forgone opportunities into account, or not. For example, in the market-exit game, when deciding in period t to exit the market or to stay in the market for one additional period, it is not clear whether a firm, when evaluating the regrets, should only

compare the potential payoff consequences of exiting and staying in the market, or if the firm should also compare these payoffs to the payoffs linked to the lost opportunity of exiting the market in previous periods. As argued by Halpern and Leung (2016), it may seem irrational to consider forgone opportunities – past is past and cannot be changed – yet people often do. Another problem is linked to local decisions (in a given period). So suppose that both firms are still in the market in period t . If a firm decides to exit the market, the regret is the lost potential monopoly payoff $\frac{M}{(1+r)^t} \cdot \frac{1+r}{r}$. If the firm stays in the market for one additional period, it at most suffers from the duopoly loss $\frac{D}{(1+r)^t}$, given that it can exit the market one period later (so the regret is only $\frac{-D}{(1+r)^t}$). This may induce a firm to stay in the market in period t because $M(1+r)/r > -D$. Yet, for the same reason, this induces it to also stay in the market in period $t+1$. It follows from this fact that, as mentioned by Halpern and Leung (2016), a “sophisticated” player should, in period t , take into account that her way of taking decisions will lead her to not exiting the market in period $t+1$. So, by staying in the market in period t , she may get at least two times the duopoly loss, and this may change her decision. It follows from these observations that the definition of minimax regret in extensive form games is still a work in progress, which explains our choice to work on the reduced normal form.

4. Minimax regret, mixed-strategy Nash equilibrium and maximin payoff: different behavior philosophies

It follows from figures 1 and 3 and from propositions 1, 2, 4 and 5 that the structures of the mixed-strategy NE distribution and the minimax regret distribution are completely different.

To go into these differences, we restrict attention, in this section, to the game in continuous time with $t^* < T$.

First, in the mixed-strategy NE, T is a mass point, whereas T is not even in the support of the minimax regret distribution. Second, the probabilities decrease from 0 to T^- in the mixed-strategy NE distribution, at rate Dr/M , whereas they increase from 0^+ to t^* in the minimax regret distribution, at rate $r(M-D)/M$.

This strong divergence in behavior derives from the different philosophies underlying both concepts.

The mixed-strategy NE distribution follows from the differential equation $D(1 - F(t)) + \frac{M}{r}f(t) = 0$. This equation says that switching from t to $t+dt$, so leaving at t or at $t+dt$, does

not change firm i 's expected payoff, when firm j ($\neq i$) behaves according to the density and cumulative distributions $f(\cdot)$ and $F(\cdot)$. As a matter of fact, staying in the market up to time $t+dt$ instead of up to time t lowers firm i 's payoff by D each time firm j leaves after t (hence firm i suffers from the additional loss Ddt with probability $1-F(t)$ because it stays in “ dt ” time longer), but it increases firm i 's payoff with the endless monopoly payoff M/r each time firm j leaves at time t (i.e. with probability $f(t)dt$). Given that firm i earns a null payoff when exiting the market at time 0, this equation also ensures that each firm gets a null payoff with each strategy in the support of the mixed-strategy NE. Hence firm i gets a null expected payoff at equilibrium.

Observe that the functions $f(\cdot)$ and $F(\cdot)$, that allow us to comment on firm i 's indifference between leaving at t and leaving at $t+dt$, are firm j 's probability distributions. So for example, in the numerical example, firm 1 leaves the market in period 6 with the large probability 0.3349 not because this strategy is in firm 1's interests, but because it ensures that firm 2 is indifferent to all the pure strategies: as a matter of fact, leaving in period 6 is a very efficient strategy for firm 2, except if firm 1 also chooses this strategy with a strong probability. So, rather strangely, firm 1 stays in the market up to period 6 with a large probability not in order to win a nice payoff, but only to avoid this strategy becoming the (only) best reply for the opponent.

It is rather strange to assume that a player's sole objective is to make the opponent indifferent among his strategies. In other terms, the NE distribution is logical (because it is in accordance with a best-reply fixed-point logic) but it does not convince from a behavioral viewpoint.

According to the minimax-regret philosophy, a player does not try to anticipate the strategies played by the others but he builds a probability distribution on his own pure strategies so as to never suffer from too large a regret, regardless of the strategies chosen by the others. The differential equation that leads to the minimax regret distribution is $\frac{M}{r}f(t) + F(t)(D - M) = 0$. This equation, quite different from the one obtained for the NE, says that firm i , by behaving according to the distribution $f(\cdot)$, has the same regret when firm j switches from leaving at time t to leaving at time $t+dt$, with $t < t^*$. The changes induced by this switch are the following. Nothing changes if firm i stays in for longer than $t+dt$ (it has no regret). When exiting at time d , with d from 0 to t , it does not suffer from the duopoly loss at time $t+dt$ and it does not regret the monopoly profit at time t , hence its regret decreases by the amount $F(t)(M-D)dt$. And when it leaves at time $t+dt$, it has no regret when firm j exists at time t , but regrets

the endless monopoly profit (M/r) when firm j leaves at time $t+dt$. Hence it has the additional regret $(M/r)f(t)dt$. Given that its total regret does not change, we get $\frac{M}{r}f(t) + F(t)(D - M) = 0$. Observe that $f(\cdot)$ and $F(\cdot)$ are the density and cumulative distributions of firm i . A firm, by playing according to these distributions, is sure to always have the regret $g(0)M/r$, regardless of the other firm's exit time, and therefore gets the best-reply payoff to this exit time minus the regret $g(0)M/r$.

The minimax-regret philosophy shares with the maximin-payoff philosophy the fact that each player builds a probability function for himself, in order to protect himself from the others' behavior. But the link between both philosophies stops here. With the maximin payoff philosophy, a player builds a probability distribution on his actions that ensures him a payoff that he tries to maximize, regardless of what is played by the others. So the maximin payoff philosophy is very pessimistic in that a player, when choosing a strategy, only focuses on the worst thing that may happen with this strategy. This explains why each time a firm chooses to stay in the market up to the time t , it fears that the opponent will stay in longer, and thus that it will lose money. Therefore the best thing a firm can do is to leave the market immediately, which is the only way to be sure to not get a negative payoff. This explains that the maximin payoff strategy leads to a null payoff (which is also the mixed-strategy NE expected payoff). The minimax regret philosophy is less pessimistic: a player, when choosing a strategy, focuses on the fact that it may not be the best response to the opponent's behavior. Given the opponent's unknown way of playing, another strategy than the chosen one may lead to a larger payoff. So the philosophy is "I could have done better" which is much less pessimistic than the maximin payoff philosophy.

This also explains that the minimax regret philosophy is much less "one side" focalized than the maximin payoff philosophy. Like the maximin payoff philosophy, the minimax regret philosophy takes into account that the opponent may exit the market late, but it also takes into account that the opponent may exit the market early, which generates a different regret. The more nuanced strategy that ensues is especially fruitful, in that it leads to a strictly positive expected payoff when both firms play this strategy (see proposition 6 below).

Despite this main difference, there is a technical link worth mentioning between the minimax regret strategy and the maximin payoff strategy. If we transform the regret matrix into a two-player zero-sum game, where firm 1's payoffs are the opposite of the regrets, and player 2 is an artificial player who obtains the regrets, and if we look for firm 1's maximin payoff

strategy in this new game, then we get its minimax regret strategy in the original game. This follows from the fact that it is equivalent to maximize function $h(\cdot)$ and to minimize function $-h(\cdot)$, under the same set of constraints. Moreover, given that, in two-player zero-sum games, the maximin payoff behavior is a mixed-strategy NE, we get a link between the minimax regret behavior and the mixed-strategy NE (in another game). This link has been observed by Renou and Schlag (2010).

5. Minimax regret, a positive payoff for both firms

Proposition 6

If the opponent plays the minimax regret strategy, then, for each firm, the minimax regret strategy leads to a strictly positive expected payoff, $E(g)$. $E(g)$ is the sum of the best-reply payoffs to each exit time t , with t from $0+$ to $\min(t^, T)$ in the continuous setting, from 1 to $\min(t^*, T)$ in the discrete setting, weighted by the probabilities of exiting at time t .*

In the continuous model, for $t^ \leq T$, $E(g)$ is equal to $\frac{M}{r} \left(1 - \frac{g(0)(M-2D)}{-D} \right)$.*

In the continuous model, for $t^ > T$, $E(g)$ is equal to*

$$\frac{g(0)(M-2D)}{D} \cdot \frac{M}{r} + \frac{g(0)}{Dr} \left(D^2 e^{\frac{(M-D)rT}{M}} - (M-D)^2 e^{-\frac{DrT}{M}} \right).$$

Proof: see Appendix G

The expected payoff is positive by construction. A firm, for each opponent's exit time, gets the best-reply payoff minus the regret, equal to $p_0 M(1+r)/r$ in the discrete version, to $g(0)M/r$ in the continuous version. So the expected sum of regrets is this regret. Yet the best-reply payoff, when the opponent exits the market at time 0, weighted by the probability of doing so, is $p_0 M(1+r)/r$ in the discrete game, $g(0)M/r$ in the continuous game. It follows that $E(g)$ is the sum of the best-reply payoffs to the opponent's exit times t , with t from 1 to $\min(t^*, T)$ in the discrete-time game, from $0+$ to $\min(t^*, T)$ in the continuous-time game, weighted by the probability of exiting the market at time t . This sum is strictly positive by definition.

