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Vector spaces spanned by Tutte polynomials

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Abstract

Kung exhibited two bases for the subspace of bivariate polynomials spanned by the Tutte polynomials of matroids of size n and rank r , thereby determining its dimension, and asked what dimension subspace of bivariate polynomials is spanned by the Tutte polynomials of matroids of size n and rank r drawn from a fixed class \mathcal{C} of matroids (such as cycle matroids of graphs or binary matroids). We give a sufficient condition for this space to be of the maximum possible dimension $r(n - r) + 1$, i.e. the same as that spanned by the Tutte polynomials of all matroids. In particular, we produce a basis composed of the Tutte polynomials of $r(n - r) + 1$ graphs, each of which is a series-parallel graph with a number of bridges and loops added.

We also determine the dimension of the subspace of bivariate polynomials spanned by the Tutte polynomials of connected matroids of given size and rank. Again, an explicit basis is provided: it is composed only of Tutte polynomials of planar 2-connected loopless graphs. To obtain this, we make the intermediate step of providing an explicit basis of the subspace generated by the Tutte polynomials of all matroids of given size and rank that are connected after the removal of any loops. We additionally obtain, as a by-product, an explicit basis for the class of all loopless matroids of given size and rank.

1 Introduction

The Tutte polynomial of a graph is a bivariate polynomial invariant, specializations of which include the chromatic polynomial, reliability polynomial, partition function of the Potts model in statistical physics, and the Jones polynomial of an alternating knot (encoded as a plane graph). After a simple transformation of variables the Tutte polynomial of a graph coincides with the Whitney rank-size generating function for spanning subgraphs and accordingly depends only on the cycle matroid of the graph. Indeed the Tutte polynomial thereby extends to an invariant of matroids generally, and has served to bridge disparate areas of combinatorics (such as hyperplane arrangements and coding theory) and appears in various disciplines (such as the random cluster model in physics and DNA sequencing in biology). A voluminous handbook devoted to the Tutte polynomial has recently been published [6].

How “generic” is the Tutte polynomial among bivariate polynomials? Certain constraints arising from the combinatorial definition of the polynomial impose restrictions on what can be a Tutte polynomial of some matroid: for example, the coefficients of the Tutte polynomial are non-negative integers (indeed Tutte interprets these coefficients combinatorially [11]), and the Tutte polynomial of a connected matroid, as an element of $\mathbf{Z}[x, y]$, is irreducible [9]. Over a given field \mathbf{F} , we approach the question from a linear algebra perspective: via the ring homomorphism $\phi : \mathbf{Z} \rightarrow \mathbf{F}$ defined by $\phi(1) = 1_{\mathbf{F}}$ (ϕ the inclusion map if for example $\mathbf{F} = \mathbf{Q}$ or reduction modulo p if $\mathbf{F} = \mathbf{Z}/p\mathbf{Z}$), we view the Tutte polynomial as an element of $\mathbf{F}[x, y]$. The opening question of this paragraph for us takes the specific form of determining the dimension of the subspace of $\mathbf{F}[x, y]$ spanned by Tutte polynomials of matroids.

For a matroid M of size n and rank r , the Tutte polynomial $T_M(x, y)$ has degree in x equal to r and degree in y equal to $n - r$. Polynomials over \mathbf{F} of degree r in x and degree $n - r$ in y span a subspace of $\mathbf{F}[x, y]$ of dimension $(r + 1)(n - r + 1)$. Kung [8] shows that the subspace spanned by Tutte polynomials of matroids of size n and rank r has dimension $r(n - r) + 1$. This corresponds precisely to the reduction in

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dimension due to the fact that the coefficients of the Tutte polynomial of a matroid of size n and rank r satisfy n independent linear relations (namely the Brylawski relations [4], quoted as Theorem 2.2 below) and no linear relation independent of these holds for all matroids of given size and rank (Kung shows this by giving a basis of $r(n-r)+1$ freedom matroids; we give a basis of series-parallel graphs in Section 3 below).¹

If a matroid M of size n and rank r has no coloops, then the coefficient of $x^i y^{n-r}$ for $1 \leq i \leq r$ is equal to zero (a coloop contributes a factor of x to the Tutte polynomial); dually, if M has no loops, then the coefficient of $x^r y^j$ for $1 \leq j \leq n-r$ is equal to zero (a loop contributes a factor of y to the Tutte polynomial). These additional linear relations on the coefficients are independent of the Brylawski relations, and in fact the dimension of the subspace spanned by Tutte polynomials of matroids without coloops (respectively, without loops) is equal to $r(n-r-1)+1$ (respectively, $(r-1)(n-r)+1$), i.e. a reduction of the dimension of the space spanned by Tutte polynomials of all matroids by r (respectively, by $n-r$). This follows Kung's result [8, Corollary 6.9] that the dimension of the subspace of $\mathbf{F}[x, y]$ spanned by Tutte polynomials of matroids of girth (i.e. minimum circuit size) at least g is $(r-g+1)(n-r)+1$, the case of loopless matroids corresponding to $g=2$.

Kung [8, §9] asks for the dimension for other “interesting” classes of matroids. In this paper our interest lies in connected matroids, whose Tutte polynomials are irreducible in $\mathbf{Z}[x, y]$: we show (Theorem 5.1) that, for given size n and rank r , the subspace spanned by Tutte polynomials of connected matroids has dimension equal to $(r-1)(n-r-1)+1$, which is to say $n-1$ less than that of the space spanned by Tutte polynomials of all matroids (except when $n=r+2 \geq 4$ or $r=2 \leq n-2$, in which case the dimension is reduced by an additional one to $(r-1)(n-r-1)$). We establish this in two steps, first exhibiting a basis for the subspace spanned by Tutte polynomials of “connected matroids with loops” (Section 4) before using this basis to derive by a series of “loop elimination” identities, a basis for the subspace spanned by Tutte polynomials of connected matroids (Section 5). For a matroid without coloops or loops the coefficients mentioned at the beginning of the previous paragraph are all zero and account for $n-1$ independent linear relations (the coefficient of $x^r y^{n-r}$ is zero both for coloopless and for loopless matroids). Since connected matroids of size at least 2 have no coloops or loops, it follows from our result that the subspace spanned by the Tutte polynomials of matroids with neither coloops nor loops coincides with that spanned by the Tutte polynomials of connected matroids except when $n=r+2 \geq 4$ or $r=2 \leq n-2$, when the subspace spanned by Tutte polynomials of connected matroids has dimension one less.

Kung gives two bases of Tutte polynomials of freedom matroids for the subspace spanned by Tutte polynomials of matroids of given size and rank: the first [8, Theorem 5.1] consists of Tutte polynomials of uniform matroids with added loops and coloops, while the second [8, Theorem 5.8] consists of Tutte polynomials of matroids containing exactly one cyclic flat, or exactly two cyclic flats one of which is the entire ground set. The basis we construct for the subspace spanned by Tutte polynomials of matroids consists of series-parallel graphs with loops and coloops allowed (Section 4.2), and that of the subspace spanned by Tutte polynomials of connected matroids consists of 2-connected series parallel graphs together with one graph obtained from K_4 by series and parallel extensions (Section 5.2). Surprisingly, then, the linear relations satisfied by Tutte polynomials of matroids of given size and rank are essentially accounted for by those satisfied by Tutte polynomials of series-parallel graphs alone (along with the graph built from K_4 when restricting to connected matroids). Consequently, the dimension of the space spanned by Tutte polynomials of (2-connected) graphs does not differ from that of the space spanned by the Tutte polynomials of all (connected) matroids. This answers one of the specific questions raised by Kung [8, §9].

2 Preliminaries

The terms “graph” and “multigraph” are used interchangeably. We shall explicitly mention if loops or multiple edges are forbidden in the graphs considered, in particular using the term “simple graph” if both

¹Beke et al. [1] have shown that for a set E any subset expansion of the form

$$\sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}, \quad (1)$$

in which $r : 2^E \rightarrow \mathbf{N}$ satisfies $0 \leq r(A) \leq r(E)$ and $0 \leq r(A) \leq |A|$ for all $A \subseteq E$ (the weakest conditions needed to ensure that the expression (1) defines a polynomial in x and y) satisfies the Brylawski relations. Thus no further linear constraints on coefficients are imposed by specifying that the function $r : 2^E \rightarrow \mathbf{N}$ also satisfies the conditions

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B), \quad \text{and} \quad r(A \cup \{e\}) \leq r(A) + 1$$

for all $e \in E$ and $A, B \subseteq E$ that make r a matroid rank function and the expression (1) equal to the Tutte polynomial.

are forbidden. Every undefined term related to graph theory can be found, e.g., in the book by Bondy and Murty [3]. A connected graph G naturally gives rise to a matroid, called the *cycle matroid* of G , with ground set the set of edges of G and with bases exactly corresponding to the edge sets of the spanning trees of G .

In a matroid M , a *loop* is an element contained in no basis of M , while a *coloop* is an element contained in every basis of M . So if M is the cycle matroid of a graph G , then the loops of M are exactly those of G , while the coloops of M are the bridges of G . The matroid M is *connected* if any two of its elements are contained in a circuit. The *dual matroid* of M is the matroid M^* with the same ground set E as M , and bases those subsets B^* of E such that $E \setminus B^*$ is a basis of M . For positive integers n and $r \leq n$, we write $U_{n,r}$ for the uniform matroid with rank r and size n , that is, the unique matroid with n elements such that every subset of r elements forms a basis. We need only basic notions related to matroids, and we follow the standard terminology as used, e.g., in the book by Oxley [10].

Fix a field \mathbf{F} with neutral multiplicative element $1_{\mathbf{F}}$. Via the ring homomorphism $\phi: \mathbf{Z} \rightarrow \mathbf{F}$ defined by $\phi(1) = 1_{\mathbf{F}}$, we will identify a positive integer n with $\phi(n) = n \cdot 1_{\mathbf{F}}$. Given a matroid M with ground set E , and an element $e \in E$, the matroid obtained by *deleting* e is the matroid $M - e$ with ground set $E \setminus \{e\}$ and independent sets those subsets of $E \setminus \{e\}$ that are independent in M . If e is not a loop, then the matroid obtained by *contracting* e , is the matroid M/e with ground set $E \setminus \{e\}$ and bases those subsets B of $E \setminus \{e\}$ such that $B \cup \{e\}$ is a basis of M . If e is a loop, then $M/e = M - e$, and the same is true if e is a coloop. Via $\phi: \mathbf{Z} \rightarrow \mathbf{F}$, the *Tutte polynomial* of M in x and y , which we write $T(M; x, y)$ or $T_M(x, y)$, can be seen as the element of $\mathbf{F}[x, y]$ defined by the following “deletion-contraction” formula:

$$T(M; x, y) = \begin{cases} 1 = 1_{\mathbf{F}} & \text{if } M \text{ has no elements,} \\ y \cdot T(M - e; x, y) & \text{if } e \in E \text{ is a loop in } M, \\ x \cdot T(M/e; x, y) & \text{if } e \in E \text{ is a coloop in } M, \\ T(M - e; x, y) + T(M/e; x, y) & \text{if } e \in E \text{ is neither a loop nor a coloop in } M. \end{cases}$$

In long formulæ, we liberally omit the variables x and y to ease the reading.

If \mathcal{B} is a set of matroids, or of graphs, then we write \mathcal{B}_T for the corresponding set of Tutte polynomials, that is, $\mathcal{B}_T := \{T(M) : M \in \mathcal{B}\}$. Given a class \mathcal{C} of matroids, we define the *Tutte space* $\mathcal{T}(\mathcal{C})$ of \mathcal{C} to be the subspace of the vector space of all bivariate polynomials over \mathbf{F} generated by \mathcal{C}_T . We define $\mathcal{M}(n, r)$ to be the class of all matroids of size n and rank r . Using an inductive argument, it follows from the definition of the Tutte polynomial that if $M \in \mathcal{M}(n, r)$, then the x -degree of $T(M; x, y)$ is r , while the y -degree is $n - r$. In particular, the dimension of the Tutte space of $\mathcal{M}(n, r)$ is at most $(r + 1)(n - r + 1)$.

We tackle a general question of Kung [8], who asked for the dimensions of spaces spanned by the Tutte polynomials of “interesting” classes of matroids. We are interested in the dimension of the Tutte space for several classes of matroids with (arbitrarily) fixed rank and size, including that of all matroids and that of all connected matroids (with given rank and size). Not only do we determine these dimensions, but we also provide explicit bases. A striking fact is that all the bases we give are formed of Tutte polynomials of cycle matroids arising from plane graphs. Further, these graphs are all series-parallel for the class of all matroids, while all but (at most) one are series-parallel for the class of connected matroids. In addition, the dimensions of Tutte spaces, and the bases we construct, do not depend on the ground field \mathbf{F} .

We introduce notation to single out coefficients of monomials in bivariate polynomials.

Definition 2.1. Let i and j be non-negative integers.

- If $P \in \mathbf{F}[x, y]$, we define $[x^i y^j]P$ to be the coefficient of the monomial $x^i y^j$ in $P(x, y)$. We simply write $[x^i]P$ if $j = 0$, and $[y^j]P$ if $i = 0$. We say that P has no monomial $x^i y^j$ if $[x^i y^j]P = 0$.
- If M is a matroid or a graph, we define $t_{i,j}(M)$ to be $[x^i y^j]T(M)$, that is, the coefficient of the monomial $x^i y^j$ in the Tutte polynomial of M .

A first natural property to consider in this scope are the Brylawski relations, satisfied by the coefficients of any Tutte polynomial.

Theorem 2.2 (Brylawski [4]). *Fix a positive integer n . The Tutte polynomial $T(M; x, y)$ of a matroid M of size n satisfies the following linear relations, for $k \in \{0, \dots, n - 1\}$,*

$$\sum_{i=0}^k \sum_{j=0}^{k-i} (-1)^j \binom{k-i}{j} t_{i,j}(M) = 0. \quad (2)$$

Theorem 2.2 implies that the dimension of the Tutte space of the class of all matroids of size n and rank r is at most $r(n - r) + 1$ because the n linear relations in Theorem 2.2 are linearly independent.

The basic properties of coefficients of the Tutte polynomial singled out in the following lemma will be useful, for example, in Section 4.1. They are standardly proved by using the fact that the coefficients $t_{i,j}$ of the Tutte polynomial give the number of spanning forests of internal activity i and external activity j (see e.g. [2]). As we do not require this combinatorial interpretation of the coefficients, we give direct proofs by induction.

Lemma 2.3. *Let $M \in \mathcal{M}(n, r)$ have Tutte polynomial*

$$T(M; x, y) = \sum_{i,j} t_{i,j}(M) x^i y^j \in \mathbf{Z}[x, y].$$

Then $t_{i,j}(M) \geq 0$ for each i, j , and

1. *if $n \geq 1$, then $t_{0,0}(M) = 0$;*
2. *if M has no loops, then $t_{1,0}(M) \neq 0$ if and only if M is connected; dually, if M has no coloops, then $t_{0,1}(M) \neq 0$ if and only if M is connected;*
3. *if $n \geq 2$, then $t_{1,0}(M) = t_{0,1}(M)$;*
4. *the monomial x^k divides $T(M; x, y)$ if and only if M has at least k coloops, and the monomial y^ℓ divides $T(M; x, y)$ if and only if M has at least ℓ loops;*
5. *given M has exactly k coloops and exactly ℓ loops, if $i \geq r$ or $j \geq n - r$, then $t_{i,j}(M) = 0$ except when $i = r$ and $j = \ell$, or $i = k$ and $j = n - r$, where we have $t_{r,\ell}(M) = 1 = t_{k,n-r}(M)$.*

Proof. That the $t_{i,j}(M)$ are non-negative integers follows from the deletion-contraction formula.

1. This follows by induction on $n \geq 1$. A matroid of size $n = 1$ either consists of a single coloop or a single loop, whose Tutte polynomials are respectively x and y . The deletion-contraction formula gives the induction step.
2. If $M = M_1 \oplus M_2$ for two matroids M_1 and M_2 , each of size at least 1, then $T(M; x, y) = T(M_1; x, y)T(M_2; x, y)$, and $t_{1,0}(M) = t_{1,0}(M_1)t_{0,0}(M_2) + t_{0,0}(M_1)t_{1,0}(M_2)$ is equal to zero by Item 1. The only connected matroid M with a coloop has size 1, and in this case $t_{1,0}(M) = 1$. If M of size at least 2 is connected, then at least one of M/e and $M \setminus e$ is connected. By the deletion-contraction formula $T(M; x, y) = T(M - e; x, y) + T(M/e; x, y)$, it follows that $t_{1,0}(M) \geq \max\{t_{1,0}(M - e), t_{1,0}(M/e)\}$. This allows the result to be established by induction on n . A similar argument establishes the dual result.
3. The equality holds trivially if M of size at least 2 has a coloop or loop e , as in this case $T(M; x, y) = xT(M/e; x, y)$ or $T(M; x, y) = yT(M - e; x, y)$, respectively, and M/e and $M - e$ have size at least 1, so their Tutte polynomials have zero constant term. If M has no coloops or loops and is not connected, then by Item 2 we have $t_{1,0}(M) = 0 = t_{0,1}(M)$. It remains to establish the equality for connected matroids M , and this follows by induction using the deletion-contraction formula $T(M; x, y) = T(M - e; x, y) + T(M/e; x, y)$ along with our previous remarks.
4. If M has ℓ loops, then by deletion-contraction $T(M; x, y) = y^\ell T(M'; x, y)$, where M' is the matroid obtained from M by deleting ℓ loops; if M has k coloops, then $T(M; x, y) = x^k T(M''; x, y)$, where M'' is the matroid obtained from M by contracting k coloops. Suppose conversely that M is a matroid of size at least 2 such that x^k divides $T(M; x, y)$ for some $k \geq 1$. Then $t_{1,0}(M) = t_{0,1}(M) = 0$ and so by Item 2. the matroid M is not connected and $T(M; x, y) = T(M_1; x, y)T(M_2; x, y)$ for matroids M_1, M_2 of smaller size, such that x^{k_1} divides $T(M_1; x, y)$ and x^{k_2} divides $T(M_2; x, y)$ for some k_1, k_2 with $k_1 + k_2 = k$. Inductively, M_1 has at least k_1 coloops and M_2 at least k_2 coloops, whence M has at least k coloops. The argument for loops is similar.
5. For this argument it is convenient to introduce the *co-rank* of a matroid, defined as the difference between the size of the matroid and its rank. We proceed by induction on the size of the matroid M . First, observe that the statement holds if every element of M is a loop or a coloop, or if the size of M is at most 2. Suppose now that M has $n \geq 3$ elements, including an element e that is

neither a loop nor a coloop, and that the statement holds for matroids with fewer elements than M . Let r be the rank of M . Let (i, j) be such that $i \geq r$ or $j \geq n - r$. By the deletion-contraction formula, $T(M) = T(M/e) + T(M - e)$. The induction hypothesis applies to each of M/e and $M - e$. Consequently, if $i \geq r$, then $t_{i,j}(M/e) = 0$ because the rank of M/e is $r - 1$, and $t_{i,j}(M - e) = 0$ unless the rank of $M - e$ is r , and $i = r$ while j is the number of loops of $M - e$, which is equal to ℓ . In this latter case $t_{i,j}(M - e) = 1$, so this proves the first part of the statement. If $j \geq n - r$, then $t_{i,j}(M - e) = 0$ because $M - e$ has rank r and size $n - 1$, so its co-rank is smaller than $n - r$. Moreover, $t_{i,j}(M/e) = 0$ unless the co-rank of M/e is equal to $n - r$, and $j = n - r$ while i is the number of coloops of M/e , which is k . In this latter case, $t_{i,j}(M/e) = 1$, so this concludes the proof. \square

Before going further, we formalise a tool from linear algebra that will be useful to us in Section 4.1.

Proposition 2.4. *Let E_0 be an \mathbf{F} -vector space and let f_1, \dots, f_k be \mathbf{F} -linearly independent vectors of E_0 . We fix an \mathbf{F} -subspace F_0 of E_0 such that*

$$E_0 = \text{Vect}_{\mathbf{F}}\{f_1, \dots, f_k\} \bigoplus F_0,$$

and for $j \in \{1, \dots, k\}$ we let p_j denote the projection $E_0 \rightarrow \mathbf{F} \cdot f_j$ relative to this splitting.

Let $d \in \mathbf{N}$ and let E be a d -dimensional \mathbf{F} -subspace of E_0 equipped with an \mathbf{F} -basis B . For $j \in \{1, \dots, k\}$ let

$$X_j = \{e \in B : e \equiv f_j \pmod{F_0}\},$$

and set $s = \#\{j \in \{1, \dots, k\} : X_j \neq \emptyset\}$. Assume that $B \setminus (\cup_{j=1}^n X_j) \subset F_0$. Then

$$\dim_{\mathbf{F}} \left(E \cap \left(\bigcap_{i=1}^k \ker p_i \right) \right) \leq d - s.$$

Proof. Let $G = \bigcap_{1 \leq i \leq k} \ker p_i$. For each j such that X_j is non empty, we let e_j be an element of B such that $e_j \equiv f_j \pmod{F_0}$. By assumption we may split E as follows

$$E = \text{Vect}_{\mathbf{F}}\{e_j : e_j \equiv f_j \pmod{F_0} \text{ for some } j\} \bigoplus \text{Vect}_{\mathbf{F}}\{b : b \in B \setminus (\cup_{j=1}^n X_j)\}.$$

By the definition of s , the left-hand summand has \mathbf{F} -dimension equal to s and thus the right-hand summand has \mathbf{F} -dimension equal to $d - s$. Using this direct sum decomposition, we let $v \in E \cap G$ and decompose it as follows

$$v = \sum_{e_j \equiv f_j \pmod{F_0}} \lambda_j e_j + \sum_{b_k \in B \setminus (\cup_{j=1}^n X_j)} \mu_k b_k \quad (\lambda_j, \mu_k \in \mathbf{F}).$$

Applying the projections p_j , for $j \in \{1, \dots, k\}$, we obtain $\lambda_j = 0$ for all j and thus $G \cap E$ can be seen as a subspace of $\text{Vect}_{\mathbf{F}}\{b : b \in B \setminus (\cup_{j=1}^n X_j)\}$, which is $(d - s)$ -dimensional. \square

3 Tutte spaces of all matroids

Given positive integers r and n with $n \geq r$, we consider $\mathcal{T}(\mathcal{M}(n, r))$, the Tutte space of $\mathcal{M}(n, r)$, and we define $d(n, r)$ to be the dimension of $\mathcal{T}(\mathcal{M}(n, r))$. Our goal is to prove the following.

Theorem 3.1. *For non-negative integers n and r with $n \geq r$,*

$$d(n, r) = r(n - r) + 1.$$

Furthermore, we can construct a basis for $\mathcal{T}(\mathcal{M}(n, r))$ composed of Tutte polynomials of series-parallel multigraphs. In particular, the Tutte space of series-parallel graphs with n edges and rank r (loops allowed) equals the Tutte space of all matroids of size n and rank r .

We now explicitly define a family of series-parallel graphs that form a basis of $\mathcal{T}(\mathcal{M}(n, r))$. To this end, we start by introducing some notation.

Definition 3.2. Let k and m be positive integers. We define $C_k(m)$ to be the multigraph obtained from a cycle $v_0 \cdots v_{k-1} v_0$ on k vertices by adding $m - 1$ edges between v_0 and v_1 . We also define $C_k(0)$ as the k -vertex path $P_k = v_0 \cdots v_{k-1}$.

Thus $C_k(m)$ has $k + m - 1$ edges and rank $k - 1$, and $C_k(1) = C_k$ is the cycle of length k . In particular, $C_1(1) = C_1$ is the graph with 1 vertex and 1 loop, and more generally $C_1(m)$ is the graph with 1 vertex and m loops. Note that $C_2(2)$ is thus the graph composed of 2 vertices joined by 3 edges, while $C_2(0)$ consists of a single bridge.

Using the deletion-contraction recurrence for the Tutte polynomial, a straightforward induction gives, for $k \geq 2$ and $m \geq 1$,

$$T(C_k(m); x, y) = x^{k-1} + (y + x + \cdots + x^{k-2}) \cdot (1 + y + \cdots + y^{m-1}). \quad (3)$$

The multigraph $C_2(m)$ consists of $m + 1$ parallel edges joining 2 vertices, for which

$$T(C_2(m); x, y) = x + y + \cdots + y^m,$$

consistent with Equation (3) with $k = 2$. The multigraph $C_k(0)$ consists of $k - 1$ bridges, so $T(C_k(0)) = x^{k-1}$, consistent with Equation (3) with $m = 0$, as the rightmost sum, equal to $\frac{y^m - 1}{y - 1}$, vanishes. The multigraph $C_1(m)$ consists of m loops on a vertex, and $T(C_1(m); x, y) = y^m$.

Definition 3.3. Fix $n \geq r \geq 0$. We define $r(n - r) + 1$ multigraphs as follows.

- For $(i, \ell) \in \{1, \dots, r\} \times \{0, \dots, n - r - 1\} \cup \{(0, n - r)\}$, let $G_{i, \ell}^{n, r}$ be the graph obtained from $C_{i+1}(n - r - \ell)$ by adding $r - i$ bridges and ℓ loops.
- We set $\mathcal{G}(n, r) := \{G_{i, \ell}^{n, r} : (i, \ell) \in \{1, \dots, r\} \times \{0, \dots, n - r - 1\} \cup \{(0, n - r)\}\}$.

Let us underline some immediate properties of the graphs introduced in Definition 3.3. For each (i, ℓ) , the multigraph $G_{i, \ell}^{n, r}$ has size n and rank r . The multigraph $G_{0, n-r}^{n, r}$ consists of r bridges and $n - r$ loops, and has Tutte polynomial

$$T(G_{0, n-r}^{n, r}; x, y) = x^r y^{n-r}. \quad (4)$$

The multigraph $G_{1, \ell}^{n, r}$ consists of $r - 1$ bridges, one class of $n - r - \ell + 1$ parallel edges, and ℓ loops, and has Tutte polynomial

$$T(G_{1, \ell}^{n, r}; x, y) = x^{r-1} y^\ell (x + y + \cdots + y^{n-r-\ell}).$$

Proposition 3.4. The planar dual $(G_{i, \ell}^{n, r})^*$ of $G_{i, \ell}^{n, r}$ satisfies

$$(G_{i, \ell}^{n, r})^* \cong G_{n-r-\ell, r-i}^{n, n-r},$$

for each $(i, \ell) \in \{1, \dots, r\} \times \{0, \dots, n - r - 1\} \cup \{(0, n - r)\}$.

By Equation (3),

$$T(G_{i, \ell}^{n, r}; x, y) = x^{r-i} y^\ell [x^i + (y + x + \cdots + x^{i-1}) \cdot (1 + y + \cdots + y^{n-r-\ell-1})]. \quad (5)$$

We define a linear order on the monomial basis for $\mathbf{F}[x, y]$ so as to be able to represent a polynomial of maximum degree r in x and maximum degree $n - r$ in y by the sequence of $(r + 1)(n - r + 1)$ coefficients of $x^a y^b$, a vector in $\mathbf{F}^{(r+1)(n-r+1)}$. Define the linear order \triangleleft on pairs of integers by $(a, b) \triangleleft (a', b')$ if $b < b'$, or $b = b'$ and $a < a'$.

Lemma 3.5. For $(i, \ell) \in \{1, \dots, r\} \times \{0, \dots, n - r - 1\}$, the Tutte polynomial $T(G_{i, \ell}^{n, r}; x, y)$ has coefficient sequence

$$0^{(\ell+1)(r+1)-i} 1^i (0^{r-i} 1^i 0)^{n-r-\ell-1} 0^{r-i} 10^i$$

in which the (a, b) -entry, where $0 \leq a \leq r$ and $0 \leq b \leq n - r$, is the coefficient of $x^a y^b$, the pairs (a, b) being linearly ordered by \triangleleft .

For $(i, \ell) = (0, n - r)$, the coefficient sequence is $0^{(r+1)(n-r+1)-1} 1$.

Proof. If $(i, \ell) = (0, n - r)$ then the statement follows directly from (4). For the other cases, the statement is a translation of Equation (5). In the monomial order given by \triangleleft , the first term with non-zero coefficient is $x^{r-i+1} y^\ell$. This implies that the sequence starts with $\ell(r + 1) + r - i + 1 = (\ell + 1)(r + 1) - i$ zeroes, since the predecessors of $x^{r-i+1} y^\ell$ are precisely the monomials $x^i y^j$ with either $j \in \{0, \dots, \ell - 1\}$ or $(j = \ell$ and $i \in \{0, \dots, r - i\})$. Further, one sees that a monomial of the form $x^k y^\ell$ has a non-zero coefficient if and only if $k \in \{r - i + 1, \dots, r\}$. Therefore, the sequence continues with i ones. Then a monomial of the form $x^k y^j$ with $j \in \{\ell + 1, \dots, n - r - \ell - 1\}$ has a non-zero coefficient if and only if $k \in \{r - i + 1, \dots, r - 1\}$. Finally, $x^k y^{n-r}$ has a non-zero coefficient if and only if $k = r - i$. \square

The vector space of polynomials in $\mathbf{F}[x, y]$ of maximum degree r in x and maximum degree $n - r$ in y has dimension $(r + 1)(n - r + 1)$. The Tutte polynomials of matroids in a given class (such as cycle matroids of graphs) of size n and rank r belong to this vector space. Our next statement implies Theorem 3.1.

Theorem 3.6. *The set of $r(n - r) + 1$ Tutte polynomials $\mathcal{G}_T(n, r)$ forms a basis for $\mathcal{T}(\mathcal{M}(n, r))$ over \mathbf{F} .*

Proof. In the coefficient sequence of $T(G_{i,\ell}^{n,r}; x, y)$ given by Lemma 3.5, the initial sequence of zeros has length $(r + 1)\ell + i - 1$, which is followed by coefficient 1 in the $(r + 1)\ell + i$ position. If $1 \leq i, i' \leq r$ and $0 \leq \ell, \ell' < n - r$, then $(r + 1)\ell + i < (r + 1)\ell' + i'$ if and only if $(i, \ell) \triangleleft (i', \ell')$. This establishes that the coefficient sequences of $T(G_{i,\ell}^{n,r}; x, y)$ are linearly independent over \mathbf{F} (indeed, when taken as rows of a matrix they form a triangular matrix with non-zero diagonal coefficients). On the other hand, by Brylawski's relations, given in Theorem 2.2, the nullity of the space of Tutte polynomials is at least $n = (r + 1)(n - r + 1) - [r(n - r) + 1]$. \square

By Lemma 2.3-2 and 3, a connected matroid M with Tutte polynomial $\sum_{i,j} t_{i,j} x^i y^j$ has the property that $t_{1,0} = t_{0,1} \neq 0$, i.e. the coefficient of x (equal to the coefficient of y) is non-zero. The argument in the proof of Theorem 3.6 therefore applies to any choice of connected matroid M of size $n - r + i - \ell$ and rank i to which are added $r - i$ coloops and ℓ loops (and with $t_{1,0}(M)$ non-zero modulo p when \mathbf{F} has characteristic p) in place of $G_{i,\ell}^{n,r}$.

