The renormalisation group and the renormalisation group equations

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We describe the rigorous renormalisation group in the framework of the euclidean path integral, restricting (largely) to scalar fields. We start from a historical and conceptual overview leading from block spin transformations to the continuum description, with hindsight to critical phenomena. On the technical side we introduce some of the basic properties of Gaussian measures and perturbations thereof. The decomposition of the covariance of a Gaussian measure and its derivative with respect to a parameter lead to discrete (finite) or continuous (infinitesimal) renormalisation group transformations. The latter constitute the differential flow equations of renormalisation group. We comment on some generic results obtained with these methods. Finally we derive the Callan-Symanzik equations.

Keywords:
Renormalisation group, flow equations, perturbed Gaussian measures, renormalisation theory, Callan-Symanzik equations.

Keypoints:

- Universality and the renormalisation group
- Gaussian measures and renormalisation group transformations
- Rigorous results on Gaussian and non-Gaussian fixed points
- Flow equations and renormalisability
- Callan-Symanzik equations

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1 Introduction

It is not obvious how to give a satisfactory general definition of the renormalisation group (RG) for the purposes of mathematical physics. When trying to do so, one would probably be led to aspects of probability measure theory. Such a procedure would however not capture simultaneously the mathematical and physical aspects of the RG. It is probably more fruitful to first describe the type of problems one would like to solve, and then to show how the RG proceeds and (in some cases) succeeds in solving the problem. In fact the way in which the RG is implemented turns out to depend in an essential way on the problem to be analysed.

Historically renormalisation group equations (RGEs) have their origin both in quantum field theory, mainly in quantum electrodynamics, and (somewhat later) in the theory of critical phenomena in statistical physics. The name goes back to [Stueckelberg and Petermann, 1953], and to [Gell-Mann and Low, 1954]. Their work was based on the fact that (perturbative) relativistic field theories are parametrised through renormalisation conditions. The renormalisation group equations then analyse the relation between different parametrisations. In statistical physics, Wilson [Wilson, 1971a], [Wilson, 1971b] introduced the RG as a method of relating hamiltonians describing the same class of physical phenomena, on successively integrating out degrees of freedom. The Wilson RG is conceptually more general than the now so-called old RG from quantum field theory, and comprises it conceptually (under certain caveats) as we will show.

The Wilson RG is naturally defined in euclidean space or on lattices. Problems raised by the Minkowski metric governing relativistic physics can partially be circumvented by analytic continuation techniques. So far the theories constituting the standard model of particle physics are not mathematically controllable by the Wilson RG. In fact the standard model consists of several sectors which are not decoupled from each other. Still it makes sense to regard them separately in a first analysis. For the sector containing nonabelian gauge fields only, in particular quantum chromodynamics, renormalised perturbation theory says that the theory has a Gaussian (i.e. interaction free) ultraviolet fixed point whose neighbourhood could be controlled by the RG. But quantum chromodynamics behaves in a completely different way in the infrared (or low energy) regime which is (and probably will be for a long time) out of control of rigorous analysis. For the other sectors one expects that the only way in which they can be given rigorous meaning when taking away the ultraviolet cutoff, is as interaction free theories. This is the triviality conjecture which could be made rigorous for the one- (and to some degree two-)component scalar field theory with lattice regularisation [Fröhlich, 1982], [Aizenman, 1982], [Aizenman and Duminil-Copin, 2021]. Nevertheless the standard model is extremely successful in physics when considered as a renormalisable model in perturbation theory. We shall describe how the results on the renormalisability of the standard model can be recovered from the differential RGEs.

The Wilson RG is intimately linked to a prominent problem of contemporary theoretical physics, the so-called N-body problem. It adresses the physical properties of systems comprising a large number of particles, or more generally a large number N of degrees of freedom, which interact with each other via some interaction assumed to be (essentially) known.

In relativistic quantum physics, the N-body problem is implicitly always present since relativistic kinematics allows to transform energy into particles, and quantum physics implies that all degrees of freedom of the theory (possibly restricted by symmetries) are virtually present due to the uncertainty relations. The systems studied in quantum field theory are introduced

through lagrangian densities which are functionals of the physical fields and which depend on a number of constants related in particular to the interaction strengths and to the masses of the particles appearing in the theory. In quantum electrodynamics formulated for electrons and positrons, these would be the electron charge and the electron mass. In our short review we do not enter into the symmetry aspects of gauge and spinor fields, but rather limit ourselves to scalar fields. In the standard model of particle physics there appears such a scalar field, called the (Brout-Englert-)Higgs field. In a certain parametrisation, the (initial) lagrangian density within the standard model, associated with this field only, is given (when continued to the euclidean metric) by

$$\mathscr{L}_{\text{BEH}}(\varphi(x)) = \frac{1}{2}\, z\, (\partial_\mu \varphi(x))(\partial_\mu \varphi(x)) + \frac{1}{2}\, m^2\, \varphi^2(x) + \frac{g}{4}\, \varphi^4(x)\,, \tag{1}$$

where the sum over repeated indices (here μ) is implicit. Physical theories are defined on four-dimensional spacetime in which case $x \in \mathbb{R}^4$. In the euclidean lagrangian density (1), the constants we mentioned are z, related to the normalisation of the field variable, then m, (related to) the particle mass, and finally g, which in dimension $d = 4$ is a pure number and measures the scalar particle's selfinteraction strength.

In nonrelativistic physics, the N-body problem is at the heart of explanations of macroscopic physics in terms of the microscopic constituents of matter. A typical example is a system of N charged particles interacting pairwise via the Coulomb interaction. A simpler system, which nevertheless exhibits the complexity of the problem, is the Ising spin system. In its simplest form it is described by the hamiltonian

$$H(\{s_i\}) = -\sum_{\langle i,j\rangle} J\, s_i\, s_j\,, \tag{2}$$

where J is a real constant, and the spin variables s_i, $i = 1,\ldots,N$, can take the values ± 1. The symbol $\langle\,,\rangle$ indicates nearest neighbours. In order to make it well-defined, one has to fix the geometric disposition of the spin variables. For example the spins may be associated with sites on a cubic lattice in dimension d. It then turns out that the behaviour of the system depends in an important way on this geometry, in particular on the dimension. Physically interesting questions concern for example the energy spectrum of the system. More generally one is interested in its properties at thermal equilibrium which can be derived from the partition function

$$Z(\beta;\{h_i\}) = \sum_{\{s_i\}} e^{-\beta H(\{s_i\}) + \beta \sum_i h_i s_i}\,. \tag{3}$$

The logarithm of Z times $-\beta^{-1}$ plays the role of the free energy of the system. Here $\beta = \frac{1}{T}$ is the inverse temperature in units of Boltzmann's constant, and h_i denotes the magnetic field at site i in suitable units, which interacts with the spin s_i.

One way of analysing Z (be it analytically or numerically) with hindsight to the macroscopic properties of the system, is to regroup the spins into blocks of, say, n spins. Their appropriate definition depends on the geometry of the system. The spin variable of a given block, S_j, now takes values $n, n-1, \ldots, -n+1, -n$. The values $n-\nu$ or $-n+\nu$ ($\nu \geq 0$) are realized by $\binom{n}{\nu}$ microscopic spin configurations. To rewrite Z for vanishing magnetic field in terms of the new

variables while keeping its form (3), one defines a new hamiltonian $H^{(1)}$ such that

$$e^{-\beta H^{(1)}(\{S_j\})} = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})} \prod_j \delta_{S_j, \sum_{i\, s.t.\, s_i \in S_j} s_i} \, .$$

In this case

$$Z(\beta;0) = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})} = \sum_{\{s_i\}} \underbrace{\sum_{\{S_j\}} \prod_j \delta_{S_j, \sum_{i\, s.t.\, s_i \in S_j} s_i}}_{=1} e^{-\beta H(\{s_i\})} = \sum_{\{S_j\}} e^{-\beta H^{(1)}(\{S_j\})} \, . \tag{4}$$

The motivation for this way of proceeding is not evident, since the hamiltonian, written in terms of the block spin variables, becomes immediately complicated. Generically blocks interact via their boundaries with neighbouring blocks, which in turn interact with new neighbours etc. so that already after the first block transformation we will get a hamiltonian (for vanishing field) of the form

$$H^{(1)}(\{S_j\}) = \sum_{k \geq 2} \sum_{j_1, \ldots, j_k} K_{j_1, \ldots, j_k} \, S_{j_1} \ldots S_{j_k} \, , \tag{5}$$

where the constants $K_{j_1,\ldots,j_k}$ can (in principle) be calculated from the original hamiltonian. The hamiltonian in terms of the block variables has become nonpolynomial and nonlocal. Still the symmetry under $\{s_i\} \to \{-s_i\}$ of the original hamiltonian (2) implies that $H^{(1)}(\{S_j\})$ is even in $\{S_i\}$, and the constants $K_{j_1,\ldots,j_k}$ will have an expansion in J.

