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Criteria for a split real polynomial

J.-M. Billiot E. Fontenas *

Abstract

In this article, we establish necessary and sufficient conditions for a polynomial of degree n to have exactly n real roots. A complete study of polynomials of degree five is carried out. The results are compared with those obtained using Sturm sequences.

Keywords: Polynomials with only real roots, Polynomial sequences, Interlacing method, Sturm's theorem, Euclidean division.

1 Introduction

The study of polynomials with only real roots is the subject of much research. [8] proposes results concerning sequences of polynomials and obtains sufficient conditions for these polynomials to have only real roots. Their result is based on the notion of interlacing roots. Recently, [5], [6] have studied cubic, quartic and quintic polynomials and have proposed conditions on the coefficients associated with the Sturm sequences which will determine the multiplicities of the real and complex roots as well as the order of the real roots with respect to the multiplicity.

In this article, we give necessary and sufficient conditions on the coefficients of a polynomial of degree n so that this polynomial has n distinct real roots. The idea here is to construct a sequence of polynomials with real roots, introducing the same assumption about the last constant of the polynomial constructed. We take as an example the case of a polynomial of degree three and then explain the general idea. A complete study of fifth-order polynomials is carried out, highlighting the importance of multiple roots to obtain bounds on the coefficients of polynomials. Our method is then compared with that proposed by Sturm [9]. Sturm's sequences allow us to obtain conditions on the parameters of a polynomial (very effective when the coefficients are known). Unlike the result proposed here, these conditions are not solved for any polynomial.

2 Method

For simplicity's sake (but our method also works in the general case), we are interested in polynomials where the coefficient of the highest monomial is equal to one (otherwise, we simply

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divide the polynomial by this coefficient). We also assume that the coefficient of the second-highest monomial is zero: an appropriate change of variable makes this possible. This means that the sum of the roots is zero. Depending on the context, the notation P_n or $P_n(x)$ will be used to designate the same polynomial of degree n in the variable x .

We recall the notion of interlaced polynomial roots:

Definition 1 Given both polynomials P and Q of order n and $n-1$ respectively and $\{\alpha_i\}_{1 \leq i \leq n}$ and $\{\beta_j\}_{1 \leq j \leq n-1}$ be all real roots of P and Q in nonincreasing order respectively. The roots of Q are interlaced with the roots of P if

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \dots \leq \beta_{n-2} \leq \alpha_{n-1} \leq \beta_{n-1} \leq \alpha_n.$$

2.1 Discussion for polynomials of degree two and three

1. Consider $P_2(x) = x^2 + p/3$. This polynomial has two distinct real roots as soon as $p < 0$.
2. For the polynomial $P_3(x) = x^3 + px + q$, by the Euclidean division by $P_3'(x) = 3P_2(x)$, it comes then

$$P_3(x) = \frac{x}{3}P_3'(x) - R_1(x) = xP_2(x) - R_1(x)$$

with $R_1(x) = -(\frac{2}{3}px + q)$. The polynomial P_3 has three real roots if and only if P_2 has two real roots and if R_1 has a real root interlaced with those of P_2 . Because of the previous remark, P_2 has two real roots if $p < 0$.

It then remains to fix the constant q so that the root of R_1 is interlaced with those of P_2 . The root of R_1 is $\beta_1^{(1)} = -\frac{3q}{2p}$ and is interlaced with those of P_2 if

$$-\frac{3q}{2p} \in \left] \alpha_1^{(2)} = -\left(-\frac{p}{3}\right)^{1/2}; \alpha_2^{(2)} = \left(-\frac{p}{3}\right)^{1/2} \right[$$

or

$$q \in \left] -\frac{2}{3}p\alpha_1^{(2)}; -\frac{2}{3}p\alpha_2^{(2)} \right[.$$

If we call R_1^0 the function defined by $R_1^0(x) = R_1(x) + q = -\frac{2}{3}px$, we have that

$$q \in]R_1^0(\alpha_1^{(2)}); R_1^0(\alpha_2^{(2)})[.$$

This means that $R_1(\alpha_1^{(2)}) < 0$ and that $R_1(\alpha_2^{(2)}) > 0$. Note here that the result means that the discriminant of P_3 namely $\Delta(P_3) = -(4p^3 + 27q^2)$ is not negative. Moreover, if $p < 0$, it is well known that only cases of multiple roots for P_3 correspond to $\beta_1^{(1)} = \alpha_1^{(2)}$ or $\beta_1^{(1)} = \alpha_2^{(2)}$ according with the sign of q . Remark that this real double root is also a root of P_2 and R_1 .