Theorem 3.7. *Fix $n \geq r \geq 0$. Let $\{M_{i,\ell} : (i, \ell) \in \{1 \dots, r\} \times \{0 \dots, n - r - 1\} \cup \{(0, n - r)\}\}$ be a set of connected matroids such that $M_{i,\ell}$ has size $n - r + i - \ell$ and rank i , and $t_{1,0}(M_{i,\ell}) \neq 0$ in \mathbf{F} [automatically true for characteristic 0]. Let $B_{s,t}$ denote the sum of s coloops and t loops. Then the set of $r(n - r) + 1$ Tutte polynomials*

$$\{T(B_{r-i,\ell} \oplus M_{i,\ell}; x, y) : (i, \ell) \in \{1 \dots, r\} \times \{0 \dots, n - r - 1\} \cup \{(0, n - r)\}\}$$

forms a basis for $\mathcal{T}(\mathcal{M}(n, r))$.

The use of coloops and loops in the construction of a basis for $\mathcal{T}(\mathcal{M}(n, r))$ (giving a conveniently triangular system of coefficient vectors upon choosing the right order of basis monomials for $\mathbf{F}[x, y]$) prompts the question as to whether they are needed. The answer, as we shall see, is yes.

4 Tutte spaces of connected matroids with loops

For connectivity of graphs, we follow the (standard) terminology of Bondy and Murty [3]: a non-trivial graph is 2-connected if any two distinct vertices are linked by at least two internally vertex-disjoint paths. In particular, a graph with a loop may still be 2-connected, and a graph composed of two vertices with at least two edges between them is 2-connected, as is any graph with only one vertex. Recall that a matroid M is connected, if for any two of its elements, there is a circuit in M containing them both. In particular, the cycle matroid of a graph G is connected if and only if G is 2-connected and loopless.

Our next main goal is to determine the dimension of Tutte spaces of connected matroids of given size and rank. To this end, we first study, in this section, a superclass, which a sense can be considered as that of “connected matroids with loops allowed”.

We let $\mathcal{M}^{\ell c}(n, r)$ be the set of all the matroids M in $\mathcal{M}(n, r)$ satisfying the following properties:

- at least two elements of M are not loops; and
- whenever e and f are two non-loop elements of M , there exists a circuit of M containing both e and f .

Note in particular that if $M \in \mathcal{M}^{\ell c}(n, r)$ and $e \in M$, then $M - e$ has rank r . We define $d^{\ell c}(n, r)$ to be the dimension of the Tutte space $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ over the arbitrary field \mathbf{F} , which we recall is fixed throughout this work. We prove the following statement.

Theorem 4.1. *For positive integers n and r with $n \geq r + 1$,*

$$d^{\ell c}(n, r) = \begin{cases} r(n - r - 1) & \text{if } n = r + 2 \geq 4 \text{ or } r = 2 \leq n - 2, \\ r(n - r - 1) + 1 & \text{otherwise.} \end{cases}$$

Furthermore, we can construct a basis for $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ composed of Tutte polynomials of plane 2-connected multigraphs.

We prove Theorem 4.1 by first establishing the upper bound (Proposition 4.2) and next the matching lower bound (Proposition 4.4). The argument for the upper bound relies on finding monomials the coefficients of which vanish for each elements of $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$, and then using our explicit basis for the space $\mathcal{T}(\mathcal{M}(n, r))$ to prove that the dimension must decrease by at least r . In the special cases (i.e., $n = r + 2 \geq 4$ or $r = 2 \leq n - 2$), we will need to prove that the dimension decreases by at least $r + 1$. The lower bound is obtained by constructing, inductively, a family of plane 2-connected multigraphs the Tutte polynomials of which are linearly independent over \mathbf{F} , with size matching the upper bound.

The case where $n = r + 1$ is trivial, as $\mathcal{M}^{\ell c}(r + 1, r)$ contains only the uniform matroid $U_{r, r+1}$, which happens to be the cycle matroid of the cycle C_{r+1} on $r + 1$ vertices. We henceforth assume that $n \geq r + 2$ in the rest of this section.

4.1 The upper bound

Our goal is to prove the following statement.

Proposition 4.2. *For positive integers n and r with $n \geq r + 2$,*

$$d^{\ell c}(n, r) \leq \begin{cases} r(n - r - 1) & \text{if } n = r + 2 \geq 4 \text{ or } r = 2 \leq n - 2, \\ r(n - r - 1) + 1 & \text{otherwise.} \end{cases}$$

For the next lemma we recall the coefficient notation from Definition 2.1.

Lemma 4.3. *Fix two positive integers n and r with $n \geq r + 1$.*

1. *If $P \in \mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ then $[x^i y^{n-r}]P = 0$ for each $i \in \{1, \dots, r\}$.*
2. *If $M \in \mathcal{M}^{\ell c}(r + 2, r)$, then $t_{r,0}(M) = 1 - t_{r,1}(M) \in \{0, 1\}$. Moreover, $t_{r,1}(M) = 1$ if and only if M has a loop.*
3. *If $M \in \mathcal{M}^{\ell c}(n, 2)$, then $t_{0,n-2}(M) = 1$.*

Proof. By the definition of $\mathcal{M}^{\ell c}(n, r)$, the space $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ is generated by Tutte polynomials of matroids with no coloops and less than $n - r$ loops. We can thus apply Lemma 2.3-5 with $k = 0$, which directly yields Items 1 and 3.

For Item 2 we additionally notice that a matroid in $\mathcal{M}^{\ell c}(r + 2, r)$ has either 0 or 1 loop. \square

Lemma 4.3-1 tells us that $d^{\ell c}(n, r) \leq r(n - r) + 1 - r = r(n - r - 1) + 1$ because $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ is a subspace of the space $\mathcal{T}(\mathcal{M}(n, r))$. While any basis of $\mathcal{T}(\mathcal{M}(n, r))$ spans $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$, to generate an element in $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ the linear combinations of elements of the basis must satisfy a set of r linearly independent equations, which ensures that $\dim_{\mathbf{F}}(\mathcal{T}(\mathcal{M}^{\ell c}(n, r)))$ is at most $\dim_{\mathbf{F}}(\mathcal{T}(\mathcal{M}(n, r))) - r$. This is captured by Proposition 2.4, which it suffices to apply. Lemma 4.3-2 and 3 are useful to handle the specific behaviour of the cases where $n = r + 2 \geq 4$ or $r = 2 \leq n - 2$.

Proof of Proposition 4.2. We will apply Proposition 2.4. The monomials $f_j = x^j y^{n-r}$ for $j \in \{1, \dots, r\}$ form a linearly independent set of the vector space E_0 of all bivariate polynomials in $\mathbf{F}[x, y]$ with degree at most r in x and at most $n - r$ in y , of which $E = \mathcal{T}(\mathcal{M}(n, r))$ is a subspace of dimension $r(n - r) + 1$. We adopt the notation in Proposition 2.4 for F_0 and the projections p_j . By Theorem 3.6, the set of polynomials $\mathcal{G}_T(n, r)$ (Definition 3.3) is a basis of $\mathcal{T}(\mathcal{M}(n, r))$, and it follows from Lemma 3.5 that $\mathcal{G}_T(n, r)$ can be partitioned into $r + 1$ non-empty parts X_0, \dots, X_r such that

- $X_i = \{P \in \mathcal{G}_T(n, r) : \forall j, p_j(P) = \delta_{i,j}\}$ for each $i \in \{1, \dots, r\}$, that is, a polynomial in X_i has exactly one monomial with degree $n - r$ in y , which is $x^i y^{n-r}$ with coefficient 1; and
- $X_0 = \{P \in \mathcal{G}_T(n, r) : \forall j, p_j(P) = 0\}$, that is, a polynomial in X_0 has no monomial $x^j y^{n-r}$ where j is positive — actually, even if not relevant, one sees from Lemma 3.5 that a polynomial in X_0 has y^{n-r} as the unique monomial of top degree in y , with coefficient 1.

In particular, X_0 is contained in F_0 . Proposition 2.4 thus implies that

$$\begin{aligned} \dim_{\mathbf{F}} \left(E \cap \left(\bigcap_{j=1}^r \ker p_j \right) \right) &\leq \dim(E) - r \\ &= r(n - r) + 1 - r \\ &= r(n - r - 1) + 1. \end{aligned} \tag{6}$$

Recalling that $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ is a subspace of $\mathcal{T}(\mathcal{M}(n, r))$, Lemma 4.3-1 implies that $p_j(P) = [x^j y^{n-r}]P = 0$ for each $P \in \mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ and each $j \in \{1, \dots, r\}$. This means that $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$, which by definition is contained in E , is also contained in $\cap_{1 \leq j \leq r} \ker p_j$, so that $d^{\ell c}(n, r) \leq r(n - r - 1) + 1$ by (6). This concludes the proof if $n \neq r + 2$.

If $n = r + 2 \geq 4$ or $r = 2 \leq n - 2$, we need to work a little more to show a decrease of $r + 1$ rather than of r only. To this end, we just push the argument used in the proof of Proposition 2.4 a little further, which is made possible by Items 2 and 3 of Lemma 4.3. Our first step is to make the above reasoning a little more explicit, and then extend it with a new relation.

For the case where $n = r + 2 \geq 4$, Lemma 3.5 further implies that $|X_j| = 2$ for each $j \in \{0, \dots, r - 1\}$, while $|X_r| = 1$. Specifically, and writing $T_{i,\ell}$ for the Tutte polynomial of $G_{i,\ell}^{r+2,r}$ for convenience, we have $X_r = \{T_{0,2}\}$ and $X_j = \{T_{r-j,0}, T_{r-j,1}\}$ if $0 \leq j \leq r - 1$.

Consider an arbitrary element $M \in \mathcal{M}^{\ell c}(r + 2, r)$. Then $T_M = \sum_{(i,\ell) \in I} \alpha_{i,\ell} \cdot T_{i,\ell}$, where $I = \{1, \dots, r\} \times \{0, \dots, n - r - 1\} \cup \{(0, n - r)\}$ and $\alpha_{i,\ell} \in \mathbf{F}$. We combine the previous observation with Lemma 4.3-1 first and then with Lemma 4.3-2. First, we deduce that $\alpha_{0,2} = 0$ and $\alpha_{i,0} = -\alpha_{i,1}$ for each $i \in \{1, \dots, r - 1\}$. Second, observing that $[x^r]T_{i,\ell}$ is 1 if $\ell = 0$ and 0 otherwise, while $[x^r y]T_{i,\ell}$ is 1 if $\ell = 1$ and 0 otherwise, we infer that

$$\sum_{i=1}^r \alpha_{i,0} = 1 - \sum_{i=1}^r \alpha_{i,1},$$

which given the previous deductions implies that $\alpha_{r,0} = 1 - \alpha_{r,1}$. These $r + 1$ conditions all together tell us that $d^{\ell c}(r + 2, r) \leq |\mathcal{G}_T(r + 2, r)| - (r + 1) = r$. Explicitly, let $f: \mathbf{F}^r \rightarrow E$ be such that

$$f(c^1, \dots, c^r) = \sum_{i=1}^r c^i (T_{i,0} - T_{i,1}) + T_{r,1}.$$

The dimension of the image of f is at most that of \mathbf{F}^r , that is r . It thus suffices to show that $\mathcal{T}(\mathcal{M}^{\ell c}(r + 2, r))$ is contained in the image of f . To this end, it suffices by linearity to show that the Tutte polynomial T_M of any matroid $M \in \mathcal{M}^{\ell c}(r + 2, r)$ belongs to the image of f . There exists $(\alpha_{i,\ell})_{(i,\ell) \in I}$ such that $T_M = \sum_{(i,\ell) \in I} \alpha_{i,\ell} \cdot T_{i,\ell}$. Now, the previous paragraph implies that $\alpha_{0,2} = 0$ and $\alpha_{i,1} = -\alpha_{i,0}$ for each $i \in \{1, \dots, r - 1\}$ while $\alpha_{r,1} = 1 - \alpha_{r,0}$. Consequently, $T_M = f(\alpha_{1,0}, \dots, \alpha_{r,0})$, which concludes the proof in this case.

For the case where $r = 2 \leq n - 2$, Lemma 3.5 implies that $|X_j| = n - 2$ for each $j \in \{0, 1\}$, while $X_2 = \{T(G_{0,n-2}^{n,2})\}$. Consider an arbitrary element $M \in \mathcal{M}^{\ell c}(n, 2)$, and write $T_M = \sum_{(i,\ell) \in I} \alpha_{i,\ell} \cdot T_{i,\ell}$ where for convenience $T_{i,\ell}$ now stands for the Tutte polynomial of $G_{i,\ell}^{n,2}$. We combine the previous observation with Lemma 4.3-1 first and then with Lemma 4.3-3. First, we deduce that $\alpha_{0,n-2} = 0$ and $\sum_{\ell=0}^{n-3} \alpha_{1,\ell} = 0$, so $\alpha_{1,0} = -\sum_{\ell=1}^{n-3} \alpha_{1,\ell}$. Second, observing that $[y^{n-2}]T_{i,\ell}$ is 1 if $i = 2$ and 0 otherwise, we infer that $\sum_{\ell=0}^{n-3} \alpha_{2,\ell} = 1$, and consequently that $\alpha_{2,0} = 1 - \sum_{\ell=1}^{n-3} \alpha_{2,\ell} = \alpha_{2,\ell}$. Therefore, the dimension of $\mathcal{T}(\mathcal{M}^{\ell c}(n, 2))$ is at most $d(n, 2) - 3$. Explicitly, set $I' = I \setminus \{(0, n - 2), (1, 0), (2, 0)\}$ and let $f: \mathbf{F}^{|I'|} \rightarrow E$ be such that

$$f(c^{1,1}, \dots, c^{1,n-3}, c^{2,1}, \dots, c^{2,n-3}) = \sum_{\ell=1}^{n-3} c^{1,\ell} \cdot (T_{1,\ell} - T_{1,1}) + T_{2,0} + \sum_{\ell=1}^{n-3} c^{2,\ell} \cdot (T_{2,\ell} - T_{2,0}).$$

The dimension of the image of f is at most that of $\mathbf{F}^{|I'|}$, that is $2(n - 3)$. It thus suffices to show that $\mathcal{T}(\mathcal{M}^{\ell c}(n, 2))$ is contained in the image of f . As in the previous case, linearity reduces this to showing that T_M belongs to the image of f for any $M \in \mathcal{M}^{\ell c}(n, 2)$. If we write $T_M = \sum_{(i,\ell) \in I} \alpha_{i,\ell} \cdot T_{i,\ell}$, then by what has just been shown $\alpha_{0,n-2} = 0$, $\alpha_{1,0} = -\sum_{\ell=1}^{n-3} \alpha_{1,\ell}$ and $\alpha_{2,0} = -\sum_{\ell=1}^{n-3} \alpha_{2,\ell}$. In other words, $T_M = f((\alpha_{i,\ell})_{(i,\ell) \in I'})$. This concludes the proof. \square

4.2 The lower bound

We now turn to establishing the matching lower bound on $d^{\ell c}(n, r)$, thereby completing the proof of Theorem 4.1. To this end, we split $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ into two subspaces and find, using induction on n and then on r , two families of graphs the Tutte polynomials of which are independent.

Proposition 4.4. *For positive integers n and r with $n \geq r + 2$,*

$$d^{\ell c}(n, r) \geq \begin{cases} r(n - r - 1) & \text{if } n = r + 2 \geq 4 \text{ or } r = 2 \leq n - 2, \\ r(n - r - 1) + 1 & \text{otherwise.} \end{cases}$$

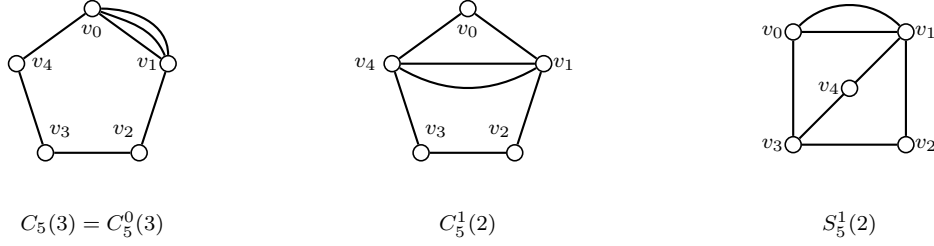


Figure 1: The graphs composing the set $\tilde{\mathcal{S}}(7, 4)$.

We recall that if \mathcal{B} is a set of graphs, then we write \mathcal{B}_T for $\{T(G) : G \in \mathcal{B}\}$. We also mention that the rank of the cycle matroid of a connected graph equals the number of vertices of the graph minus 1. Accordingly, a *connected graph of rank r* is a connected graph with $r + 1$ vertices.

To prove Proposition 4.4, we find an explicit basis $\mathcal{B}_T(n, r)$ composed of Tutte polynomials of 2-connected graphs with rank r and n edges, which is conveniently defined recursively. We make a straightforward, but key remark: if a connected graph (of rank r) has a loop, then the projection of its Tutte polynomial on $X := \{x, x^2, \dots, x^r\}$ is the zero vector. This hints at splitting the sought set of graphs into two parts $\mathcal{L}(n, r)$ and $\mathcal{S}(n, r)$: every graph in $\mathcal{L}(n, r)$ contains one or more loops, while every graph in $\mathcal{S}(n, r)$ is loopless. We establish some properties of the elements in these sets that allow us to deal with them separately. Finally, the fact that $\mathcal{L}_T(n, r)$ forms a linearly independent set is guaranteed by induction, while the fact that $\mathcal{S}_T(n, r)$ forms a linearly independent set is obtained directly by studying its projection on X , in the generic case (that is, $n \neq r + 2$ or $r \neq 2$ or $n = 3$).

We obtain $\mathcal{L}(n, r)$ almost directly using induction on n , as $\mathcal{L}(n, r)$ is obtained by adding a loop to every graph in $\mathcal{B}(n - 1, r)$. It follows that $\mathcal{L}_T(n, r) = \{y \cdot P(x, y) : P \in \mathcal{B}_T(n - 1, r)\}$, and hence $\mathcal{L}_T(n, r)$ is a set of size $d^{\text{lc}}(n - 1, r)$, composed of Tutte polynomials of 2-connected graphs with rank r and with n edges, at least one of which is a loop. The polynomials in $\mathcal{L}_T(n, r)$ are linearly independent over \mathbf{F} because those in $\mathcal{B}_T(n - 1, r)$ are. The graphs in $\mathcal{S}(n, r)$ are defined explicitly, and we thus obtain an explicit basis for $\mathcal{T}(\mathcal{M}^{\text{lc}}(n, r))$. Let us see all of this precisely. We start by setting the notation for extra classes of graphs, consistently with Definition 3.2.

Definition 4.5. Fix positive integers m, m' and $k \geq 3$. (To ease the reading, we note that k will always be the number of vertices in the graphs we define.) Let $C_k(m, m')$ be the graph obtained from $C_k(m)$ by adding $m' - 1$ edges between v_1 and v_2 .

If $k \geq 4$, then for each $i \in \{1, \dots, k - 3\}$ let $C_k^i(m)$ be the graph obtained from C_k by adding m chords between v_{k-1} and v_i .

If $k \geq 5$, then for each $i \in \{1, \dots, k - 4\}$ let $S_k^i(m)$ be the graph obtained from $C_{k-1}^i(1)$ by adding $m - 1$ new edges between v_0 and v_1 , and by subdividing the chord between v_{k-2} and v_i once, creating in this way the vertex v_{k-1} .

We use the notation from Definition 4.5 to start defining our set of graphs.

Definition 4.6. Let $r \geq 1$ and $n \geq r + 2$ be integers. We set $\mathcal{D}(n, r)$ to be $\{C_{r+1}(n - r)\} \cup \{C_{r+1}^i(n - r - 1) : 1 \leq i \leq \lceil r/2 \rceil - 1\}$, so in particular if $r \in \{1, 2\}$, then $\mathcal{D}(n, r) = \{C_{r+1}(n - i)\}$. We set $\tilde{\mathcal{D}}(n, r)$ to be $\{S_{r+1}^i(n - r - 1) : 1 \leq i \leq \lceil r/2 \rceil - 1\}$, so in particular $\tilde{\mathcal{D}}(n, r)$ is empty if $r \leq 3$. Finally, let $\tilde{\mathcal{S}}(n, r)$ be $\mathcal{D}(n, r) \cup \tilde{\mathcal{D}}(n, r)$ and note that $|\tilde{\mathcal{S}}(n, r)| = r - 1$ for all values of $r \geq 2$ and $n \geq r + 2$.

We first proceed with the case where $r = 1$, which is much simpler and quicker to establish. For any graph G , we let G° be a graph obtained from G by adding a loop, attached to an arbitrary vertex — since we are interested in Tutte polynomials, the choice of the vertex is irrelevant.

Definition 4.7. Let $\mathcal{B}(2, 1) = \{C_2\}$. For $n \geq 3$, let $\mathcal{B}(n, 1) = \mathcal{L}(n, 1) \cup \tilde{\mathcal{S}}(n, 1)$, where $\mathcal{L}(n, 1) = \{G^\circ : G \in \mathcal{B}(n - 1, 1)\}$.

One quickly verifies that $\mathcal{B}_T(n, 1)$ is linearly independent over \mathbf{F} , thereby proving Proposition 4.4 when $r = 1$, because $|\mathcal{B}_T(n, 1)| = n - 1$ for each $n \geq 2$.

Lemma 4.8. *The matrix obtained by projecting the elements in $\mathcal{B}_T(n, 1)$ on $\{x, xy, \dots, xy^{n-2}\}$ has rank $n - 1$.*

Proof. First we observe that $\mathcal{B}(n, 1)$ is composed of exactly one graph with ℓ loops for each $\ell \in \{0, \dots, n-2\}$. Proposition 2.3-5 implies that writing the rows of the matrix in increasing order with respect to the number of loops of the corresponding graphs yields the identity matrix, which concludes the proof. \square

We now assume that $r \geq 2$. Our first goal is to compute the rank of the $(r-1) \times r$ -matrix \tilde{A}_X obtained by projecting the elements in $\tilde{\mathcal{S}}_T(n, r)$ on $X(r) := \{x, \dots, x^r\}$. To illustrate the next two lemmas, we here provide the matrix $\tilde{A}_{X(r)}$ for $r = 11$ (and any value of $n \geq 12$ since, as we shall see in the next lemma, $\tilde{A}_{X(r)}$ does not depend on n) with the rows of \tilde{A}_X given by $C_{r+1}^0(n-r), C_{r+1}^1(n-r-1), \dots, C_{r+1}^{\lceil r/2 \rceil - 1}(n-r-1), S_{r+1}^1(n-r-2), \dots, S_{r+1}^{\lfloor r/2 \rfloor - 1}(n-r-2)$, in this order.

$$\tilde{A}_{X(11)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 1 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 2 & 1 \\ 1 & 3 & 4 & 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 4 & 5 & 6 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

We are thus interested in the coefficient of x^j in the Tutte polynomials of graphs in $\tilde{\mathcal{S}}_T(n, r)$ for $j \in \{1, \dots, r\}$.

Lemma 4.9. Fix positive integers k and m . If $k \geq 4$, then, for $i \in \{1, \dots, \lceil \frac{k-1}{2} \rceil - 1\}$ and $j \geq 0$,

$$t_{j,0}(C_k^i(m)) = t_{j,0}(C_k^i) = \begin{cases} 0 & \text{if } j = 0 \text{ or } j \geq k, \\ j & \text{if } 1 \leq j \leq i, \\ i+1 & \text{if } i+1 \leq j \leq k-1-i, \\ k-j & \text{if } k-i \leq j \leq k-1, \end{cases}$$

and if $k \geq 5$, then, for $i \in \{1, \dots, \lfloor \frac{k}{2} \rfloor - 1\}$ and $j \geq 0$,

$$t_{j,0}(S_k^i(m)) = t_{j,0}(S_k^i) = \begin{cases} 0 & \text{if } j = 0 \text{ or } j \geq k, \\ 1 & \text{if } j = 1, \\ j+1 & \text{if } 2 \leq j \leq i, \\ i+2 & \text{if } i+1 \leq j \leq k-i-2, \\ k-j & \text{if } k-i-1 \leq j \leq k-1. \end{cases}$$

In addition, $t_{j,0}(C_k(m)) = t_{j,0}(C_k) = 1$ for $k \geq 2$ and $j \in \{1, \dots, k-1\}$.

Proof. First we observe that by the deletion-contraction formula for the Tutte polynomial, if G is a graph with at least two different edges e and e' between some pair of vertices, then $t_{j,0}(G) = t_{j,0}(G - e)$ for any positive integer j . This yields the first equality in each of the three statements, and hence the full last one since $T(C_k) = x^{k-1} + \dots + x + y$ for each integer $k \geq 2$.

The second equality in either of the first two statements then follows by straightforward induction using the deletion-contraction formula on the edge $v_{i-1}v_i$.

Indeed, to initialise the induction we have $T(C_4^1; x, y) = x^3 + 2x^2 + 2xy + y^2 + x + y$. Now, fixing $k \geq 5$ and assuming the statement holds for $k-1$, the aforementioned deletion-contraction yields that

$$T(C_k^i; x, y) = x^i \cdot T(C_{k-i}; x, y) + T(H),$$

where H stands for C_{k-1}^{i-1} if $i \geq 2$, and for C_{k-1} if $i = 1$. We deduce that

$$t_{j,0}(C_k^i) = t_{j-i,0}(C_{k-i}) + t_{j,0}(H).$$

Recalling that $T(C_t) = x^{t-1} + \dots + x + y$ for each integer $t \geq 2$, the statement directly follows if $i = 1$, while if $i \geq 2$ then it follows after applying the induction hypothesis, which is possible since then $1 \leq i-1 \leq \lceil (k-2)/2 \rceil - 1$ as $2 \leq i \leq \lceil (k-1)/2 \rceil - 1$.

The argument for S_k^i is similar, and we omit it. \square

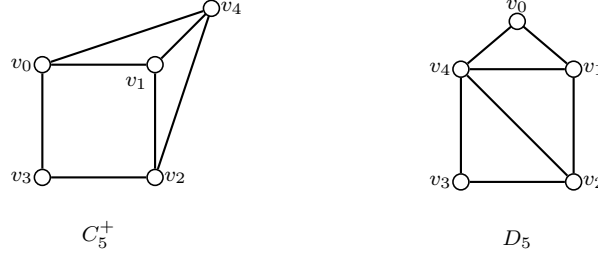


Figure 2: The two graphs added to $\tilde{\mathcal{S}}(7, 4)$ to form $\mathcal{S}(7, 4)$.

Lemma 4.10. *For each $r \geq 2$ and each $n \geq r + 2$, the $(r - 1) \times r$ -matrix \tilde{A}_X obtained by projecting the elements in $\tilde{\mathcal{S}}_T(n, r)$ on $X(r) = \{x, \dots, x^r\}$ has rank $r - 1$.*

Proof. We first note that the statement directly follows from the last statement of Lemma 4.9 if $r = 2$, since it implies that the unique row of \tilde{A}_X is non-zero. We assume now that $r \geq 3$.

Without loss of generality, let the rows of \tilde{A}_X be given by $C_{r+1}^0, C_{r+1}^1, \dots, C_{r+1}^{\lceil r/2 \rceil - 1}, S_{r+1}^1, \dots, S_{r+1}^{\lceil r/2 \rceil - 1}$, in this order.

We use Lemma 4.9 to prove that for each $j \in \{2, \dots, r - 1\}$, there are two columns A_1^j and A_2^j of \tilde{A}_X whose first j coordinates differ only in the j -th one, that is,

$$A_1^j(j) \neq A_2^j(j) \quad \text{and} \quad A_1^j(i) = A_2^j(i) \quad \text{if } 1 \leq i \leq j - 1. \quad (7)$$

This is enough to prove that the rank of \tilde{A}_X is $r - 1$. Indeed, if $(\alpha_1, \dots, \alpha_{r-1}) \cdot \tilde{A}_X$ is the zero vector, then (7) applied successively for each j from $r - 1$ down to 2 implies that $\alpha_j = 0$ if $j \in \{2, \dots, r - 1\}$. In addition, the first column of \tilde{A}_X is the all-one vector by Lemma 4.9, and hence α_1 is also zero.

It remains to establish (7). If $j \in \{2, \dots, \lceil r/2 \rceil\}$, the columns $j - 1$ and j can be chosen as A_1^j and A_2^j : indeed, by Lemma 4.9, the first j entries of the column $j - 1$ are $1, 2, \dots, j - 1, j - 1$, while those of the column j are $1, 2, \dots, j - 1, j$, and we note that $j - 1 \neq j$ in every field. If $j \in \{\lceil r/2 \rceil + 1, \dots, r - 1\}$ then we set $j' := j + 1 - \lceil r/2 \rceil$ and we observe that the columns j' and $r + 1 - j'$ can be chosen as A_1^j and A_2^j . This concludes the proof. \square

A direct consequence of Lemma 4.10 is that $\tilde{\mathcal{S}}_T(r + 2, r)$ along with the Tutte polynomial of the graph obtained from the cycle on $r + 1$ vertices by adding a loop to one of its vertices form a linearly independent subset of $\mathcal{T}(\mathcal{M}^{\ell c}(r + 2, r))$ of size r . By Proposition 4.2 this set, which we call $\mathcal{B}(r + 2, r)$, thus provides a basis of $\mathcal{T}(\mathcal{M}^{\ell c}(r + 2, r))$, namely $\mathcal{B}_T(r + 2, r)$. Let us formalise this.