The interest of the RG method of which the previous block spin transformation is a conceptually simple, though technically awkward, realisation, stems in particular from the fact that the behaviour of physical systems when they approach second order phase transitions, turns out to be quite universal. The critical exponents, which caracterise the singularity of response functions, like the susceptibility or the specific heat near the critical temperature, are independent of most details of the microscopic hamiltonians or of the specific physical systems studied, as for example magnetic systems, systems undergoing a superfluid Lambda transition, systems of alloys ... What matters, is the dimensionality of the system, its symmetries and the number and nature of the field variables, but not the details of the underlying microscopic description of the system. This universality was observed experimentally and partially recovered in the mean field description of the systems, and later understood in a convincing way with the aid of the RG. We note in passing that the formulation of the RG in terms of successive block spin transformations shows that the RG is generally only a semi-group, namely two successive RG block spin transformations (RGTs) can be fused into a single one, but there is no inverse block spin transformation.

The partition function (3) is the generating functional of the correlation functions of the system. The simplest one, the mean value of spin i, is obtained as

$$M_i(\{h_j\}, T) = \langle s_i \rangle = \frac{1}{\beta} \, \partial_{h_i} \log Z \, . \tag{6}$$

Assuming the field to be homogeneous: $h_i = h \; \forall i$, and assuming translation invariance of the model (which is related to the assumptions on the boundary conditions imposed), we get

$$M(h,T) = \frac{1}{N} \sum_i \langle s_i \rangle = \frac{1}{N} \, \frac{1}{\beta} \, \partial_h \log Z(\beta, h) \, . \tag{7}$$

One observes experimentally that $M(h=0,T)$ vanishes at the critical temperature T_c characterising the transition from the ferromagnetic to the paramagnetic state, as

$$M \propto |T - T_c|^{\beta} \,, \tag{8}$$

which defines the critical exponent β (not to be confounded with the inverse temperature).
The zero field susceptibility χ is defined as

$$\chi(T) = \frac{1}{N}\frac{1}{\beta^2}\sum_{ij}\partial_{h_i}\partial_{h_j}\log Z\big|_{\{h_k\}=0} = \frac{1}{N}\sum_{ij}\big(\langle s_i\, s_j\rangle - \langle s_i\rangle\langle s_j\rangle\big)\big|_{\{h_k\}=0} \,. \tag{9}$$

If the system is translation invariant, we may write (independently of the choice of j)

$$\chi(T) = \frac{1}{\beta}\,\partial_h M(h,T)\big|_{h=0} = \sum_i \big(\langle s_i\, s_j\rangle - \langle s_i\rangle\langle s_j\rangle\big)\big|_{h=0} = \sum_i \langle s_i\, s_j\rangle_c\big|_{h=0} \,. \tag{10}$$

Here the lower index c stands for connected or cumulant. The physical expectation is that the susceptibility diverges at the critical temperature T_c as

$$\chi(T) \propto |T - T_c|^{-\gamma} \,, \tag{11}$$

which defines the critical exponent γ.
Connected correlation functions are obtained through derivatives of $\log Z$. They should stay well-defined in the thermodynamic limit $N \to \infty$, and they can depend on the boundary conditions imposed on the system. The correlation length ξ describes the large distance decay of the connected two-point function. It is defined as

$$\frac{1}{\xi(T)} := -\lim_{\mathrm{dist}(i,j)\to\infty} \frac{\log|\langle s_i\, s_j\rangle_c(T)\,|}{\mathrm{dist(i,j)}} \,, \tag{12}$$

where the distance is measured in lattice units. If this limit vanishes, the correlation length is infinite. One expects that the correlation length diverges close to the critical temperature as

$$\xi(T) \propto |T - T_c|^{-\nu} \,, \tag{13}$$

which defines the critical exponent ν. At the critical temperature the correlation length is infinite, and the correlations have a power law falloff, parametrised as

$$|\langle s_i\, s_j\rangle_c| \sim \mathrm{dist(i,j)}^{-(d-2+\eta)} \quad \text{for } \mathrm{dist(i,j)} \to \infty \,, \tag{14}$$

which defines the critical exponent η, d being the number of space dimensions.

Critical behaviour was analysed by Landau in 1937 in the mean field approximation. Here the following values for the above mentioned critical exponents of Ising type models were found :

$$\beta = 1/2 \,, \quad \gamma = 1 \,, \quad \nu = 1/2 \,, \quad \eta = 0 \,. \tag{15}$$

The divergence of the correlation length at the critical temperature hints to the fact that microscopic details become unimportant. Therefore iterating block transformations may shed light on the problem, despite of transforming simple hamiltonians into complicated ones. For the

Ising model the block variables take values in larger and larger intervals, of size $2n^k$ after k iterations. After suitable rescaling of the variables one typically ends, on taking limits, with a continuous distribution in a compact or noncompact interval of values. Passing to a continuum description, we write $\varphi(x)$ instead S_i, the continuous variable x now taking the role of the discrete index i. When performing a Taylor expansion of the monomials $\varphi(x_1)\dots\varphi(x_k)$ around the same point x, we formally get from (5) families of hamiltonians of the type[1]

$$H(\varphi) = \int_V \sum_{k\geq 2} \sum_{n_1,\nu_1,\dots,n_k,\nu_k} A_{n_1,\nu_1,\dots,n_k,\nu_k}(x)\ (\partial^{\nu_1}\varphi^{n_1}(x))\dots(\partial^{\nu_k}\varphi^{n_k}(x))\ d^dx\ , \tag{16}$$

where $n_i \in \mathbb{N}$, $\nu_i \in \mathbb{N}_0$, and the $A_{n_1,\nu_1,\dots n_k,\nu_k}(x)$ are real. If translation invariance in euclidean space holds, the coefficients $A_{n_1,\nu_1,\dots n_k,\nu_k}$ are constants independent of x. The (Lebesgue) integral extends over the volume V of the system, with hindsight to the infinite volume limit.

Considering general hamiltonians of the type (16), the universality hypothesis stated above implies that a large class of hamiltonians of this form should lead, under suitable assumptions, to the same critical behaviour as the original lattice hamiltonian (2). The infinite set of coefficients $\{A_{n_1,\nu_1,\dots n_k,\nu_k}\}$ characterises a specific theory. A RGT $\mathscr{R}$ maps (on suitably rescaling the field variables) the infinite dimensional space of coefficients $\{A_{n_1,\nu_1,\dots n_k,\nu_k}\}$ onto itself. In this setting critical behaviour (in the strict sense) is associated to fixed points under the RG

$$\mathscr{R}(A_{n_1,\nu_1,\dots n_k,\nu_k}) = A_{n_1,\nu_1,\dots n_k,\nu_k} \quad \forall n_1,\nu_1,\dots n_k,\nu_k\ .$$

It then turns out, in approximate analysis, that when one stays sufficiently close to a fixed point, only a few coefficients $A_{n_1,\nu_1,\dots n_k,\nu_k}$ have to be fine-tuned to converge to a fixed point - these coefficents are called relevant (or marginal) parameters -, whereas the others, which are called irrelevant, only modify the fine-tuned path to the fixed point under the RGT $\mathscr{R}$ in a qualitatively inessential way.

So the universality idea is that irrelevant parameters are not needed to describe the macroscopic behaviour of the system near the fixed point. For (5) heuristic analysis shows that in $d=4$, terms $\propto A_{n_1,\nu_1,\dots n_k,\nu_k}$ for which $\sum n_i + \nu_i > 4$, are irrelevant. The explanation of universality thorugh the RG is then tantamount to show that the set of relevant parameters is the same for many physical systems. For a review in theoretical physics see [Zinn-Justin, 2007], [Zinn-Justin, 2010]. From the mathematical point of view the universality hypothesis is in general only a heuristic guiding principle. But rigorous statements can be made under sufficiently strong assumptions on the parameter values, as will be detailed below in section 3.1.

