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# Proximal optimization of OT-regularized cost functions for multichannel signal recovery

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## 1 Introduction

### 1.1 Problem Statement

We consider a multichannel signal viewed as a sequence of monovariate signals  $(\mathbf{y}_n)_{n \in \llbracket 1, N \rrbracket}$  with  $N$  the number of channels. Each signal in the sequence  $\mathbf{y}_n \in \mathbb{R}^K$  can be represented in dictionary  $A \in \mathbb{R}^{K \times M}$ . The inverse problem aims to retrieve  $(\mathbf{x}_n)_{n \in \llbracket 1, N \rrbracket}$  from given signal  $(\mathbf{y}_n)_{n \in \llbracket 1, N \rrbracket}$  and dictionary  $A$ .

We propose two variational formulations which make use of Optimal Transport (OT) regularization. These are Quadratic Programs that we solve using the Condat-Vu's Primal-Dual algorithm.

### 1.2 Definition & notations

**Notation:**

- $\mathbf{1}_M$  is the vector of ones of size  $M \times 1$
- $\langle \cdot, \cdot \rangle_F$  is the Frobenius inner product
- $A \otimes B$  is the Kronecker product between two matrices of arbitrary sizes
- $\mathbb{A} \bar{\times}_n \mathbf{b}$  is the  $n$ -mode vector product of tensor  $\mathbb{A} \in \mathbb{R}^{I_1 \times \dots \times I_n \times \dots \times I_N}$  (of order  $N$ ) with vector  $\mathbf{b} \in \mathbb{R}^{I_n}$  [2].

$$(\mathbb{A} \bar{\times}_n \mathbf{b})_{i_1 \dots i_{n-1} i_{n+1} \dots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 i_2 \dots i_K} \mathbf{b}_{i_n}.$$

Each mode- $n$  fiber of tensor  $\mathbb{A}$  is multiplied by  $\mathbf{b}$  (inner product). The order of the resulting tensor  $\mathbb{A} \bar{\times}_n \mathbf{b}$  is  $N - 1$ .

**Definition 1.1** (Spectral Norm). *Let  $A$  be a matrix. The spectral norm  $\|\cdot\|$  of  $A$  is defined as:*

$$\begin{aligned} \|A\| &= \sup \{ \|Ax\|_2 : \|x\|_2 \leq 1 \} \\ \|A\| &= \sqrt{\lambda_{\max}(A^\top A)} \end{aligned}$$

with  $\lambda_{\max}(A^\top A)$  the largest eigenvalue of  $A^\top A$  (always positive because  $A^\top A$  is a PSD matrix).

**Basic Properties:**

- Given  $A, B$  matrices :  $\|A \otimes B\| = \|A\| \|B\|$ .
- Let  $A$  be a matrix,  $Q$  and  $P$  permutation matrices:  $\|QAP\| = \|A\|$
- Let  $A, B, C, D$  be matrices with consistent size  $(A \otimes B)(C \otimes D) = (AC \otimes BD)$

See reviews [1] & [2] for more details.

## 2 Condat-Vu's Algorithm

Condat-Vu's algorithm [3] is a Primal-Dual algorithm to solve the following type of problem:

$$\min_{\mathbf{z}} h(\mathbf{z}) + f(\mathbf{z}) + \sum_{i=0}^{I-1} g_i(L_i \mathbf{z}), \quad (1)$$

with the following terms in each part of the splitting:

- $h$  is convex and differentiable
- $f$  and  $(g_i)_{i \in \llbracket 0, I-1 \rrbracket}$  are convex
- $(L_i)_{i \in \llbracket 0, I-1 \rrbracket}$  are linear operators

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### Algorithm 1 Condat-Vu's algorithm

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**Initialize:**  $z^{(0)} = \mathbf{0}, \quad \forall i, u_i^{(0)} = \mathbf{0}$   
**for**  $k = 1, 2, \dots$  **do**  

$$z^{(k+1)} = \text{prox}_{\tau f}(z^{(k)} - \tau \nabla h(z^{(k)}) - \tau \sum_{i=1}^I L_i^* u_i^{(k)})$$
  
**for**  $i = 0, 1, \dots, I - 1$  **do**  

$$u_i^{(k+1)} = \text{prox}_{\sigma_i g_i^*} \left( u_i^{(k)} + \sigma_i L_i (2z^{(k+1)} - z^{(k)}) \right)$$
  
**end for**  
**end for**

---

Nesterov acceleration technique can be added to the algorithm.