This expected payoff is equal to 1 in our numerical example, but it can become very large. For example, when $-D=M$, the expected payoff becomes $M/(4r)$ and is linearly increasing in M .

In the game in continuous time, for $t^* < T$ (or T infinite), we make the following comments.

We set $x = -D/M$, $g(0) = \left(-\frac{D}{M-D} \right)^{\frac{M-D}{M}} = \left(\frac{x}{1+x} \right)^{1+x}$, $t^* = \frac{\ln\left(\frac{M-D}{-D}\right)}{r} = \frac{\ln\left(\frac{1+x}{x}\right)}{r}$ and

$$E(g) = \frac{M}{r} \left(1 - \frac{g(0)(M-2D)}{-D} \right) = \frac{M}{r} \left(1 - \left(\frac{x}{1+x} \right)^x - \left(\frac{x}{1+x} \right)^{1+x} \right)$$

$g(0)$ is depicted in figure 4³. For a fixed value M , $g(0) \rightarrow 0$ when $D \rightarrow 0$, given that a firm never loses money by staying in the market if $D=0$. The more D becomes negative, hence the more $-D/M$ is large, the more we expect a firm to exit the market early, so $g(0)$ increases as expected. Yet $g(0)$ has an upper bound. We show in Appendix G that even if D is very negative for a fixed value M , i.e. if $-D/M \rightarrow +\infty$, $g(0)$ does not go to 1. It only tends towards e^{-1} , i.e. 0.368, a rather surprising result. This result derives from the fact that a firm, by leaving at time 0, still regrets the large amount M/r it could obtain by staying in the market when the opponent leaves the market at time 0. That is why a probability of 1 of immediately leaving the market is not acceptable from a regret viewpoint. It follows from this fact, in the continuous setting, that even in the worst scenario, the minimax regret strategy does not converge to the maximin payoff strategy (which consists in exiting the market at time 0 with probability 1).

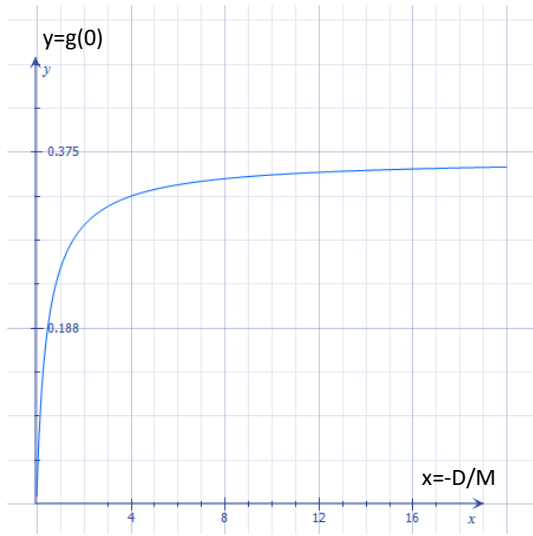


Figure 4: $g(0)$

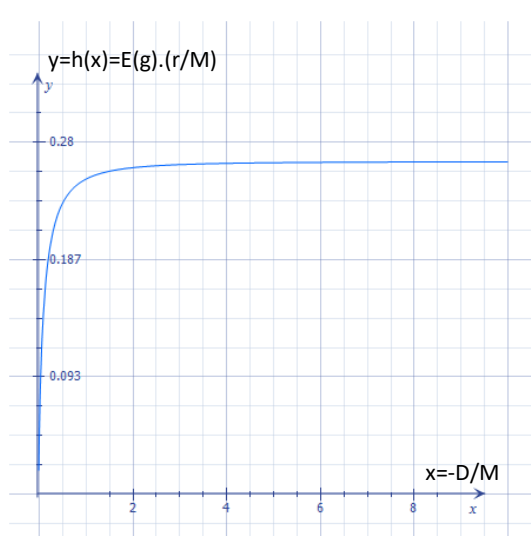


Figure 5: $E(g)r/M$

Function $h(x)=E(g)r/M$ is represented in figure 5⁴. It follows that:

- For a fixed value $x = -D/M$, $E(g)$ increases in M , which shows that the minimax regret criterion, contrary to the NE concept, allows both firms to get a nice expected payoff.

³ Figure 4 makes sense only as long as $t^* < T$. So, when T is bounded, the part of the curve corresponding to $\frac{\ln\left(\frac{1+x}{x}\right)}{r} > T$ should not be taken in account, in that it is inappropriate.

⁴ Figure 5 makes sense only as long as $t^* < T$. So, when T is bounded, the part of the curve corresponding to $\frac{\ln\left(\frac{1+x}{x}\right)}{r} > T$ should not be taken in account, in that it is inappropriate.

- $E(g)$ decreases in r , which expresses that the firms benefit better from the infinite monopoly payoff when the actualization rate is low. When r tends towards 0, the expected payoff tends towards $+\infty$.
- For fixed values M and r , $E(g)$ increases in $x = -D/M$. It goes to 0 when x goes to 0 because, on the one hand, no firm leaves the market if $D=0$ (staying in the market generates no regret), but, on the other hand, the firms cannot make a positive profit because they earn the null duopoly profit forever.
- $E(g)$ tends towards $(1-2e^{-1})M/r = 0.264M/r$ when $-D/M$ tends towards $+\infty$: even with very bad duopoly payoffs, the minimax regret expected payoff is positive. Moreover, for a fixed value of M , $E(g)$ grows in $-D/M$, so becomes larger when D becomes more negative. This rather surprising result stems from two facts. When D is very low ($-D$ is large), a firm is induced to exit the market faster ($g(0)$ grows). So it does not earn the bad duopoly payoffs for very long. Moreover, the opponent also leaves the market very fast, so there is a large probability that a firm will be a monopolist sooner and earn the monopoly payoff for a longer time. Putting things together induces a larger expected payoff. Observe that $E(g)$ rapidly converges to $0.264M/r$, given that $E(g) > 0.25M/r$ for $x > 1$.

We can here observe a discontinuity between the discrete-time and the continuous-time approaches. In the discrete-time setting, things become different when x becomes large. The payoff function, for a fixed M and r , is a step-wise defined function. It is continuous in x , and starts by growing in x but decreases in x for large values of x . This is due to the fact that, in the discrete-time game, when a firm has to leave fast because x is large, it is impossible to leave at time $0+\varepsilon$, with ε going to 0, in that a firm can only leave at time 0 or in period 1. It derives from this fact that $g(0)$ will converge to 1 and that the payoff goes to 0 when x goes to $+\infty$.

Let us make an additional comment on payoffs. The aim, in this section, is not to claim that minimax regret is the best way to behave in order to get a positive payoff. As a matter of fact, minimax regret is not the best response to a player who plays the minimax regret strategy. In the numerical example, it is obvious that a firm exiting in period 6, by exploiting the fact that nobody goes out after $t^*=4$, gets a much larger payoff (2.153) than a firm playing the minimax regret strategy (0.7645).

And if we consider a random player playing like the students, the best strategy consists in exiting in period 6 in that it leads to the payoff 0.069, whereas the minimax regret leads to the

negative payoff -0.41. Even the mixed-strategy Nash equilibrium does better, leading to the less negative payoff -0.1828. So of course the mixed minimax regret behavior may perform poorly, if confronted with a given random strategy, just as it may perform well if confronted with another random strategy.

But this is not what we want to highlight. The point is that, if both firms' strategy consists in minimizing their regrets, rather than trying to best reply to an unknown strategy, then they always get positive expected payoffs, that may even be quite large when M is large (for a fixed ratio $-D/M$). And this result is not an exception. Halpern and Pass (2012) popularized the minimax regret concept by showing its favorable consequences in the traveler's dilemma. Linhart and Radner (1989) put into light that minimax regret performs well in sealed-bid bargaining situations, and similar results have been obtained for the Bertrand duopoly.

6. Conclusion, limit exit-time and mixed strategies

We have shown in this paper how minimax regret leads to a different behavior from the mixed-strategy NE in the duopoly-exit game. We have also established that this new behavior, if adopted by both firms, ensures both firms potentially obtain large payoffs whereas the mixed-strategy NE leads to a null expected payoff.

We open the discussion on the limit time T and on the notion of mixed strategy.