Coming back to the universality class of the Ising model, a specific fixed point of the theory in $d=4$, characterised by an infinite correlation length and vanishing effective interaction, turns out to be associated with a specific choice of relevant initial parameter values for the simple continuum hamiltonian, taken here as a starting point of our RG analysis

$$H^0(\varphi) = \int_V \mathscr{H}^0(\varphi(x))\ d^dx\ ,\ \ \mathscr{H}^0(\varphi(x)) = \frac{1}{2}\, z^0\, (\partial_\mu\varphi(x))^2 + \frac{1}{2}\,(m^0)^2\,\varphi^2(x) + \frac{g^0}{4}\,\varphi^4(x)\ , \tag{17}$$

[1] This form of the hamiltonian is based on the assumption that the constants $K_{j_1,\dots,j_k}$ from (5) decay sufficiently rapidly, when the distance between any pair of blocks becomes large. In any case at this stage the reasoning is formal.

which also appeared in the field theory context (1). For mathematical consistency (or physical stability) $\mathscr{H}^0$ should be bounded from below. We require that

$$\mathscr{H}^0(\varphi) \geq -K \quad \text{for some} \quad K > 0\,,$$

which implies $g^0 \geq 0$. The dimension $d = 4$ is of particular interest not only in quantum field theory, but also in statistical physics, since theoretical physics has developed calculational techniques permitting a continuation in the number of dimensions (the so-called ε-expansion) which allow to draw conclusions on the critical behaviour at $d = 3$ from that at $d = 4$.
Continuing on this line of reasoning the partition function corresponding to (16), (17) is written as

$$Z(j) = \int \mathscr{D}(\varphi)\; e^{-\int_V \mathscr{H}^0(\varphi(x))\; d^dx + \int_V j(x)\,\varphi(x)\; d^dx}\,. \tag{18}$$

Here we absorbed the factor of β in the constants appearing in $\mathscr{H}^0$, respectively in $j = \beta h$. The notation $\mathscr{D}(\varphi)$, common in the physics literature, should indicate a (continuously) infinite product of translationally invariant Lebesgue measures, since the field variable at any point x can take any value. The corresponding integrals are called path integrals. Such measures are known not to exist mathematically. So we will revise the mathematical setting in section 2.

In the contiunuum description the expression for the zero field susceptibility becomes

$$\chi = \int_V \langle \varphi(x)\,\varphi(y)\rangle_c\; d^dx = \int_V \frac{\delta}{\delta j(x)}\,\frac{\delta}{\delta j(y)}\;\log Z(j)\Big|_{j\equiv 0}\; d^dx\,, \tag{19}$$

where we have introduced functional (Fréchet) derivatives w.r.t. j. Due to translation invariance χ should not depend on y, but it depends on the parameters of $\mathscr{H}^0$. We are interested in the infinite volume limit. For the correlation length we get using translation invariance

$$\frac{1}{\xi} := -\lim_{x\to\infty} \frac{\log|\langle \varphi(x)\;\varphi(0)\rangle_c|}{|x|}\,. \tag{20}$$

In the next section 2 we present a mathematically well-defined expression, corresponding to a regularized version of (18), which is based on the definition of Gaussian measures on infinite dimensional function spaces. These are characterised by their covariances. Using decompositions of the covariance of a Gaussian measure into positive parts, we can define the action of finite RGTs on the corresponding Hamiltonians or effective actions in section 3. In subsection 3.1 we comment on rigorous work performed in this framework and describe more explicitly a few recent results. In subsection 3.2 we describe work on the Gross-Neveu model as a prominent example of one (of the few) rigorous RG ultraviolet constructions.
If the covariance depends smoothly on a parameter t, deriving w.r.t. this parameter generates a functional differential equation, which is one form of the RG differential equations and describes the evolution of the effective action under infinitesimal RGTs, see section 4. In the theoretical physics literature the name of RG equations (RGEs) refers to differential equations for Green functions or Feynman amplitudes in renormalized perturbation theory, which permit to analyse their behaviour under scaling transformations. We derive one form of those equations, the Callan-Symanzik equations.

In closing this introductory overview we would like to apologise that for lack of space we do not touch upon many important RG achievements. We privilege issues minimising supplementary knowledge and prerequisites, and also to some degree historical priority. Let us just

refer to important work, mostly on fermionic models in solid state physics. Here there appears another relevant parameter related to the particle density, the so-called chemical potential μ. Reviews are [Salmhofer, 1999], [Mastropietro, 2008], [Giuliani et al., 2021]. A large series of papers on two-dimensional Fermi liquids appeared in 2003/2004, see [Feldman et al., 2004] and references given there. Another similarly monumental endeavour on bosonic theories, with hindsight to the Bose-Einstein condensation, still going on, can be found from the reference [Balaban et al., 2017].

2 Perturbed Gaussian measures

Finite dimensional Gaussian integrals are well-known to mathematicians and physicists. We recall

$$\frac{1}{\sqrt{2\pi a}} \int_{\mathbb{R}} e^{-\frac{1}{2}\frac{x^2}{a}}\, dx = 1 \;\text{ for }\; a>0\,, \qquad \frac{1}{\sqrt{(2\pi)^n \det A}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,A^{-1}x)}\, d^n x = 1\,.$$

In the n-dimensional case we write $x=(x_1,\dots,x_n)$, $(x,A^{-1}x) = \sum_{i,j} x_i (A^{-1})_{ij} x_j$, and $(\ ,\ )$ denotes the standard scalar product in $\mathbb{R}^n$. The matrix (or operator) A is supposed to be positve and symmetric:

$$(x,Ax) > 0 \;\; \forall x \neq 0\,, \quad (x,Ay) = (Ax,y) \quad \forall x,y \in \mathbb{R}^n\,.$$

We then denote for $A>0$ on $\mathbb{R}^n$

$$d\mu_A(x) = \frac{1}{\sqrt{(2\pi)^n \det A}}\, e^{-\frac{1}{2}(x,A^{-1}x)}\, d^n x$$

to be the (normalised) Gaussian measure of covariance A.
For the generating functional $Z_0(h)$ of the moments of the Gaussian measure we then obtain

$$Z_0(h) := \int e^{(h,x)}\, d\mu_A(x) \;=\; e^{\frac{1}{2}(h,Ah)}\,,$$

$$\partial_{h_{i_1}} \dots \partial_{h_{i_{2n}}} Z_0(h)\big|_{h=0} = \int x_{i_1} \dots x_{i_{2n}}\, d\mu_A(x) \;=\; \partial_{h_{i_1}} \dots \partial_{h_{i_{2n}}}\, e^{\frac{1}{2}(h,Ah)}\big|_{h=0}\,.$$

This expression can be evaluated to give

$$\partial_{h_{i_1}} \dots \partial_{h_{i_{2n}}} Z_0(h)\big|_{h=0} = \sum_{\text{pairings}} A_{\alpha_1,\alpha_2} \dots A_{\alpha_{2n-1},\alpha_{2n}} \qquad \text{(Wick's theorem)}\,,$$

where the sum is over the $(2n-1)!!$ pairings such that we have $\{\alpha_1,\dots,\alpha_{2n}\} = \{i_1,\dots,i_{2n}\}$ (for sets with repetitions).
The following rules of calculation will be used later [2].
Let $F(x)$ be differentiable and bounded. Then we have

1. Integration by parts : $$\int x_i\, F(x)\, d\mu_A(x) = \int A_{ik}\, [\partial_{x_k} F(x)]\, d\mu_A(x)\,. \tag{21}$$

[2] Summation over doubly occurring indices is always understood.

2. Translation of a Gaussian measure : Under a change of variable $x = x' + x_0$ we find

$$d\mu_A(x) = e^{-\frac{1}{2}(x_0, A^{-1}x_0)}\, e^{-(A^{-1}x_0, x')}\, d\mu_A(x') \,. \tag{22}$$

3. Decomposition of the covariance : Assume that we have the decomposition

$$A = A_1 + A_2 \,, \quad A_i > 0 \text{ and symmetric } .$$

Then

$$\int F(x)\, d\mu_A(x) = \int \Big(\int F(x_1 + x_2)\, d\mu_{A_1}(x_1) \Big) d\mu_{A_2}(x_2) \quad \forall\, F \in L^1(d\mu_A(x)) \,. \tag{23}$$

4. Infinitesimal Change of covariance : Let $A(t)$ depend differentiably on a real parameter t. Then

$$\frac{d}{dt} \int F(x)\, d\mu_{A(t)}(x) = \frac{1}{2} \int [\partial_{x_i} \dot{A}_{ik}(t) \partial_{x_k} F](x)\, d\mu_{A(t)}(x) \,, \text{ where } \dot{A} := \frac{dA}{dt} \,. \tag{24}$$

5. Finite change of covariance :

$$\int F(x)\, e^{-\frac{1}{2}(x, a^{-1}x)} d\mu_A(x) = \frac{1}{\sqrt{\det(1 + A\, a^{-1})}} \int F(x)\, d\mu_{A+a}(x) \quad \text{for } A + a > 0 \,. \tag{25}$$

The definition of Gaussian measures can be extended to infinite dimensional spaces of functions with the aid of the Bochner-Minlos theorem [Glimm and Jaffe, 1987]. The operator A now is supposed to be a positive linear operator acting on some linear space of functions. The preceding formulae can be straightforwardly translated to this context and remain valid. The most important caveat is that the support of Gaussian measures on infinite dimensional spaces is fixed by the covariance of the measure, i.e. by A. It becomes more restricted if the covariance has better regularity and fall-off properties.