Definition 4.11. For $r \geq 2$ we define $\mathcal{B}(r + 2, r)$ to be $\{(C_{r+1})^\circ\} \cup \tilde{\mathcal{S}}(r + 2, r)$.

We have thus proved the following.

Corollary 4.12. *For $r \geq 2$ the set $\mathcal{B}_T(r + 2, r)$ is a basis of $\mathcal{T}(\mathcal{M}^{\ell c}(r + 2, r))$, composed of Tutte polynomials of 2-connected series-parallel multigraphs.*

We follow the same pattern for the general case, adding yet one more graph to the set $\tilde{\mathcal{S}}(n, r)$ to form $\mathcal{S}(n, r)$, except that when $n = r + 3$ a second graph needs to be added to compensate for the boundary effect observed for $d^{\ell c}(r + 2, r)$. These graphs are defined as follows.

Definition 4.13. For integers $k \geq 4$ and $m \geq 1$, we define $C_k^+(m)$ to be the graph obtained from the cycle C_{k-1} on $k - 1$ vertices v_0, \dots, v_{k-2} by adding a new vertex v_{k-1} joined by 1 edge to each of v_0 and v_1 , and by m edges to v_2 .

If $k \geq 5$, then we define D_k to be obtained from C_k^1 by adding an edge between v_{k-1} and v_2 . We also write D_4 for the graph $C_4(2, 2)$.

For $n \geq r + 3$, we now define the sets $\mathcal{S}(n, r)$ and $\mathcal{B}(n, r)$ as follows, recalling that $\mathcal{B}(r + 2, r)$ is defined in Definition 4.11.

Definition 4.14.

- For $n \geq 5$ we set $\mathcal{S}(n, 2) := \tilde{\mathcal{S}}(n, 2) \cup \{C_3(n - 3, 2)\} = \{C_3(n - 2), C_3(n - 3, 2)\}$.

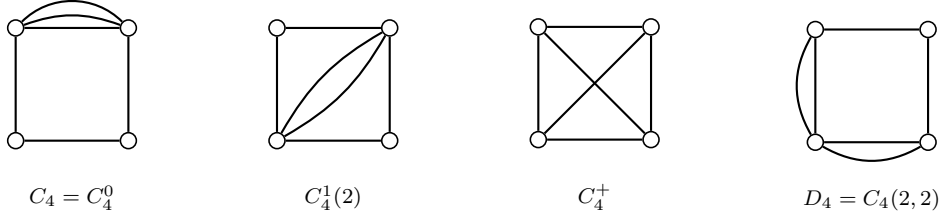


Figure 3: The graphs in the set $\mathcal{S}(6, 3)$.

- For $r \geq 3$,
 - we set $\mathcal{S}(r+3, r) := \tilde{\mathcal{S}}(r+3, r) \cup \{C_{r+1}^+, D_{r+1}\}$; and
 - for $n \geq r+4$ we set $\mathcal{S}(n, r) := \tilde{\mathcal{S}}(n, r) \cup \{C_{r+1}^+(n-r-2)\}$.
- For $n \geq r+3 \geq 5$ we let $\mathcal{L}(n, r) = \{G^\circ : G \in \mathcal{B}(n-1, r)\}$ and $\mathcal{B}(n, r) = \mathcal{L}(n, r) \cup \mathcal{S}(n, r)$.

We need the following statements about the ranks of some projection matrices for the extension of Lemma 4.10 to Corollary 4.18 below.

Lemma 4.15. *For positive integers m and $k \geq 4$,*

$$t_{1,0}(C_k^+(m)) = 2 \neq 1 = t_{k-1,0}(C_k^+(m)).$$

In particular, for $r \geq 3$ and $n \geq r+3$ the matrix A_X obtained by projecting the elements in $\mathcal{S}_T(n, r)$ on $X = \{x, \dots, x^r\}$ has rank r .

Proof. The second statement follows from the first one by Lemma 4.10 because every polynomial P in (the span of) $\tilde{\mathcal{S}}_T(n, r)$ satisfies $t_{1,0}(P) = t_{k-1,0}(P)$ by Lemma 4.9. (Note that when $n = r+3$, the matrix A_X has $r+1$ rows but only r columns.)

The deletion-contraction formula implies that for each non-negative integer j ,

$$t_{j,0}(C_k^+(m)) = t_{j,0}(C_k^+),$$

and hence it suffices to prove the statement when $m = 1$. Indeed, if $m \geq 2$, then deleting-contracting one of the m edges between v_k and v_2 yields that

$$T(C_k^+(m); x, y) = T(C_k^+(m-1); x, y) + y^{m-1} \cdot T(C_{k-1}^1; x, y),$$

and thus $t_{j,0}(C_k^+(m)) = t_{j,0}(C_k^+(m-1))$, which gives the announced conclusion by a direct induction on $m \geq 1$.

We have $t_{k-1,0}(C_k^+) = 1$ by Lemma 2.3-5 applied with $\ell = 0$, since C_k^+ has rank $k-1$. We now proceed by induction on $k \geq 4$ to prove that $t_{1,0}(C_k^+) = 2$, which implies the statement as $2 \neq 1$ in every field.

Note that C_4^+ is the complete graph K_4 on 4 vertices, and hence the statement is true in this case as $T(K_4; x, y) = x^3 + y^3 + 3x^2 + 4xy + 3y^2 + 2x + 2y$.

Assume now that $k \geq 5$. By the deletion-contraction formula,

$$T(C_k^+; x, y) = T(C_{k-1}^+; x, y) + x^{k-4} \cdot T(K_4^-; x, y), \quad (8)$$

where K_4^- is the graph obtained from K_4 by deleting an edge. Because $T_{K_4^-}$ has no constant term and $k \geq 5$, the second summand in (8) cannot influence the coefficient of x . Therefore, $t_{1,0}(C_k^+)$ equals $t_{1,0}(C_{k-1}^+)$, which by induction is 2. \square

The following fact is useful to settle the case where $r = 2$.

Lemma 4.16. *If $n \geq 5$, then $t_{0,2}(C_3(n-2)) = 1 \neq 2 = t_{0,2}(C_3(n-3, 2))$.*

Proof. By deletion-contraction of an edge between v_0 and v_1 in $C_3(n-2)$, we see that $t_{0,2}(C_3(n-2)) = t_{0,2}(C_3(n-3))$ as soon as $n \geq 5$ (as no Tutte polynomial other than that of an empty graph has a non-zero constant term by Lemma 2.3-1). Moreover, it also tells us that $t_{0,2}(C_3(2)) = t_{0,1}(C_2) = 1$, which establishes that $t_{0,2}(C_3(n-2)) = 1$ for each $n \geq 4$. We now apply deletion-contraction on an edge between v_0 and v_1 in $C_3(n-3, 2)$. This yields that

$$T(C_3(n-3, 2); x, y) = T(C_3(n-4, 2); x, y) + y^{n-4} \cdot T(C_2(2); x, y)$$

for each $n \geq 5$, where we recall that $C_2(2)$ is the graph composed of 2 vertices joined by 3 edges. We deduce that $t_{0,2}(C_3(n-3, 2)) = t_{0,2}(C_3(3, 2))$ for each $n \geq 6$ by Lemma 2.3-1, and also that $t_{0,2}(C_3(3, 2)) = t_{0,2}(C_3(2, 2)) = 2$, which concludes the proof. \square

We need one more lemma to deal with the particular case where $n = r + 3$.

Lemma 4.17. *Fix an integer $r \geq 3$.*

1. *If $G \in \mathcal{D}(r+3, r)$ then $t_{0,2}(G) = 1$, while if $G \in \tilde{\mathcal{D}}(r+3, r) \cup \{D_{r+1}\}$ then $t_{0,2}(G) = 2$. Moreover, $t_{0,2}(C_4^+) = 3$.*
2. *If $G \in \mathcal{D}(r+3, r) \setminus \{C_{r+1}(3)\}$, then $t_{2,0}(G) = 2$, while if $G \in \tilde{\mathcal{D}}(r+3, r) \cup \{D_{r+1}\}$, then $t_{2,0}(G) = 3$ unless $r = 3$ and $G = D_4$. Moreover, $t_{2,0}(D_4) = 1 = t_{2,0}(C_{r+1}(3))$.*
3. *If $G \in \tilde{\mathcal{S}}(r+3, r) \setminus \{C_{r+1}(3)\}$, then $t_{r-1,0}(G) = 2$, while $t(C_{r+1}(3)) = 1$ and $t_{r-1,0}(D_{r+1}) = 3$ if $r \geq 4$.*
4. *We have $t_{1,0}(D_{r+1}) = 1 = t_{r,0}(D_{r+1})$.*
5. *If $G \in \mathcal{B}(r+3, r)$ has exactly 1 loop, then $t_{0,2}(G) = 1$.*

Proof. According to Definition 4.6, we have

$$\tilde{\mathcal{S}}(r+3, r) = \{C_{r+1}(3)\} \cup \mathcal{D}(r+3, r) \cup \mathcal{D}(r+3, r)$$

with

$$\mathcal{D}(r+3, r) = \{C_{r+1}^i(2) : 1 \leq i \leq \lceil r/2 \rceil - 1\} \quad \text{and} \quad \tilde{\mathcal{D}}(r+3, r) = \{S_{r+1}^i : 1 \leq i \leq \lfloor r/2 \rfloor - 1\}.$$

1. First, we note that $t_{0,2}(C_{r+1}(3)) = t_{0,2}(C_{r+1}(2))$ by deleting-contracting an edge between v_0 and v_1 , thanks to Lemma 2.3-1. Moreover, $T(C_{r+1}(2)) = T(C_{r+1}) + y \cdot T(C_r)$ so $T_{0,2}(C_{r+1}(3)) = t_{0,1}(C_r) = 1$. Consider $C_{r+1}^i(2) \in \mathcal{D}(r+3, r)$. We have

$$T(C_{r+1}^i(2)) = T(C_{r+1}^i) + y \cdot T(C_{i+1})T(C_{r-i}), \tag{9}$$

and hence $t_{0,2}(C_{r+1}^i(2)) = t_{0,2}(C_{r+1}^i)$. On the other hand,

$$T(C_{r+1}^i) = T(C_{r+1}) + T(C_{i+1})T(C_{r-i}), \tag{10}$$

and $t_{0,2}(C_{r+1}) = 0$. Since $t_{j,j'}(C_t) = 0$ if $j, j' \geq 1$, we deduce that $[y^2](T(C_{i+1})T(C_{r-i})) = 1$, and consequently that $t_{0,2}(C_{r+1}^i(2)) = 1$.

Let $S_{r+1}^i(2) \in \tilde{\mathcal{D}}(r+3, r)$, so that $r \geq 4$. We have

$$T(S_{r+1}^i(2)) = x \cdot T(C_r(2)) + C_{r+1}^i(1, 2) \tag{11}$$

and $T(C_{r+1}^i(1, 2)) = T(C_{r+1}^i) + y \cdot T(H)$, where H is C_{r-1}^{i-1} if $i \geq 2$, and $C_{r-1}(2)$ otherwise. Either way, $t_{0,1}(H) = 1$, and since $t_{0,2}(C_{r+1}^i) = 1$ by the previous paragraph, we infer that $t_{0,2}(S_{r+1}^i(2)) = 2$.

Recall that $D_4 = C_4(2, 2)$. We observe that $T(C_4(2, 2)) = T(C_4) + y \cdot (T(C_3) + T(C_3(2)))$, which is obtained by first deleting-contracting an edge between v_0 and v_1 in $C_4(2, 2)$, and then in the graph isomorphic to $C_4(2)$ obtained thereby. Consequently, $t_{0,2}(D_4) = t_{0,1}(T(C_3)) + t_{0,1}(C_3(2)) = 2$, as announced. If $r \geq 6$, then $T(D_{r+1}) = x^{r-3} \cdot T(C_4^1) + T(D_r)$ by deleting-contracting the edge v_2v_3 . Consequently, the statement will directly follow by induction if we prove it for D_5 . As $T(D_5) = x \cdot T(C_4^1) + T(C_4^1(1, 2))$, and $T(C_4^1(1, 2)) = T(C_4^1) + y \cdot T(C_3(2))$, we see that $t_{0,2}(D_5) = 2$ indeed.

Finally, since C_4^+ is the complete graph on 4 vertices, we see that $t_{0,2}(C_4^+) = 3$, which concludes the proof of Item 1.

2. For every graph G , if G contains several edges between two vertices and e is one of these, then the deletion-contraction formula using the edge e directly implies that $t_{j,0}(G) = t_{j,0}(G - e)$ for any positive integer j . Therefore, we see that $t_{2,0}(C_{r+1}(3)) = t_{2,0}(C_{r+1}) = 1$.

Using (10), we see that

$$t_{2,0}(C_{r+1}^i) = t_{2,0}(C_{r+1}) + [x^2](T(C_{i+1})T(C_{r-i})) = 1 + 1 = 2,$$

as $t_{0,0}(C_t) = 0$ for every positive integer t . Analogously to (11), we have

$$T(S_{r+1}^i) = x \cdot T(C_r) + T(C_{r+1}^i), \quad (12)$$

from which we deduce that

$$t_{2,0}(S_{r+1}^i) = t_{1,0}(C_r) + t_{2,0}(C_{r+1}^i) = 1 + 2 = 3.$$

Finally, still using the expression from the proof of Item 1, we see that $t_{2,0}(D_4) = t_{2,0}(C_4) = 1$.

3. As noticed earlier, it suffices to establish the equalities for the underlying simple graphs. First, we have $t_{r-1,0}(C_{r+1}(3)) = 1$. Second, $t_{r-1,0}(C_{r+1}^i) = 2$ by (10), since $t_{a,0}(C_{i+1})t_{b,0}(C_{r-i}) = 0$ unless $(a, b) = (i, r - i - 1)$. Third, (12) implies that

$$t_{r-1,0}(S_{r+1}^i) = t_{r-2,0}(C_r) + t_{r-1,0}(C_r^i) = 1 + 1 = 2,$$

using Lemma 2.3-5. Finally, using induction on $r \geq 4$ one proves similarly as for Item 1 that $t_{r-1,0}(D_{r+1}) = 3$, because $t_{2,0}(C_4^1) = 2$ by (2) and $t_{r-1,0}(D_r) = 1$ by Lemma 2.3-5.

4. The second equality directly follows from Lemma 2.3-5. For the first equality, we start with the special case $r = 3$, where $D_4 = C_4(2, 2)$: by the equality shown earlier, $t_{j,0}(D_4) = t_{j,0}(C_4)$ for any integer j , and the statement follows for this case. The cases where $r \geq 4$ are proved by induction on r . Again, we deduce from previous computations that $t_{1,0}(D_5) = t_{1,0}(C_3) = 1$.

Next, if $r \geq 5$ then the same deletion-contraction as before yields that

$$T(D_{r+1}) = x^{r-3}T(C_4^1) + T(D_r).$$

The first term on the right side does not contribute to x , and we thus infer the statement using the induction hypothesis.

5. By definition, the graphs in $\mathcal{L}(r+3, r)$ with exactly 1 loop are precisely those obtained by adding a loop to $C_{r+1}(2)$, C_{r+1}^i with $1 \leq i \leq \lceil r/2 \rceil - 1$ and S_{r+1}^i with $1 \leq i \leq \lfloor r/2 \rfloor - 1$. We are thus interested in the coefficient of y for the Tutte polynomials of these graphs. We have $t_{0,1}(C_{r+1}(2)) = t_{0,1}(C_{r+1}) = 1$ using deletion-contraction and Lemma 2.3-1. Moreover, $t_{1,0}(C_{r+1}^i) = t_{1,0}(C_{r+1}) + [y](T(C_{i+1})T(C_{r-i})) = 1$, and $t_{1,0}(S_{r+1}^i) = t_{0,0}(C_r) + t_{1,0}(C_r^i) = 1$. \square

We can now deduce the following property, useful when $n = r + 3$.

Corollary 4.18. *For every integer $r \geq 3$, the matrix $A_{X'}$ obtained by projecting the elements in $\mathcal{S}_T(r+3, r)$ on $X' := \{x, \dots, x^r, y^2\}$ has rank $r + 1$.*

Proof. Moreover, the rows of $A_{X'}$ correspond to the elements of $\mathcal{S}(r+3, r)$ starting with those in $\tilde{\mathcal{S}}(r+3, r)$ listed in the order already mentioned in the proof of Lemma 4.10, and next C_{r+1}^+ followed by D_{r+1} for the last row.

Suppose that $(\alpha_1, \dots, \alpha_{r+1}) \cdot A_{X'}$ is the zero vector. To avoid particular cases and thus ease the reading, let us first deal with the case where $r = 3$. Then $A_{X'}$ is a 4×4 -matrix, with rows corresponding to the graphs $C_4(3)$, $C_4^1(2)$, C_4^+ and D_4 , respectively. Recalling that C_4^+ is the complete graph on 4 vertices, so $T(C_4^+) = x^3 + y^3 + 3x^2 + 4xy + 3y^2 + 2x + 2y$, Lemmas 4.9, 4.15 and 4.17 imply that

$$A_{X'} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

which is quickly seen to have rank 4. (We recall that an integer n is always identified with $n \cdot 1_{\mathbf{F}}$, and we observe that this convention has no impact on the arguments presented.)

We now assume that $r \geq 4$. Lemmas 4.9, 4.15 and 4.17-4 imply that $\alpha_r = 0$. Specifically, the column corresponding to x^r is the all-1 vector, which implies that $\sum_{1 \leq i \leq r+1} \alpha_i = 0$, while the column corresponding to x has a 1 everywhere except in the row corresponding to C_{r+1}^+ , where it has a 2, which now implies that $\alpha_r = 0$ and $\sum_{i \neq r} \alpha_i = 0$.

Our next goal is to show that $\alpha_{r+1} = 0$. Because $\alpha_r = 0 = \sum_{i=1}^{r+1} \alpha_i$, we deduce from Lemma 4.17-1 that $\sum_{i=\lceil r/2 \rceil+1}^{r-1} \alpha_i + \alpha_{r+1} = 0$, recalling that $|\mathcal{D}(r+3, r)| = \lceil r/2 \rceil - 1$ while $|\tilde{\mathcal{D}}(r+3, r)| = \lfloor r/2 \rfloor - 1$. Indeed, the corresponding column has a 1 in the first $\lceil r/2 \rceil$ rows and a 2 in the remaining rows except the last but one, which we can ignore since $\alpha_r = 0$.

It immediately follows that $\sum_{i=1}^{\lceil r/2 \rceil} \alpha_i = 0$. As a consequence of the two previous relations, we infer from Lemma 4.17-2 that $\sum_{i=2}^{\lceil r/2 \rceil} \alpha_i = 0$, and hence $\alpha_1 = 0$. Indeed, the column corresponding to x^2 has a 1 in the first row, a 2 in the next $\lceil r/2 \rceil - 1$ rows (which correspond to the graphs $C_{r+1}^i(2)$), a 3 in the next $\lfloor r/2 \rfloor - 1$ rows (which correspond to the graphs $S_{r+1}^i(2)$), and a 3 in the last row. This implies the statement — recalling that $\alpha_r = 0 = \sum_{i=1}^{r+1} \alpha_i$.

Therefore, since Lemma 4.17-3 along with the fact that $\alpha_r = 0$ imply that

$$\alpha_1 + 2 \sum_{i=2}^{r-1} \alpha_i + 3\alpha_{r+1} = 0,$$

we deduce that $\alpha_{r+1} = 0$. (In characteristic 3, we deduce that $\sum_{i=2}^{r-1} \alpha_i = 0$, and since $\sum_{i=1}^{\lceil r/2 \rceil} \alpha_i = 0$, this implies that $\sum_{i=\lceil r/2 \rceil+1}^{r-1} \alpha_i = 0$, and therefore $\alpha_{r+1} = 0$ since $\sum_{i=1}^{\lceil r/2 \rceil} \alpha_i + \alpha_{r+1} = 0$.)

It follows that $(\alpha_1, \dots, \alpha_{r-1}) \cdot A_X^T = 0$, and therefore that $\alpha_i = 0$ for each $i \in \{1, \dots, r-1\}$ by Lemma 4.10, which concludes the proof. \square

We are now in a position to establish Proposition 4.4.

Proof of Proposition 4.4. The case where $r = 1$ is already dealt with by Lemma 4.8. Fix an integer $r \geq 2$. We proceed by induction on $n \geq r+1$, the statement being true if $n \leq r+2$ as reported earlier: the Tutte polynomial of the cycle on $r+1$ vertices spans a subspace of dimension 1 in $\mathcal{T}(\mathcal{M}^{\ell c}(r+1, r))$ and $\mathcal{B}_T(r+2, r)$ is a linearly independent system in $\mathcal{T}(\mathcal{M}^{\ell c}(r+2, r))$ by Corollary 4.12. We now assume that $n \geq r+3$ and that $\mathcal{B}(n-1, r)$ is a basis of $\mathcal{T}(\mathcal{M}^{\ell c}(n-1, r))$. In particular, $d^{\ell c}(n-1, r) = r(n-r-1) - r$ if $n = r+2$ or $r = 2 \leq n-2$, whereas $d^{\ell c}(n-1, r) = r(n-r-1) + 1 - r$ otherwise.

The definition of $\mathcal{L}(n, r)$ implies that $\mathcal{L}_T(n, r)$ is a set of $d^{\ell c}(n-1, r)$ linearly independent polynomials in $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$. We want to show that $\mathcal{L}_T(n, r) \cup \mathcal{S}_T(n, r)$ spans a subspace of dimension $d^{\ell c}(n-1, r) + r + \varepsilon$ in $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$, where $\varepsilon = 1$ if $r = 2 \leq n-3$ or $n = r+3 \geq 4$, and $\varepsilon = 0$ otherwise. Suppose that

$$\sum_{P \in \mathcal{L}_T(n, r)} \alpha_P \cdot P + \sum_{P \in \mathcal{S}_T(n, r)} \beta_P \cdot P = 0 \quad (13)$$

in $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$, where $\alpha_P, \beta_P \in \mathbf{F}$. Our goal is to prove that $\beta_P = 0$ for each $P \in \mathcal{S}_T(n, r)$, which will imply that $\alpha_P = 0$ for each $P \in \mathcal{L}_T(n, r)$ thanks to the induction hypothesis.

Let us first deal with the case where $r = 2$. We can partition $\mathcal{B}(n, 2)$ into $n-2$ parts $\mathcal{B}^0, \dots, \mathcal{B}^{n-3}$, such that \mathcal{B}^i contains exactly the elements of $\mathcal{B}(n, 2)$ with i loops, for $i \in \{0, \dots, n-3\}$. In particular, $\mathcal{B}^0(n, 2) = \mathcal{S}(n, 2) = \{C_3(n-2), C_3(n-3, 2)\}$. For each $G \in \mathcal{B}(n, 2)$ and $i \in \{0, \dots, n-3\}$, Lemma 2.3-5 implies that $t_{2,i}(G) = 1$ if and only if $G \in \mathcal{B}^i$. We have $t_{0,2}(G) = 0$ if $G \in \mathcal{B}^2(n, 2) \cup \dots \cup \mathcal{B}^{n-3}(n, 2)$ by Lemma 2.3-4 (combined with (1) for $\mathcal{B}(n, 2)$), whereas $t_{0,2}(G) = 1$ if $G \in \mathcal{B}^1(n, 2) \cup \{C_3(n-2)\}$ and $t_{0,2}(C_3(n-3, 2)) = 2$ by Lemmas 2.3-5 and 4.16. Consequently, the projections of (13) on x^2y , on x^2 and on y^2 respectively imply the following three equalities:

$$\begin{aligned} \sum_{P \in \mathcal{B}_T^1(n, 2)} \alpha_P &= 0, \\ \beta_{T(C_3(n-2))} + \beta_{T(C_3(n-3, 2))} &= 0, \\ \beta_{T(C_3(n-2))} + 2\beta_{T(C_3(n-3, 2))} + \sum_{P \in \mathcal{B}_T^1(n, 2)} \alpha_P &= 0. \end{aligned}$$

Consequently, $\beta_{T(C_3(n-3,2))} = 0 = \beta_{T(C_3(n-2))}$. Therefore, Equation (13) implies that $\sum_{P \in \mathcal{L}_T(n,2)} \alpha_P \cdot P = 0$, which further implies that $\alpha_P = 0$ for each $P \in \mathcal{L}_T(n,2)$ thanks to the induction hypothesis. This concludes the proof when $r = 2$.

We now assume that $r \geq 3$. By the definition of $\mathcal{L}(n,r)$, the coefficient of each monomial in $X := \{x, x^2, \dots, x^r\}$ vanishes in every polynomial P in $\mathcal{L}_T(n,r)$, that is, $[x^j]P = 0$. Consequently, projecting (13) on x^j implies that

$$\forall j \in \{1, \dots, r\}, \quad \sum_{P \in \mathcal{S}_T(n,r)} \beta_P \cdot [x^j]P = 0. \quad (14)$$

We first deal with the case where $n \geq r + 4$, so $\varepsilon = 0$. Then $|\mathcal{S}_T(n,r)| = r$, and by Lemma 4.15, the matrix associated to the linear system of r equations given by (14) has rank r . We deduce that this linear system has a unique solution, which is therefore $\beta_P = 0$ for each $P \in \mathcal{S}_T(n,r)$. Consequently, (13) yields that $\sum_{P \in \mathcal{L}_T(n,r)} \alpha_P \cdot P = 0$, which implies that $\alpha_P = 0$ for each $P \in \mathcal{L}_T(n,r)$ because $\mathcal{B}_T(n-1,r)$ is linearly independent over \mathbf{F} . We conclude that $\mathcal{B}_T(n,r) = \mathcal{L}_T(n,r) \cup \mathcal{S}_T(n,r)$ spans in $\mathcal{T}(\mathcal{M}^{\ell c}(n,r))$ a subspace of dimension $d^{\ell c}(n-1,r) + r = r(n-r-1) + 1$, as announced.

It remains to deal with the case where $n = r + 3$. The set $\mathcal{S}_T(r+3,r)$ contains $r + 1$ polynomials, so looking at the projection Z_X of the elements in $\mathcal{B}_T(r+3,r)$ on X is not sufficient, and this is where the additional properties on coefficients of other monomials are important. Specifically, let $X' := X \cup \{y^2\}$. We define $Z_{X'}$ and $A_{X'}$ to be the respective projections of the elements in $\mathcal{B}_T(r+3,r)$ and in $\mathcal{S}_T(r+3,r)$ on X' . Let G_0 be the unique graph in $\mathcal{L}(r+3,r)$ with 2 loops. Projecting (13) on $x^r y^2$ tells us that $\alpha_{T(G_0)} = 0$ thanks to Lemma 2.3-5. Similarly, writing $\mathcal{L}_T^1(n,r)$ for the set $\mathcal{L}_T(n,r) \setminus \{G_0\}$, projecting (13) on $x^r y$ tells us that $\sum_{P \in \mathcal{L}_T^1(r+3,r)} \alpha_P = 0$.

Now let Z_{y^2} be the column vector obtained by projecting the elements in $\mathcal{L}_T(n,r)$ on y^2 . By Lemma 4.17-5, the entries of Z_{y^2} are all 1, bar the one corresponding to G_0 . We thus deduce that

$$(\alpha_P : P \in \mathcal{L}_T(r+3,r)) \cdot Z_{y^2} = 0.$$

It follows that if A_{y^2} is the column vector obtained by projecting the elements in $\mathcal{B}_T(r+3,r)$ on y^2 , then

$$(\beta_P : P \in \mathcal{S}_T(r+3,r)) \cdot A_{y^2} = 0. \quad (15)$$

As noted earlier, all entries in Z_X corresponding to elements in $\mathcal{L}_T(n,r)$ are 0, and therefore we conclude that (15) extends to the whole matrix $A_{X'}$, i.e., (15) holds for each column of $A_{X'}$ and not just the one corresponding to y^2 . Lemma 4.18 now gives the conclusion. \square

5 Tutte spaces of connected matroids (no loops)

We now wish to eliminate loops: what is the dimension $d^c(n,r)$ of the Tutte space $\mathcal{T}(\mathcal{M}^c(n,r))$ of the class $\mathcal{M}^c(n,r)$ of (loopless) connected matroids with size n and rank r ? We prove the following.

Theorem 5.1. *For positive integers n and r with $n \geq r + 1$,*

$$d^c(n,r) = \begin{cases} (r-1)(n-r-1) & \text{if } n = r+2 \geq 4 \text{ or } r = 2 \leq n-2, \\ (r-1)(n-r-1) + 1 & \text{otherwise.} \end{cases}$$

Furthermore, we can construct a basis for $\mathcal{T}(\mathcal{M}^c(n,r))$ composed of Tutte polynomials of plane 2-connected loopless multigraphs, all but at most one being in addition series-parallel.

The case where $n = r + 1$ is, as usual, trivial, because for each $r \geq 1$ the only connected matroid with rank r and size $r + 1$ is the uniform matroid $U_{r,r+1}$, which is the cycle matroid of the cycle C_{r+1} on $r + 1$ vertices. Notice that the case $r = 1$ of Theorem 5.1 is also trivial, as there is only one connected matroid with rank 1 and n elements for each $n \geq r + 1$, namely the cycle matroid of the graph composed of two vertices joined by n edges. Further, there is also, up to isomorphism, only one connected matroid with rank r and $r + 2$ elements, namely the cycle matroid of the cycle graph of length $r + 1$ with exactly one edge of multiplicity 2. We thus assume that $n > r + 2 \geq 4$ in the rest of Section 5.

We heavily exploit the explicit basis for $\mathcal{T}(\mathcal{M}^{\ell c}(n,r))$ exhibited in Section 4.2, and our approach is as follows. By the definition, $\mathcal{T}(\mathcal{M}^{\ell c}(n,r))$ contains $\mathcal{T}(\mathcal{M}^c(n,r))$. We shall consider a particular proper subspace of $\mathcal{T}(\mathcal{M}^{\ell c}(n,r))$, called $\text{Pol}^z(n,r)$, that still contains $\mathcal{T}(\mathcal{M}^c(n,r))$ and has dimension $d^z(n,r) =$

$d^{\ell c}(n, r) - (n - r - 1)$. Next, we shall define a set of plane 2-connected loopless multigraphs, called $\mathcal{C}(n, r)$, which has size precisely $d^z(n, r)$ and is such that $\mathcal{C}_T(n, r)$ generates the whole space $\text{Pol}^z(n, r)$. This will mean that $\text{Pol}^z(n, r) = \mathcal{T}(\mathcal{M}^c(n, r))$ and $\mathcal{C}_T(n, r)$ is a basis for $\mathcal{T}(\mathcal{M}^c(n, r))$, thereby proving Theorem 5.1. We start by defining $\text{Pol}^z(n, r)$ and proving it satisfies the required properties.

