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# Split Covariance Intersection with Correlated Components for Distributed Estimation

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**Abstract**—This paper introduces a new conservative fusion method to exploit the correlated components within the estimation errors. Fusion is the process of combining multiple estimates of a given state to produce a new estimate with a smaller MSE. To perform the optimal linear fusion, the (centralized) covariance associated with the errors of all estimates is required. If it is partially unknown, the optimal fusion cannot be computed. Instead, a solution is to perform a conservative fusion. A conservative fusion provides a gain and a bound on the resulting MSE matrix which guarantees that the error is not underestimated. A well-known conservative fusion is the Covariance Intersection fusion. It has been modified to exploit the uncorrelated components within the errors. In this paper, it is further extended to exploit the correlated components as well. The resulting fusion is integrated into standard distributed algorithms where it allows exploiting the process noise observed by all agents. The improvement is confirmed by simulations.

**Index Terms**—Conservative fusion, Covariance Intersection, Distributed estimation, Linear fusion

## I. INTRODUCTION

Distributed estimation is a recurrent problem in sensor networks. It consists in estimating the state of a dynamical system using a network of agents, i.e., nodes equipped with sensors and with communication capabilities. Each agent is performing independent measurements of the state, and shares its estimate with its neighbors in the network. The fusion of the estimates received by an agent is a complex task, especially when the communication between the agents are limited. To optimally fuse several estimates, i.e., with the smallest resulting Mean Square Error (MSE) matrix, the agent needs to know the covariances of the errors of each estimate and their cross-covariances. For example, the optimal fusion of two estimates is given by the Bar-Shalom-Campo's formulas [3]. However, in cooperating networks, each agent has only local knowledge; it can estimate the covariance of its own error but cannot estimate the cross-covariances with its neighbors' errors, as this would require knowledge of the whole network topology. Without these cross-covariances, the agent cannot calculate the MSE matrix of the fused estimate and must use *conservative* bounds. A conservative fusion provides a bound

on the MSE matrix which guarantees that the estimation error is not underestimated. The first conservative fusion proposed was the Covariance Intersection fusion (CI) [8]. CI provides conservative bounds by considering that the estimation errors may be correlated to any degree. Generally, this assumption is very loose and tighter fusions have been derived when refined assumptions can be made. When the errors contain independent components, the Split CI (SCI) fusion provides tighter bounds [7]. Furthermore, if the errors are partitioned into two components with only the cross-covariances between the first components unknown, another fusion rule called Partitioned CI (PCI) was proposed in [14] and improved in [1]. Another extension is the Inverse CI (ICI) which considers that the estimates were all obtained by combining some initial estimate with independent estimates [2], [13]. All these improvements of the CI fusion use the structure of the errors to reduce the set of admissible cross-covariances and tighten the bounds. They have been applied to wide range of problems: e.g., SLAM [9], cooperative localization [10], or cooperative perception [11].

In distributed estimation, the fusion of the estimates can be performed using CI. However, two elements can be used to produce tighter bounds. First, the independent measurements induce independent components in the errors. They can be exploited by the SCI fusion, as in [7]. If the measurements are transmitted to the neighbors, the agents can optimally fuse them with their estimate, this method is known as Diffusion Kalman Filtering (DKF) with CI [4], [6]. The second element is the process noise which is observed by all the agents. It reduces as well the set of admissible covariances: e.g., the errors cannot be perfectly negatively correlated. The current fusions do not take advantage from such common noises as they do not consider correlated components.

In this paper, an extension of the SCI fusion, called Extended SCI (ESCI), is introduced. It is motivated by the distributed estimation problems in which it can exploit both the uncorrelated components of the errors (induced by the measurements) and their correlated components (induced by the process noise). It is integrated into standard algorithms and

applied to an example inspired by Search-And-Rescue (SAR) missions.

The rest of the paper is organized as follows. Section II recalls the definitions of a conservative fusion and of the SCI fusion. Then, Section III presents the new ESCI fusion. Section IV details its integration into standard distributed estimation algorithms. The application is presented in Section V. Finally, Section VI concludes the paper.