2.2 General idea

We construct a sequence of polynomials $(P_n)_{n \geq 2}$ so that $\forall n \geq 3, P_{n-1}(x) = \frac{1}{n}P_n'(x)$:

$$\begin{cases} P_2(x) = x^2 + c_0/3 \\ P_3(x) = x^3 + c_0x + c_1 \\ P_4(x) = x^4 + \frac{4!}{2!3!}c_0x^2 + \frac{4!}{1!3!}c_1x + \frac{4!}{3!0!}c_2 \\ P_5(x) = x^5 + \frac{5!}{3!3!}c_0x^3 + \frac{5!}{3!2!}c_1x^2 + \frac{5!}{1!3!}c_2x + \frac{5!}{0!3!}c_3 \\ \dots \\ P_n(x) = x^n + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \dots + a_0 \end{cases}$$

where

$$\forall k \in \{0, \dots, n-2\}, \quad a_k = \frac{n!}{k!3!} c_{n-2-k}.$$

From a more general point of view, the main result is based on the following result:

Proposition 1

1. Set $P_n(x) = xQ_{n-1}(x) - R_{n-2}(x)$. If Q_{n-1} has $n-1$ real roots and if R_{n-2} has $n-2$ real roots interlaced with those of Q_{n-1} , then P_n has n real roots.
2. For a polynomial of degree n , to have n real roots, its derivative must have $n-1$ distinct real roots.

Now consider our sequence of polynomials and divide P_n Euclidean by $P_{n-1}(x) = P'_n(x)/n$: $P_n(x) = \frac{x}{n}P'_n(x) - R_{n-2}(x)$. P_n has n distinct real roots as soon as its derivative has $n-1$ distinct real roots and those of R_{n-2} denoted by $\beta_1^{(n-2)} \dots \beta_{n-2}^{(n-2)}$ are interlaced with those of P'_n .

The coefficient of highest degree of R_{n-2} is $-\frac{2}{n}a_{n-2}$. If $a_{n-2} < 0$ (as the sign of the discriminant of P_2) and if R_{n-2} has $n-2$ real distinct and interlaced roots with P'_n then P_n has n distinct real roots. If we write:

$$R_{n-2}(x) = \frac{1}{n-2}(x - \beta_1^{(1)})R'_{n-2}(x) - T_{n-4}(x),$$

as $R'_{n-2}(x) = (n-1)R_{n-3}(x)$, R'_{n-2} has $n-3$ distinct real roots. It is therefore necessary to ensure that T_{n-4} has $n-4$ distinct real roots denoted by $\gamma_1^{(n-4)} \dots \gamma_{n-4}^{(n-4)}$ and besides interlaced with those $\beta_1^{(n-3)} \dots \beta_{n-3}^{(n-3)}$ of R'_{n-2} and the highest degree coefficient of T_{n-4} is positive and so on.

We will specify this construction later when studying polynomials of degree five when we will explain how to choose the last coefficient a_0 of the polynomial P_n iteratively. This leads us now to the presentation of our main result.

3 Main result

First of all, we present a characteristic property of interlacing roots. Then, we explain how choosing the integration constant a_0 using the extrema of P_n and the remainder of Euclidean division of P_n by P'_n . At last, we propose some necessary and sufficient assumptions on the coefficients to obtain n distinct real roots for P_n .

Proposition 2 *Let P_n be a polynomial and let*

$$P_n(x) = \frac{x}{n}P'_n(x) - R_{n-2}(x) = xP_{n-1}(x) - R_{n-2}(x).$$

Denote by $\alpha_1^{(n-1)} \dots \alpha_{n-1}^{(n-1)}$, the $n-1$ distinct real roots of P_{n-1} and $\beta_1^{(n-2)} \dots \beta_m^{(n-2)}$, the $n-2$ distinct real roots of R_{n-2} .

If n is even,

$$\alpha_1^{(n-1)} < \beta_1^{(n-2)} < \alpha_2^{(n-1)} < \beta_2^{(n-2)} < \dots < \alpha_{n-2}^{(n-1)} < \beta_{n-2}^{(n-2)} < \alpha_{n-1}^{(n-1)}$$

$$\Leftrightarrow \sup_{k \in \{1, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) < 0 < \inf_{k \in \{0, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}).$$

If n is odd,

$$\alpha_1^{(n-1)} < \beta_1^{(n-2)} < \alpha_2^{(n-1)} < \beta_2^{(n-2)} < \dots < \alpha_{n-2}^{(n-1)} < \beta_{n-2}^{(n-2)} < \alpha_{n-1}^{(n-1)}$$

$$\Leftrightarrow \sup_{k \in \{0, \dots, \frac{n-3}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}) < 0 < \inf_{k \in \{1, \dots, \frac{n-1}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}).$$

Proof: we only present here the proof for n even. Suppose that R_{n-2} admits $n-2$ distinct roots $\beta_1^{(n-2)} \dots \beta_{n-2}^{(n-2)}$ interlaced with those of P_{n-1} :

$$\alpha_1^{(n-1)} < \beta_1^{(n-2)} < \alpha_2^{(n-1)} < \beta_2^{(n-2)} < \dots < \alpha_{n-2}^{(n-1)} < \beta_{n-2}^{(n-2)} < \alpha_{n-1}^{(n-1)}.$$

So,

$$\sup_{k \in \{0, \dots, \frac{n-2}{2}\}} P_n(\alpha_{2k+1}^{(n-1)}) < 0 < \inf_{k \in \{1, \dots, \frac{n-2}{2}\}} P_n(\alpha_{2k}^{(n-1)}).$$

which is also written

$$\sup_{k \in \{1, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) < 0 < \inf_{k \in \{0, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}).$$

Conversely,

$$\sup_{k \in \{1, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) < 0 < \inf_{k \in \{0, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)})$$

$$\Leftrightarrow \forall k \in \{1, \dots, n-2\}, R_{n-2}(\alpha_k^{(n-1)}) \times R_{n-2}(\alpha_{k+1}^{(n-1)}) < 0$$

This implies that R_{n-2} has $n-2$ interlacing roots with those of P_{n-1} .

