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## ► To cite this version:

Yousif Alyousifi, Kamarulzaman Ibrahim, Mahmud Othamn, Wan Zawiah Wan Zin, Nicolas Vergne, et al.. Bayesian Information Criterion for Fitting the Optimum Order of Markov Chain Models: Methodology and Application to Air Pollution Data. *Mathematics* , 2022, 10 (13), pp.2280. 10.3390/math10132280 . hal-04485219

**HAL Id: hal-04485219**

**<https://hal.science/hal-04485219v1>**

Submitted on 4 Apr 2024

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## Article

# Bayesian Information Criterion for Fitting the Optimum Order of Markov Chain Models: Methodology and Application to Air Pollution Data

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**Abstract:** The analysis of air pollution behavior is becoming crucial, where information on air pollution behavior is vital for managing air quality events. Many studies have described the stochastic behavior of air pollution based on the Markov chain (MC) models. Fitting the optimum order of MC models is essential for describing the stochastic process. However, uncertainty remains concerning the optimum order of such models for representing and characterizing air pollution index (API) data. In this study, the optimum order of the MC models for hourly and daily API sequences from seven stations in the central region of Peninsular Malaysia is identified, based on the Bayesian information criteria (BIC), contributing to exploring an adequate explanation of the probabilistic dependence of air pollution. A summary of the statistics for the API was calculated prior to the analysis. The Markov property and the divergence for the empirically estimated transition matrix of an MC sequence are also investigated. It is found from the analysis that the optimum order varies from one station to another. At most stations, for both observed and simulated API data, the second and third orders of the MC models are found to be optimum for hourly API occurrences, while the first-order MC is found to be most fitting for describing the dynamics of the daily API. Overall, fitting the optimum order of the MC model for the API data sequence captured the delay effect of air pollution. Accordingly, we concluded that the air quality standard lies within controllable limits, except for some infrequent occurrences of API values exceeding the unhealthy level.

**Keywords:** chi-squared test; high-order Markov chain; log-likelihood function; Markov property; maximum likelihood estimation; R software

**MSC:** 60J10



**Citation:** Alyousifi, Y.; Ibrahim, K.; Othamn, M.; Zin, W.Z.W.; Vergne, N.; Al-Yaari, A. Bayesian Information Criterion for Fitting the Optimum Order of Markov Chain Models: Methodology and Application to Air Pollution Data. *Mathematics* **2022**, *10*, 2280. <https://doi.org/10.3390/math10132280>

Academic Editor: Victor Leiva

Received: 30 May 2022

Accepted: 25 June 2022

Published: 29 June 2022

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## 1. Introduction

Clean air is necessary for healthy living; however, much of the air in the atmosphere is polluted. Air pollution is an environmental phenomenon that occurs as a result of natural disasters and anthropogenic activities [1]. It is one of the most serious environmental issues that have negative impacts on people's health and quality of life [2]. The Department of Environment (DOE) defines the air pollution index (API) as a generalized metric for characterizing the status of the air quality in the environment. The API is computed using

the average indices for five major pollution variables: ozone ( $O_3$ ), nitric oxide ( $NO_2$ ), particulate matter ( $PM_{10}$ ), carbon monoxide (CO), and sulphur dioxide ( $SO_2$ ), and the highest value from these five subindices at a particular hour is chosen as the API value [3,4].

There are many applications of the MC model in various areas of research, where it is considered a useful tool to describe probabilistic dependence in a model for a stochastic process. MC models are widely used in many different disciplines such as in economics, environment, education, medical, business engineering, queuing networks, and manufacturing systems [5]. For example, Zhou et al. [6] have used the MC model for predicting the transition probabilities of bike rental and returns for a bike-sharing system in China. Choji et al. [7] have employed the MC model for predicting the long run of share prices for two banks in Nigeria. Saad et al. [8] have investigated the track movements of lecturers in universities using the MC model. In [9], a discrete-time MC model has been applied for analyzing job transitions in Mexico.

Among the statistical models that are widely used in environmental research, the MC model is considered a powerful model, particularly for describing the probabilistic behavior in environmental problems, as reported in the studies by [10–13]. They found that the MC is a superior model in characterizing the probabilistic behavior of environmental problems. Particularly, MC models have been used for modeling the air quality conditions in the environment and implemented to represent air pollution occurrences. For example, Larsen et al. [14] modeled air quality using ozone data for a given level of threshold based on the MC. They worked with the high-order chains that can be estimated by examining correlation plots, and transition probabilities have been estimated using the maximum likelihood method. Hoyos et al. [15] employed the MC model for modeling  $O_3$  and  $SO_2$  for evaluating the effect of air pollution events in Mexico City. Rodrigues and Achcar [16] have proposed a discrete-time MC model for ozone air pollution based on the maximum daily observations. They investigated the case of air pollution problems with maximum daily measurements and found that ozone behavior did not indicate a time-homogeneous property. Asadollahfardi et al. [17] have predicted the air pollution of  $PM_{2.5}$  index using a combination method of MC and artificial neural network methods.

Recently, Nebenzal and Fishbain [18] predicted the long run of air pollution using discrete-time MC models, which have precisely explained the distribution of the nitrogen levels. Mohamad et al. [19] have used the first-order MC to model the daily  $PM_{10}$  concentrations in Malaysia. Alyousifi et al. [20] have proposed the use of the discrete-time MC model for describing the probabilistic behavior of air pollution in Klang, Malaysia, based on API data. Alyousifi et al. [21] have improved the estimation of the transition probability matrix of the MC model, based on the Bayesian-based method under three different priors. Chen and Wu [22] have applied the MC model for online air quality monitoring data for predicting the air quality and determining the main air pollutants for a certain area in Taiwan. Alyousifi et al. [23] have introduced the spatial MC model for exploring the potential regional impact of air pollution in Malaysia. Gao [20] has implemented the discrete-time MC model for identifying the stochastic behavior of Air Quality Index data in China.

The uncertainty of the optimum order of the MC model may lead to a lack of sureness about the prediction of the future state, such as the air pollution state, and whether it depends only on the previous state, some or all past states. Uncertainty may range from falling short of certainty to an almost complete lack of conviction or knowledge, especially about an outcome or result.

Based on the reviewed studies mentioned above, as well as to the best of our knowledge, fitting the optimum order of the MC model for the API has not yet been conducted. Thus, this study aims to fit the optimum order of the MC for modeling the stochastic dependence of the API. This study considers an extension of the analysis of the stochastic dependence of the API conducted by [20,21,23], contributing to an adequate description of the probabilistic behaviors of the API in Malaysia, and estimating the transition probabilities from one state of air pollution to another. Based on a chi-squared test, the Markov property test to analyze the serial dependence or independence in the time series through

testing and a divergence test for the estimated transition matrices have been investigated. The high-order MC model has been fitted to the API data in order to determine the optimum order of the MC model for describing API data. This might suggest that the future transition of the state of air pollution relies on the current k-state and is unaffected by the past states of the process, which contribute to providing an appropriate illustration of air pollution transitions that would improve the prediction of air quality status. The rest of this paper is organized as follows. Section 2 includes the research framework and the approach used for the analysis. In Section 3, a description of the dataset as well as the results and discussion are presented. Finally, the conclusions of the paper are presented in Section 4.