5.1 The space $\text{Pol}^z(n, r)$

We recall from Section 4 that $\mathcal{B}_T(n, r)$ is a basis for $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$, and in particular it spans $\mathcal{T}(\mathcal{M}^c(n, r))$. Consequently, any polynomial in $\mathcal{T}(\mathcal{M}^c(n, r))$ can be expressed as a linear combination of polynomials in $\mathcal{B}_T(n, r)$. In addition, by Lemma 2.3-5 the Tutte polynomial of an arbitrary loopless 2-connected graph with n edges and rank r has no monomial of the form $x^r y^j$ for $j \in \{1, \dots, n - r - 1\}$. Since such polynomials also generate $\mathcal{T}(\mathcal{M}^c(n, r))$, we will infer that the coefficients of any linear combination of elements of $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ that yields an element of $\mathcal{T}(\mathcal{M}^c(n, r))$ must satisfy $n - r - 1$ linear equations, which turn out to be linearly independent.

Recall that the set $\mathcal{B}(n, r)$ is the disjoint union of $\mathcal{L}(n, r)$ and $\mathcal{S}(n, r)$, where the latter contains only loopless 2-connected graphs while the former contains only 2-connected graphs with at least 1 loop. Specifically, recall from the inductive construction of $\mathcal{B}(n, r)$ that this set, if partitioned according to the number of loops of its elements, is the disjoint union of $\mathcal{B}^0(n, r) = \mathcal{S}(n, r), \mathcal{B}^1(n, r), \dots, \mathcal{B}^{n-r-1}(n, r)$ where, for each $i \geq 1$, the set $\mathcal{B}^i(n, r)$ consists of the elements in $\mathcal{B}(n - i, r)$, each with exactly i loops added.

This means, using Lemma 2.3-5, that for each graph $G \in \mathcal{B}(n, r)$, and each $j \in \{0, \dots, n - r - 1\}$, we have $t_{r,j}(G) = \mathbf{1}_{\mathcal{B}^j(n, r)}(G)$, where $\mathbf{1}_X(g)$ is 1 if $g \in X$ and 0 otherwise. Since no polynomial $Q \in \mathcal{T}(\mathcal{M}^c(n, r))$ has a monomial of the form $x^r y^j$ with $j \geq 1$, we infer that every polynomial $Q \in \mathcal{T}(\mathcal{M}^c(n, r))$ is a linear combination of the form

$$Q = \sum_{G \in \mathcal{B}(n, r)} \alpha_G \cdot T_G = \sum_{j=0}^{n-r-1} \sum_{G \in \mathcal{B}^j(n, r)} \alpha_G \cdot T_G,$$

where, for each $j \in \{1, \dots, n - r - 1\}$, we necessarily have

$$\sum_{G \in \mathcal{B}^j(n, r)} \alpha_G = 0,$$

since Q has no monomial $x^r y^j$ with $j \geq 1$.

This is why we define $\text{Pol}^z(n, r)$ to be the subspace of $\mathcal{T}(\mathcal{M}^{\ell c}(n, r))$ generated by such linear combinations: we have just proved that $\mathcal{T}(\mathcal{M}^c(n, r))$ is contained in $\text{Pol}^z(n, r)$.

Furthermore, we observe that the dimension $d^z(n, r)$ of $\text{Pol}^z(n, r)$ is

$$|\mathcal{B}_T(n, r)| - (n - r - 1) = d^{\ell c}(n, r) - (n - r - 1).$$

Indeed, choosing an arbitrary element in each set $\mathcal{B}^i(n, r)$ for $i \in \{1, \dots, n - r - 1\}$ and removing it from $\mathcal{B}(n, r)$ yields a set $\mathcal{B}^z(n, r)$ of graphs such that $\mathcal{B}_T^z(n, r)$ generates $\text{Pol}^z(n, r)$. On the other hand, $\mathcal{B}_T^z(n, r)$ is linearly independent over \mathbf{F} because $\mathcal{B}_T(n, r)$ is, and has size $d^{\ell c}(n, r) - (n - r - 1)$ because the sets $\mathcal{B}^i(n, r)$, for $i \in \{1, \dots, n - r - 1\}$, are pairwise disjoint.

We therefore have proved that $\mathcal{T}(\mathcal{M}^c(n, r))$ is contained in the space $\text{Pol}^z(n, r)$, which is composed of all the linear combinations of polynomials in $\mathcal{B}_T(n, r)$ with coefficients corresponding to elements in $\mathcal{B}_T^i(n, r)$ summing to 0, for each $i \in \{1, \dots, n - r - 1\}$.

Our next step is to define a set $\mathcal{C}(n, r)$ of plane 2-connected loopless multigraphs, each bar at most one being series-parallel, such that $\mathcal{C}_T(n, r)$ generates $\text{Pol}^z(n, r)$. (The size of $\mathcal{C}(n, r)$ must then be precisely $d^z(n, r)$.)

5.2 Realising $\text{Pol}^z(n, r)$ as the Tutte space of 2-connected loopless graphs

We introduce some notation: in what follows, k is a positive integer, which will always be the number of vertices of the graphs we define. Most of them are illustrated in Figure 4.

We first extend our notation for cycles consistently with Definitions 3.2 and 4.5. Let $C_k = C_k(1, \dots, 1)$ be the cycle of length k with vertices v_0, \dots, v_{k-1} , in this (cyclic) order, where $C_1 = C_1(1)$ means the graph with 1 vertex and 1 loop. For positive integers m_1, \dots, m_k , we define $C_k(m_1, \dots, m_k)$ to be the graph

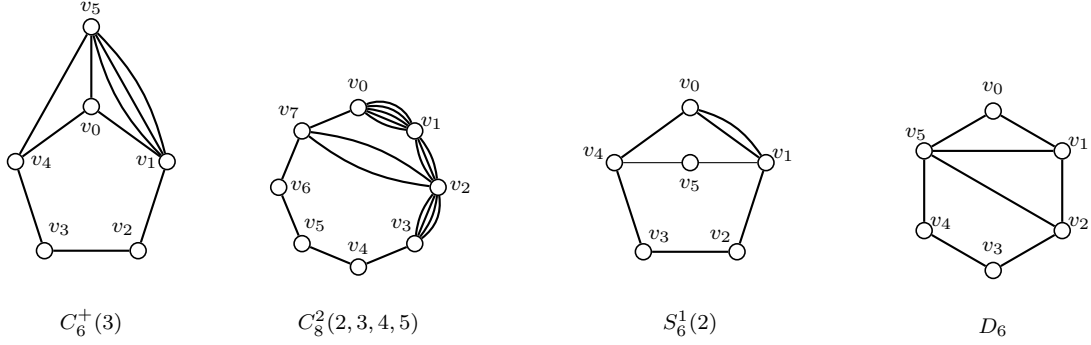


Figure 4: Illustrations of notation for graphs.

obtained from the cycle C_k by adding $m_k - 1$ edges between v_{k-1} and v_0 , and $m_i - 1$ edges between v_{i-1} and v_i for each $i \in \{1, \dots, k-1\}$. Note that if $k \geq 3$, then m_i is exactly the number of edges between v_{i-1} and v_i for $i \in \{0, \dots, k-1\}$ considered modulo k , whereas if $k = 2$ then the number of edges between the 2 vertices v_0 and v_1 is $m_1 + m_2$.

If $k \geq 4$, then let $C_k^+(m_1)$ be the graph obtained from C_{k-1} by adding a vertex v_{k-1} adjacent precisely to v_0 , v_1 and v_2 , with exactly m_1 edges between v_{k-1} and v_2 . If $i \in \{1, \dots, k-3\}$, then we define $C_k^i(m_1, m_2, m_3, m_4)$ to be the graph obtained from C_k by adding m_1 edges between v_{k-1} and v_i , then $m_2 - 1$ edges between v_i and v_{i-1} , next $m_3 - 1$ edges between v_i and v_{i+1} and, finally, $m_4 - 1$ edges between v_{i-1} and v_{i-2} (where $i-2$ is considered modulo k , hence equal to $k-1$ when $i=1$).

Similarly, if $k \geq 5$ and $i \in \{1, \dots, k-4\}$, then we define $S_k^i(m_1)$ to be the k -vertex graph obtained from C_{k-1} by adding $m_1 - 1$ copies of the edge $v_{i-1}v_i$, and a vertex v_{k-1} adjacent precisely to v_{k-2} and v_i , that is, we add a subdivided chord between v_{k-2} and v_i .

We recall that $D_4 = C_4(2, 2)$, and that for $k \geq 5$, the graph D_k is obtained from C_k by adding the edge $\{v_{k-1}, v_1\}$ and the edge $\{v_{k-1}, v_2\}$. We also recall that for any graph G , we let G° be a graph obtained from G by adding a loop, attached to an arbitrary vertex. Finally, and consistently with our notation $C_k = C_k(1, \dots, 1)$, for any sequence of integers we only write the shortest sequence containing a value different from 1, so that S_k^i stands for $S_k^i(1)$ and $C_k^i(1, 1, 2)$ stands for $C_k^i(1, 1, 2, 1)$, for example.

5.2.1 A general reduction

We recall that our goal is now to prove the existence of a set $\mathcal{C}(n, r)$ of loopless 2-connected graphs with rank r and n edges such that every element of $\text{Pol}^2(n, r)$ is a linear combination of Tutte polynomials of graphs in $\mathcal{C}(n, r)$. To this end, we shall proceed as follows. Fixing $i \in \{1, \dots, n-r-1\}$, consider any linear combination $\sum_{G \in \mathcal{B}^i(n, r)} \alpha_G \cdot P_G$ such that $\sum_{G \in \mathcal{B}^i(n, r)} \alpha_G = 0$. It is enough to prove, for every $G \in \mathcal{B}^i(n, r)$, the existence of a linear combination L_G of the Tutte polynomials of the graphs in $\mathcal{C}(n, r)$ such that:

$$\forall G, G' \in \mathcal{B}^i(n, r), \quad T_G - L_G = T_{G'} - L_{G'}.$$

The desired linear combination will then be $\sum_{G \in \mathcal{B}^i(n, r)} \alpha_G \cdot L_G$. Indeed,

$$\begin{aligned} \sum_{G \in \mathcal{B}^i(n, r)} \alpha_G \cdot T_G - \sum_{G \in \mathcal{B}^i(n, r)} \alpha_G \cdot L_G &= \sum_{G \in \mathcal{B}^i(n, r)} \alpha_G \cdot (T_G - L_G) \\ &= 0, \quad \text{since } T_G - L_G \text{ is independent of } G. \end{aligned}$$

We now proceed along this road, seeking to establish the following.

Proposition 5.2. *Fix $r \geq 2$ and $n \geq r + 2$. For each $\ell \in \{1, \dots, n-r-1\}$, let $\mathcal{B}^\ell(n, r)$ be the graphs in $\mathcal{B}(n, r)$ that contain exactly ℓ loops. There exists a set $\mathcal{C}(n, r)$ of loopless 2-connected graphs with rank r and n edges such that for every $\ell \in \{1, \dots, n-r-1\}$, if $(\alpha_G)_{G \in \mathcal{B}^\ell(n, r)}$ is a sequence of elements of \mathbf{F} summing to 0, then $\sum_{G \in \mathcal{B}^\ell(n, r)} \alpha_G \cdot T_G$ is equal to a linear combination L (over \mathbf{F}) of Tutte polynomials of graphs in $\mathcal{C}(n, r)$, with scalars also summing to 0.*

Let us make a few remarks on the statement of Proposition 5.2. First, note that $|\mathcal{B}^{n-r-1}(n, r)| = 1$ by definition, so there is nothing to prove when $\ell = n-r-1$, the statement being trivial. Second, the property

that the scalars of the linear combination L sum to 0 is automatic, i.e. it is a by-product of the first part of the statement. To see this, simply consider the projection on the monomial y^{n-r} . The projection of every element of $\mathcal{B}_T(n, r)$ is 1, and so is that of the Tutte polynomial of any loopless 2-connected graph. Therefore, the projections on y^{n-r} of $\sum_{G \in \mathcal{B}^\ell(n, r)} \alpha_G \cdot T_G$ and of L are simply the sum of the scalars involved in the respective linear combinations, which gives the conclusion since $\sum_{G \in \mathcal{B}^\ell(n, r)} \alpha_G = 0$.

As our next step towards a proof of Proposition 5.2, we now show that, with a little additional care, it suffices to establish it when $\ell = 1$. For each $r \geq 2$, we let $\mathcal{C}(r+1, r) = \mathcal{S}(r+1, r) = \{C_{r+1}\}$, so $\mathcal{C}_T(r+1, 1)$ is a basis for $\mathcal{T}(\mathcal{M}^c(r+1, r)) = \mathcal{T}(\mathcal{M}^{\ell c}(r+1, r))$.

Lemma 5.3. *Assume that for all $r \geq 2$ and all $n \geq r+2$, there exists a set $\mathcal{C}(n, r)$ of loopless 2-connected graphs with rank r and n edges such that for each set $X \in \{\mathcal{B}^1(n, r), \mathcal{C}(n-1, r)^\circ\}$, whenever $(\alpha_G)_{G \in X}$ is a sequence of elements of \mathbf{F} summing to 0, it holds that $\sum_{G \in X} \alpha_G \cdot T_G$ is equal to a linear combination (over \mathbf{F}) of Tutte polynomials of elements of $\mathcal{C}(n, r)$, with scalars also summing to 0.*

Then, for all $r \geq 2$ and $n \geq r+2$, and for each $\ell \in \{1, \dots, n-r-1\}$, if $(\alpha_G)_{G \in \mathcal{B}^\ell(n, r)}$ is a sequence of elements of \mathbf{F} summing to 0, then $\sum_{G \in \mathcal{B}^\ell(n, r)} \alpha_G \cdot T_G$ is equal to a linear combination (over \mathbf{F}) of Tutte polynomials of elements of $\mathcal{C}(n, r)$, with scalars also summing to 0.

Proof. We proceed by induction on $n \geq r+2$, the statement being trivially true if $n = r+2$, since $\mathcal{B}^\ell(r+2, r) = \emptyset$ for $\ell \geq 2$. Assume now that $n \geq r+3$. We use a finite induction on $\ell \in \{1, \dots, n-r-1\}$, the conclusion being true by hypothesis if $\ell = 1$. Assume that $\ell \in \{2, \dots, n-r-1\}$. Let $(\alpha_G)_{G \in \mathcal{B}^\ell(n, r)}$ be a sequence of scalars summing to 0. By the definition of $\mathcal{B}(n, r)$, the elements of $\mathcal{B}^\ell(n, r)$ are exactly the elements of $\mathcal{B}^{\ell-1}(n-1, r)$ with an additional loop added to each (recalling that the choice of the vertex to which the loop is added is irrelevant for our purposes), and we can thus let f be the corresponding bijection from $\mathcal{B}^{\ell-1}(n-1, r)$ to $\mathcal{B}^\ell(n, r)$. Then

$$\sum_{G \in \mathcal{B}^\ell(n, r)} \alpha_G \cdot T(G; x, y) = y \cdot \sum_{G \in \mathcal{B}^{\ell-1}(n-1, r)} \alpha_{f(G)} \cdot T(G; x, y).$$

Note that $\sum_{G \in \mathcal{B}^{\ell-1}(n-1, r)} \alpha_{f(G)} = 0$ by definition. Now, using the induction hypothesis on n , and the hypothesis or the induction hypothesis on ℓ depending on whether $\ell = 2$ or $\ell \geq 3$, we have

$$\sum_{G \in \mathcal{B}^{\ell-1}(n-1, r)} \alpha_{f(G)} \cdot T_G = \sum_{H \in \mathcal{C}(n-1, r)} \beta_H \cdot T_H,$$

noting that $n-1 \geq r+2$ and $\ell-1 \in \{1, \dots, (n-1)-r-1\}$. For each graph $H \in \mathcal{C}(n-1, r)$, let H° be the 2-connected graph with $r+1$ vertices and n edges obtained by adding a single loop to H , and set $\beta_{H^\circ} = \beta_H$. The set of graphs thus obtained is $\mathcal{C}(n-1, r)^\circ$. Combining the last two equalities yields that

$$\sum_{G \in \mathcal{B}^\ell(n, r)} \alpha_G \cdot T_G = \sum_{H^\circ \in \mathcal{C}(n-1, r)^\circ} \beta_{H^\circ} \cdot T_{H^\circ},$$

Since $\sum_{H^\circ \in \mathcal{C}(n-1, r)^\circ} \beta_{H^\circ} = 0$, our hypothesis implies that $\sum_{H^\circ \in \mathcal{H}^\circ} \beta_{H^\circ} \cdot T_{H^\circ}$ is a linear combination of loopless 2-connected graphs with n edges and rank r , with scalars summing to 0. This concludes the proof. \square

It thus remains to prove that the assumptions of Lemma 5.3 are satisfied. To this end, we provide an explicit definition for the sets $\mathcal{C}(n, r)$, which therefore gives us explicit bases for Tutte spaces of (loopless) connected matroids with given size and rank. We show that the difference of any pair of polynomials in $\mathcal{B}_T^1(n, r)$ can be expressed as a linear combination of elements of $\mathcal{C}_T(n, r)$. We point out that not every basis for Tutte spaces of (loopless) connected matroids satisfies this property. The set $\mathcal{C}(n, r)$ consists of 2-connected $(r+1)$ -vertex plane multigraphs with n edges, as is explained next.

5.2.2 Basis for Tutte spaces of connected matroids

We first directly define a basis for the trivial values of size and rank for which there is a single connected matroid: for $n \geq 2$ we let $\mathcal{C}(n, 1) = \{C_2(n-1)\}$ and we recall that for each $r \geq 2$ we set $\mathcal{C}(r+1, r) = \mathcal{S}(r+1, r) = \{C_{r+1}\}$.

Let us fix integers $r \geq 2$ and $n \geq r+2$. We construct a set, named $\mathcal{C}(n, r)$, such that $\mathcal{C}_T(n, r)$ forms a basis for $\mathcal{T}(\mathcal{M}^c(n, r))$. To ease the description, we split $\mathcal{C}(n, r)$ into sub-families of $(r+1)$ -vertex plane

multigraphs with n edges. We use the notation introduced at the beginning of Section 5.2, and in the rest of this section m_1 and m_2 are positive integers.

We let $\mathcal{C}_A(n, r)$ be the set composed of the graphs $C_{r+1}(m_1, m_2)$ where $m_1 + m_2 = n + 1 - r$ and $m_1 \in \{ \lceil (n + 1 - r)/2 \rceil, \dots, n - r \}$. Note that $C_{r+1}(m_1, m_2)$ and $C_{r+1}(m_2, m_1)$ are actually isomorphic, so as long as $m_1 + m_2 = n + 1 - r$ and both m_1 and m_2 are positive integers, the graph $C_{r+1}(m_1, m_2)$ does belong to $\mathcal{C}_A(n, r)$. We note that $|\mathcal{C}_A(n, r)| = \lfloor (n + 1 - r)/2 \rfloor$.

If $n \geq 6$, then we let $\mathcal{C}_{A'}(n, 2)$ be the set composed of the graphs $C_3(2, m_1, m_2)$ where $m_1 + m_2 = n - 2$ and $m_1 \in \{ \lfloor (n - 3)/2 \rfloor + 1, \dots, n - 4 \}$. We note that $\mathcal{C}_{A'}(n, 2)$ and $\mathcal{C}_A(n, 2)$ are disjoint, and $|\mathcal{C}_{A'}(n, 2)| = \lceil (n - 3)/2 \rceil - 1$.

We define $\mathcal{C}(n, 2) = \mathcal{C}_A(n, 2) \cup \mathcal{C}_{A'}(n, 2)$ if $n \geq 6$, noting that $|\mathcal{C}(n, 2)| = n - 3$ in this case, and we define $\mathcal{C}(n, 2) = \mathcal{C}_A(n, 2)$ if $n \in \{4, 5\}$. We observe that if $n \in \{4, 5\}$, then $|\mathcal{C}_A(n, 2)| = \lfloor (n + 1 - r)/2 \rfloor = n - 3$, so that $\mathcal{C}(n, 2) = n - 3$ for all $n \geq 4$.

We now assume that $r \geq 3$. Let $\mathcal{C}_B(n, r)$ be composed of the graphs $S_{r+1}^i(m_1, m_2)$ with $m_1 + m_2 = n - r$ and $i \in \{1, \dots, \lfloor r/2 \rfloor - 1\}$, and satisfying the following additional restriction: if r is even and $i = r/2 - 1$, then $m_1 \geq \lceil (n - r)/2 \rceil$. This last restriction comes from the fact that if r is even, then $S_{r+1}^{r/2-1}(a, b)$ and $S_{r+1}^{r/2-1}(b, a)$ are isomorphic graphs. We notice that $|\mathcal{C}_B(n, r)|$ is equal to $(n - r - 1) \cdot \lfloor (r - 1)/2 \rfloor - 1$ if r is odd, and to $(n - r - 1) \cdot (r/2 - 2) + \lfloor (n - r)/2 \rfloor$ if r is even. In particular, $\mathcal{C}_B(n, 3)$ is empty.

Let $\mathcal{C}_C(n, r)$ be composed of the graphs $C_{r+1}^i(m_1, m_2)$ with $m_1 + m_2 = n - r$ and $i \in \{1, \dots, \lceil r/2 \rceil - 1\}$, and the following additional restriction: if r is odd and $i = (r + 1)/2 - 1$, then $m_1 \geq \lceil (n - r)/2 \rceil$. We notice that $|\mathcal{C}_C(n, r)|$ is equal to $(n - r - 1) \cdot \lfloor (r/2) - 1 \rfloor$ if r is even, and to $(n - r - 1) \cdot ((r + 1)/2 - 2) + \lfloor (n - r)/2 \rfloor$ if r is odd.

We let $\mathcal{C}_D(n, r)$ be the set composed of the graphs $C_{r+1}^1(1, m_1, m_2)$ with $m_1 + m_2 = n - r$ and $m_1 \in \{ \lfloor (n - r - 1)/2 \rfloor, \dots, n - r - 2 \}$ if $r \geq 4$. If $r = 3$, then we require that $m_1 \in \{ \lfloor (n - 4)/2 \rfloor + 1, \dots, n - 5 \}$, and to compensate, if $n \geq 6$ then we then also add to $\mathcal{C}_D(n, 3)$ the graph $C_4^1(1, n - 4)$. Indeed if n is even then allowing both $m_1 = (n - 4)/2$ and $m_1 = (n - 4)/2 + 1$ would produce two isomorphic graphs (namely $C_4^1(1, (n - 4)/2, (n - 4)/2 + 1)$ and $C_4^1(1, (n - 4)/2 + 1, (n - 4)/2)$). (When n is odd, although it would not produce a pair of isomorphic graphs, it can be seen that it would create a set that is not linearly independent; moreover, we need no compensation when $n = 5$, and as a matter of fact $C_4(1, 1)$ already belongs to $\mathcal{C}_C(5, 3)$.) Note that, in any case, $\mathcal{C}_D(n, r)$ and $\mathcal{C}_C(n, r)$ are disjoint, and $|\mathcal{C}_D(n, r)| = \lfloor (n - r - 1)/2 \rfloor$.

We let $\mathcal{C}_E(n, r)$ be the set composed of the graphs $C_{r+1}^1(1, m_1, 1, m_2)$ with $m_1 + m_2 = n - r$ and $m_1 \in \{ \lfloor (n - r)/2 \rfloor, \dots, n - r - 2 \}$. Note that $\mathcal{C}_E(n, r)$ and $\mathcal{C}_C(n, r) \cup \mathcal{C}_D(n, r)$ are disjoint, and $|\mathcal{C}_E(n, r)| = \lfloor (n - r)/2 \rfloor - 1$.

Let $\mathcal{C}_F(n, r)$ be composed of the graph $C_{r+1}^+(n - r - 2)$ if $n \geq r + 3$, and set $\mathcal{C}_F(r + 2, r) = \emptyset$ if $n = r + 2$; so $|\mathcal{C}_F(n, r)| = 1$ if $n \geq r + 3$ and 0 if $n = r + 2$.

We set

$$\mathcal{C}(n, r) = \mathcal{C}_A(n, r) \cup \mathcal{C}_B(n, r) \cup \mathcal{C}_C(n, r) \cup \mathcal{C}_D(n, r) \cup \mathcal{C}_E(n, r) \cup \mathcal{C}_F(n, r).$$

We observe that whatever the parity of r (and even if $r = 3$), we have

$$|\mathcal{C}_B(n, r)| + |\mathcal{C}_C(n, r)| = \lfloor (n - r)/2 \rfloor + (n - r - 1) \cdot (r - 3).$$

It follows that if $n \geq r + 3$, then

$$\begin{aligned} |\mathcal{C}(n, r)| &= \left\lfloor \frac{n + 1 - r}{2} \right\rfloor + \left\lfloor \frac{n - r}{2} \right\rfloor + (n - r - 1) \cdot (r - 3) + \left\lfloor \frac{n - 1 - r}{2} \right\rfloor + \left\lfloor \frac{n - r}{2} \right\rfloor \\ &= (n - r) + (n - r - 1) \cdot (r - 3) + (n - r - 1) \\ &= 1 + (n - r - 1) \cdot (r - 1), \end{aligned}$$

while $|\mathcal{C}(r + 2, r)| = (r - 1)$.

5.2.3 Proving Proposition 5.2

First, we note that it suffices to establish the assumptions of Lemma 5.3 for $n \geq r + 3$, as they are trivial for $n = r + 2$ since $|\mathcal{B}^1(r + 2, r)| = 1 = |\mathcal{C}(r + 1, r)^\circ|$.

We introduce the following equivalence relation over matroids (and graphs).

Definition 5.4. Fix a field \mathbf{F} . Let X be a set of graphs or matroids. Two graphs or two matroids M and M' are *equivalent modulo X (over \mathbf{F})*, written $M \equiv M' \pmod{X}$, if there exist scalars $(\alpha_N)_{N \in X} \in \mathbf{F}^X$ such that

$$T(M; x, y) - T(M'; x, y) = \sum_{N \in X} \alpha_N T(N; x, y). \quad (16)$$

Recall that if the graphs or matroids M, M' and the members of X have all the same size and rank, then necessarily the scalars in (16) satisfy $\sum_{N \in X} \alpha_N = 0$.

To conclude the proof of Proposition 5.2, it suffices to establish the following statement, dealing with the case where $n \geq r + 3$.

Lemma 5.5. *For $r \geq 2$ and $n \geq r + 3$, if G and G' are two graphs in $\mathcal{B}^1(n, r)$, or two graphs in $\mathcal{C}(n - 1, r)^\circ$, then G and G' are equivalent modulo $\mathcal{C}(n, r)$.*

Since Lemma 5.5 guarantees that the assumptions of Lemma 5.3 are satisfied, it follows that Lemma 5.5 combined with Lemma 5.3 imply Proposition 5.2. Indeed, let X be either $\mathcal{B}^1(n, r)$ or $\mathcal{C}(n - 1, r)^\circ$, and let $(\alpha_G)_{G \in X}$ be scalars summing to 0, and fix an arbitrary element $G_0 \in X$. For $G \in X$, we define $L_G(x, y)$ to be the right side of (16) when $M = G$ and $M' = G_0$. Then

$$\begin{aligned} \sum_{G \in X} \alpha_G T(G; x, y) &= \sum_{G \in X} \alpha_G (T(G_0; x, y) + L_G(x, y)) \\ &= T(G_0; x, y) \sum_{G \in X} \alpha_G + \sum_{G \in X} \alpha_G \cdot L_G(x, y) \\ &= \sum_{G \in X} \alpha_G \cdot L_G(x, y), \end{aligned}$$

as required. It thus remains to prove Lemma 5.5. This is deferred to the appendix.

6 Tutte spaces of loopless matroids

For any given size n and rank r , we started in Section 3 from the class $\mathcal{M}(n, r)$, continued in Section 4 with the class $\mathcal{M}^{\ell c}(n, r)$, and next in Section 5 with the class $\mathcal{M}^c(n, r)$. Lying between $\mathcal{M}^c(n, r)$ and $\mathcal{M}(n, r)$ is the class of loopless matroids of size n and rank r , which we denote by $\mathcal{M}^s(n, r)$. As mentioned in the introduction, a result of Kung gives the dimension of $\mathcal{T}(\mathcal{M}^s(n, r))$. Here we derive this result in another way, and moreover exhibit a basis extending our basis for $\mathcal{T}(\mathcal{M}^c(n, r))$. We define $\bar{d}(n, r)$ to be the dimension of $\mathcal{T}(\mathcal{M}^s(n, r))$. The basis for $\mathcal{T}(\mathcal{M}(n, r))$ given in Section 3 allows us to quickly derive a basis for $\mathcal{T}(\mathcal{M}^s(n, r))$, following an argument already used in Section 5.1.

Theorem 6.1. *For positive integers n and r with $n \geq r + 1$,*

$$\bar{d}(n, r) = (r - 1)(n - r) + 1.$$

Furthermore, we can construct a basis for $\mathcal{T}(\mathcal{M}^s(n, r))$ composed of Tutte polynomials of plane loopless multigraphs, all but at most one being in addition series-parallel.

We will use the following definitions and observations to establish Theorem 6.1 when $n = r + 2 \geq 5$ or $r = 2 \leq n - 2$.

Definition 6.2.

- For every integer $k \geq 4$, let $C_{k-1} \cdot C_2$ be the graph obtained from C_{k-1} by adding a new vertex v_{k-1} linked to the vertex v_0 by 2 edges.
- For every integer $n \geq 4$, let $C_2 \cdot C_2(n - 3)$ be the graph obtained from C_2 by adding a new vertex v_2 linked to the vertex v_0 by $n - 2$ edges.

Lemma 6.3.

1. If $r \geq 3$, then $t_{0,1}(C_{r+1} \cdot C_2) = 0$ and $t_{0,2}(C_{r+1} \cdot C_2) = 1$.
2. If $r \geq 2$ and $G \in \mathcal{C}(r + 2, r)$, then $t_{0,1}(G) = 1 = t_{0,2}(G)$.