For the covariance (or the free propagator) appearing in euclidean scalar field theory which in momentum space takes the form

$$\tilde{C}(p) = \frac{1}{p^2 + m^2} \,, \tag{26}$$

the support is distributional in space dimensions $d \geq 2$. This is related to the nonintegrability of $\tilde{C}(p)$ for p large, or to the singularity at $x = y$ of

$$C(x - y) = \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{ip(x-y)}}{p^2 + m^2} \,. \tag{27}$$

In the infinite dimensional case the generating functional takes the form

$$Z_0(j) := \int e^{(j,\varphi)}\, d\mu_C(\varphi) \;=\; e^{\frac{1}{2}\langle j, C\, j\rangle} \,,$$

and we have again Wick's theorem

$$\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_{2n})}\, Z_0(j)\big|_{j\equiv 0} = \int \varphi(x_1) \ldots \varphi(x_{2n})\, d\mu_C(\varphi) = \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_{2n})}\, e^{\frac{1}{2}(j, C\, j)}\big|_{j\equiv 0}$$

$$= \sum_{\text{pairings}} C(x_{\alpha_1} - x_{\alpha_2}) \dots C(x_{\alpha_{2n-1}}, x_{\alpha_{2n}}) \ ,$$

The generalisation of the expressions (21) to (25) to the infinite dimensional case can be written

1. Integration by parts : $\int \varphi(x)\, F(\varphi)\, d\mu_C(\varphi) = \int \big(C * \frac{\delta F(\varphi)}{\delta \varphi}\big)\, d\mu_C(\varphi)$, (28)

with the definition $C * \frac{\delta}{\delta \varphi} := \int C(x-y) \frac{\delta}{\delta \varphi(y)}\, d^d y$.

2. Translation of a Gaussian measure : Under a change of variable $\phi = \varphi + \psi$ we find

$$d\mu_C(\phi) = e^{-\frac{1}{2}(\psi, C^{-1}\psi)}\, e^{-(C^{-1}\psi, \varphi)}\, d\mu_C(\varphi) \ . \tag{29}$$

3. Decomposition of the measure : Assume that we have the decomposition

$$C = C_1 + C_2 \ , \quad C_i > 0 \ .$$

Then for $F(\varphi) \in L^1(d\mu_C(\varphi))$

$$\int F(\varphi)\, d\mu_C(\varphi) = \int \big(\int F(\varphi_1 + \varphi_2)\, d\mu_{C_1}(\varphi_1) \big)\, d\mu_{C_2}(\varphi_2) \ . \tag{30}$$

One can show that $\operatorname{supp} d\mu_{C_1}$, $\operatorname{supp} d\mu_{C_2} \subset \operatorname{supp} d\mu_C$.

4. Infinitesimal change of covariance :

$$\frac{d}{dt} \int F(\varphi)\, d\mu_{C(t)}(\varphi) = \frac{1}{2} \int [\langle \frac{\delta}{\delta \varphi}, \dot{C}(t) \frac{\delta}{\delta \varphi} \rangle F](\varphi)\, d\mu_{C(t)}(\varphi) \ , \quad \text{where } \dot{C} = \frac{d}{dt} C \ . \tag{31}$$

Here we use the notation

$$\langle \frac{\delta}{\delta \varphi}, \dot{C}(t) \frac{\delta}{\delta \varphi} \rangle = \int d^d x\, d^d y\, \frac{\delta}{\delta \varphi(x)} \dot{C}(t; x-y) \frac{\delta}{\delta \varphi(y)} \ .$$

The functional $F(\varphi)$ is supposed to be smoooth, and integrable w.r.t. $d\mu_{C(t)}\ \forall t$.

5. Finite change of covariance :

$$\int F(\varphi)\, e^{-\frac{1}{2}(\varphi, \delta C^{-1} \varphi)}\, d\mu_C(\varphi) = \frac{1}{\sqrt{\det(1 + C\, \delta C^{-1})}} \int F(\varphi)\, d\mu_{C + \delta C}(\varphi) \tag{32}$$

for $C + \delta C > 0$ and $C\, \delta C^{-1}$ traceclass.

We will be interested in the correlation functions of perturbed Gaussian measures of the form

$$e^{-L_V^0(\varphi)}\, d\mu_C(\varphi) \ . \tag{33}$$

Here L_V^0 is supposed to be of the general form of H^0 from (17),

$$L_V^0(\varphi) = \int_V \mathscr{L}^0(\varphi(x))\, d^d x \ , \ \ \mathscr{L}^0(\varphi(x)) = \frac{1}{2} \nu^0\, \varphi^2(x) + \frac{1}{2} b^0\, (\partial_\mu \varphi)^2(x) + \frac{g^0}{4}\, \varphi^4(x) \ . \tag{34}$$

We have changed the naming of the coefficients of the quadratic terms since they have been partially transferred to the Gaussian measure $d\mu_C(\varphi)$. The integration in (34) has been restricted to a finite volume V. Mathematical consistency requires that the coefficients of these (absorbed) terms are positive so that the covariance of the Gaussian measure (possibly after rescaling) takes the form (26). Generally one wants to show that connected correlation functions exist in the infinite volume limit, otherwise stated, they do not depend on V, up to corrections vanishing when V exhausts $\mathbb{R}^4$.
We then define as before the correlation functions

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle := \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_n)} Z(j)\big|_{j\equiv 0}\,, \quad Z(j) = \int e^{(j,\varphi)}\, e^{-L_V^0(\varphi)}\, d\mu_C(\varphi)\,.$$

The connected correlation functions are defined as the moments of $\log Z(j)$:

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle_c := \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_n)} \log Z(j)\big|_{j\equiv 0}\,.$$

The connected free propagator amputated correlation functions are defined as the moments of $\log Z(C^{-1} j)$. All these systems of functions contain in principle the full information about the theory. The connected ones are free from certain divergences which appear in the infinite volume limit. They are thus more suitable for studying the theory. The free propagator amputated correlation functions appear for example when writing scattering cross sections in quantum field theory.

Going back to the hamiltonians of type (17), we will suppose that our theory is symmetric under $\varphi \to -\varphi$, as is the case for the Ising model w.r.t. the set of variables $\{s_i\}$. We also assume that it is symmetric under the euclidean group. This assumption is generally admitted for continuum models supposed to have the same critical behaviour as the Ising model, the physical reason being that lattice details become invisible when the correlation length becomes infinite. In this case the lowest order terms w.r.t. the number of fields φ and the number of derivatives $\partial_\mu \varphi$ are indeed

$$\varphi^2(x)\,,\ (\partial_\mu \varphi)^2(x)\,,\ \varphi^4(x)\ .$$

The rigorous analysis close to a critical point starts from bare functionals of the form (34). The constant ν^0 has to be chosen suitably as a function of g^0 for the theory to be (close to) critical so that $T - T_c$ is tuned via ν^0.

