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► **To cite this version:**

Rabah Labbas, Stéphane Maingot, Alexandre Thorel. Solvability of a fourth order elliptic problem in a bounded sector, part II. *Bollettino dell'Unione Matematica Italiana*, 2024, 10.1007/s40574-024-00406-x . hal-04487972

**HAL Id: hal-04487972**

**<https://hal.science/hal-04487972>**

Submitted on 4 Mar 2024

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# Solvability of a fourth order elliptic problem in a bounded sector, part II

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## Abstract

After different variables and functions changes, the generalized dispersal problem, recalled in (1) below and considered in part I, see Labbas, Maingot and Thorel [14], leads us to consider, to study and to invert the sum of linear operators (4) below in a suitable Banach space by using two strategies: namely the theory of sums of operators in Banach spaces as developed by Da Prato-Grisvard [4] and successfully improved by Dore-Venni [5].

**Key Words and Phrases:** Sum of linear operators, second and fourth order boundary value problem, functional calculus, bounded imaginary powers, maximal regularity

**2020 Mathematics Subject Classification:** 34G10, 35B65, 35C15, 35R20, 47A60.

## 1 Introduction and main result

This work is a natural continuation of Labbas, Maingot and Thorel [14], where we have considered, for  $k > 0$ , the following problem

$$\begin{cases} \Delta^2 u - k\Delta u = f & \text{in } S_{\omega, \rho} \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_0 \cup \Gamma_{\omega, \rho} \\ u = \frac{\partial^2 u}{\partial n^2} = 0 & \text{on } \Gamma_\rho, \end{cases} \quad (1)$$

with

$$\begin{cases} S_{\omega, \rho} &= \{(x, y) = (r \cos \theta, r \sin \theta) : 0 < r < \rho \text{ and } 0 < \theta < \omega\} \\ \Gamma_0 &= (0, \rho) \times \{0\} \\ \Gamma_\omega &= \{(r \cos \omega, r \sin \omega) : 0 < r < \rho\} \\ \Gamma_\rho &= \{(\rho \cos \theta, \rho \sin \theta) : 0 < \theta < \omega\}, \end{cases}$$

for given  $\rho > 0$  and  $\omega \in (0, 2\pi]$ .

In part I, see Labbas, Maingot and Thorel [14], to study problem (1), we have done many variables and functions changes to write it as a sum of linear operators. To this end, we have introduced the following functions for  $t > 0$  and  $(r, \theta) \in S_{\rho, \omega}$ :

$$\begin{cases} v(r, \theta) = u(r \cos \theta, r \sin \theta) \\ G(t)(\theta) := G(t, \theta) = g(\rho e^{-t}, \theta) = f(\rho e^{-t} \cos \theta, \rho e^{-t} \sin \theta) \\ \phi(t)(\theta) := \phi(t, \theta) = \frac{v(\rho e^{-t}, \theta)}{\rho e^{-t}} \\ H(t)(\theta) := H(t, \theta) = e^{-3t} G(t)(\theta), \end{cases} \quad (2)$$

and the two abstract vector-valued functions

$$V(t) = \begin{pmatrix} e^{\nu t} \phi(t) \\ e^{\nu t} \phi''(t) \end{pmatrix}, \quad \mathcal{F}_\nu(t) = \begin{pmatrix} 0 \\ \rho^3 e^{\nu t} H(t) \end{pmatrix}, \quad \nu = 3 - \frac{2}{p} \in (1, 3) \text{ with } p > 1. \quad (3)$$

By considering the Banach space

$$X = W_0^{2,p}(0, \omega) \times L^p(0, \omega),$$

and after the changes indicated above in (2) and (3), we have rewritten problem (1) in the space  $L^p(0, +\infty; X)$ , see Labbas, Maingot and Thorel [14], in the following form

$$(\mathcal{L}_{1,\nu} + \mathcal{L}_2)V + k\rho^2(\mathcal{P}_1 + \mathcal{P}_{2,\nu})V = \mathcal{F}_\nu,$$

where

$$\begin{cases} D(\mathcal{L}_{1,\nu}) &= \{V \in W^{2,p}(0, +\infty; X) : V(0) = V(+\infty) = 0\} \\ [\mathcal{L}_{1,\nu}(V)](t) &= (\partial_t - \nu I)^2 V(t) = V''(t) - 2\nu V'(t) + \nu^2 V(t), \\ D(\mathcal{L}_2) &= \{V \in L^p(0, +\infty; X) : \text{for a.e. } t \in (0, +\infty), V(t) \in D(\mathcal{A})\} \\ [\mathcal{L}_2(V)](t) &= -\mathcal{A}V(t), \end{cases}$$

with

$$\begin{cases} D(\mathcal{A}) &= [W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega)] \times W_0^{2,p}(0, \omega) \subset X \\ \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} \psi_2 \\ -\left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 \psi_1 - 2\left(\frac{\partial^2}{\partial \theta^2} - 1\right) \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}), \end{cases}$$

and

$$\begin{cases} D(\mathcal{P}_1) &= \{V \in L^p(0, +\infty; X) : \text{for a.e. } t \in (0, +\infty), V(t) \in D(\mathcal{A}_0)\} \\ [\mathcal{P}_1(V)](t) &= -e^{-2t} \mathcal{A}_0 V(t), \end{cases}$$

with

$$\begin{cases} D(\mathcal{A}_0) &= W_0^{2,p}(0, \omega) \times L^p(0, \omega) = X \\ \mathcal{A}_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ \left(\frac{\partial^2}{\partial \theta^2} + 1\right) \psi_1 + \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}_0), \end{cases}$$

and

$$\begin{cases} D(\mathcal{P}_{2,\nu}) &= W^{1,p}(0, +\infty; X) \\ [\mathcal{P}_{2,\nu}(V)](t) &= -e^{-2t} (\mathcal{B}_{2,\nu} V)(t), \end{cases}$$

with

$$\mathcal{B}_{2,\nu} = \begin{pmatrix} 0 & 0 \\ -2(\partial_t - \nu I) & 0 \end{pmatrix}.$$

In the present paper, we will focus ourselves on the resolution of the following abstract equation

$$(\mathcal{L}_{1,\mu} + \mathcal{L}_2)V + k\rho^2(\mathcal{P}_1 + \mathcal{P}_{2,\mu})V = \mathcal{F}, \quad (4)$$

where  $\mu \in \mathbb{R}$  is a general parameter and  $\mathcal{F} \in L^p(0, +\infty; X)$ .

Note that, in part I, see Labbas, Maingot and Thorel [14], subsection 3.4, we have worked with  $\mu = \nu = 3 - \frac{2}{p}$  which comes from the variable change concerning the weighted Sobolev space. Here, in this second part, we consider a more general  $\mu \in \mathbb{R}$ .

The aim of this work is to show that there exists a unique classical solution of (4) that is a function  $V$  such that

$$V \in W^{2,p}(0, +\infty; X) \cap L^p(0, +\infty; D(\mathcal{A})).$$

This regularity is necessary to deduce all those of the function  $v$  stated in Theorem 2.2 in Labbas, Maingot and Thorel [14].

To this end, we will use the Da Prato-Grisvard sum theory in order to invert  $\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}$ . Then, we solve the following equation

$$\left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}\right) V + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu}) V = \mathcal{F},$$

by using a perturbation argument. Next, we use some convexity inequalities to prove that  $V$  belongs to a more suitable space, more precisely

$$V \in W^{1,p}(0, +\infty; X) \cap L^p\left(0, +\infty; \left[W^{3,p}(0, \omega) \cap W_0^{2,p}(0, \omega)\right] \times L^p(0, \omega)\right).$$

At this step,  $V$  is the unique strong solution of (4), see (46). To obtain that  $V$  is a classical solution, we use the Dore-Venni sum theory, see Section 6.

Among the results that we will use is the fact that the roots of the equation

$$(\sinh(z) + z)(\sinh(z) - z) = 0,$$

in  $\mathbb{C}_+ := \{w \in \mathbb{C} : \operatorname{Re}(w) > 0\}$ , constitute a family of complex numbers  $(z_j)_{j \geq 1}$  such that

$$\tau := \min_{j \geq 1} |\operatorname{Im}(z_j)| > 0 \quad \text{and} \quad |z_j| \longrightarrow +\infty.$$

These roots are computed in Fädle [7] with  $\tau \simeq 4.21239$ .

Our main result is the following.

**Theorem 1.1.** Let  $\mathcal{F} \in L^p(0, +\infty; X)$  and assume that

$$\omega\mu < \tau. \tag{5}$$

Then, there exists  $\rho_0 > 0$  such that for all  $\rho \in (0, \rho_0]$ , the abstract equation

$$(\mathcal{L}_{1,\mu} + \mathcal{L}_2) V + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu}) V = \mathcal{F},$$

has a unique classical solution  $V \in L^p(0, +\infty; X)$ , that is

$$V \in W^{2,p}(0, +\infty; X) \cap L^p(0, +\infty; D(\mathcal{A})).$$

In particular,  $\mathcal{L}_{1,\mu} + \mathcal{L}_2$  is closed and  $V \in D(\mathcal{L}_{1,\mu} + \mathcal{L}_2)$ .

This second part is organized as follows. Section 2 is devoted to recalling some needed results. In Section 3, we analyze the spectral properties of operators  $\mathcal{L}_{1,\mu}$  and  $\mathcal{L}_2$  in view to study the invertibility of  $\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}$  in Section 4. In Section 5, by considering that operator  $k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu})$  is a perturbation, we deduce the existence and the uniqueness of a strong solution of equation (4). Finally, Section 6 is devoted to the proof of our main result given in Theorem 1.1.

## 2 Definitions and prerequisites

### 2.1 The class of Bounded Imaginary Powers of operators

**Definition 2.1.** A Banach space  $E$  is a UMD space if and only if for all  $p \in (1, +\infty)$ , the Hilbert transform is bounded from  $L^p(\mathbb{R}, E)$  into itself (see Bourgain [2] and Burkholder [3]).

**Definition 2.2.** Let  $\alpha \in (0, \pi)$ .  $\operatorname{Sect}(\alpha)$  denotes the space of closed linear operators  $T_1$  which satisfying

- i)  $\sigma(T_1) \subset \overline{S_\alpha}$ ,
- ii)  $\forall \alpha' \in (\alpha, \pi), \quad \sup \left\{ \|\lambda(\lambda I - T_1)^{-1}\|_{\mathcal{L}(X)} : \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}} \right\} < +\infty,$

where

$$S_\alpha := \begin{cases} \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \alpha\} & \text{if } \alpha \in (0, \pi] \\ (0, +\infty) & \text{if } \alpha = 0, \end{cases} \tag{6}$$

see Haase [9], p. 19. Such an operator  $T_1$  is called sectorial operator of angle  $\alpha$ .

**Remark 2.3.** From Komatsu [11], p. 342, we know that any injective sectorial operator  $T_1$  admits imaginary powers  $T_1^{is}$  for all  $s \in \mathbb{R}$ ; but in general,  $T_1^{is}$  is not bounded.

**Definition 2.4.** Let  $\kappa \in [0, \pi)$ . We denote by  $\text{BIP}(E, \kappa)$ , the class of sectorial injective operators  $T_2$  such that

- i)  $\overline{D(T_2)} = \overline{R(T_2)} = E$ ,
- ii)  $\forall s \in \mathbb{R}, \quad T_2^{is} \in \mathcal{L}(E)$ ,
- iii)  $\exists C \geq 1, \forall s \in \mathbb{R}, \quad \|T_2^{is}\|_{\mathcal{L}(E)} \leq Ce^{|s|\kappa}$ ,

see Prüss and Sohr [19], p. 430.

## 2.2 Recall on the sum of linear operators

Let us fix a pair of two closed linear densely defined operators  $\mathcal{M}_1$  and  $\mathcal{M}_2$  in a general Banach space  $\mathcal{E}$ . We note their domains by  $D(\mathcal{M}_1)$  and  $D(\mathcal{M}_2)$  respectively. Then we can define their sum by

$$\begin{cases} \mathcal{M}_1 w + \mathcal{M}_2 w \\ w \in D(\mathcal{M}_1) \cap D(\mathcal{M}_2). \end{cases}$$

We assume the following hypotheses

(H<sub>1</sub>) There exist  $\theta_{\mathcal{M}_1} \in [0, \pi)$ ,  $\theta_{\mathcal{M}_2} \in [0, \pi)$ ,  $C > 0$  and  $R > 0$  such that

$$\begin{cases} \rho(\mathcal{M}_1) \supset \Sigma_{1,R} = \{z \in \mathbb{C} \setminus \{0\} : |z| \geq R \text{ and } |\arg(z)| < \pi - \theta_{\mathcal{M}_1}\} \\ \forall z \in \Sigma_{1,R}, \quad \|(\mathcal{M}_1 - zI)^{-1}\| \leq \frac{C}{|z|}, \end{cases}$$

and

$$\begin{cases} \rho(\mathcal{M}_2) \supset \Sigma_{2,R} = \{z \in \mathbb{C} \setminus \{0\} : |z| \geq R \text{ and } |\arg(z)| < \pi - \theta_{\mathcal{M}_2}\} \\ \forall z \in \Sigma_{2,R}, \quad \|(\mathcal{M}_2 - zI)^{-1}\| \leq \frac{C}{|z|}, \end{cases}$$

with

$$\theta_{\mathcal{M}_1} + \theta_{\mathcal{M}_2} < \pi.$$

(H<sub>2</sub>)  $\sigma(\mathcal{M}_1) \cap \sigma(-\mathcal{M}_2) = \emptyset$ .

(H<sub>3</sub>) The resolvents of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  commute, that is

$$(\mathcal{M}_1 - \lambda_1 I)^{-1} (\mathcal{M}_2 - \lambda_2 I)^{-1} = (\mathcal{M}_2 - \lambda_2 I)^{-1} (\mathcal{M}_1 - \lambda_1 I)^{-1},$$

for all  $\lambda_1 \in \rho(\mathcal{M}_1)$  and all  $\lambda_2 \in \rho(\mathcal{M}_2)$ .

**Remark 2.5.** Note that from (H<sub>2</sub>), we have  $\rho(\mathcal{M}_1) \cup \rho(-\mathcal{M}_2) = \mathbb{C}$  and in particular  $\mathcal{M}_1$  or  $\mathcal{M}_2$  is boundedly invertible.