## 2. Methodology

In this section, the concepts and the main definitions of the high-order MC and the maximum likelihood estimation (MLE) of the transition probability matrix of the model are introduced. In addition, the statistical criteria for testing the Markovian property and the goodness of fit for the MC models are presented.

### 2.1. Discrete-Time Markov Chain Model

A stochastic process  $\{X_t, t = 0, 1, 2, \dots, T\}$  with the state space  $S = \{1, 2, \dots, k\}$  that satisfies the Markov property is called a discrete-time Markov chain (DTMC), meaning that the Markov process  $X_{t+1}$  at time  $t + 1$  depends only on its present value at time  $t$  or  $X_t$ , regardless of how  $X_t$  is obtained. For every  $t$  and all states  $j, i, i_0, i_1, \dots, i_{t-1}$  in  $S$ , we have

$$P(X_{t+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_t = i_t) = P(X_{t+1} = j | X_t = i_t) \tag{1}$$

Suppose that  $p_{ij} = Pr(X_{t+1} = j | X_t = i)$  is the transition probability from state  $i$  to state  $j$  at time  $t$  and  $t + 1$ , respectively, which can be represented by the components of the transition probability matrix given by  $P = [p_{ij}]$ , where  $0 \leq p_{ij} \leq 1$ , for all  $i, j = 1, 2, \dots, k$ ,  $\sum_{j=1}^k p_{ij} = 1$  for all  $j = 1, 2, \dots, k$ , and  $k$  is the number of states [24]. Each row of  $P$  is a multinomial distribution. If the transition probabilities between the individual state pairs do not depend on time, the DTMC is called time-homogeneous, which is given by

$$P(X_{t+1} = j | X_t = i_t) = P(X_1 = j | X_0 = i) = p_{ij} \text{ for all } i \text{ and } j \text{ in } S \tag{2}$$

where  $p_{ij}$  is the one-step transition probability from state  $i$  to state  $j$  in the sequence of discrete-time states, indicating that the Markov process has satisfied the Markov property. The values of  $p_{ij}$ 's are the probabilities that describe the cells in the transition probability matrix  $P_{k \times k}$  of MC [25–27]. The quantities  $p_{ij}$  must satisfy the conditions  $p_{ij} \geq 0$ , and for all  $i, j, \sum_{j=1}^k p_{ij} = 1$ . The above requirements exist because, given that we are in state  $i$ , the next state must be one of the possible states. Therefore, the sum of the rows of any state transition matrix must equal one. The MC is usually revealed by a state transition diagram in which the arrows from each state to other states illustrate transition probabilities  $p_{ij}$ . If there is no arrow from state  $i$  to state  $j$ , then it indicates that  $p_{ij} = 0$  [26,28].

Although the first-order MC is found to be a powerful model in representing the stochastic behavior of air pollution, it cannot be assumed that the order of the model is always one because sometimes the first-order model is inadequate for explaining the data [25,29]. Thus, it could be beneficial to examine the suitability of the MC model with a higher order of dependency on its past data. In this study, a  $d$ th-order MC model for  $d = 2$  or more was fitted to the API data. If the  $d$ th-order MC model is fitted well to the data, this indicates that the upcoming air pollution state is affected by all the previous states when the last  $d$  states of the sequence are identified. If this result is found, then the MC of order  $d$  is the optimum order of the model for describing the API data. The high-order MC model may be carried out for describing the sequence of the API to be compared with the first-order MC. The  $d$ th order of the MC model can be written as follows:

First-order:

$$P(X_{t+1} | X_t, \dots, X_1, X_0) = P(X_{t+1} | X_t) \tag{3}$$

Second-order:

$$P(X_{t+1}|X_t, \dots, X_1, X_0) = P(X_{t+1}|X_t, X_{t-1}) \tag{4}$$

Similarly, for the MC model of the  $d$ th order, the general transition probabilities can be given as

$$p_{i,i_1, \dots, i_d} = \Pr(X_{t+1} = i | X_t = i_t, X_{t-1} = i_{t-1}, \dots, X_1 = i_1, X_0 = i_0) = \Pr(X_{t+1} = i | X_t = i_t, X_{t-1} = i_{t-1}, \dots, X_{t-d-1} = i_d) \tag{5}$$

where  $i, i_1, \dots, i_d \in S$ ;  $S$  is the state space [30]. In other words, the current state of the process depends on  $d$  past states, such that

$$\sum_{i=1}^S p_{i,i_1, \dots, i_d} = 1, 1 \leq i, i_1, \dots, i_d \leq S \tag{6}$$

In the case of  $d = 1$ , it is the first-order MC. The joint probability distribution of  $X = \{X_t; t = 0, 1, 2, \dots, T, \dots, U\}$ , which denotes the  $u$  variables that represent the air pollution events in terms of the hourly API are given by  $\Pr(X_u = i_u, \dots, X_1 = i_1) = P(i_u, i_{u-1}, \dots, i_1)$ , as we denote  $\Pr(\cdot)$  by  $P(\cdot)$ . Then,

$$P(i_u, i_{u-1}, \dots, i_1) = P(i_u | i_{u-1}, \dots, i_{u-d}) P(i_{u-1} | i_{u-2}, \dots, i_{u-d-1}) \dots P(i_{d+1} | i_d, \dots, i_1) P(i_d | i_{d-1}, \dots, i_1) \dots P(i_2 | i_1) P(i_1) = P_{i_1} \prod_{j=1}^{u-d} P_{i_j, \dots, i_{d+j}} \tag{7}$$

The transition probabilities of the MC model can be estimated based on the maximum likelihood estimate (MLE) for  $p_{ij}$ , which is given by

$$\hat{p}_{ij} = \frac{y_{ij}}{\sum_j y_{ij}} = \frac{y_{ij}}{y_i} \tag{8}$$

where  $y_{ij}$  represents the number of transitions in states  $i$  and  $j$  at time  $t - 1$  and  $t$ , respectively, and  $y_i$  denotes the total number of transitions from  $i$ . For more details, see [31]. For estimating the parameters of the high-order MC model, the maximum likelihood function of the  $d$ th order MC can be given by

$$L_d(X_1, \dots, X_u) = \prod_{s_1, \dots, s_d} \hat{P}_{s_1, \dots, s_d}^{y_{s_1, \dots, s_d}} L_d(X_1, \dots, X_u) = P_{i_1} \prod_{j=1}^{u-d} P_{i_j, \dots, i_{d+j}} = P_{i_j, \dots, i_{d+j}} \prod_{s_1, \dots, s_d} \hat{P}_{s_1, \dots, s_{d+1}}^{y_{s_1, \dots, s_{d+1}}} \tag{9}$$

where  $\hat{P}_{s_1, \dots, s_{d+1}}^{y_{s_1, \dots, s_{d+1}}}$  is the estimated transition probabilities of the random process that goes from state  $s_{d-1}$  to  $s_d$ , where  $s_d$  is the state of the most recent observation. The superscript  $y_{s_1, \dots, s_{d+1}}$  denotes the associated transition counts. The MLE estimators of the transition probabilities of Equation (5) can be given by

$$\hat{p}_{s_1, \dots, s_{d+1}} = \frac{y_{s_1, \dots, s_{d+1}}}{\sum_{s_d} y_{s_1, \dots, s_d}} \tag{10}$$

For further detail see [32–34].