We stated already that the support of the measure $d\mu_C$ is distributional in $d \geq 2$. This means that expressions of the type $\varphi^2(x)\,,\ (\partial_\mu \varphi)^2(x)\,,\ \varphi^4(x) \dots$ are not well-defined. One can show that for covariances falling off more rapidly in p-space, the support of the measure becomes more regular in position space. If they fall off more rapidly than any power of $|p|$, the support is contained in the set of smooth functions in position space, see e.g. [Reed, 1973], [Colella, P., Lanford, O.E., 1973]. We will use subsequently the following regularised covariance:

$$\tilde{C}^{\Lambda,\Lambda_0}(p) = \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} d\lambda\ e^{-\lambda(p^2+m^2)} = \frac{e^{-\frac{p^2+m^2}{\Lambda_0^2}} - e^{-\frac{p^2+m^2}{\Lambda^2}}}{p^2+m^2}\,, \quad 0 \leq \Lambda \leq \Lambda_0 < \infty\,. \tag{35}$$

We have, uniformly in p : $\lim_{\Lambda\to 0}\lim_{\Lambda_0\to\infty}\tilde{C}^{\Lambda,\Lambda_0}(p)=\tilde{C}(p)$ [3]. There is a general conviction that interesting results do not depend on the choice of the regularisation. But in practice interesting results can mostly be obtained only for judicious choices. The previous choice has the advantage of having good analyticity properties.

Our mathematically well-defined starting expression corresponding to (18) is then

$$Z^{\Lambda,\Lambda_0}(j)=\int e^{-L_V^0(\varphi)+(j(x),\varphi)}\,d\mu_{C^{\Lambda,\Lambda_0}}(\varphi)=e^{\frac{1}{2}\langle j,C^{\Lambda,\Lambda_0}j\rangle}\int e^{-L_V^0(\phi+C^{\Lambda,\Lambda_0}*j)}\,d\mu_{C^{\Lambda,\Lambda_0}}(\phi)\;, \quad (36)$$

where we performed a translation of variables $\varphi=\phi+C^{\Lambda,\Lambda_0}*j$, see (29).

For $\varphi=C^{\Lambda,\Lambda_0}*j$ we define

$$e^{-L_V^{\Lambda,\Lambda_0}(\varphi)}:=\frac{1}{Z^{\Lambda,\Lambda_0}(0)}\,e^{-\frac{1}{2}\langle\varphi,(C^{\Lambda,\Lambda_0})^{-1}\varphi\rangle}\,Z^{\Lambda,\Lambda_0}((C^{\Lambda,\Lambda_0})^{-1}\varphi)$$

or equivalently

$$L_V^{\Lambda,\Lambda_0}(\varphi)=\log Z^{\Lambda,\Lambda_0}(0)-\log Z((C^{\Lambda,\Lambda_0})^{-1}\varphi)+\frac{1}{2}\langle\varphi,((C^{\Lambda,\Lambda_0})^{-1}\varphi\rangle\;. \quad (37)$$

We thus find

$$e^{-L_V^{\Lambda,\Lambda_0}(\varphi)}=\frac{1}{Z^{\Lambda,\Lambda_0}(0)}\int e^{-L_V^0(\phi+\varphi)}\,d\mu_{C^{\Lambda,\Lambda_0}}(\phi)\;. \quad (38)$$

The functional $L^{\Lambda,\Lambda_0}(\varphi)$ is called the Wilson effective action at scale Λ. Its moments are the connected amputated Schwinger functions (CASFs) of the theory which give access to its physical contents.

3 Renormalisation group transformations

There are two classes of RG problems to be distinguished. The construction of models with an ultraviolet (uv) fixed point or of models with an infrared (ir) fixed point [4]. The first situation is the one from quantum field theory where one wants to take the ultraviolet limit $\Lambda_0\to\infty$. In this case we will assume that the theory is infrared safe which means in our case that $m^2\neq 0$. A convenient decomposition of the covariance C^{0,Λ_0} (35) is

$$C^0_{\rm uv}(m;x-y)=\int\frac{d^dp}{(2\pi)^4}\,e^{ip(x-y)}\int_1^\infty d\lambda\;e^{-\lambda(p^2+m^2)}\;, \quad (39)$$

$$C^l_{\rm uv}(m;x-y)=\int\frac{d^dp}{(2\pi)^4}\,e^{ip(x-y)}\int_{M^{-2l}}^{M^{-2(l-1)}}d\lambda\;e^{-\lambda(p^2+m^2)}\;,\quad 1\le l\le N\;, \quad (40)$$

where $M>1$, $\Lambda_0=M^{2N}$, so that $\sum_{=0}^N C^l_{\rm uv}=C^{0,\Lambda_0}$. Often one requires $M\gg 1$ for technical reasons. The aim of an RG analysis is then to obtain statements uniform in the uv limit $N\to\infty$

[3] Λ_0 is sufficient as a regulator, the parameter Λ is related to the RG flow, see below.

[4] Generally one analyses the neighbourhood of one fixed point. [Abdesselam, 2007] presents a rigorous construction of a complete RG trajectory between two fixed points for a field theory in 3 dimensions.

or $\Lambda_0 \to \infty$.

In the case of an infrared problem, like the one of the critical behaviour of Ising type models, we are interested in the case $m^2 = 0$ for $\Lambda_0 < \infty$ fixed, since this choice of parameters corresponds to an infinite correlation length (of the Gaussian theory) in the presence of an uv cutoff. In the microscopic Ising type hamiltonian this cutoff is present through the lattice. Choosing in this case units such that $\Lambda_0 = 1$, we decompose

$$C^l_{\rm ir}(x-y) = \int \frac{d^d p}{(2\pi)^4}\, e^{ip(x-y)} \int_{M^{2l}}^{M^{2(l+1)}} d\lambda\; e^{-\lambda p^2}\ , \tag{41}$$

$$\text{where}\ \ 0 \le l \le N-1\ ,\ \ \Lambda = M^{-2N}\ ,\ \ \Lambda_0 = 1\ ,\ \ \sum_{l=0}^{N-1} C^l_{\rm ir} = C^{\Lambda,1}\ .$$

The aim is to control the ir limit $N \to \infty$ or $\Lambda \to 0$. It is straightforward to see that $C^l_{\rm ir}(x-y)$ has exponential decay of scale M^{-l}:

$$|C^l_{\rm ir}(x-y)| \le K\, e^{-M^{-l}|x-y|} \quad \text{for suitable}\ \ K > 0\ . \tag{42}$$

We note that recent important work on the RG which we are going to cite below, was performed using decompositions with even better decay properties, corresponding to compact support in position space [Brydges et al., 2003], i.e. such that

$$C^l_{\rm ir}(x-y) = 0 \quad \forall x, y\ \text{ with }\ |x-y| \ge K\, M^l\ . \tag{43}$$

3.1 Critical models

Starting from our decompositions we may now define corresponding effective actions. In the infrared case we choose the decomposition (41). We then define

$$e^{-L_V^{l+1}(\varphi)} = \frac{1}{Z_V^l(0)} \int e^{-L_V^l(\phi+\varphi)}\, d\mu_{C^l_{\rm ir}}(\phi)\ , \quad 0 \le l \le N-1\ , \tag{44}$$

which means

$$L_V^{l+1}(\varphi) = -\log\left(\int e^{-L_V^l(\phi+\varphi)}\, d\mu_{C^l_{\rm ir}}(\phi) \right) + \log Z_V^l(0)\ , \tag{45}$$

where

$$Z_V^l(0) = \int e^{-L_V^l(\phi)}\, d\mu_{C^l_{\rm ir}}(\phi)\ . \tag{46}$$

The task is to control the sequence of functionals $L_V^l\ \ \forall l \le N-1$, $N \to \infty$, and also to control the infinite volume (or thermodynamic) limit for the moments of L_V^l, i.e. for the CASFs.

For the Ising type universality class described by lagrangians of the type (34), the critical behaviour may depend on the number of fields and on the symmetry properties of the models. So we include a slight generalisation, assuming the field φ to be of vectorial nature, having n components, $\varphi = (\varphi_1, \ldots, \varphi_n)$ so that $\varphi^2 = \sum_{i=1}^n \varphi_i^2$ and we substitute $\varphi^4 \to |\varphi|^4 := (\varphi^2)^2$.

Then L_V^0 is invariant under the rotation group $O(n)$ for $n \geq 2$; for $n = 1$, it is invariant under $\mathbb{Z}_2$, i.e. $\varphi \to -\varphi$. We now write[5]

$$L_V^l(\varphi) = \int_V \big[\frac{1}{2} \nu_l \varphi^2(x) + \frac{1}{2} b_l (\partial_\mu \varphi)^2(x) + \frac{g_l}{4} |\varphi|^4(x) \big] \, d^d x + \delta L_V^l(\varphi) \tag{47}$$

and want to control

$$\nu_i \,,\; b_l \,,\; g_l \,,\; \delta L_V^l \quad \text{for} \;\; l \to \infty$$

under suitable conditions for the initial values

$$\nu_0 \,,\; b_0 \,,\; g_0 \;\; \text{and} \;\; \delta L_V^0 = 0 \,.$$

For the moments of the L_V^l, we aim at statements uniform in sequences of volumes V_j exhausting $\mathbb{R}^d$ for $j \to \infty$. The L_V^l themselves are expected to diverge linearly with the volume, due to translation invariance[6]. Correspondingly Z_V^l is expected to depend exponentially with V. Analysing CASFs, i.e. moments of $L_V^l(\varphi)$ in the infinite volume limit, then requires to factor out the denominator from (44) in the expression of the integral, in order to be able to divide it out.