**Theorem 2.6** (Da Prato and Grisvard [4], Grisvard [8]). Assume that (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>3</sub>) hold. Then, operator  $\mathcal{M}_1 + \mathcal{M}_2$  is closable. Its closure  $\overline{\mathcal{M}_1 + \mathcal{M}_2}$  is boundedly invertible and

$$\left(\overline{\mathcal{M}_1 + \mathcal{M}_2}\right)^{-1} = \frac{-1}{2i\pi} \int_{\Gamma} (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} dz; \quad (7)$$

where  $\Gamma$  is a path which separates  $\sigma(\mathcal{M}_1)$  and  $\sigma(-\mathcal{M}_2)$  and joins  $\infty e^{-i\theta_0}$  to  $\infty e^{i\theta_0}$  with  $\theta_0$  such that

$$\theta_{\mathcal{M}_1} < \theta_0 < \pi - \theta_{\mathcal{M}_2}.$$

This Theorem is proved in Da Prato and Grisvard [4] (Theorem 3.7, p. 324), when  $R = 0$  and has been extended to the case  $R \geq 0$  in Grisvard [8] (Theorem 2.1, p. 7). In this last case, the curve  $\Gamma$  does not need to be connected.

**Corollary 2.7.** Assume that  $(H_1)$ ,  $(H_2)$  and  $(H_3)$  hold. Let  $i = 1, 2$  and  $\mathcal{E}_i$  a Banach space with  $D(\mathcal{M}_i) \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}$ . We suppose that there exist  $C > 0$  and  $\delta \in (0, 1)$  such that

$$\begin{cases} \|w\|_{\mathcal{E}_i} \leq C \left( \|w\|_{\mathcal{E}} + \|w\|_{\mathcal{E}}^{1-\delta} \|\mathcal{M}_i w\|_{\mathcal{E}}^{\delta} \right) \\ \text{for every } w \in D(\mathcal{M}_i). \end{cases} \quad (8)$$

Then  $D(\overline{\mathcal{M}_1 + \mathcal{M}_2}) \subset \mathcal{E}_i$ .

*Proof.* It is enough to prove that the integral in (7) converges in  $\mathcal{E}_1$ . For all  $\xi \in \mathcal{E}$ , we have

$$\left\| \int_{\Gamma} (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi dz \right\|_{\mathcal{E}_1} \leq \int_{\Gamma} \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}_1} |dz|,$$

then, applying (8), we obtain

$$\begin{aligned} & \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}_1} \\ & \leq C \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}} \\ & + C \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{1-\delta} \left\| \mathcal{M}_1 (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{\delta}. \end{aligned}$$

Now, for all  $z \in \Gamma$ , we have

$$\begin{aligned} & \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{1-\delta} \left\| \mathcal{M}_1 (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{\delta} \\ & \leq \frac{[C_1(\theta_{\mathcal{M}_1})C_2(\theta_{\mathcal{M}_2})]^{1-\delta} [C_1(\theta_{\mathcal{M}_1})]^{\delta} [C_2(\theta_{\mathcal{M}_2})]^{\delta}}{|z|^{2(1-\delta)} |z|^{\delta}} \|\xi\|_{\mathcal{E}} = \frac{C_1(\theta_{\mathcal{M}_1})C_2(\theta_{\mathcal{M}_2})}{|z|^{1+(1-\delta)}} \|\xi\|_{\mathcal{E}}, \end{aligned}$$

from which we deduce the convergence of the integral in (7).  $\square$

### 3 Spectral study of operators

In all the sequel, in view to apply the above results, we will consider the following particular Banach space

$$\mathcal{E} = L^p(0, +\infty; X) \quad \text{with} \quad X = W_0^{2,p}(0, \omega) \times L^p(0, \omega),$$

equipped with its natural norm.

#### 3.1 Study of operator $\mathcal{L}_{1,\mu}$

We study the spectral equation

$$\mathcal{L}_{1,\mu} V - \lambda V = R,$$

where  $V \in D(\mathcal{L}_{1,\mu})$ ,  $R \in \mathcal{E}$  and  $\lambda \in \mathbb{C}$  (which will be precised below), that is

$$\begin{cases} V''(t) - 2\mu V'(t) + (\mu^2 - \lambda)V(t) = R(t), & t > 0 \\ V(0) = 0, \quad V(+\infty) = 0. \end{cases} \quad (9)$$

We set

$$\Pi_{\mu} = \{z \in \mathbb{C} \setminus \mathbb{R}_- : \operatorname{Re}(\sqrt{z}) > \mu\}.$$

Now, let us specify  $\Pi_{\mu}$ . For all  $z = x + iy \in \mathbb{C} \setminus \mathbb{R}_-$ , we have

$$\operatorname{Re}(\sqrt{z}) > \mu \iff \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} > \mu \iff \sqrt{x^2 + y^2} > 2\mu^2 - x.$$

- First case : if  $x > \mu^2$ , we have  $\sqrt{x^2 + y^2} + x \geq 2x > 2\mu^2$ , then  $\operatorname{Re}(\sqrt{z}) > \mu$ .
- Second case : if  $x \leq \mu^2$ , then  $y^2 + 4\mu^2x - 4\mu^4 > 0$ . Thus, we deduce that  $\Pi_\mu$  is strictly outside the parabola of equation

$$y^2 + 4\mu^2x - 4\mu^4 = 0,$$

turned towards the negative real axis and passing through the points  $(\mu^2, 0)$ ,  $(0, 2\mu^2)$  and  $(0, -2\mu^2)$ .

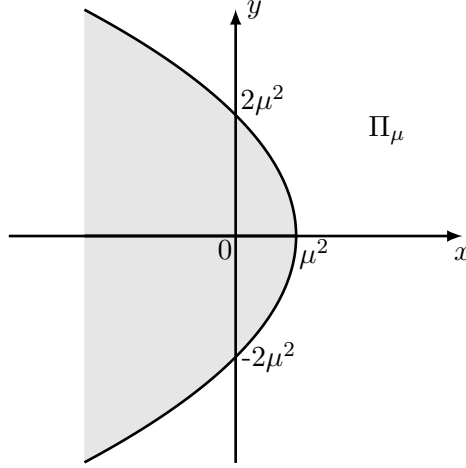


Figure 1: In this figure,  $\Pi_\mu$  is the entire uncolored area.

Now, let  $\varepsilon_{\mathcal{L}_{1,\mu}}$  be a small fixed positive number and consider the following set

$$\Sigma_{\mathcal{L}_{1,\mu}} := \left\{ \lambda \in \Pi_\mu, \quad |\arg(\lambda)| \leq \pi - 2\varepsilon_{\mathcal{L}_{1,\mu}} \quad \text{and} \quad |\lambda| \geq \frac{4\mu^2}{\sin^2(\varepsilon_{\mathcal{L}_{1,\mu}})} \right\}. \quad (10)$$

We then obtain the following proposition.

**Proposition 3.1.** The linear operator  $\mathcal{L}_{1,\mu}$  is closed and densely defined in  $\mathcal{E}$ . Moreover, there exists a constant  $M_{\mathcal{L}_{1,\mu}} > 0$  such that for all  $\lambda \in \Sigma_{\mathcal{L}_{1,\mu}}$ , operator  $\mathcal{L}_{1,\mu} - \lambda I$  is invertible and

$$\|(\mathcal{L}_{1,\mu} - \lambda I)^{-1}\|_{\mathcal{L}(\mathcal{E})} \leq \frac{M_{\mathcal{L}_{1,\mu}}}{|\lambda|}.$$

Therefore, assumption  $(H_1)$  in Section 2.2 is verified for  $\mathcal{L}_{1,\mu}$  with

$$\theta_{\mathcal{L}_{1,\mu}} = 2\varepsilon_{\mathcal{L}_{1,\mu}}. \quad (11)$$

*Proof.* Let  $\lambda \in \Pi_\mu$ . From Eltaief and Maingot [6], Theorem 2, p. 712, there exists a unique solution  $V \in W^{2,p}(0, +\infty; X)$  of problem (9), given by

$$\begin{aligned} V(t) &= \frac{e^{t(\mu - \sqrt{\lambda})}}{2\sqrt{\lambda}} \int_0^{+\infty} e^{-s(\mu + \sqrt{\lambda})} R(s) ds \\ &\quad - \frac{1}{2\sqrt{\lambda}} \left( \int_0^t e^{(t-s)(\mu - \sqrt{\lambda})} R(s) ds + \int_t^{+\infty} e^{-(s-t)(\mu + \sqrt{\lambda})} R(s) ds \right), \end{aligned} \quad (12)$$

see formula (15) in Eltaief and Maingot [6] where  $L_1 := -\mu I - \sqrt{\lambda} I$  and  $L_2 := \mu I - \sqrt{\lambda} I$ . It follows that  $\Pi_\mu \subset \rho(\mathcal{L}_{1,\mu})$ . This proves that  $\mathcal{L}_{1,\mu}$  is closed. The boundary conditions are verified by using Lemma 8, p. 718 in Eltaief and Maingot [6].

Moreover, from (12), we obtain

$$\begin{aligned} \|V\|_{\mathcal{E}} &\leq \frac{1}{2\sqrt{|\lambda|}} \left( \int_0^{+\infty} e^{-tp(\operatorname{Re}(\sqrt{\lambda})-\mu)} dt \right)^{1/p} \int_0^{+\infty} e^{-s(\mu+\operatorname{Re}(\sqrt{\lambda}))} \|R(s)\|_X ds \\ &\quad + \sup_{t \in \mathbb{R}_+} \left( \int_0^t |e^{(t-s)(\mu-\sqrt{\lambda})}| ds + \int_t^{+\infty} |e^{-(s-t)(\mu+\sqrt{\lambda})}| ds \right) \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}}, \end{aligned}$$

hence, noting  $q$  the conjugate exponent of  $p$ , we have

$$\begin{aligned} \|V\|_{\mathcal{E}} &\leq \left( \frac{1}{p(\operatorname{Re}(\sqrt{\lambda})-\mu)} \right)^{1/p} \left( \int_0^{+\infty} e^{-sq(\mu+\operatorname{Re}(\sqrt{\lambda}))} ds \right)^{1/q} \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\quad + \sup_{t \in \mathbb{R}_+} \left( \frac{1 - e^{-t(\operatorname{Re}(\sqrt{\lambda})-\mu)}}{\operatorname{Re}(\sqrt{\lambda})-\mu} + \frac{1}{\operatorname{Re}(\sqrt{\lambda})+\mu} \right) \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\leq \frac{1}{p^{1/p}(\operatorname{Re}(\sqrt{\lambda})-\mu)^{1/p}} \frac{1}{q^{1/q}(\operatorname{Re}(\sqrt{\lambda})+\mu)^{1/q}} \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\quad + \frac{2}{\operatorname{Re}(\sqrt{\lambda})-\mu} \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\leq \frac{2}{\operatorname{Re}(\sqrt{\lambda})-\mu} \frac{\|R\|_{\mathcal{E}}}{\sqrt{|\lambda|}}. \end{aligned}$$

Let  $\lambda = |\lambda|e^{i\arg(\lambda)} \in \Sigma_{\mathcal{L}_1, \mu}$ , then  $|\arg(\lambda)| \leq \pi - 2\varepsilon_{\mathcal{L}_1, \mu}$ . Thus

$$\begin{aligned} \operatorname{Re}(\sqrt{\lambda}) - \mu &\geq \sqrt{|\lambda|} \cos\left(\frac{\arg(\lambda)}{2}\right) - \mu \geq \sqrt{|\lambda|} \cos\left(\frac{\pi}{2} - \varepsilon_{\mathcal{L}_1, \mu}\right) - \mu \\ &\geq \sqrt{|\lambda|} \sin(\varepsilon_{\mathcal{L}_1, \mu}) - \frac{\sqrt{|\lambda|}}{2} \sin(\varepsilon_{\mathcal{L}_1, \mu}) \geq \frac{\sqrt{|\lambda|}}{2} \sin(\varepsilon_{\mathcal{L}_1, \mu}). \end{aligned}$$

We then obtain

$$\|V\|_{\mathcal{E}} \leq \frac{M_{\mathcal{L}_1, \mu}}{|\lambda|} \|R\|_{\mathcal{E}}, \quad (13)$$

where  $M_{\mathcal{L}_1, \mu} = \frac{4}{\sin(\varepsilon_{\mathcal{L}_1, \mu})}$ . □

### 3.2 Study of operator $\mathcal{L}_2$

The spectral properties of  $\mathcal{L}_2$  are the same as those of its realization  $-\mathcal{A}$ .

In all this subsection, we assume that

$$\lambda \leq 0.$$

We have to solve the following spectral equation in  $X$

$$\mathcal{A}\Psi - \lambda\Psi = F, \quad (14)$$

that is

$$\left( \begin{array}{cc} 0 & 1 \\ -\left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 & -2\left(\frac{\partial^2}{\partial \theta^2} - 1\right) \end{array} \right) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

with

$$F_1 \in W_0^{2,p}(0, \omega) \quad \text{and} \quad F_2 \in L^p(0, \omega).$$



We have to find the unique couple  $(\psi_1, \psi_2) \in \left(W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega)\right) \times W_0^{2,p}(0, \omega)$ , which satisfies the following system

$$\begin{cases} \psi_2 - \lambda\psi_1 & = F_1 \\ -\psi_1^{(4)} - 2\psi_1'' - \psi_1 - 2\psi_2'' + 2\psi_2 - \lambda\psi_2 & = F_2. \end{cases}$$

Thus, we first have to solve

$$\begin{cases} -\psi_1^{(4)} - 2\psi_1'' - \psi_1 - 2(\lambda\psi_1 + F_1)'' + (2 - \lambda)(\lambda\psi_1 + F_1) = F_2 \\ \psi_1 \in W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega), \end{cases}$$

that is

$$\begin{cases} -\psi_1^{(4)} - 2(1 + \lambda)\psi_1'' - (\lambda - 1)^2\psi_1 = F_2 + 2F_1'' + (\lambda - 2)F_1 \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{cases}$$

Set  $G_\lambda = -F_2 - 2(F_1'' - F_1) - \lambda F_1$ , it follows that the previous system writes

$$\begin{cases} \psi_1^{(4)} + 2(\lambda + 1)\psi_1'' + (\lambda - 1)^2\psi_1 = G_\lambda \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{cases} \quad (15)$$

Then, the characteristic equation

$$\chi^4 + 2(1 + \lambda)\chi^2 + (\lambda - 1)^2 = 0,$$

admits, for  $\lambda < 0$ , the following four distinct solutions

$$\begin{cases} \alpha_1 = \sqrt{-\lambda} + i, & \alpha_3 = -\alpha_1 \\ \alpha_2 = \sqrt{-\lambda} - i, & \alpha_4 = -\alpha_2, \end{cases} \quad (16)$$

and for  $\lambda = 0$ , two double solutions  $i$  and  $-i$ .

We have to distinguish below two cases :  $\lambda = 0$  and  $\lambda < 0$ .

### 3.2.1 Case $\lambda = 0$ : Invertibility of $\mathcal{A}$

**Proposition 3.2.**  $\mathcal{A}$  is boundedly invertible. Then :  $\exists \varepsilon_0 > 0 : \overline{B(0, \varepsilon_0)} \subset \rho(\mathcal{A})$ .