**Notation.** In the sequel, vectors are denoted in lowercase boldface letters e.g.,  $\mathbf{x} \in \mathbb{R}^n$ , and matrices in uppercase boldface variables e.g.,  $\mathbf{M} \in \mathbb{R}^{n \times n}$ . The notation  $\mathbb{E}[\cdot]$  denotes the expected value. For two matrices  $\mathbf{A}$  and  $\mathbf{B}$ , the notation  $\mathbf{A} \preceq \mathbf{B}$  means that the difference  $\mathbf{B} - \mathbf{A}$  is positive semi-definite. The unit simplex of  $\mathbb{R}^n$  is denoted as  $\mathcal{K}^n \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid \forall i, x_i \geq 0, \mathbf{x}^\top \mathbf{1} = 1\}$ .

## II. BACKGROUND

Consider  $N$  unbiased estimates  $\hat{\mathbf{x}}_i$  for  $i \in \{1, \dots, N\}$  of a random variable  $\mathbf{x} \in \mathbb{R}^d$ . The estimation errors are denoted as  $\tilde{\mathbf{x}}_i \triangleq \hat{\mathbf{x}}_i - \mathbf{x}$  and their covariances as  $\tilde{\mathbf{P}}_i \triangleq \mathbb{E}[\tilde{\mathbf{x}}_i \tilde{\mathbf{x}}_i^\top]$ . A linear fusion is defined by a matrix of gains  $\mathbf{K} = [\mathbf{K}_1 \ \dots \ \mathbf{K}_N] \in \mathbb{R}^{d \times Nd}$ , with  $\mathbf{K}_i \in \mathbb{R}^{d \times d}$  and  $\sum_i \mathbf{K}_i = \mathbf{I}_d$ , as:

$$\hat{\mathbf{x}}_F(\mathbf{K}) \triangleq \sum_{i=1}^N \mathbf{K}_i \hat{\mathbf{x}}_i = \mathbf{K} \hat{\mathbf{x}}_c, \quad (1)$$

where  $\hat{\mathbf{x}}_c \triangleq (\hat{\mathbf{x}}_1^\top \ \dots \ \hat{\mathbf{x}}_N^\top)^\top \in \mathbb{R}^{Nd}$ . The covariance of the error of the fused estimate depends on the gain  $\mathbf{K}$  and on the covariance of the error of  $\hat{\mathbf{x}}_c$ ,  $\tilde{\mathbf{P}}_c$ :

$$\tilde{\mathbf{P}}_F(\mathbf{K}, \tilde{\mathbf{P}}_c) = \mathbf{K} \tilde{\mathbf{P}}_c \mathbf{K}^\top. \quad (2)$$

If  $\tilde{\mathbf{P}}_c$  is not entirely known but is only assumed to belong to some subset of admissible covariance matrices  $\mathcal{A}$ , then  $\tilde{\mathbf{P}}_F(\mathbf{K}, \tilde{\mathbf{P}}_c)$  cannot be computed. In this case, an alternative is to provide a *conservative* bound. A couple  $(\mathbf{K}, \mathbf{B}_F)$  is said to generate a conservative fusion for the set  $\mathcal{A}$  if:

$$\forall \mathbf{P}_c \in \mathcal{A}, \quad \tilde{\mathbf{P}}_F(\mathbf{K}, \mathbf{P}_c) \preceq \mathbf{B}_F. \quad (3)$$

In other words, fusing the estimates with the gain  $\mathbf{K}$  ensures that the covariance of the error is bounded by  $\mathbf{B}_F$ .

CI considers that the covariances of the errors  $\tilde{\mathbf{P}}_i$  are known but not their cross-covariances  $\tilde{\mathbf{P}}_{i,j} \triangleq \mathbb{E}[\tilde{\mathbf{x}}_i \tilde{\mathbf{x}}_j^\top]$ . For any  $\omega \in \mathcal{K}^N$ , CI provides a conservative fusion defined as:

$$\hat{\mathbf{x}}_F = \mathbf{B}_F \sum_{i=1}^N \omega_i \tilde{\mathbf{P}}_i^{-1} \hat{\mathbf{x}}_i, \quad \mathbf{B}_F^{-1} = \sum_{i=1}^N \omega_i \tilde{\mathbf{P}}_i^{-1}. \quad (4)$$

CI considers that the errors may be completely correlated. In distributed estimation, the estimators integrate independent measurements  $\mathbf{z}_i$  and have the following structure:

$$\hat{\mathbf{x}}_i = (\mathbf{I} - \mathbf{K} \mathbf{H}_i) \hat{\mathbf{x}}_i^- + \mathbf{K} \mathbf{z}_i.$$

