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# On weaker notions for Kähler-Ricci solitons

Nefton PALI

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## Abstract

We show that shrinking Kähler-Ricci solitons over a compact Kähler manifold are gradient shrinking Kähler-Ricci solitons. The proof relies on a remarkable identity on the kernels of a real and a complex elliptic operator proved in our solution of the variational stability problem for gradient shrinking Kähler-Ricci solitons in [Pal6].

## 1 Introduction

The notion of Ricci soliton has been introduced by D.H. Friedon in [Fri]. It is a natural generalization of the notion of Einstein metric. The terminology is justified by the fact that the pull back of the Ricci soliton metric via the flow of automorphisms generated by its vector field is a solution to Hamilton's Ricci flow [Ham].

A remarkable and very useful generalization of this flow and his solitons has been introduced by Moroianu-Murcia-Shahbazi in [Mo-Mu-Sh1, Mo-Mu-Sh2].

An other very important generalization of Ricci solitons has been introduced by Nurowski-Randall in [Nur-Ran]. Indeed the later generalization is not only very important in pure mathematics but also quite fundamental in theoretical physics.

In this article we will restrict our considerations to the case of compact Kähler manifolds. In this case the notion of gradient shrinking Kähler-Ricci soliton is quite standard nowadays (see for example [Cao, Pal5, Pal6] among a very large number of other authors).

In this article we will call them standard Kähler-Ricci solitons, for terminology simplicity.

We will consider below two weaker definitions of Kähler-Ricci solitons.

The second one is usually called shrinking Kähler-Ricci soliton in the literature on the subject.

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**Key words** : Kähler manifold, Kähler-Ricci soliton. Holomorphic vector fields.

**AMS Classification** : 53C25, 53C55, 32J15.

Given a Kähler structure  $(g, J)$  over a complex manifold  $X$ , we will denote by  $T_{X,J}$  the real tangent bundle  $T_X$  equipped with the complex structure  $J$  over its fibers.

Then  $\mathcal{O}(X, T_{X,J})$  will denote the space of real  $J$ -holomorphic vector fields over  $X$ .

**Definition 1** *A Kähler structure  $(g, J)$  over a complex manifold  $X$  is called a very weak Kähler Ricci soliton if there exists a vector field  $\xi \in C^\infty(X, T_X)$  such that  $g - \text{Ric}(g) = L_\xi g$ .*

**Definition 2** *A Kähler structure  $(g, J)$  over a complex manifold  $X$  is called a weak Kähler Ricci soliton if there exists a real holomorphic vector field  $\xi \in \mathcal{O}(X, T_{X,J})$  such that  $g - \text{Ric}(g) = L_\xi g$ .*

We will show the following result.

**Theorem 1** *Let  $(g, J)$  be a weak Kähler Ricci soliton over a compact complex manifold  $X$  and let  $\omega := gJ$  be the associated symplectic form. Then there exists a unique vector field  $\hat{\xi} \in \mathcal{O}(X, T_{X,J})$  such that  $g - \text{Ric}(g) = L_{\hat{\xi}} g$  and such that  $L_{J\hat{\xi}}\omega = 0$ . In particular  $(X, J)$  is a Fano variety.*

It is known (see Remarks 2 and 3 in Section 4 in [Pal2]) that a Kähler structure which satisfies the two equations in the statement of Theorem 1 with respect to a vector field  $\xi \in \mathcal{O}(X, T_{X,J})$ , is a standard Kähler Ricci soliton.

We notice now that in the case of a very weak Kähler Ricci soliton, the symplectic form  $\omega$  satisfies the equation

$$\omega J - \text{Ric}_J(\omega)J = L_\xi \omega J + \omega L_\xi J,$$

since the Chern-Ricci form  $\text{Ric}_J(\omega)$  satisfies the identity

$$\text{Ric}_J(\omega) = \text{Ric}(g)J, \tag{1}$$

in the Kähler case (see for example [Pal4]). Therefore the form  $L_\xi \omega$  is  $J$ -invariant if and only if

$$L_\xi J = 0, \tag{2}$$

since the bilinear form  $\omega L_\xi J$  is  $J$ -anti-invariant. (The later is because the endomorphism  $L_\xi J$  is  $J$ -anti-linear). The condition 2 is equivalent to the holomorphy condition

$$\xi \in \mathcal{O}(X, T_{X,J}),$$

in the case of an integrable almost complex structure  $J$ , (see the identity (19.5) in Lemma 28 in the Section 19.4 of the Appendix in [Pal4]).

Thus in the case of a weak Kähler Ricci soliton, we deduce that the soliton equation  $g - \text{Ric}(g) = L_\xi g$ , is equivalent to the equation

$$\omega - \text{Ric}_J(\omega) = L_\xi \omega. \tag{3}$$

## 2 Proof of Theorem 1

**STEP I. The Fano condition and the potential of the holomorphic vector field  $\xi$ .**

The fact that the form  $\omega$  is symplectic implies

$$L_\xi \omega = d(\xi \lrcorner \omega).$$

Then the Equation (3) implies that the manifold  $(X, J)$  is a Fano variety, by definition. We recall that  $H_d^1(X, \mathbb{C}) = 0$  in the case of Fano manifolds.

Then by Hodge decomposition, the identity  $H_{\bar{\partial}}^{0,1}(X, \mathbb{C}) = 0$  holds.

For any function  $u \in C^\infty(X, \mathbb{C})$  we define the  $J$ -complex  $g$ -gradient as the real vector field

$$\nabla_{g,J} u := \nabla_g \operatorname{Re} u + J \nabla_g \operatorname{Im} u \in C^\infty(X, T_X).$$

With this notations, the complex decomposition formula holds

$$\nabla_{g,J} u \lrcorner g = \partial_J \bar{u} + \bar{\partial}_J u. \quad (4)$$

We notice (see the proof of Lemma 1 below or the proof of Lemma 10 in Section 13 in [Pal6]), that over a connected manifold (in this proof we will always assume is the case) the condition

$$\nabla_{g,J} u = 0,$$

is equivalent to  $u$  being constant. Let  $\Omega > 0$  be an arbitrary volume form and let  $C_\Omega^\infty(X, \mathbb{R})_0$  be the space of smooth functions with zero integral with respect to  $\Omega$ .