*Proof.* Here  $\lambda = 0$ . We have to solve (14). This is equivalent to solve (15) that is

$$\begin{cases} \psi_1^{(4)} + 2\psi_1'' + \psi_1 = -F_2 - 2(F_1'' - F_1) \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{cases} \quad (17)$$

From Thorel [22], Theorem 2.8, statement 2., there exists a unique classical solution of problem (17) that is  $(\psi_1, \psi_2) = (\psi_1, F_1) \in \left(W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega)\right) \times W_0^{2,p}(0, \omega)$ . We then deduce that there exists  $C_1 > 0$  such that

$$\|\psi_1''\|_{L^p(0, \omega)} \leq C_1 \left( \|F_2\|_{L^p(0, \omega)} + 2\|F_1\|_{W^{2,p}(0, \omega)} \right) \leq 2C_1\|F\|_X,$$

and from the Poincaré inequality, there exists  $C_\omega > 0$  such that

$$\|\psi_1\|_{W_0^{2,p}(0, \omega)} \leq C_\omega\|\psi_1''\|_{L^p(0, \omega)} \leq 2C_1C_\omega\|F\|_X.$$

Finally, since  $\psi_2 = F_1$ , we have

$$\|\psi_2\|_{L^p(0, \omega)} = \|F_1\|_{L^p(0, \omega)} \leq \|F_2\|_{L^p(0, \omega)} + \|F_1\|_{W^{2,p}(0, \omega)} = \|F\|_X.$$

□

### 3.2.2 Case $\lambda < 0$ : Spectral study of $\mathcal{A}$

In order to prove Proposition 3.9, we first have to state the following technical results.

**Lemma 3.3.** Let  $\alpha \in \mathbb{C} \setminus \{0\}$ ,  $a, b \in \mathbb{R}$  with  $a < b$  and  $f \in W_0^{2,p}(a, b)$ . For all  $x \in [a, b]$ , we set

$$K(x) = \int_a^x e^{-(x-s)\alpha} f(s) ds + \int_x^b e^{-(s-x)\alpha} f(s) ds.$$

Then, we have

$$K(x) = \frac{2}{\alpha} f(x) + \frac{1}{\alpha^2} \int_a^x e^{-(x-s)\alpha} f''(s) ds + \frac{1}{\alpha^2} \int_x^b e^{-(s-x)\alpha} f''(s) ds.$$

*Proof.* The result is easily obtained by two integrations by parts.  $\square$

Now, let us solve explicitly problem (15) for  $\lambda < 0$ .

**Proposition 3.4.** Problem (15) has a unique solution which can be written in the following form

$$\begin{aligned} \psi_1(\theta) &:= e^{-\theta\alpha_2}(\beta_1 + \beta_2 + \beta_3 + \beta_4) + e^{-(\omega-\theta)\alpha_2}(\beta_3 + \beta_4 - \beta_1 - \beta_2) + S(\theta) \\ &+ \left( e^{-\theta\alpha_1} - e^{-\theta\alpha_2} \right) (\beta_2 + \beta_4) + \left( e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2} \right) (\beta_4 - \beta_2), \end{aligned} \quad (18)$$

where the constants  $\beta_i$ ,  $i = 1, 2, 3, 4$ , and the particular solution  $S$  will be explicitly given below.

*Proof.* In order to apply results obtained in Labbas, Lemrabet, Maingot and Thorel [12] and Labbas, Maingot, Manceau and Thorel [13], we set

$$L_- = -\alpha_1 I, \quad M = -\alpha_2 I, \quad r_- = \alpha_1^2 - \alpha_2^2, \quad a = 0 \quad \text{and} \quad b = \omega.$$

Then, problem (15) can be written as

$$\begin{cases} \psi_1^{(4)} - (L_-^2 + M^2)\psi_1'' + L_-^2 M^2 \psi_1 = G_\lambda \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{cases} \quad (19)$$

From Labbas, Maingot, Manceau and Thorel [13], there exists a unique classical solution  $\psi_1$  of problem (19). Its representation formula, in the form (18), is explicitly given in Labbas, Lemrabet, Maingot and Thorel [12] by (14)-(15)-(16) or also in a more general framework in Labbas, Maingot and Thorel [15] by (22)-(30)-(31).

For the reader convenience, we will prove, by a long calculus, that

$$\begin{cases} \beta_1 &= \frac{1}{4i} U_1^{-1} \frac{1 - e^{-\omega\alpha_1}}{1 - e^{-\omega\alpha_2}} (J(0) - J(\omega)) \\ \beta_2 &= -\frac{1}{4i} U_1^{-1} (J(0) - J(\omega)) \\ \beta_3 &= -\frac{1}{4i} U_2^{-1} \frac{1 + e^{-\omega\alpha_1}}{1 + e^{-\omega\alpha_2}} (J(0) + J(\omega)) \\ \beta_4 &= \frac{1}{4i} U_2^{-1} (J(0) + J(\omega)), \end{cases} \quad (20)$$

with

$$\begin{cases} U_1 &:= 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda} e^{-\omega\sqrt{-\lambda}} \\ U_2 &:= 1 - e^{-2\omega\sqrt{-\lambda}} + 2\omega\sqrt{-\lambda} e^{-\omega\sqrt{-\lambda}}, \end{cases} \quad (21)$$

and for all  $\theta \in [0, \omega]$

$$\begin{aligned} S(\theta) &:= \frac{e^{-\theta\alpha_2}}{2\alpha_2(1-e^{-2\omega\alpha_2})} (J(0) - e^{-\omega\alpha_2}J(\omega)) - \frac{\lambda}{\alpha_1^2\alpha_2^2} F_1(\theta) \\ &\quad + \frac{e^{-(\omega-\theta)\alpha_2}}{2\alpha_2(1-e^{-2\omega\alpha_2})} (J(\omega) - e^{-\omega\alpha_2}J(0)) - \frac{1}{2\alpha_2} J(\theta), \end{aligned} \quad (22)$$

with

$$J(\theta) := \int_0^\theta e^{-(\theta-s)\alpha_2} v(s) ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} v(s) ds, \quad (23)$$

where

$$\begin{aligned} v(\theta) &:= \frac{e^{-\theta\alpha_1}}{2\alpha_1(1-e^{-2\omega\alpha_1})} (I(0) - e^{-\omega\alpha_1}I(\omega)) + \frac{\lambda}{\alpha_1^2\alpha_2^2} F_1''(\theta) \\ &\quad + \frac{e^{-(\omega-\theta)\alpha_1}}{2\alpha_1(1-e^{-2\omega\alpha_1})} (I(\omega) - e^{-\omega\alpha_1}I(0)) - \frac{1}{2\alpha_1} I(\theta), \end{aligned} \quad (24)$$

and

$$\begin{aligned} I(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1} \left( -F_2 - 2(F_1'' - F_1) + \frac{\lambda}{\alpha_1^2} F_1'' \right) (s) ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1} \left( -F_2 - 2(F_1'' - F_1) + \frac{\lambda}{\alpha_1^2} F_1'' \right) (s) ds. \end{aligned} \quad (25)$$

Let us begin our proof. Here, we are inspired by the formulas given by (15) and (16), p. 2943, in Labbas, Lemrabet, Maingot and Thorel [12], where the authors have used the notations  $F_-$ ,  $f_-$ ,  $U_-$  and  $V_-$  which are replaced respectively here by  $S$ ,  $G_\lambda$ ,  $U_1$  and  $U_2$ . Consequently, we have

$$\begin{aligned} S(\theta) &= \frac{Ze^{-\theta\alpha_2}}{2\alpha_2} \int_0^\omega e^{-s\alpha_2} v_0(s) ds + \frac{Ze^{-(\omega-\theta)\alpha_2}}{2\alpha_2} \int_0^\omega e^{-(\omega-s)\alpha_2} v_0(s) ds \\ &\quad - \frac{1}{2\alpha_2} \int_0^\theta e^{-(\theta-s)\alpha_2} v_0(s) ds - \frac{1}{2\alpha_2} \int_\theta^\omega e^{-(s-\theta)\alpha_2} v_0(s) ds \\ &\quad - \frac{Ze^{-\omega\alpha_2}e^{-\theta\alpha_2}}{2\alpha_2} \int_0^\omega e^{-(\omega-s)\alpha_2} v_0(s) ds \\ &\quad - \frac{Ze^{-\omega\alpha_2}e^{-(\omega-\theta)\alpha_2}}{2\alpha_2} \int_0^\omega e^{-s\alpha_2} v_0(s) ds, \end{aligned} \quad (26)$$

where

$$\begin{aligned} v_0(\theta) &:= \frac{We^{-\theta\alpha_1}}{2\alpha_1} \int_0^\omega e^{-s\alpha_1} G_\lambda(s) ds + \frac{We^{-(\omega-\theta)\alpha_1}}{2\alpha_1} \int_0^\omega e^{-(\omega-s)\alpha_1} G_\lambda(s) ds \\ &\quad - \frac{We^{-\omega\alpha_1}e^{-\theta\alpha_1}}{2\alpha_1} \int_0^\omega e^{-(\omega-s)\alpha_1} G_\lambda(s) ds \\ &\quad - \frac{We^{-\omega\alpha_1}e^{-(\omega-\theta)\alpha_1}}{2\alpha_1} \int_0^\omega e^{-s\alpha_1} G_\lambda(s) ds - \frac{1}{2\alpha_1} I_1(\theta), \end{aligned} \quad (27)$$

with  $Z := (1 - e^{-2\omega\alpha_2})^{-1}$ ,  $W := (1 - e^{-2\omega\alpha_1})^{-1}$  and

$$I_1(\theta) = \int_0^\theta e^{-(\theta-s)\alpha_1} G_\lambda(s) ds + \int_\theta^\omega e^{-(s-\theta)\alpha_1} G_\lambda(s) ds. \quad (28)$$

Then, since  $G_\lambda = -F_2 - 2(F_1'' - F_1) - \lambda F_1$ , we have

$$\begin{aligned} I_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1} (-F_2 - 2(F_1'' - F_1)) (s) ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1} (-F_2 - 2(F_1'' - F_1)) (s) ds \\ &\quad - \lambda \int_0^\theta e^{-(\theta-s)\alpha_1} F_1(s) ds - \lambda \int_\theta^\omega e^{-(s-\theta)\alpha_1} F_1(s) ds. \end{aligned}$$

From Lemma 3.3 and the fact that  $F_1 \in W_0^{2,p}(0, \omega)$ , it follows that

$$\begin{aligned}
I_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1} (-F_2 - 2(F_1'' - F_1))(s) ds \\
&\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1} (-F_2 - 2(F_1'' - F_1))(s) ds \\
&\quad - \frac{2\lambda}{\alpha_1} F_1(\theta) + \frac{\lambda}{\alpha_1^2} \left( \int_0^\theta e^{-(\theta-s)\alpha_1} F_1''(s) ds + \int_\theta^\omega e^{-(s-\theta)\alpha_1} F_1''(s) ds \right) \\
&= -\frac{2\lambda}{\alpha_1} F_1(\theta) + \int_0^\theta e^{-(\theta-s)\alpha_1} \left( -F_2 - 2(F_1'' - F_1) + \frac{\lambda}{\alpha_1^2} F_1'' \right) (s) ds \\
&\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1} \left( -F_2 - 2(F_1'' - F_1) + \frac{\lambda}{\alpha_1^2} F_1'' \right) (s) ds.
\end{aligned}$$

Thus  $I$ , given by (25), satisfies

$$I(\theta) = I_1(\theta) + \frac{2\lambda}{\alpha_1} F_1(\theta). \quad (29)$$

Note that, from (28) and (29), we have

$$\int_0^\omega e^{-s\alpha_1} G_\lambda(s) ds = I_1(0) = I(0) \quad \text{and} \quad \int_0^\omega e^{-(\omega-s)\alpha_1} G_\lambda(s) ds = I_1(\omega) = I(\omega).$$

Therefore, from (27), we deduce that for all  $\theta \in [0, \omega]$

$$\begin{aligned}
v_0(\theta) &= \frac{W e^{-\theta\alpha_1}}{2\alpha_1} (I(0) - e^{-\omega\alpha_1} I(\omega)) \\
&\quad + \frac{W e^{-(\omega-\theta)\alpha_1}}{2\alpha_1} (I(\omega) - e^{-\omega\alpha_1} I(0)) - \frac{1}{2\alpha_1} I_1(\theta) \\
&= \frac{W e^{-\theta\alpha_1}}{2\alpha_1} (I(0) - e^{-\omega\alpha_1} I(\omega)) + \frac{\lambda}{\alpha_1^2} F_1(\theta) \\
&\quad + \frac{W e^{-(\omega-\theta)\alpha_1}}{2\alpha_1} (I(\omega) - e^{-\omega\alpha_1} I(0)) - \frac{1}{2\alpha_1} I(\theta).
\end{aligned}$$

Set

$$v_1 = v_0 - \frac{\lambda}{\alpha_1^2} F_1, \quad (30)$$

and

$$J_1(\theta) = \int_0^\theta e^{-(\theta-s)\alpha_2} v_0(s) ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} v_0(s) ds; \quad (31)$$

then, due to Lemma 3.3, we obtain

$$\begin{aligned}
J_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_2} v_1(s) ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} v_1(s) ds \\
&\quad + \frac{\lambda}{\alpha_1^2} \left( \int_0^\theta e^{-(\theta-s)\alpha_2} F_1(s) ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} F_1(s) ds \right) \\
&= \int_0^\theta e^{-(\theta-s)\alpha_2} \left( v_1(s) + \frac{\lambda}{\alpha_1^2 \alpha_2} F_1''(s) \right) ds \\
&\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_2} \left( v_1(s) + \frac{\lambda}{\alpha_1^2 \alpha_2} F_1''(s) \right) ds + \frac{2\lambda}{\alpha_1^2 \alpha_2} F_1(\theta).
\end{aligned}$$

From (30), for all  $\theta \in [0, \omega]$ , we deduce that  $v$  given by (24) and  $J$  given by (23), satisfy

$$v(\theta) = v_1(\theta) + \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1''(\theta),$$

and

$$J(\theta) = J_1(\theta) - \frac{2\lambda}{\alpha_1^2 \alpha_2} F_1(\theta). \quad (32)$$

Note that, from (31) and (32), we have

$$\int_0^\omega e^{-s\alpha_2} v_0(s) ds = J_1(0) = J(0) \quad \text{and} \quad \int_0^\omega e^{-(\omega-s)\alpha_2} v_0(s) ds = J_1(\omega) = J(\omega).$$

Finally, from (23), (26), (32) and for all  $\theta \in [0, \omega]$ , we deduce that

$$\begin{aligned} S(\theta) &= \frac{Ze^{-\theta\alpha_2}}{2\alpha_2} (J(0) - e^{-\omega\alpha_2} J(\omega)) \\ &\quad + \frac{Ze^{-(\omega-\theta)\alpha_2}}{2\alpha_2} (J(\omega) - e^{-\omega\alpha_2} J(0)) - \frac{1}{2\alpha_2} J_1(\theta), \end{aligned}$$

which is (22).