### 2.2. Testing the Markov Property and Divergence for Empirically Estimated Transition Matrix of MC Sequence

In this subsection, the Markov property can be tested by using the chi-square  $\chi^2$  test [31,35,36]. The test is used to assess the Markovian property, which verifies whether a

given MC holds the Markov property, whereby the transition probability of the next state is dependent only on the current state. Let  $X_1, X_2, \dots, X_t$  be a set of observations where  $1 \leq t \leq N - 2$  and  $y_{ijm}$  is the number of transitions from state  $i$  to state  $j$  and then to state  $m$ , written as  $X_{t-1} = i, X_t = j, X_{t+1} = m$ . If the Markov property is verified,  $y_{ijm}$  follows a Binomial distribution with the parameters  $y_{ij}$  and  $p_{ij}$ . Let

$$p_{ijm} = Pr(X_{t+1} = m | X_t = j, X_{t-1} = i) = Pr(X_{t+1} = m | X_t = j) \tag{11}$$

Specifically, Equation (11) involves testing the null hypothesis  $H_0 =$  (verify the Markov property,  $p_{ijm} = p_{jm}$ ) for  $j, m = 1, 2, \dots, k$  against the alternative hypothesis  $H_1 =$  (does not verify the Markov property). Then the test statistic is

$$Q = \sum_i \sum_j \sum_m \frac{(y_{ijm} - y_{ij} \hat{p}_{jm})}{y_{ij} \hat{p}_{jm}} \sim \chi^2(|S|^3) \tag{12}$$

where  $S$  is the state space [37]. Thus, if the  $H_0$  is true,  $Q$  has a chi-square distribution with  $|k|^3$  degrees of freedom. In addition, as applied by [38], the chi-square  $\chi^2$  test can be utilized to test the divergence for an empirically estimated transition matrix of the MC sequence. Suppose  $y_{ij}$  is the raw count matrix and its test statistics follows a chi-square law. Then the divergence test can be written as

$$D = 2 * \sum_i \sum_j y_{ij} \ln \frac{y_{ij}}{y_i p_{ij}} \sim \chi^2(k(k - 1)) \tag{13}$$

Concerning the Markovian property, fitting the optimum order of the MC model of the API will be determined and discussed in the following section.

### 2.3. Fitting the Optimum Order of the MC Model

If the Markovian property of the sequence is determined, it is then important to investigate which order fits best to the sequence under study. For fitting the most appropriate order of the MC model, the most commonly used measurements of goodness-of-fit are computed, which are the Bayesian information criterion (BIC) and Akaike’s information criterion (AIC). The statistical criterion that is widely used for selecting the appropriate orders of MC models is the Bayesian information criterion (BIC), due to its advantages in identifying the appropriate order of the MC model [38,39]. Here, BIC measures the information lost once a given model is employed to describe the API data. Hence, the BIC’s least-loss function is considered in this study for selecting the optimum order of the MC model for the hourly and daily API data. This is because AIC tends to overestimate the optimum order and may produce inconsistent results compared to the BIC [33,40,41]. Moreover, utilizing the BIC gives a mathematical formulation with a principle of parsimony in model building [42]. Therefore, according to the smallest BIC value, the optimum order of the MC model that best describes the sequence of the hourly and daily API will be determined.

The log-likelihood functions for transition probabilities are used in this criterion. The log-likelihoods for s-state MCs of order 1, 2, . . . ,  $d$  are [43]:

$$L_1 = \sum_{i=0}^{k-1} \sum_{j=2}^{k-1} y_{ij} \ln(\hat{p}_{ij}) \tag{14}$$

$$L_2 = \sum_{i_1=0}^{k-1} \sum_{i_2=2}^{k-1} \sum_{i_3=0}^{k-1} y_{i_1 i_2 i_3} \ln(\hat{p}_{i_1 i_2 i_3}) \tag{15}$$

$$L_d = \sum_{i_1=0}^{k-1} \sum_{i_2=2}^{k-1} \dots \sum_{i_d=0}^{k-1} y_{i_1 i_2 \dots i_d} \ln(\hat{p}_{i_1 i_2 \dots i_d}) \tag{16}$$



The formula for BIC is given by

$$BIC = -2\ln(L_d) + v\ln(Y) \quad (17)$$

where  $L_d$  is the likelihood function,  $Y$  is the sample size, and  $v$  is the number of parameters in the MC model.

### 3. Results and Discussion

This section presents a description of the air pollution data and the study area considered in this study. In addition, the trend analysis for air pollution, which includes the time series plots and the descriptive statistics of the air pollution data are depicted. In addition, the results of the MC modeling and the investigation of the optimum order of MC models are introduced.

#### 3.1. Application to Air Pollution Data

The hourly and daily API data collected from seven air-monitoring stations located in the central region of Peninsular Malaysia for three years (1 January 2012–31 December 2014) were considered in modeling the transition behavior of air pollution states in this study.

Particularly, the studied stations are located in seven major cities, namely, Kuala Lumpur, Cheras, Klang, Petaling Jaya, Shah Alam, Banting, and Kuala Selangor. Figure 1 exhibits the sites of these stations in the Peninsula. For modeling the dynamics of air pollution using a discrete-time MC model, the air pollution data was divided into three categories, namely, ( $0 < API \leq 100$ ), ( $100 < API \leq 200$ ), and ( $API > 200$ ) that are denoted by numbers from 1 to 3, representing the moderate, unhealthy, and very unhealthy states of air pollution, respectively [18].

#### 3.2. Trend Analysis for the API Data

When the API value exceeds 100, the air pollution states become unhealthy, suggesting that the air quality status is harmful to public health. Thus, assessing the risk of an API that is greater than 100 is critical, particularly in urban settings. Before going into the study in-depth, some descriptive statistics and time series plots of the API time series are determined, to provide basic information about the variables in a dataset and to derive insights into the air pollution data used in this study, which might support the results of the analysis. Table 1 presents a descriptive statistics summary of the hourly API data; it is observed that the maximum API values in the year 2013 were 495 and 323 for Klang and Banting, respectively. The API's mean, for all the stations considered, ranged from 40 to 57, indicating that the hourly API's mean is at a moderate level for all stations. In addition, the standard deviation of the API ranged from 16 to 25, and proportions of the API values exceed  $u$ , where  $u = 100, 200, \text{ and } 300$ , indicating that some unhealthy air pollution episodes occurred in the studied area, varying from station to station.