The following theorem allows to choose a_0 so that the new polynomial P_n has n distinct real roots:

Theorem 1 Let P_n be a polynomial and set

$$P_n(x) = \frac{x}{n} P_n'(x) - R_{n-2}(x) = x P_{n-1}(x) - R_{n-2}(x).$$

We call $R_{n-2}^0(x) = R_{n-2}(x) + a_0$.

Denote by $\alpha_1^{(n-1)} \dots \alpha_{n-1}^{(n-1)}$ the $n-1$ distinct real roots of P_{n-1} .

For n even, if

$$\left[\begin{array}{l} \sup_{k \in \{1, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) < 0 < \inf_{k \in \{0, \dots, \frac{n-2}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}) \\ a_0 \in \left[\sup_{k \in \{1, \dots, \frac{n-2}{2}\}} R_{n-2}^0(\alpha_{2k}^{(n-1)}); \inf_{k \in \{0, \dots, \frac{n-2}{2}\}} R_{n-2}^0(\alpha_{2k+1}^{(n-1)}) \right] \end{array} \right],$$

then P_n has n distinct real roots.

For n odd, if

$$\left[\begin{array}{l} \sup_{k \in \{0, \dots, \frac{n-3}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}) < 0 < \inf_{k \in \{1, \dots, \frac{n-1}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) \\ a_0 \in \left[\sup_{k \in \{0, \dots, \frac{n-3}{2}\}} R_{n-2}^0(\alpha_{2k+1}^{(n-1)}); \inf_{k \in \{1, \dots, \frac{n-1}{2}\}} R_{n-2}^0(\alpha_{2k}^{(n-1)}) \right] \end{array} \right],$$

then P_n has n distinct real roots.

The following corollary introduces the notion of degenerate cases in the context of our study. We consider only the case where n is odd (a similar result can be established for n even):

Corollary 1 *In the case of n odd,*

1. if $a_0 = \sup_{k \in \{0, \dots, \frac{n-3}{2}\}} R_{n-2}^0(\alpha_{2k+1}^{(n-1)})$ or $a_0 = \inf_{k \in \{1, \dots, \frac{n-1}{2}\}} R_{n-2}^0(\alpha_{2k}^{(n-1)})$, the polynomial P_n as a double root which is a root of P_{n-1} and R_{n-2} .
2. if $a_0 = \sup_{k \in \{0, \dots, \frac{n-3}{2}\}} R_{n-2}^0(\alpha_{2k+1}^{(n-1)}) = \inf_{k \in \{1, \dots, \frac{n-1}{2}\}} R_{n-2}^0(\alpha_{2k}^{(n-1)})$, the polynomial P_n has either two double roots or one triple root. So these two real distinct roots are also roots of R_{n-2} and therefore of P_n .

Applying recursively Theorem 1, the following theorem allows choosing all coefficients a_l , $0 \leq l < n-2$,

Theorem 2 *Let $P_n(x) = x^n + a_{n-2}x^{n-2} + \dots + a_1x + a_0$ and define the sequence $[P_i, R_i, R_i^0]$, $i \in \{3, \dots, n\}$, such that*

$$\begin{cases} P_{i-1}(x) = \frac{1}{i}P_i'(x) \\ P_i(x) = xP_{i-1}(x) - R_{i-2}(x) \end{cases}$$

and, $\forall i \in \{3, \dots, n\}$, $R_{i-2}(x) = R_{i-2}(x) + a_{n-i}$.

We assume that the intervals below of the form $] \sup; \inf [$ have a meaning, i.e., the \sup must always be less than the \inf .

If n is even, P_n has n distinct real roots if and only if, for all $l \in \{0, \dots, \frac{n}{2} - 2\}$,

$$\left[\begin{array}{l} \bullet a_{n-2} < 0 \\ \bullet a_{2l} \in \left[\sup_{k \in \{1, \dots, \frac{n-2}{2} - l\}} R_{n-2l-2}^0(\alpha_{2k}^{(n-2l-1)}); \inf_{k \in \{0, \dots, \frac{n-2}{2} - 1\}} R_{n-2l-2}^0(\alpha_{2k+1}^{(n-2l-1)}) \right] \\ \bullet a_{2l+1} \in \left[\sup_{k \in \{0, \dots, \frac{n-4}{2} - l\}} R_{n-2l-3}^0(\alpha_{2k+1}^{(n-2l-2)}); \inf_{k \in \{1, \dots, \frac{n-2}{2} - 1\}} R_{n-2l-3}^0(\alpha_{2k}^{(n-2l-2)}) \right] \end{array} \right].$$

If n is odd, P_n has n distinct real roots if and only if,

$$\begin{array}{l} \bullet a_{n-2} < 0. \\ \bullet \forall l \in \{0, \dots, \frac{n-3}{2}\}, a_{2l} \in \left[\sup_{k \in \{0, \dots, \frac{n-3}{2} - l\}} R_{n-2l-2}^0(\alpha_{2k+1}^{(n-2l-1)}); \inf_{k \in \{0, \dots, \frac{n-1}{2} - 1\}} R_{n-2l-2}^0(\alpha_{2k}^{(n-2l-1)}) \right] \\ \bullet \forall l \in \{0, \dots, \frac{n-5}{2}\}, a_{2l+1} \in \left[\sup_{k \in \{1, \dots, \frac{n-3}{2} - l\}} R_{n-2l-3}^0(\alpha_{2k}^{(n-2l-2)}); \inf_{k \in \{0, \dots, \frac{n-3}{2} - 1\}} R_{n-2l-3}^0(\alpha_{2k+1}^{(n-2l-2)}) \right]. \end{array}$$

Remark: The hypothesis that intervals of the form $] \sup; \inf [$ have a meaning will be studied in the context of a polynomial of degree five.