Now, in order to compute  $\beta_i$ ,  $i = 1, \dots, 4$ , we must explicit  $U_1$ ,  $U_2$  and  $S'(0) \pm S'(\omega)$ .

$$\begin{aligned} U_1 &= 1 - e^{-\omega(\alpha_1 + \alpha_2)} - (\alpha_1^2 - \alpha_2^2)^{-1} (\alpha_1 + \alpha_2)^2 (e^{-\omega\alpha_2} - e^{-\omega\alpha_1}) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} + i\sqrt{-\lambda} (e^{-\omega(\sqrt{-\lambda}-i)} - e^{-\omega(\sqrt{-\lambda}+i)}) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} - 2\sqrt{-\lambda} e^{-\omega\sqrt{-\lambda}} \sin(\omega), \end{aligned}$$

and

$$\begin{aligned} U_2 &= 1 - e^{-\omega(\alpha_1 + \alpha_2)} + (\alpha_1^2 - \alpha_2^2)^{-1} (\alpha_1 + \alpha_2)^2 (e^{-\omega\alpha_2} - e^{-\omega\alpha_1}) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} - i\sqrt{-\lambda} (e^{-\omega(\sqrt{-\lambda}-i)} - e^{-\omega(\sqrt{-\lambda}+i)}) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} + 2\sqrt{-\lambda} e^{-\omega\sqrt{-\lambda}} \sin(\omega). \end{aligned}$$

From (22), it follows that

$$\begin{aligned} S'(\theta) &= -\frac{Ze^{-\theta\alpha_2}}{2} (J(0) - e^{-\omega\alpha_2} J(\omega)) - \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1'(\theta) \\ &\quad + \frac{Ze^{-(\omega-\theta)\alpha_2}}{2} (J(\omega) - e^{-\omega\alpha_2} J(0)) - \frac{1}{2\alpha_2} J'(\theta) \\ &= -\frac{Ze^{-\theta\alpha_2}}{2} (J(0) - e^{-\omega\alpha_2} J(\omega)) + \frac{Ze^{-(\omega-\theta)\alpha_2}}{2} (J(\omega) - e^{-\omega\alpha_2} J(0)) \\ &\quad + \frac{1}{2} \left( \int_0^\theta e^{-(\theta-s)\alpha_2} v(s) ds - \int_\theta^\omega e^{-(s-\theta)\alpha_2} v(s) ds \right) - \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1'(\theta), \end{aligned}$$

then

$$S'(0) + S'(\omega) = -\frac{J(0) - J(\omega)}{(1 - e^{-\omega\alpha_2})} \quad \text{and} \quad S'(0) - S'(\omega) = -\frac{J(0) + J(\omega)}{(1 + e^{-\omega\alpha_2})}.$$

from which we deduce the constants  $\beta_i$ ,  $i = 1, 2, 3, 4$ , written in (20).  $\square$

**Remark 3.5.** Since  $0 < \sin(\omega) < \omega$ , for all  $\omega > 0$ , then we have

$$U_1 = 1 - e^{-2\omega\sqrt{-\lambda}} - 2\sqrt{-\lambda} e^{-\omega\sqrt{-\lambda}} \sin(\omega) \geq 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda} e^{-\omega\sqrt{-\lambda}},$$

and

$$U_2 = 1 - e^{-2\omega\sqrt{-\lambda}} + 2\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}}\sin(\omega) \geq 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}}.$$

For  $x > 0$ , we consider the following function

$$h(x) = 1 - e^{-2x} - 2xe^{-x};$$

we then have

$$h'(x) = 2e^{-2x} - 2e^{-x} + 2xe^{-x} = 2e^{-x}(e^{-x} + x - 1) > 0.$$

It follows that  $h(x) > h(0) = 0$ , for all  $x > 0$ . Finally, we deduce that

$$U_1 \geq 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}} = h(\omega\sqrt{-\lambda}) > 0, \quad (33)$$

and

$$U_2 \geq 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}} = h(\omega\sqrt{-\lambda}) > 0. \quad (34)$$

**Lemma 3.6.** For all  $\lambda \in (-\infty, -\varepsilon_0]$ , where  $\varepsilon_0$  is defined in Proposition 3.2, functions  $J$ ,  $v$  and  $I$ , given by (23), (24) and (25), satisfy the following estimates

1.  $\|I\|_{L^p(0,\omega)} \leq \frac{2}{\sqrt{-\lambda}} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right).$
2.  $|I(0)| + |I(\omega)| \leq \frac{2}{\sqrt{-\lambda}^{1-1/p}} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right).$
3.  $\|v\|_{L^p(0,\omega)} \leq \frac{M_1}{-\lambda} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right),$   
where  $M_1 = 2 + \frac{2}{1 - e^{-2\omega\sqrt{\varepsilon_0}}}$ .
4.  $\|J\|_{L^p(0,\omega)} \leq \frac{2}{\sqrt{-\lambda}} \|v\|_{L^p(0,\omega)}.$
5.  $|J(0)| + |J(\omega)| \leq \frac{2}{\sqrt{-\lambda}^{1-1/p}} \|v\|_{L^p(0,\omega)}.$

*Proof.*

1. From (16) and (25), we obtain

$$\begin{aligned} \|I\|_{L^p(0,\omega)} &\leq \sup_{\theta \in [0,\omega]} \left( \int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}} ds + \int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}} ds \right) \|2F_1 - F_2\|_{L^p(0,\omega)} \\ &\quad + \sup_{\theta \in [0,\omega]} \left( \int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}} ds + \int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}} ds \right) \left\| \left( \frac{\lambda}{\alpha_1^2} - 2 \right) F_1'' \right\|_{L^p(0,\omega)} \\ &\leq \sup_{\theta \in [0,\omega]} \left( \frac{1 - e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}} + \frac{1 - e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}} \right) \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} \right) \\ &\quad + 3 \sup_{\theta \in [0,\omega]} \left( \frac{1 - e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}} + \frac{1 - e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}} \right) \|F_1''\|_{L^p(0,\omega)} \\ &\leq \frac{2}{\sqrt{-\lambda}} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right). \end{aligned}$$

2. Due to (16), (25) and the Hölder inequality, it follows

$$\begin{aligned}
|I(0)| + |I(\omega)| &\leq \int_0^\omega e^{-s\sqrt{-\lambda}} \left| -F_2(s) - 2(F_1''(s) - F_1(s)) + \frac{\lambda}{\alpha_1^2} F_1''(s) \right| ds \\
&\quad + \int_0^\omega e^{-(\omega-s)\sqrt{-\lambda}} \left| -F_2(s) - 2(F_1''(s) - F_1(s)) + \frac{\lambda}{\alpha_1^2} F_1''(s) \right| ds \\
&\leq \left( \int_0^\omega e^{-q(\omega-s)\sqrt{-\lambda}} ds \right)^{1/q} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) \\
&\quad + \left( \int_0^\omega e^{-sq\sqrt{-\lambda}} ds \right)^{1/q} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) \\
&\leq \frac{2(1 - e^{-\omega q\sqrt{-\lambda}})^{1/q}}{q^{1/q}\sqrt{-\lambda}^{1/q}} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) \\
&\leq \frac{2}{\sqrt{-\lambda}^{1-1/p}} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right).
\end{aligned}$$

3. From (16), (24) and the fact that  $|\alpha_1| = |\alpha_2| = \sqrt{1-\lambda} > \sqrt{-\lambda}$ , we have

$$\begin{aligned}
\|v\|_{L^p(0,\omega)} &\leq \frac{|I(0)| + |I(\omega)|}{2\sqrt{1-\lambda}(1 - e^{-2\omega\sqrt{\varepsilon_0}})} \left( \int_0^\omega e^{-p\theta\sqrt{-\lambda}} d\theta \right)^{1/p} + \frac{-\lambda}{(1-\lambda)^2} \|F_1''\|_{L^p(0,\omega)} \\
&\quad + \frac{|I(0)| + |I(\omega)|}{2\sqrt{1-\lambda}(1 - e^{-2\omega\sqrt{\varepsilon_0}})} \left( \int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}} d\theta \right)^{1/p} + \frac{1}{2\sqrt{1-\lambda}} \|I\|_{L^p(0,\omega)} \\
&\leq \frac{(|I(0)| + |I(\omega)|)}{\sqrt{1-\lambda}\sqrt{-\lambda}^{1/p} p^{1/p} (1 - e^{-2\omega\sqrt{\varepsilon_0}})} + \frac{1}{2\sqrt{1-\lambda}} \|I\|_{L^p(0,\omega)} \\
&\quad + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)},
\end{aligned}$$

hence

$$\begin{aligned}
\|v\|_{L^p(0,\omega)} &\leq \frac{2 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{\sqrt{1-\lambda}\sqrt{-\lambda}^{1/p+1/q} q^{1/q} p^{1/p} (1 - e^{-2\omega\sqrt{\varepsilon_0}})} \\
&\quad + \frac{\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}}{\sqrt{1-\lambda}\sqrt{-\lambda}} + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)} \\
&\leq \frac{M_1}{\sqrt{1-\lambda}\sqrt{-\lambda}} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right).
\end{aligned}$$

4. From (16) and (23), we have

$$\begin{aligned}
\|J\|_{L^p(0,\omega)} &\leq \sup_{\theta \in [0,\omega]} \left( \int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}} ds + \int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}} ds \right) \|v\|_{L^p(0,\omega)} \\
&\leq \sup_{\theta \in [0,\omega]} \left( \frac{1 - e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}} + \frac{1 - e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}} \right) \|v\|_{L^p(0,\omega)} \\
&\leq \frac{2}{\sqrt{-\lambda}} \|v\|_{L^p(0,\omega)}.
\end{aligned}$$

5. Due to (16), (23) and the Hölder inequality, we deduce that

$$\begin{aligned}
|J(0)| + |J(\omega)| &\leq \int_0^\omega e^{-s\sqrt{-\lambda}} |v(s)| ds + \int_0^\omega e^{-(\omega-s)\sqrt{-\lambda}} |v(s)| ds \\
&\leq \left( \left( \int_0^\omega e^{-sq\sqrt{-\lambda}} ds \right)^{1/q} + \left( \int_0^\omega e^{-(\omega-s)q\sqrt{-\lambda}} ds \right)^{1/q} \right) \|v\|_{L^p(0,\omega)} \\
&\leq \frac{2 \left(1 - e^{-\omega q\sqrt{-\lambda}}\right)^{1/q}}{q^{1/q} \sqrt{-\lambda}^{1/q}} \|v\|_{L^p(0,\omega)} \\
&\leq \frac{2}{\sqrt{-\lambda}^{1-1/p}} \|v\|_{L^p(0,\omega)}.
\end{aligned}$$

□

**Lemma 3.7.** Let  $\lambda < 0$ . Then, we have

$$\left( \int_0^\omega \left| e^{-\theta\alpha_1} - e^{-\theta\alpha_2} \right|^p d\theta \right)^{1/p} \leq \frac{4}{\sqrt{-\lambda}^{1+1/p}},$$

and

$$\left( \int_0^\omega \left| e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2} \right|^p d\theta \right)^{1/p} \leq \frac{4}{\sqrt{-\lambda}^{1+1/p}}.$$

*Proof.* For  $x \geq 0$ , we have  $e^{-\frac{px}{2}} x^p < 1$ , so

$$\int_0^{+\infty} e^{-px} x^p dx = \int_0^{+\infty} e^{-\frac{px}{2}} e^{-\frac{px}{2}} x^p dx \leq \int_0^{+\infty} e^{-\frac{px}{2}} dx = \frac{2}{p}.$$

Then, from (16), we have

$$\int_0^\omega \left| e^{-\theta\alpha_1} - e^{-\theta\alpha_2} \right|^p d\theta = \int_0^\omega \left| e^{-\theta\sqrt{-\lambda}} \left( e^{-\theta i} - e^{\theta i} \right) \right|^p d\theta = 2^p \int_0^\omega e^{-p\theta\sqrt{-\lambda}} |\sin(\theta)|^p d\theta,$$

hence, setting  $x = \theta\sqrt{-\lambda}$ , it follows that

$$\begin{aligned}
\int_0^\omega \left| e^{-\theta\alpha_1} - e^{-\theta\alpha_2} \right|^p d\theta &= 2^p \int_0^{\omega\sqrt{-\lambda}} e^{-px} \left| \sin\left(\frac{x}{\sqrt{-\lambda}}\right) \right|^p \frac{dx}{\sqrt{-\lambda}} \\
&\leq \frac{2^p}{\sqrt{-\lambda}} \int_0^{\omega\sqrt{-\lambda}} e^{-px} \left(\frac{x}{\sqrt{-\lambda}}\right)^p dx \\
&\leq \frac{2^p}{\sqrt{-\lambda}^{p+1}} \int_0^{+\infty} e^{-px} x^p dx \\
&\leq \frac{2^{p+1}}{p\sqrt{-\lambda}^{p+1}} \leq \frac{2^{2p}}{\sqrt{-\lambda}^{p+1}}.
\end{aligned}$$

The second estimate is obtained by change of variable, taking  $\omega - \theta$  instead of  $\theta$ . □

**Lemma 3.8.** For all  $\lambda \in (-\infty, -\varepsilon_0]$ , where  $\varepsilon_0$  is defined in Proposition 3.2, the constants  $\beta_1, \beta_2, \beta_3$  and  $\beta_4$ , defined by (20), satisfy

$$\max(|\beta_1 + \beta_2|, |\beta_3 + \beta_4|) \leq \frac{M_1 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right) \left( 1 - e^{-\omega\sqrt{\varepsilon_0}} \right)},$$



and

$$\max(|\beta_2|, |\beta_4|) \leq \frac{M_1 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right)},$$

where  $M_1 = 2 + \frac{2}{1 - e^{-2\omega\sqrt{\varepsilon_0}}}$ .