3. If $n \geq 4$, then $t_{0,1}(C_2 \cdot C_2(n-3)) = 0$.

Proof.

1. As $T(C_{r+1} \cdot C_2)$ is the product of $(x+y)$ and $T(C_{r+1})$, and $t_{0,0}(C_{r+1}) = 0$ while $t_{0,1}(C_{r+1}) = 1$, the statement follows.
2. By definition, $\mathcal{C}(r+2, r) = \mathcal{C}_A(r+2, r) \cup \mathcal{C}_B(r+2, r) \cup \mathcal{C}_C(r+2, r)$, where we recall that $\mathcal{C}_B(5, 3) = \emptyset$. The statement follows by using the deletion-contraction formula as follows. First, for elements in $\mathcal{C}_A(r+2, r)$ we delete-contraction one of the two edges between v_0 and v_1 , and conclude using the expressions of the Tutte polynomial for cycles. Second, we establish the statement for elements of $\mathcal{C}_C(r+2, r)$ by deleting-contraction the chord, that is, the edge between v_r and v_i . Third, we consider the elements of $\mathcal{C}_B(r+2, r)$, for which we delete-contraction the edge between v_{r-1} and v_r , and use the properties for the elements of $\mathcal{C}_C(r+2, r)$.
3. As $T(C_2 \cdot C_2(n-3))$ is the product of $(x+y)$ and $T(C_2(n-3))$, and $t_{0,0}(C_2(n-3)) = 0$ while $t_{0,1}(C_2(n-3)) = t_{0,1}(C_2) = 1$, the statement follows. \square

Proof of Theorem 6.1. Since $\mathcal{M}^s(n, r)$ is a subset of $\mathcal{M}(n, r)$, Theorem 3.1 ensures that $\mathcal{G}_T(n, r)$ generates $\mathcal{T}(\mathcal{M}^s(n, r))$. Let us partition $\mathcal{G}(n, r)$ according to the number of loops of its elements: we obtain $n - r + 1$ parts, and we let $Y^j(n, r)$ be the part composed of the elements with exactly j loops, for each $j \in \{0, \dots, n - r\}$. Since a polynomial in $\mathcal{T}(\mathcal{M}^s(n, r))$ has no monomial $x^j y^{n-r}$ with $j \geq 1$, we infer that if a linear combination $\sum_{G \in \mathcal{G}(n, r)} \alpha_G T_G$, where $\alpha_G \in \mathbf{F}$ for each $G \in \mathcal{G}(n, r)$, belongs to $\mathcal{T}(\mathcal{M}^s(n, r))$, then for each $j \in \{1, \dots, n - r\}$,

$$\sum_{G \in Y^j} \alpha_G T_G = 0.$$

As a result, letting G_j be an arbitrary element in $Y^j(n, r)$ for each $j \in \{1, \dots, n - r\}$, we see that the set of Tutte polynomials of the graphs in $\bigcup_{j=0}^{n-r} Y^j(n, r) \setminus \{G_j\}$ generates $\mathcal{T}(\mathcal{M}^s(n, r))$. (Notice that $|Y^{n-r}(n, r)| = 1$, so we may as well take the preceding union up to $j = n - r - 1$ only.) This proves that $\bar{d}(n, r) \leq d(n, r) - (n - r) = (r - 1)(n - r) + 1$.

To obtain the lower bound, we notice that the set $\mathcal{C}_T(n, r)$ generates a subspace of $\mathcal{T}(\mathcal{M}^s(n, r))$ of dimension $d^c(n, r)$. (When $n = 3$ and $r = 2$, we recall that $\mathcal{C}(3, 2)$ is composed only of C_3 , the unique 2-connected 3-vertex graph with 3 edges.) We consider the set $\mathcal{O}(n, r)$ defined to be the union of $\mathcal{C}(n, r)$ with $\{G_{i,0}^{n,r} : 1 \leq i \leq r - 1\}$, recalling that the graphs $G_{i,0}^{n,r}$ have been introduced in Definition 3.3. The set $\mathcal{O}_T(n, r)$ has thus size $d^c(n, r) + r - 1$, and is linearly independent over \mathbf{F} . Indeed, we already know that each of the two sets forming $\mathcal{O}_T(n, r)$ is linearly independent over \mathbf{F} , and, by Lemma 2.3-5, for each $k \in \{1, \dots, r - 1\}$, the only element P in $\mathcal{O}_T(n, r)$ satisfying $[x^k y^{n-r}]P \neq 0$ is $T(G_{r-k,0}^{n,r})$, for which this coefficient is 1. We therefore deduce that $\bar{d}(n, r) \geq d^c(n, r) + r - 1$. This matches the upper bound from the previous paragraph, unless $n = r + 2 \geq 4$ or $r = 2 \leq n - 2$.

Let us assume that $r = 2$ and $n \geq 4$. Notice that $\mathcal{O}(n, 2) = \mathcal{C}(n, 2) \cup \{G_{1,0}^{n,2}\}$. We add to $\mathcal{O}(n, 2)$ the graph $C_2 \cdot C_2(n-3)$ obtained from C_2 by adding a new vertex v_2 linked by $n-2$ edges to the vertex v_0 . The new set $\mathcal{O}^+(n, 2)$ thus obtained has size $d^c(n, 2) + 2 = n - 1 = (r - 1)(n - r) + 1$. Moreover, $\mathcal{O}_T^+(n, 2)$ is linearly independent over \mathbf{F} . Indeed, suppose that

$$\sum_{P \in \mathcal{O}_T^+(n, 2)} \alpha_P \cdot P = 0, \quad (17)$$

with $\alpha_P \in \mathbf{F}$ for each $P \in \mathcal{O}_T^+(n, 2)$. Since we already know that $\mathcal{O}_T(n, 2)$ is linearly independent over \mathbf{F} , it suffices to prove that $\alpha_P = 0$ for $P = T(C_2 \cdot C_2(n-3))$.

Lemma 2.3-5 guarantees that $[y^{n-2}]P = 1$ for $P \in \mathcal{O}_T^+(n, 2)$, unless $P = T(G_{1,0}^{n,2})$, in which case we have $[y^{n-2}]P = 0$. By Lemmas 3.5 and 6.3-2 and 3, we know that $[y]P = 1$ unless $P \in \{T(G_{1,0}^{n,2}), T(C_2 \cdot C_2(n-3))\}$, in which case $[y]P = 0$. Therefore, the difference of the projections of (17) on the monomial y^{n-2} and on the monomial y gives that $\alpha_{T(C_2 \cdot C_2(n-3))} = 0$, as desired.

It remains to deal with the case where $n = r + 2 \geq 5$. In this case, we add to the set $\mathcal{O}(r+2, r)$ the graph $C_{r+1} \cdot C_2$ obtained from C_{r+1} by adding a new vertex v_{r+1} linked to the vertex v_0 by 2 edges. The new set $\mathcal{O}^+(r+2, r)$ thus obtained has size $d^c(r+2, r) + r = 2r - 1 = (r - 1)(n - r) + 1$. Moreover, it is linearly independent over \mathbf{F} . Indeed, suppose that

$$\sum_{P \in \mathcal{O}_T^+(r+2, r)} \alpha_P \cdot P = 0, \quad (18)$$

with $\alpha_P \in \mathbf{F}$ for each $P \in \mathcal{O}_T^+(r+2, r)$. Recall that, as argued above, $\mathcal{O}_T(r+2, r)$ is linearly independent. Therefore, it suffices to prove that $\alpha_P = 0$ if $P = T(C_{r+1} \cdot C_2)$.

First, we note that if $i \in \{1, \dots, r-1\}$, then $t_{0,k}(G_{i,0}^{r+2,r}) = 0$ for any non-negative integer k , since $G_{i,0}^{r+2,r}$ has at least one bridge, and hence its Tutte polynomial is divisible by x — this can also be seen directly from Lemma 3.5.

Now, by Lemma 6.3-1, the projection of (18) on the monomial y yields that $\sum_{P \in \mathcal{C}(r+2,r)} \alpha_P = 0$. On the other hand, Lemma 6.3-1 also guarantees that the projection of (18) on the monomial y^2 yields that $\sum_{P \in \mathcal{C}(r+2,r) \cup \{C_{r+1} \cdot C_2\}} \alpha_P = 0$. Consequently, $\alpha_P = 0$ if $P = T(C_{r+1} \cdot C_2)$, which concludes the proof. \square

A Dictionary of identities

In order to ease reading the relations that follow, we adopt the convention of identifying a graph with its Tutte polynomial. Thus, for an ordinary edge e , the fundamental deletion-contraction recurrence in this notation is

$$G = G \setminus e + G / e.$$

(Note that this convention is consistent with the notation $C_{k-1} \cdot C_2$ and $C_2 \cdot C_2(n-3)$ for the graphs from Definition 6.2, since the respective Tutte polynomials of these graphs are $T(C_{k-1}) \cdot T(C_2)$ and $T(C_2) \cdot T(C_2(n-3))$.) In order to derive the various relations below, it helps to have a compression of this deletion-contraction recurrence for edges in series (forming an induced path) and in parallel. For an edge e of a graph G , let G_e^{-k} denote the graph obtained by replacing e by an induced path of k edges (k -stretching the edge) and $G_e^{\equiv k}$ the graph obtained by replacing e by k edges in parallel (k -thickening the edge). Then, for an ordinary edge e and integer $k \geq 1$,

$$G_e^{-k} = (k)_x G \setminus e + G / e,$$

$$G_e^{\equiv k} = G \setminus e + (k)_y G / e,$$

where, using quantum integer notation,

$$(k)_x = 1 + x + \dots + x^{k-1} = \frac{x^k - 1}{x - 1}, \quad (k)_y = 1 + y + \dots + y^{k-1} = \frac{y^k - 1}{y - 1}.$$

We in particular note the following identity, coming directly from the definition of quantum integers.

Identity A.1. *For non-negative integers i and j ,*

$$x^i - x^j = (1 - x) \cdot [(j)_x - (i)_x].$$

Further, the following two identities between quantum integers will also be useful later.

Identity A.2. *For positive integers a , b and $c \in \{0, \dots, b-1\}$,*

$$(a+c)_y(b-c)_y - (a)_y(b)_y = (c)_y[(b-c)_y - (a)_y],$$

Identity A.3. *For positive integers a and b ,*

$$\sum_{j=1}^b [(a+b+1-j)_y - (j)_y] = (a)_y[(b+1)_y - (1)_y] = y(a)_y(b)_y.$$

For future reference, we now express the Tutte polynomial of a cycle with a thickened edge in terms of quantum integers.

Identity A.4. *For positive integers a and $s \geq 2$,*

$$C_s(a) = (s-1)_x \cdot [(a)_y - 1 + x] + y^a.$$

In particular,

$$C_2(a) = (a+1)_y - 1 + x.$$

If G and G' are connected graphs of rank r with the same number n of edges, then $G - G'$ is divisible by $x + y - xy$. This can be seen by considering the fact that $T_G(x, y) = (x - 1)^r y^n$ modulo the ideal $((x - 1)(y - 1) - 1)$, so that $T_G(x, y) - T_{G'}(x, y)$ is zero modulo $(x + y - xy)$. We derive a set of relations expressing the difference of two 2-connected graphs of the same size n and rank r each with a loop added to them (effectively multiplying the difference by y) as a linear combination of differences of loopless 2-connected graphs of size $n + 1$ and rank r . To do this, we will use the following expressions for differences between our basis graphs. We recall from Definition 3.2 that $C_2(0)$ is defined as the graph with 2 vertices and 1 edge. Consistently with our previous notation, we let $C_2(a) \cdot C_2(b)$ be the (Tutte polynomial of the) path $v_0 v_1 v_2$ on 3 vertices where $v_0 v_1$ has multiplicity $a + 1$ and $v_1 v_2$ has multiplicity $b + 1$.

Identity A.5. For integers $a, b \geq 2$ and $c \in \{0, \dots, b\}$,

$$\frac{C_2(a) \cdot C_2(b) - C_2(a + c) \cdot C_2(b - c)}{x + y - xy} = (c)_y \cdot [(a + 1)_y - (b + 1 - c)_y].$$

Identity A.6. For positive integers $a, b, s \geq 3$ and $c \in \{0, \dots, b - 1\}$,

$$\frac{C_s(a, b) - C_s(a + c, b - c)}{x + y - xy} = (s - 2)_x (c)_y [(a)_y - (b - c)_y].$$

Identity A.7. For positive integers a, b, c and $d \in \{0, \dots, b - 1\}$,

$$\frac{C_3(a, b, c) - C_3(a + d, b - d, c)}{x + y - xy} = (d)_y \cdot [(a)_y - (b - d)_y].$$

Identity A.8. For positive integers $a, b, s \geq 4, i \in \{1, \dots, s - 3\}$ and $c \in \{0, \dots, a - 1\}$,

$$\frac{C_s^i(a, b) - C_s^i(a - c, b + c)}{x + y - xy} = (s - 1 - i)_x (i)_x (c)_y [(b)_y - (a - c)_y] - (i)_x [(a)_y - (a - c)_y].$$

Identity A.9. For positive integers $a, b, s \geq 4, i \in \{1, \dots, s - 3\}$ and $c \in \{0, \dots, b - 1\}$,

$$\begin{aligned} \frac{C_s^i(1, a, b) - C_s^i(1, a + c, b - c)}{x + y - xy} &= (s - 2)_x (c)_y [(a)_y - (b - c)_y] + (i)_x (s - 2 - i)_x (c)_y [(a)_y - (b - c)_y] \\ &\quad - (i)_x [(b)_y - (b - c)_y] - (s - 2 - i)_x [(a)_y - (a + c)_y]. \end{aligned}$$

Identity A.10. For positive integers $a, b, s \geq 5$ and $i \in \{2, \dots, s - 3\}$,

$$\frac{C_s^i(a, b) - C_s^{i-1}(a, b)}{x + y - xy} = (a)_y \cdot \left((i - 1)_x - (i)_x + (b)_y \cdot [(s - 1 - i)_x - (i - 1)_x] \right).$$

Identity A.11. For positive integers a, b, c and $s \geq 4$,

$$\frac{C_s^1(1, a, 1, b + c) - C_s^1(1, a + c, 1, b)}{x + y - xy} = (s - 2)_x \cdot (c)_y \cdot [(a)_y - (b)_y].$$

Identity A.12. For positive integers a, b and $s \geq 4$,

$$\frac{C_s^1(a, b) - C_s^1(1, a, 1, b)}{x + y - xy} = -y(a - 1)_y.$$

Identity A.13. For positive integers $a, b, s \geq 4$ and $c \in \{0, \dots, b - 1\}$,

$$\begin{aligned} \frac{C_s^1(1, a, b) - C_s^1(1, a + c, 1, b - c)}{x + y - xy} &= (s - 2)_x \cdot (c)_y \cdot [(a)_y - (b - c)_y] \\ &\quad + y \cdot (b - 1)_y \cdot [(s - 3)_x \cdot (a)_y - 1]. \end{aligned}$$

Identity A.14. For positive integers $a, b, s \geq 5, i \in \{1, \dots, \lfloor (s - 1)/2 \rfloor - 1\}$ and $c \in \{0, \dots, b - 1\}$,

$$\begin{aligned} \frac{S_s^i(a, b) - S_s^i(a + c, b - c)}{x + y - xy} &= (c)_y \cdot [(1 + x)(s - 3)_x + (i)_x (s - 3 - i)_x] \cdot [(a)_y - (b - c)_y] \\ &\quad + (i)_x [(b - c)_y - (b)_y] + (s - 3 - i)_x [(a + c)_y - (a)_y]. \end{aligned}$$

Using Identities A.3, A.8 and A.14–A.16, we obtain the following two identities.

Identity A.15. For positive integers $a, b, s \geq 4$ and $i \in \{1, \dots, \lceil (s-1)/2 \rceil - 1\}$,

$$\begin{aligned} \sum_{j=1}^b C_s^i(j, a+b+1-j) - C_s^i(a+b+1-j) &= y(a)_y(b)_y \cdot (i)_x, \\ &= \sum_{j=1}^a C_s^i(j, a+b+1-j) - C_s^i(a+b+1-j). \end{aligned}$$

Identity A.16. For positive integers $a, b, s \geq 5$ and $i \in \{1, \dots, \lfloor (s-1)/2 \rfloor - 1\}$,

$$\begin{aligned} \sum_{j=1}^b S_s^i(j, a+b+1-j) - S_s^i(a+b+1-j) &= y(a)_y(b)_y \cdot [(s-3-i)_x - (i)_x], \\ &= \sum_{j=1}^a S_s^i(j, a+b+1-j) - S_s^i(a+b+1-j). \end{aligned}$$

Identity A.17. For positive integers $a, b, s \geq 5$ and $i \in \{1, \dots, \lfloor (s-1)/2 \rfloor - 1\}$,

$$\frac{S_s^i(a, b) - C_s^i(b, a)}{x + y - xy} = (s-3-i)_x \cdot (a)_y \cdot [(1+x) \cdot (b)_y - 1].$$

B Proof of Lemma 5.5

As reported in Section 5.2.3, we can now assume that $n \geq r+3 \geq 5$. We recall from Section 5.2.2 the notation for the set of graphs $\mathcal{C}(n, r)$ and its various subfamilies.

A useful remark for the proof of Lemma 5.5.

Observation B.1. Let $r \geq 2$ and $n \geq r+2$ be integers. The set $\mathcal{B}^1(n, r)$ is contained in the set $\mathcal{C}(n-1, r)^\circ = \{G^\circ : G \in \mathcal{C}(n-1, r)\}$ unless $n = r+4 \geq 8$, in which case $\mathcal{B}^1(n, r) \setminus \mathcal{C}(n-1, r)^\circ = \{D_{r+1}^\circ\}$.

To make the above statement more obvious, we recall that $\mathcal{B}^1(n, r) = \{G^\circ : G \in \mathcal{S}(n-1, r)\}$, where we set $\mathcal{S}(r+1, r) = \{C_{r+1}\}$ for convenience. For instance,

$$\begin{aligned} \mathcal{B}^1(5, 3) &= \{C_4^\circ\}, \\ \mathcal{B}^1(6, 3) &= \{C_4(2)^\circ, (C_4^1)^\circ\}, \\ \mathcal{B}^1(7, 3) &= \{C_4(3)^\circ, C_4^1(2)^\circ, (C_4^+)^{\circ}, (C_4(2, 2))^\circ\}, \\ \mathcal{B}^1(8, 3) &= \{C_4(4)^\circ, C_4^1(3)^\circ, C_4^+(2)^\circ\}; \end{aligned}$$

while

$$\begin{aligned} \mathcal{B}^1(6, 4) &= \{C_5^\circ\}, \\ \mathcal{B}^1(7, 4) &= \{C_5(2)^\circ, (C_5^1)^\circ, (S_5^1)^\circ\}, \\ \mathcal{B}^1(8, 4) &= \{C_5(3)^\circ, C_5^1(2)^\circ, S_5^1(2)^\circ, (C_5^+)^{\circ}, (D_5)^\circ\}, \\ \mathcal{B}^1(9, 4) &= \{C_5(4)^\circ, C_5^1(3)^\circ, S_5^1(3)^\circ, C_5^+(2)^\circ\}. \end{aligned}$$

By Observation B.1, when $n \neq r+4$ or $r = 3$, it suffices to prove that every pair of graphs in $\mathcal{C}(n-1, r)^\circ$ are equivalent modulo $\mathcal{C}(n, r)$, while if $n = r+4 \geq 8$, we in addition have to prove that D_{r+1}° is equivalent to a graph in $\mathcal{C}(n-1, r)^\circ$. All equivalences between two elements are established by proving a more general identity between Tutte polynomials. For each of these, it should be checked that the right side involves Tutte polynomials of elements of $\mathcal{C}(n, r)$ only.

For two sets of graphs A and B , *reducing A to B* means proving that every graph in A is equivalent (modulo $\mathcal{C}(n, r)$) to a graph in B . If we know that all the graphs in B are equivalent to one another (which is in particular the case if B is a singleton), then this also proves that all the graphs in $A \cup B$ are equivalent. We often use a “cascading” approach: imagining the graphs in A to be listed as G_1, \dots, G_t according to some parameter, we would then prove that G_i is equivalent to G_{i-1} for each $i \in \{2, \dots, t\}$, and that G_1 is equivalent to a member of B .

Let us start by taking care of the unique graph in $\mathcal{B}^1(n, r) \setminus \mathcal{C}(n-1, r)^\circ$, which exists when $n = r+4 \geq 8$. We recall that we now identify a graph and its Tutte polynomial.

We reduce $\{D_{r+1}^\circ\}$ to $\{C_{r+1}^2(1, 2)^\circ\}$, which is contained in $\mathcal{C}_C(r+3, r)^\circ$ if $r \geq 6$; we reduce $\{D_5^\circ\}$ to $\{C_5^1(1, 1, 2)^\circ\}$, which belongs to $\mathcal{C}_D(7, 4)^\circ$, and we reduce $\{D_6^\circ\}$ to $\{S_6^1(1, 2)^\circ\}$, which belongs to $\mathcal{C}_B(8, 5)^\circ$.

We deal with D_6° separately, and first take care of the other cases.

Relation B.2. For every integer $r \geq 6$,

$$D_{r+1}^\circ \equiv C_{r+1}^2(1, 2)^\circ \pmod{\mathcal{C}(r+4, r)}.$$

Further,

$$D_5^\circ \equiv C_5^1(1, 1, 2)^\circ \pmod{\mathcal{C}(8, 4)}.$$

Proof. We observe that it suffices to prove that for each $s \geq 5$,

$$D_s^\circ - C_s^2(1, 2)^\circ = C_s^1(2, 2) - C_s^1(1, 3). \quad (19)$$

Indeed, if $r = s - 1$ and $n = s + 3 = r + 4$, then all the graphs appearing in the right side of (19) belong to $\mathcal{C}(n, r)$, so (19) clearly establishes the first identity when $s \geq 7$ (and we note that $C_s^2(1, 2)$ then belongs to $\mathcal{C}_C(r+3, r)$). Moreover, (19) also implies the second identity when $s = 5$, by noticing that $C_5^1(1, 1, 2)$ is actually isomorphic to $C_5^2(1, 2)$ (and we note that $C_5^1(1, 1, 2) \in \mathcal{C}_D(7, 4)$).

First, Identity A.6, applied with $(a, b, c) = (2, 2, 1)$, yields that

$$\begin{aligned} \frac{C_s^1(2, 2) - C_s^1(1, 3)}{x + y - xy} &= (s-2)_x \cdot [(2)_y - 1] - [(2)_y - 1] \\ &= y \cdot [(s-2)_x - 1]. \end{aligned} \quad (20)$$

Now, deleting-contracting the edge $v_1 v_s$ in D_s , and an edge between v_1 and v_2 in $C_s^2(1, 2)$, and using Identity A.4, yields that

$$\begin{aligned} D_s - C_s^2(1, 2) &= C_2 \cdot C_{s-2}(2) - y \cdot C_{s-1}^1 \\ &= (x+y) \cdot [(s-2)_x - 1 + y] - y \cdot [(s-1)_x - 1 + y] \\ &= (x+y) \cdot (s-2)_x + (x+y) \cdot (y-1) - y \cdot [x \cdot (s-2)_x] - y^2 \\ &= (s-2)_x \cdot (x+y-xy) + (xy-x-y) \\ &= (x+y-xy) \cdot [(s-2)_x - 1], \end{aligned} \quad (21)$$

where the third line uses that $(t)_x = x(t-1)_x + 1$ for every positive integer t . Comparing (20) and (21) concludes the proof. \square

We now handle the case of D_6° .

Relation B.3. It holds that

$$D_6^\circ \equiv S_6^1(1, 2)^\circ \pmod{\mathcal{C}(9, 5)}.$$

Proof. It suffices to prove that

$$D_6^\circ - S_6^1(1, 2)^\circ = S_6^1(3) - S_6^1(1, 3) + C_6^1(2, 2) - C_6^1(1, 3). \quad (22)$$

Notice that all the graphs appearing in the right side of (22) belong to $\mathcal{C}(9, 5)$.

Let us prove that the right side of (22) is equal to $(x+y-xy)xy \cdot [(3)_x - (2)_y]$. By Identity A.14 applied with $s = 6, i = 1, a = 1$ and $b = 3$,

$$\begin{aligned} \frac{S_6^1(3) - S_6^1(1, 3)}{x + y - xy} &= (3)_y - 1 - (1+x) \cdot [(3)_x - 1] \\ &= -x[(3)_y - 1] \\ &= -xy(2)_y. \end{aligned} \quad (23)$$

On the other hand, Identity A.10 applied with $s = 6, i = 1$ and $a = b = 2$ yields that

$$\begin{aligned} \frac{C_6^1(2, 2) - C_6^1(1, 3)}{x + y - xy} &= (4)_x \cdot [(2)_y - 1] - [(2)_y - 1] \\ &= [(4)_x - 1] \cdot [(2)_y - 1] \\ &= xy(3)_x, \end{aligned} \quad (24)$$

as $(a)_x - 1 = x(a-1)_x$ for every positive integer a . Adding (23) and (24) gives the announced expression for the right side of (22).

Showing that the left side of (22) is also equal to this expression can be done quickly thanks to the deletion-contraction formula, and we omit this straightforward computation. \square

All members of $\mathcal{C}_A(n-1, r)^\circ$ are equivalent to one another.

Relation B.4. Fix $r \geq 2$ and $n \geq r+3$. If $a \in \{[(n-r)/2], \dots, n-1-r\}$ and $b = n-r-a$, then

$$C_{r+1}(a, b)^\circ \equiv C_{r+1}(a+1, b-1)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. It suffices to show that for positive integers a and $b \geq 2$,

$$C_3(a, b)^\circ - C_3(a+1, b-1)^\circ = C_3(a, b+1) - C_3(a+2, b-1) + C_3(2, a+1, b-1) - C_3(2, a, b), \quad (25)$$

and if $s \geq 4$, then

$$C_s(a, b)^\circ - C_s(a+1, b-1)^\circ = C_s(a, b+1) - C_s(a+2, b-1) + C_s^1(1, a+1, 1, b-1) - C_s^1(1, a, 1, b). \quad (26)$$

Indeed, setting $r = s-1$ and $n = a+b+r$, if a and b satisfy the assumptions of the statement, then all the graphs appearing in the right sides of (25) and of (26) belong to $\mathcal{C}(n, r)$ (when relevant).

By Identity A.6, for each $s \geq 3$,

$$\frac{C_s(a, b) - C_s(a+1, b-1)}{x+y-xy} = (s-2)_x \cdot [(a)_y - (b-1)_y], \quad (27)$$

On the other hand, Identity A.6 also yields that

$$\frac{C_s(a, b+1) - C_s(a+2, b-1)}{x+y-xy} = (s-2)_x \cdot (1+y) \cdot [(a)_y - (b-1)_y]. \quad (28)$$

If $s \geq 4$, then Identity A.11 implies that

$$\frac{C_s^1(1, a+1, 1, b-1) - C_s^1(1, a, 1, b)}{x+y-xy} = (s-2)_x \cdot [(b-1)_y - (a)_y], \quad (29)$$

while if $s = 3$, then we deduce from Identity A.7 that

$$C_3(b-1, a+1, 2) - C_3(b, a, 2) = (1)_y \cdot [(b-1)_y - (a)_y]. \quad (30)$$

Therefore, (28), (29) and (30) conclude the proof thanks to (27). \square

Relation B.4 is sufficient to finish the proof of Lemma 5.5 in the case where $r = 2$ and $n \in \{5, 6\}$, since then $\mathcal{C}(n-1, 2) = \mathcal{C}_A(n-1, 2)$. Our next reduction concludes this proof for the case $r = 2$, by showing that for $n \geq 7$, every graph in $\mathcal{C}_{A'}(n-1, 2)^\circ$ is equivalent (modulo $\mathcal{C}(n, 2)$) to a graph in $\mathcal{C}_A(n-1, 2)^\circ$ (which, by the previous, are all equivalent to one another).

We reduce $\mathcal{C}_{A'}(n-1, 2)^\circ$ to $\mathcal{C}_A(n-1, 2)^\circ$.

Relation B.5. For $n \geq 7$, if $a \in \{[(n-4)/2] + 1, \dots, n-5\}$ and $b = n-3-a$, then

$$C_3(a, b, 2)^\circ \equiv C_3(a+1, b)^\circ \pmod{\mathcal{C}(n, 2)}.$$

Proof. It suffices to prove that for positive integers a and b ,

$$C_3(a, b, 2)^\circ - C_3(a+1, b)^\circ = C_3(a+b-1, 3) - C_3(a+b-1, 2, 2) + C_3(a+1, b, 2) - C_3(a+2, b). \quad (31)$$

Indeed, we observe that if a and b satisfy the assumptions of the statement, with $n = a+b+3 \geq 7$, so $C_3(a, b, 2) \in \mathcal{C}_{A'}(n-1, 2)^\circ$, then $C_3(a+1, b)^\circ \in \mathcal{C}_A(n-1, 2)^\circ$ and all the graphs appearing in the right side of (31) belong to $\mathcal{C}(n, 2) = \mathcal{C}_A(n, 2) \cup \mathcal{C}_{A'}(n, 2)$.

Applying Identity A.7 with $d = 1$ and $c = b$ yields that

$$C_3(a, b, 2)^\circ - C_3(a+1, b)^\circ = -y(x+y-xy) \cdot [(a)_y - 1]. \quad (32)$$

Moreover, applying it with $a = b = 2$, $c = a + b - 1$ and $d = 1$ implies that

$$C_3(a + b - 1, 3)^\circ - C_3(a + b - 1, 2, 2)^\circ = y(x + y - xy), \quad (33)$$

while with $d = 1$ and $c = b$, we deduce that

$$C_3(a + 1, b, 2)^\circ - C_3(a + 2, b)^\circ = -(x + y - xy) \cdot [(a + 1)_y - 1]. \quad (34)$$

By (32), adding (33) to (34) concludes the proof because $(a + 1)_y - 1 = y \cdot (a)_y$. \square

For now on we assume that $r \geq 3$ (and we recall that $n \geq r + 3$ by assumption).

We reduce $\mathcal{C}_E(n - 1, r)^\circ$ to $\{C_{r+1}^1(1, n - 1 - r)^\circ\}$, which is contained in $\mathcal{C}_C(n - 1, r)^\circ$ if $r \geq 4$, and in $\mathcal{C}_D(n - 1, r)^\circ$ if $r = 3$.