Is it interesting, from an economic viewpoint, to include a limit time T ? Very often, in many countries, it is forbidden to stay in a market when losing money. As regards the mixed-strategy NE concept, the question does not matter, in that both firms have a null expected payoff regardless of T (and whether T be finite or infinite). This is not the case with the minimax regret criterion.

We observe, on the one hand, that $T > t^*$ has no impact on the firms' behavior in that they exit the market latest at $t^* < T$. On the other hand, for $T < t^*$, the more T is low, the more the expected payoff $E(g)$ is low. As a matter of fact, for $T < t^*$, $E(g)$ is increasing in T , and equal to the expected payoff obtained for $T > t^*$ only when $T = t^*$. For example, for $T = 2$ and $-D = M = 1$ and $r = 0.25$, we get $E(g) = 0.837$ whereas the expected payoff for $T > t^*$ is equal to 1.

We illustrate this fact in figure 6, for $-D = M = 1$, $r = 0.25$ (hence $t^* = 2.77$ and $E(g) = 1$ for $T > t^*$).

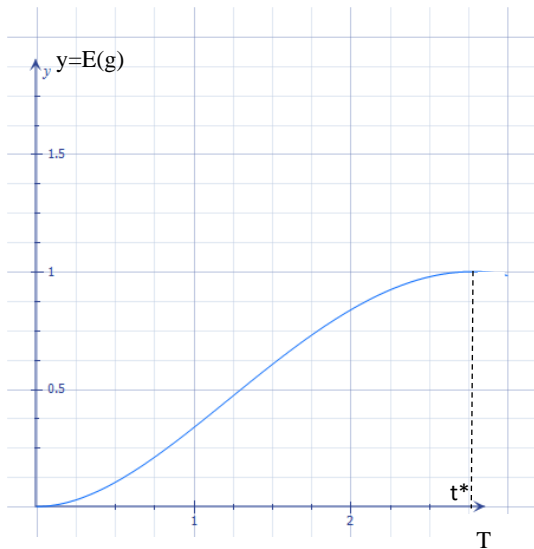


Figure 6: $E(g)$ for $T < t^*$, $-D=M=1$, $r=0.25$

It follows from this observation that the minimax regret criterion suggests to not introduce a limit exit time T (or to introduce a large enough one, so that $T > t^*$).

A last remark is about mixed strategies. In a mixed-strategy NE, the probabilities of a player only make sense for the other players in that they stabilize their behavior; by contrast, the minimax regret probabilities of a player make sense for himself, in that they help the player to balance his potential regrets. In this sense the minimax regret criterion belongs to the behavioral approaches of mixed strategies, in that it gives a behavioral meaning to the mixed strategies. Other approaches, for example Best Reply Matching (see Kosfeld et al. (2002)) also aim to give more behavioral meaning to mixed strategies. According to the Best Reply Matching philosophy, a strategy is played as often as it is a best reply: so, if player 1's action A is a best reply to player 2's action B and player 2 plays B with probability p , then player 1 plays A with the same probability p . Applying this criterion to the market-exit game would also induce increasing probabilities, from p_1 to p_{t^*} , but it would also put positive probabilities on p_t , for t from t^*+1 to T . In other terms, coping with mixed strategies in a different way from the Nash equilibrium allows us to give new insights on old but important economic topics.

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Appendix A

We look for a symmetric mixed-strategy NE with a mass point on T .

Suppose that firm 2's strategy is given by the density function $f(t)$ and the cumulative probability function $F(t)$ defined on $[0, T]$.

By leaving in period d , firm 1 gets:

$$\int_d^{T^-} (\int_0^d De^{-rs} ds) f(t) dt + (\int_0^d De^{-rs} ds) g(T) + \int_0^d (\int_0^t De^{-rs} ds + \int_t^\infty Me^{-rs} ds) f(t) dt$$

This payoff is constant over $[0, T]$, so the derivative in d has to be equal to 0, from 0 to T^- .

$$\text{It follows: } -(\int_0^d De^{-rs} ds) f(d) + \int_d^{T^-} De^{-rd} f(t) dt + De^{-rd} g(T) + (\int_0^d De^{-rs} ds + \int_d^\infty Me^{-rs} ds) f(d) = 0$$

$$\text{Developing this equation leads to } D - DF(d) + \frac{M}{r} f(d) = 0.$$

$$\text{Solving this differential equation leads to } F(t) = 1 - e^{\frac{Dr}{M}t} \text{ with } f(t) = -\frac{Dr}{M} e^{\frac{Dr}{M}t} \text{ and } g(T) = 1 - F(T^-) = e^{\frac{Dr}{M}T}.$$

The mean payoff obtained by leaving in period d is the payoff obtained by leaving at time 0, i.e. 0. We check that the same payoff is obtained by leaving in period T :

$$\begin{aligned}
& \left(\int_0^T D e^{-rs} ds \right) g(T) + \int_0^{T^-} \left(\int_0^t D e^{-rs} ds + \int_t^\infty M e^{-rs} ds \right) f(t) dt \\
&= \frac{D - D e^{-rT}}{r} e^{\frac{Dr}{M}T} + \int_0^{T^-} \left(\frac{D - D e^{-rt}}{r} + \frac{M}{r} e^{-rt} \right) \left(-\frac{Dr}{M} \right) e^{\frac{Dr}{M}t} dt \\
&= \frac{D - D e^{-rT}}{r} e^{\frac{Dr}{M}T} + \int_0^{T^-} \left(\frac{(M-D)}{r} \right) \left(-\frac{Dr}{M} \right) e^{\frac{(D-M)r}{M}t} dt + \int_0^{T^-} \left(-\frac{D^2}{M} \right) e^{\frac{Dr}{M}t} dt \\
&= \frac{D e^{\frac{Dr}{M}T} - D e^{\frac{(D-M)r}{M}T}}{r} + \frac{D}{r} \left(e^{\frac{(D-M)r}{M}T} - 1 \right) - \frac{D}{r} \left(e^{\frac{Dr}{M}T} - 1 \right) = 0
\end{aligned}$$

$f(t)$ is decreasing in t given that $f'(t) = -\left(\frac{Dr}{M}\right)^2 e^{\frac{Dr}{M}t} < 0$.

Appendix B

The only symmetric NE is a full-support mixed NE. First, it is not possible to only exit the market in (some) periods up to k (with $k < T$) because, if so, leaving in period $k+1$ leads to a higher payoff than leaving in period k . As a matter of fact, it yields the same payoff as leaving in period k when confronted with an opponent leaving before period k , and it yields a larger payoff with an opponent leaving in period k (by ensuring the monopoly payoff in all periods later than k). Second, it is not possible to have a hole in the support, for example between the periods k and r , with $r > k+1$ (which means that the firm does not leave the market in the periods from $k+1$ to $r-1$). As a matter of fact, exiting in period $k+1$ is a better reply than exiting in period r because it provides the same payoff when confronted with an opponent leaving from time 0 to period k , and a larger payoff with an opponent leaving in period r or later. Finally, only exiting the market in periods from k (> 0) to M cannot be an equilibrium, because leaving in period k always provides a negative payoff, lower than the null payoff obtained when leaving at time 0. So we have a full-support NE.

The discrete-time NE is similar to the continuous-time one. We construct it step by step. Exiting the market at time 0 and in period 1 gives rise to the same payoff if and only if:

$$0 = \frac{M(1+r)}{r} p_0 + D(1 - p_0). \text{ So } p_0 = -\frac{rD}{M+r(M-D)}$$

Leaving in period T and leaving in period $T-1$ give rise to the same payoff, except if the opponent also leaves in period T or $T-1$, so we need :

$$p_{T-1} \sum_{t=0}^{T-2} \frac{D}{(1+r)^t} + p_T \sum_{t=0}^{T-2} \frac{D}{(1+r)^t} = p_{T-1} \left(\sum_{t=0}^{T-2} \frac{D}{(1+r)^t} + \frac{\frac{M}{(1+r)^{T-1}(1+r)}}{r} \right) + p_T \sum_{t=0}^{T-1} \frac{D}{(1+r)^t}$$

$$\text{Hence } p_T = \frac{M(1+r)}{-rD} p_{T-1} \text{ and } p_{T-1} + p_T = \frac{M+r(M-D)}{-rD} p_{T-1}$$