This analysis requires to perform several types of expansions in the functional integral. Each of these expansions is relatively transparent individually. What makes the task intricate is that the expansions have to be performed iteratively in a harmonised way at each renormalisation group step, while keeping the outcome under inductive control. In modern work these expansions have been partially fused and adapted to each other [Brydges et al., 2019]. Note also that the expressions (44) and (45) indicate that each step involves taking a logarithm and a re-exponentiation. These operations have been adapted to the expansions by defining a new (so-called circle) product [Brydges et al., 2019] in this context. Iterating the expansions and keeping control requires the couplings g_i to stay uniformly (very !) small. In our overview we now shortly describe the three expansions separately in their original form, treating L_V^l as if it were of the same form as L_V^0, i.e. neglecting the technically hard part, the control of δL_V^l. Then we have the following expansions, see for example [Brydges, 1984], [Rivasseau, 1991] :

The cluster expansion is intended to make visible the decay of the correlations. This decay is in particular required for performing the infinite volume limit. One decomposes the volume into cubes, the side length of which $\sim M^l$ corresponds to the decay of the covariance C^l (42). The expansion is w.r.t. the situation where all these cubes are factorised in the expansion. It permits to present a partition function Z_V as a sum over products of factorised contributions called polymer amplitudes whose mutually disjoint supports are unions of cubes. Due to the fall-off of (42) the polymer amplitudes decrease exponentially with the polymer size.

The fact that the polymer amplitudes produced by the cluster expansion have disjoint supports, can be expressed by introducing hardcore interactions between the polymers

$$V(Y_r, Y_s) = 0 \quad \text{if} \quad Y_r \cap Y_s = \emptyset \,, \quad V(Y_r, Y_s) = \infty \quad \text{otherwise} \;\; ,$$

where Y_r and Y_s are the supports of a given pair of polymers, i.e. unions of (open) cubes.

[5] We suppressed the lower indices V on $\nu_l \,,\; b_l \,,\; g_l$.

[6] Being related to the Helmholtz free energy, they describe extensive quantities in the terminology of thermodynamics.

The Mayer expansion then (maybe astonishingly) expands the situation of the polymer gas with hardcore constraints around the situation, where these polymers overlap, by writing for the hardcore constraints

$$e^{-V(Y_r,Y_s)} = 1 + (e^{-V(Y_r,Y_s)} - 1) .$$

The outcome of cluster and successive Mayer expansion permits to factorise the denominator in (44) in the numerator and then to take the infinite volume limit for CASFs.

The expansion in effective couplings ν_l, b_l, g_l is related to the perturbative expansion w.r.t. g_0. But this expansion does not even converge for the one-dimensional integral $\int e^{-g_0 x^4 - \frac{1}{2}x^2}\, dx$. The aim is therefore to write an expansion to low orders in the couplings, but with sufficiently strong bounds on the remainder, which permit to control the flow of the couplings and of $\delta L_V^l(\varphi)$ simultaneously. The bounds have to be strong enough to gain inductive control on the iterated RGTs. This requires that the Gaussian measure is only weakly perturbed, that is to say that $g_0 > 0$ is very small. The most important couplings are the $\{g_l\}$, since $g_0 > 0$ is at origin of the evolution of all other couplings. In $d = 4$ we find at leading order[7]

$$g_{l+1} = g_l - \beta_2 g_l^2 , \quad \beta_2 > 0 . \tag{48}$$

Corrections are of order g_l^3, but also depend on the other constants and on V. Note also that β_2 is (weakly) l- and volume dependent, and depends on the number of field components n. Iterating this equation N times gives at leading order in g_0

$$g_N \sim \frac{g_0}{1 + N\beta_2 g_0} , \tag{49}$$

which for $N \to \infty$ leads to the interaction free Gaussian fixed point. This fact that the g_l stay uniformly small, is a crucial prerequisite for the convergence of the expansion. Following the law of the constants ν_l shows that they are expected to increase at leading order proportional to gM^2 when going away from the critical point, i.e. going from $l+1$ to l. This says that $\nu_0 < 0$ has to be fine-tuned to a specific value ν_c in order that $\nu_N \to 0$ for $N \to \infty$, the latter being required to obtain an infinite correlation length for the critical theory. The ν_i act as mass terms, finite mass being associated with finite correlation length. The flow of the b_l which at leading order is of order g_l^2, stays under control when approaching the critical point, if the g_l are under control and are very small, as is the case for the law (48), (49) if $0 \le g_0 \ll 1$.

As we said the technical difficulty consists in controlling the corrections to this leading behaviour and to make the different expansions compatible. The control of δL_V^l which is nonlocal and nonpolynomial in the fields requires to introduce sophisticated norms to measure its size in a way appropriate for the construction, which makes visible the fact that the sum over all irrelevant contributions stays irrelevant on iteration.

We note that the heavy apparatus of iterated RGTs is generally only required if the non-gaussian couplings (like g in our case) are relevant or marginal,i.e. contract only logarithmically under RGTs as in (49), near the fixed point. This is not the case for the construction

[7] The complete exact expression of which (48) is the lowest order approximation, is of the form

$$g_{l+1} = \mathscr{F}_l(g_l, \nu_l, b_l, \delta\mathscr{L}_l; n, V) ,$$

where the functional $\mathscr{F}_i$ has to be controlled, which is possible for g_0 very small.

of the critical theory in $d > 4$. In this case it has been shown rigorously that the φ^4 model for $d > 4$ exhibits the same critical exponents η, γ as the mean field approximation, see (15), [Fröhlich, 1982], [Aizenman, 1982]. In particular it is known that with $\nu = \nu_c + \varepsilon$ and as $\varepsilon \to +0$,

$$\chi(g,\nu) \asymp O(\frac{1}{\varepsilon}) ,$$

where $A \asymp B$ means $C^{-1}A \leq B \leq CA$ for a universal constant $C > 0$. This means that the critical exponent γ takes the value $\gamma = 1$.

We now want to cite a few relatively recent rigorous results from [Brydges et al., 2019] obtained with the aid of the RG. We concentrate on the susceptibility defined in (9). The results have been obtained using a lattice uv cutoff and a compact support decomposition of the measure (43). In our context they can be phrased as

Theorem :
Let

$$L^0(\varphi) = \int_V \big[\frac{1}{2}\nu_0\,\varphi^2(x) + \frac{1}{2}b_0\,(\partial_\mu\varphi)^2(x) + \frac{1}{4}g_0\,|\varphi|^4(x)\big]\,d^dx\,.$$

Let $d = 4$, $n \geq 1$ and $g_0 > 0$ sufficiently small. In the (suitably defined) infinite volume limit there exists $\nu_c = \nu_c(g,n) < 0$ such that, with $\nu_0 = \nu_c + \varepsilon$ and as $\varepsilon \to +0$,

$$\chi(g,\nu) \sim A_{g,n}\,\frac{1}{\varepsilon}(\log\varepsilon^{-1})^{(n+2)/(n+8)}\,, \quad \xi(g,\nu) \sim C_{g,n}\frac{1}{\varepsilon^{1/2}}\,(\log\varepsilon^{-1})^{\frac{1}{2}(n+2)/(n+8)}\,, \tag{50}$$

and, for $|x| \to \infty$

$$\langle\varphi^1(0)\,\varphi^1(x)\rangle_{g,\nu_c} \sim \frac{A'_{g,n}}{|x|^2}\,, \tag{51}$$

where $A(x) \sim B(x)$ means $A(x) = B(x)(1+o(x))$. The constant $A_{g,n}$ satisfies for $g \to +0$

$$A_{g,n} \sim \Big[\frac{(n+8)g}{16\pi^2}\Big]^{\frac{n+2}{n+8}}\,. \tag{52}$$

The results (50) say that the critical exponent γ equals 1, and that the critical exponent ν equals $1/2$. But in both cases the mean field theory behaviour undergoes precisely calculated logarithmic corrections. It is in particular remarkable that the constant $A_{g,n}$ in (50) can be calculated to this precision. From (51) we see that the critical exponent η stays zero as in the free theory.