We consider now the  $J$ -anti-linear component of the complex Hessian map

$$\mathcal{H}_{g,J}^{0,1} : C_\Omega^\infty(X, \mathbb{C})_0 \longrightarrow C^\infty(X, \Lambda_J^{0,1} T_X^* \otimes_{\mathbb{C}} T_{X,J})$$

$$u \longmapsto \bar{\partial}_{T_{X,J}} \nabla_{g,J} u.$$

**Lemma 1** *Let  $(X, J, g)$  be a compact connected Kähler manifold with  $H_d^1(X, \mathbb{C}) = 0$ . Then the map*

$$\operatorname{Ker} \mathcal{H}_{g,J}^{0,1} \longrightarrow H^0(X, T_{X,J})$$

$$u \longmapsto \nabla_{g,J} u,$$

*is an isomorphism of complex vector spaces.*

**Proof** We observe first the injectivity. Using the complex decomposition (4) we infer the formula

$$d(\nabla_{g,J} u \lrcorner g) = \partial_J \bar{\partial}_J (u - \bar{u}),$$

which in the case  $\nabla_{g,J}u = 0$  implies  $\text{Im } u = 0$  and thus  $\text{Re } u = 0$ . In order to show the surjectivity we consider an arbitrary  $\xi \in H^0(X, T_{X,J})$ . Then the identity (5) below implies

$$\bar{\partial}_J(\xi_J^{1,0} \lrcorner \omega) = 0.$$

The identity  $H_{\bar{\partial}}^{0,1}(X, \mathbb{C}) = 0$ , implies the existence of a unique function  $u \in C_{\Omega}^{\infty}(X, \mathbb{C})_0$  such that

$$i\bar{\partial}_J u = \xi_J^{1,0} \lrcorner \omega = i\xi_J^{1,0} \lrcorner g.$$

Thus  $\xi = \nabla_{g,J}u$  thanks to the complex decomposition (4).  $\square$

**Lemma 2** *Let  $(X, J)$  be a complex manifold and let  $\omega \in C^{\infty}(X, \Lambda_J^{1,1}T_X^*)$ ,  $\xi \in C^{\infty}(X, T_X^{1,0})$ . Then the following identity holds*

$$\bar{\partial}_J(\xi \lrcorner \omega) = \bar{\partial}_{T_{X,J}^{1,0}} \xi \lrcorner \omega - \xi \lrcorner \bar{\partial}_J \omega. \quad (5)$$

**Proof** Let  $\eta, \mu \in C^{\infty}(X, T_X^{0,1})$ . The identities (see [Pal1])

$$\bar{\partial}_J(\xi \lrcorner \omega)(\eta, \mu) = \eta \cdot \omega(\xi, \mu) - \mu \cdot \omega(\xi, \eta) - \omega(\xi, [\eta, \mu]),$$

$$\begin{aligned} \bar{\partial}_J \omega(\eta, \xi, \mu) &= \eta \cdot \omega(\xi, \mu) + \mu \cdot \omega(\eta, \xi) \\ &\quad - \omega([\eta, \xi], \mu) + \omega([\eta, \mu], \xi) - \omega([\xi, \mu], \eta) \\ &= \bar{\partial}_J(\xi \lrcorner \omega)(\eta, \mu) - \omega(\bar{\partial}_{T_{X,J}^{1,0}} \xi(\eta), \mu) + \omega(\bar{\partial}_{T_{X,J}^{1,0}} \xi(\mu), \eta) \\ &= \bar{\partial}_J(\xi \lrcorner \omega)(\eta, \mu) - \omega(\bar{\partial}_{T_{X,J}^{1,0}} \xi(\eta), \mu) - \omega(\eta, \bar{\partial}_{T_{X,J}^{1,0}} \xi(\mu)) \\ &= [\bar{\partial}_J(\xi \lrcorner \omega) - \bar{\partial}_{T_{X,J}^{1,0}} \xi \lrcorner \omega](\eta, \mu), \end{aligned}$$

imply the required identity.  $\square$

Therefore with the notations in the proof of Lemma 1 we have in our case

$$\begin{aligned} L_{\xi} \omega &= d(\xi \lrcorner \omega) \\ &= \partial_J(\xi_J^{1,0} \lrcorner \omega) + \partial_J(\xi_J^{0,1} \lrcorner \omega) + \bar{\partial}_J(\xi_J^{1,0} \lrcorner \omega) + \bar{\partial}_J(\xi_J^{0,1} \lrcorner \omega) \\ &= \partial_J \alpha + i\partial_J \bar{\partial}_J u + \bar{\partial}_J \bar{\alpha} - i\bar{\partial}_J \partial_J \bar{u} \\ &= i\partial_J \bar{\partial}_J (u + \bar{u}). \end{aligned}$$

This combined with (3) implies the identity

$$\omega - \text{Ric}_J(\omega) = i\partial_J\bar{\partial}_J(u + \bar{u}). \quad (6)$$

We write now  $2u = f + ih$ , with  $f, h$  real valued and we set  $\Omega := e^f dV_g$ . Then (6) rewrites as

$$\omega = \text{Ric}_J(\Omega), \quad (7)$$

where  $\text{Ric}_J(\Omega)$ , is the Chern-Ricci form. We will denote by

$$\text{Ric}_J^*(\Omega)_\omega := \omega^{-1} \text{Ric}_J(\Omega),$$

the associated endomorphism.

**STEP II. Combination of the soliton equation with the weighted complex Bochner Formula.**

We use the convention  $\Delta_g := -\text{div} \nabla_g$ . We consider the Laplacian operators acting on any  $v \in C^\infty(X, \mathbb{C})$ ,

$$\Delta_g^\Omega v := \Delta_g v + df(\nabla_g v).$$

$$\Delta_{g,-J}^\Omega v := \Delta_g^\Omega v + ig(\nabla_g v, J\nabla_g f).$$

We notice the identity

$$\Delta_{g,-J}^\Omega f = \Delta_g^\Omega f.$$

We observe now that the fact that the condition

$$\xi = \nabla_{g,J} u \in \mathcal{O}(X, T_{X,J}),$$

implies the identity

$$\nabla_{g,J} \Delta_{g,-J}^\Omega u = 2 \text{Ric}_J^*(\Omega)_\omega \nabla_{g,J} u, \quad (8)$$

thanks to the weighted complex Bochner formula (12) in the next section or formula (13.4) in [Pal6]. Then (7) combined with (8) implies

$$\nabla_{g,J} (\Delta_{g,-J}^\Omega - 2\mathbb{I}) u = 0,$$

(where  $\mathbb{I}$  is the identity operator), which means

$$(\Delta_{g,-J}^\Omega - 2\mathbb{I}) u = c_1 + ic_2,$$

for some real constants  $c_1$  and  $c_2$ . The later is equivalent to the system

$$\begin{cases} \Delta_g^\Omega f - g(\nabla_g h, J\nabla_g f) - 2f = c_1, \\ \Delta_g^\Omega h - 2h = c_2. \end{cases}$$

The function  $2\hat{h} := 2h + c_2$ , satisfies the identity

$$(\Delta_g^\Omega - 2\mathbb{I})\hat{h} = 0.$$

We infer

$$J\nabla_g h \in \mathcal{O}(X, T_{X,J}),$$

by the remarkable identity (18.1) in Corollary 5, Section 18 of [Pal6] i.e. identity (27) in Corollary 2 in the next section, and by the identity (18) in the next section. We deduce

$$\nabla_g f \in \mathcal{O}(X, T_{X,J}).$$

The fact that

$$L_{J\nabla_g v}\omega = 0,$$

for any smooth function, (see Remark 2 in Section 4 of [Pal2]) implies

$$2L_\xi\omega = L_{\nabla_g f}\omega.$$

Thus

$$\hat{\xi} := 2^{-1}\nabla_g f,$$

is the required vector field in the statement of Theorem 1 and it is unique by the identity (6). This concludes the proof of Theorem 1.