Leaving in period $T-1$ and leaving in period $T-2$ give rise to the same payoff, except if the opponent leaves in period T , $T-1$, or $T-2$, so we need:

$$p_{T-2} \sum_{t=0}^{T-3} \frac{D}{(1+r)^t} + p_{T-1} \sum_{t=0}^{T-3} \frac{D}{(1+r)^t} + p_T \sum_{t=0}^{T-3} \frac{D}{(1+r)^t} = p_{T-2} \left(\sum_{t=0}^{T-3} \frac{D}{(1+r)^t} + \frac{\frac{M}{(1+r)^{T-2}(1+r)}}{r} \right) +$$

$$p_{T-1} \sum_{t=0}^{T-2} \frac{D}{(1+r)^t} + p_T \sum_{t=0}^{T-2} \frac{D}{(1+r)^t}$$

$$\text{Hence } p_{T-2} \left(\frac{\frac{M}{(1+r)^{T-2}(1+r)}}{r} \right) + p_{T-1} \left(\frac{D}{(1+r)^{T-2}} \right) + p_T \left(\frac{D}{(1+r)^{T-2}} \right) = 0$$

$$\text{So we get } p_{T-1} = \frac{M(1+r)}{M+r(M-D)} p_{T-2} \text{ and } p_{T-2} + p_{T-1} + p_T = \frac{M+r(M-D)}{-rD} p_{T-2}$$

More generally, if we assume $\sum_{i=t}^T p_i = \frac{M+r(M-D)}{-rD} p_t$, then leaving in period $t-1$ leads to the same payoff as leaving in period t if and only if:

$$\sum_{i=t-1}^T (p_i \sum_{j=0}^{t-2} \frac{D}{(1+r)^j}) = p_{t-1} (\sum_{j=0}^{t-2} \frac{D}{(1+r)^j} + \frac{\frac{M}{(1+r)^{t-1}(1+r)}}{r}) + \sum_{i=t}^T (p_i \sum_{j=0}^{t-1} \frac{D}{(1+r)^j})$$

Hence $p_{t-1} \left(\frac{\frac{M}{(1+r)^{t-1}(1+r)}}{r} \right) + \left(\frac{D}{(1+r)^{t-1}} \right) \sum_{i=t}^T p_i = 0$. It follows from the above assumption:

$$p_{t-1} \left(\frac{M(1+r)}{r} \right) + D \left(\frac{M+r(M-D)}{-rD} \right) p_t = 0$$

So we get $p_t = \frac{M(1+r)}{M+r(M-D)} p_{t-1}$ and $\sum_{i=t-1}^T p_i = \frac{M+r(M-D)}{-rD} p_{t-1}$

The same relation holds until $t = 2$, hence $p_2 = \frac{M(1+r)}{M+r(M-D)} p_1$ and $p_t = \left(\frac{M(1+r)}{M+r(M-D)} \right)^{t-1} p_1$ for t from 2 to $T-1$.

Exiting in period 1 and exiting in period 2 lead to the same payoff if and only if:

$$D(1 - p_0) = p_1 \left(D + \frac{\frac{M}{(1+r)(1+r)}}{r} \right) + (1 - p_0 - p_1) \left(D + \frac{D}{1+r} \right)$$

Given that $p_0 = -\frac{rD}{M+r(M-D)}$, we get $p_1 = -\frac{rDM(1+r)}{(M+r(M-D))^2} = \frac{M(1+r)}{M+r(M-D)} p_0$ and $p_t = \left(\frac{M(1+r)}{M+r(M-D)} \right)^t p_0$ for t from 1 to $T-1$.

Hence $p_t = -\left(\frac{M(1+r)}{M+r(M-D)} \right)^t \frac{rD}{M+r(M-D)}$ for t from 1 to $T-1$ and $p_T = \frac{M(1+r)}{-rD} p_{T-1} = \left(\frac{M(1+r)}{M+r(M-D)} \right)^T$.

We get a geometric sequence from p_0 to p_{T-1} and a different probability for p_T . It can be checked that the sum of all the probabilities is equal to 1.

When there is no limit period T , that is to say when $T \rightarrow +\infty$, we simply get $p_t = -\left(\frac{M(1+r)}{M+r(M-D)} \right)^t \frac{rD}{(M+r(M-D))}$ for t from 0 to $+\infty$.

The expected payoff is 0.

Appendix C

To establish the maximin payoff y , we solve the optimization problem:

$$\begin{aligned} & \max_{p_0 \dots p_T} y \\ & \text{u.c.} \\ & \sum_{i=0}^T p_i P(i, j) \geq y \quad j \text{ from } 0 \text{ to } T \\ & \sum_{i=0}^T p_i = 1 \\ & p_i \geq 0 \quad i \text{ from } 0 \text{ to } T \end{aligned}$$

where p_i is the probability firm 1 assigns to the strategy “leaving in period i ” and $P(i, j)$ is firm 1’s payoff when it leaves in period i and the opponent leaves in period j . At least one inequation constraint is checked with equality because $\sum_{i=0}^T p_i = 1$ and the payoffs are finite.

If $i > j' > j$, then $P(i, j) > P(i, j')$ because firm 1 gets the monopoly payoff faster and the duopoly payoff less often when the opponent exits in period j than when he leaves in period j' .

If $j' \geq i > j$, then $P(i, j) \geq P(i, j')$, because firm 1 never gets the monopoly payoff when meeting an opponent exiting in period j' , and it also gets the duopoly payoff for longer than when it meets an opponent leaving in period j .

If $j' > j \geq i$, then $P(i, j) = P(i, j')$ because firm 1 gets the duopoly payoff during i periods when its opponent exits the market later than itself.

So, when the last constraint $\sum_0^T p_i P(i, T) \geq y$ is satisfied, all the other constraints are satisfied too. So this last constraint is checked with equality. $P(i, T)$ is lower than $P(0, T)$ for any i from 1 to T because it is better to leave the market immediately when the opponent leaves the market in period T . So $\sum_0^T p_i P(i, T)$, hence y , is maximal for $p_0 = 1$ and $p_i = 0$ for i from 1 to T . It follows that leaving the market at time 0 and getting a null payoff is the maximum of this program.

Appendix D

We first assume $t^* \leq T$ and consider the game in discrete time.

For any strategy that consists in leaving in period t , with $t \leq t^*-1$, the maximal regret is observed when the other firm exits in period t . If so, the firm gets t times the negative duopoly payoff instead of t times the negative duopoly payoff plus the positive monopoly payoff forever, from period $t+1$ onwards, by leaving one period later. Hence the maximal regret is the monopoly payoff obtained from period $t+1$ onwards. As a matter of fact, if the opponent leaves earlier than the firm has no regret, and if he leaves in period d , with d higher than t but lower than t^* , the best decision consists in exiting in period $d+1$, which leads to more duopoly losses and less monopoly profits, hence to a lower regret. Finally, if the opponent leaves in period d , with $d \geq t^*$, the best reply is to leave immediately and the regret is the opposite of the duopoly losses $-\sum_{i=0}^{t-1} D/(1+r)^i$, which is lower than the monopoly payoff obtained from period $t+1$ onwards. Hence the maximal regret is the monopoly payoff obtained from period $t+1$ onwards and the lowest regret of this kind is obtained for $t=t^*-1$.

For any strategy that consists in leaving in period t , with t larger than or equal to t^* , the regret is equal to t times the opposite of the duopoly payoff if the opponent leaves in period t or later. If the opponent leaves in period d , with d lower than t but larger than t^*-1 , the firm gets the duopoly payoff less often and gets the monopoly payoff from period $d+1$ onwards, so its regret is lower. If the opponent leaves before period t^* , the firm has no regret. Hence the maximal regret, when leaving in period t later than t^*-1 , is t times the opposite of the duopoly payoff and the lowest regret of this type is obtained for $t=t^*$.

So the pure minimax regret strategy consists in leaving in period t^*-1 or in period t^* , depending on whether $-\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i}$ is larger than $\frac{M}{(1+r)^{t^*-1}} \frac{(1+r)}{r}$ or not.

The previous reasoning easily adapts to the continuous setting. In this setting, we have $-\int_0^{t^*} D e^{-rt} dt = \int_{t^*}^{+\infty} M e^{-rt} dt$ and the pure minimax regret strategy simply consists in leaving at time t^* .

We now assume $t^* > T$.

In the discrete-time game, as above, leaving in period t , with $t \leq T-1$, leads to the maximal regret when the opponent leaves in period t . This regret is the monopoly payoff obtained from period $t+1$ onwards (and the lowest regret of this kind is obtained for $t=T-1$). The maximal regret, when leaving in period T , is obtained when the opponent also leaves in period T , and it is equal to $-\sum_{i=0}^{T-1} \frac{D}{(1+r)^i}$. Given that $-\sum_{i=0}^{T-1} \frac{D}{(1+r)^i} < \frac{M}{(1+r)^{T-1}} \frac{(1+r)}{r}$ because $T < t^*$, the pure minimax regret strategy consists in leaving in period T .

The same result holds for the game in continuous time, given that $-\int_0^T D e^{-rt} dt < \int_T^{+\infty} M e^{-rt} dt$ for $T < t^*$.

Appendix E

Study with $t^* \leq T$.

We call column t the column of firm 1's regrets when firm 2 leaves in period t . The regrets in each column have a regular pattern.