4 Polynomial of order three and four

The polynomial $P_3(x) = x^3 + px + q$ has three roots if $p < 0$ and $q \in]-2(-\frac{p}{3})^{3/2}; 2(-\frac{p}{3})^{3/2}[\Leftrightarrow \Delta(P_3) = -4p^3 - 27q^2 > 0$.

Proposition 3 $P_4(x) = x^4 + 2px^2 + 4qx + 4r$ has four distinct real roots if we choose:

$$p < 0, \quad q \in]-2(-\frac{p}{3})^{3/2}; 2(-\frac{p}{3})^{3/2}[, \quad 4r \in]R_2^0(\alpha_2^{(3)}); \inf_{i \in \{1,3\}} R_2^0(\alpha_i^{(3)})[\subset]R_2^0(\beta_1^{(1)}); +\infty[$$

where

$$\begin{cases} \beta_1^{(1)} = -\frac{3q}{2p} \text{ is the root of } R_1(x) = -\frac{2}{3}px - q, \\ \alpha_i^{(3)}, i = 1..3, \text{ the roots of, } P_3(x) = x^3 + px + q, \\ R_2^0(x) = -px^2 - 3qx. \end{cases}$$

More precisely, P_4 has four distinct roots if

$$p < 0, \quad q \in]-2(-\frac{p}{3})^{3/2}; 2(-\frac{p}{3})^{3/2}[, \quad 4r \in]R_2^0(x_3); R_2^0(x_1)[$$

where

$$\begin{cases} R_2^0(x_3) = \frac{4p^2}{9} \cos(\theta/3) [\cos(\theta/3) + \cos(\theta)] \\ R_2^0(x_1) = \frac{4p^2}{9} \cos(\theta/3 + \pi/3) [\cos(\theta/3 + \pi/3) + \cos(\theta)] \\ \cos(\theta) = -\frac{|q|}{2\sqrt{-p/3}} \end{cases}$$

Remarks:

1. Say $4r \in]R_2^0(\beta_1^{(1)}); +\infty[$ means R_2 has two distinct roots $\beta_1^{(2)}$ and $\beta_2^{(2)}$ (the discriminant of R_2 is given by $\Delta R_2 = 9q^2 - 16pr > 0$). Clearly:

$$\Delta_2 > 0 \Leftrightarrow R_2(\beta_1^{(1)}) = R_2^0(\beta_1^{(1)}) - 4r < 0.$$

2. Say $4r \in]R_2^0(\alpha_2^{(3)}); \inf_{i \in \{1,3\}} R_2^0(\alpha_i^{(3)})[$ means the roots of R_2 are interlaced with $\alpha_1^{(3)}$, $\alpha_2^{(3)}$, $\alpha_3^{(3)}$, the roots of P_3 .

Study of multiple roots for a polynomial of order four

In the proposition 3, the maximum or minimum values reached by r give very particular cases concerning the roots of the polynomial P_4 .

1. The bound $r = p^2/4$ ($q = 0$) corresponds to the fact that the polynomial P_4 has two double roots $-\sqrt{-p}$ and $\sqrt{-p}$.
2. The bound $r = -p^2/12$ corresponds to a double root $-\sqrt{-p/3}$ for the polynomial P_3 (which implies $q^2 = -4p^3/27$): so, according with the sign of q , P_4 has one triple root $\pm\sqrt{-p/3}$.

5 Polynomial of order five

In this section, we focus on the fifth order. In particular, we explain our assumptions on the choice of the last parameter s . Different cases related to multiple roots are studied.

5.1 Our assumptions

Let's consider the following polynomial

$$P_5(x) = x^5 + \frac{10p}{3}x^3 + 10qx^2 + 20rx + 20s = \frac{xP_4(x)}{5} - R_3(x)$$

with $R_3(x) = -(\frac{4}{3}px^3 + 6qx^2 + 16rx + 20s)$. For our method, recall the assumptions of order four:

$$p < 0, \quad q \in \left] -2 \left(-\frac{p}{3}\right)^{3/2}; 2 \left(-\frac{p}{3}\right)^{3/2} \right[, \quad 4r \in \left] R_2^0(\alpha_2^{(3)}); \inf_{k \in \{1,3\}} R_2^0(\alpha_k^{(3)}) \right[$$

where $\alpha_i^{(3)}$, $i = 1..3$, are the roots of $P_3(x) = x^3 + px + q$ and $R_2^0(x) = -px^2 - 3qx$. Applying Theorem 1, we explain how to choose the parameter s .