First, note that $\mathcal{C}_E(n - 1, r)$ is empty if $n \geq r + 4$, so we can here assume that $n \geq r + 5$. The following is enough to establish the announced reduction.

Relation B.6. Fix $r \geq 3$ and $n \geq r + 5$. If $a \in \{[(n - 1 - r)/2], \dots, n - r - 3\}$ and $b = n - 1 - r - a$, then

$$C_{r+1}^1(1, a, 1, b)^\circ \equiv C_{r+1}^1(1, a + 1, 1, b - 1)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. It suffices to prove that if $s \geq 4$, then for all integers $a \geq 1$ and $b \geq 2$,

$$C_s^1(1, a, 1, b)^\circ - C_s^1(1, a + 1, 1, b - 1)^\circ = C_s(a + 1, b + 1) - C_s(a + 2, b). \quad (35)$$

We observe that, setting $r = s - 1$ and $n = a + b + r + 1$, if n , a and b satisfy the assumptions of the statement, then the two graphs in the right side of (35) belong to $\mathcal{C}(n, r)$.

By repeatedly deleting-contracting the edges along the paths $v_1 v_2 \dots v_{s-1}$, we see that the left side of (35) equals $(s - 2)_x \cdot (C_3(a, b) - C_3(a + 1, b - 1))$, which by Identity A.6 equals $(x + y - xy)(s - 2)_x((a)_y - (b - 1)_y)$.

On the other hand, and again by Identity A.6, the right side of (35) equals $(x + y - xy)(s - 2)_x \cdot [(a + 1)_y - (b)_y]$, which thus equals the left side because $(c + 1)_y = y \cdot (c)_y + 1$ for every non-negative integer c . \square

We reduce $\mathcal{C}_D(n - 1, r)^\circ$ to $\mathcal{C}_A(n - 1, r)^\circ \cup \mathcal{C}_C(n - 1, r)^\circ$.

Let us deal separately with the case where $r = 3$. Note that $\mathcal{C}_D(n - 1, 3)$ is empty if $n \leq 6$ and $\mathcal{C}_D(n - 1, 3) = \{C_4^1(1, n - 5)\}$ if $n \in \{7, 8\}$. The next statement in particular handles the case of $C_4^1(1, n - 5)$ for all values of $n \geq 7$.

Relation B.7. For $n \geq 8$, if $a \in \{[(n - 5)/2] + 1, \dots, n - 5\}$ and $b = n - 4 - a$, then

$$C_4^1(1, a, b)^\circ \equiv C_4^1(a, b)^\circ \pmod{\mathcal{C}(n, 3)}.$$

Moreover,

$$C_4^1(1, 2)^\circ \equiv C_4(2, 2)^\circ \pmod{\mathcal{C}(7, 3)},$$

Proof. First, we prove that for positive integers a and b , if $a + b \geq 4$ then

$$C_4^1(1, a, b)^\circ - C_4^1(a, b)^\circ = C_4^1(1, a, b + 1) - C_4^1(1, a, 1, b + 1). \quad (36)$$

We note that if $C_4^1(1, a, b) \in \mathcal{C}_D(n - 1, 3)$, so in particular $n = a + b + 4$, then $C_4^1(a, b) \in \mathcal{C}_C(n - 1, 3)$ and all the graphs appearing in the right side of (36) belong to $\mathcal{C}(n, 3)$, unless $a = b + 1$ (in which case $C_4^1(1, a, b + 1)$ does not belong to $\mathcal{C}(n, 3)$, so we will prove a different identity for this).

By Identity A.13 applied with $s = 4$ and $c = 0$,

$$\frac{C_4^1(1, a, b + 1) - C_4^1(1, a, 1, b + 1)}{x + y - xy} = y(b)_y \cdot [(a_y) - 1]. \quad (37)$$

Now considering the left side of (36), we delete-contract the edge v_0v_3 in $C_4^1(1, a, b)$ and the edge v_2v_3 in $C_4^1(a, b)$ to see that

$$\begin{aligned}
C_4^1(1, a, b)^\circ - C_4^1(a, b)^\circ &= y \cdot (C_2(a-1)C_3(b) + C_3(a+1, b) - x \cdot C_3(a, b) - C_3(a+1, b)) \\
&= y \cdot (x \cdot C_2(a-1)C_2(b-1) + C_2(a-1)C_2(b) \\
&\quad - x \cdot C_1(a-1)C_2(b-1) - x \cdot C_2(a+b-1)) \\
&= y \cdot (C_2(a-1)C_2(b) - x \cdot C_2(a+b-1)) \\
&= y \cdot (x+y-xy) \cdot (b)_y \cdot [(a)_y - 1],
\end{aligned} \tag{38}$$

where the last line uses Identity A.5, recalling that $C_2(0) = x$. Consequently, (37) and (38) establish (36).

It remains to deal with the case where $a = b + 1$. We show that for each integer $a \geq 2$,

$$C_4^1(1, a, a-1)^\circ - C_4^1(a, a-1)^\circ = C_4^1(a, a) - C_4^1(a+1, a-1) + C_4^1(1, a+1, a-1) - C_4^1(1, a, 1, a). \tag{39}$$

Setting $n = 2a + 3$, this concludes the proof if $n \geq 8$, since then $C_4^1(a, a-1) \in \mathcal{C}(n-1, 3)$ and all the graphs appearing in the right side of (39) belong to $\mathcal{C}(n, 3)$. (We note that $C_4^1(1, 2, 1, 2)$ does not belong to $\mathcal{C}(7, 3)$, which justifies why the case $n = 7$ is dealt with separately.)

We observe that taking $b = a - 1$ in (36), and comparing the obtained equation to (39), it suffices to establish that

$$C_4^1(a, a) - C_4(a+1, a-1) = C_4^1(1, a, a) - C_4^1(1, a+1, a-1). \tag{40}$$

Identity A.8 yields that

$$\begin{aligned}
C_4^1(a, a) - C_4^1(a+1, a-1) &= (1+x) \cdot [(a)_y - (a-1)_y] - [(a)_y - (a-1)_y] \\
&= (1+x)y^{a-1} - y^a.
\end{aligned} \tag{41}$$

Moreover, Identity A.9 implies that

$$\begin{aligned}
C_4^1(1, a, a) - C_4^1(1, a+1, a-1) &= (1+x) \cdot [(a)_y - (a-1)_y] + [(a)_y - (a-1)_y] \\
&\quad - [(a)_y - (a-1)_y] - [(a)_y - (a+1)_y] \\
&= (1+x)y^{a-1} - y^a,
\end{aligned} \tag{42}$$

which, combined to (41) concludes the proof.

Finally, we now conclude by establishing the second statement, for which it is enough to show that

$$C_4^1(1, 2)^\circ - C_4(2, 2)^\circ = C_4^1(2, 2) - C_4^1(1, 3), \tag{43}$$

noting that $C_4(2, 2) \in \mathcal{C}_A(6, 3)$ while the graphs in the right side of (43) belong to $\mathcal{C}(7, 3)$.

Deleting-contracting the edge v_3v_1 in $C_4^1(1, 2)$, and the edge v_1v_2 in $C_4(2, 2)$ shows the left side of (43) to be equal to $y \cdot [C_2 \cdot C_2(2) - y \cdot C_3(2)]$, which by Identity A.7 is equal to $xy(x+y-xy)$. On the other hand, Identity A.9 applied with $s = 4, i = 1, c = 1$ and $a = b = 2$ yields that $C_4^1(2, 2) - C_4^1(1, 3) = xy \cdot (x+y-xy)$, which thus concludes the proof of the second statement. \square

The following handles the case where $r \geq 4$ (recalling that $\mathcal{C}_D(n-1, r)$ is then empty if $n = r+3$).

Relation B.8. Fix $r \geq 4$ and $n \geq r+4$. If $a \in \{[(n-r-2)/2], \dots, n-r-3\}$ and $b = n-1-r-a$, then

$$C_{r+1}^1(1, a, b)^\circ \equiv C_{r+1}^1(1, a+1, b-1)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. It suffices to prove that if $s \geq 5$, then for all integers $a \geq 1$ and $b \geq 2$,

$$\begin{aligned}
y \cdot (C_s^1(1, a, b) - C_s^1(1, a+1, b-1)) &= C_s(a+1, b+1) - C_s(a+2, b) \\
&\quad + S_s^1(b-1, a+2) - S_s^1(a+1, b) \\
&\quad + C_s^1(a+1, b) - C_s^1(a+2, b-1) \\
&\quad + C_s^1(1, a, b+1) - C_s^1(1, a+2, b-1) \\
&\quad + C_s^1(1, a+2, 1, b-1) - C_s^1(1, a, 1, b+1).
\end{aligned} \tag{44}$$

We observe that, setting $r = s - 1$ and $n = a + b + 1 + r$, if a satisfies the assumptions of the statement, so $C_s^1(1, a, b)$ belongs to $\mathcal{C}_D(n-1, r)$, then $C_s^1(1, a+1, b-1)$ belongs to $\mathcal{C}_D(n-1, r)$ if $b \geq 3$ and to $\mathcal{C}_C(n-1, r)$ when $b = 2$. Moreover, the graphs appearing in the right side of (44) then belong to $\mathcal{C}(n, r)$.

By Identity A.9 applied with $(i, c) = (1, 1)$, we see that

$$\begin{aligned}
\frac{C_s^1(1, a, b) - C_s^1(1, a+1, b-1)}{x+y-xy} &= (s-2)_x \cdot [(a)_y - (b-1)_y] + (s-3)_x \cdot [(a)_y - (b-1)_y] \\
&\quad - (b)_y + (b-1)_y - (s-3)_x \cdot [(a)_y - (a+1)_y] \\
&= (1+x \cdot (s-3)_x) \cdot [(a)_y - (b-1)_y] + (s-3)_x \cdot [(a)_y - (b-1)_y] \\
&\quad - (b)_y + (b-1)_y + y^a \cdot (s-3)_x \cdot [(a)_y - (a+1)_y] \\
&= (s-3)_x \cdot (x \cdot [(a)_y - (b-1)_y] + [(a)_y - (b-1)_y] + y^a) \\
&\quad + (a)_y - (b-1)_y - (b)_y + (b-1)_y \\
&= (a)_y - (b)_y + (s-3)_x \cdot [y^a + (1+x) \cdot [(a)_y - (b-1)_y]].
\end{aligned} \tag{45}$$

We now proceed with the five differences of two polynomials appearing on the right side of (44). First, by Identity A.6,

$$\begin{aligned}
\frac{C_s(a+1, b+1) - C_s(a+2, b)}{x+y-xy} &= (s-2)_x \cdot [(a+1)_y - (b)_y] \\
&= y \cdot (s-2)_x \cdot [(a)_y - (b-1)_y].
\end{aligned} \tag{46}$$

By Identity A.14,

$$\begin{aligned}
\frac{S_s^1(b+1, a+2) - S_s^1(a+1, b)}{x+y-xy} &= ((1+x) \cdot (s-3)_x + (s-4)_x) \cdot (a+2-b)_y \cdot [(b-1)_y - (b)_y] \\
&\quad + (b)_y - (a+2)_y + (s-4)_x \cdot [(a+1)_y - (b-1)_y] \\
&= ((1+x) \cdot (s-3)_x + (s-4)_x) \cdot (a+2-b)_y \cdot (-y^{b-1}) \\
&\quad + (b)_y - (a+2)_y + (s-4)_x \cdot [(a+1)_y - (b-1)_y] \\
&= -(1+x) \cdot (s-3)_x \cdot [(a+1)_y - (b-1)_y] \\
&\quad - (s-4)_x \cdot [(a+1)_y - (b-1)_y] \\
&\quad + (b)_y - (a+2)_y + (s-4)_x \cdot [(a+1)_y - (b-1)_y] \\
&= y \cdot [(1+x) \cdot (s-3)_x \cdot [(b-2)_y - (a)_y] + (b-1)_y - (a+1)_y].
\end{aligned} \tag{47}$$

Next, by Identity A.6,

$$\begin{aligned}
\frac{C_s^1(a+1, b) - C_s^1(a+2, b-1)}{x+y-xy} &= (s-2)_x \cdot [(a+1)_y - (b-1)_y] + (a+2)_y - (a+1)_y \\
&= y \cdot [(s-2)_x \cdot [(a)_y - (b-2)_y] + y^a].
\end{aligned} \tag{48}$$

Moreover, Identity A.9 implies that

$$\begin{aligned}
\frac{C_s^1(1, a, b+1) - C_s^1(1, a+2, b-1)}{x+y-xy} &= (s-2)_x \cdot (1+y) \cdot [(a)_y - (b-1)_y] \\
&\quad + (s-3)_x \cdot (1+y) \cdot [(a)_y - (b-1)_y] \\
&\quad + (b-1)_y - (b+1)_y - (s-3)_x \cdot [(a)_y - (a+2)_y] \\
&= y \cdot ((1+y) \cdot [(a-1)_y - (b-2)_y] \cdot [(s-2)_x + (s-3)_x] \\
&\quad - y^{b-1} - y^{b-2} + (s-3)_x \cdot (y^a + y^{a-1})),
\end{aligned} \tag{49}$$

and Identity A.11 implies that

$$\frac{C_s^1(1, a+2, 1, b-1) - C_s^1(1, a, 1, b+1)}{x+y-xy} = (s-2)_x \cdot (1+y) \cdot [(b-1)_y - (a)_y]. \tag{50}$$

Now, the right sides of (46) and (50) sum to

$$(s-2)_x \cdot [(b-1)_y - (a)_y] = y \cdot [(s-2)_x \cdot [(b-2)_y - (a-1)_y]],$$

so that adding the right side of (49) yields

$$\begin{aligned} & y \cdot ((s-2)_x \cdot (y \cdot [(a-1)_y - (b-2)_y]) + (s-3)_x \cdot (y^a + y^{a-1} + (1+y) \cdot [(a-1)_y - (b-2)_y]) \\ & \quad - y^{b-1} - y^{b-2}) \\ &= y \cdot ((s-2)_x \cdot (y \cdot [(a-1)_y - (b-2)_y]) + (s-3)_x \cdot (y^a + y^{a-1} + (a-1)_y - (b-2)_y + (a)_y - (b-1)_y) \\ & \quad - y^{b-1} - y^{b-2}). \end{aligned}$$

Further adding the right side of (47), we obtain

$$\begin{aligned} & y \cdot ((s-2)_x \cdot [(a)_y - (b-1)_y] + (s-3)_x \cdot (y^a + y^{a-1} + (a-1)_y - (b-1)_y + x \cdot [(b-2)_y - (a)_y]) \\ & \quad - y^{b-1} - y^{b-2} + (b-1)_y - (a+1)_y). \end{aligned}$$

Finally, considering (48) and using that $(s-2)_x = x \cdot (s-3)_x + 1$, we now deduce that the right side of (44), once divided by $(x+y-xy)$, is equal to

$$\begin{aligned} & y \cdot ((s-2)_x \cdot [2(a)_y - (b-1)_y - (b-2)_y] + (s-3)_x \cdot (y^a + (a)_y - (b-1)_y + x \cdot [(b-2)_y - (a)_y]) \\ & \quad - y^{b-1} - y^{b-2} + (b-1)_y - (a+1)_y + y^a) \\ &= y \cdot ((s-3)_x \cdot (y^a + (a)_y - (b-1)_y + x \cdot [(a)_y - (b-1)_y]) \\ & \quad - y^{b-1} - y^{b-2} + (b-1)_y - (a)_y + 2(a)_y - (b-1)_y - (b-2)_y) \\ &= y \cdot ((s-3)_x \cdot (y^a + (1+x) \cdot [(a)_y - (b-1)_y]) + (a)_y - (b)_y), \end{aligned}$$

which by (45) finishes the proof. \square

We reduce $\mathcal{C}_F(n-1, r)^\circ$ to $\mathcal{C}_B(n-1, r)^\circ$ when $r \geq 4$ and $n \geq r+5$, and to $\mathcal{C}_C(n-1, 3)^\circ$ when $n = r+4 \geq 8$ or $r = 3$ and $n \geq 8$, and to $\mathcal{C}_A(6, 3)^\circ$ when $r = 3$ and $n = 7$.

We recall that $\mathcal{C}_F(n-1, r) = \{C_{r+1}^+(n-r-3)\}$ for each $n \geq r+4$, while $\mathcal{C}_F(n-1, r)$ is empty if $n \leq r+3$. We deal separately with the two boundary cases, namely if $n = r+4$ first, and then if $r = 3 < n-4$.

Relation B.9. *If $r \geq 4$, then*

$$(C_{r+1}^+)^\circ \equiv C_{r+1}^1(1, 2)^\circ \pmod{\mathcal{C}(r+4, r)},$$

while

$$(C_4^+)^\circ \equiv C_4(2, 2)^\circ \pmod{\mathcal{C}(r+4, r)}.$$

Proof. Assume first that $r \geq 4$. Then it suffices to prove that

$$y \cdot (C_{r+1}^+ - C_{r+1}^1(1, 2)) = C_{r+1}^1(1, 2, 2) - C_{r+1}^1(1, 3). \quad (51)$$

Indeed $C_{r+1}^1(1, 2)$ belongs to $\mathcal{C}_C(r+3, r)$, while the graphs in the right side of (51) all belong to $\mathcal{C}(r+4, r)$ for $r \geq 4$ (but not if $r = 3$).

Deleting the edge $v_r v_2$ in C_{r+1}^+ yields a graph isomorphic to C_{r+1}^1 , so deleting-contracting this edge, and an edge between v_0 and v_1 in $C_4^1(1, 2)$ implies that

$$C_{r+1}^+ - C_{r+1}^1(1, 2) = C_r^1(1, 2) - yC_r(2). \quad (52)$$

On the other hand, deleting-contracting an edge between v_1 and v_2 in $C_{r+1}^1(1, 2, 2)$, and an edge between v_0 and v_1 in $C_{r+1}^1(1, 3)$ yields that

$$C_{r+1}^1(1, 2, 2) - C_{r+1}^1(1, 3) = yC_r^1(1, 2) - y^2C_r(2),$$

which by (52) concludes the proof of (51).

It remains to deal with the case where $r = 3$ and $n = 7$. It suffices to prove that

$$y \cdot (C_4^+ - C_4(2, 2)) = C_4^1(2, 2) - C_4^1(3) + C_4^1(2, 2) - C_4^1(1, 3). \quad (53)$$

Notice that $C_4(2, 2)$ belongs to $\mathcal{C}_A(6, 3)$, and all the graphs in the right side of (53) belong to $\mathcal{C}(7, 3)$.

Identity A.6 implies that

$$C_4^1(2, 2) - C_4^1(3) = y(x + y - xy)(1 + x + y),$$

and also that

$$\begin{aligned} C_4^1(2, 2) - C_4^1(1, 3) &= y(x + y - xy)x, \\ C_4^1(2, 2) - C_4^1(3) + C_4^1(2, 2) - C_4^1(1, 3) &= y(x + y - xy)(1 + 2x + y). \end{aligned} \quad (54)$$

On the other hand, deleting-contracting an edge in C_4^+ (which is isomorphic to the complete graph on 4 vertices), and the edge v_0v_3 in $C_4(2, 2)$ yields that

$$\begin{aligned} C_4^+ - C_4(2, 2) &= C_4^1 + C_3(2, 2) - xC_2C_2 - C_3(2, 2) \\ &= C_4 + (1 - x)C_2C_2 \\ &= 2x^2 + x + y + 2xy + y^2 - 2x^2y - xy^2 \\ &= 2x(x + y - xy) + y(1 + y - xy) + x \\ &= 2x(x + y - xy) + y(x + y - xy) + y - xy + x \\ &= (x + y - xy)(2x + y + 1), \end{aligned}$$

which by (54) concludes the proof. \square

We now tackle the case where $r = 3$ and $n \geq 8$.

Relation B.10. *If $n \geq 8$, then*

$$C_4^+(n - 6)^\circ \equiv C_4^1(n - 5)^\circ \pmod{\mathcal{C}(n, 3)}.$$

Proof. It suffices to establish that for every positive integer a ,

$$y \cdot (C_4^+(a) - C_4^1(a + 1)) = C_4^1(1, a + 1, 2) - C_4^1(1, a + 2) + C_4^1(1, a + 1, 2) - C_4^1(1, a + 1, 1, 2). \quad (55)$$

Indeed, if $n = a + 6$ then $C_4^1(a + 1) \in \mathcal{C}_C(n - 1, 3)$, and all the graphs appearing in the right side of (55) belong to $\mathcal{C}(n, 3)$.

We start by considering the left side of (55). Observe first that deleting the edge v_3v_1 in $C_4^+(a)$ yields a graph isomorphic to $C_4^1(1, a)$. Therefore, by deleting-contracting this edge in $C_4^+(a)$, and an edge between v_0 and v_1 in $C_4^1(a + 1)$, we deduce that

$$\begin{aligned} C_4^+(a) - C_4^1(a + 1) &= C_4^1(1, a) - C_4^1(a) + C_3(a + 1, 2) - y^a C_2C_2 \\ &= (x + y - xy) \cdot [(a)_y - 1] + C_3(a + 1, 2) - y^a C_2C_2, \end{aligned} \quad (56)$$

using Identity A.6 and the fact that $(t)_y - 1 = y(t - 1)_y$ for every positive integer t to obtain the second equality. Since $C_3(a + 1, 2) = C_2(2) \cdot (a + 1)_y + x \cdot C_2$ by repeatedly deleting-contracting an edge between v_0 and v_1 , and $C_2(2) = y^2 + C_2 = y^2 + (y + x)$, we have

$$\begin{aligned} C_3(a + 1, 2) - y^a C_2C_2 &= (y^2 + C_2) \cdot (a + 1)_y + xC_2 - y^a(x + y)^2 \\ &= y^2(a)_y - y^a x(x + y) - y^{a+1}x + (x + y) \cdot [(a + 1)_y + x] \\ &= y^{a+1}(1 - x) + y^2 \cdot (a - 1)_y + (x + y) \cdot [(a + 1)_y + x(1 - y^a)] \\ &= y^{a+1}(1 - x) + y^2 \cdot (a - 1)_y + (x + y) \cdot [1 + (a)_y \cdot (x + y - xy)], \end{aligned} \quad (57)$$

where the last equality uses that $1 - y^t = (1 - y) \cdot (t)_y$ for every non-negative integer t . The same identity allows us to deduce that

$$\begin{aligned} y^{a+1}(1 - x) + y^2 \cdot (a - 1)_y &= (1 - x) - (1 - x - y + xy) \cdot (a + 1)_y + (a + 1)_y - 1 - y \\ &= -(x + y) - (1 - x - y + xy) \cdot (a + 1)_y + (a + 1)_y \\ &= -(x + y) - (xy - x - y) \cdot (a + 1)_y, \end{aligned}$$

and consequently (56) and (57) yield that

$$\begin{aligned}
y \cdot (C_4^+(a) - C_4^1(a+1)) &= y \cdot \left((x+y-xy) \cdot [(a)_y - 1] - (x+y) - (xy-x-y) \cdot (a+1)_y \right. \\
&\quad \left. + (x+y) + (x+y)(x+y-xy) \cdot (a)_y \right) \\
&= y(x+y-xy) \cdot [(1+x+y) \cdot (a)_y + (a+1)_y - 1] \\
&= y(x+y-xy) \cdot (1+x+2y) \cdot (a)_y,
\end{aligned} \tag{58}$$

As for the right side of (55), first note that Identity A.9 and Identity A.1 imply that

$$\begin{aligned}
\frac{C_4^1(1, a+1, 2) - C_4^1(1, a+2)}{x+y-xy} &= (1+x) \cdot [(a+1)_y - 1] + [(a+1)_y - 1] - y + y^{a+1} \\
&= y[(2+x) \cdot (a)_y + y^a - 1] \\
&= y(1+x+y) \cdot (a)_y.
\end{aligned} \tag{59}$$

Second, Identity A.13 yields that

$$\frac{C_4^1(1, a+1, 2) - C_4^1(1, a+1, 1, 2)}{x+y-xy} = y^2(a)_y,$$

which, combined with (59), yields that the right side of (55) equals

$$y(x+y-xy)[(1+x+2y) \cdot (a)_y].$$

This establishes (55) thanks to (58). \square

It remains to deal with the case where $r \geq 4$.

Relation B.11. Fix $r \geq 4$. If $n \geq r+5$, then

$$C_{r+1}^+(n-r-3)^\circ \equiv S_{r+1}^1(n-r-2)^\circ \pmod{\mathcal{C}(n, r)},$$

while

$$(C_{r+1}^+)^\circ \equiv C_{r+1}(1, 2)^\circ \pmod{\mathcal{C}(r+4, 4)}.$$

Proof. For the second statement, it suffices to prove that for each integer $s \geq 5$,

$$y \cdot (C_s^+ - C_s^1(1, 2)) = C_s^1(1, 2, 2) - C_s^1(1, 3). \tag{60}$$

We note that if $r = s-1$, then $C_{r+1}^1(1, 2) \in \mathcal{C}_C(r+3, r)$, and the graphs in the right side of (60) belong to $\mathcal{C}(r+4, r)$.

By deleting-contracting the edge v_0v_2 in C_s^+ , and an edge between v_0 and v_1 in $C_s^1(1, 2)$, we see that

$$\begin{aligned}
C_s^+ - C_s^1(1, 2) &= C_s^1 + C_{s-1}^1(1, 2) - C_s^1 - yC_{s-1}(2) \\
&= C_{s-1}^1(1, 2) - yC_{s-1}(2).
\end{aligned} \tag{61}$$

On the other hand, by deleting-contracting an edge between v_1 and v_2 in $C_s^1(1, 2, 2)$, and an edge between v_0 and v_1 in $C_s^1(1, 3)$, we deduce that

$$\begin{aligned}
C_s^1(1, 2, 2) - C_s^1(1, 3) &= C_s^1(1, 2) + yC_{s-1}^1(1, 2) - C_s^1(1, 2) - y^2C_{s-1}(2) \\
&= yC_{s-1}^1(1, 2) - y^2C_{s-1}(2),
\end{aligned}$$

which by (61) establishes (60).

We now establish the first statement. It suffices to prove that for integers $s \geq 5$ and $a \geq 2$,

$$\begin{aligned}
y \cdot (C_s^+(a) - S_s^1(a+1)) &= C_s^1(a+1, 2) - C_s^1(a+2) + C_s^1(2, a+1) - C_s^1(3, a) \\
&\quad + C_s^1(1, a, 3) - C_s^1(1, a+1, 2).
\end{aligned} \tag{62}$$

Indeed, setting $r = s - 1$ and $n = a + r + 3 \geq r + 5$, we see that $S_s^1(a + 1) \in \mathcal{C}_B(n - 1, r)$, and all the graphs appearing in the right side of (62) belong to $\mathcal{C}(n, r)$. (This last assertion is not true if $n = r + 4$, which explains why this case was dealt with separately.)

First, we show that

$$\begin{aligned} & C_s^1(a + 1, 2) - C_s^1(a + 2) + C_s^1(2, a + 1) - C_s^1(3, a) + C_s^1(1, a, 3) - C_s^1(1, a + 1, 2) \\ &= (x + y - xy) \cdot y \cdot ((s - 3)_x \cdot [(1 + x) \cdot (a)_y - 1] + (a + 1)_y). \end{aligned} \quad (63)$$

We consider each of the three differences of two terms appearing in the left side of (63). First, Identity A.8 implies that

$$\begin{aligned} \frac{C_s^1(a + 1, 2) - C_s^1(a + 2)}{x + y - xy} &= (s - 2)_x \cdot [(a + 1)_y - 1] + (a + 2)_y - (a + 1)_y \\ &= y \cdot [(s - 2)_x \cdot (a)_y + y^a], \end{aligned} \quad (64)$$

and also that

$$\frac{C_s^1(2, a + 1) - C_s^1(3, a)}{x + y - xy} = (s - 2)_x \cdot [(2)_y - (a)_y] + y^2. \quad (65)$$

Moreover, Identity A.9 gives

$$\frac{C_s^1(1, a, 3) - C_s^1(1, a + 1, 2)}{x + y - xy} = (s - 2)_x \cdot [(a)_y - (2)_y] + (s - 3)_x \cdot [(a)_y - (2)_y] - y^2 + (s - 3)_x \cdot y^a. \quad (66)$$

Now, summing (65) and (66) yields a right side equal to

$$(s - 3)_x \cdot [(a + 1)_y - (2)_y] = y \cdot (s - 3)_x \cdot [(a)_y - 1].$$

Further adding (64), and using that $(s - 2)_x = x \cdot (s - 3)_x + 1$, we deduce that the left side of (63) is equal to

$$\begin{aligned} & y(x + y - xy) \cdot ((s - 3)_x \cdot [(a)_y - 1 + x \cdot (a)_y] + (a)_y + y^a) \\ &= y(x + y - xy) \cdot ((s - 3)_x \cdot [(1 + x) \cdot (a)_y - 1] + (a + 1)_y), \end{aligned} \quad (67)$$

as announced.

It remains to show that the expression (67) is equal to the left side of (62). We first note that by repeatedly deleting-contracting along the path from v_1 to v_{s-1} ,

$$C_s^+(a) - S_s^1(a + 1) = (s - 3)_x \cdot [C_2(a) \cdot C_2 - y^a \cdot C_3] + C_3(a + 1, 2) - C_2(a + 1) \cdot C_2, \quad (68)$$

recalling that $C_2(a) \cdot C_2$ stands for the (Tutte polynomial of the) the path v_0, v_1, v_2 where $v_0 v_1$ has multiplicity $a + 1$ and $v_1 v_2$ has multiplicity 2.