We would like to address now the following question.

**Question.** Over a compact Kähler manifold, is a very weak Kähler Ricci soliton a standard one?

### 3 Real eigenfunctions of the weighted complex Laplacian

For the readers convenience we include in this section the statement and a self contained proof of the remarkable identity (18.1) in Corollary 5, Section 18 of [Pal4], i.e. identity (27) in Corollary 2 below, with

$$\Delta_{g,J}^\Omega := \overline{\Delta_{g,-J}^\Omega}.$$

We start with a few notations from [Pal4]. We equip the set of smooth Riemannian metrics  $\mathcal{M}$  over  $X$  with the scalar product

$$(u, v) \mapsto \int_X \langle u, v \rangle_g \Omega, \tag{9}$$

for all  $u, v \in \mathcal{H} := L^2(X, S_{\mathbb{R}}^2 T_X^*)$ . Let  $P_g^*$  be the formal adjoint of some operator  $P$  with respect to a metric  $g$ . We observe that the operator

$$P_g^{*\Omega} := e^f P_g^*(e^{-f} \bullet),$$

with  $f := \log \frac{dV_g}{\Omega}$ , is the formal adjoint of  $P$  with respect to the scalar product (9). With the above notation we can define the general  $\Omega$ -Laplacian operator  $\Delta_g^\Omega$  as

$$\Delta_g^\Omega = \nabla_g^{*\Omega} \nabla_g = \Delta_g + \nabla_g f \lrcorner \nabla_g.$$

We define the  $\Omega$ -divergence operator acting on vector fields as follows;

$$\operatorname{div}^\Omega \xi := \frac{d(\xi \lrcorner \Omega)}{\Omega} = e^f \operatorname{div}_g(e^{-f} \xi) = \operatorname{div}_g \xi - g(\xi, \nabla_g f).$$

We infer in particular the identity  $\operatorname{div}^\Omega \nabla_g u = -\Delta_g^\Omega u$ , for all functions  $u$ . We observe also the integration by parts formula

$$-\int_X u \operatorname{div}^\Omega \xi \Omega = \int_X g(\nabla_g u, \xi) \Omega.$$

We consider also the linear operator

$$\begin{aligned} B_{g,J}^\Omega : C^\infty(X, \mathbb{R}) &\longrightarrow C_\Omega^\infty(X, \mathbb{R})_0, \\ B_{g,J}^\Omega u &:= \operatorname{div}^\Omega(J \nabla_g u). \end{aligned}$$

This is a first order differential operator. Indeed

$$\begin{aligned} B_{g,J}^\Omega u &= \operatorname{Tr}_{\mathbb{R}}(J \nabla_g^2 u) - df \cdot J \nabla_g u \\ &= g(\nabla_g u, J \nabla_g f), \end{aligned}$$

since  $J$  is  $g$ -anti-symmetric. If we extend  $B_{g,J}^\Omega$  over  $C^\infty(X, \mathbb{C})$  by complex linearity, then

$$\begin{aligned} \Delta_{g,J}^\Omega &= \Delta_g^\Omega - i B_{J,g}^\Omega \\ &= -2 \operatorname{div}^\Omega \nabla_{g,J}^{0,1}. \end{aligned}$$

We denote by

$$\Lambda_{g,J}^\Omega := \operatorname{Ker}(\Delta_{g,J}^\Omega - 2\mathbb{I}) \subset C_\Omega^\infty(X, \mathbb{C})_0,$$

and by

$$\Lambda_{g,J}^{\Omega,\perp} := [\operatorname{Ker}(\Delta_{g,J}^\Omega - 2\mathbb{I})]^\perp \subseteq C_\Omega^\infty(X, \mathbb{C})_0,$$

its  $L_\Omega^2$ -orthogonal inside  $C_\Omega^\infty(X, \mathbb{C})_0$ . Let also

$$2d_J^c u := i(\bar{\partial}_J - \partial_J)u = -du \cdot J.$$



We define now the set  $\mathcal{V}_1 := \{\Omega > 0 \mid \int_X \Omega = 1\}$ . Notice that the tangent space of  $\mathcal{M} \times \mathcal{V}_1$  is given by

$$T_{\mathcal{M} \times \mathcal{V}_1} = C^\infty(X, S^2 T_X^*) \oplus C^\infty(X, \Lambda^{2n} T_X^*)_0,$$

where  $n := \dim_{\mathbb{C}} X$  and

$$C^\infty(X, \Lambda^{2n} T_X^*)_0 := \left\{ V \in C^\infty(X, \Lambda^{2n} T_X^*) \mid \int_X V = 0 \right\}.$$

We denote by  $\text{End}_g(T_X)$  the bundle of  $g$ -symmetric endomorphisms of  $T_X$ . We will systematically use the fact that for any  $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ , the tangent space  $T_{\mathcal{M} \times \mathcal{V}_1, (g, \Omega)}$  identifies with  $C^\infty(X, \text{End}_g(T_X)) \oplus C^\infty(X, \mathbb{R})_0$  via the isomorphism

$$(v, V) \mapsto (v_g^*, V_\Omega^*) := (g^{-1}v, V/\Omega).$$

We consider now the pseudo-Riemannian structure over the space  $\mathcal{M} \times \mathcal{V}_1$  given by the formula  $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1 \mapsto G_{g, \Omega}$ , with

$$G_{g, \Omega}(u, U; v, V) = \int_X [(u, v)_g - 2U_\Omega^* V_\Omega^*] \Omega,$$

for all  $(u, U), (v, V) \in T_{\mathcal{M} \times \mathcal{V}_1}$ . We consider the vector space

$$\mathbb{F}_{g, \Omega} := \{(v, V) \in T_{\mathcal{M} \times \mathcal{V}_1} \mid \nabla_g^* v_g^* + \nabla_g V_\Omega^* = 0\}.$$