*Proof.* Recall that  $\beta_i$ ,  $i = 1, 2, 3, 4$ , depends on  $U_1^{-1}$  and  $U_2^{-1}$ . From (33) and (34), it follows

$$U_1 \geq h(\omega\sqrt{-\lambda}) \geq h(\omega\sqrt{\varepsilon_0}) = 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} > 0,$$

and

$$U_2 \geq h(\omega\sqrt{-\lambda}) \geq h(\omega\sqrt{\varepsilon_0}) = 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} > 0.$$

Thus, we deduce

$$U_1^{-1} \leq \frac{1}{1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}}} \quad \text{and} \quad U_2^{-1} \leq \frac{1}{1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}}}.$$

Therefore, from (20) and Lemma 3.6, we have

$$\begin{aligned} |\beta_1 + \beta_2| &\leq \frac{|J(0)| + |J(\omega)|}{4 \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right)} \left| \frac{1 - e^{-\omega\alpha_1}}{1 - e^{-\omega\alpha_2}} - 1 \right| \\ &\leq \frac{\|v\|_{L^p(0,\omega)}}{2\sqrt{-\lambda}^{1-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right)} \left| \frac{e^{-\omega\alpha_2} - e^{-\omega\alpha_1}}{1 - e^{-\omega\alpha_2}} \right| \\ &\leq \frac{M_1 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right)} \frac{2e^{-\omega\sqrt{-\lambda}}}{1 - e^{-\omega\sqrt{\varepsilon_0}}} \\ &\leq \frac{M_1 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right) \left( 1 - e^{-\omega\sqrt{\varepsilon_0}} \right)} \end{aligned}$$

and similarly

$$\begin{aligned} |\beta_3 + \beta_4| &\leq \frac{|J(0)| + |J(\omega)|}{4 \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right)} \left| 1 - \frac{1 + e^{-\omega\alpha_1}}{1 + e^{-\omega\alpha_2}} \right| \\ &\leq \frac{M_1 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right) \left( 1 - e^{-\omega\sqrt{\varepsilon_0}} \right)}. \end{aligned}$$

In the same way, we obtain

$$|\beta_2| \leq \frac{M_1 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right)},$$

and

$$|\beta_4| \leq \frac{M_1 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p} \left( 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0} e^{-\omega\sqrt{\varepsilon_0}} \right)}.$$

□

**Proposition 3.9.**  $\mathcal{A}$  is closed and densely defined in  $X$ . Moreover, there exists a constant  $M > 0$  such that for all  $\lambda \leq 0$ , operator  $\mathcal{A} - \lambda I$  is invertible with bounded inverse and

$$\|(\mathcal{A} - \lambda I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|}.$$

**Remark 3.10.** This result implies, in particular, that  $-\sqrt{\mathcal{A}}$  is well-defined and generates a uniformly bounded analytic semigroup  $(e^{-s\sqrt{\mathcal{A}}})_{s \geq 0}$ .

*Proof.* Recall that

$$D(\mathcal{A}) = [W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega)] \times W_0^{2,p}(0, \omega).$$

It is clear that

$$\mathcal{D}(0, \omega) \times \mathcal{D}(0, \omega) \subset D(\mathcal{A}) \subset X = W_0^{2,p}(0, \omega) \times L^p(0, \omega),$$

where  $\mathcal{D}(0, \omega)$  is the set of  $C^\infty$ -functions with compact support in  $(0, \omega)$ . Since  $\mathcal{D}(0, \omega)$  is dense in each spaces  $W_0^{2,p}(0, \omega)$  and  $L^p(0, \omega)$  for their respective norms, then  $D(\mathcal{A})$  is dense in  $X$ .

From Proposition 3.2,  $0 \in \rho(\mathcal{A})$ , thus  $\mathcal{A}$  is closed. From Proposition 3.4, for all  $\lambda < 0$ , there exists a unique couple

$$(\psi_1, \psi_2) \in (W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega)) \times W_0^{2,p}(0, \omega)$$

which satisfies

$$\begin{cases} \psi_2 & = \lambda\psi_1 + F_1 \\ \psi_1^{(4)} + 2(\lambda + 1)\psi_1'' + (\lambda - 1)^2\psi_1 & = G_\lambda, \end{cases} \quad (35)$$

where  $G_\lambda = -F_2 - 2(F_1'' - F_1) - \lambda F_1$ . Then  $\mathbb{R}_- \subset \rho(\mathcal{A})$  and

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (\mathcal{A} - \lambda I)^{-1} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = (\mathcal{A} - \lambda I)^{-1} F,$$

where  $\psi_1$  is given by (18)-(20)-(22) and

$$\begin{aligned} \psi_2(\theta) &:= \lambda e^{-\theta\alpha_2}(\beta_1 + \beta_2 + \beta_3 + \beta_4) + \lambda e^{-(\omega-\theta)\alpha_2}(\beta_3 + \beta_4 - \beta_1 - \beta_2) \\ &+ \lambda (e^{-\theta\alpha_1} - e^{-\theta\alpha_2})(\beta_2 + \beta_4) + \lambda (e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2})(\beta_4 - \beta_2) \\ &+ \lambda S(\theta) + F_1(\theta), \end{aligned} \quad (36)$$

with  $\beta_i$ ,  $i = 1, 2, 3, 4$  are defined in (20)-(21). From (22), we have

$$\begin{aligned} \lambda S(\theta) + F_1(\theta) &= \frac{\lambda}{2\alpha_2(1 - e^{-2\omega\alpha_2})} e^{-\theta\alpha_2} (J(0) - e^{-\omega\alpha_2} J(\omega)) \\ &+ \frac{\lambda}{2\alpha_2(1 - e^{-2\omega\alpha_2})} e^{-(\omega-\theta)\alpha_2} (J(\omega) - e^{-\omega\alpha_2} J(0)) \\ &- \frac{\lambda^2}{\alpha_1^2 \alpha_2^2} F_1(\theta) + F_1(\theta) - \frac{\lambda}{2\alpha_2} J(\theta), \end{aligned} \quad (37)$$

where  $J(\theta)$  is given by (23). Our aim is to prove that, for all  $\lambda \leq 0$ , there exists  $M > 0$ , such that

$$\|(\mathcal{A} - \lambda I)^{-1} F\|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|} \|F\|_X,$$

with

$$\|F\|_X = \left\| \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \right\|_X = \|F_1\|_{W_0^{2,p}(0, \omega)} + \|F_2\|_{L^p(0, \omega)}. \quad (38)$$

To this end, we consider that  $\lambda \in (-\infty, -\varepsilon_0]$ , where  $\varepsilon_0$  is defined in Proposition 3.2. We first study  $S''$  and  $\psi_1''$ . From (22), for *a.e.*  $\theta \in [0, \omega]$ , we have

$$\begin{aligned} S''(\theta) &= \frac{\alpha_2 e^{-\theta\alpha_2}}{2(1 - e^{-2\omega\alpha_2})} (J(0) - e^{-\omega\alpha_2}J(\omega)) - \frac{\lambda}{\alpha_1^2\alpha_2^2} F_1''(\theta) \\ &\quad + \frac{\alpha_2 e^{-(\omega-\theta)\alpha_2}}{2(1 - e^{-2\omega\alpha_2})} (J(\omega) - e^{-\omega\alpha_2}J(0)) - \frac{1}{2\alpha_2} J''(\theta), \end{aligned}$$

and from (23), we obtain  $J''(\theta) = \alpha_2^2 J(\theta) - 2\alpha_2 v(\theta)$ , hence

$$\begin{aligned} S''(\theta) &= \frac{\alpha_2 e^{-\theta\alpha_2}}{2(1 - e^{-2\omega\alpha_2})} (J(0) - e^{-\omega\alpha_2}J(\omega)) - \frac{\lambda}{\alpha_1^2\alpha_2^2} F_1''(\theta) \\ &\quad + \frac{\alpha_2 e^{-(\omega-\theta)\alpha_2}}{2(1 - e^{-2\omega\alpha_2})} (J(\omega) - e^{-\omega\alpha_2}J(0)) - \frac{\alpha_2}{2} J(\theta) + v(\theta). \end{aligned}$$

Then, since  $\alpha_1 = \sqrt{-\lambda} + i$  and  $\alpha_2 = \sqrt{-\lambda} - i$ , we have  $|e^{-\omega\alpha_1}| = |e^{-\omega\alpha_2}| = e^{-\omega\sqrt{-\lambda}} \leq 1$  with  $-\lambda \geq \varepsilon_0$ , thus

$$\begin{aligned} \|S''\|_{L^p(0,\omega)} &\leq \frac{\sqrt{1-\lambda} (|J(0)| + |J(\omega)|)}{2(1 - e^{-2\omega\sqrt{\varepsilon_0}})} \left( \int_0^\omega e^{-p\theta\sqrt{-\lambda}} d\theta \right)^{1/p} \\ &\quad + \frac{\sqrt{1-\lambda} (|J(0)| + |J(\omega)|)}{2(1 - e^{-2\omega\sqrt{\varepsilon_0}})} \left( \int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}} d\theta \right)^{1/p} \\ &\quad + \frac{-\lambda}{(1-\lambda)^2} \|F_1''\|_{L^p(0,\omega)} + \frac{\sqrt{1-\lambda}}{2} \|J\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} \\ &\leq \frac{\sqrt{1-\lambda} (|J(0)| + |J(\omega)|)}{\sqrt{-\lambda}^{1/p} (1 - e^{-2\omega\sqrt{\varepsilon_0}})} + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)} \\ &\quad + \frac{\sqrt{1-\lambda}}{2} \|J\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)}. \end{aligned}$$

From Lemma 3.6, we have

$$\begin{aligned} \|S''\|_{L^p(0,\omega)} &\leq \frac{2\sqrt{1-\lambda}}{\sqrt{-\lambda}^{1-1/p+1/p} (1 - e^{-2\omega\sqrt{\varepsilon_0}})} \|v\|_{L^p(0,\omega)} \\ &\quad + \frac{\sqrt{1-\lambda}}{\sqrt{-\lambda}} \|v\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)} \\ &\leq \frac{2M_1}{-\lambda (1 - e^{-2\omega\sqrt{\varepsilon_0}})} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) \\ &\quad + \frac{2M_1}{-\lambda} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)}. \end{aligned}$$

Finally, we obtain

$$\|S''\|_{L^p(0,\omega)} \leq \frac{M_2}{-\lambda} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right), \quad (39)$$

where  $M_2 = \frac{2M_1}{1 - e^{-2\omega\sqrt{\varepsilon_0}}} + 2M_1 + 1$ .

Now, we have

$$\begin{aligned}\psi_1''(\theta) - S''(\theta) &= \alpha_2^2 e^{-\theta\alpha_2}(\beta_1 + \beta_2 + \beta_3 + \beta_4) + \alpha_2^2 e^{-(\omega-\theta)\alpha_2}(\beta_3 + \beta_4 - \beta_1 - \beta_2) \\ &\quad + \alpha_2^2 \left( e^{-\theta\alpha_1} - e^{-\theta\alpha_2} \right) (\beta_2 + \beta_4) + \alpha_2^2 \left( e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2} \right) (\beta_4 - \beta_2).\end{aligned}$$

Then

$$\begin{aligned}\|\psi_1'' - S''\|_{L^p(0,\omega)} &\leq (1-\lambda)(|\beta_1 + \beta_2| + |\beta_3 + \beta_4|) \left( \int_0^\omega e^{-p\theta\sqrt{-\lambda}} d\theta \right)^{1/p} \\ &\quad + (1-\lambda)(|\beta_1 + \beta_2| + |\beta_3 + \beta_4|) \left( \int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}} d\theta \right)^{1/p} \\ &\quad + (1-\lambda)(|\beta_2| + |\beta_4|) \left( \int_0^\omega |e^{-\theta\alpha_1} - e^{-\theta\alpha_2}|^p d\theta \right)^{1/p} \\ &\quad + (1-\lambda)(|\beta_2| + |\beta_4|) \left( \int_0^\omega |e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2}|^p d\theta \right)^{1/p}.\end{aligned}$$

Using the fact that

$$\frac{1-\lambda}{-\lambda} = 1 + \frac{1}{-\lambda} \leq 1 + \frac{1}{\varepsilon_0},$$

with Lemma 3.7 and Lemma 3.8, we obtain

$$\begin{aligned}\|\psi_1'' - S''\|_{L^p(0,\omega)} &\leq \frac{2(1-\lambda)(|\beta_1 + \beta_2| + |\beta_3 + \beta_4|)}{\sqrt{-\lambda}^{1/p}} + \frac{8(1-\lambda)(|\beta_2| + |\beta_4|)}{\sqrt{-\lambda}^{1+1/p}} \\ &\leq \frac{M_3 \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right)}{-\lambda},\end{aligned}$$

where  $M_3 = \frac{4M_1(1+\frac{1}{\omega})\left(1+\frac{1}{\varepsilon_0}\right)}{(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}e^{-\omega\sqrt{\varepsilon_0}})(1-e^{-\omega\sqrt{\varepsilon_0}})}$ .

Due to (39), it follows that

$$\begin{aligned}\|\psi_1''\|_{L^p(0,\omega)} &\leq \|\psi_1'' - S''\|_{L^p(0,\omega)} + \|S''\|_{L^p(0,\omega)} \\ &\leq \frac{M_3 + M_2}{-\lambda} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) \\ &\leq \frac{3(M_2 + M_3)}{-\lambda} \|F\|_X.\end{aligned}$$

From the Poincaré inequality, there exists  $C_\omega > 0$  such that

$$\|\psi_1\|_{W_0^{2,p}(0,\omega)} \leq C_\omega \|\psi_1''\|_{L^p(0,\omega)} \leq \frac{3C_\omega(M_2 + M_3)}{-\lambda} \|F\|_X. \quad (40)$$

Now, we focus ourselves on  $\|\psi_2\|_{L^p(0,\omega)}$ . As previously, we obtain

$$\begin{aligned}\|\psi_2\|_{L^p(0,\omega)} &\leq |\lambda|(|\beta_1 + \beta_2| + |\beta_3 + \beta_4|) \left( \int_0^\omega e^{-p\theta\sqrt{-\lambda}} d\theta \right)^{1/p} \\ &\quad + |\lambda|(|\beta_1 + \beta_2| + |\beta_3 + \beta_4|) \left( \int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}} d\theta \right)^{1/p} \\ &\quad + |\lambda|(|\beta_2| + |\beta_4|) \left( \int_0^\omega |e^{-\theta\alpha_1} - e^{-\theta\alpha_2}|^p d\theta \right)^{1/p} \\ &\quad + |\lambda|(|\beta_4| + |\beta_2|) \left( \int_0^\omega |e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2}|^p d\theta \right)^{1/p} + \|\lambda S + F_1\|_{L^p(0,\omega)} \\ &\leq \frac{M_2 + M_3}{-\lambda} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) \\ &\quad + \|\lambda S + F_1\|_{L^p(0,\omega)}.\end{aligned}$$

Moreover, from (37) and Lemma 3.6, we deduce that

$$\begin{aligned}
\|\lambda S + F_1\|_{L^p(0,\omega)} &\leq \frac{-\lambda(|J(0)| + |J(\omega)|)}{2\sqrt{1-\lambda}(1 - e^{-2\omega\sqrt{\varepsilon_0}})} \left( \int_0^\omega e^{-p\theta\sqrt{-\lambda}} d\theta \right)^{1/p} \\
&\quad + \frac{-\lambda(|J(0)| + |J(\omega)|)}{2\sqrt{1-\lambda}(1 - e^{-2\omega\sqrt{\varepsilon_0}})} \left( \int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}} d\theta \right)^{1/p} \\
&\quad + \left| 1 - \frac{\lambda^2}{\alpha_1^2\alpha_2^2} \right| \|F_1\|_{L^p(0,\omega)} + \frac{-\lambda}{2\sqrt{1-\lambda}} \|J\|_{L^p(0,\omega)} \\
&\leq \frac{-\lambda(|J(0)| + |J(\omega)|)}{\sqrt{-\lambda}^{1+1/p}(1 - e^{-2\omega\sqrt{\varepsilon_0}})} + \left| 1 - \frac{\lambda^2}{(1-\lambda)^2} \right| \|F_1\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} \\
&\leq \frac{-2\lambda}{\sqrt{-\lambda}^2(1 - e^{-2\omega\sqrt{\varepsilon_0}})} \|v\|_{L^p(0,\omega)} + \frac{1-2\lambda}{(1-\lambda)^2} \|F_1\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} \\
&\leq \left( \frac{2}{1 - e^{-2\omega\sqrt{\varepsilon_0}}} + 1 \right) \|v\|_{L^p(0,\omega)} + \left( \frac{1}{\lambda^2} + \frac{1}{-\lambda} \right) \|F_1\|_{L^p(0,\omega)} \\
&\leq \left( \frac{2}{1 - e^{-2\omega\sqrt{\varepsilon_0}}} + 1 \right) \|v\|_{L^p(0,\omega)} + \frac{\frac{1}{\varepsilon_0} + 1}{-\lambda} \|F_1\|_{L^p(0,\omega)}.
\end{aligned}$$

Then, from Lemma 3.6, we obtain

$$\|\lambda S + F_1\|_{L^p(0,\omega)} \leq \frac{M_4}{-\lambda} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right),$$

where  $M_4 = \left( \frac{2}{1 - e^{-2\omega\sqrt{\varepsilon_0}}} + 1 \right) M_1 + \frac{1}{\varepsilon_0} + 1$ .