5.2 Choice of s

As $P_5(x) = \frac{xP_4(x)}{5} - R_3(x)$, $R_3(x)$ must have three distinct real roots and then they have to be interlaced with the four roots $\alpha_i^{(4)}$, $i = 1 \dots 4$, of P_4 . According to the result of order three, we find

$$20s \in]R_3^0(\beta_2^{(2)}), R_3^0(\beta_1^{(2)})[$$

where $\beta_i^{(2)}$, $i = 1..2$, roots of $R_2(x) = -px^2 - 3qx - 4r$.

The Euclidean division of R_3 by R_3' give as a remainder whose sign we change

$$T_1(x) = \frac{2(16pr - 9q^2)}{3p}x + 20s - \frac{8qr}{p}$$

and this remainder vanishes at

$$\gamma_1^{(1)} = \frac{3(-10sp + 4qr)}{16pr - 9q^2}$$

It is enough now that $\gamma_1^{(1)}$ is interlaced with $\beta_1^{(2)}$ and $\beta_2^{(2)}$:

$$\beta_1^{(2)} < \gamma_1^{(1)} < \beta_2^{(2)}.$$

We obtain :

$$20s \in]R_3^0(\beta_2^{(2)}); R_3^0(\beta_1^{(2)})[= \left] \frac{-9q^3}{p^2} + \frac{24qr}{p} - \frac{(\Delta_2)^{3/2}}{3p^2}; \frac{-9q^3}{p^2} + \frac{24qr}{p} + \frac{(\Delta_2)^{3/2}}{3p^2} \right[.$$

With the assumption $\Delta(R_2) = 9q^2 - 16pr > 0$, we deduce that R_3 will have three distinct real roots.

Or, as

$$-T_1(\beta_1^{(2)}) = R_3(\beta_1^{(2)}) = R_3^0(\beta_1^{(2)}) - 20s$$

and

$$-T_1(\beta_2^{(2)}) = R_3(\beta_2^{(2)}) = R_3^0(\beta_2^{(2)}) - 20s.$$

The condition can be written as $R_3(\beta_1^{(2)}) > 0$ and $R_3(\beta_2^{(2)}) < 0$ or $T_1(\beta_1^{(2)}) < 0$ and $T_1(\beta_2^{(2)}) > 0$. Now, if the three roots of R_3 are interlaced with those of P_4 , then

$$20s \in \left] \sup_{k \in \{1,3\}} R_3^0(\alpha_k^{(4)}); \inf_{k \in \{2,4\}} R_3^0(\alpha_k^{(4)}) \right[\subset]R_3^0(\beta_2^{(2)}); R_3^0(\beta_1^{(2)})[$$

under the assumption

$$\sup_{k \in \{1,3\}} R_3^0(\alpha_k^{(4)}) < \inf_{k \in \{2,4\}} R_3^0(\alpha_k^{(4)}).$$

Theorem 3 *Under the condition*

$$\sup_{k \in \{1,3\}} R_3^0(\alpha_k^{(4)}) < \inf_{k \in \{2,4\}} R_3^0(\alpha_k^{(4)}). \quad (1)$$

the polynomial $P_5(x) = x^5 + \frac{10p}{3}x^3 + 10qx^2 + 20rx + 20s$ has five distinct real roots if and only if

$$p < 0, \quad q \in \left] -2 \left(-\frac{p}{3}\right)^{3/2}; 2 \left(-\frac{p}{3}\right)^{3/2} \right[, \quad 4r \in \left] R_2^0(\alpha_2^{(3)}); \inf_{k \in \{1,3\}} R_2^0(\alpha_k^{(3)}) \right[$$

and

$$20s \in \left] \sup_{k \in \{1,3\}} R_3^0(\alpha_k^{(4)}); \inf_{k \in \{2,4\}} R_3^0(\alpha_k^{(4)}) \right[$$

where

$$\begin{cases} \alpha_i^{(3)}, i = 1..3, \text{ are the roots of } P_3(x) = x^3 + px + q \\ R_2^0(x) = -px^2 - 3qx \\ \alpha_i^{(4)}, i = 1..4, \text{ are the roots of } P_4(x) = x^4 + 2px^2 + 4qx + 4r \\ R_3^0(x) = -\frac{4}{3}px^3 - 6qx^2 - 16rx. \end{cases}$$

5.3 Study of the hypothesis of the theorem 3

We must then examine the conditions under which the previous hypothesis (1) is satisfied. The assumption (1) may not be satisfied if $R_3^0(a) > R_3^0(b)$ with $a = \alpha_1^{(4)}$ the smallest root and $b = \alpha_4^{(4)}$ the biggest root of P_4 (as the sum of the roots is zero, $a < 0$ and $b > 0$). Taking $X = ab \leq -2\sqrt{|r|}$, the inequality (1) can be written as

$$\left(\frac{6r}{p} - p\right)X(X^2 - 4r) + \left(\frac{9q^2}{p} + 2r\right)X^2 - 8r^2 < 0. \quad (2)$$

Under the assumptions

$$p < 0, \quad q \in \left] -2\left(-\frac{p}{3}\right)^{3/2}; 2\left(-\frac{p}{3}\right)^{3/2} \right[, \quad 4r \in \left] R_2^0(\alpha_2^{(3)}); \inf_{k \in \{1,3\}} R_2^0(\alpha_k^{(3)}) \right[, \quad (3)$$