In dimensions below 4, the ir limit becomes even harder. Dimensions $2 < d < 4$ are studied in the physics literature using expansions in dimension [Wilson and Fisher, 1972] and number of components. An alternative to the expansion in $\varepsilon = 4 - d$ is to consider so-called long-range interactions decaying with distance as $r^{-(d+\alpha)}$ for $\alpha \in (0,2)$. In contrast to the above theorem, the long-range results involve a non-Gaussian RG fixed point, with corrections to mean-field scaling that are power law rather than logarithmic. An example of a result of this type is the following theorem which pertains to the $|\varphi|^4$-model defined with the lattice laplacian operator replaced by the fractional power $(-\Delta)^{\alpha/2}$, with $\alpha = \frac{1}{2}(d+\varepsilon)$ and small $\varepsilon > 0$. The kernel of this operator decays at large distance as

$$(-\Delta)^{\alpha/2}_{xy} \asymp |x-y|^{-(d+\alpha)}\,. \tag{53}$$

Theorem [Slade, 2018] :
For $d = 1, 2, 3$, $n \geq 1$, V sufficiently large, and $\varepsilon = 2\alpha d > 0$ sufficiently small, there exists $\bar{s} \asymp \varepsilon$ such that, for $g \in [\frac{63}{64}\bar{s}, \frac{65}{64}\bar{s}]$, there exist $\nu_c = \nu_c(g,n)$ and $C > 0$ such that for $\nu = \nu_c + t$ with $t \to +0$, the susceptibility of the long-range model obeys

$$C^{-1}\, t^{(1+\frac{n+2}{n+8}\,\frac{\varepsilon}{\alpha} - C\varepsilon^2)} \leq \chi(g,\nu;n) \leq C\, t^{(1+\frac{n+2}{n+8}\,\frac{\varepsilon}{\alpha} + C\varepsilon^2)} \,. \tag{54}$$

This statement implies that the critical exponent γ exists, and undergoes finite corrections of order ε, as compared to mean field theory :

$$\gamma = 1 + \frac{n+2}{n+8}\,\frac{\varepsilon}{\alpha} + O(\varepsilon^2) \,. \tag{55}$$

3.2 Ultraviolet limits

The method we have exposed is sufficiently general to treat also quantum field theories in four euclidean dimensions, as they appear in the standard model. In this case we may use a decomposition of the covariance as in (40) and (39). Considering a massive scalar field theory, the challenge consists in performing the uv limit. It then turns out that at leading order the flow of the effective coupling as compared to (48), is of the form[8]

$$g_l = g_{l+1} - \beta_2\, g_{l+1}^2 \,, \quad \beta_2 > 0 \,, \quad 0 \leq l \leq N-1 \,. \tag{56}$$

Starting from a small bare coupling g_N this leads to a vanishing physical (renormalised) coupling $g = g_0$ in the uv limit $N \to \infty$. As mentioned in the introduction, the sign of β changes when considering nonabelian gauge theories. But in these theories gauge symmetry forbids a mass terms. Then the uv and ir constructions would have to be joined together, and the effective coupling would (presumably) grow arbitrarily in the ir region. Generally rigorous RG constructions require that the flowing couplings stay globally very small, since otherwise there is no control of the enormous combinatorics generated by successive RG integrations.

One may then consider larger classes of field theories, other types of fields, admit euclidean dimensions $d \neq 4$, other geometries, new symmetry structures or covariances deviating from the standard form as e.g. in (53). Generally this is a vast field which we cannot enter on these few pages. The simplest option is to consider the uv limit of the massive $|\varphi|^4$-models in $d = 1, 2, 3$. These are so-called superrenormalisable models whose effective couplings vanish according to a power of the scale when going in the uv direction. As a consequence their construction does not fully require the iterative RG apparatus, even if in $d = 3$ it stays challenging.

Looking for models in euclidean dimensions $d \neq 4$, for which the flow of the relevant and marginal couplings under RGTs is similar to the one of the critical $|\varphi|^4$ theory in $d = 4$, a prominent example for which the construction could be achieved is the massive Gross-Neveu model in euclidean dimensions $d = 2$ [Gawedzki and Kupiainen, 1985], [Feldman et al., 1986]. Its (bare) lagrangian is given by

$$L_V^0(\psi, \bar{\psi}) = \int_V \left[\bar{\psi}(x)(i z_N \not{\partial} + m_N)\,\psi(x) - \lambda_N\,(\bar{\psi}\,\psi)^2(x)\right] d^2x \,, \tag{57}$$

[8] In the uv context it is more adapted to label the starting point of the RG construction by N, and then to go down to smaller values reaching the physical value at index 0.

where $\psi, \bar{\psi}$ are two-component fermionic Grassmann spinor fields. The regularized (free) fermionic propagator in momentum space is assumed to be

$$S^{0,\Lambda_0}(p) = \frac{\not{p}+m}{p^2+m^2}\, e^{-\Lambda_0^{-2}p^2}\ , \quad \Lambda_0 = M^N\ .$$

For the construction to be feasible, one has to assume that there exist at least $n \geq 2$ flavour components $\psi^1, \ldots, \psi^n$. The definition of the term $\bar{\psi}\psi$ in (57) is thus

$$\bar{\psi}\psi = \sum_{a=1}^{n} \sum_{\alpha=1,2} \bar{\psi}^a_\alpha \psi^a_\alpha\ .$$

The perturbative analysis of this model is quite similar to the one of the $|\varphi|^4$-model. Both have a four-field vertex. They also have similar power counting, since the fermion propagator has slower fall-off in momentum space, which is compensated by lowering the dimension from 4 to 2. It turns out that due to the sign changes occurring for Grassmann variables, the sign in (56) is now reversed so that at leading order the flow is

$$\lambda_l = \lambda_{l+1} + \beta_{2,n}\, \lambda_{l+1}^2\ , \quad \beta_{2,n} > 0\ . \tag{58}$$

Assuming the bare coupling λ_N to behave as $1/(N+C)$, for $N \to \infty$, $C \gg 1$, the solutions λ_i of (58) stay globally very small. The sign cancellations present for fermionic fields, also tame to some degree the problems linked to the infinite volume limit, and those related to establishing bounds on the remainder terms, in particular on the δL_V^l in the inductive construction. Unfortunately these improvements can only partially be maintained under the RG iterations. The final result for the uv RG construction of the Gross-Neveu model can be stated as follows: *Theorem : Choosing*

$$\lambda_N = \frac{1}{\beta_{2,n} N + \frac{1}{\pi}\log N + C}\ , \quad m_N = m\, N^{-\gamma_n}\ , \quad z_N = 0\ ,$$

where

$$\beta_{2,n} = \frac{2(n-1)}{\pi}\log M\ , \quad \gamma_n = \frac{n-\frac{1}{2}}{n-1}\ , \quad C\,,\ M \textit{ (from (40)) sufficiently large}\ ,$$

the uv limit of the Gross-Neveu-model as a two-dimensional interacting euclidean field theory exists. The CASFs of the theory thus constructed are the Borel sums of their renormalised perturbative expansion.

As stated, the Gross-Neveu model serves as an illustration of the uv behaviour suspected to hold also in nonabelian gauge theories. The fact that the ir limit of those is not under control, can be bypassed by studying them in finite volume. We mention monumental work by Balaban [Balaban, 1988] who studied lattice gauge theory on a ("small") torus in a long series of papers.

4 The renormalisation group equations

As we said the quantum field theories constituting the standard model of particle physics are so far not known to exist as mathematical models beyond perturbation theory. On the other

hand the formal perturbative loop expansion often gives extremely precise results, in agreement with experiment. What can be done on the mathematical side, is to show that perturbation theory gives order by order finite results for the CASFs. To implement perturbation theory, we introduce an expansion parameter generally called $\hbar$ and rewrite (38) as[9]

$$e^{-\frac{1}{\hbar}L^{\Lambda,\Lambda_0}(\varphi)} = \frac{1}{Z^{\Lambda,\Lambda_0}(0)} \int e^{-\frac{1}{\hbar}L^0(\phi+\varphi)}\, d\mu_{\hbar C^{\Lambda,\Lambda_0}}(\phi)\ . \tag{59}$$

The moments of L^{Λ,Λ_0} can then be expanded in a formal power series w.r.t. $\hbar$. Setting $\hbar = 1$ we get back (37), (38). This expansion is called the loop expansion. In simple cases as symmetric φ^4-theory it is equivalent to perturbation theory.