Despite it is not needed here we recall (see [Pal6]) that  $\mathbb{F}_{g, \Omega}$  is the orthogonal space, with respect to  $G$ , to the tangent space at the point  $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$  of the orbit

$$[g, \Omega] = \text{Diff}_0(X) \cdot (g, \Omega),$$

of the point  $(g, \Omega)$  under the action of the identity component of the group of smooth diffeomorphisms  $\text{Diff}_0(X)$  of  $X$ . In formal terms the following equality holds

$$T_{[g, \Omega], (g, \Omega)}^{\perp G} = \mathbb{F}_{g, \Omega}.$$

Let now  $\mathcal{J}_{\text{int}}$  be the space of smooth integrable complex structures over  $X$ . We consider the space of  $\omega$ -compatible complex structures

$$\mathcal{J}_\omega := \{J \in \mathcal{J}_{\text{int}} \mid \omega = J^* \omega J, \omega J < 0\},$$

and let  $\mathcal{M}_\omega := -\omega \cdot \mathcal{J}_\omega \subset \mathcal{M}$ . Over a Fano manifold, the Chern-Ricci map provides a natural embedding of  $\mathcal{J}_\omega$  inside  $\mathcal{M} \times \mathcal{V}_1$ . The image  $\mathcal{S}_\omega \subset \mathcal{M} \times \mathcal{V}_1$  of this embedding is

$$\mathcal{S}_\omega := \{(g, \Omega) \in \mathcal{M}_\omega \times \mathcal{V}_1 \mid \omega = \text{Ric}_J(\Omega), J = g^{-1}\omega\}.$$

We consider now the vector space

$$\mathbb{D}_g^J := \{v \in C^\infty(X, S_{\mathbb{R}}^2 T_X^*) \mid v_g^* = \bar{\partial}_{T_X, J} \nabla_{g, J} \bar{\psi}_v, \psi_v \in \Lambda_{g, J}^{\Omega, \perp}\},$$

where  $v'_J$  and  $v''_J$  denote respectively the  $J$ -invariant and  $J$ -anti-invariant parts of  $v$ . We define also the vector space

$$\mathbb{T}_{g,\Omega}^J[0] := \left\{ (v, V) \in \mathbb{D}_g^J \times T_{\mathcal{V}_1} \mid d \left[ (\nabla_g^{*\Omega} v_g^* + \nabla_g V_\Omega^*) \lrcorner \omega \right] = 0 \right\}.$$

We consider also the sub-space of  $\mathbb{T}_{g,\Omega}^J[0]$

$$\mathbb{F}_{g,\Omega}^J := \{ (v, V) \in \mathbb{F}_{g,\Omega} \mid v \in \mathbb{D}_g^J \}.$$

For any  $(g, \Omega) \in \mathcal{S}_\omega$  we denote by

$$[g, \Omega]_\omega := \text{Symp}^0(X, \omega) \cdot (g, \Omega) \subset \mathcal{S}_\omega,$$

the orbit of the point  $(g, \Omega)$  under the action of the identity component of the group of smooth symplectomorphisms  $\text{Symp}^0(X, \omega)$  of  $X$ . With these notations, the following property holds.

**Lemma 3** *For any point  $(g, \Omega) \in \mathcal{S}_\omega$  we have*

$$T_{[g,\Omega]_\omega, (g,\Omega)}^{\perp \mathcal{G}} \cap \mathbb{T}_{g,\Omega}^J[0] = \mathbb{F}_{g,\Omega}^J. \quad (10)$$

We need now a few preliminaries.

### 3.1 The weighted complex Bochner identity

The weighted complex Bochner identity plays a crucial role in all this article. For the reader convenience we recall here its statement and proof from [Pal6].

**Lemma 4** *Let  $(X, J, g)$  be a Kähler manifold with symplectic form  $\omega := gJ$  and let  $\Omega > 0$  be a smooth volume form. Then for all  $u \in C^\infty(X, \mathbb{R})$  and  $v \in C^\infty(X, \mathbb{C})$  the complex Bochner type formulas hold*

$$2\partial_{T_X, J}^{*g, \Omega} \partial_{T_X, J}^g \nabla_g u = \nabla_{g, J} \Delta_{g, J}^\Omega u - 2\bar{\partial}_{T_X, J} \nabla_g f \nabla_g u, \quad (11)$$

$$2\bar{\partial}_{T_X, J}^{*g, \Omega} \bar{\partial}_{T_X, J} \nabla_{g, J} \bar{v} = \nabla_{g, J} \overline{\Delta_{g, J}^\Omega v} - 2\text{Ric}_J^*(\Omega)_\omega \nabla_{g, J} \bar{v}. \quad (12)$$

**Proof** Let  $\xi \in C^\infty(X, T_X)$  and observe that for bi-degree reasons the following identity holds

$$\begin{aligned} 2\partial_{T_X, J}^{*g, \Omega} \partial_{T_X, J}^g \xi &= 2\nabla_g^{*\Omega} \partial_{T_X, J}^g \xi \\ &= \Delta_g^\Omega \xi - \nabla_g^{*\Omega} (J\nabla_{g, J \bullet} \xi) \\ &= \Delta_g^\Omega \xi - \nabla_g^* (J\nabla_{g, J \bullet} \xi) - J\nabla_{g, J\nabla_g f} \xi. \end{aligned}$$

Let  $(e_k)_{k=1}^{2n}$  be a local  $g$ -orthonormal frame over a neighborhood of an arbitrary point  $p$  such that  $\nabla_g e_k(p) = 0$ . Then at the point  $p$  the following equalities hold

$$\begin{aligned} -\nabla_g^*(J\nabla_{g,J\bullet}\xi) &= J\nabla_{g,e_k}\nabla_{g,Je_k}\xi \\ &= \frac{1}{2}(J\nabla_{g,e_k}\nabla_{g,Je_k}\xi - J\nabla_{g,Je_k}\nabla_{g,e_k}\xi), \end{aligned}$$

since  $(Je_k)_{k=1}^{2n}$  is also a local  $g$ -orthonormal frame. Then the fact that  $[e_k, Je_k](p) = 0$  implies

$$-\nabla_g^*(J\nabla_{g,J\bullet}\xi) = \frac{1}{2}J\mathcal{R}_g(e_k, Je_k)\xi = \text{Ric}^*(g)\xi.$$