Thus, it follows that

$$\begin{aligned}
\|\psi_2\|_{L^p(0,\omega)} &\leq \frac{M_2 + M_3 + M_4}{-\lambda} \left( \|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)} \right) \\
&\leq \frac{3(M_2 + M_3 + M_4)}{-\lambda} \|F\|_X.
\end{aligned}$$

Finally, from (40), we have

$$\|(\mathcal{A} - \lambda I)^{-1}F\|_X = \|\psi_1\|_{W_0^{2,p}(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)} \leq \frac{M}{|\lambda|} \|F\|_X,$$

where  $M = 3((C_\omega + 1)(M_2 + M_3) + M_4)$ . □

Since  $-\mathcal{A}$  is the realization of  $\mathcal{L}_2$ , we deduce the following corollary.

**Corollary 3.11.** There exist  $\varepsilon_{\mathcal{L}_2} \in (0, \pi)$  small enough and  $M_{\mathcal{L}_2} > 0$  such that

$$\forall z \in \Sigma_{\mathcal{L}_2} := \overline{B(0, \varepsilon_0)} \cup \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \varepsilon_{\mathcal{L}_2}\},$$

we have

$$\left\| (\mathcal{L}_2 - zI)^{-1} \right\|_{\mathcal{L}(X)} \leq \frac{M_{\mathcal{L}_2}}{1 + |z|}.$$

Therefore, assumption  $(H_1)$  in Section 2.2 is verified for  $\mathcal{L}_2$  with

$$\theta_{\mathcal{L}_2} = \pi - \varepsilon_{\mathcal{L}_2}. \tag{41}$$

**Remark 3.12.**  $\mathcal{A}$  is anti-compact; since  $\sigma(-\mathcal{L}_2) = \sigma(\mathcal{A})$  then  $\sigma(-\mathcal{L}_2)$  is uniquely composed by isolated eigenvalues  $(\lambda_j)_{j \geq 1}$  such that  $|\lambda_j| \rightarrow +\infty$ , see Kato [10], Theorem 6.29, p. 187. More precisely, the calculus of the resolvent operator  $(\mathcal{A} - \lambda I)^{-1}$  requires that, for all  $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ ,  $U_1$  and  $U_2$  defined by (21) do not vanish. Since  $U_1 U_2 = 0$  is equivalent to

$$\left( \sinh(\omega\sqrt{-\lambda}) - \omega\sqrt{-\lambda} \right) \left( \sinh(\omega\sqrt{-\lambda}) + \omega\sqrt{-\lambda} \right) = 0,$$

then, using  $(z_j)_{j \geq 1}$  defined in Section 1, we deduce that

$$\forall j \geq 1, \quad \lambda_j = -\frac{z_j^2}{\omega^2} \in \mathbb{C} \setminus \mathbb{R}_+.$$

Now, we prove that operator  $\mathcal{A}$  has Bounded Imaginary Powers, see Definition 2.4.

**Proposition 3.13.**  $\mathcal{A} \in \text{BIP}(X, \theta_{\mathcal{A}})$ , for any  $\theta_{\mathcal{A}} \in (0, \pi)$ .

*Proof.* We will be inspired by the method used in Labbas and Moussaoui [16] or in Labbas and Sadallah [17].

Let  $\varepsilon > 0$  and  $r \in \mathbb{R}$ . For all  $\lambda > 0$ ,  $F_1 \in W_0^{2,p}(0, \omega)$  and  $F_2 \in L^p(0, \omega)$ , we have

$$\begin{aligned} \left[ (\mathcal{A} + I)^{-\varepsilon+ir} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \right] (\theta) &= \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \lambda^{-\varepsilon+ir} \left[ (\mathcal{A} + I + \lambda I)^{-1} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \right] (\theta) d\lambda \\ &= \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \lambda^{-\varepsilon+ir} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\theta) d\lambda \end{aligned}$$

where  $\Gamma_{\varepsilon,r} = \Gamma(1 - \varepsilon + ir)\Gamma(\varepsilon - ir)$ , see for instance Triebel [23], (6), p. 100.

Now, let us focus ourselves on the first component  $\psi_1$  for instance. Due to Seeley [21], we only consider the convolution term in  $\psi_1$ , which is the most singular term. In our case, this term is given by

$$\begin{aligned} I_S(\theta) &= \frac{1}{4\alpha_1\alpha_2} \int_0^\omega e^{-|\theta-s|\alpha_2} \int_0^\omega e^{-|s-t|\alpha_1} G_{\lambda+1}(t) dt ds \\ &= \frac{1}{4\alpha_1\alpha_2} \int_0^\omega e^{-|\theta-s|\alpha_2} \int_0^\omega e^{-|s-t|\alpha_1} (-F_2 - 2(F_1'' - F_1) + (\lambda + 1)F_1)(t) dt ds, \end{aligned}$$

see (26) and (27). We will use the two following extensions

$$\widetilde{G}_0(\theta) = \begin{cases} -F_2(\theta) - 2F_1''(\theta) + 2F_1(\theta), & \text{if } \theta \in [0, \omega] \\ 0, & \text{else,} \end{cases} \quad \text{with} \quad \widetilde{F}_1(\theta) = \begin{cases} F_1(\theta), & \text{if } \theta \in [0, \omega] \\ 0, & \text{else,} \end{cases}$$

and

$$E_\alpha(\theta) = e^{-|\theta|\alpha}.$$

Now, we will use the Fourier transform denoted by  $\mathcal{F}_t$ . We then have

$$\begin{aligned} I_{\varepsilon,r}(\theta) &:= \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \lambda^{-\varepsilon+ir} I_S(\theta) d\lambda \\ &= \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \left( E_{\alpha_2} \star \left( E_{\alpha_1} \star \left( \widetilde{G}_0 + (\lambda + 1)\widetilde{F}_1 \right) \right) \right) (\theta) d\lambda \\ &= \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t^{-1} \left( \mathcal{F}_t \left( E_{\alpha_2} \star \left( E_{\alpha_1} \star \left( \widetilde{G}_0 + (\lambda + 1)\widetilde{F}_1 \right) \right) \right) (\xi) \right) (\theta) d\lambda, \end{aligned}$$

hence

$$\begin{aligned}
I_{\varepsilon,r}(\theta) &= \mathcal{F}_t^{-1} \left( \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) \mathcal{F}_t(\widetilde{G}_0 + (\lambda+1)\widetilde{F}_1)(\xi) d\lambda \right) (\theta) \\
&= \mathcal{F}_t^{-1} \left( \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) d\lambda \mathcal{F}_t(\widetilde{G}_0)(\xi) \right) (\theta) \\
&\quad + \mathcal{F}_t^{-1} \left( \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) d\lambda \mathcal{F}_t(\widetilde{F}_1)(\xi) \right) (\theta).
\end{aligned}$$

We recall that

$$\mathcal{F}_t(E_\alpha)(\xi) = \frac{2\alpha}{\alpha^2 + 4\pi^2\xi^2},$$

here  $\alpha_1 = \sqrt{\lambda+1} + i$  and  $\alpha_2 = \sqrt{\lambda+1} - i$ . Hence

$$\begin{aligned}
\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) &= \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \frac{4\alpha_1\alpha_2}{(\alpha_1^2 + 4\pi^2\xi^2)(\alpha_2^2 + 4\pi^2\xi^2)} \\
&= \frac{\lambda^{-\varepsilon+ir}}{\alpha_1^2\alpha_2^2 + 4\pi^2\xi^2(\alpha_1^2 + \alpha_2^2) + 16\pi^4\xi^4} \\
&= \frac{\lambda^{-\varepsilon+ir}}{\lambda^2 + 4(1 + 2\pi^2\xi^2)\lambda + 4(1 + 4\pi^4\xi^4)} \\
&= \frac{\lambda^{-\varepsilon+ir}}{(\lambda + \lambda_1)(\lambda + \lambda_2)},
\end{aligned}$$

where

$$\begin{cases} \lambda_1 = 2 + 4\pi\xi + 4\pi^2\xi^2 = 4\pi^2 \left( \xi - \frac{(1+i)}{2\pi} \right) \left( \xi - \frac{(1-i)}{2\pi} \right) \\ \lambda_2 = 2 - 4\pi\xi + 4\pi^2\xi^2 = 4\pi^2 \left( \xi + \frac{(1+i)}{2\pi} \right) \left( \xi + \frac{(1-i)}{2\pi} \right). \end{cases}$$

Thus, since

$$\frac{1}{(\lambda + \lambda_1)(\lambda + \lambda_2)} = \frac{1}{\lambda_1 - \lambda_2} \left( -\frac{1}{\lambda + \lambda_1} + \frac{1}{\lambda + \lambda_2} \right),$$

it follows that

$$\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) = \frac{1}{8\pi\xi} \left( -\frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_1} + \frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_2} \right).$$

Then, setting

$$\sigma_1 = \frac{\lambda}{\lambda_1} \quad \text{and} \quad \sigma_2 = \frac{\lambda}{\lambda_2},$$

we obtain

$$\begin{aligned}
\int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) d\lambda &= \frac{1}{8\pi\xi} \left( -\int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_1} d\lambda + \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_2} d\lambda \right) \\
&= -\frac{\lambda_1^{-\varepsilon+ir}}{8\pi\xi} \int_0^{+\infty} \frac{\sigma_1^{-\varepsilon+ir}}{\sigma_1 + 1} d\sigma_1 \\
&\quad + \frac{\lambda_2^{-\varepsilon+ir}}{8\pi\xi} \int_0^{+\infty} \frac{\sigma_2^{-\varepsilon+ir}}{\sigma_2 + 1} d\sigma_2.
\end{aligned}$$

Moreover, for all  $z \in \mathbb{C} \setminus \mathbb{N}^-$ , where  $\mathbb{N}^-$  is the set of negative integer, we have

$$\int_0^{+\infty} \frac{\sigma^{-z}}{\sigma + 1} d\sigma = \Gamma(z)\Gamma(1-z). \quad (42)$$

It follows that

$$\int_0^{+\infty} \frac{\sigma^{-\varepsilon+ir}}{\sigma+1} d\sigma = \Gamma(\varepsilon - ir)\Gamma(1 - \varepsilon + ir) = \Gamma_{\varepsilon,r}, \quad (43)$$

hence

$$\begin{aligned} \frac{1}{\Gamma_{\varepsilon,r}} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) d\lambda &= \frac{1}{8\pi\xi \Gamma_{\varepsilon,r}} \left( \lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir} \right) \Gamma_{\varepsilon,r} \\ &= \frac{1}{8\pi\xi} \left( \lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir} \right). \end{aligned}$$

In the same way, we have

$$\begin{aligned} \frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) &= \frac{1}{\lambda_1 - \lambda_2} \left( -\frac{(\lambda+1)\lambda^{-\varepsilon+ir}}{\lambda + \lambda_1} + \frac{(\lambda+1)\lambda^{-\varepsilon+ir}}{\lambda + \lambda_2} \right) \\ &= \frac{1}{8\pi\xi} \left( -\frac{\lambda^{1-\varepsilon+ir}}{\lambda + \lambda_1} + \frac{\lambda^{1-\varepsilon+ir}}{\lambda + \lambda_2} \right) \\ &\quad + \frac{1}{8\pi\xi} \left( -\frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_1} + \frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_2} \right). \end{aligned}$$

Then, setting

$$\sigma_1 = \frac{\lambda}{\lambda_1} \quad \text{and} \quad \sigma_2 = \frac{\lambda}{\lambda_2},$$

we obtain

$$\begin{aligned} \Upsilon &= \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2} \mathcal{F}_t(E_{\alpha_2})(\xi) \mathcal{F}_t(E_{\alpha_1})(\xi) d\lambda \\ &= -\frac{1}{8\pi\xi} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_1} d\lambda + \frac{1}{8\pi\xi} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{\lambda + \lambda_2} d\lambda \\ &\quad - \frac{1}{8\pi\xi} \int_0^{+\infty} \frac{\lambda^{1-\varepsilon+ir}}{\lambda + \lambda_1} d\lambda + \frac{1}{8\pi\xi} \int_0^{+\infty} \frac{\lambda^{1-\varepsilon+ir}}{\lambda + \lambda_2} d\lambda \\ &= -\frac{\lambda_1^{-\varepsilon+ir}}{8\pi\xi} \int_0^{+\infty} \frac{\sigma_1^{-\varepsilon+ir}}{\sigma_1+1} d\sigma_1 + \frac{\lambda_2^{-\varepsilon+ir}}{8\pi\xi} \int_0^{+\infty} \frac{\sigma_2^{-\varepsilon+ir}}{\sigma_2+1} d\sigma_2 \\ &\quad - \frac{\lambda_1^{1-\varepsilon+ir}}{8\pi\xi} \int_0^{+\infty} \frac{\sigma_1^{1-\varepsilon+ir}}{\sigma_1+1} d\sigma_1 + \frac{\lambda_2^{1-\varepsilon+ir}}{8\pi\xi} \int_0^{+\infty} \frac{\sigma_2^{1-\varepsilon+ir}}{\sigma_2+1} d\sigma_2. \end{aligned}$$