Figure 2 illustrates the time series plots of the observed API values. The study areas of Petaling Jaya, Banting, Shah Alam, and particularly, Klang showed a more volatile fluctuation of API values because these areas hosted more observations beyond the threshold limit of 200, whereas for the areas of Cheras, Kuala Lumpur, and Kuala Selangor, the observations are found to fluctuate around a constant mean and almost all API values are below the threshold limit of 200. Most of these figures indicate that the air pollution condition in the studied areas is not severe; nonetheless, some areas require more attention than others. Furthermore, the time series plots in Figure 2 show that the areas of Klang, Shah Alam, and Banting had experienced a severe episode of haze, with serious air pollution levels among the other cities. Because of transboundary pollution, the Klang region had the worst experiences of all.

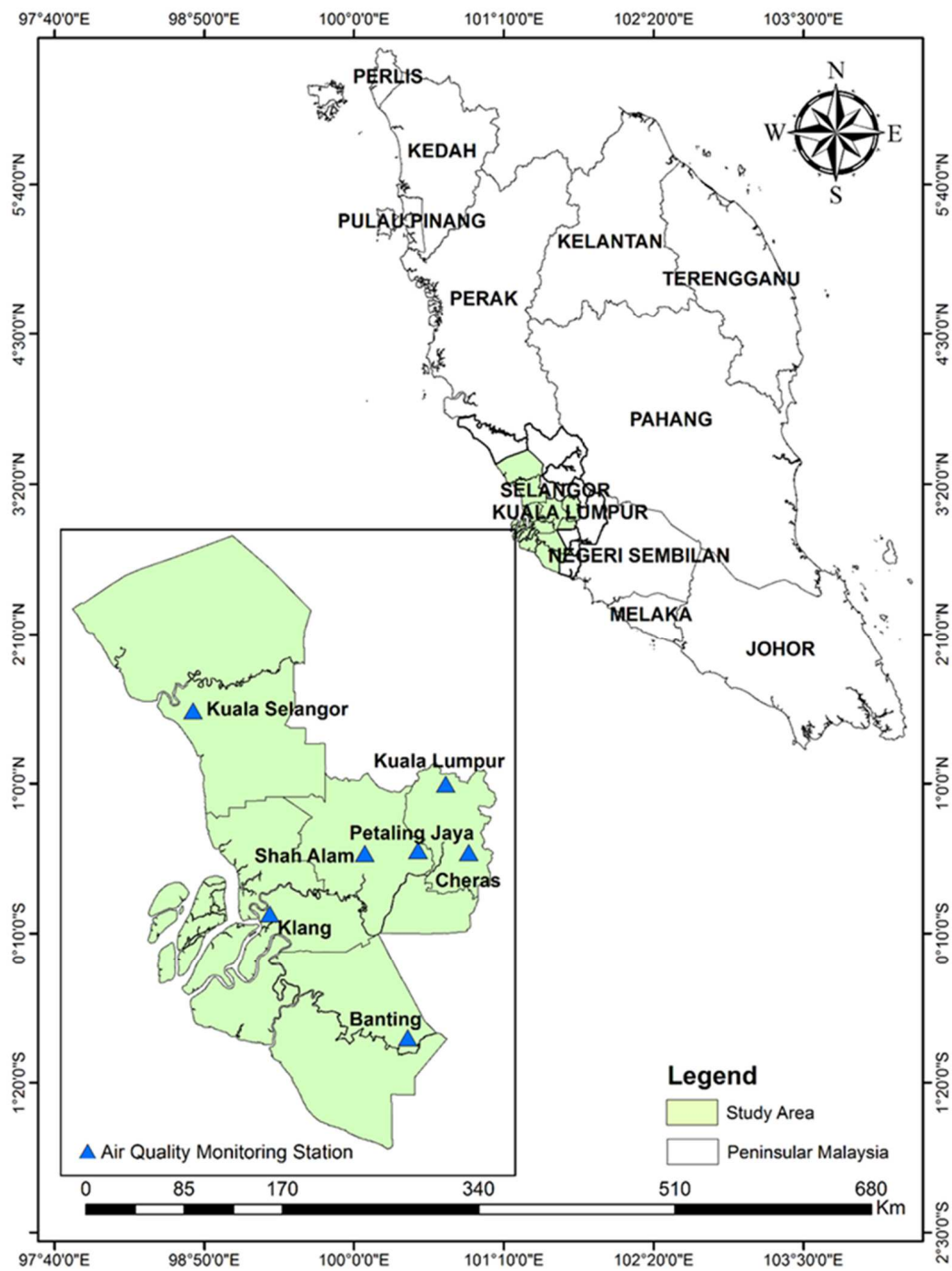


Figure 1. Sites of the studied stations in Peninsular Malaysia.

Table 1. Descriptive statistics of hourly air pollution index (API) in the study areas, 2012–2014.

Station	Area	Statistics							
		Max	Mean	Standard Deviation	Prop. API > 100	Prop. API > 200	Prop. API > 300	Kurtosis	Skewness
A1	Kuala Lumpur	203	46.91	19.05	0.0146	0.00009	0.0000	7.8650	1.84787
A2	Klang	495	56.90	25.13	0.0333	0.00372	0.0024	90.333	7.18003



Table 1. Cont.

Station	Area	Statistics							
		Max	Mean	Standard Deviation	Prop. API > 100	Prop. API > 200	Prop. API > 300	Kurtosis	Skewness
A3	Cheras	201	49.85	17.25	0.0141	0.00003	0.0000	6.3822	1.58419
A4	Petaling Jaya	231	47.37	16.53	0.0106	0.00061	0.0000	19.469	2.88533
A5	Banting	323	51.35	19.73	0.0220	0.00182	0.0005	35.720	1.58419
A6	Shah Alam	301	47.32	18.41	0.0146	0.0008	0.00004	20.606	2.72789
A7	Kuala Selangor	247	40.16	19.09	0.0135	0.00057	0.0000	13.121	2.47169