We infer the complex Bochner type formula

$$2\partial_{T_{X,J}}^{*g,\Omega}\partial_{T_{X,J}}^g\xi = \Delta_g^\Omega\xi + \text{Ric}^*(g)\xi - J\nabla_{g,J\nabla_g f}\xi. \quad (13)$$

In a similar way we obtain

$$2\bar{\partial}_{T_{X,J}}^{*g,\Omega}\bar{\partial}_{T_{X,J}}\xi = \Delta_g^\Omega\xi - \text{Ric}^*(g)\xi + J\nabla_{g,J\nabla_g f}\xi. \quad (14)$$

Using formulas (13) and

$$\nabla_g\Delta_g^\Omega u = \Delta_g^\Omega\nabla_g u + \text{Ric}_g^*(\Omega)\nabla_g u, \quad (15)$$

we deduce the expressions

$$\begin{aligned} 2\partial_{T_{X,J}}^{*g,\Omega}\partial_{T_{X,J}}^g\nabla_g u &= \nabla_g\Delta_g^\Omega u - \nabla_g^2 f\nabla_g u - J\nabla_g^2 u J\nabla_g f \\ &= \nabla_g\Delta_g^\Omega u - (\nabla_g^2 f + J\nabla_g^2 f J)\nabla_g u \\ &\quad - J(\nabla_g^2 u J\nabla_g f - \nabla_g^2 f J\nabla_g u) \\ &= \nabla_g\Delta_g^\Omega u - 2\bar{\partial}_{T_{X,J}}\nabla_g f\nabla_g u - J\nabla_g[g(\nabla_g u, J\nabla_g f)]. \end{aligned}$$

Using the first order expression of  $B_{J,g}^\Omega$  we obtain

$$2\partial_{T_{X,J}}^{*g,\Omega}\partial_{T_{X,J}}^g\nabla_g u = [\nabla_g\Delta_g^\Omega - J\nabla_g B_{g,J}^\Omega]u - 2\bar{\partial}_{T_{X,J}}\nabla_g f\nabla_g u.$$

We infer the complex differential Bochner type formula (11). In a similar way using formulas (14) and (15) we deduce

$$\begin{aligned}
2\bar{\partial}_{T_{X,J}}^{*g,\Omega}\bar{\partial}_{T_{X,J}}\nabla_g u &= \nabla_g\Delta_g^\Omega u - 2\text{Ric}_g^*(\Omega)\nabla_g u + \nabla_g^2 f\nabla_g u + J\nabla_g^2 u J\nabla_g f \\
&= \nabla_g\Delta_g^\Omega u - 2\text{Ric}_g^*(\Omega)\nabla_g u + (\nabla_g^2 f + J\nabla_g^2 f J)\nabla_g u \\
&+ J(\nabla_g^2 u J\nabla_g f - \nabla_g^2 f J\nabla_g u) \\
&= \nabla_g\Delta_g^\Omega u - 2\text{Ric}_g^*(\Omega)\nabla_g u + 2\bar{\partial}_{T_{X,J}}\nabla_g f\nabla_g u \\
&+ J\nabla_g[g(\nabla_g u, J\nabla_g f)].
\end{aligned}$$

Using the first order expression of  $B_{J,g}^\Omega$  we obtain

$$2\bar{\partial}_{T_{X,J}}^{*g,\Omega}\bar{\partial}_{T_{X,J}}\nabla_g u = [\nabla_g\Delta_g^\Omega + J\nabla_g B_{g,J}^\Omega]u - 2\text{Ric}_J^*(\Omega)_\omega\nabla_g u.$$

We infer the complex differential Bochner type formula

$$2\bar{\partial}_{T_{X,J}}^{*g,\Omega}\bar{\partial}_{T_{X,J}}\nabla_g u = \nabla_{g,J}\overline{\Delta_{g,J}^\Omega}u - 2\text{Ric}_J^*(\Omega)_\omega\nabla_g u. \quad (16)$$

More generally for all  $v \in C^\infty(X, \mathbb{C})$  this writes as (12).  $\square$

Notice that for bi-degree reasons the identity (15) decomposes as

$$\begin{aligned}
2\bar{\partial}_{T_{X,J}}^{*g,\Omega}\partial_{T_{X,J}}^g\nabla_g u + 2\bar{\partial}_{T_{X,J}}^{*g,\Omega}\bar{\partial}_{T_{X,J}}\nabla_g u &= \nabla_{g,J}\Delta_{g,J}^\Omega u + \nabla_{g,J}\overline{\Delta_{g,J}^\Omega}u \\
&- 2\bar{\partial}_{T_{X,J}}\nabla_g f\nabla_g u - 2\text{Ric}_J^*(\Omega)_\omega\nabla_g u.
\end{aligned}$$

Then we can obtain (16) from (11) and vice versa. We observe also that in the case  $\text{Ric}_J(\Omega) = \omega$  the complex Bochner identity (12) writes as

$$2\bar{\partial}_{T_{X,J}}^{*g,\Omega}\bar{\partial}_{T_{X,J}}\nabla_{g,J}\bar{v} = \nabla_{g,J}\overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})v}, \quad (17)$$

for all  $v \in C^\infty(X, \mathbb{C})$ . We obtain now an integration by parts formula.

Let  $\xi \in C^\infty(X, T_X)$ ,  $A \in C^\infty(X, T_{X,-J}^* \otimes T_{X,J})$  and observe that the comparison between Riemannian and Hermitian norms of  $T_X$ -valued 1-forms in the Appendix [Pal3] implies

$$\begin{aligned}
\int_X \langle \bar{\partial}_{T_{X,J}}\xi, A \rangle_g \Omega &= \frac{1}{2} \int_X [\langle \bar{\partial}_{T_{X,J}}\xi, A \rangle_\omega + \langle A, \bar{\partial}_{T_{X,J}}\xi \rangle_\omega] \Omega \\
&= \frac{1}{2} \int_X [\langle \xi, \bar{\partial}_{T_{X,J}}^{*g,\Omega} A \rangle_\omega + \langle \bar{\partial}_{T_{X,J}}^{*g,\Omega} A, \xi \rangle_\omega] \Omega \\
&= \int_X \langle \xi, \bar{\partial}_{T_{X,J}}^{*g,\Omega} A \rangle_g \Omega.
\end{aligned}$$

Using this and multiplying both sides of (17) by  $\nabla_{g,J}\bar{v}$  we obtain the identity

$$2 \int_X |\bar{\partial}_{T_{X,J}} \nabla_{g,J} \bar{v}|_g^2 \Omega = \int_X \left\langle \nabla_{g,J} \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})v}, \nabla_{g,J} \bar{v} \right\rangle_g \Omega, \quad (18)$$

in the case  $\text{Ric}_J(\Omega) = \omega$ . This implies that the first eigenvalue estimate  $\lambda_1(\Delta_{g,J}^\Omega) \geq 2$ . On the other hand the identities (17) and (18) show that in the case  $\text{Ric}_J(\Omega) = \omega$  the following identity holds

$$\overline{\Lambda_{g,J}^\Omega} = \text{Ker } \mathcal{H}_{g,J}^{0,1}. \quad (19)$$

We infer the following well known result. (See the Appendix B in [Pal6] for a more complete statement.)