P_4 has four distinct real roots $a < b < c < d$. Let $Y = a + b$. We have

$$\begin{cases} Y + c + d = 0 \\ (c + d)Y + X + cd = 2p \\ (c + d)X + cdY = -4q \\ Xcd = 4r \end{cases}$$

It follows that

$$\begin{cases} -Y^2 + X + \frac{4r}{X} = 2p \\ -YX + \frac{4r}{X}Y = -4q \end{cases}$$

We deduce that

$$16q^2 X^3 = (4r - X^2)^2 (X^2 - 2pX + 4r) < 0$$

with $r < \frac{p^2}{4}$ and $p - \sqrt{p^2 - 4r} < X < -2\sqrt{|r|}$. The condition $R^3(a) < R^3(b)$ is written

$$(X^2 - 4r)[X(6r - p^2) + 2rp] + 9q^2 X^2 > 0.$$

Multiplying this inequality by X and noting that

$$16q^2 X^3 = (4r - X^2)^2(X^2 - 2pX + 4r) < 0,$$

we get

$$(X - x_1)(X - x_2)(X - x_3)(X - x_4) < 0 \quad (4)$$

where

$$\begin{cases} x_1 = \frac{4p+2\sqrt{4p^2-27r}}{3} \\ x_2 = \frac{4p-2\sqrt{4p^2-27r}}{3} \\ x_3 = \frac{-p+\sqrt{p^2+12r}}{3} \\ x_4 = \frac{-p-\sqrt{p^2+12r}}{3} \end{cases}$$

From this, we can deduce:

Proposition 4 *Under the hypothesis (3), the assumption*

$$\sup_{k \in \{1,3\}} R_3^0(\alpha_k^{(4)}) < \inf_{k \in \{2,4\}} R_3^0(\alpha_k^{(4)}) \quad (5)$$

is satisfied as soon as:

1. $r < p^2/9$,
2. $\frac{p^2}{9} \leq r \leq \frac{5p^2}{36}$ and $X < x_1 = \frac{4p+2\sqrt{4p^2-27r}}{3}$
3. $\frac{5p^2}{36} \leq r \leq \frac{4p^2}{27}$ and $x_2 = \frac{4p-2\sqrt{4p^2-27r}}{3} < X < x_1 = \frac{4p+2\sqrt{4p^2-27r}}{3}$

with $p < 0$ and

$$16q^2 X^3 = (4r - X^2)^2(X^2 - 2pX + 4r) < 0.$$

Remarks: 1. The hypothesis (5) reduces the interval of values of the parameter r , $r \in] - p^2/12; p^2/4[$, obtained by the conditions (3), conditions for the polynomial P_4 to have four real roots.

2. The proposition 4 also states that for $p < 0$, $q \in] -2(-\frac{p}{3})^{3/2}; 2(-\frac{p}{3})^{3/2}[$, it is not possible to construct a polynomial of degree five with five real roots with $r \in]4p^2/27; p^2/4[$.

5.4 Notes: degenerate cases

Three degenerate cases can be distinguished:

1. $20s = \sup_{k \in \{1,3\}} R_3^0(\alpha_k^{(4)}) = R_3^0(\alpha_1^{(4)}) = R_3^0(\alpha_3^{(4)})$;
2. $20s = \inf_{k \in \{2,4\}} R_3^0(\alpha_k^{(4)}) = R_3^0(\alpha_2^{(4)}) = R_3^0(\alpha_4^{(4)})$;
3. $20s = \sup_{k \in \{1,3\}} R_3^0(\alpha_k^{(4)}) = \inf_{k \in \{2,4\}} R_3^0(\alpha_k^{(4)})$.

These correspond to the fact that the polynomial P_5 has two double roots (see corollary 1). If we denote by a, b, c the roots of P_5 with a and b the double roots, the first two cases correspond to the situations $a < b < c$ or $c < a < b$ and the last case to $a < c < b$. This is the subject of the two following paragraphs.

5.4.1 Analysis of the first two cases

For example, here is a polynomial of degree 5 (see Figure 1) corresponding to the first two cases:

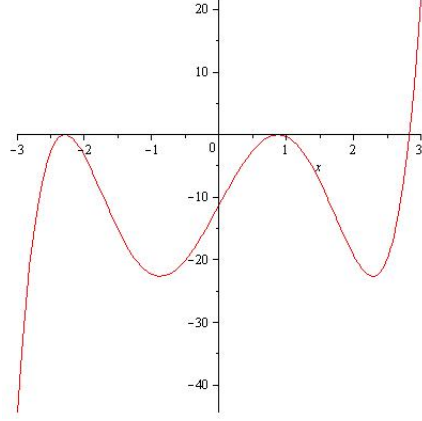


Figure 1: an example of degenerate case with $a < b < c$

The polynomial P_5 has two double roots a and b which are roots of P_4 and R_3 . The Euclidean division of $P_5(x)$ by $(x - a)^2(x - b)^2$ gives a remainder that is identically zero if

$$\begin{cases} -3Y^2 + 2X = \frac{10p}{3} \\ 2Y^3 + 2XY = 10q \\ -X(4Y^2 - X) = 20r \\ 2X^2Y = 20s \end{cases}$$

where $X = ab$ and $Y = a + b$. It comes

$$\begin{cases} Y^2 = \frac{2}{3}(X - \frac{5p}{3}) \\ Y = \frac{3q}{X - \frac{2p}{3}} \\ X^2 - \frac{8p}{3}X + 12r = 0 \end{cases}$$