Let $A := (s - 3)_x \cdot [C_2(a) \cdot C_2 - y^a \cdot C_3]$ and $B := C_3(a + 1, 2) - C_2(a + 1) \cdot C_2$. Using Identity A.4, we observe that

$$\begin{aligned} B &= C_2(a + 3) - y^{a+1} \cdot C_2(2) \\ &= (a + 3)_y - 1 + x - y^{a+1} \cdot ((2)_y - 1 + x) \\ &= (x - 1) \cdot (1 - y^{a+1}) + (a + 3)_y - [(a + 3)_y - (a + 1)_y] \\ &= (x - 1) \cdot (1 - y) \cdot (a + 1)_y + (a + 1)_y \\ &= (a + 1)_y \cdot ((x - 1) \cdot (1 - y) + 1) \\ &= (x + y - xy) \cdot (a + 1)_y. \end{aligned} \quad (69)$$

Moreover, using again Identity A.4,

$$\begin{aligned} \frac{A}{(s - 3)_x} &= (x + y) \cdot [(a + 1)_y - 1 + x] - y^a \cdot [(3)_y - 1 + y] \\ &= (x + y) \cdot (x - 1) + x \cdot (a + 1)_y + y \cdot [(a + 1)_y - y^a] - y^a \cdot (x^2 + x) \\ &= (xy - x - y) + x^2 + x \cdot [(a + 1)_y - y^a] + y \cdot (a)_y - y^a \cdot x^2 \\ &= (xy - x - y) + (x + y) \cdot (a)_y + x^2 \cdot (1 - y^a) \\ &= (xy - x - y) + (x + y) \cdot (a)_y + x^2 \cdot (1 - y) \cdot (a)_y \\ &= (xy - x - y) + (a)_y \cdot (x + y + x^2(1 - y)) \\ &= (xy - x - y) + (a)_y \cdot (x + y - xy)(1 + x) \\ &= (x + y - xy) \cdot ((1 + x) \cdot (a)_y - 1). \end{aligned} \quad (70)$$

It thus follow from (68), (69) and (70) that

$$C_s^+(a)^\circ - S_s^1(a+1)^\circ = y \cdot (x+y-xy) \cdot [(a+1)_y + (s-3)_x \cdot [(1+x) \cdot (a)_y - 1]].$$

which by (63) concludes the proof. \square

We reduce $\mathcal{C}_C(n-1, r)^\circ$ to $\mathcal{C}_A(n-1, r)^\circ$.

We proceed in two steps: we first prove that it is enough to consider the elements $C_{r+1}^1(m_1, m_2) \in \mathcal{C}_C(n-1, r)^\circ$, and then we prove that each such element is equivalent to an element of $\mathcal{C}_A(n-1, r)^\circ$. By definition, if $G \in \mathcal{C}(n-1, r)$ with $r \in \{3, 4\}$, then G is isomorphic to $C_{r+1}^1(m_1, m_2)$ for some positive integers m_1 and m_2 . So the first step need only be done when $r \geq 5$.

Relation B.12. Fix integers $r \geq 5$ and $n \geq r+3$. For $i \in \{2, \dots, \lceil r/2 \rceil - 1\}$, if a and b are positive integers such that $C_{r+1}^i(a, b) \in \mathcal{C}_C(n-1, r)$, then

$$C_{r+1}^i(a, b)^\circ \equiv C_{r+1}^{i-1}(a, b)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. We first deal with the case where $i \geq 3$. It is enough to prove that for positive integers $a, b, s \geq 5$ and $i \in \{3, \dots, \lceil (s-1)/2 \rceil - 1\}$,

$$\begin{aligned} C_s^i(a, b)^\circ - C_s^{i-1}(a, b)^\circ &= \sum_{j=1}^b S_s^{i-1}(j, a+b+1-j) - S_s^{i-1}(a+b+1-j, j) \\ &\quad + \sum_{j=1}^b C_s^{i-1}(j, a+b+1-j) - C_s^{i-1}(a+b+1-j, j) \\ &\quad + \sum_{j=1}^b C_s^{i-2}(a+b+1-j, j) - C_s^{i-2}(j, a+b+1-j) \\ &\quad + S_s^{i-1}(a+b) - S_s^{i-1}(b, a+1) + S_s^{i-2}(b, a+1) - S_s^{i-2}(a+b) \\ &\quad + C_s^{i-2}(1, a+b) - C_s^{i-2}(a+b) + C_s^{i-2}(b, a+1) - C_s^{i-2}(a+1, b) \\ &\quad + C_s^{i-1}(a+b) - C_s^{i-1}(b, a+1) + C_s^i(a+1, b) - C_s^i(1, a+b). \end{aligned} \tag{71}$$

We note that if $r = s-1$ and $C_s^i(a, b) \in \mathcal{C}_C(n-1, r)$, so in particular $n = a+b+r+1$, then $C_s^{i-1}(a, b)$ belongs to $\mathcal{C}_C(n-1, r)$ and all the graphs in the right side of (71) belong to $\mathcal{C}(n, r)$.

By Identity A.10,

$$\frac{C_s^i(a, b) - C_s^{i-1}(a, b)}{x+y-xy} = (a)_y \cdot [(i-1)_x - (i)_x] + (b)_y \cdot [(s-1-i)_x - (i-1)_x]. \tag{72}$$

To ease the exposition, we write Σ_ℓ for the sum appearing in line ℓ of (71) divided by $(x+y-xy)$, so for example we have $(x+y-xy)\Sigma_3 = \sum_{j=1}^b C_s^{i-2}(a+b+1-j, j) - C_s^{i-2}(j, a+b+1-j)$.

By Identities A.15 and A.16,

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = y(a)_y(b)_y \cdot [(s-2-i)_x - (i-2)_x]. \tag{73}$$

Moreover, Identities A.2 and A.14 combined with the fact that $y^t(k)_y = (k+t)_y - (t)_y$ for non-negative integers k and t , yield that

$$\begin{aligned} \Sigma_4 &= -y(a)_y(b-1)_y \cdot [(1+x)(s-3)_x + (i-1)_x(s-2-i)_x] + y(a)_y \cdot (i-1)_x - y^b(a)_y(s-2-i)_x \\ &\quad + y(a)_y(b-1)_y \cdot [(1+x)(s-3)_x + (i-2)_x(s-1-i)_x] \\ &\quad - y(a)_y(i-2)_x + y^b(a)_y(s-1-i)_x \\ &= y(a)_y(b-1)_y \cdot [(i-2)_x(s-1-i)_x - (i-1)_x(s-2-i)_x] \\ &\quad + y(a)_y \cdot [(i-1)_x - (i-2)_x + y^{b-1}(s-1-i)_x - y^{b-1}(s-2-i)_x] \\ &= y(a)_y(b-1)_y \cdot [(i-2)_x - (s-2-i)_x] \\ &\quad + y(a)_y \cdot [(i-1)_x - (i-2)_x + y^{b-1}(s-1-i)_x - y^{b-1}(s-2-i)_x] \\ &= y(a)_y(b-1)_y(i-2)_x - y(a)_y(b)_y(s-2-i)_x \\ &\quad + y(a)_y \cdot [(i-1)_x - (i-2)_x + y^{b-1}(s-1-i)_x]. \end{aligned} \tag{74}$$

Further, Identities A.2 and A.8 imply that

$$\begin{aligned}\Sigma_5 &= (i-2)_x \cdot [(a+b)_y - 1 - (b)_y + (a+1)_y] \\ &= y(a)_y(i-2)_x + y^b(a)_y(i-2)_x,\end{aligned}$$

so that by (74)

$$\begin{aligned}\Sigma_4 + \Sigma_5 &= y(a)_y(b)_y(i-2)_x - y(a)_y(b)_y(s-2-i)_x \\ &\quad + y(a)_y \cdot [(i-1)_x + y^{b-1}(s-1-i)_x],\end{aligned}$$

and thus by (73)

$$\Sigma_1 + \dots + \Sigma_5 = y(a)_y(i-1)_x + y^b(a)_y(s-1-i)_x. \quad (75)$$

Finally, Identity A.8, combined with Identity A.2, implies that

$$\begin{aligned}\Sigma_6 &= (s-i)_x(i-1)_x(a)_y \cdot [1 - (b)_y] - (i-1)_x \cdot [(a+b)_y - (b)_y] \\ &\quad + (a)_y(i)_x(s-1-i)_x \cdot [(b)_y - 1] - (i)_x \cdot [(a+1)_y - 1] \\ &= y(a)_y(b-1)_y \cdot [(i)_x(s-1-i)_x - (i-1)_x(s-i)_x] \\ &\quad - y^b(a)_y(i-1)_x - y(a)y(i)_x \\ &= y(a)_y(b-1)_y \cdot [(s-1-i)_x - (i-1)_x] - y^b(a)_y(i-1)_x - y(a)y(i)_x \\ &= y(a)_y(b-1)_y(s-1-i)_x - y(a)_y(b)_y(i-1)_x - y(a)y(i)_x,\end{aligned}$$

which by (75) yields that

$$\Sigma_1 + \dots + \Sigma_6 = y(a)_y(b)_y \cdot [(s-1-i)_x - (i-1)_x] + y(a)_y \cdot [(i-1)_x - (i)_x].$$

It remains to deal with the case where $i = 2$. We split the analysis according as $a \geq b$ or $a < b$. First, we show that for all integers $a \geq b \geq 1$ and $s \geq 6$,

$$\begin{aligned}C_s^2(a, b)^\circ - C_s^1(a, b)^\circ &= \sum_{j=1}^b S_s^1(j, a+b+1-j) - S_s^1(a+b+1-j, j) \\ &\quad + S_s^1(a+b) - S_s^1(b, a+1) \\ &\quad + C_s^1(1, a, b+1) - C_s^1(b+1, a) + C_s^2(a+1, b) - C_s^2(1, a+b).\end{aligned} \quad (76)$$

We note that if $r = s-1$ and $n = a+b+r+1$, then $C_s^1(a, b)$ belongs to $\mathcal{C}_C(n-1, r)$ and all the graphs in the right side of (76) belong to $\mathcal{C}(n, r)$. (This would not be the case if $a < b$, as then $C_s^1(1, a, b+1) \notin \mathcal{C}(n, r)$.)

Identity A.10 implies that,

$$\frac{C_s^2(a, b) - C_s^1(a, b)}{x+y-xy} = (a)_y \cdot (-x + (b)_y \cdot [(s-3)_x - 1]), \quad (77)$$

and Identity A.8 that

$$\begin{aligned}\frac{C_s^2(a+1, b) - C_s^2(1, a+b)}{x+y-xy} &= (a)_y(2)_x(s-3)_x \cdot [(b)_y - 1] - (2)_x \cdot [(a+1)_y - 1] \\ &= y(a)_y \cdot [(b-1)_y \cdot (1+x)(s-3)_x - (1+x)].\end{aligned} \quad (78)$$

By Identity A.16,

$$\frac{\sum_{j=1}^b S_s^1(j, a+b+1-j) - S_s^1(a+b+1-j, j)}{x+y-xy} = y(a)_y(b)_y \cdot [(s-4)_x - 1]. \quad (79)$$

Moreover, using twice that $y^t(k)_y = (k+t)_y - (t)_y$ for non-negative integers k and t , Identity A.14 implies that

$$\frac{S_s^1(a+b) - S_s^1(b, a+1)}{x+y-xy} = -y(a)_y(b-1)_y \cdot [(1+x)(s-3)_x + (s-4)_x] + y(a)_y - y^b(a)_y(s-4)_x. \quad (80)$$

By deleting-contracting the edge $v_0 v_{s-1}$ in each of $C_s^1(1, a, b+1)$ and $C_s^1(b+1, a)$, we see that

$$\begin{aligned} C_s^1(1, a, b+1) - C_s^1(b+1, a) &= x^a C_{s-1}(b+1) + C_{s-1}(a+1, b+1) - x^a C_{s-1}(b+1) - C_{s-1}(a+b+1) \\ &= C_{s-1}(a+1, b+1) - C_{s-1}(a+b+1) \\ &= y(x+y-xy) \cdot (a)_y \cdot (b)_y \cdot (s-3)_x, \end{aligned} \quad (81)$$

where the last equality follows from Identity A.6.

Summing the right sides of (78)–(81), all divided by $y(a)_y(x+y-xy)$ yields that the right side of (76) equals

$$\begin{aligned} &(b-1)_y(s-3)_x(1+x) - (1+x) + (b)_y \cdot [(s-4)_x - 1] \\ &\quad - (b-1)_y \cdot [(1+x)(s-3)_x + (s-4)_x] + 1 - y^{b-1}(s-4)_x + (b)_y(s-3)_x \\ &= -x + (b)_y \cdot [(s-4)_x - 1] - (b)_y(s-4)_x + (b)_y(s-3)_x \\ &= -x - (b)_y + (b)_y(s-3)_x, \end{aligned}$$

which by (77) establishes (76).

To conclude, we observe that our previous computations directly imply that for all integers $b > a \geq 1$ and $s \geq 6$,

$$\begin{aligned} C_s^2(a, b)^\circ - C_s^1(a, b)^\circ &= \sum_{j=1}^a S_s^1(j, a+b+1-j) - S_s^1(a+b+1-j, j) \\ &\quad + S_s^1(a+b) - S_s^1(b, a+1) \\ &\quad + C_s^1(1, b, a+1) - C_s^1(a+1, b) + C_s^2(a+1, b) - C_s^2(1, a+b). \end{aligned} \quad (82)$$

We note that if $r = s-1$ and $n = a+b+r+1$, then $C_s^1(a, b)$ belongs to $\mathcal{C}_C(n-1, r)$ and all the graphs in the right side of (82) belong to $\mathcal{C}(n, r)$. (This would not be the case if $a > b$, as then $C_s^1(1, b, a+1) \notin \mathcal{C}(n, r)$.) \square

We now proceed with the second step.

Relation B.13. Fix $r \geq 3$ and $n \geq r+3$. If a and b are positive integers such that $C_{r+1}^1(a, b) \in \mathcal{C}_C(n-1, r)$, then

$$C_{r+1}^1(a, b)^\circ \equiv C_{r+1}(a, b+1)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. It is enough to prove the following identity for $s \geq 4$ and positive integers a and b .

$$\begin{aligned} C_s^1(a, b)^\circ - C_s(a, b+1)^\circ &= C_s(a+b, 2) - C_s(a, b+2) + C_s^1(a+1, b) - C_s^1(1, a+b) \\ &\quad + C_s^1(1, a, 1, b+1) - C_s^1(1, a+1, 1, b). \end{aligned} \quad (83)$$

We note that if $C_s^1(a, b) \in \mathcal{C}_C(n-1, r)$, so in particular $r = s-1$ and $n = s+a+b$, then $C_s(a, b+1) \in \mathcal{C}_A(n-1, r)$ and all the graphs in the right side of (83) belong to $\mathcal{C}(n, r)$.

By repeatedly deleting-contracting the edges along the path $v_1 \dots v_{s-1}$, we see that

$$C_s^1(a, b) = (s-2)_x \cdot C_3(a, b) + y^a \cdot C_2(b+1)$$

and

$$C_s(a, b+1) = (s-2)_x \cdot C_2(a-1) \cdot C_2(b) - C_2(a+b+1),$$

where $C_2(0) \cdot C_2(b)$ is defined to be the graph obtained from $C_2(b)$ by adding a new vertex linked to v_0 .

Consequently, the left side of (83) equals

$$\begin{aligned}
& (s-2)_x \cdot [C_3(a, b) - C_2(a-1) \cdot C_2(b)] + y^a \cdot [(b+1)_y - 1 + x] - (a+b+1)_y + 1 - x \\
&= (s-2)_x \cdot [C_2(a+b) - y^b \cdot C_2(a)] + (1-x) \cdot (1-y^a) + (a+b+1)_y - (a)_y - (a+b+1)_y \\
&= (s-2)_x \cdot \left((a+b)_y - 1 + x - y^b \cdot [(a)_y - 1 + x] \right) + (1-x) \cdot (1-y^a) \\
&\quad + (a+b+1)_y - (a)_y - (a+b+1)_y \\
&= (s-2)_x \cdot \left((x-1) \cdot (1-y^b) + (a+b)_y - (a+b)_y + (b)_y \right) + (1-x)(1-y) \cdot (a)_y - (a)_y \\
&= (s-2)_x \cdot \left((x-1)(1-y) \cdot (b)_y + (b)_y \right) + (1-x)(1-y) \cdot (a)_y - (a)_y \\
&= (s-2)_x \cdot (x+y-xy) \cdot (b)_y + (xy-x-y) \cdot (a)_y \\
&= (x+y-xy) \cdot [(s-2)_x \cdot (b)_y - (a)_y].
\end{aligned} \tag{84}$$

We consider now the differences appearing in the right side of (83). First, Identity A.6 yields that

$$\frac{C_s(a+b, 2) - C_s(a, b+2)}{x+y-xy} = (s-2)_x \cdot (b)_y \cdot [(2)_y - (a)_y]. \tag{85}$$

Next, Identity A.8 implies that

$$\begin{aligned}
\frac{C_s^1(a+1, b) - C_s^1(1, a+b)}{x+y-xy} &= (s-2)_x \cdot (a)_y \cdot [(b)_y - 1] - [(a+1)_y - 1] \\
&= (s-2)_x \cdot (a)_y \cdot [(b)_y - 1] - y(a)_y.
\end{aligned} \tag{86}$$

Last, Identity A.11 implies that

$$\frac{C_s^1(1, a, 1, b+1) - C_s^1(1, a+1, 1, b)}{x+y-xy} = (s-2)_x \cdot [(a)_y - (b)_y]. \tag{87}$$

Consequently, the right side of the sum of (85), (86) and (87) is

$$\begin{aligned}
& -y(a)_y + (s-2)_x \cdot \left((b)_y \cdot [(2)_y - (a)_y] + (a)_y \cdot [(b)_y - 1] + (a)_y - (b)_y \right) \\
&= -y(a)_y + (s-2)_x \cdot \left((1+y)(b)_y - (b)_y(a)_y + (a)_y(b)_y - (a)_y + (a)_y - (b)_y \right) \\
&= -y(a)_y + (s-2)_x \cdot (y \cdot (b)_y),
\end{aligned}$$

which by (84) concludes the proof. \square

We reduce $\mathcal{C}_B(n-1, r)^\circ$ to $\mathcal{C}_C(n-1, r)^\circ$.

Recall that $\mathcal{C}_B(n-1, r)$ is empty if $r = 3$, so we assume that $r \geq 4$. Each graph in $\mathcal{C}_B(n-1, r)$ is isomorphic to $S_{r+1}^i(a, b)$ for some positive integers i, a, b , and it will be reduced to $C_{r+1}^i(b, a)$. However, due to different behaviours in a number of “boundary cases” (with respect with the values of the parameter i , and also to edges multiplicities), we have to prove a number of different identities. This is why the reduction is split into several cases.

We now observe once and for all that if $S_{r+1}^i(a, b)$ belongs to $\mathcal{C}_B(n-1, r)$, then $C_{r+1}^i(b, a)$ belongs to $\mathcal{C}_C(n-1, r)$.

We first assume that $i \geq 2$, which implies that $r \geq 6$, since $i \leq \lfloor r/2 \rfloor - 1$ by definition. We start with the case $b = 1$.

Relation B.14. Fix $r \geq 6$ and $n \geq r+3$. For $i \in \{2, \dots, \lfloor r/2 \rfloor - 1\}$ and every positive integer a ,

$$S_{r+1}^i(a) \equiv C_{r+1}^i(1, a) \pmod{\mathcal{C}(n, r)}.$$

Proof. We use two different identities, regarding whether i is 2 or at least 3. We note once and for all that if $S_s^i(a)$ belongs to $\mathcal{C}_B(n-1, r)$, then $r = s-1$ and $n = r+a+2$. The left side will always be $S_s^i(a)^\circ - C_s^i(1, a)^\circ$, and we note, using Identity A.17, that

$$S_s^i(a) - C_s^i(1, a) = (a)_y \cdot x(s-3-i)_x. \quad (88)$$

We start with the case where $i = 2$, which is a direct application of Identity A.14. Specifically, we prove that for each $s \geq 7$,

$$S_s^2(a)^\circ - C_s^2(1, a)^\circ = S_s^1(1, a+1) - S_s^1(a+1). \quad (89)$$

We note that if $S_s^2(2) \in \mathcal{C}(n-1, r)$, then $r = s-1 \geq 6$, and all the graphs in the right side of (89) belong to $\mathcal{C}(n, r)$.

Identity A.14 yields that

$$\begin{aligned} \frac{S_s^1(1, a+1) - S_s^1(a+1)}{x+y-xy} &= 1 - (a+1)_y + (s-4)_x \cdot [(a+1)_y - 1] \\ &= y(a)_y \cdot x(s-5)_x, \end{aligned}$$

which by (88) proves (89).

It remains to deal with the case where $i \geq 3$. We show that if $s \geq 9$ and $i \in \{3, \dots, \lfloor (s-1)/2 \rfloor - 1\}$, then

$$\begin{aligned} S_s^i(a)^\circ - C_s^i(1, a)^\circ &= S_s^{i-1}(1, a+1) - S_s^{i-1}(a+1) \\ &\quad + C_s^1(a+1) - C_s^1(1, a+1) + C_s^{i-1}(1, a+1) - C_s^{i-1}(a+1). \end{aligned} \quad (90)$$

We note that if $S_s^i(a) \in \mathcal{C}(n-1, r)$, then all the graphs in the right side of (90) belong to $\mathcal{C}(n, r)$. By Identity A.14,

$$\begin{aligned} \frac{S_s^{i-1}(1, a+1) - S_s^{i-1}(a+1)}{x+y-xy} &= -(i-1)_x \cdot [(a+1)_y - 1] + (s-2-i)_x \cdot [(a+1)_y - 1] \\ &= y(a)_y \cdot [(s-2-i)_x - (i-1)_x]. \end{aligned} \quad (91)$$

Moreover, Identity A.8 (applied twice) implies that

$$\begin{aligned} \frac{C_s^1(a+1) - C_s^1(1, a+1) + C_s^{i-1}(1, a+1) - C_s^{i-1}(a+1)}{x+y-xy} &= -[(a+1)_y - 1] + (i-1)_x \cdot [(a+1)_y - 1] \\ &= y(a)_y \cdot [(i-1)_x - 1]. \end{aligned} \quad (92)$$

Consequently, summing (91) and (92) yields (90), since $(s-2-i)_x - 1 = x(s-3-i)_x$. This ends the proof. \square

We now deal with the case where $2 \leq i \leq \lfloor r/2 \rfloor - 2$ and $b \geq 2$.

Relation B.15. For each $r \geq 6$ and $n \geq r+3$. If $2 \leq i \leq \lfloor r/2 \rfloor - 2$, then for positive integers a and $b \geq 2$ such that $S_s^i(a, b) \in \mathcal{C}_B(n-1, r)$,

$$S_s^i(a, b)^\circ \equiv C_s^i(b, a)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. We prove the following identity for $s \geq 7$, positive integers $a, b \geq 2$, and $i \in \{2, \dots, \lfloor (s-1)/2 \rfloor - 1\}$.

$$\begin{aligned}
S_s^i(a, b)^\circ - C_s^i(b, a)^\circ &= \sum_{j=1}^b S_s^{i-1}(j, a+b+1-j) - S_s^{i-1}(a+b+1-j, j) \\
&\quad + 2 \sum_{j=1}^b S_s^i(j, a+b+1-j) - S_s^i(a+b+1-j, j) \\
&\quad + \sum_{j=1}^b S_s^{i+1}(j, a+b+1-j) - S_s^{i+1}(a+b+1-j, j) \\
&\quad + 2 \sum_{j=1}^b C_s^1(a+b+1-j, j) - C_s^1(j, a+b+1-j) \\
&\quad + \sum_{j=1}^b C_s^{i-1}(j, a+b+1-j) - C_s^{i-1}(a+b+1-j, j) \\
&\quad + 2 \sum_{j=1}^b C_s^i(j, a+b+1-j) - C_s^i(a+b+1-j, j) \\
&\quad + \sum_{j=1}^b C_s^{i+1}(j, a+b+1-j) - C_s^{i+1}(a+b+1-j, j) \\
&\quad + S_s^i(a+b) - S_s^i(1, a+b) + S_s^i(a+1, b) - S_s^i(b, a+1) \\
&\quad + C_s^1(1, a+b) - C_s^1(a+b) + C_s^1(b, a+1) - C_s^1(1, b, 1, a+1) \\
&\quad + C_s^i(a+b) - C_s^i(1, a+b) + C_s^i(a+1, b) - C_s^i(b, a+1) \\
&\quad + S_s^{i+1}(a+b) - S_s^{i+1}(b, a+1) + C_s^{i+1}(a+1, b) - C_s^{i+1}(1, a+b).
\end{aligned} \tag{93}$$

We note that if $S_s^i(a, b) \in \mathcal{C}_B(n-1, r)$, so in particular $r = s-1$ and $n = r + a + b + 1$, then all the graphs in the right side of (93) belong to $\mathcal{C}(n, r)$. (We recall that the graphs $C_s^1(1, b, 1, a+1)$ and $C_s^1(1, a+1, 1, b)$ are isomorphic.)

To ease the exposition, let Σ_ℓ denote the sum appearing in line ℓ of the right side of (93) divided by $(x+y-xy)$; so we have, for example, $(x+y-xy) \cdot \Sigma_3 = \sum_{j=1}^b S_s^{i+1}(j, a+b+1-j) - S_s^{i+1}(a+b+1-j, j)$.

Using Identities A.8 and A.14–A.16, we obtain the following expressions

$$\begin{aligned}
\Sigma_1 + \dots + \Sigma_7 &= y(a)_y(b)_y \cdot [(s-2-i)_x + 2(s-3-i)_x + (s-4-i)_x - 2], \\
\Sigma_8 &= (y(a)_y + y^b(a)_y) \cdot [(i)_x - (s-3-i)_x], \\
\Sigma_9 &= (a+b)_y - 1 - y(b-1)_y = (a+b)_y - (b)_y = y^b(a)_y, \\
\Sigma_{10} &= -(i)_x \cdot [y(a)_y + y^b(a)], \\
\Sigma_{11} &= -y(a)_y(b)_y(s-4-i)_x - y(a)_y(b-1)_y(s-3-i)_x + y(a)_y(b-1)_y.
\end{aligned}$$

Therefore,

$$\Sigma_8 + \Sigma_{10} = -(s-3-i)_x \cdot [y^b(a)_y + y(a)_y],$$

so

$$\Sigma_8 + \Sigma_{10} + \Sigma_{11} = -y(a)_y(b)_y \cdot [(s-4-i)_x + (s-3-i)_x] + y(a)_y(b-1)_y - y(a)_y(s-3-i)_x,$$

and hence

$$\Sigma_8 + \dots + \Sigma_{11} = -y(a)_y(b)_y(s-4-i)_x - y(a)_y(b)_y(s-3-i)_x - y(a)_y(s-3-i)_x + y(a)_y(b)_y.$$

It thus follows that

$$\begin{aligned}
\Sigma_1 + \dots + \Sigma_{11} &= y(a)_y(b)_y(s-3-i)_x + y(a)_y(b)_y(s-2-i)_x - y(a)_y(b)_y - y(a)_y(s-3-i)_x \\
&= y(a)_y(b)_y \cdot [(s-2-i)_x - 1 + (s-3-i)_x] - y(a)_y(s-3-i)_x \\
&= y(a)_y(b)_y \cdot [(1+x)(s-3-i)_x] - y(a)_y(s-3-i)_x \\
&= y(a)_y(s-3-i)_x \cdot [(1+x)(b)_y - 1],
\end{aligned}$$

which by Identity A.17 ends the proof of (93).

We now turn to the case where $a = 1 = b$. Specifically, we prove that for each $i \in \{2, \dots, \lfloor r/2 \rfloor - 1\}$,

$$\begin{aligned} y \cdot (S_s^i - C_s^i) &= S_s^{i-1}(1, 2) - S_s^{i-1}(2) + C_s^1(2) - C_s^1(1, 2) \\ &\quad + C_s^{i-1}(1, 2) - C_s^{i-1}(2). \end{aligned} \quad (94)$$

We note that if $r = s - 1$, then $C_s^i \in \mathcal{C}_C(r + 2, r)$ and all the graphs appearing in the right side of (94) belong to $\mathcal{C}(r + 3, r)$.

By Identity A.17, we have

$$S_s^i - C_s^i = x(x + y - xy) \cdot (s - 3 - i)_x. \quad (95)$$

On the other hand, Identities A.6 and A.14 imply that

$$\begin{aligned} \frac{S_s^{i-1}(1, 2) - S_s^{i-1}(2) + C_s^1(2) - C_s^1(1, 2) + C_s^{i-1}(1, 2) - C_s^{i-1}(2)}{x + y - xy} &= [-y \cdot (i - 1)_x + y(s - 2 - i)_x] \\ &\quad + [1 - (2)_y] + [(i - 1)_x \cdot [(2)_y - 1]] \\ &= y \cdot [(s - 2 - i)_x - 1] \\ &= yx \cdot (s - 3 - i)_x, \end{aligned}$$

which by (95) establishes (94). \square

We now take care of the case where $i = \lfloor r/2 \rfloor - 1$ and $b \geq 2$.

Relation B.16. For each $r \geq 6$ and $n \geq r + 3$. If $i = \lfloor r/2 \rfloor - 1$, then for positive integers a and $b \geq 2$ such that $S_s^i(a, b) \in \mathcal{C}_B(n - 1, r)$,

$$S_s^i(a, b)^\circ \equiv C_s^i(b, a)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. Fix $s \geq 7$ and $i = \lfloor (s - 1)/2 \rfloor - 1$. We first prove that if s is odd, then

$$\begin{aligned} y \cdot (S_s^i(a, b) - C_s^i(b, a)) &= 2 \sum_{j=1}^b S_s^{i-1}(j, a + b + 1 - j) - S_s^{i-1}(a + b + 1 - j, j) \\ &\quad + 2 \sum_{j=1}^b C_s^1(a + b + 1 - j, j) - C_s^1(j, a + b + 1 - j) \\ &\quad + 3 \sum_{j=1}^b C_s^{i-1}(j, a + b + 1 - j) - C_s^{i-1}(a + b + 1 - j, j) \\ &\quad + \sum_{j=1}^b C_s^i(j, a + b + 1 - j) - C_s^i(a + b + 1 - j, j) \\ &\quad + S_s^{i-1}(a + b) - S_s^{i-1}(b, a + 1) \\ &\quad + C_s^1(1, a + b) - C_s^1(a + b) \\ &\quad + C_s^{i-1}(a + b) - C_s^{i-1}(1, a + b) + C_s^{i-1}(a + 1, b) - C_s^{i-1}(b, a + 1) \\ &\quad + C_s^i(a + 1, b) - C_s^i(1, a + b) + C_s^1(b, a + 1) - C_s^1(1, b, 1, a + 1). \end{aligned} \quad (96)$$

We note that if $S_s^i(a, b) \in \mathcal{C}_B(n - 1, r)$, so in particular $r = s - 1$ and $n = r + a + b + 1$, then all the graphs appearing in the right side of (96) belong to $\mathcal{C}(n, r)$.