**Corollary 1** *Let  $(X, J)$  be a Fano manifold and let  $g$  be a  $J$ -invariant Kähler metric such that  $\omega := gJ \in 2\pi c_1(X, [J])$ . Let also  $\Omega > 0$  be the unique smooth volume form with  $\int_X \Omega = 1$  such that  $\text{Ric}_J(\Omega) = \omega$ . Then the map*

$$\begin{aligned} \overline{\Lambda_{g,J}^\Omega} &\longrightarrow H^0(X, T_{X,J}) \\ u &\longmapsto \nabla_{g,J} u, \end{aligned}$$

is well defined and it represents an isomorphism of complex vector spaces. Moreover the first eigenvalue  $\lambda_1(\Delta_{g,J}^\Omega)$  of the operator  $\Delta_{g,J}^\Omega$  satisfies the estimate  $\lambda_1(\Delta_{g,J}^\Omega) \geq 2$ , with equality in the case  $H^0(X, T_{X,J}) \neq 0$ .

### 3.2 Proof of Lemma 3

**Proof** For any  $(v, V) \in \mathbb{D}_g^J \times T_{\mathcal{V}_1}$  we consider the expression

$$v_g^* = \bar{\partial}_{T_{X,J}} \nabla_{g,J} \overline{\psi_v},$$

with unique  $\psi_v \in \Lambda_{g,J}^{\Omega, \perp}$ . Then the weighted complex Bochner identity (17), implies the equality

$$\bar{\partial}_{T_{X,J}}^{*g, \Omega} v_g^* + \nabla_g V_\Omega^* = \frac{1}{2} \nabla_{g,J} \left[ \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi_v} + 2V_\Omega^* \right]. \quad (20)$$

Thus

$$\mathbb{F}_{g, \Omega}^J = \{(v, V) \in \mathbb{D}_g^J \times T_{\mathcal{V}_1} \mid (\Delta_{g,J}^\Omega - 2\mathbb{I})\psi_v = -2V_\Omega^*\}. \quad (21)$$

Let

$$R_{\psi_v} := \text{Re}[(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi_v],$$

$$I_{\psi_v} := \text{Im}[(\Delta_{g,J}^\Omega - 2\mathbb{I})\psi_v],$$

(for any  $z \in \mathbb{C}$  we write  $z = \operatorname{Re} z + i \operatorname{Im} z$ ) and observe that (20) implies the identity

$$(\nabla_g^* v_g^* + \nabla_g V_\Omega^*) \lrcorner \omega = \frac{1}{2} dI_{\psi_v} + d_J^c(R_{\psi_v} + 2V_\Omega^*),$$

for any  $(v, V) \in \mathbb{D}_g^J \times T_{\mathcal{V}_1}$ . Thus

$$\mathbb{T}_{g,\Omega}^J[0] = \left\{ (v, V) \in \mathbb{D}_g^J \times T_{\mathcal{V}_1} \mid R_{\psi_v} = -2V_\Omega^* \right\}. \quad (22)$$

We notice now the equalities

$$\begin{aligned} T_{[g,\Omega]_\omega,(g,\Omega)} &= \{ (L_\xi g, L_\xi \Omega) \mid \xi \in C^\infty(X, T_X) : L_\xi \omega = 0 \} \\ &= \{ (L_{J\nabla_g u} g, L_{J\nabla_g u} \Omega) \mid u \in C_\Omega^\infty(X, \mathbb{R})_0 \} \\ &= \{ (2g J \bar{\partial}_{T_X, J} \nabla_g u, \operatorname{div}^\Omega(J\nabla_g u) \Omega) \mid u \in C_\Omega^\infty(X, \mathbb{R})_0 \}, \end{aligned}$$

indeed

$$(L_{J\nabla_g u} g)_g^* = J\nabla_g^2 u - \nabla_g^2 u J = 2J \bar{\partial}_{T_X, J} \nabla_g u.$$

We deduce that  $(v, V) \in T_{[g,\Omega]_\omega,(g,\Omega)}^{\perp G}$  if and only if for all  $u \in C_\Omega^\infty(X, \mathbb{R})_0$  the following equalities hold

$$\begin{aligned} 0 &= 2 \int_X [ \langle J \bar{\partial}_{T_X, J} \nabla_g u, v_g^* \rangle_g - \operatorname{div}^\Omega(J\nabla_g u) \cdot V_\Omega^* ] \Omega \\ &= -2 \int_X \langle \nabla_g u, J(\bar{\partial}_{T_X, J}^* v_g^* + \nabla_g V_\Omega^*) \rangle_g \Omega \\ &= 2 \int_X u \cdot \operatorname{div}^\Omega [ J(\bar{\partial}_{T_X, J}^* v_g^* + \nabla_g V_\Omega^*) ] \Omega. \end{aligned}$$

If we assume  $(v, V) \in \mathbb{T}_{g,\Omega}^J[0]$  then

$$\bar{\partial}_{T_X, J}^* v_g^* + \nabla_g V_\Omega^* = -\frac{1}{2} J \nabla_g I_{\psi_v}, \quad (23)$$

thanks to (20) and (22). Thus if  $(v, V) \in T_{[g,\Omega]_\omega,(g,\Omega)}^{\perp G} \cap \mathbb{T}_{g,\Omega}^J[0]$  then

$$0 = - \int_X u \cdot \Delta_g^\Omega I_{\psi_v} \Omega,$$

for all  $u \in C_\Omega^\infty(X, \mathbb{R})_0$ , i.e.  $\Delta_g^\Omega I_{\psi_v} = 0$ , which is equivalent to the condition  $I_{\psi_v} = 0$ . We infer

$$T_{[g,\Omega]_\omega,(g,\Omega)}^{\perp G} \cap \mathbb{T}_{g,\Omega}^J[0] \subseteq \mathbb{F}_{g,\Omega}^J.$$

The reverse inclusion is obvious. We deduce the identity (10).  $\square$

### 3.3 Positivity of the metric $G_{g,\Omega}$ over the space $\mathbb{T}_{g,\Omega}^J[0]$

We show first the following integration by parts formulas.