Consider a period \tilde{t} with $\tilde{t} \leq t^* - 1$, and suppose that the opponent (firm 2) exits the market in period \tilde{t} . By leaving in period $t < \tilde{t}$, firm 1 gets the negative payoff $\sum_{i=0}^{t-1} \frac{D}{(1+r)^i}$ and the best payoff it could obtain is $\sum_{i=0}^{\tilde{t}-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{\tilde{t}}} \cdot \frac{1+r}{r}$. So its regret is $\sum_{i=t}^{\tilde{t}-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{\tilde{t}}} \cdot \frac{1+r}{r}$. Given that D is negative, this regret is increasing in t ($< \tilde{t}$). So the regret is maximum for $t = \tilde{t}$, where it is equal to $\frac{M}{(1+r)^{\tilde{t}}} \cdot \frac{1+r}{r}$.

By comparing two adjacent columns, where firm 2 leaves in period $\tilde{t} - 1$ or \tilde{t} , firm 1's regret when leaving in period t , with $t < \tilde{t} - 1$, is $\sum_{i=t}^{\tilde{t}-2} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{\tilde{t}-1}} \cdot \frac{1+r}{r}$ when firm 2 exits in period $\tilde{t}-1$, and $\sum_{i=t}^{\tilde{t}-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{\tilde{t}}} \cdot \frac{1+r}{r}$ when firm 2 exits in period \tilde{t} . So its regret is larger when firm 2 leaves in period $\tilde{t} - 1$ (firm 1 has greater regret upon leaving the market earlier than the opponent when the opponent leaves the market faster). A similar observation holds for $t = \tilde{t} - 1$.

In contrast, when firm 2 leaves in period \tilde{t} , with $\tilde{t} \geq t^*$ the best thing firm 1 can do is to exit the market immediately, so, when it exits in period $t \leq \tilde{t}$, its regret is $-\sum_{i=0}^{t-1} \frac{D}{(1+r)^i}$. This regret is growing in t , and constant in \tilde{t} with $\tilde{t} \geq t^*$. When firm 1 exits after period \tilde{t} , its regret is $-\sum_{i=0}^{\tilde{t}-1} \frac{D}{(1+r)^i} - \frac{M}{(1+r)^{\tilde{t}}} \cdot \frac{1+r}{r}$. This regret is growing in \tilde{t} given that firm 1 earns the negative duopoly profit longer and the monopoly profit later when \tilde{t} is larger.

Finally, the regrets in column t^*-1 are larger than the regrets in column t^* , when firm 1 leaves in period t , with $t \leq t^*-1$. If so, when the opponent leaves in period t^*-1 , firm 1's regret is $\sum_{i=t}^{t^*-2} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{t^*-1}} \cdot \frac{1+r}{r}$. We have $\sum_{i=0}^{t^*-2} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{t^*-1}} \cdot \frac{1+r}{r} > 0$ so $\sum_{i=t}^{t^*-2} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{t^*-1}} \cdot \frac{1+r}{r} > -\sum_{i=0}^{t-1} \frac{D}{(1+r)^i}$. So this regret is larger than the regret with an opponent leaving in period t^* . In contrast, firm 1 has no regret when it leaves in period t^* and firm 2 leaves in period t^*-1 , whereas its regret is $-\sum_{i=0}^{t^*-1} \frac{D}{(1+r)^i}$ when firm 2 leaves in period t^* too.

We write p_t to denote the probability firm 1 assigns to the strategy "leaving in period t ", $\text{Regret}(\tilde{t})$ for firm 1's expected regret when its opponent leaves in period \tilde{t} , and X_{it} for firm 1's regret when it exits in period i and firm 2 exits in period t . The properties of the regrets ensure that the program

$$\begin{aligned} & \min_{p_0 \dots p_T} y \\ & \text{u.c. } \text{Regret}(t) \leq y \quad \text{for } t \text{ from } 0 \text{ to } T \\ & \sum_{t=0}^T p_t = 1 \\ & p_t \geq 0 \quad \text{for } t \text{ from } 0 \text{ to } T \end{aligned}$$

has a unique solution that checks $p_t > 0$ for t from 0 to t^* and $p_t = 0$ for $t > t^*$. This property ensures that $\text{Regret}(t) = \text{Regret}(t^*)$ for $t > t^*$.

To prove this result, we assume that, at the optimum, $\text{Regret}(t) = y$ for t from 0 to t^* . The positivity of the optimal p_t , t from 0 to t^* , results from the fact that the regret, when leaving in

period t (before the opponent), is lower when the opponent leaves the market later. As a matter of fact, for two columns \tilde{t} and $\tilde{t} + 1$, with $\tilde{t} + 1 < t^*$, we get: $\text{Regret}(\tilde{t}) = p_0 X_{0\tilde{t}} + \dots + p_{\tilde{t}} X_{\tilde{t}\tilde{t}}$ and $\text{Regret}(\tilde{t} + 1) = p_0 X_{0\tilde{t}+1} + \dots + p_{\tilde{t}} X_{\tilde{t}\tilde{t}+1} + p_{\tilde{t}+1} X_{\tilde{t}+1, \tilde{t}+1}$. Suppose that $p_i > 0$ for i from 0 to \tilde{t} . We get:

$p_0 X_{0\tilde{t}} + \dots + p_{\tilde{t}} X_{\tilde{t}\tilde{t}} = y = p_0 X_{0\tilde{t}+1} + \dots + p_{\tilde{t}} X_{\tilde{t}\tilde{t}+1} + p_{\tilde{t}+1} X_{\tilde{t}+1, \tilde{t}+1} \Rightarrow p_{\tilde{t}+1} > 0$ because $X_{i\tilde{t}} > X_{i\tilde{t}+1}$ for any i from 0 to \tilde{t} . And $p_{t^*} > 0$ given the structure of the regrets in the columns t^*-1 and t^* . So when p_0 is positive, p_t is positive too, for any t from 1 to t^* .

The way the probabilities are determined ensures unicity. The first column leads to $p_0 X_{00} = y$, so $p_0 = \frac{y}{X_{00}}$, then the second column leads to $p_0 X_{01} + p_1 X_{11} = y$ so $p_1 = \frac{y - p_0 X_{01}}{X_{11}}$ and so on, up to p_{t^*} .

The Karush Kuhn Tucker (KKT) function is $y + \sum_{t=0}^T \lambda_t (\text{Regret}(t) - y) - \sum_{t=0}^T \mu_t p_t + \lambda (\sum_{t=0}^T p_t - 1)$. The multipliers λ_i for i from 0 to T have to be positive or null, and the multipliers μ_t have to be null for t from 0 to t^* (given the strict positivity of p_t) and positive or null for t from t^*+1 to T . These conditions are satisfied thanks to the structure of the regrets.

As a matter of fact, the derivative of the KKT function in p_0 leads to $\sum_{i=0}^{t^*-1} \lambda_i X_{0i} + \lambda = 0$. More generally, the derivative in p_t leads to $\sum_{i=t}^T \lambda_i X_{ti} + \lambda = 0$ for any t from 1 to t^* and $\sum_{i=t^*}^T \lambda_i X_{ti} + \lambda - \mu_t = 0$ for any t from t^*+1 to T .

We first look at the equations for t and k larger than or equal to t^* . First, X_{tk} does not depend on t for $t > k$, and $X_{tk} < X_{tk+1}$ for $t > k+1$. So the two last equations $\sum_{i=t^*}^T \lambda_i X_{T-1i} + \lambda - \mu_{T-1} = 0$ and $\sum_{i=t^*}^T \lambda_i X_{Ti} + \lambda - \mu_T = 0$ lead to $\lambda_{T-1} X_{T-1T-1} + \lambda_T X_{T-1T} - \mu_{T-1} = \lambda_{T-1} X_{TT-1} + \lambda_T X_{TT} - \mu_T$, given $X_{T-1t} = X_{Tt}$ for t from t^* to $T-2$.

We set $\mu_T = \mu_{T-1} = 0$ and we suppose $\lambda_T > 0$. Given $X_{T-1T-1} > X_{TT-1}$ and $X_{T-1T} < X_{TT}$, it follows $\lambda_{T-1} > 0$. We get $\lambda_{T-1} = \alpha_{T-1} \lambda_T$ with $\alpha_{T-1} > 0$.

We now compare the adjacent equations for $T-1$ and $T-2$. We have $X_{T-2t} = X_{T-1t}$ for any t from t^* to $T-3$ and we set $\mu_{T-2} = 0$. So we get: $\lambda_{T-2} X_{T-2T-2} + \lambda_{T-1} X_{T-2T-1} + \lambda_T X_{T-2T} = \lambda_{T-2} X_{T-1T-2} + \lambda_{T-1} X_{T-1T-1} + \lambda_T X_{T-1T}$.

$$X_{T-1T-1} = X_{T-1T} \text{ and } X_{T-2T-2} = X_{T-2T-1} = X_{T-2T} < X_{T-1T-1} = X_{T-1T}$$

$$\text{imply } \lambda_{T-2} X_{T-2T-2} + (\lambda_{T-1} + \lambda_T) X_{T-2T-1} = \lambda_{T-2} X_{T-1T-2} + (\lambda_{T-1} + \lambda_T) X_{T-1T-1}.$$

Given $X_{T-1T-2} < X_{T-2T-2}$ and $X_{T-2T-1} < X_{T-1T-1}$, the positivity of $(\lambda_{T-1} + \lambda_T)$ implies the positivity of λ_{T-2} . It follows $\lambda_{T-2} = \alpha_{T-2} \lambda_T$, with $\alpha_{T-2} > 0$. And so on, down to t^* .