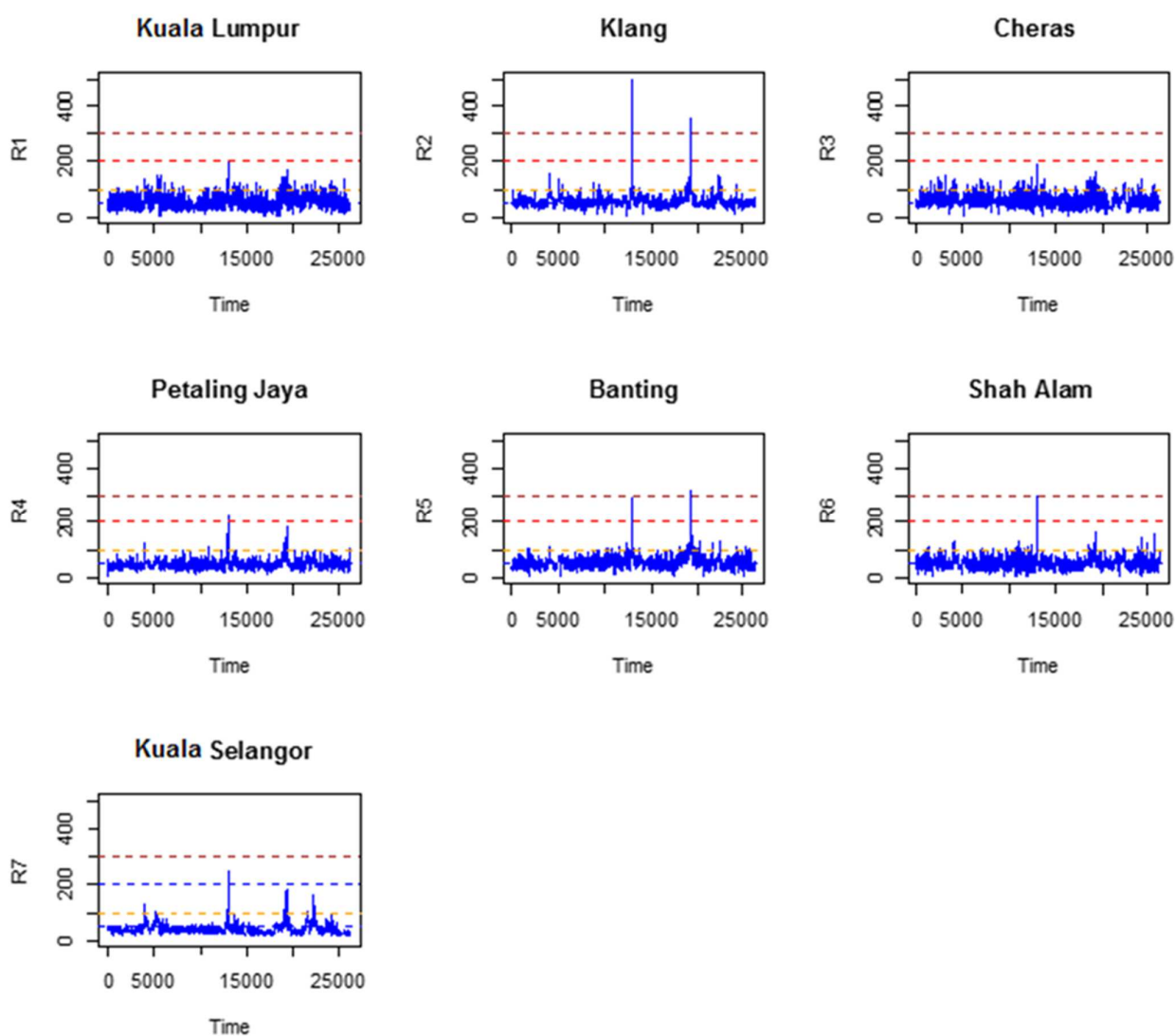


Figure 2. Temporal trend of hourly APIs for the stations considered in this study.

### 3.3. The Performance of the Markov Chain Modeling

With reference to the studies by [20,21,23], air pollution behavior may be depicted as a stochastic process  $X = \{X_t, t = 0, 1, 2, \dots, T\}$ , where  $X_t$  is the API value of the air pollution state at time  $t$ . The random variable  $X_t$  has values in the state space  $S$ , where  $S = \{1, 2, 3\}$ , equivalent to  $S = \{[0, 100], (100, 200], (200, \infty)\}$ , representing all three states. Particularly, if the process is in state 2 at time  $t$ , then  $X_t = 2$  (or  $100 < API \leq 200$ );

if it is in state 3 at time  $t$ , then  $X_t = 3$  (or  $API > 200$ ), which can be represented by Equation (18),

$$X_t = \begin{cases} 1 \text{ Moderate state at time } t \\ 2 \text{ Unhealthy state at time } t \\ 3 \text{ Very Unhealthy state at time } t \end{cases} \quad (18)$$

Based on the studies by [20,21,23], for API data, the first-order MC can derive appropriate insights into the transition probability for the behavior of air pollution, and offers a description of the transition probability's dependency structure in terms of time. However, it is important to examine the appropriateness of the MC with a higher order of dependency on its past data. Accordingly, in addition to the first-order MC, it might be useful to fit the high-order MC to investigate the dependency of the preceding states on the current state. This may reveal information about the dependence structures among the  $d$ -lag times congruous to the hourly API transition frequency. Thus, for a preliminary examination, the autocorrelation function (ACF) of the  $d$ -time lag effect is applied, which is given by

$$r_d = \frac{\sum_{t=1}^{n-d} (x_t - \bar{x})(x_{t+d} - \bar{x})}{\sum_{t=1}^n (x_t - \bar{x})^2} \quad (19)$$

A  $d$ -order of the MC indicates that the API state is independent of its previous states when the present  $d$ -lag states of the chain are identified [44,45]. Figure 3 demonstrates the ACF plot for the API sequence for each station. It is obvious that when the  $d$ -time lag increases, the ACF values decrease. Hence, it will be useful to fit the high-order MC model to describe the transition behavior of the APIs. Furthermore, to introduce an accurate analysis regarding the MC model for air pollution, the first-order MC model would be compared with the high-order MC model.

Although the first order of the MC model was a flexible model in describing the stochastic behavior of air pollution, as reported in the studies by [20,21,23], it cannot be assumed that the order is always one because sometimes it is inadequate to give an appropriate model [46]. In addition, the influence of being exposed to the API is more than 24 h, as reported by the World Health Organization [47]. It could be useful and a good idea to examine the fittingness of an MC with a higher order of dependency on its past data. Accordingly, fitting the high-order MC model of the API data implies that the next air pollution state is identified based on all the past states, once the present  $k$ -state of the chain is known. Therefore, investigating the optimum order of the MC model for describing the air pollution index data is considered.

### 3.4. Assessment of the Markov Property and Divergence of Markov Chain Sequence

The decision of the test hypothesis on the Markov property and the divergence of the MC can be made using the calculated chi-squared value and the  $p$ -value. If the  $p$ -value is greater than the given significance level, we cannot reject the hypothesis that the sequence satisfies the Markov property. From Table 2, it can be seen that the assumption of the Markovian property is satisfied for only three stations, since the  $p$ -values found for these stations are greater than the 0.05 level of significance, while the other two stations have  $p$ -values less than 0.05. Thus, based on the analysis of the Markov property test, these four stations will be considered for further analysis in the future based on the high-order MC. In addition, all the  $p$ -values of the divergence test are greater than the 0.05 level of significance, implying that the empirical transition matrix is consistent with the theoretical transition matrix.