Moreover, from (42) and (43), we deduce that

$$\begin{aligned} \Upsilon &= \left( \frac{\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}}{8\pi\xi} \right) \Gamma_{\varepsilon,r} + \left( \frac{\lambda_2^{1-\varepsilon+ir} - \lambda_1^{1-\varepsilon+ir}}{8\pi\xi} \right) \Gamma(\varepsilon - ir - 1)\Gamma(1 - (\varepsilon - ir - 1)) \\ &= \left( \frac{\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}}{2\pi\xi} \right) (\Gamma_{\varepsilon,r} + \Gamma(\varepsilon - ir - 1)\Gamma(1 - (\varepsilon - ir - 1))). \end{aligned}$$

For all  $z \in \mathbb{C} \setminus \mathbb{Z}$ , we have

$$\Gamma(z-1)\Gamma(1-(z-1)) = \frac{\pi}{\sin(\pi(z-1))} = -\frac{\pi}{\sin(\pi z)} = -\Gamma(z)\Gamma(1-z),$$

Setting  $z = \varepsilon - ir$ , with  $\varepsilon \in (0, 1)$ , it follows that

$$\Gamma(\varepsilon - ir - 1)\Gamma(1 - (\varepsilon - ir - 1)) = -\Gamma(\varepsilon - ir)\Gamma(1 - \varepsilon + ir) = -\Gamma_{\varepsilon,r},$$



hence  $\Upsilon = 0$ . Finally, we obtain that

$$I_{\varepsilon,r}(\theta) = \mathcal{F}_t^{-1} \left( m_\varepsilon(\xi) \mathcal{F}_t \left( \widetilde{G}_0 \right) (\xi) \right) (\theta),$$

where

$$m_\varepsilon(\xi) = \frac{\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}}{8\pi\xi}.$$

Setting

$$m(\xi) := \lim_{\varepsilon \rightarrow 0} m_\varepsilon(\xi) = \frac{\lambda_2^{ir} - \lambda_1^{ir}}{8\pi\xi},$$

due to the Lebesgue's dominated convergence Theorem, it follows that

$$I_{0,r}(\theta) := \lim_{\varepsilon \rightarrow 0} I_{\varepsilon,r}(\theta) = \mathcal{F}_t^{-1} \left( m(\xi) \mathcal{F}_t \left( \widetilde{G}_0 \right) (\xi) \right) (\theta).$$

Moreover, for all  $x_1, x_2 \in \mathbb{R}$ , we have

$$\left| e^{ix_1} - e^{ix_2} \right| \leq |x_1 - x_2|,$$

then, for all  $\xi \in \mathbb{R} \setminus \{0\}$ , we deduce that

$$|m(\xi)| = \frac{|\lambda_2^{ir} - \lambda_1^{ir}|}{8\pi|\xi|} = \frac{|e^{ir \ln(\lambda_2)} - e^{ir \ln(\lambda_1)}|}{8\pi|\xi|} \leq \frac{|r| |\ln(\lambda_2) - \ln(\lambda_1)|}{8\pi|\xi|} \leq \frac{|r| \left| \ln \left( \frac{2+\sqrt{2}}{2-\sqrt{2}} \right) \right|}{8\pi\xi}.$$

Thus

$$\sup_{\xi \in \mathbb{R}} |m(\xi)| = \lim_{\xi \rightarrow 0} |m(\xi)| = \left| \lim_{\xi \rightarrow 0} \frac{\lambda_2^{ir} - \lambda_1^{ir}}{8\pi\xi} \right|,$$

and

$$\begin{aligned} \lim_{\xi \rightarrow 0} \frac{\lambda_2^{ir} - \lambda_1^{ir}}{8\pi\xi} &= \lim_{\xi \rightarrow 0} 2^{ir} \left( \frac{1 + ir(-2\pi\xi + 2\pi^2\xi^2) + 2ir(ir-1)\pi^2\xi^2 + o(\xi^2)}{8\pi\xi} \right) \\ &\quad - \lim_{\xi \rightarrow 0} 2^{ir} \left( \frac{1 + ir(2\pi\xi + 2\pi^2\xi^2) + 2ir(ir-1)\pi^2\xi^2 + o(\xi^2)}{8\pi\xi} \right) \\ &= \lim_{\xi \rightarrow 0} 2^{ir} \left( \frac{-4ir\pi\xi + o(\xi^2)}{8\pi\xi} \right) \\ &= -2^{ir-1}ir. \end{aligned}$$

Then

$$\sup_{\xi \in \mathbb{R}} |m(\xi)| = \frac{|r|}{2}.$$

We have

$$\begin{aligned} \xi m'(\xi) &= \frac{2\pi\xi^2 \left( ir\lambda_2^{ir-1}(-4\pi + 8\pi^2\xi) - ir\lambda_1^{ir-1}(4\pi + 8\pi^2\xi) \right) - 2\pi\xi(\lambda_2^{ir} - \lambda_1^{ir})}{64\pi^2\xi^2} \\ &= \frac{ir}{32} \left( \lambda_2^{ir-1}(-1 + 2\pi\xi) - \lambda_1^{ir-1}(1 + 2\pi\xi) \right) - \frac{\lambda_2^{ir} - \lambda_1^{ir}}{32\pi\xi}, \end{aligned}$$

and in the same way we obtain

$$\sup_{\xi \in \mathbb{R}} |\xi m'(\xi)| = \lim_{\xi \rightarrow 0} |\xi m'(\xi)| = \left| \lim_{\xi \rightarrow 0} \xi m'(\xi) \right|,$$

with

$$\lim_{\xi \rightarrow 0} \xi m'(\xi) = \frac{ir}{32} \lim_{\xi \rightarrow 0} \left( \lambda_2^{ir-1}(-1 + 2\pi\xi) - \lambda_1^{ir-1}(1 + 2\pi\xi) \right) - \frac{\lambda_2^{ir} - \lambda_1^{ir}}{32\pi\xi},$$

where

$$\lim_{\xi \rightarrow 0} \lambda_2^{ir-1} = \lim_{\xi \rightarrow 0} 2^{ir-1} \left( 1 + (ir-1) \left( -2\pi\xi + 2\pi^2\xi^2 \right) + 4ir(ir-1)\pi^2\xi^2 + o(\xi^2) \right) = 2^{ir-1},$$

and

$$\lim_{\xi \rightarrow 0} \lambda_1^{ir-1} = \lim_{\xi \rightarrow 0} 2^{ir-1} \left( 1 + (ir-1) \left( 2\pi\xi + 2\pi^2\xi^2 \right) + 4ir(ir-1)\pi^2\xi^2 + o(\xi^2) \right) = 2^{ir-1}.$$

Thus

$$\begin{aligned} \lim_{\xi \rightarrow 0} \xi m'(\xi) &= \frac{1}{4} \lim_{\xi \rightarrow 0} \frac{ir}{8} \left( - \left( \lambda_2^{ir-1} + \lambda_1^{ir-1} \right) + 2\pi\xi \left( \lambda_2^{ir-1} - \lambda_1^{ir-1} \right) \right) - \frac{\lambda_2^{ir} - \lambda_1^{ir}}{8\pi\xi} \\ &= \frac{1}{4} \left( -\frac{2^{ir}ir}{8} + 2^{ir-1}ir \right) \\ &= 3 \times 2^{ir-5}ir. \end{aligned}$$

Then

$$\sup_{\xi \in \mathbb{R}} |\xi m'(\xi)| = \left| 3 \times 2^{ir-5}ir \right| = \frac{3}{32}|r|.$$

Therefore, we deduce that

$$\sup_{\xi \in \mathbb{R}} |m(\xi)| + \sup_{\xi \in \mathbb{R}} |\xi m'(\xi)| = \frac{|r|}{2} + \frac{3}{32}|r| = \frac{19}{32}|r|,$$

From the Mihlin Theorem, see Mihlin [18], for all  $\gamma > 0$ , there exists  $C_{\gamma,p} > 0$ , such that, for all  $r \in \mathbb{R}$ , we have

$$\|I_{0,r}(\cdot)\|_{\mathcal{L}(X)} = \left\| \mathcal{F}_t^{-1} \left( m(\xi) \mathcal{F}_t(\widetilde{G}_0)(\xi) \right) (\cdot) \right\|_{\mathcal{L}(X)} \leq C_{\gamma,p} e^{\gamma|r|}.$$

Finally, for all  $\gamma > 0$ , there exists a constant  $C_{\gamma,p} > 0$  such that for all  $r \in \mathbb{R}$ , we obtain

$$\left\| (\mathcal{A} + I)^{ir} \right\|_{\mathcal{L}(X)} \leq C_{\gamma,p} e^{\gamma|r|}.$$

Therefore, taking  $\theta_{\mathcal{A}} = \gamma > 0$ , we have  $\mathcal{A} + I \in \text{BIP}(X, \theta_{\mathcal{A}})$  and from Theorem 2.3, p. 69 in Arendt, Bu and Haase [1], we deduce that  $\mathcal{A} = \mathcal{A} + I - I \in \text{BIP}(X, \theta_{\mathcal{A}})$ .  $\square$

## 4 Study of the sum $\mathcal{L}_{1,\mu} + \mathcal{L}_2$

### 4.1 Invertibility of the closure of the sum

In this section, we will apply the results described in Section 2.2. We take  $\mathcal{L}_{1,\mu} = \mathcal{M}_1$  and  $\mathcal{L}_2 = \mathcal{M}_2$ .

**Theorem 4.1.** Assume that (5) holds. Then  $\mathcal{L}_{1,\mu} + \mathcal{L}_2$  is closable and its closure  $\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}$  is invertible.

*Proof.* Assumption  $(H_1)$  is satisfied from Proposition 3.1 and Corollary 3.11, with

$$\theta_{\mathcal{M}_1} + \theta_{\mathcal{M}_2} = \varepsilon_{\mathcal{L}_{1,\mu}} + \pi - \varepsilon_{\mathcal{L}_2},$$

where it suffices to take  $\varepsilon_{\mathcal{L}_2} > \varepsilon_{\mathcal{L}_{1,\mu}}$  in order to obtain  $\theta_{\mathcal{M}_1} + \theta_{\mathcal{M}_2} < \pi$ .

For assumption  $(H_2)$ , due to Proposition 3.2, it follows that  $0 \notin \sigma(\mathcal{L}_{1,\mu}) \cap \sigma(-\mathcal{L}_2)$ . Moreover, from Proposition 3.1, we have

$$\sigma(\mathcal{L}_{1,\mu}) = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \pi \text{ and } \operatorname{Re}(\sqrt{\lambda}) \leq \mu\},$$

and from Remark 3.12, it follows that

$$\sigma(-\mathcal{L}_2) = \{\lambda \in \mathbb{C} \setminus \mathbb{R}_+ : \sinh(\omega\sqrt{-\lambda}) = \pm\omega\sqrt{-\lambda}\} = \left\{ -\frac{z_j^2}{\omega^2} \in \mathbb{C} \setminus \mathbb{R}_+ : j \in \mathbb{N} \setminus \{0\} \right\}.$$

Then, since

$$\operatorname{Re} \left( \sqrt{-\frac{z_j^2}{\omega^2}} \right) = \frac{1}{\omega} |\operatorname{Im}(z_j)|,$$

the condition  $\sigma(\mathcal{L}_{1,\mu}) \cap \sigma(-\mathcal{L}_2) = \emptyset$  is fulfilled if (5) holds.

The commutativity assumption  $(H_3)$  is clearly verified since the actions of operators  $\mathcal{L}_{1,\mu}$  and  $\mathcal{L}_2$  are independent.

Now, applying Theorem 2.6, we obtain the result.  $\square$

**Remark 4.2.** We can conjecture that, for the critical case  $\omega\mu = \tau$ , the sum  $\mathcal{L}_{1,\mu} + \mathcal{L}_2$  is not closable.

## 4.2 Convexity inequalities

In view to apply Corollary 2.7, we are going to verify inequality (8) in two situations.

**Proposition 4.3.** Let

$$\mathcal{E}_1 = W^{1,p}(0, +\infty; X) \subset \mathcal{E} = L^p(0, +\infty; X),$$

and

$$\mathcal{E}_2 = L^p \left( 0, +\infty; \left[ W^{3,p}(0, \omega) \cap W_0^{2,p}(0, \omega) \right] \times L^p(0, \omega) \right) \subset \mathcal{E}.$$

Then, we have

$$D \left( \overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right) \subset \mathcal{E}_1 \cap \mathcal{E}_2. \quad (44)$$

*Proof.* Let  $V \in D(\mathcal{L}_{1,\mu})$ . We must prove that there exists  $\delta \in (0, 1)$  such that

$$\|V\|_{\mathcal{E}_1} \leq C \left[ \|V\|_{\mathcal{E}} + \|V\|_{\mathcal{E}}^{1-\delta} \|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}}^{\delta} \right].$$

For all  $V \in W^{2,p}(0, +\infty; X)$ , from Kato [10], inequality (1.15), p. 192, we have the convexity inequality

$$\|V'\|_{\mathcal{E}} \leq 2\sqrt{2} \|V\|_{\mathcal{E}}^{1/2} \|V''\|_{\mathcal{E}}^{1/2}.$$

Thus, we deduce that

$$\|V\|_{\mathcal{E}_1} = \|V\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} \leq \|V\|_{\mathcal{E}} + 2\sqrt{2} \|V\|_{\mathcal{E}}^{1/2} \|V''\|_{\mathcal{E}}^{1/2}.$$

Since  $\mathcal{L}_{1,\mu}$  is not invertible, we will estimate  $\|V''\|_{\mathcal{E}}$  by  $\|\mathcal{L}_{1,\mu}(V) - \lambda_0 V\|_{\mathcal{E}}$ , where  $\lambda_0 \in \rho(\mathcal{L}_{1,\mu})$ . We have

$$V'' - 2\mu V' + (\mu^2 - \lambda_0)V = \mathcal{L}_{1,\mu}(V) - \lambda_0 V.$$

Then, there exists a constant  $C > 0$  such that

$$\|V''\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} + \|V\|_{\mathcal{E}} \leq C \|\mathcal{L}_{1,\mu}(V) - \lambda_0 V\|_{\mathcal{E}},$$

hence

$$\|V''\|_{\mathcal{E}} \leq C \|\mathcal{L}_{1,\mu}(V) - \lambda_0 V\|_{\mathcal{E}} \leq C \|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}} + |\lambda_0| C \|V\|_{\mathcal{E}}.$$