For t^*-1 and t^* , we have the equations: $\sum_{i=t^*-1}^T \lambda_i X_{t^*-1i} + \lambda = 0$ and $\sum_{i=t^*}^T \lambda_i X_{t^*i} + \lambda = 0$ i.e. $\lambda_{t^*-1} X_{t^*-1t^*-1} + (\lambda_{t^*} + \dots + \lambda_T) X_{t^*-1t^*} + \lambda = 0$ and $(\lambda_{t^*} + \dots + \lambda_T) X_{t^*t^*} + \lambda = 0$. Given the positivity of λ_t for t from t^* to T , and given that $X_{t^*-1t^*} < X_{t^*t^*}$, we get $\lambda_{t^*-1} > 0$, so $\lambda_{t^*-1} = \alpha_{t^*-1} \lambda_T > 0$, with $\alpha_{t^*-1} > 0$.

We now turn to t^*-2 and t^*-1 . The two adjacent equations become: $\lambda_{t^*-2} X_{t^*-2t^*-2} + \lambda_{t^*-1} X_{t^*-2t^*-1} + (\lambda_{t^*} + \dots + \lambda_T) X_{t^*-2t^*} + \lambda = 0$ and $\lambda_{t^*-1} X_{t^*-1t^*-1} + (\lambda_{t^*} + \dots + \lambda_T) X_{t^*-1t^*} + \lambda = 0$. Given that $X_{t^*-2t^*-1}$ and $X_{t^*-2t^*}$ are respectively lower than $X_{t^*-1t^*-1}$ and $X_{t^*-1t^*}$, it follows $\lambda_{t^*-2} > 0$, and $\lambda_{t^*-2} = \alpha_{t^*-2} \lambda_T$, with $\alpha_{t^*-2} > 0$. Proceeding in the same way downwards ensures the positivity of each value λ_t down to $t = 0$, as soon as $\lambda_T > 0$.

The derivative of the KKT function in y leads to $1 - \sum_{i=0}^T \lambda_i = 0$, hence $\lambda_T = 1 / (1 + \sum_{i=0}^{T-1} \alpha_i)$, which is positive, and the derivative in p_0 gives the value λ .

Moreover, given that $\lambda_t > 0$ for t from 0 to T , all the constraints are checked with equality, due to the exclusion relations, which is our starting assumption.

So, given the convexity of the problem, $y = \frac{p_0 M(1+r)}{r}$ will be the solution of the minimization program (global minimum), and the obtained values of p_t , t from 0 to T , will be the optimal probabilities. To get these probabilities, we equalize the regrets in the first t columns, t going from 0 to t^* .

$$X_{00} = \frac{M(1+r)}{r} \text{ so Regret}(0) = \frac{p_0 M(1+r)}{r} = y$$

$$\text{Regret}(1) = p_0 \left(D + \frac{M}{1+r} \cdot \frac{1+r}{r} \right) + p_1 \left(\frac{M}{1+r} \cdot \frac{1+r}{r} \right)$$

$$\text{The equality of both regrets leads to } p_1 = \frac{r(M-D)}{M} p_0$$

$$\text{Regret}(2) = p_0 \left(D + \frac{D}{1+r} + \frac{M}{(1+r)^2} \cdot \frac{1+r}{r} \right) + p_1 \left(\frac{D}{1+r} + \frac{M}{(1+r)^2} \cdot \frac{1+r}{r} \right) + p_2 \frac{M}{(1+r)^2} \cdot \frac{1+r}{r}$$

$$\text{Regret}(1) = \text{Regret}(2) \Rightarrow (p_0 + p_1)(M-D)r = p_2 M$$

$$\text{Given } Mp_1 = r(M-D)p_0, \text{ the previous equation becomes: } p_2 = \frac{M+r(M-D)}{M} \cdot p_1$$

More generally, suppose $p_t = \left(\frac{M+r(M-D)}{M} \right)^{t-1} p_1$ and $\sum_{i=0}^{t-1} p_i (M-D)r = p_t M$ for $t < t^*-1$.

$$\text{We get: Regret}(t) = \sum_{i=0}^{t-1} p_i \left(\frac{M(1+r)}{(1+r)^{tr}} + \sum_{j=i}^{t-1} \frac{D}{(1+r)^j} \right) + p_t \frac{M}{(1+r)^t} \cdot \frac{1+r}{r}$$

$$\text{Regret}(t+1) = \sum_{i=0}^{t-1} p_i \left(\frac{M(1+r)}{(1+r)^{t+1r}} + \sum_{j=i}^{t-1} \frac{D}{(1+r)^j} + \frac{D}{(1+r)^t} \right) + p_t \left(\frac{D}{(1+r)^t} + \frac{M}{(1+r)^{t+1}} \cdot \frac{1+r}{r} \right) + p_{t+1} \frac{M}{(1+r)^{t+1}} \cdot \frac{1+r}{r}$$

$$\text{Regret}(t) = \text{Regret}(t+1) \Rightarrow r \sum_{i=0}^t p_i (M-D) = p_{t+1} M, \text{ so } p_t M + p_t (M-D)r = p_{t+1} M$$

$$\text{and } p_{t+1} = \frac{M+r(M-D)}{M} \cdot p_t = \left(\frac{M+r(M-D)}{M} \right)^t p_1$$

$$\text{Finally we get Regret}(t^*) = -p_1 D - p_2 \left(D + \frac{D}{1+r} \right) - \dots - p_{t^*-1} \left(D + \frac{D}{1+r} + \dots + \frac{D}{(1+r)^{t^*-2}} \right) - p_{t^*} \left(D + \frac{D}{1+r} + \dots + \frac{D}{(1+r)^{t^*-1}} \right)$$

$$= -D(p_1 + p_2 + \dots + p_{t^*-1}) - \frac{D}{1+r} \cdot (p_2 + \dots + p_{t^*-1}) - \dots - \frac{D}{(1+r)^{t^*-2}} \cdot p_{t^*-1} - p_{t^*} \left(D + \frac{D}{1+r} + \dots + \frac{D}{(1+r)^{t^*-1}} \right)$$

$$= -\frac{Dp_1(1-B^{t^*-1})}{(1-B)} - \frac{DBp_1(1-B^{t^*-2})}{(1+r) \cdot (1-B)} - \frac{DB^{t^*-2}p_1(1-B)}{(1+r)^{t^*-2} \cdot (1-B)} - p_{t^*} D \left(\frac{1 - \left(\frac{1}{1+r} \right)^{t^*}}{1 - \frac{1}{1+r}} \right) \text{ with } B = \frac{M+r(M-D)}{M}$$

$$= -\frac{Dp_1}{1-B} \cdot \left((1 - B^{t^*-1}) + \frac{B(1-B^{t^*-2})}{(1+r)} + \dots + \frac{B^{t^*-2}(1-B)}{(1+r)^{t^*-2}} \right) - p_{t^*} D \left(\frac{1 - \left(\frac{1}{1+r} \right)^{t^*}}{1 - \frac{1}{1+r}} \right)$$

$$= -\frac{Dp_1}{1-B} \cdot \left(\frac{1 - \left(\frac{B}{1+r} \right)^{t^*-1}}{1 - \frac{B}{1+r}} - B^{t^*-1} \cdot \frac{1 - \left(\frac{1}{1+r} \right)^{t^*-1}}{1 - \frac{1}{1+r}} \right) - p_{t^*} D \left(\frac{1 - \left(\frac{1}{1+r} \right)^{t^*}}{1 - \frac{1}{1+r}} \right)$$

We equalize Regret(t^*) with Regret(0) and we get:

$$\frac{p_0 M(1+r)}{r} = Dp_0 \left(\frac{1 - \left(\frac{B}{1+r} \right)^{t^*-1}}{1 - \frac{B}{1+r}} - B^{t^*-1} \cdot \frac{1 - \left(\frac{1}{1+r} \right)^{t^*-1}}{1 - \frac{1}{1+r}} \right) - p_{t^*} D \left(\frac{1 - \left(\frac{1}{1+r} \right)^{t^*}}{1 - \frac{1}{1+r}} \right) \text{ because } -\frac{Dp_1}{1-B} = Dp_0$$

$$\text{We get } \frac{p_0 M(1+r)}{r} \cdot \left(\left(\frac{B}{1+r} \right)^{t^*-1} + \frac{D}{M} \cdot B^{t^*-1} \left(1 - \left(\frac{1}{1+r} \right)^{t^*-1} \right) \right) + p_{t^*} D \left(\frac{1 - \left(\frac{1}{1+r} \right)^{t^*}}{1 - \frac{1}{1+r}} \right) = 0$$

$$\text{So } -p_{t^*} D \left(1 - \left(\frac{1}{1+r} \right)^{t^*} \right) = p_0 B^{t^*-1} \left((M-D) \left(\frac{1}{1+r} \right)^{t^*-1} + D \right)$$

and we get $p_{t^*} = \frac{p_0 B^{t^*-1} \left((M-D) \left(\frac{1}{1+r} \right)^{t^*-1} + D \right)}{-D \left(1 - \left(\frac{1}{1+r} \right)^{t^*} \right)}$

which is positive because $D + (M-D) \left(\frac{1}{1+r} \right)^{t^*-1} = \frac{\left(D + \frac{D}{1+r} + \dots + \frac{D}{(1+r)^{t^*-2}} + \frac{M}{(1+r)^{t^*-1}} \cdot \frac{1+r}{r} \right) r}{r+1}$ is positive by definition of t^* .