**Proposition 1.** *Gradient step of algorithm should verify the condition expressed in [3, Th. 9.7] to ensure convergence of iterates:*

$$\forall i \in \llbracket 1, I \rrbracket, \quad \sigma_i \geq 0, \quad \tau \geq 0, \quad (2)$$

$$\tau \leq \frac{1}{\left\| Q + \sum_{i=0}^{I-1} \sigma_i L_i^\top L_i \right\|} \quad (3)$$

Using the triangular inequality, we propose a slightly more restrictive condition on gradient steps, easier to compute:

$$\tau \leq \frac{1}{\|Q\| + \sum_{i=0}^{I-1} \sigma_i \|L_i^\top L_i\|} \quad (4)$$

Clearly, (4) implies (3)

### 3 Problem adaptation to Condat-Vu's framework: First order

The first OT regularized problem results in the following Quadratic Program:

$$\min_{\substack{\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}_+^M \\ P_1, \dots, P_{N-1} \in \mathbb{R}_+^{M \times M}}} \sum_{n=1}^N \|\mathbf{y}_n - A\mathbf{x}_n\|_2^2 + \omega \sum_{n=1}^{N-1} \langle P_n, C \rangle_F, \quad (5a)$$

$$\mathbf{s.t.} \quad \forall n \in \llbracket 1, N \rrbracket, \quad \mathbf{1}_M^\top \mathbf{x}_n = 1, \quad (5b)$$

$$\forall n \in \llbracket 1, N-1 \rrbracket, \quad P_n \bar{\times}_2 \mathbf{1}_M = \mathbf{x}_n, \quad P_n \bar{\times}_1 \mathbf{1}_M = \mathbf{x}_{n+1}. \quad (5c)$$

Let us introduce global column vector  $\mathbf{z}$ , which is an unfolded version gathering all variables together:

$$\begin{aligned} \mathbf{x} &= \text{Vect}(\mathbf{x}_1, \dots, \mathbf{x}_N) \\ \mathbf{p} &= \text{Vect}(P_1, \dots, P_{N-1}) \\ \mathbf{z} &= \text{Vect}(\mathbf{x}, \mathbf{p}) \end{aligned}$$

**Proposition 2.** *Criteria of problem (5) can be reformulated with respect to global variable  $\mathbf{z}$  as  $h(\mathbf{z})$ :*

$$h(\mathbf{z}) = \frac{1}{2} \mathbf{z}^\top Q \mathbf{z} + q^\top \mathbf{z} + cst, \quad (6)$$

with  $h$  quadratic and:

$$\begin{aligned} Q &= \begin{pmatrix} I_N \otimes 2A^\top A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ q &= \text{Vect}(-2A^\top \mathbf{y}_1, \dots, -2A^\top \mathbf{y}_N, \omega \mathbf{c}, \dots, \omega \mathbf{c}) \\ cst &= \sum_{n=1}^N \|\mathbf{y}_n\|_2 \end{aligned}$$

**Proposition 3.** *Problem (5) can be reformulated into:*

$$\min_z h(z), \quad (7a)$$

$$\mathbf{s.t.} \quad z \geq 0, \quad L_1 z = \mathbf{1}_N, \quad L_0 z = \mathbf{0}, \quad (7b)$$

with  $L_1$  and  $L_0$  linear operators. Then, it can be easily expressed in the Condat-Vu's framework:

$$\min_z h(z) + f(z) + g_0(L_0 z) + g_1(L_1 z), \quad (8)$$

with:

- $I = 2$ ,
- $g_0 = \iota_{\{\mathbf{0}\}}$ ,  $g_1 = \iota_{\{\mathbf{1}_N\}}$  indicator functions on vectors,
- $f = \iota_{\mathbb{R}_+^z}$  indicator function on the positive orthant.