Therefore, the errors  $\tilde{\mathbf{x}}_i$  cannot be perfectly correlated and SCI produces tighter bounds. It considers that the estimation errors are split into a correlated and an uncorrelated component as:

$$\tilde{\mathbf{x}}_i = \tilde{\mathbf{x}}_i^{(1)} + \tilde{\mathbf{x}}_i^{(2)}, \quad (5)$$

where the components  $\tilde{\mathbf{x}}_i^{(1)}$  are correlated to an unknown degree while the components  $\tilde{\mathbf{x}}_i^{(2)}$  are uncorrelated between each other and with the  $\tilde{\mathbf{x}}_i^{(1)}$ . The covariances of  $\tilde{\mathbf{x}}_i^{(1)}$  and  $\tilde{\mathbf{x}}_i^{(2)}$  are denoted as  $\tilde{\mathbf{P}}_i^{(1)}$  and  $\tilde{\mathbf{P}}_i^{(2)}$  (and are assumed known). For any  $\omega \in \mathcal{K}^N$ , SCI provides a conservative fusion defined as:

$$\hat{\mathbf{x}}_F = \mathbf{B}_F \sum_{i=1}^N \omega_i \left( \tilde{\mathbf{P}}_i^{(1)} + \omega_i \tilde{\mathbf{P}}_i^{(2)} \right)^{-1} \hat{\mathbf{x}}_i, \quad (6a)$$

$$\mathbf{B}_F^{-1} = \sum_{i=1}^N \omega_i \left( \tilde{\mathbf{P}}_i^{(1)} + \omega_i \tilde{\mathbf{P}}_i^{(2)} \right)^{-1}. \quad (6b)$$

The parameter  $\omega$  must be chosen: optimized or empirically tuned with e.g., the methods in [12] or [5].

## III. EXTENDED SCI

### A. Motivation: Limits of the SCI fusion

The CI fusion considers that the errors can be correlated to any degree. In distributed estimation problems, the estimates incorporate independent measurements, and therefore their errors contain independent components. The SCI fusion has been defined to exploit these independent terms.

Moreover, the state to estimate is often disturbed by an additive process noise  $\mathbf{w}$ . This noise is added to all the estimation errors during the prediction step. All the errors share a common component, which also reduces the space of admissible centralized covariance matrices  $\mathcal{A}$ . For example, the errors cannot be perfectly negatively correlated. However, SCI cannot exploit this correlated component as it can only handle uncorrelated components. The new ESCI fusion is defined to overcome this limitation.

### B. Definition of the ESCI fusion

Consider that the estimation errors are split into two components as in (5). The first components  $\tilde{\mathbf{x}}_i^{(1)}$  are still correlated to an unknown degree. The second components  $\tilde{\mathbf{x}}_i^{(2)}$  are not assumed uncorrelated, but are assumed to have known second moments. Introduce the centralized errors,

$$\tilde{\mathbf{x}}_c^{(l)} \triangleq \left( \tilde{\mathbf{x}}_1^{(l)\top} \ \dots \ \tilde{\mathbf{x}}_N^{(l)\top} \right)^\top, \quad l \in \{1, 2\},$$

whose covariances and cross-covariances are denoted as  $\tilde{\mathbf{P}}_c^{(l)} \triangleq \mathbb{E}[\tilde{\mathbf{x}}_c^{(l)} \tilde{\mathbf{x}}_c^{(l)\top}]$  and  $\tilde{\mathbf{P}}_c^{(1,2)} \triangleq \mathbb{E}[\tilde{\mathbf{x}}_c^{(1)} \tilde{\mathbf{x}}_c^{(2)\top}]$ . The matrices  $\tilde{\mathbf{P}}_c^{(2)}$  and  $\tilde{\mathbf{P}}_c^{(1,2)}$  are known, but only the diagonal blocks of  $\tilde{\mathbf{P}}_c^{(1)}$  (corresponding to the covariances  $\tilde{\mathbf{P}}_i^{(1)}$ ) are known. If  $\tilde{\mathbf{P}}_c^{(1,2)} \neq \mathbf{0}$ , the errors (5) are virtually re-splittable to set  $\tilde{\mathbf{P}}_c^{(1,2)} = \mathbf{0}$  by letting:

$$\tilde{\mathbf{x}}_c^{(1)} \leftarrow \tilde{\mathbf{x}}_c^{(1)} - \tilde{\mathbf{P}}_c^{(1,2)} (\tilde{\mathbf{P}}_c^{(2)})^{-1} \tilde{\mathbf{x}}_c^{(2)}, \quad (7a)$$

$$\tilde{\mathbf{x}}_c^{(2)} \leftarrow \tilde{\mathbf{x}}_c^{(2)} + \tilde{\mathbf{P}}_c^{(1,2)} (\tilde{\mathbf{P}}_c^{(2)})^{-1} \tilde{\mathbf{x}}_c^{(2)}. \quad (7b)$$

The errors  $\tilde{\mathbf{x}}_c^{(1)}$  and  $\tilde{\mathbf{x}}_c^{(2)}$  satisfy the same properties: only the off-diagonal blocks of  $\tilde{\mathbf{P}}_c^{(1)}$  are unknown. We therefore assume without loss of generality that  $\tilde{\mathbf{P}}_c^{(1,2)} = \mathbf{0}$ . In this splitting,  $\tilde{\mathbf{x}}_i^{(2)}$  contains all *known* components. In distributed estimation,

it will contain the independent measurement noise plus the common process noise as illustrated in the next section.

For any  $\omega \in \mathcal{K}^N$ , the ESCI fusion is defined as:

$$\hat{x}_F = B_F B_c^{-1} H \hat{x}_c, \quad B_F^{-1} = H^\top B_c^{-1} H, \quad (8a)$$

with:

$$H = \mathbf{1}_N \otimes I_d, \quad (8b)$$

$$B_c^{(1)} = \text{diag} \left( \frac{1}{\omega_1} \tilde{P}_1^{(1)}, \dots, \frac{1}{\omega_N} \tilde{P}_N^{(1)} \right), \quad (8c)$$

$$B_c = B_c^{(1)} + \tilde{P}_c^{(2)}. \quad (8d)$$

The ESCI is a generalization of the SCI in the sense that if  $P_c^{(2)}$  is block diagonal, then (8) and (6) define the same fusion.

**Theorem 1.** For any  $\omega \in \mathcal{K}^N$ , the ESCI fusion defined in (8) is conservative for the set:

$$\mathcal{A}_{\text{ESCI}} \triangleq \left\{ P_c^{(1)} + \tilde{P}_c^{(2)} \mid P_c^{(1)} \succeq \mathbf{0} \text{ and } \forall i \in \{1, \dots, N\}, P_i^{(1)} = \tilde{P}_i^{(1)} \right\}. \quad (9)$$

*Proof.* The proof is the same as for the SCI fusion [7]. For any  $\omega \in \mathcal{K}^N$ , the matrix  $B_c^{(1)}$  is a conservative bound on the centralized covariance of  $\tilde{x}_c^{(1)}$  [7]. Therefore,  $B_c$  is a conservative bound for the centralized covariance of  $\tilde{x}_c$ . The fusion is then obtained by applying the gain:  $K = (H^\top B_c^{-1} H)^{-1} H^\top B_c^{-1}$ .  $\square$

### C. Special case of a common noise

When the correlated components of the errors come from a common noise  $w$ , the ESCI expressions can be simplified. Consider that the estimation errors are split as :

$$\tilde{x}_i = \tilde{x}_i^{(1)} + \tilde{x}_i^{(\text{ind})} + M_i w, \quad (10)$$

where the components  $\tilde{x}_i^{(1)}$  are correlated to an unknown degree, the components  $\tilde{x}_i^{(\text{ind})}$  are uncorrelated between each other, with the  $\tilde{x}_i^{(1)}$  and with  $w$ , the matrices  $M_i$  are known, and  $w$  is a common independent noise. The covariances of each component are known and denoted as  $P_i^{(\text{ind})}$  for  $\tilde{x}_i^{(\text{ind})}$  and  $Q$  for  $w$ . In this case the fusion (8) becomes:

$$\hat{x}_F = B_F \sum_{i=1}^N \omega_i (I_d - S_1 S_0^{-1} M_i^\top) \tilde{P}_i'^{-1} \tilde{x}_i, \quad (11a)$$

$$B_F^{-1} = \sum_{i=1}^N \omega_i \tilde{P}_i'^{-1} - S_1 S_0^{-1} S_1^\top, \quad (11b)$$

with  $\tilde{P}_i' \triangleq \tilde{P}_i^{(1)} + \omega_i \tilde{P}_i^{(\text{ind})}$  and :

$$S_0 = \sum_{i=1}^N \omega_i M_i^\top \tilde{P}_i'^{-1} M_i + Q^{-1}, \quad (11c)$$

$$S_1 = \sum_{i=1}^N \omega_i \tilde{P}_i'^{-1} M_i. \quad (11d)$$

The advantage of (11) over (8) is that (11) requires to invert  $N + 1$  matrices of size  $d$  while (8) requires the inversion of