Summing to 1, the probabilities lead to:

$$p_0 + p_1(1 + B + \dots + B^{t^*-2}) + \frac{p_0 B^{t^*-1} \left((M-D) \left(\frac{1}{1+r} \right)^{t^*-1} + D \right)}{-D \left(1 - \left(\frac{1}{1+r} \right)^{t^*} \right)} = 1$$

which leads to $p_0 = -D \cdot \frac{(1+r)^{t^*-1}}{(M(1+r)-Dr) \left(\frac{M+r(M-D)}{M} \right)^{t^*-1}} = -\left(\frac{D}{M} \right) \cdot \frac{(1+r)^{t^*-1}}{\left(\frac{M+r(M-D)}{M} \right)^{t^*}}$

All the other probabilities follow.

Study with $t^* > T$

Nothing changes as regards the equations, by replacing t^* by T . The only change is that p_T may be large.

Appendix F

Study with $t^* \leq T$

For the same reasons as in Appendix E, to get the minimax regret density function $f(t)$, we equalize the regrets in the columns from 0 to t^* , where $t^* = \ln \left(-\frac{M-D}{D} \right) / r$.

We look for a density function $f(t)$ with a mass point on 0, $g(0)$ being the probability assigned to 0, and we get $g(0) + \int_{0+}^{t^*} f(t) dt = 1$.

$F(t)$ is the cumulative distribution function, with $F(0) = g(0)$ and $F(t^*)=1$.

In the continuous-time game, the regrets in column t^* can be calculated in two ways: either by comparing the payoffs with the null payoff obtained by leaving the market at time t^* , $\int_0^{t^*} D e^{-rt} dt + \int_{t^*}^{\infty} M e^{-rt} dt = 0$, or by comparing the payoffs with the null payoff obtained by leaving at time 0.

We equalize the regrets and we get:

$$\text{Regret}(0) = g(0) \int_0^{\infty} M e^{-rt} dt = g(0)M/r$$

By exiting the market at time s ($< t^*$) when the opponent exits at time d , with $d > s$, firm 1 gets $\int_0^s D e^{-rt} dt$, whereas it could obtain $\int_0^d D e^{-rt} dt + \int_d^{\infty} M e^{-rt} dt$, so its regret is $\int_s^d D e^{-rt} dt + \int_d^{\infty} M e^{-rt} dt$.

We get: $\text{Regret}(d) = g(0) \left(\int_0^d D e^{-rt} dt + \int_d^{\infty} M e^{-rt} dt \right) + \int_{0+}^d \left(\int_s^d D e^{-rt} dt + \int_d^{\infty} M e^{-rt} dt \right) f(s) ds$

$\text{Regret}(d)$ is constant over $[0, t^*]$, so we need $\text{Regret}'(d)=0$, for d from $0+$ to t^* . We get:

$$\begin{aligned} & g(0) (D e^{-rd} - M e^{-rd}) + \left(\int_d^{\infty} M e^{-rt} dt \right) f(d) + \int_{0+}^d (D e^{-rd} - M e^{-rd}) f(s) ds \\ &= g(0) (D e^{-rd} - M e^{-rd}) + \frac{M e^{-rd}}{r} f(d) + (F(d) - g(0)) (D e^{-rd} - M e^{-rd}) \\ &= \frac{M e^{-rd}}{r} f(d) + F(d) (D e^{-rd} - M e^{-rd}). \end{aligned}$$

We need $\frac{M}{r} f(d) + F(d) (D - M) = 0$, with $F(0)=g(0)$ and $F(t^*)=1$.

This differential equation leads to $F(t) = g(0)e^{\frac{(M-D)rt}{M}}$, $f(t) = \frac{g(0)(M-D)r}{M}e^{\frac{(M-D)rt}{M}}$ and

$$g(0) = \left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}}$$

We check that $\text{Regret}(0) = \text{Regret}(d)$ for d from $0+$ to t^* .

$$\text{Regret}(0) = g(0)M/r$$

$$\begin{aligned} \text{Regret}(d) &= g(0)\left(\int_0^d De^{-rt} dt + \int_d^\infty Me^{-rt} dt\right) + \int_{0+}^d \left(\int_s^d De^{-rt} dt + \int_d^\infty Me^{-rt} dt\right) f(s) ds \\ &= \frac{g(0)(D+(M-D)e^{-rd})}{r} + g(0) \int_0^d \left(\frac{M-D}{r} e^{-rd} + \frac{D}{r} e^{-rs}\right) \cdot \frac{M-D}{M} r e^{\frac{(M-D)rs}{M}} ds = \frac{g(0)M}{r} \end{aligned}$$

after development.

Study with $t^* > T$

When $t^* > T$, the distribution has two mass points 0 and T , played with probability $g(0)$ and $g(T)$, and $F(T) = F(T-) + g(T)$.

$$\text{We have: } \text{Regret}(T) = \int_{0+}^{T-} \left(\int_0^s -De^{-rt} dt\right) f(s) ds + g(T) \int_0^T -De^{-rt} dt.$$

$\text{Regret}(s)$ is constant over $[0, T]$, so again we get the differential equation

$$\frac{M}{r} f(d) + F(d)(D-M) = 0, \text{ with } F(0)=g(0) \text{ and } F(T)=1, \text{ which leads to } F(t) = g(0)e^{\frac{(M-D)rt}{M}} \text{ hence } f(t) = \frac{g(0)(M-D)r}{M} e^{\frac{(M-D)rt}{M}}.$$

Given $F(T) = F(T-) + g(T)$, we get $g(0)$ and $g(T)$ by requiring $\text{Regret}(0) = \text{Regret}(T)$. We get:

$$\begin{aligned} \frac{g(0)M}{r} &= \int_{0+}^{T-} \left(\int_0^s -De^{-rt} dt\right) f(s) ds + g(T) \int_0^T -De^{-rt} dt \\ \int_{0+}^{T-} \left(\int_0^s -De^{-rt} dt\right) f(s) ds + g(T) \int_0^T -De^{-rt} dt &= \int_{0+}^{T-} \frac{(De^{-rs}-D)}{r} f(s) ds + \frac{g(T)(De^{-rT}-D)}{r} \\ &= -\frac{D}{r} (1 - g(0) - g(T)) + \frac{D}{r} \int_{0+}^{T-} \frac{g(0)(M-D)r}{M} e^{\frac{-Drs}{M}} ds + \frac{g(T)(De^{-rT}-D)}{r} \\ &= -\frac{D}{r} (1 - g(0)) + \frac{g(0)D(M-D)}{M} \int_{0+}^{T-} e^{\frac{-Drs}{M}} ds + \frac{g(T)(De^{-rT})}{r} \\ &= -\frac{D}{r} + \frac{g(0)M}{r} - \frac{g(0)(M-D)e^{\frac{-DrT}{M}}}{r} + \frac{g(T)(De^{-rT})}{r} \\ \frac{g(0)M}{r} &= -\frac{D}{r} + \frac{g(0)M}{r} - \frac{g(0)(M-D)e^{\frac{-DrT}{M}}}{r} + \frac{g(T)(De^{-rT})}{r} \Rightarrow \\ g(T) &= e^{rT} - \frac{g(0)(M-D)}{-D} e^{\frac{(M-D)rT}{M}}. \end{aligned}$$

$$F(T-) + g(T) = 1 \text{ hence } g(0)e^{\frac{(M-D)rT}{M}} + e^{rT} - \frac{g(0)(M-D)}{-D} e^{\frac{(M-D)rT}{M}} = 1.$$

$$\text{We get } g(0) = (e^{rT} - 1) \left(\frac{-D}{M}\right) e^{-(M-D)rT/M} > 0 \text{ and } g(T) = 1 + \frac{e^{rT}D}{M} - \frac{D}{M}.$$

$g(T)$ is decreasing in T and it is easily checked that for $0 < T < t^*$ we get $1 > g(T) > 0$, and that

$$\text{for } T=t^* = \frac{\ln\left(\frac{M-D}{-D}\right)}{r} \text{ we get } g(T)=0 \text{ and } g(0) = \left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}}.$$

$g(0)$ is increasing in T :

$$\begin{aligned} g'_T(0) &= re^{rT} \left(\frac{-D}{M}\right) e^{-\frac{(M-D)rT}{M}} + (e^{rT} - 1) \left(\frac{-D}{M}\right) \left(-\frac{(M-D)r}{M}\right) e^{-(M-D)rT/M} \\ &= r \left(-\frac{D}{M}\right) e^{-\frac{(M-D)rT}{M}} \left(\frac{D}{M} e^{rT} + 1 - \frac{D}{M}\right) = r \left(-\frac{D}{M}\right) e^{-\frac{(M-D)rT}{M}} g(T) > 0. \end{aligned}$$

Given that $g(0)$ is equal to $\left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}}$ for $T=t^*$, $g(0)$ is always lower than the probability assigned to 0 in the model with $T > t^*$. Hence, when firms are forced to leave before t^* , they less often leave at time 0.