#### 3.1 Expression of linear operators $L_1$ and $L_0$

Constraints (5b) can be expressed with respect to  $\mathbf{z}$  introducing operator  $L_1$ :

$$L_1 \mathbf{z} = \begin{bmatrix} I_N \otimes \mathbf{1}_M^\top & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \mathbf{1}_N,$$

Constraints (5c) are reformulated:

$$\begin{aligned} & \forall n \in \llbracket 1, N-1 \rrbracket, \quad P_n \bar{\times}_2 \mathbf{1}_M = \mathbf{x}_n, \quad P_n \bar{\times}_1 \mathbf{1}_M = \mathbf{x}_{n+1}, \\ \Leftrightarrow & \forall n \in \llbracket 1, N-1 \rrbracket, \quad S_2 \mathbf{p}_n = \mathbf{x}_n, \quad S_1 \mathbf{p}_n = \mathbf{x}_{n+1}, \\ \Leftrightarrow & \quad \quad \quad S_2 \mathbf{p} = M_2 \mathbf{x}, \quad S_1 \mathbf{p} = M_1 \mathbf{x}, \end{aligned}$$

with  $M_k$  selecting operators on  $\mathbf{x}$ :

$$M_1 = [\mathbf{0}, I_{(N-1)M}], \quad M_2 = [I_{(N-1)M}, \mathbf{0}],$$

with  $S_1$  sum operator on the first dimension (rows) and  $S_2$  sum operator on the second dimension (columns).

$$S_2 \mathbf{p} = \text{Vect}(S_2 \mathbf{p}_1, \dots, S_2 \mathbf{p}_{N-1}) \quad S_1 \mathbf{p} = \text{Vect}(S_1 \mathbf{p}_1, \dots, S_1 \mathbf{p}_{N-1})$$

Constraints (5c) are expressed with respect to  $\mathbf{z}$  introducing operator  $L_0$ :

$$L_0 \mathbf{z} = \begin{bmatrix} -M_2 & S_2 \\ -M_1 & S_1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = \mathbf{0}_{2M(N-1)},$$

$S_2$  and  $M_2$  are columns permutation of matrix  $S_1$  and  $M_1$  respectively. We have:

$$S_1 = I_{(N-1)M} \otimes \mathbf{1}_M^\top$$

Introducing  $H = \begin{bmatrix} -M_1 & S_1 \end{bmatrix}$ , there exist  $R$  a permutation matrix such that:

$$L_0 = \begin{bmatrix} HR \\ H \end{bmatrix}$$

### 3.2 Condition on gradient step

The condition (4) is adapted for the above problem (8):

$$\tau \leq \frac{1}{\|Q\| + \sigma_0 \|L_0^\top L_0\| + \sigma_1 \|L_1^\top L_1\|}. \quad (9)$$

#### 3.2.1 Computation of $\|Q\|$

$$\|Q\| = \left\| \begin{pmatrix} I_N \otimes 2A^\top A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right\| = \|I_N \otimes 2A^\top A\| = \|I_N\| \|2A^\top A\| = 2 \|A^\top A\|.$$

#### 3.2.2 Computation of $\|L_1^\top L_1\|$

**Lemma 1.** Let  $\boldsymbol{\rho}_M \in \{-1, 1\}^M$  be a vector of size  $M$  whose components are either  $-1$  or  $1$ . then:

$$\|\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top\| = M$$

*Proof.* Notice that:

$$\forall \mathbf{x}, \quad \|\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top \mathbf{x}\|_2 = \|\boldsymbol{\rho}_M \langle \boldsymbol{\rho}_M, \mathbf{x} \rangle\|_2 = |\langle \boldsymbol{\rho}_M, \mathbf{x} \rangle| \|\boldsymbol{\rho}_M\|_2$$

Using the Cauchy-Schwarz inequality, we get:

$$\|\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top \mathbf{x}\|_2 \leq \|\boldsymbol{\rho}_M\|_2^2 \|\mathbf{x}\|_2$$

As  $\|\boldsymbol{\rho}_M\|_2^2 = M$ , we have:

$$\|\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top \mathbf{x}\|_2 \leq M \|\mathbf{x}\|_2 \quad (10)$$

Clearly,  $\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top$  is a PSD matrix. Thus, all eigenvalues of  $\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top$  are positive. According to (10), the eigenvalues are bound by  $M$ . Also, note that:

$$\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top \boldsymbol{\rho}_M = \boldsymbol{\rho}_M \langle \boldsymbol{\rho}_M, \boldsymbol{\rho}_M \rangle = \boldsymbol{\rho}_M \|\boldsymbol{\rho}_M\|_2^2 = M \boldsymbol{\rho}_M$$