**Lemma 5** *For any  $u, v \in C^\infty(X, \mathbb{C})$  the following integration by parts identity holds*

$$\int_X \left[ \Delta_{g,J}^\Omega u \cdot \bar{v} + \overline{\Delta_{g,J}^\Omega u} \cdot v \right] \Omega = 2 \int_X g(\nabla_{g,J} \bar{u}, \nabla_{g,J} v) \Omega. \quad (24)$$

If  $u \in C^\infty(X, \mathbb{R})$  then also the integration by parts identity holds

$$\int_X \left[ \Delta_{g,J}^\Omega u \cdot v + \overline{\Delta_{g,J}^\Omega u} \cdot \bar{v} \right] \Omega = 2 \int_X g(\nabla_g u, \nabla_{g,J} v) \Omega. \quad (25)$$

**Proof** Using the complex decomposition (4) and the fact that Hermitian product  $\langle \cdot, \cdot \rangle_\omega$  on  $T_X^* \otimes_{\mathbb{R}} \mathbb{C}$  is the sesquilinear extension of the dual of  $g$ , we deduce

$$\begin{aligned} g(\nabla_{g,J} u, \nabla_{g,J} v) &= \langle \partial_J \bar{u} + \bar{\partial}_J u, \partial_J v + \bar{\partial}_J v \rangle_g \\ &= \langle \partial_J \bar{u} + \bar{\partial}_J u, \partial_J v + \bar{\partial}_J v \rangle_\omega \\ &= \langle \partial_J \bar{u}, \partial_J v \rangle_\omega + \langle \bar{\partial}_J u, \bar{\partial}_J v \rangle_\omega \\ &= \langle \partial_J \bar{u}, \partial_J v \rangle_\omega + \overline{\langle \partial_J \bar{u}, \partial_J v \rangle_\omega}. \end{aligned}$$

Integrating by parts and taking the conjugate we infer the identity

$$2 \int_X g(\nabla_{g,J} u, \nabla_{g,J} v) \Omega = \int_X \left[ \Delta_{g,J}^\Omega u \cdot v + \overline{\Delta_{g,J}^\Omega u} \cdot \bar{v} \right] \Omega. \quad (26)$$

Replacing  $u$  with  $\bar{u}$ ,  $v$  with  $\bar{v}$  in (26) we obtain (24). In the case  $u \in C^\infty(X, \mathbb{R})$  formula (26) implies directly (25).  $\square$

**Lemma 6** *For any  $(g, \Omega) \in \mathcal{S}_\omega$  the restriction of the symmetric form  $G_{g,\Omega}$  to the vector space  $\mathbb{T}_{g,\Omega}^J[0]$ , with  $J := g^{-1}\omega$ , is positive definite.*

**Proof** Let  $(u, U), (v, V) \in \mathbb{T}_{g,\Omega}^J[0]$ . Using the expression (22) for the space  $\mathbb{T}_{g,\Omega}^J[0]$  we have

$$\begin{aligned} u_g^* &= \bar{\partial}_{T_X, J} \nabla_{g,J} \bar{\varphi}, \\ -2U_\Omega^* &= \operatorname{Re} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \right], \end{aligned}$$

and

$$\begin{aligned} v_g^* &= \bar{\partial}_{T_X, J} \nabla_{g,J} \bar{\psi}, \\ -2V_\Omega^* &= \operatorname{Re} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi \right], \end{aligned}$$

with unique  $\varphi, \psi \in \Lambda_{g,J}^{\Omega, \perp}$ . We decompose now the term

$$\begin{aligned} G_{g,\Omega}(u, U; v, V) &= \int_X [\langle u, v \rangle_g - 2U_\Omega^* \cdot V_\Omega^*] \Omega \\ &= \int_X \left[ \left\langle \bar{\partial}_{T_{X,J}} \nabla_{g,J} \bar{\varphi}, \bar{\partial}_{T_{X,J}} \nabla_{g,J} \bar{\psi} \right\rangle_g \right] \Omega \\ &\quad - \frac{1}{2} \int_X \operatorname{Re} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \right] \operatorname{Re} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi \right] \Omega. \end{aligned}$$

Integrating by parts and using the weighted complex Bochner formula (17) we transform the integral

$$\begin{aligned} I_1 &:= \int_X \left\langle \bar{\partial}_{T_{X,J}} \nabla_{g,J} \bar{\varphi}, \bar{\partial}_{T_{X,J}} \nabla_{g,J} \bar{\psi} \right\rangle_g \Omega \\ &= \int_X \left\langle \bar{\partial}_{T_{X,J}}^* \bar{\partial}_{T_{X,J}} \nabla_{g,J} \bar{\varphi}, \nabla_{g,J} \bar{\psi} \right\rangle_g \Omega \\ &= \frac{1}{2} \int_X \left\langle \nabla_{g,J} \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi}, \nabla_{g,J} \bar{\psi} \right\rangle_g \Omega. \end{aligned}$$

Using the integration by parts formula (25) we deduce

$$\begin{aligned} I_1 &= \frac{1}{4} \int_X \left[ \Delta_{g,J}^\Omega (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \cdot \bar{\psi} + \overline{\Delta_{g,J}^\Omega (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi} \cdot \psi \right] \Omega \\ &= \frac{1}{4} \int_X \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \cdot \overline{\Delta_{g,J}^\Omega \psi} + \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi} \cdot \Delta_{g,J}^\Omega \psi \right] \Omega. \end{aligned}$$

Adding and subtracting  $2\bar{\psi}$  to the factor  $\overline{\Delta_{g,J}^\Omega \psi}$  and respectively  $2\psi$  to the factor  $\Delta_{g,J}^\Omega \psi$ , we infer

$$\begin{aligned} I_1 &= \frac{1}{2} \int_X \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \cdot \bar{\psi} + \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi} \cdot \psi \right] \Omega \\ &\quad + \frac{1}{4} \int_X \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \cdot \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi} + \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi} \cdot (\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi \right] \Omega \\ &= \frac{1}{2} \int_X \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \cdot \bar{\psi} + \bar{\varphi} \cdot (\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi \right] \Omega \\ &\quad + \frac{1}{4} \int_X \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \cdot \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi} + \overline{(\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi} \cdot (\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi \right] \Omega. \end{aligned}$$

We deduce the general formula

$$\begin{aligned} G_{g,\Omega}(u, U; v, V) &= \int_X \left\{ \frac{1}{2} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \cdot \bar{\psi} + (\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi \cdot \bar{\varphi} \right] \right\} \Omega \\ &\quad + \frac{1}{2} \int_X \operatorname{Im} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \varphi \right] \operatorname{Im} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I}) \psi \right] \Omega. \end{aligned}$$



In particular

$$\begin{aligned} G_{g,\Omega}(u, U; u, U) &= \int_X \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I})\varphi \cdot \bar{\varphi} \right] \Omega \\ &+ \frac{1}{2} \int_X \left\{ \operatorname{Im} \left[ (\Delta_{g,J}^\Omega - 2\mathbb{I})\varphi \right] \right\}^2 \Omega \geq 0, \end{aligned}$$

with equality if and only if  $\varphi = 0$ , i.e.  $(u, U) = (0, 0)$ , thanks to the variational characterization of the first eigenvalue  $\lambda_1(\Delta_{g,J}^\Omega) \geq 2$  of the elliptic operator  $\Delta_{g,J}^\Omega$ .  $\square$

We show now that the identity (18.1) in Corollary 5 Section 18 of [Pal 4] (i.e. Corollary 2 below) is a consequence of the above positivity result.