The discriminant of $X^2 - \frac{8p}{3}X + 12r = 0$ is $\Delta = 64p^2/9 - 48r$. The latter is positive for $r < 4p^2/27$. Let $x_1 = (8p/3 + \sqrt{\Delta})/2$ and $x_2 = (8p/3 - \sqrt{\Delta})/2$ be the two roots of this polynomial. If $r < 0$, $x_1 > 0$ (reached in $r=0$) and $x_2 < 5p/3$. This amounts to $X > 0$ either two positive or negative double roots. We can then establish the following result:

Proposition 5 *The polynomial P_5 has two consecutive double roots a and b ($c < a < b$ or $a < b < c$) if*

$$\begin{cases} p < 0 \\ -\frac{p^2}{12} \leq r \leq \frac{4p^2}{27} \\ x_1 \leq -\frac{p}{3} \\ 729q^2 = -8[11p^3 - 81pr + 2(4p^2 - 27r)^{3/2}] \\ 10s = \frac{3q}{x_1 - \frac{2p}{3}} x_1^2 \end{cases}$$

The parameter q is in the interval $\left[-\sqrt{\frac{-4p^3}{27}} ; \sqrt{\frac{-4p^3}{27}} \right]$.

5.4.2 The last case

The last case of degeneration corresponds to the following example (see Figure 2):

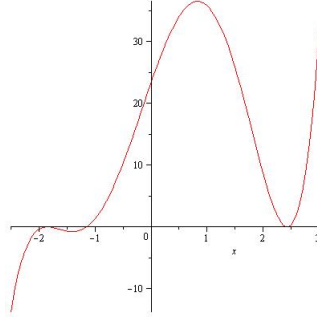


Figure 2: an example of the last degenerate case $a < c < b$

By using the result of the proposition 4, we obtain the following result

Proposition 6 *The polynomial P_5 has two double roots a and b ($a < c < b$) when*

$$\begin{cases} p < 0 \\ \frac{5p^2}{36} \leq r \leq \frac{4p^2}{27} \\ x_2 = \frac{4p}{3} - \frac{2}{3}\sqrt{4p^2 - 27r} \\ q^2 = -\frac{8}{729}[11p^3 - 81pr - 2(4p^2 - 27r)^{3/2}] \\ 10s = \frac{3q}{X - \frac{2p}{3}} x_2^2 \end{cases}$$

The parameter q is in the interval $\left] -\sqrt{\frac{-8p^3}{729}} ; \sqrt{\frac{-8p^3}{729}} \right[$.

It is possible to find a triplet (p, q, r) for which the propositions 5 and 6 are satisfied at the same time by choosing $r = 4p^2/27$ (i.e. $X = 4p/3$), $q^2 = -8p^3/729$, $s = 4pq/5$. This situation corresponds to a triple root and a double root for P_5 of the form $\pm \frac{2}{3}\sqrt{-2p}$ and $\mp \sqrt{-2p}$.

6 Comparison with Sturm's method

Let us first recall Sturm's method. Let P_n be a polynomial and denote its derivative by P_{n-1} . Proceed as in Euclid's theorem to find :

$$\begin{cases} S_n = P_n \\ S_{n-1} = P_{n-1}' \\ S_n = Q_1 S_{n-1} - S_{n-2} \\ \dots \\ S_2 = Q_{n-1} S_1 - S_0. \end{cases}$$

Theorem 4 (Sturm's Theorem) *The number of distinct real roots of a polynomial in $[a; b]$ is equal to the excess of the number of changes of sign in the sequence $S_n(a), \dots, S_0(a)$ over the number of changes of sign in the sequence $S_n(b), \dots, S_0(b)$.*

Since we are interested only in polynomials of degree n with n distinct real roots, a corollary of Sturm's theorem is the following result:

Corollary 2 *A polynomial of degree n has n distinct real roots if and only if all the coefficients of the monomial of the greatest degree of the polynomials $S_n(x), \dots, S_0(x)$ are strictly positive.*

We compare our results with those obtained using Sturm's method for polynomials of degrees 3, 4 and 5.

Sturm sequence associated with P_3

In our case, the polynomial P_3 has three roots if $p < 0$ and $\Delta(P_3) > 0$. By Sturm's method,

$$\begin{cases} S_3(x) = P_3(x) = x^3 + px + q \\ S_2(x) = P_3'(x) = 3x^2 + p \\ S_1(x) = R_1(x) = -\left(\frac{2}{3}px + q\right) \\ S_0 = \frac{-4p^3 - 27q^2}{4p^2} = \frac{\Delta(P_3)}{4p^2} \end{cases}$$

Sturm sequence associated with P_4

Sturm's sequence is given by

$$\begin{cases} S_4(x) = P_4(x) = x^4 + 2px^2 + 4qx + 4r \\ S_3(x) = P_4'(x) = 4x^3 + 4px + 4q \\ S_2(x) = R_2(x) = -px^2 - 3qx - 4r \\ S_1(x) = \frac{-4}{p^2}[(-4pr + p^3 + 9q^2)x + q(12r + p^2)] \\ S_0 = \frac{p^2\Delta(P_4)}{256(-4pr + p^3 + 9q^2)^2} \end{cases}$$

Using Sturm's theorem, we have four distinct real roots if the coefficients of the term of the highest degree of S_3, S_2, S_1 and S_0 are strictly positive. These assumptions are the following:

$$p < 0, -4pr + p^3 + 9q^2 < 0, \Delta(P_4) > 0.$$

In contrast to the results of Sturm's algorithm, we obtained the exact solutions for a polynomial of degree 4 to have exactly four distinct real roots (see Proposition 3).