Thus,  $M$  is the largest eigenvalue of  $\boldsymbol{\rho}_M \boldsymbol{\rho}_M^\top$ . ■

$$\begin{aligned} \|L_1^\top L_1\| &= \left\| \begin{bmatrix} (I_N \otimes \mathbf{1}_M^\top)^\top I_N \otimes \mathbf{1}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\| \\ &= \|(I_N \otimes \mathbf{1}_M^\top)^\top (I_N \otimes \mathbf{1}_M)\| \end{aligned}$$

Using Lemma 1:

$$\|(I_N \otimes \mathbf{1}_M^\top)^\top I_N \otimes \mathbf{1}_M\| = \|I_N\| \|\mathbf{1}_M \mathbf{1}_M^\top\| = \|\mathbf{1}_M \mathbf{1}_M^\top\| = M,$$

consequently:

$$\|L_1^\top L_1\| = M. \quad (11)$$

### 3.2.3 Computation of $\|L_0^\top L_0\|$

Let  $E \in \mathbb{N}$  be an arbitrary dimension. We introduce matrix:  $H_{E,M} = \begin{bmatrix} I_E & I_E \otimes -\mathbf{1}_M^\top \\ I_E \otimes -\mathbf{1}_M & I_E \otimes \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix}$ .

**Lemma 2.** *The spectral norm of matrix  $H_{E,M}$  is equal to  $M + 1$ .*

*Proof.* There exist permutation matrices  $R_1$  and  $R_2$  on rows and columns respectively such that:

$$\begin{aligned} H_{E,M} &= \begin{bmatrix} I_E & I_E \otimes -\mathbf{1}_M^\top \\ I_E \otimes -\mathbf{1}_M & I_E \otimes \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix} \\ H_{E,M} &= \begin{bmatrix} I_E & I_E \otimes -\mathbf{1}_M^\top \\ I_E \otimes -\mathbf{1}_M & I_E \otimes \mathbf{1}_M \otimes \mathbf{1}_M^\top \end{bmatrix} \\ H_{E,M} &= \begin{bmatrix} I_E & I_E \otimes -\mathbf{1}_M^\top \\ R_1(-\mathbf{1}_M \otimes I_E) & R_1(\mathbf{1}_M \otimes I_E \otimes \mathbf{1}_M^\top) \end{bmatrix} \\ H_{E,M} &= \begin{bmatrix} I_E & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \begin{bmatrix} I_E & I_E \otimes -\mathbf{1}_M^\top \\ -\mathbf{1}_M \otimes I_E & \mathbf{1}_M \otimes I_E \otimes \mathbf{1}_M^\top \end{bmatrix} \\ H_{E,M} &= \begin{bmatrix} I_E & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \begin{bmatrix} I_E & (-\mathbf{1}_M^\top \otimes I_E) R_2 \\ -\mathbf{1}_M \otimes I_E & (\mathbf{1}_M \otimes \mathbf{1}_M^\top \otimes I_E) R_2 \end{bmatrix} \\ H_{E,M} &= \begin{bmatrix} I_E & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \begin{bmatrix} I_E & -\mathbf{1}_M^\top \otimes I_E \\ -\mathbf{1}_M \otimes I_E & \mathbf{1}_M \mathbf{1}_M^\top \otimes I_E \end{bmatrix} \begin{bmatrix} I_E & \mathbf{0} \\ \mathbf{0} & R_2 \end{bmatrix} \end{aligned}$$

As spectral norm is invariant by permutation:

$$\begin{aligned} \|H_{E,M}\| &= \left\| \begin{bmatrix} I_E & \mathbf{0} \\ \mathbf{0} & R_1 \end{bmatrix} \begin{bmatrix} I_E & -\mathbf{1}_M^\top \otimes I_E \\ -\mathbf{1}_M \otimes I_E & \mathbf{1}_M \mathbf{1}_M^\top \otimes I_E \end{bmatrix} \begin{bmatrix} I_E & \mathbf{0} \\ \mathbf{0} & R_2 \end{bmatrix} \right\| \\ \|H_{E,M}\| &= \left\| \begin{bmatrix} I_E & -\mathbf{1}_M^\top \otimes I_E \\ -\mathbf{1}_M \otimes I_E & \mathbf{1}_M \mathbf{1}_M^\top \otimes I_E \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 1 & -\mathbf{1}_M^\top \\ -\mathbf{1}_M & \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix} \otimes I_E \right\| \\ &= \left\| \begin{bmatrix} 1 & -\mathbf{1}_M^\top \\ -\mathbf{1}_M & \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix} \right\| \|I_E\| \end{aligned}$$