When T goes to 0, $g(0)$ goes to 0 and $g(T)$ goes to 1. So, when T becomes small, firms are more and more incited to exit the market as late as possible (as with the pure minimax regret strategy).

Appendix G

Game in discrete time

$$E(g) = \sum_{t=0}^{\min(t^*, T)} (\text{best-reply payoff to } t) p_t - \sum_{t=0}^{\min(t^*, T)} (p_0 M(1+r)/r) p_t =$$

$$\frac{p_0 M(1+r)}{r} + \sum_{t=1}^{\min(t^*, T)} (\text{best-reply payoff to } t) p_t - \frac{p_0 M(1+r)}{r} =$$

$$\sum_{t=1}^{\min(t^*, T)} (\text{best-reply payoff to } t) p_t > 0 \text{ (because the best-reply payoff is larger than 0 for } t < \min(t^*, T) \text{ and equal to 0 for } t = \min(t^*, T)).$$

Game in continuous time with $t^* \leq T$

$$E(g) = \left(\frac{M}{r} - \frac{M}{r} g(0) \right) g(0) + \int_{0+}^{t^*} \left(\int_0^t D e^{-rs} ds + \int_t^\infty M e^{-rs} ds - \frac{g(0)M}{r} \right) f(t) dt =$$

$$\frac{M}{r} g(0) + \int_{0+}^{t^*} \left(\int_0^t D e^{-rs} ds + \int_t^\infty M e^{-rs} ds \right) f(t) dt - \frac{M}{r} g(0) =$$

$$\int_{0+}^{t^*} \left(\int_0^t D e^{-rs} ds + \int_t^\infty M e^{-rs} ds \right) f(t) dt > 0$$

Developing $\int_{0+}^{t^*} \left(\int_0^t D e^{-rs} ds + \int_t^\infty M e^{-rs} ds \right) f(t) dt > 0$ leads to

$$E(g) = \int_{0+}^{t^*} \frac{D + (M-D)e^{-rt}}{r} f(t) dt = g(0) \left(\frac{D}{r} \left(e^{\frac{(M-D)rt^*}{M}} - 1 \right) + \frac{(M-D)^2}{Dr} \left(1 - e^{-\frac{Drt^*}{M}} \right) \right)$$

$$= g(0) \left(\frac{(M-2D)}{D} \cdot \frac{M}{r} + \frac{D^2 e^{\frac{(M-D)rt^*}{M}} - (M-D)^2 e^{-\frac{Drt^*}{M}}}{Dr} \right)$$

$$= g(0) \left(\frac{(M-2D)}{D} \cdot \frac{M}{r} + \frac{D^2 \left(\frac{M-D}{-D} \right)^{\frac{(M-D)}{M}} - (M-D)^2 \left(\frac{M-D}{-D} \right)^{-\frac{D}{M}}}{Dr} \right)$$

$$= \frac{M}{r} \left(1 - \frac{g(0)(M-2D)}{-D} \right) = \frac{M}{r} \left(1 - \left(\frac{x}{1+x} \right)^x - \left(\frac{x}{1+x} \right)^{1+x} \right) \text{ with } x = -D/M$$

We now study $g(0)$, t^* and $E(g)$.

$g(0) = 0$ for $x = 0$ and $g(x) \rightarrow e^{-1} = 0.368$ when $x \rightarrow +\infty$ because $\left(\frac{x}{1+x} \right)^{1+x} = e^{(1+x) \ln \left(1 - \frac{1}{1+x} \right)}$, which tends towards e^{-1} .

$t^* = \frac{\ln \left(\frac{1+x}{x} \right)}{r} \rightarrow +\infty$ when $r \rightarrow 0$ and/or $x \rightarrow 0$. $t^* \rightarrow 0$ when $x \rightarrow +\infty$ for a given r .

$E(g)$ is increasing in M and decreasing in r for a fixed value $-D/M$. $E(g) \rightarrow 0$ when $x \rightarrow 0$,

$$E(g) \rightarrow (1-2e^{-1})M/r \text{ when } x \rightarrow +\infty. \text{ This result follows from: } E(g) = \frac{M}{r} \left(1 - \left(\frac{x}{1+x} \right)^x - \left(\frac{x}{1+x} \right)^{1+x} \right) =$$

$$\frac{M}{r} \left(1 - e^{x \ln \left(\frac{x}{1+x} \right)} - e^{(1+x) \ln \left(\frac{x}{1+x} \right)} \right) \rightarrow \frac{M}{r} \left(1 - e^{x \left(-\frac{1}{1+x} \right)} - e^{(1+x) \left(\frac{-1}{1+x} \right)} \right) \rightarrow$$

$$(1-2e^{-1})M/r \text{ when } x \rightarrow +\infty.$$

Game in continuous time with $t^* > T$

$$E(g) = \left(\frac{M}{r} - \frac{M}{r}g(0)\right)g(0) + \int_{0+}^{T-} \left(\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds - \frac{g(0)M}{r}\right)f(t)dt + g(T)(0 - g(0)M/r) = \frac{M}{r}g(0) + \int_{0+}^{T-} \left(\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds\right)f(t)dt - \frac{M}{r}g(0) =$$

$$\int_{0+}^{T-} \left(\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds\right)f(t)dt > 0$$

Developing this expression leads to:

$$E(g) = g(0)\left(\frac{D}{r}\left(e^{\frac{(M-D)rT}{M}} - 1\right) + \frac{(M-D)^2}{Dr}\left(1 - e^{-\frac{DrT}{M}}\right)\right) \\ = \frac{g(0)(M-2D)}{D} \cdot \frac{M}{r} + \frac{g(0)}{Dr} \left(D^2 e^{\frac{(M-D)rT}{M}} - (M-D)^2 e^{-\frac{DrT}{M}}\right)$$

We show that $E(g)$ is increasing in T over $]0, t^*[$. So the firm's expected payoff, when it is forced to exit before t^* , is lower than the expected payoff obtained when the firm is free to leave at a time later than t^* (or at any time).

$$\text{We have } E'(g) = \frac{g'(0)(M-2D)}{D} \cdot \frac{M}{r} + \frac{g'(0)}{Dr} \left(D^2 e^{\frac{(M-D)rT}{M}} - (M-D)^2 e^{-\frac{DrT}{M}}\right) + \\ \frac{g(0)}{Dr} \left(\frac{D^2(M-D)r}{M} e^{\frac{(M-D)rT}{M}} - \frac{(M-D)^2(-Dr)}{M} e^{-\frac{DrT}{M}}\right).$$

$$\frac{g'(0)(M-2D)}{D} \cdot \frac{M}{r} + \frac{g'(0)}{Dr} \left(D^2 e^{\frac{(M-D)rT}{M}} - (M-D)^2 e^{-\frac{DrT}{M}}\right) > 0 \text{ because } g'(0) > 0 \text{ and } E(g) > 0. \text{ So it}$$

remains to show that $\frac{g(0)}{Dr} \left(\frac{D^2(M-D)r}{M} e^{\frac{(M-D)rT}{M}} - \frac{(M-D)^2(-Dr)}{M} e^{-\frac{DrT}{M}}\right)$ is larger than 0.

$$\text{We have } \frac{g(0)}{Dr} \left(\frac{D^2(M-D)r}{M} e^{\frac{(M-D)rT}{M}} - \frac{(M-D)^2(-Dr)}{M} e^{-\frac{DrT}{M}}\right) = \frac{g(0)}{Dr} \left(\frac{Dr}{M}\right) (M-D) e^{-\frac{DrT}{M}} (De^{rT} + M - D) > 0 \text{ because } T < t^*.$$