Thus

$$\begin{aligned}
\|V\|_{\mathcal{E}_1} = \|V\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} &\leq \|V\|_{\mathcal{E}} + 2\sqrt{2}\|V\|_{\mathcal{E}}^{1/2}\|V''\|_{\mathcal{E}}^{1/2} \\
&\leq \|V\|_{\mathcal{E}} + 2\sqrt{2C}\|V\|_{\mathcal{E}}^{1/2}(\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}} + |\lambda_0|\|V\|_{\mathcal{E}})^{1/2} \\
&\leq \|V\|_{\mathcal{E}} + 2\sqrt{2C}\|V\|_{\mathcal{E}}^{1/2}\left(\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}}^{1/2} + |\lambda_0|^{1/2}\|V\|_{\mathcal{E}}^{1/2}\right) \\
&\leq \left(1 + 2\sqrt{2C}|\lambda_0|^{1/2}\right)\|V\|_{\mathcal{E}} + 2\sqrt{2C}\|V\|_{\mathcal{E}}^{1/2}\|\mathcal{L}_{1,\mu}(V)\|_{\mathcal{E}}^{1/2}.
\end{aligned}$$

Therefore, inequality (8) is satisfied for  $\delta = 1/2$  and  $\mathcal{M}_1 = \mathcal{L}_{1,\mu}$ . Using Corollary 2.7, we obtain

$$D\left(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}\right) \subset \mathcal{E}_1.$$

Now, we must show that, for all  $V \in D(\mathcal{L}_2)$ , we have

$$\|V\|_{\mathcal{E}_2} \leq C \left[ \|V\|_{\mathcal{E}} + \|V\|_{\mathcal{E}}^{1/2}\|\mathcal{L}_2(V)\|_{\mathcal{E}}^{1/2} \right].$$

To this end, it suffices to do it for  $\mathcal{A}$ . Set

$$\mathcal{G}_1 = \left[ W^{3,p}(0, \omega) \cap W_0^{2,p}(0, \omega) \right] \times L^p(0, \omega) \subset X.$$

We must prove that

$$\forall \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}), \quad \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} \leq C \left[ \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X + \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2} \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2} \right].$$

Here, we have

$$\begin{aligned}
\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} &= \|\psi_1\|_{W^{3,p}(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)} \\
&= \|\psi_1\|_{L^p(0,\omega)} + \|\psi_1'\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1'''\|_{L^p(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)}.
\end{aligned}$$

Set  $\varphi = \psi_1''$ . Then, for all  $\eta > 0$ , from Kato [10], inequality (1.12), p. 192, taking  $n = \eta + 1$  and  $b - a = \omega$ , we obtain

$$\|\varphi'\|_{L^p(0,\omega)} \leq \frac{\omega}{\eta}\|\varphi''\|_{L^p(0,\omega)} + \frac{2}{\omega}\left(\eta + 3 + \frac{2}{\eta}\right)\|\varphi\|_{L^p(0,\omega)}.$$

It is not difficult to see that the second member is minimal when

$$\eta = \frac{\sqrt{2}\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}}{\|\varphi\|_{L^p(0,\omega)}^{1/2}}.$$

Therefore, we deduce that

$$\begin{aligned}
\|\varphi'\|_{L^p(0,\omega)} &\leq \frac{\omega\sqrt{2}\|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi''\|_{L^p(0,\omega)}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} + \frac{4}{\omega}\frac{\sqrt{2}\|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi\|_{L^p(0,\omega)}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} \\
&\quad + \frac{\sqrt{2}\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}\|\varphi\|_{L^p(0,\omega)}}{\omega\|\varphi\|_{L^p(0,\omega)}^{1/2}} + \frac{6}{\omega}\|\varphi\|_{L^p(0,\omega)} \\
&\leq \frac{\sqrt{2}}{\omega}\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}\|\varphi\|_{L^p(0,\omega)}^{1/2} + \frac{6}{\omega}\|\varphi\|_{L^p(0,\omega)} \\
&\quad + \left(\frac{4}{\omega}\|\varphi\|_{L^p(0,\omega)} + \omega\|\varphi''\|_{L^p(0,\omega)}\right)\frac{\sqrt{2}\|\varphi\|_{L^p(0,\omega)}^{1/2}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} \\
&\leq C\left(\|\varphi\|_{L^p(0,\omega)} + \|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi''\|_{L^p(0,\omega)}^{1/2}\right).
\end{aligned}$$

Then, we have

$$\|\psi_1'''\|_{L^p(0,\omega)} \leq C_\omega \left( \|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)}^{1/2} \|\psi_1^{(4)}\|_{L^p(0,\omega)}^{1/2} \right).$$

Hence

$$\begin{aligned} \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} &\leq \|\psi_1\|_{L^p(0,\omega)} + \|\psi_1'\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)} \\ &\quad + C_\omega \left( \|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)}^{1/2} \|\psi_1^{(4)}\|_{L^p(0,\omega)}^{1/2} \right) + \|\psi_2\|_{L^p(0,\omega)} \\ &\leq (1 + C_\omega) \|\psi_1\|_{W_0^{2,p}(0,\omega)} + C_\omega \|\psi_1\|_{W_0^{2,p}(0,\omega)}^{1/2} \|\psi_1^{(4)}\|_{L^p(0,\omega)}^{1/2} + \|\psi_2\|_{L^p(0,\omega)} \end{aligned}$$

Now, since  $\mathcal{A}$  is invertible, see Proposition 3.2, we have proved that there exists a constant  $C'_\omega$  depending only on  $\omega$  such that

$$\|\psi_1^{(4)}\|_{L^p(0,\omega)} \leq C'_\omega \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X.$$

Moreover, it follows

$$\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} \leq (1 + C_\omega) \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X + C_\omega C'_\omega \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2} \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2}.$$

Therefore, inequality (8) is satisfied for  $\delta = 1/2$  and  $\mathcal{M}_2 = \mathcal{L}_2$ . Using Corollary 2.7, we obtain

$$D(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}) \subset \mathcal{E}_2,$$

which gives the expected result.  $\square$

## 5 Back to the abstract problem

Now, we are in position to solve the following equation

$$\left( \overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right) V + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu}) V = \mathcal{F}. \quad (45)$$

**Theorem 5.1.** Let  $\mathcal{F} \in L^p(0, +\infty; X)$  and assume that (5) holds. Then, there exists  $\rho_0 > 0$  such that for all  $\rho \in (0, \rho_0]$ , equation (45) has a unique strong solution  $V \in L^p(0, +\infty; X)$ , that is

$$\left\{ \begin{array}{l} \exists (V_n)_{n \geq 0} \in D(\mathcal{L}_{1,\mu}) \cap D(\mathcal{L}_2) : \\ V_n \xrightarrow[n \rightarrow +\infty]{} V \text{ in } L^p(0, +\infty; X) \text{ and} \\ (\mathcal{L}_{1,\mu} + \mathcal{L}_2) V_n + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu}) V_n \xrightarrow[n \rightarrow +\infty]{} \mathcal{F} \text{ in } L^p(0, +\infty; X), \end{array} \right. \quad (46)$$

satisfying

$$V \in W^{1,p}(0, +\infty; X) \cap L^p\left(0, +\infty; \left[ W^{3,p}(0, \omega) \cap W_0^{2,p}(0, \omega) \right] \times L^p(0, \omega)\right). \quad (47)$$

*Proof.* Due to Theorem 4.1, if (5) holds, then  $\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}$  is invertible. Thus, it follows that

$$\left[ I + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu}) \left( \overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right)^{-1} \right] \left( \overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right) V = \mathcal{F}.$$

From (44), we deduce that  $V \in D(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2}) \subset \mathcal{E}_1 \cap \mathcal{E}_2$ , that is (47) which involves that

$$(\mathcal{P}_1 + \mathcal{P}_{2,\mu})(\overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2})^{-1} \in \mathcal{L}(X).$$

Then, there exists  $\rho_0 > 0$  small enough such that, for all  $\rho \in (0, \rho_0]$ , we have

$$V = \left( \overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right)^{-1} \left[ I + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu}) \left( \overline{\mathcal{L}_{1,\mu} + \mathcal{L}_2} \right)^{-1} \right]^{-1} \mathcal{F}, \quad (48)$$

which means that  $V$  is the unique strong solution of (45).  $\square$

## 6 Proof of Theorem 1.1

From Theorem 5.1, there exists  $\rho_0 > 0$  such that for all  $\rho \in (0, \rho_0]$ , equation (45) has a unique strong solution  $V \in L^p(0, +\infty; X)$  satisfying (47). Then, due to (46), there exists a sequence  $(V_n)_{n \in \mathbb{N}} \in D(\mathcal{L}_{1,\mu} + \mathcal{L}_2)$  such that  $V_n \xrightarrow[n \rightarrow +\infty]{} V$  and

$$\lim_{n \rightarrow +\infty} (\mathcal{L}_{1,\mu} + \mathcal{L}_2) V_n + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_{2,\mu}) V_n = \mathcal{F}.$$

Since  $(V_n)_{n \in \mathbb{N}} \in D(\mathcal{L}_{1,\mu} + \mathcal{L}_2)$ , then the previous equality can be written as

$$\begin{cases} \lim_{n \rightarrow +\infty} (V_n''(t) - \mathcal{A}V_n(t) - \mathcal{F}_n(t)) = 0 \\ \lim_{n \rightarrow +\infty} V_n(0) = 0, \quad \lim_{n \rightarrow +\infty} V_n(+\infty) = 0, \end{cases} \quad (49)$$

where

$$\mathcal{F}_n(t) = k\rho^2 e^{-2t} \mathcal{A}_0 V_n(t) + k\rho^2 e^{-2t} [(\mathcal{B}_{2,\mu} V_n)](t) + 2\mu V_n'(t) - \mu^2 V_n(t) + \mathcal{F}(t).$$

Since  $V_n \xrightarrow[n \rightarrow +\infty]{} V$  in  $\mathcal{E}$  and  $V$  satisfies (47), we deduce that

$$\lim_{n \rightarrow +\infty} V_n(0) = V(0) = 0 \quad \text{with} \quad \lim_{n \rightarrow +\infty} V_n(+\infty) = V(+\infty) = 0,$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{F}_n(t) = \mathcal{F}_\infty(t) \in L^p(0, +\infty; X),$$

where

$$\mathcal{F}_\infty(t) = k\rho^2 e^{-2t} \mathcal{A}_0 V(t) + k\rho^2 e^{-2t} [(\mathcal{B}_{2,\mu} V)](t) + 2\mu V'(t) - \mu^2 V(t) + \mathcal{F}(t).$$

Thus, problem (49) can be written as follows

$$\begin{cases} \lim_{n \rightarrow +\infty} (V_n''(t) - \mathcal{A}V_n(t)) = \mathcal{F}_\infty(t) \\ V(0) = 0, \quad V(+\infty) = 0. \end{cases}$$

Moreover, from Proposition 3.13,  $\mathcal{A} \in \text{BIP}(X, \theta_{\mathcal{A}})$ , with  $\theta_{\mathcal{A}} \in (0, \pi)$  and due to Haase [9], Proposition 3.2.1, e), p. 71, it follows that  $\sqrt{\mathcal{A}} \in \text{BIP}(X, \theta_{\mathcal{A}}/2)$  with  $\theta_{\mathcal{A}}/2 \in (0, \pi/2)$ . Therefore, due to Eltaief and Maingot [6], Theorem 2, p. 712, with  $L_1 = L_2 = -\sqrt{\mathcal{A}}$ , there exists a unique classical solution to the following problem

$$\begin{cases} \mathcal{V}''(t) - \mathcal{A}\mathcal{V}(t) = \mathcal{F}_\infty(t) \\ \mathcal{V}(0) = 0, \quad \mathcal{V}(+\infty) = 0, \end{cases}$$

that is

$$\mathcal{V} \in W^{2,p}(0, +\infty; X) \cap L^p(0, +\infty; D(\mathcal{A})).$$

Thus, it follows that

$$\lim_{n \rightarrow +\infty} (V_n''(t) - \mathcal{A}V_n(t)) = \mathcal{V}''(t) - \mathcal{A}\mathcal{V},$$

hence

$$\lim_{n \rightarrow +\infty} \left[ (V_n(t) - \mathcal{V}(t))'' - \mathcal{A}(V_n(t) - \mathcal{V}(t)) \right] = 0.$$

Now, set

$$\begin{cases} D(\delta_2) &= \{ \varphi \in W^{2,p}(0, +\infty; X) : \varphi(0) = \varphi(+\infty) = 0 \} \\ \delta_2 \varphi &= \varphi'', \quad \varphi \in D(\delta_2). \end{cases}$$

Then

$$0 = \lim_{n \rightarrow +\infty} \left[ (V_n(t) - \mathcal{V}(t))'' - \mathcal{A}(V_n(t) - \mathcal{V}(t)) \right] = \lim_{n \rightarrow +\infty} -(-\delta_2 + \mathcal{A})(V_n(t) - \mathcal{V}(t)). \quad (50)$$

From Prüss and Sohr [20], Theorem C, p. 166-167, it follows that  $-\delta_2 \in \text{BIP}(X, \theta_{\delta_2})$ , for every  $\theta_{\delta_2} \in (0, \pi)$  and due to Proposition 3.13,  $\mathcal{A} \in \text{BIP}(X, \theta_{\mathcal{A}})$ , for all  $\theta_{\mathcal{A}} \in (0, \pi)$ . Thus, since  $-\delta_2$  and  $\mathcal{A}$  are resolvent commuting with  $\theta_{\delta_2} + \theta_{\mathcal{A}} < \pi$ , from Prüss and Sohr [19], Theorem 5, p. 443, we obtain that

$$-\delta_2 + \mathcal{A} \in \text{BIP}(X, \theta), \quad \theta = \max(\theta_{\delta_2}, \theta_{\mathcal{A}}).$$

Moreover, due to Proposition 3.2, we have  $0 \in \rho(\mathcal{A})$ , then we deduce from Prüss and Sohr [19], remark at the end of p. 445, that  $0 \in \rho(\delta_2 + \mathcal{A})$ . Therefore, due to (50), we obtain that

$$\lim_{n \rightarrow +\infty} V_n(t) - \mathcal{V}(t) = 0,$$

hence, since  $V_n \xrightarrow[n \rightarrow +\infty]{} V$ , by uniqueness of the limit, we deduce that

$$V = \mathcal{V} \in W^{2,p}(0, +\infty; X) \cap L^p(0, +\infty; D(\mathcal{A})).$$

This prove that  $\mathcal{L}_{1,\mu} + \mathcal{L}_2$  is closed and that  $V \in D(\mathcal{L}_{1,\mu} + \mathcal{L}_2)$ .

## Acknowledgments

We would like to thank the referee for its valuable comments and corrections which have helped us to improve this second part.

## Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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