Using Lemma 1:

$$\|H_{E,M}\| = \left\| \begin{bmatrix} 1 & -\mathbf{1}_M^\top \\ -\mathbf{1}_M & \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix} \right\| = \|\boldsymbol{\rho}_{M+1} \boldsymbol{\rho}_{M+1}^\top\| = M + 1$$

■

$$\begin{aligned} \|L_0^\top L_0\| &= \left\| \begin{bmatrix} H^\top & (HR)^\top \end{bmatrix} \begin{bmatrix} H \\ HR \end{bmatrix} \right\| \\ \|L_0^\top L_0\| &= \|H^\top H + R^\top H^\top HR\| \\ \|L_0^\top L_0\| &\leq \|H^\top H\| + \|R^\top H^\top HR\| \end{aligned}$$

The spectral norm is invariant by permutation. Thus, we have:

$$\|L_0^\top L_0\| \leq 2 \|H^\top H\|$$

$$\begin{aligned} H^\top H &= \begin{bmatrix} M_1^\top M_1 & -M_1^\top S_1 \\ -S_1^\top M_1 & S_1^\top S_1 \end{bmatrix} \\ \|H^\top H\| &= \left\| \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{(N-1)M} & -I_{(N-1)M} \otimes \mathbf{1}_M^\top \\ \mathbf{0} & -I_{(N-1)M} \otimes \mathbf{1}_M & I_{(N-1)M} \otimes \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix} \right\| \\ \|H^\top H\| &= \left\| \begin{bmatrix} I_{(N-1)M} & I_{(N-1)M} \otimes -\mathbf{1}_M^\top \\ I_{(N-1)M} \otimes -\mathbf{1}_M & I_{(N-1)M} \otimes \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix} \right\| \end{aligned}$$

We have:

$$\|H^\top H\| = \|H_{E,M}\| \quad \text{with } E = (N-1)M$$

Using Lemma 2:

$$\|H^\top H\| = \|H_{E,M}\| = M + 1$$

and:

$$\|L_0^\top L_0\| \leq 2(M + 1) \tag{12}$$

### 3.2.4 Condition on gradient step for problem (5)

Condition below is a weaker sufficient condition than (9)

$$\tau \leq \frac{1}{2 \|A^\top A\| + \sigma_1 M + 2\sigma_0(M + 1)} \tag{13}$$

## 4 Second order problem

$$\min_{\substack{\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}_+^M \\ P_1, \dots, P_{N-1} \in \mathbb{R}_+^{M \times M} \\ \mathbb{P}_1, \dots, \mathbb{P}_{N-2} \in \mathbb{R}_+^{M \times M \times M}}} \sum_{n=1}^N \|\mathbf{y}_n - A\mathbf{x}_n\|_2^2 + \omega \sum_{n=1}^{N-2} \langle \mathbb{P}_n, \mathbf{C} \rangle_F, \tag{14a}$$

$$\text{s.t. } \forall n \in \llbracket 1, N \rrbracket, \quad \mathbf{1}_M^\top \mathbf{x}_n = 1, \tag{14b}$$

$$\forall n \in \llbracket 1, N-1 \rrbracket, \quad P_n \bar{\times}_2 \mathbf{1}_M = \mathbf{x}_n, \quad P_n \bar{\times}_1 \mathbf{1}_M = \mathbf{x}_{n+1}. \tag{14c}$$

$$\forall n \in \llbracket 1, N-2 \rrbracket, \quad \mathbb{P}_n \bar{\times}_3 \mathbf{1}_M = P_n, \quad \mathbb{P}_n \bar{\times}_1 \mathbf{1}_M = P_{n+1}. \tag{14d}$$

## 5 Problem adaptation to Condat-Vu's framework: Second order

Let us introduce global variable  $\mathbf{z}$ :

$$\begin{aligned}\mathbf{x} &= \text{Vect}(\mathbf{x}_1, \dots, \mathbf{x}_N) \\ \mathbf{p} &= \text{Vect}(P_1, \dots, P_{N-1}) \\ \mathbb{p} &= \text{Vect}(\mathbb{P}_1, \dots, \mathbb{P}_{N-2}) \\ \mathbf{z} &= \text{Vect}(\mathbf{x}, \mathbf{p}, \mathbb{p})\end{aligned}$$