**Corollary 2** *For any  $(g, \Omega) \in \mathcal{S}_\omega$  the following identity holds*

$$\operatorname{Ker}_{\mathbb{R}}(\Delta_g^\Omega - 2\mathbb{I}) = \operatorname{Ker}_{\mathbb{R}}(\Delta_{g,J}^\Omega - 2\mathbb{I}), \quad (27)$$

with  $J := g^{-1}\omega$ .

**Proof** Let  $u \in C_\Omega^\infty(X, \mathbb{R})_0$  and  $(\varphi_t)_{t \in \mathbb{R}} \subset \operatorname{Symp}^0(X, \omega)$  the 1-parameter subgroup generated by the symplectic vector field  $\xi := (du)_\omega^* = -J\nabla_g u$ . We set  $J_t := \varphi_t^* J$ ,  $g_t := \varphi_t^* g$ ,  $\Omega_t := \varphi_t^* \Omega$  and we compute  $\dot{g}_0 = L_\xi g$  and  $\dot{\Omega}_0 = L_\xi \Omega$ . The expression of the tangent space to the symplectic orbit  $[g, \Omega]_\omega$  in the proof of Lemma 3 implies

$$\begin{aligned} \dot{g}_0^* &= -2J\bar{\partial}_{T_{X,J}} \nabla_g u, \\ \dot{\Omega}_0^* &= -B_{g,J}^\Omega u. \end{aligned}$$

Then the weighted complex Bochner formula (17) implies

$$\begin{aligned} \bar{\partial}_{T_{X,J}}^* \dot{g}_0^* + \nabla_g \dot{\Omega}_0^* &= -2J\bar{\partial}_{T_{X,J}}^* \bar{\partial}_{T_{X,J}} \nabla_g u + \nabla_g \dot{\Omega}_0^* \\ &= -J\nabla_g (\Delta_g^\Omega - 2\mathbb{I})u + \nabla_g B_{g,J}^\Omega u + \nabla_g \dot{\Omega}_0^* \\ &= -J\nabla_g (\Delta_g^\Omega - 2\mathbb{I})u. \end{aligned}$$

We deduce that  $(\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}_{g,\Omega}^J$  if and only if  $(\Delta_g^\Omega - 2\mathbb{I})u = 0$ . On the other hand the (strict) positivity of the metric  $G_{g,\Omega}$  over  $\mathbb{T}_{g,\Omega}^J[0] \supset T_{[g,\Omega]_\omega, (g,\Omega)}$  implies

$$T_{[g,\Omega]_\omega, (g,\Omega)} \cap T_{[g,\Omega]_\omega, (g,\Omega)}^{\perp G} = \{0\}.$$

Then Lemma 3 implies

$$T_{[g,\Omega]_\omega, (g,\Omega)} \cap T_{[g,\Omega]_\omega, (g,\Omega)}^{\perp G} = T_{[g,\Omega]_\omega, (g,\Omega)} \cap \mathbb{F}_{g,\Omega}^J = \{0\},$$

So if  $(\dot{g}_0, \dot{\Omega}_0) \in \mathbb{F}_{g, \Omega}^J$  then  $(\dot{g}_0, \dot{\Omega}_0) = (0, 0)$ . We infer the inclusion

$$\text{Ker}_{\mathbb{R}}(\Delta_g^{\Omega} - 2\mathbb{I}) \subseteq \text{Ker}_{\mathbb{R}} B_{g, J}^{\Omega},$$

and thus the required identity (27).  $\square$

## References

- [Cao] CAO, H. D., *Limits of solutions to the Kähler-Ricci flow*, J. Differential Geom. 45, no. 2, 257-272, (1997).
- [Fri] FRIEDON, D. H., *Nonlinear models in  $2 + \epsilon$  dimensions*, Ann. Physics 163, no. 2, 318-419, (1985).
- [Ham] HAMILTON, R.S., *Three-manifolds with positive Ricci curvature*, J. differential Geom. 17, no. 2, 255-306, (1982).
- [Mo-Mu-Sh1] MOROIANU, A., MURCIA, A., SHAHBAZI, C.S., *The Heterotic-Ricci flow and its three-dimensional solitons*, J. Geom. Anal. 34, 122, (2024).
- [Mo-Mu-Sh2] MOROIANU, A., MURCIA, A., SHAHBAZI, C.S., *Heterotic solitons on four-manifolds*, New York J. Math. 28, 1463-1497, (2022).
- [Nur-Ran] NUROWSKI, P., RANDALL, M *Generalised Ricci Solitons*, J. Geom. Anal., 26:1280-1345, (2016).
- [Pal1] PALI, N., *Plurisubharmonic functions and positive  $(1, 1)$ -currents over almost complex manifolds*, Manuscripta Mathematica, volume 118, issue 3, pp. 311 - 337, (2005).
- [Pal2] PALI, N., *Characterization of Einstein-Fano manifolds via the Kähler-Ricci flow*, Indiana Univ. Math. J. 57, no. 7, 3241-3274, (2008).
- [Pal3] PALI, N., *A second variation formula for Perelman's  $\mathcal{W}$ -functional along the modified Kähler-Ricci flow*, arXiv:1201.0970v1, Mathematische Zeitschrift. 276 (2014), no. 1-2, 173-189, (2012).
- [Pal4] PALI, N., *The Soliton-Kähler-Ricci flow over Fano manifolds*, New York J. Math. 20, 845-919, (2014).
- [Pal5] PALI, N., *Variational stability of Kähler-Ricci solitons*, Advances in Mathematics 290, 15-35, (2016).
- [Pal6] PALI, N., *The Soliton-Kähler-Ricci flow with variable volume forms*, Complex Manifolds, Vol 3, Issue 1, 41-144, (2016).

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