Sturm sequence associated with P_5

Sturm's sequence is given by

$$\begin{cases} S_5(x) = P_5(x) = x^5 + \frac{10p}{3}x^3 + 10qx^2 + 20rx + 20s \\ S_4(x) = P_5'(x) = 5x^4 + 10px^2 + 20qx + 20r \\ S_3(x) = R_3(x) = -\left(\frac{4}{3}px^3 + 6qx^2 + 16rx + 20s\right) \\ S_2(x) = -5 \left[\frac{(8p^3 + 81q^2 - 48pr)x^2}{4p^2} + \frac{(-15ps + 4p^2q + 54qr)x}{p^2} + 4r + \frac{135qs}{2p^2} \right] \\ S_1(x) = -\frac{32p^2}{3(8p^3 - 48pr + 81q^2)^2} (a_{S_1}x + b_{S_1}) \\ S_0 = \frac{1000000}{81} \left(\frac{8p^3 - 48pr + 81q^2}{p a_{S_1}} \right)^2 \Delta(P_5) \end{cases}$$

$$\text{with } \begin{cases} a_{S_1} &= -80p^4r - 2106q^2pr + 1056p^2r^2 - 3456r^3 + 240p^2qs + 3240qsr \\ &+ 40p^3q^2 + 729q^4 - 450ps^2 \\ b_{S_1} &= -120p^4s - 1755spq^2 + 1560p^2rs - 4320r^2s + 40p^3qr + 729q^3r \\ &- 864qpr^2 + 2025s^2q. \end{cases}$$

The result of Sturm gives five distinct real roots for P_5 under the following assumptions:

$$p < 0, 8p^3 + 81q^2 - 48pr < 0, a_{S_1} < 0, \Delta(P_5) > 0.$$

Conjecture

By observing the last term S_0 of the preceding Sturm sequences, we can conjecture the following result

Conjecture 1 *Let P_n be the polynomial of order n , $P_{n-1} = P'_n/n$, such that :*

$$\begin{cases} P_n(x) = xP_{n-1}(x) - R_{n-2}(x) \\ R_{n-2}(x) = R_{n-2}^0(x) - a_0. \end{cases}$$

The last term S_0 given in the Sturm sequence associated with P_n is equal to:

$$\begin{aligned} S_0 &= K_n \Delta(P_n) = K(-1)^{n(n-1)/2} n^n \prod_{i=1}^{n-1} P_n(\alpha_i^{(n-1)}) \\ &= K_n (-1)^{(n-1)(n+2)/2} n^n \prod_{i=1}^{n-1} [R_{n-2}^0(\alpha_i^{(n-1)}) - a_0] \end{aligned}$$

where $\alpha_i^{(n-1)}$, $i = 1, \dots, n-1$, distinct real roots of P_{n-1} and K_n strictly not negative constant. $\Delta(P_n)$ represents the discriminant of P_n .

The conjecture concerns the first equality $S_0 = K_n \Delta(P_n)$ where K is a strictly positive constant. The other equalities arise from the relationships between the discriminant of a polynomial and the resultant associated with P_n and its derivative P'_n (see for example [4] for results on resultants and Sylvester's matrices). As we are looking for conditions for P_n to have n distinct real roots, we know that $\text{pgcd}(P_n, P'_n)$ is a constant. However, it has not been established, using Euclid's algorithm for example (or by constructing the Sturm sequence associated with P_n in a similar way), that the latter is equal to $K \Delta(P_n)$.

7 Concluding remarks

Our assumptions are explicit and depend on the roots of the previous order. That is why after order five, things become harder. Indeed, the exact expressions of the roots are unknown. Obviously, for cases of orders three and four, Cardano's, Ferrari's, Descartes's or Euler's formula of the roots are available.

Otherwise, Cayley [2] and more recently [3] and [7] proposed different methods for some kind of polynomials of degree six of those roots as functions of the roots for solvable quintics. For polynomials of order six, we may ask how the assumption we take in theorem 1 can be written using two double roots in particular with the product of the first and the fourth roots or the second and the fifth roots of the polynomial of order five.

Moreover, the conjecture we present for the last term in Sturm's sequence raises many questions. For example, what is the link between discriminants of polynomials of different degrees?

Besides notice the important role played by multiple roots and possible use to obtain upper and lower bounds for intervals of parameters for polynomials having only real roots.

On the other side, Ramanujan [1] solve some polynomials of degree three, four, five, six and seven. His approach is very original. It seems that he often started with roots having product one. In this paper, we assume that the sum of the roots is zero even if our result is true in the general case, but it takes a more complex form.

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