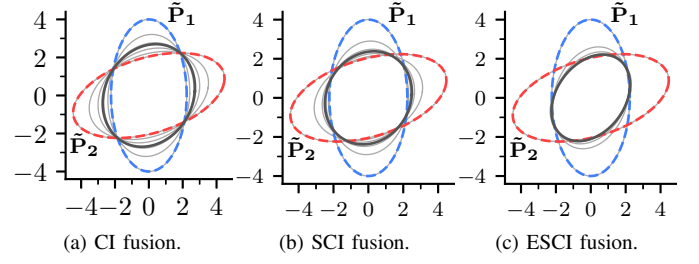


Fig. 1: Comparison of the bounds provided by CI, SCI and ESCI. The dotted ellipses represent the covariances  $\tilde{P}_1$  and  $\tilde{P}_2$ , the grey ellipses are the bound obtained with  $\omega = (2k/10 \ 1 - 2k/10)^\top$  and  $k \in \{0, \dots, 5\}$ , and the dark ellipse is the bound that minimizes the trace. The numerical values used are:  $\tilde{P}_1^{(1)} = [[1, -2], [-2, 5]]$ ,  $\tilde{P}_2^{(1)} = [[9, -1], [-1, 1]]$ ,  $\tilde{P}_1^{(\text{ind})} = [[2, 0], [0, 9]]$ ,  $\tilde{P}_2^{(\text{ind})} = [[9, 3], [3, 2]]$ ,  $Q = [[2, 2], [2, 2]]$ , and  $M_1 = M_2 = I$ .

one matrix of size  $Nd$ . As the cost of an inversion of a matrix of size  $n$  is a  $O(n^3)$ , (11) is more efficient.

If all the matrices  $M_i = I_d$ , then (11) simplifies further to:

$$\hat{x}_F = B_0 \sum_{i=1}^N \omega_i \tilde{P}_i'^{-1} \tilde{x}_i, \quad B_F = B_0 + Q, \quad (12)$$

where  $B_0^{-1} = \sum_{i=1}^N \omega_i \tilde{P}_i'^{-1}$ . This case is equivalent to first fuse the uncorrupted estimates  $\tilde{x}_i^{(1)} + \tilde{x}_i^{(\text{ind})}$  using SCI and then add the noise.

To illustrate the interest of the ESCI fusion, consider the fusion of two estimates whose errors are split according to (10). These estimates can be split using CI (without considering the splitting), using SCI (by grouping the correlated component  $M_i w$  with the first component), or using ESCI. Figure 1 compares the bounds obtained with the three fusions. It can be observed that the ESCI bounds are tighter, as expected.

## IV. DISTRIBUTED ESTIMATION ALGORITHMS

This section presents the integration of the ESCI fusion into distributed estimation algorithms.

### A. System model

Consider a system parameterized by a discrete-time state-space model. The state at time  $k \in \mathbb{N}$  is denoted as  $x(k) \in \mathbb{R}^d$ . It is assumed to follow the following linear dynamics:

$$x(0) \sim \mathcal{N}(x_0, \tilde{P}_0), \quad (13a)$$

$$\forall k \in \mathbb{N}, \quad x(k+1) = Fx(k) + w(k+1), \quad (13b)$$

where  $x_0$  is the initial state,  $\tilde{P}_0$  the covariance matrix of the initial uncertainty,  $F \in \mathbb{R}^{d \times d}$  the evolution matrix, and  $w(k)$  the process noise at time  $k$ . The system is estimated by a network of  $N$  agents. The agents are equipped with sensors to observe the state, and have (limited) communication capabilities with their neighbors in the network. As the estimation algorithms are symmetrical between the agents, a focus is made on one particular agent indexed by  $i \in \{1, \dots, N\}$ .