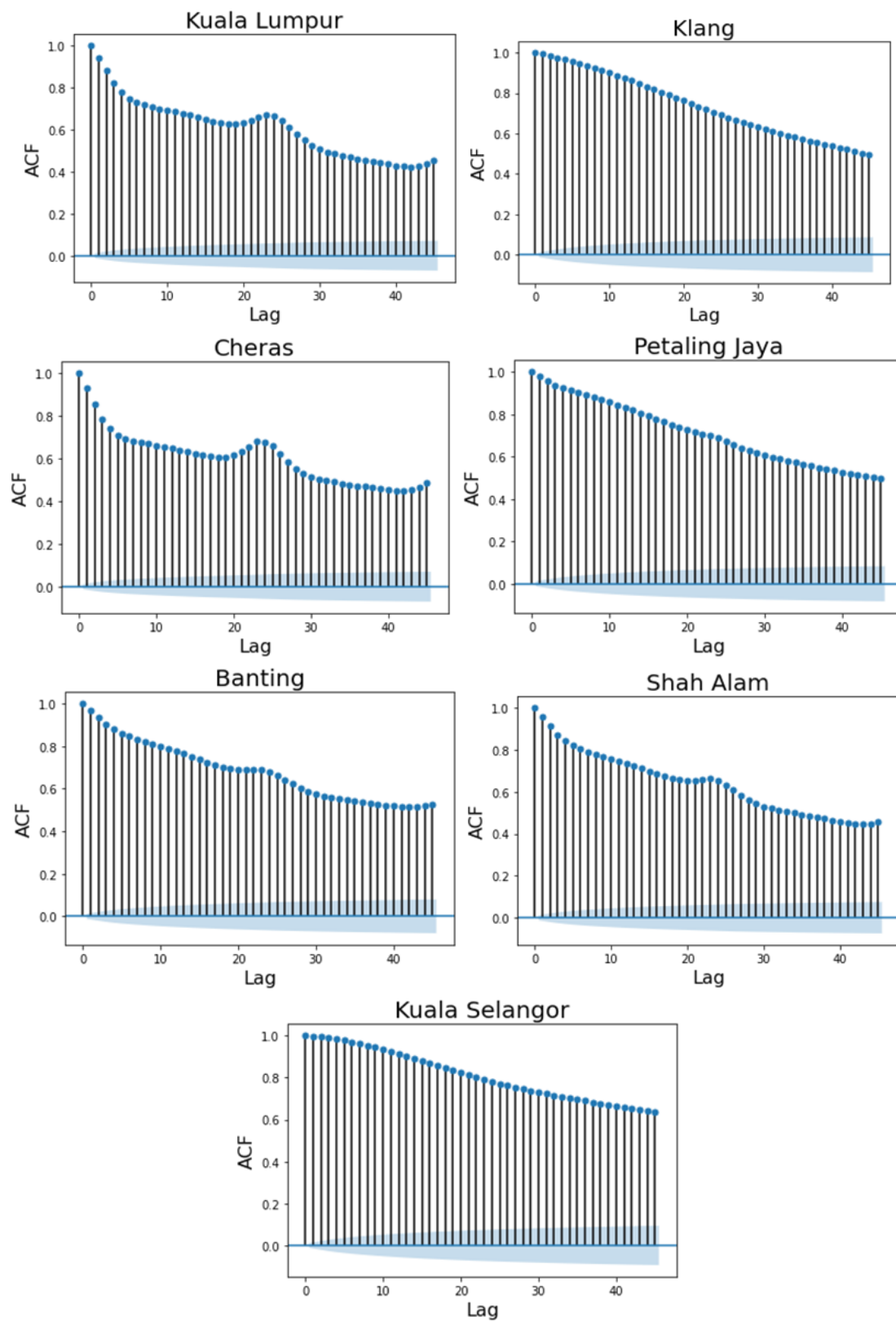


Figure 3. Visualization of the autocorrelation function (ACF) for API data.

**Table 2.** The results of the  $\chi^2$  and divergence test of MC for all the stations.

Station	Verify Markov Property			Verify Divergence		
	$\chi^2$ Statistics	df	p-Value	$\chi^2$ Statistics	df	p-Value
Kuala Lumpur	201.939	27	0.00002	1.908207	6	0.75264
Klang	47.5692	27	0.0086	0.010878	6	0.99857
Cheras	12.6243	27	0.9914	0.001336	6	0.9999
Petaling Jaya	18.8424	27	0.8856	0.009457	6	0.99998
Banting	54.1234	27	0.0015	0.066857	6	0.15348
Shah Alam	77.0592	27	0.00001	8.590637	6	0.07219
Kuala Selangor	0.04123	27	1.0000	0.036617	6	0.99983

3.5. Investigating the Optimum Order of Markov Chain Models

This study considers the optimum order found from the minimum loss function of the BIC for selecting the optimum order of the MC model for the hourly API data. Therefore, according to the smallest BIC value, the optimum order of the MC model in this study is the second order that best describes the sequence of hourly API events in the selected stations.

Table 3 shows the BIC values for the observed hourly and daily API data, indicating that the second and third order of the MC model are optimum for the BIC values, which means that the air pollution events for those stations are dependent on the events of two or three consecutive hours before the observed hours. This indicates that the concentration of air pollutants in a particular hour depends on the previous two or three hours. The computational framework employed in the manuscript has been conducted using Python and R software. In addition, the simulated APIs in Tables 3 and 4 have been generated through a simulation study that was conducted based on the Markov part of the SMM package in the R software [48].

**Table 3.** The values of AIC and BIC for each MC model using the hourly APIs.

Station Order	BIC of the Hourly API							
	KL	Klang	Cheras	PJ	Banting	ShA	KS	
Observed API	1	1696.011	788.8619	1729.459	<b>472.9546</b>	1161.643	1084.574	<b>374.2396</b>
	2	1468.189	<b>619.6946</b>	1539.279	485.638	990.832	<b>994.5611</b>	381.2125
	3	<b>1350.565</b>	671.6492	<b>1238.221</b>	690.3285	<b>967.8343</b>	1052.922	578.222
	4	2380.511	1772.959	2338.917	1791.672	2069.016	2141.571	1679.577
	5	5668.314	5072.603	5637.949	5091.351	5363.804	5441.096	4979.267
	6	15,562.45	14,967.25	15,531.98	14986.03	15,245.63	15,335.63	14,873.96
	7	45,241.6	44,646.92	45,211.03	44,665.73	44,925.17	45,015.18	44,553.67
	8	134,275.8	133,681.5	134,245.1	133,700.5	133,959.7	134,049.8	133,588.4
	9	401,375.1	400,781.4	401,344.3	400,800.3	401,059.4	401,149.5	400,688.3
	10	1,202,670	1,202,077	1,202,639	1,202,096	1,202,355	1,202,445	1,201,984

Table 3. Cont.

Station Order	BIC of the Hourly API							
	KL	Klang	Cheras	PJ	Banting	ShA	KS	
Simulated API	1	1842.167	712.794	1803.19	550.6943	1076.533	1080.534	500.2132
	2	1614.669	<b>610.7687</b>	1588.566	<b>517.9346</b>	891.2364	922.0258	<b>443.4873</b>
	3	<b>1103.086</b>	651.6381	<b>1115.187</b>	588.1897	<b>691.0557</b>	<b>689.9981</b>	578.0087
	4	2203.769	1752.948	2215.855	1689.53	1792.237	1791.181	1679.361
	5	5502.787	5052.592	5514.865	4989.205	5091.754	5090.699	4979.048
	6	15,396.82	14,947.24	15,408.88	14,883.88	14,986.28	14,985.22	14,873.74
	7	450,75.86	44,626.9	45,087.91	44,563.58	44,665.81	44,664.76	44,553.44
	8	134,109.9	133,661.6	134,122	133,598.3	133,700.4	133,699.3	133,588.2
	9	401,209.2	400,761.4	401,221.2	400,698.2	400,800.1	400,799	400,688.1
	10	1,202,504	1,202,057	1,202,516	1,201,993	1,202,095	1,202,094	1,201,983

Significant values are represented in bold.