**Proposition 4.** *Criteria  $h$  can be reformulated with respect to global variable  $\mathbf{z}$ :*

$$h(\mathbf{z}) = \frac{1}{2} \mathbf{z}^\top Q \mathbf{z} + \mathbf{q}^\top \mathbf{z} + cst, \quad (15)$$

with quadratic, linear and constant term respectively:

$$\begin{aligned}Q &= \begin{pmatrix} I_N \otimes 2A^\top A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ \mathbf{q} &= (-2\mathbf{y}_1^\top A, \dots, -2\mathbf{y}_N^\top A, 0 \dots 0, \omega \text{Vect}(\mathbb{C})^\top, \dots, \omega \text{Vect}(\mathbb{C})^\top)^\top \\ cst &= \sum_{n=1}^N \mathbf{y}_n \mathbf{y}_n^\top\end{aligned}$$

**Proposition 5.** *Problem (14) can be reformulated into:*

$$\min_z h(z), \quad (16a)$$

$$\mathbf{s.t.} \quad z \geq 0 \quad L'_1 z = \mathbf{1}_N, \quad L_{01} z = \mathbf{0}, \quad L_{02} z = \mathbf{0}, \quad (16b)$$

with  $L'_1, L_{01}$  and  $L_{02}$  linear operators. Then, it can be easily expressed in the Condat-Vu's framework:

$$\min_z h(z) + f(z) + g_1(L'_1 z) + g_0(L_{01} z) + g'_0(L_{02} z), \quad (17)$$

- $I = 3$ ,
- $g_1 = \iota_{\mathbf{1}_N}, g_0 = \iota_{\mathbf{0}}, g'_0 = \iota_{\mathbf{0}}$  indicator functions on vectors,
- $f = \iota_{\mathbb{R}_+}$  indicator function on the positive orthant.

### 5.0.1 Expression of linear operator $L'_1, L_{01}$ and $L_{02}$

Constraints (5b) can be expressed with respect to  $\mathbf{z}$  introducing operator  $L_1$ :

$$L'_1 \mathbf{z} = \begin{bmatrix} I_N \otimes \mathbf{1}_M^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \\ \mathbb{p} \end{bmatrix} = \mathbf{1}_N,$$

with:  $I_N \otimes \mathbf{1}_M^\top = I_N \otimes \mathbf{1}_M^\top$ .

Constraints (5c) are equivalent to constraints (14c). Thus  $L_{01}$  can be expressed with operator  $L_0$ :

$$L_{01} \mathbf{z} = \begin{bmatrix} L_0 & | & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \\ \mathbb{p} \end{bmatrix}$$



Constraints (14d) are reformulated:

$$\begin{aligned} & \forall n \in \llbracket 1, N-2 \rrbracket, \quad \mathbb{P}_n \bar{\times}_3 \mathbf{1}_M = P_n, \quad \mathbb{P}_n \bar{\times}_1 \mathbf{1}_M = P_{n+1}, \\ \Leftrightarrow & \forall n \in \llbracket 1, N-2 \rrbracket, \quad \Sigma_2 \mathbb{P}_n = \mathbf{p}_n, \quad \Sigma_1 \mathbb{P}_n = \mathbf{p}_{n+1}, \\ & \Leftrightarrow \quad \tilde{\Sigma}_3 \mathbb{P} = K_3 \mathbf{p}, \quad \tilde{\Sigma}_3 \mathbb{P} = K_1 \mathbf{p}. \end{aligned}$$

with:

$$K_1 = [\mathbf{0}, I_{(N-2)M^2}], \quad K_2 = [I_{(N-2)M^2}, \mathbf{0}],$$

and:

$$\tilde{\Sigma}_1 = I_{(N-2)} \otimes \Sigma_1 = I_{(N-2)M^2} \otimes \mathbf{1}_M^\top, \quad \tilde{\Sigma}_3 = I_{(N-2)} \otimes \Sigma_3 = I_{(N-2)} \otimes \mathbf{1}_M^\top \otimes I_{M^2},$$

Constraints (5c) are expressed with respect to  $\mathbf{z}$  introducing operator  $L_0$ :

$$L_{02} \mathbf{z} = \begin{bmatrix} \mathbf{0} & -K_3 & \tilde{\Sigma}_3 \\ \mathbf{0} & -K_1 & \tilde{\Sigma}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \\ \mathbb{P} \end{bmatrix} = \mathbf{0}_{2M^2(N-2)},$$