We write Σ_ℓ for the sum appearing in line ℓ of the right side of (96) divided by $(x + y - xy)$. Using Identities A.8 and A.13–A.16, we obtain the following expressions.

$$\begin{aligned} \Sigma_1 + \dots + \Sigma_4 &= y(a)_y(b)_y \cdot [(i)_x + (i - 1)_x + 2x(s - 3 - i)_x], \\ \Sigma_5 &= y(a)_y \cdot \left((b - 1)_y \cdot [-(2)_x(s - 3)_x - (i - 1)_x(s - 2 - i)_x] + (i - 1)_x - y^{b-1}(s - 2 - i)_x \right), \\ \Sigma_6 &= y(a + b - 1)_y = y^b(a)_y + y(b - 1)_y, \\ \Sigma_7 &= (i - 1)_x \cdot [1 - (a + b)_y + (b)_y - (a + 1)_y] = -y(a)_y(i - 1)_x \cdot [1 + y^{b-1}], \\ \Sigma_8 &= y(a)_y \cdot [(s - 1 - i)_x(i)_x(b - 1)_y - (i)_x] - y(b - 1)_y. \end{aligned}$$

We deduce that

$$\Sigma_6 + \Sigma_8 = y(a)_y \cdot [(s-1-i)_x(i)_x(b-1)_y - (i)_x + y^{b-1}],$$

so that, using Identity A.2,

$$\begin{aligned} \frac{\Sigma_5 + \Sigma_6 + \Sigma_8}{y(a)_y} &= (b-1)_y \cdot [(s-1-i)_x(i)_x - (2)_x(s-3)_x - (i-1)_x(s-2-i)_x] \\ &\quad - (i)_x + (i-1)_x + y^{b-1} - y^{b-1}(s-2-i)_x, \\ &= (b-1)_y \cdot [(s-3-i)_x \cdot [(i)_x - (2)_x] - (i-1)_x(s-2-i)_x] \\ &\quad - (i)_x + (i-1)_x + y^{b-1} - y^{b-1}(s-2-i)_x, \\ &= (b-1)_y \cdot [-(2)_x(s-3-i)_x + (s-3-i)_x - (i-1)_x] \\ &\quad - (i)_x + (i-1)_x - y^{b-1}x(s-3-i)_x \\ &= -(b-1)_y \cdot [x(s-3-i)_x + (i-1)_x] \\ &\quad - (i)_x + (i-1)_x - y^{b-1}x(s-3-i)_x. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\Sigma_5 + \dots + \Sigma_8}{y(a)_y} &= -(b-1)_y \cdot [x(s-3-i)_x + (i-1)_x] \\ &\quad - (i)_x - y^{b-1} \cdot [x(s-3-i)_x + (i-1)_x] \\ &= -(b)_y \cdot [x(s-3-i)_x + (i-1)_x] - (i)_x, \end{aligned}$$

so that,

$$\begin{aligned} \frac{\Sigma_1 + \dots + \Sigma_8}{y(a)_y} &= (b)_y \cdot [x(s-3-i)_x + (i)_x] - (i)_x \\ &= x(b)_y(s-3-i)_x + (i)_x \cdot [(b)_y - 1], \end{aligned}$$

which by Identity A.17 establishes (96) because $s-3-i = i$ as s is odd and $i = \lfloor (s-1)/2 \rfloor - 1$ by definition.

We now prove that if s is even and $1 \leq a < b$, then

$$\begin{aligned} y \cdot (S_s^i(a, b) - C_s^i(b, a)) &= \sum_{j=1}^a S_s^{i-1}(j, a+b+1-j) - S_s^{i-1}(a+b+1-j, j) \\ &\quad + 2 \sum_{j=1}^a S_s^i(j, a+b+1-j) - S_s^i(a+b+1-j, j) \\ &\quad + 2 \sum_{j=1}^a C_s^1(a+b+1-j, j) - C_s^1(j, a+b+1-j) \\ &\quad + \sum_{j=1}^a C_s^{i-1}(j, a+b+1-j) - C_s^{i-1}(a+b+1-j, j) \\ &\quad + 3 \sum_{j=1}^a C_s^i(j, a+b+1-j) - C_s^i(a+b+1-j, j) \\ &\quad + S_s^i(a+1, b) - S_s^i(1, a+b) + 2(S_s^i(a+b) - S_s^i(b, a+1)) \\ &\quad + C_s^1(1, a+b) - C_s^1(a+b) + C_s^1(b, a+1) - C_s^1(1, b, 1, a+1) \\ &\quad + 2(C_s^i(a+b) - C_s^i(1, a+b) + C_s^i(a+1, b) - C_s^i(b, a+1)) \\ &\quad + C_s^{i+1}(b, a+1) - C_s^{i+1}(a+b). \end{aligned} \tag{97}$$

We note that if $S_s^i(a, b)$ belongs to $\mathcal{C}_B(n-1, r)$, so in particular $r = s-1$ and $n = r + a + b + 1$, then all the graphs in the right side of (97) belong to $\mathcal{C}(n, r)$.

Similarly as before, we write Σ_ℓ for the sum appearing in line ℓ of the right side of (97) divided by $(x + y - xy)$. Using Identities A.8 and A.13–A.16, we obtain the following expressions.

$$\begin{aligned}
\Sigma_1 + \dots + \Sigma_5 &= y(a)_y(b)_y \cdot [(s-2-i)_x + 2(s-3-i)_x - 2 + (i)_x], \\
\Sigma_6 &= -y(a)_y(b-1)_y \cdot [(2)_x(s-3)_x + (i)_x(s-3-i)_x] + (i)_x \cdot [2(a+1)_y - 2 + (a+b)_y - (b)_y] \\
&\quad + (s-3-i)_x \cdot [2(b)_y - 2(a+b)_y + 1 - (a+1)_y], \\
\Sigma_7 &= (a+b)_y - 1 - y(b-1)_y = y^b(a)_y + (b)_y - (b)_y = y^b(a)_y, \\
\Sigma_8 &= 2(i)_x \cdot [(b)_y - (a+1)_y - (a+b)_y + 1], \\
\Sigma_9 &= y(a)_y(b-1)_y(s-2-i)_x(i+1)_x + y^b(a)_y(i+1)_x.
\end{aligned}$$

Consequently, using A.2 and recalling that $s-3-i = i+1$ by definition, we infer that

$$\begin{aligned}
\Sigma_6 + \dots + \Sigma_9 &= y(a)_y(b-1)_y \cdot [-(2)_x(s-3)_x - (i)_x(s-3-i)_x + (s-2-i)_x(i+1)_x] \\
&\quad + (i)_x \cdot [(b)_y - (a+b)_y] + (s-3-i)_x \cdot [2(b)_y - 2(a+b)_y + 1 - (a+1)_y] \\
&\quad + y^b(a)_y + y^b(a)_y(i+1)_x \\
&= y(a)_y(b-1)_y \cdot [(i-1)_x \cdot [(s-2-i)_x - (2)_x] - (i)_x(s-3-i)_x] \\
&\quad - y^b(a)_y(i)_x + (s-3-i)_x \cdot [-2y^b(a)_y - y(a)_y] \\
&\quad + y^b(a)_y + y^b(a)_y(i+1)_x \\
&= y(a)_y(b-1)_y \cdot [-(2)_x(i-1)_x + [(i-1)_x - (i+1)_x]] \\
&\quad - y(a)_y \cdot [y^{b-1}(i)_x + (i+1)_x \cdot [2y^{b-1} + 1] - y^{b-1} - y^{b-1}(i+1)_x] \\
&= y(a)_y(b-1)_y \cdot [1 - (i)_x - (i+1)_x] \\
&\quad - y(a)_y \cdot [y^{b-1}(i)_x + (i+1)_x \cdot [y^{b-1} + 1] - y^{b-1}] \\
&= y(a)_y \cdot ((b)_y - (i+1)_x \cdot [(b)_y + 1] - (i)_x(b)_y) \\
&= y(a)_y \cdot ((b)_y \cdot [1 - (i+1)_x - (i)_x] - (i+1)_x).
\end{aligned}$$

Moreover, since $s-3-i = i+1$,

$$\begin{aligned}
\Sigma_1 + \dots + \Sigma_5 &= y(a)_y(b)_y \cdot [(i+2)_x + 2(i+1)_x - 2 + (i)_x] \\
&= y(a)_y(b)_y \cdot [x(i+1)_x + 2(i+1)_x - 1 + (i)_x].
\end{aligned}$$

Therefore,

$$\Sigma_1 + \dots + \Sigma_9 = y(a)_y(i+1)_x \cdot [(1+x)(i+1)_x - 1],$$

which by Identity A.17 proves (98).

We end by proving that if s is even and $2 \leq b \leq a$, then

$$\begin{aligned}
y \cdot (S_s^i(a, b) - C_s^i(b, a)) &= \sum_{j=1}^b S_s^{i-1}(j, a+b+1-j) - S_s^{i-1}(a+b+1-j, j) \\
&\quad + 2 \sum_{j=1}^b S_s^i(j, a+b+1-j) - S_s^i(a+b+1-j, j) \\
&\quad + 2 \sum_{j=1}^b C_s^1(a+b+1-j, j) - C_s^1(j, a+b+1-j) \\
&\quad + \sum_{j=1}^b C_s^{i-1}(j, a+b+1-j) - C_s^{i-1}(a+b+1-j, j) \\
&\quad + 3 \sum_{j=1}^b C_s^i(j, a+b+1-j) - C_s^i(a+b+1-j, j) \\
&\quad + S_s^i(a+b) - S_s^i(1, a+b) + S_s^i(a+b) - S_s^i(b, a+1) \\
&\quad + C_s^1(1, a+b) - C_s^1(a+b) + C_s^1(b, a+1) - C_s^1(1, a+1, 1, b) \\
&\quad + 2[C_s^i(a+b) - C_s^i(1, a+b)] + C_s^i(a+1, b) - C_s^i(b, a+1) \\
&\quad + C_s^{i+1}(a+1, b) - C_s^{i+1}(a+b).
\end{aligned} \tag{98}$$

We note that if $S_s^i(a, b)$ belongs to $\mathcal{C}_B(n-1, r)$, so in particular $r = s-1$ and $n = r + a + b + 1$, then all the graphs in the right side of (98) belong to $\mathcal{C}(n, r)$. Notice that $C_s^1(1, a+1, 1, b)$ is isomorphic to $C_s^1(1, b, a+1, b)$, and it is the latter that we use to apply Identity A.13 below.

Similarly as before, we write Σ_ℓ for the sum appearing in line ℓ of the right side of (98) divided by $(x + y - xy)$. Using Identities A.8 and A.13–A.16, we obtain the following expressions.

$$\begin{aligned}\Sigma_1 + \dots + \Sigma_5 &= y(a)_y(b)_y \cdot [(s-2-i)_x + 2(s-3-i)_x - 2 + (i)_x], \\ \Sigma_6 &= -y(a)_y(b-1)_y \cdot [(2)_x(s-3)_x + (i)_x(s-3-i)_x] \\ &\quad + (i)_x \cdot [(a+b)_y + (a+1)_y - 2] - (s-3-i)_x \cdot [2(a+b)_y - (b)_y - 1], \\ \Sigma_7 &= (a+b)_y - 1 - y(b-1)_y = y^b(a)_y + (b)_y - (b)_y = y^b(a)_y, \\ \Sigma_8 &= (i)_x \cdot [(b)_y - (a+1)_y - 2(a+b)_y + 2], \\ \Sigma_9 &= y(a)_y(b-1)_y(s-2-i)_x(i+1)_x + (i+1)_x \cdot [(a+b)_y - (a+1)_y].\end{aligned}$$

Consequently, and using Identity A.2 twice as in the previous case, we infer that

$$\begin{aligned}\Sigma_6 + \Sigma_9 &= y(a)_y(b-1)_y \cdot [1 - (i+1)_x - (i)_x] + (i+1)_x \cdot [(b)_y + 1 - (a+b)_y - (a+1)_y] \\ &\quad + (i)_x \cdot [(a+b)_y + (a+1)_y - 2] \\ &= y(a)_y(b-1)_y \cdot [1 - (i+1)_x - (i)_x] - y(a)_y(i+1)_x - y^b(a)_y(i+1)_x \\ &\quad + (i)_x \cdot [(a+b)_y + (a+1)_y - 2],\end{aligned}$$

so that

$$\begin{aligned}\Sigma_6 + \dots + \Sigma_9 &= y(a)_y(b-1)_y \cdot [1 - (i+1)_x - (i)_x] - y(a)_y(i+1)_x - y^b(a)_y(i+1)_x \\ &\quad + (i)_x \cdot [(b)_y - (a+b)_y] + y^b(a)_y \\ &= y(a)_y(b-1)_y \cdot [1 - (i+1)_x - (i)_x] + y(a)_y \cdot [1 - y^{b-1}(i+1)_x - y^{b-1}(i)_x] \\ &\quad - y(a)_y(i+1)_x \\ &= y(a)_y(b)_y \cdot [1 - (i+1)_x - (i)_x] - y(a)_y(i+1)_x.\end{aligned}$$

We thus deduce that

$$\begin{aligned}\Sigma_1 + \dots + \Sigma_9 &= y(a)_y(b)_y \cdot [(1+x)(i+1)] - y(a)_y(i+1)_x \\ &= y(a)_y(i+1)_x \cdot [(1+x)(b)_y - 1],\end{aligned}$$

which by Identity A.17 concludes the proof, as $s-3-i = i+1$ by definition. \square

It remains to deal with the case where $i = 1$. To ease the exposition, the analysis is split in two, according as $r \leq 5$ or $r > 6$.

Relation B.17. Fix $r \in \{4, 5\}$ and $n \geq r+3$. If a and b are positive integers such that $S_{r+1}^1(a, b)$ belongs to $\mathcal{C}_B(n-1, r)$, then

$$S_{r+1}^1(a, b)^\circ \equiv C_{r+1}^1(b, a)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. The proof relies on several identities, depending on whether r is 4 or 5, and whether $a > b$. First, we show that for positive integers a and b , we have

$$S_5^1(a, b)^\circ - C_5^1(b, a)^\circ = \begin{cases} C_5^1(a+1, b) - C_5^1(b+1, a) + C_5^1(1, a, b+1) - C_5^1(1, a+1, 1, b) & \text{if } a \geq b, \\ C_5^1(1, b, a+1) - C_5^1(1, b, 1, a+1) & \text{otherwise.} \end{cases} \quad (99)$$

Note that if $r = 4$ and $S_5^1(a, b) \in \mathcal{C}_B(n-1, 4)$, so in particular $n = a + b + 5$, then all the graphs in the relevant (according as $a \geq b$ or $a < b$) right side of (99) belong to $\mathcal{C}(n, 4)$.

By Identity A.17,

$$\frac{S_5^1(a, b) - C_5^1(b, a)}{x + y - xy} = (a)_y \cdot [(1+x) \cdot (b)_y - 1]. \quad (100)$$

The equality when $a < b$ then follows directly from Identity A.13 applied with $c = 0$ and $s = 5$. If $a \geq b$, then by applying Identity A.8 with $c = a - b$, $i = 1$ and $s = 5$, we see that

$$\begin{aligned}\frac{C_5^1(a+1, b) - C_5^1(b+1, a)}{x + y - xy} &= -y^b \cdot (3)_x \cdot (a-b)_y - (a+1)_y + (b+1)_y \\ &= -(3)_x \cdot [(a)_y - (b)_y] - (a+1)_y + (b+1)_y.\end{aligned} \quad (101)$$

By Identity A.13 applied with $c = 1$, $i = 1$ and $s = 5$, we see that

$$\begin{aligned}
\frac{C_5^1(1, a, b+1) - C_5^1(1, a+1, 1, b)}{x+y-xy} &= (3)_x \cdot [(a)_y - (b)_y] + y \cdot (b)_y \cdot [(1+x) \cdot (a)_y - 1] \\
&= (3)_x \cdot [(a)_y - (b)_y] + y \cdot (a)_y \cdot [(1+x) \cdot (b)_y - 1] + y \cdot [(a)_y - (b)_y] \\
&= (3)_x \cdot [(a)_y - (b)_y] + y \cdot (a)_y \cdot [(1+x) \cdot (b)_y - 1] \\
&\quad + [(a+1)_y - (b+1)_y],
\end{aligned} \tag{102}$$

so that, by (100), summing (101) and (102) finishes establishing (99).

We now turn to the case where $r = 5$. By Identity A.17,

$$\frac{S_6^1(a, b) - C_6^1(b, a)}{x+y-xy} = (a)_y \cdot (1+x) \cdot [(1+x) \cdot (b)_y - 1]. \tag{103}$$

For $a < b$, we prove that

$$\begin{aligned}
S_6^1(a, b)^\circ - C_6^1(b, a)^\circ &= 2 \sum_{j=1}^a S_6^1(j, a+b+1-j) - S_6^1(a+b+1-j, j) \\
&\quad + \sum_{j=1}^a C_6^1(j, a+b+1-j) - C_6^1(a+b+1-j, j) \\
&\quad + S_6^1(a+1, b) - S_6^1(b, a+1) + S_6^1(a+b) - S_6^1(1, a+b) \\
&\quad + S_6^1(a+b) - S_6^1(b, a+1) \\
&\quad + C_6^1(a+b) - C_6^1(1, a+b) + C_6^1(a+1, b) - C_6^1(b, a+1) \\
&\quad + C_6^2(b, a+1) - C_6^2(a+b) \\
&\quad + C_6^1(1, b, a+1) - C_6^1(1, b, 1, a+1).
\end{aligned} \tag{104}$$

Note that if $S_6^1(a, b) \in \mathcal{C}_B(n-1, 5)$, so in particular $n = a+b+6$, then all the graphs in the right side of (104) belong to $\mathcal{C}(n, 5)$.

To ease the exposition, let Σ_ℓ denote the sum appearing in line ℓ of the right side of (104) divided by $(x+y-xy)$. Using Identities A.8 and A.13–A.16, we obtain the following expressions.

$$\begin{aligned}
\Sigma_1 + \Sigma_2 &= y(a)_y(b)_y \cdot (2x+1), \\
\Sigma_3 &= x \cdot [(b)_y + 1 - (a+1)_y - (a+b)_y] = x \cdot [-y(a)_y(1+y^{b-1})], \\
\Sigma_4 &= -y(a)_y(b-1)_y(2)_x(3)_x - y(a)_y(b-1)_y(2)_x + y(a)_y + (2)_x \cdot [(b)_y - (a+b)_y], \\
\Sigma_5 &= (b)_y + 1 - (a+1)_y - (a+b)_y = -y(a)_y(1+y^{b-1}), \\
\Sigma_6 &= y(a)_y(b-1)_y(2)_x(3)_x + y^b(a)_y(2)_x, \\
\Sigma_7 &= y(a)_y \cdot [(3)_x(b)_y - 1].
\end{aligned}$$

Therefore

$$\Sigma_3 + \Sigma_5 = -y(a)_y(2)_x(1+y^{b-1}) \quad \text{and} \quad \Sigma_4 + \Sigma_6 = y(a)_y \cdot [1 - (2)_x(b-1)_y],$$

so

$$\Sigma_4 + \Sigma_6 + \Sigma_7 = y(a)_y \cdot [x^2(b-1)_y + (3)_x \cdot y^{b-1}],$$

and hence

$$\begin{aligned}
\Sigma_3 + \dots + \Sigma_7 &= y(a)_y \cdot [x^2 \cdot (b-1)_y + x^2 y^{b-1} - (2)_x] \\
&= y(a)_y \cdot [x^2(b)_y - (2)_x].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Sigma_1 \dots + \Sigma_7 &= y(a)_y \cdot [x^2(b)_y - (2)_x + (2x+1)(b)_y] \\
&= y(a)_y \cdot [(x+1)^2(b)_y - (2)_x] \\
&= y(a)_y \cdot (2)_x \cdot [(2)_x(b)_y - 1],
\end{aligned}$$

which by (103) establishes (104).

It remains to deal with the case where $a \geq b$, for which we prove that

$$\begin{aligned}
S_6^1(a, b)^\circ - C_6^1(b, a)^\circ &= 2 \sum_{j=1}^b S_6^1(j, a+b+1-j) - S_6^1(a+b+1-j, j) \\
&\quad + \sum_{j=1}^b C_6^1(j, a+b+1-j) - C_6^1(a+b+1-j, j) \\
&\quad + S_6^1(a+b) - S_6^1(1, a+b) + S_6^1(a+b) - S_6^1(b, a+1) \\
&\quad + C_6^1(a+b) - C_6^1(1, a+b) + C_6^1(a+1, b) - C_6^1(b+1, a) \\
&\quad + C_6^2(a+1, b) - C_6^2(a, b) \\
&\quad + C_6^1(1, a, b+1) - C_6^1(1, a+1, 1, b).
\end{aligned} \tag{105}$$

We note that if $S_6^1(a, b) \in \mathcal{C}_B(n-1, 5)$, so in particular $n = a+b+6$, then all the graphs in the right side of (105) belong to $\mathcal{C}(n, 5)$.

Similarly as before, we let Σ_ℓ denote the sum appearing in line ℓ of the right side of (105) divided by $(x+y-xy)$. Using Identities A.8 and A.13–A.16, we obtain the following expressions.

$$\begin{aligned}
\Sigma_1 + \Sigma_2 &= y(a)_y(b)_y \cdot (2x+1), \\
\Sigma_3 &= -y(a)_y \left((b-1)_y(2)_x \cdot [(3)_x+1] + 1 \right) - y^b(a)_y(2)_x - xy(a+b-1)_y, \\
\Sigma_4 &= -y^b(4)_x(a-b)_y - (a+1)_y + (b+1)_y - y(a+b-1)_y, \\
\Sigma_5 &= y(a)_y(3)_x(2)_x(b-1)_x + y^{a+1}(2)_x(b-1)_y, \\
\Sigma_6 &= (4)_x \cdot [(a)_y - (b)_y] + y(a)_y(b)_y(3)_x - y(b)_y.
\end{aligned}$$

Now,

$$\Sigma_3 + \Sigma_5 = y(a)_y \left(1 - (2)_x(b-1)_y - y^{b-1}(2)_x \right) - xy(a+b-1)_y + y^{a+1}(b-1)_y(2)_x,$$

and since $y^b(a-b)_y = (a)_y - (b)_y$ and $(b+1)_y = 1 + y(b)_y$,

$$\begin{aligned}
\Sigma_4 + \Sigma_6 &= -(a+1)_y + 1 - y(a+b-1)_y + y(a)_y(b)_y(3)_x, \\
&= -y(a)_y + y(a)_y(b)_y(3)_x, \\
&= y(a)_y[(b)_y(3)_x - 1] - y(a+b-1)_y.
\end{aligned}$$

Therefore, using that $y^{a+1}(b-1)_y = y(a+b-1)_y - y(a)_y$,

$$\begin{aligned}
\Sigma_3 + \dots + \Sigma_6 &= y(a)_y \left(y^{b-1}(3)_x + x^2(b-1)_y - y^{b-1}(2)_x - (2)_x \right) \\
&\quad - y(a+b-1)_y(2)_x + y(a+b-1)_y(2)_x \\
&= y(a)_y \cdot [x^2(b)_y - (2)_x].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Sigma_1 + \dots + \Sigma_6 &= y(a)_y \left(x^2(b)_y - (2)_x + 2x(b)_y + (b)_y \right) \\
&= y(a)_y \left((b)_y \cdot [x^2 + 2x + 1] - (2)_x \right) \\
&= y(a)_y \left((b)_y \cdot (2)_x(2)_x - (2)_x \right) \\
&= y(a)_y(2)_x \left((2)_x(b)_y - 1 \right),
\end{aligned}$$

which by (103) concludes the proof of (105). \square

We now deal with the case where $r \geq 6$.

Relation B.18. Fix $r \geq 6$ and $n \geq r + 3$. If a and b are positive integers such that $S_{r+1}^1(a, b)$ belongs to $\mathcal{C}_B(n - 1, r)$, then

$$S_{r+1}^1(a, b)^\circ \equiv C_{r+1}^1(b, a)^\circ \pmod{\mathcal{C}(n, r)}.$$

Proof. First, we deduce from Identity A.17 that for positive integers a , b and $s \geq 7$,

$$\frac{S_s^1(a, b) - C_s^1(b, a)}{x + y - xy} = (a)_y(s - 4)_x \cdot [(2)_x(b)_y - 1]. \quad (106)$$

We now show that for positive integers a and b such that $a < b$ and $s \geq 7$,

$$\begin{aligned} y \cdot (S_s^1(a, b) - C_s^1(b, a)) &= 2 \sum_{j=1}^b S_s^1(j, a + b + 1 - j) - S_s^1(a + b + 1 - j, j) \\ &\quad + \sum_{j=1}^a S_s^2(j, a + b + 1 - j) - S_s^2(a + b + 1 - j, j) \\ &\quad + \sum_{j=1}^a C_s^2(j, a + b + 1 - j) - C_s^2(a + b + 1 - j, j) \\ &\quad + S_s^1(a + 1, b) - S_s^1(b, a + 1) + S_s^1(a + b) - S_s^1(1, a + b) \\ &\quad + S_s^2(a + b) - S_s^2(b, a + 1) + C_s^2(a + 1, b) - C_s^2(1, a + b) \\ &\quad + C_s^1(1, b, a + 1) - C_s^1(1, b, 1, a + 1). \end{aligned} \quad (107)$$

We note that if $r = s - 1$ and $S_s^1(a, b)$ belongs to $\mathcal{C}_B(n - 1, r)$, so in particular $n = r + a + b + 1$, then all the graphs in the right side of (107) belong to $\mathcal{C}(n, r)$.

Similarly as before, writing Σ_ℓ for the sum appearing in line ℓ of the right side of (107) divided by $(x + y - xy)$, we see that

$$\Sigma_5 = -y(a)_y(s - 5)_x \cdot [(2)_x(b - 1)_y - y^{b-1}],$$

and

$$\begin{aligned} \Sigma_1 + \Sigma_2 + \Sigma_3 &= y(a)_y(b)_y(s - 5)_x \cdot [2x + 1], \\ &= y(a)_y(s - 5)_x \cdot [(2x + 1)y^{b-1} + (2)_x(b - 1)_y], \end{aligned}$$

so that

$$\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_5 = xy(a)_y(s - 5)_x \cdot [(b)_y + y^{b-1}].$$

On the other hand,

$$\begin{aligned} \Sigma_4 &= [(b)_y - (a + 1)_x] \cdot [(s - 4)_x - 1] + y(a + b - 1)_y \cdot [1 - (s - 4)_x] \\ &= xy(s - 5)_x \cdot [(b - 1)_y - (a)_y - y^a(b - 1)_y - (a)_y] \\ &= xy(a)_y(s - 5)_x \cdot [(1 - y) \cdot (b - 1)_y - 2] \\ &= xy(a)_y(s - 5)_x \cdot [(b - 1)_y - (b)_y - 1]. \end{aligned}$$

It follows that

$$\begin{aligned} \Sigma_1 + \dots + \Sigma_5 &= xy(a)_y(s - 5)_x \cdot [(b)_y - 1] \\ &= y(a)_y \cdot \left((s - 4)_x \cdot [(b)_y - 1] - [(b)_y - 1] \right). \end{aligned}$$

Since

$$\begin{aligned} \Sigma_6 &= y(a)_y \cdot [(s - 3)_x(b)_y - 1] \\ &= y(a)_y \cdot [(s - 4)_x(b)_y + x^{s-4}(b)_y - 1], \end{aligned}$$

we infer that

$$\begin{aligned}
\Sigma_1 + \dots + \Sigma_6 &= y(a)_y \cdot \left((s-4)_x \cdot [(b)_y - 1] - (b)_y + (s-4)_x(b)_y + x^{s-4}(b)_y \right) \\
&= y(a)_y \cdot \left((s-4)_x \cdot [(b)_y - 1] + (b)_y \cdot [x(s-4)_x] \right) \\
&= y(a)_y(s-4)_x \cdot \left(-1 + (b)_y \cdot (1+x) \right),
\end{aligned}$$

which by (106) establishes (107).

For $a > b$ and $s \geq 7$, we prove that

$$\begin{aligned}
y \cdot (S_s^1(a, b) - C_s^1(b, a)) &= 2 \sum_{j=1}^b S_s^1(j, a+b+1-j) - S_s^1(a+b+1-j, j) \\
&\quad + \sum_{j=1}^a S_s^2(j, a+b+1-j) - S_s^2(a+b+1-j, j) \\
&\quad + \sum_{j=1}^a C_s^2(j, a+b+1-j) - C_s^2(a+b+1-j, j) \\
&\quad + S_s^1(a+1, b) - S_s^1(b, a+1) + S_s^1(a+b) - S_s^1(1, a+b) \\
&\quad + S_s^2(a+b) - S_s^2(b, a+1) + C_s^2(a+1, b) - C_s^2(1, a+b) \\
&\quad + C_s^1(a+1, b) - C_s^1(b+1, a) \\
&\quad + C_s^1(1, a, b+1) - C_s^1(1, a+1, 1, b).
\end{aligned} \tag{108}$$

We note that if $r = s-1$ and $S_s^1(a, b)$ belongs to $\mathcal{C}_B(n-1, r)$, so in particular $n = r + a + b + 1$, then all the graphs in the right side of (108) belong to $\mathcal{C}(n, r)$.

Writing Σ'_ℓ for the sum appearing in line ℓ of the right side of (108) divided by $(x+y-xy)$, we see that $\Sigma'_\ell = \Sigma_\ell$ for $\ell \in \{1, \dots, 5\}$. Consequently, and noticing that the value of $S_s^1(a, b) - C_s^1(b, a)$ is the same regardless of the comparative order of a and b , it suffices to establish that $\Sigma'_6 + \Sigma'_7 = \Sigma_6$. Recalling that $C_s^1(1, a+1, 1, b)$ and $C_s^1(1, b, 1, a+1)$ are isomorphic, it suffices to prove that

$$\Sigma'_6 = \frac{C_s^1(1, b, a+1) - C_s^1(1, a, b+1)}{x+y-xy},$$

which directly follows from Identities A.8 and A.9, as one then sees that both sides equal

$$(s-2)_x \cdot [(b)_y - (a)_y] - (a+1)_y + (b+1)_y,$$

using that $-y^b(a-b)_y = (b)_y - (a)_y = y^a(b-a)_y$. \square

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