Table 4. The values of AIC and BIC for each MC model using the daily APIs.

Station Order	BIC of the Daily API							
	KL	Klang	Cheras	PJ	Banting	ShA	KS	
Observed API	1	<b>644.1749</b>	352.2063	<b>696.8885</b>	<b>240.6926</b>	<b>448.2603</b>	<b>462.1786</b>	<b>178.31</b>
	2	663.4538	<b>332.7529</b>	698.4464	268.6937	458.9507	483.6555	239.9841
	3	827.6056	491.2444	828.3058	466.8144	636.4306	660.246	426.0242
	4	1548.058	1239.745	1539.657	1224.658	1381.723	1396.332	1184.039
	5	3778.04	3509.026	3762.801	3494.378	3635.01	3661.811	3453.944
	6	10,569.55	10,313.98	10,543.99	10,299.72	10,432.01	10,458.66	10,259.47
	7	30,942.22	30,725.85	30,933.25	30,711.95	30,841.64	30,868.32	30,670.89
	8	92,158.24	91,958.35	92,144.79	91,944.8	92,065.15	92,091.46	91,903.93
	9	275,833.6	275,651.6	275,812.2	275,639.5	275,750.7	275,770.9	275,598.9
	10	826,893.8	826,731.7	826,874	826,720	826,820.8	826,841.2	826,679.5
Simulated API	1	14,485	7672.153	15,679.53	5112.906	9859.986	10,138.47	3322.83
	2	13,159.61	5918.581	14,025.42	4393.635	8522.06	9274.675	2777.936
	3	<b>10,908.42</b>	<b>3692.711</b>	<b>11,496.32</b>	<b>2942.723</b>	<b>5883.349</b>	<b>7334.391</b>	<b>1802.015</b>
	4	11,288.58	4726.253	11,762.87	3978.854	6792.007	8114.475	2890.721
	5	13,952.63	7960.398	14,301.05	7251.089	9894.717	11,170.95	6185.456
	6	23,201.69	17,810.61	23,490.54	17,118.36	19,639.26	20,878.37	16,065.42
	7	52,314.26	47,436.42	52,518.98	46,752.08	49,142.73	50,353.15	45,730.4
	8	140,860	136,418.4	141,000.5	135,751	138,021.5	139,172.8	134,752.5
	9	407,473.1	403,455.3	407,485.1	402,795	404,986.7	406,044.4	401,849.6
	10	1,208,346	1,204,661	1,208,368	1,204,055	1,206,154	1,207,143	1,203,142

Significant values are represented in bold.

Furthermore, Table 4 shows the values of BIC for daily observed and simulated API data. Based on the smallest values of BIC, it can be concluded that the first order of the MC model is optimum for all the stations except Klang, which means that the air pollution events for all the stations are dependent on the events of one day before the observed day.

Thus, this study considers the optimum order found from the minimum loss function of the BIC for selecting the optimum order of the MC model for the API data.

Therefore, for the daily API, the optimum order of the MC model is the first order, according to the smallest BIC value. Furthermore, the results found show that the first order is the most suitable order for describing the probabilistic behavior of the daily API in Klang. This means that the concentration of air pollutants on a particular day depends on the previous day. However, as previously mentioned, the diversity between the first- and second-order MC models is not great; the interpretation of both orders may be very similar.

Figures 4 and 5 show the results of the analysis on the most appropriate order for the distribution of the observed hourly and daily API as well as the simulated hourly and daily API, respectively, for each station in the central region of Peninsular Malaysia. It can be seen from Figure 4 that it is generally found that the MC of the higher order is appropriate for describing the distribution of the hourly API at most stations. However, for the observed and simulated daily API, the first-order model is found to be appropriate at most stations. Likewise, Figure 5 displays that for the simulated hourly API, the appropriate order of the MC model is the first order for all stations except Klang. In perspective, we can use drifting Markov models, introduced by [49] and for which [50] have studied reliability and survival analysis. These models take into account the heterogeneity of the sequences.

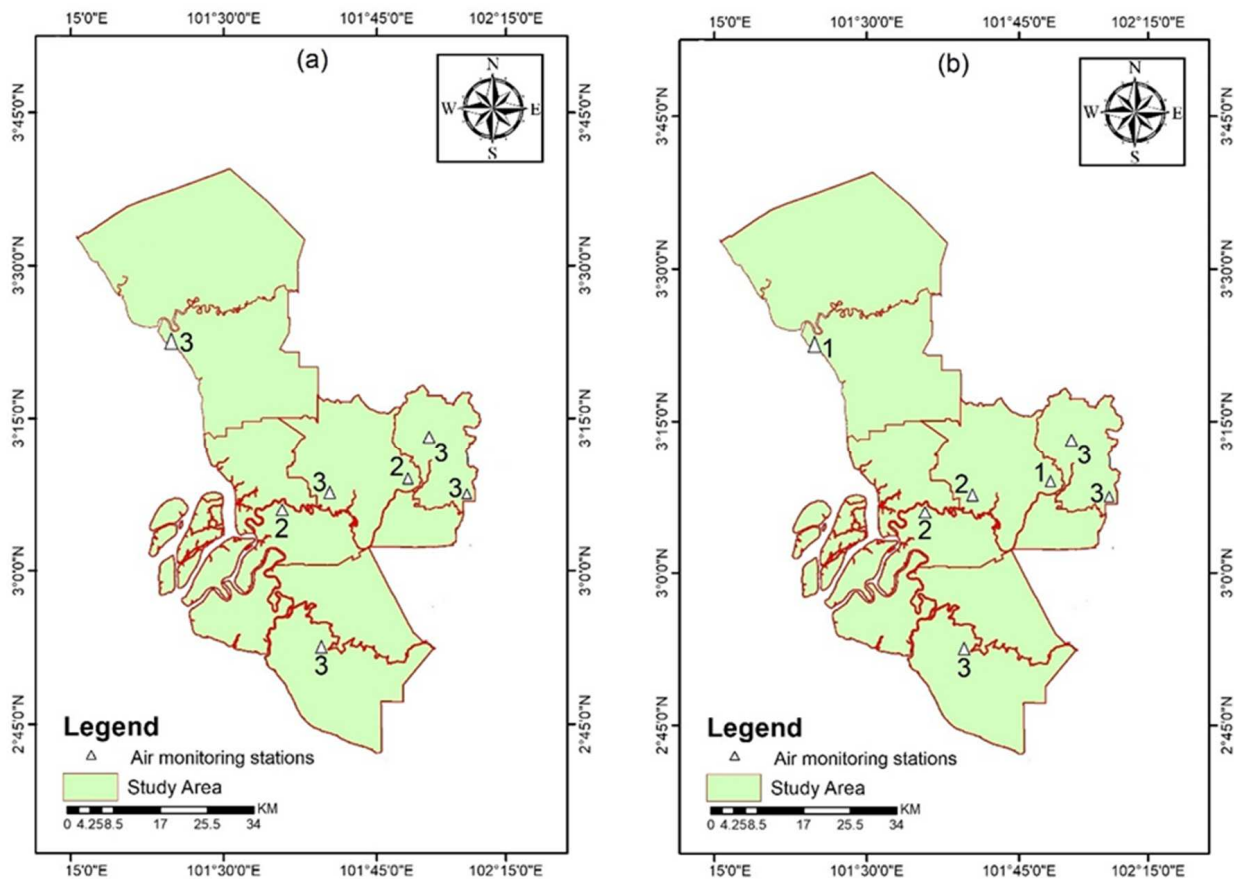


Figure 4. The optimum order of the MC model for the observed (a) hourly and (b) daily API.