$\tilde{\Sigma}_2$  and  $K_2$  are columns permutation of matrix  $\tilde{\Sigma}_1$  and  $K_1$  respectively. Introducing  $H' = \begin{bmatrix} \mathbf{0} & -K_1 & \tilde{\Sigma}_1 \end{bmatrix}$ , there exist  $R'$  a permutation matrix such that:

$$L_{02} = \begin{bmatrix} H' R' \\ H' \end{bmatrix}$$

## 5.1 Condition on gradient step

The condition (4) is adapted for problem (17):

$$\tau \leq \frac{1}{\|Q\| + \sigma_1 \|L_1'^\top L_1'\| + \sigma_{01} \|L_{01}^\top L_{01}\| + \sigma_{02} \|L_{02}^\top L_{02}\|} \quad (18)$$

### 5.1.1 Computation of $\|Q\|$

$$\|Q\| = \left\| \begin{pmatrix} I_N \otimes 2A^\top A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \right\| = 2 \|A^\top A\|,$$

### 5.1.2 Computation of $\|L_1'^\top L_1'\|$

$$\|L_1'^\top L_1'\| = \left\| \begin{bmatrix} (I_N \otimes \mathbf{1}_M^\top)^\top I_N \otimes \mathbf{1}_M^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right\| = \|(I_N \otimes \mathbf{1}_M^\top)^\top I_N \otimes \mathbf{1}_M^\top\| = M,$$

### 5.1.3 Computation of $\|L_{01}^\top L_{01}\|$

$$\|L_{01}^\top L_{01}\| = \left\| \begin{bmatrix} L_0^\top L_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\| = \|L_0^\top L_0\| \leq 2(M+1),$$

#### 5.1.4 Computation of $\|L_{02}^\top L_{02}\|$

Thus, we have:

$$\|L_{02}^\top L_{02}\| \leq 2 \|H'^\top H'\|$$

$$H'^\top H' = \begin{bmatrix} K_1^\top K_1 & -K_1^\top \tilde{\Sigma}_1 \\ -\tilde{\Sigma}_1^\top M_1 & \tilde{\Sigma}_1^\top \tilde{\Sigma}_1 \end{bmatrix}$$

$$\|H'^\top H'\| = \left\| \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{(N-2)M^2} & -I_{(N-2)M^2} \otimes \mathbf{1}_M^\top \\ \mathbf{0} & -I_{(N-2)M^2} \otimes \mathbf{1}_M & I_{(N-2)M^2} \otimes \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix} \right\|$$

$$\|H'^\top H'\| = \|H_{E,M}\| \quad \text{with } E = (N-2)M^2$$

Recall for matrix  $H_{E,M}$ :  $H_{E,M} = \begin{bmatrix} I_E & I_E \otimes -\mathbf{1}_M^\top \\ I_E \otimes -\mathbf{1}_M & I_E \otimes \mathbf{1}_M \mathbf{1}_M^\top \end{bmatrix}$

Using Lemma 2, we have:

$$\|H'^\top H'\| = \|H_{E,M}\| = M + 1$$

and:

$$\|L_{02}^\top L_{02}\| \leq 2(M + 1)$$

#### 5.1.5 Condition on gradient steps for problem (14)

Condition below is a weaker sufficient condition than (18)

$$\tau \leq \frac{1}{2\|A^\top A\| + \sigma_1 M + (\sigma_{01} + \sigma_{02})2(M + 1)} \quad (19)$$

## References

- [1] N. D. Sidiropoulos, L. De Lathauwer, X. Fu, K. Huang, E. E. Papalexakis, and C. Faloutsos, “Tensor decomposition for signal processing and machine learning”, *IEEE Transactions on Signal Processing*, vol. 65, no. 13, pp. 3551–3582, 2017.
- [2] T. G. Kolda and B. W. Bader, “Tensor decompositions and applications”, *SIAM Review*, vol. 51, no. 3, pp. 455–500, 2009.
- [3] L. Condat, D. Kitahara, A. Contreras, and A. Hirabayashi, “Proximal splitting algorithms for convex optimization: A tour of recent advances, with new twists”, *SIAM Review*, vol. 65, no. 2, pp. 375–435, 2023.