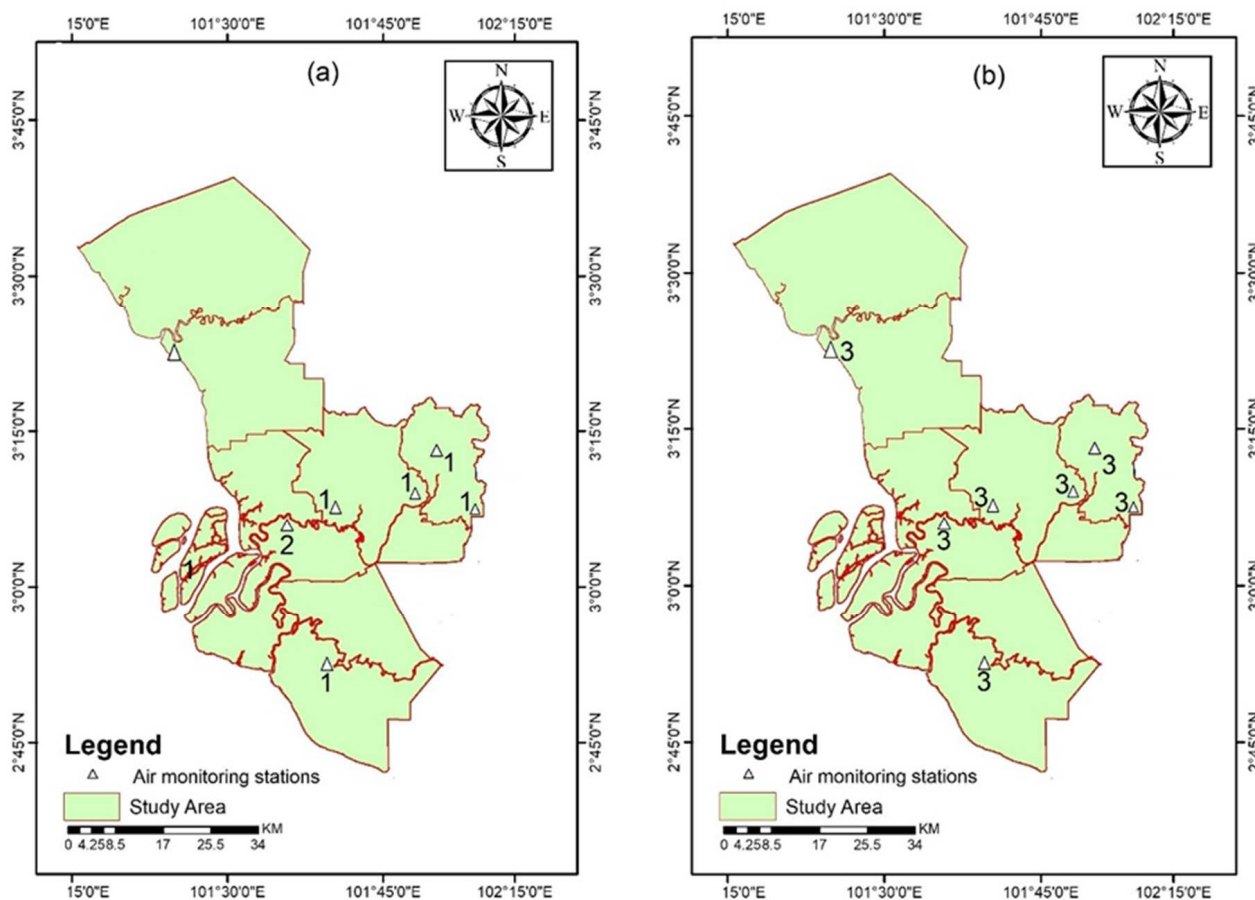


Figure 5. The optimum order of the MC model for the simulated (a) hourly and (b) daily API.

#### 4. Conclusions

In this study, the optimum order of the MC model for the hourly and daily API data, collected from seven stations in Peninsular Malaysia for a period of three years from 2012 to 2014, has been fitted in order to adequately represent the stochastic behavior of air pollution. The current study endeavors to shed some light on the intrinsic serial dependence of the hourly and daily API data through the proper choice of the MC model, based on the minimization of the BIC. The BIC has been used to determine the right order of the MC after testing the Markovian status with the chi-square test, based on the null hypothesis of serial independence. The log-likelihoods of the various orders of the MC have been determined and utilized to compute the BIC values. For the daily API, it is obvious that, according to the smallest values of BIC, the first-order MC is the most dominant in the stations. This is explained by the fact that the dependence of the concentration of the API on any given day is mainly reliant on a preceding day, and not on the long path via which that daily API is obtained. It can be also concluded from this study that the optimum order of the MC models for daily API occurrences varies from one station to another. It is observed that for both the observed and simulated hourly API, the second and the third 1order are more appropriate for most stations. This indicates that the API at any hour has dependence up to the previous two hours. In summary, this research might have significant implications for designing effective air quality management policies and promoting public health. In future work, the high-order multivariate MC model will be applied for modeling the sub-indexes of the API and some climatic factors, such as wind speed, temperature, and rainfall, in order to provide a comprehensive investigation of air pollution in the main cities of Malaysia. In addition, the MLE used in this study has a limitation in capturing the zero frequencies in the count matrices. According to this, future work should address this

limitation by using a robust empirical Bayes method and conducting a simulation study based on the MCMC method to evaluate the model performance.

**Author Contributions:** Project administration, Y.A.; supervision, K.I., W.Z.W.Z. and M.O.; coding, Y.A. and N.V.; reviewing, K.I., W.Z.W.Z., N.V., M.O. and A.A.-Y. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by the YUTP cost center, grant number (015LC0-296).

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** The datasets generated and analyzed during the current study are not publicly available because they are confidential datasets, provided by the Department of Environment Malaysia, but they are available from the corresponding author on reasonable request.

**Acknowledgments:** The authors greatly appreciate the Department of Environment Malaysia and the Institute of Climate Change (IPI) and the Earth Observation Center (EOC) at Universiti Kebangsaan Malaysia for providing the air pollution data and the shapefile dataset that made this paper possible.

**Conflicts of Interest:** The authors declare that they have no conflict of interest.

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