The functional renormalisation group equation is a functional differential equation for $L^{\Lambda,\Lambda_0}(\varphi)$ of parameter Λ. Rewritten in terms of its moments, the CASFs, it is an infinite-dimensional dynamical system of parameter Λ. Introducing the constant $\hbar$ as in (59) and using the change of covariance formula (31) we obtain writing $\dot{C}^\Lambda := \partial_\Lambda C^{\Lambda,\Lambda_0}$ and $I^{\Lambda,\Lambda_0} := -\hbar \log Z^{\Lambda,\Lambda_0}(0)$

$$\partial_\Lambda\, e^{-\frac{1}{\hbar}(L^{\Lambda,\Lambda_0}(\varphi)+I^{\Lambda,\Lambda_0})} = \frac{1}{2}\int [\langle \frac{\delta}{\delta\phi}, \hbar\dot{C}^\Lambda \frac{\delta}{\delta\phi}\rangle\, e^{-\frac{1}{\hbar}L^0(\phi+\varphi)}]\, d\mu_{\hbar C^{\Lambda,\Lambda_0}}(\phi)\ . \tag{60}$$

Since L^0 depends only on the sum $\phi+\varphi$, the r.h.s. can be rewritten as

$$\frac{1}{2}\langle \frac{\delta}{\delta\varphi}, \hbar\dot{C}^\Lambda \frac{\delta}{\delta\varphi}\rangle \int e^{-\frac{1}{\hbar}L_0(\phi+\varphi)}\, d\mu_{\hbar C^{\Lambda,\Lambda_0}}(\phi) = \frac{1}{2}\langle \frac{\delta}{\delta\varphi}, \hbar\dot{C}^\Lambda \frac{\delta}{\delta\varphi}\rangle\, e^{-\frac{1}{\hbar}(L^{\Lambda,\Lambda_0}(\varphi)+I^{\Lambda,\Lambda_0})}\ .$$

Performing the derivatives on both sides and factoring out the exponential [10] then gives

$$\partial_\Lambda L^{\Lambda,\Lambda_0} = \frac{\hbar}{2}\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \frac{\delta}{\delta\varphi}\rangle L^{\Lambda,\Lambda_0} - \frac{1}{2}\langle \frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}, \dot{C}^\Lambda \frac{\delta}{\delta\varphi}L^{\Lambda,\Lambda_0}\rangle + \hbar\,\partial_\Lambda \log Z^{\Lambda,\Lambda_0}\ , \tag{61}$$

which is the functional differential flow equation of the RG written for the Wilson effective action. The differential Wilson RGEs seem to appear first in [Wegner and Houghton, 1973]. As we said, $L^{\Lambda,\Lambda_0}(\varphi)$ is the generating functional of the CASFs in the presence of the regulators Λ, Λ_0. The task is then to prove that these functions exist in the limits $\Lambda \to 0$, $\Lambda_0 \to \infty$.

The functional L^{Λ,Λ_0} has an expansion as a formal power series in terms of Feynman diagrams with precisely l loops, n external legs, and propagator $C^{\Lambda,\Lambda_0}(p)$. As the name suggests, only connected diagrams contribute, and the (free) propagators on the external legs are removed. Analysing the functional $L^{\Lambda,\Lambda_0}(\varphi)$ in the sense of formal power series w.r.t. $\hbar$, and in moments w.r.t. φ, means that we consider the terms in the formal power series

$$L^{\Lambda,\Lambda_0}(\varphi) := \sum_{n>0}^{\infty}\sum_{l=0}^{\infty}\hbar^l \int d^4x_1 \dots d^4x_n\, \tilde{\mathscr{L}}^{\Lambda,\Lambda_0}_{n,l}(x_1,\dots,x_n)\, \varphi(x_1)\cdots\varphi(x_n)\ , \tag{62}$$

where φ is in the support of $d\mu_{\hbar C^{\Lambda,\Lambda_0}}$, where no statement is made on the convergence of the series in n, l. The objects on the right side, the perturbative CASFs, are the basic quantities in

[9] We suppress the volume index V in the notation since the infinite volume limit can be taken straightforwardly at the level of the perturbative CASFs. So we do not talk about it in this section and understand that we stay in finite volume before arriving at the perturbative CASFs.

[10] The exponential can be shown to be finite in finite volume and in the presence of regulators.

our analysis. When expanded out in φ and in powers of $\hbar$, the flow equations, deduced from (61) for the moments, read in momentum space, using translation invariance

$$\partial_\Lambda \mathscr{L}^{\Lambda,\Lambda_0}_{n,l}(p_1,\ldots,p_n) = \binom{n+2}{2} \int_k \dot{C}^\Lambda(k)\, \mathscr{L}^{\Lambda,\Lambda_0}_{n+2,l-1}(k,-k,p_1,\ldots p_n) \tag{63}$$
$$-\frac{1}{2} \sum_{\substack{l_1+l_2=l \\ n_1+n_2=n+2}} n_1 n_2\, \mathbb{S}\left[\mathscr{L}^{\Lambda,\Lambda_0}_{n_1,l_1}(p_1,\ldots,p_{n_1})\, \dot{C}^\Lambda(q)\, \mathscr{L}^{\Lambda,\Lambda_0}_{n_2,l_2}(-q,p_{n_1},\ldots,p_n)\right]$$

with $p' = p_{n_1} + \ldots + p_n = -p_1 - \ldots - p_{n_1-1}$. The symbol $\mathbb{S}$ (for symmetrization) indicates that we take the mean value over those permutations π of $(1,\ldots,n)$, for which $\pi(1) < \pi(2) < \ldots < \pi(n_1-1)$ and $\pi(n_1) < \pi(n_1+1) < \ldots < \pi(n)$.
To fix the CASFs as solutions of the flow equations, we have to impose boundary conditions. The simplest choice are so-called BPHZ conditions at $\Lambda = 0$

$$\partial^w \mathscr{L}^{0,\Lambda_0}_{n,l}(\vec{0}) = \delta_{w,0}\ \delta_{n,4}\ \delta_{l,0}\ \frac{g}{4} \quad \text{for } n+|w| \le 4\,, \tag{64}$$

together with

$$\partial^w \mathscr{L}^{\Lambda_0,\Lambda_0}_{n,l}(\vec{p}) = 0 \quad \text{for } n+|w| > 4\,. \tag{65}$$

Here w is a multiindex indicating $|w|$ derivatives w.r.t. the four-momentum components.
The task is now to prove that that perturbation theory gives order by order finite results for the CASFs on taking the limits $\Lambda_0 \to \infty$, $\Lambda \to 0$. Polchinski [Polchinski, 1984] first realised that the system (63) can be analysed inductively using a scheme going up $N = n+2l$, and for fixed N going up in l, which permits to bound the CASFs $\mathscr{L}^{\Lambda,\Lambda_0}_{n,l}$ efficiently, and thus to base renormalisation theory entirely on this system of equations.

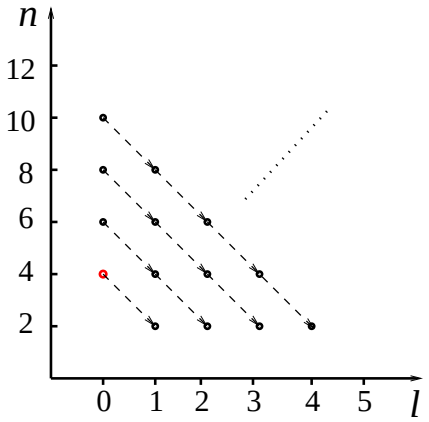

Figure 1: *Polchinski's inductive scheme starting at* $(n,l) = (4,0)$

Using this scheme it is straightforward to show the following
Theorem : The perturbative CASFs for the massive scalar field theory obey the following bounds [Keller et al., 1992] which are uniform in the cutoff Λ_0*:*

$$\left|\mathscr{L}^{\Lambda,\Lambda_0}_{n,l}(p_1,\ldots,p_n)\right| \le (\Lambda+m)^{4-n}\ \mathscr{P}(\log\frac{\Lambda+m}{m})\ \mathscr{F}(\frac{|\vec{p}|}{\Lambda+m})\,.$$

Here $\mathscr{P}$ is a polynomial of degree $\leq l$, and the function $\mathscr{F}$ can be bounded more or less stringently with more or less effort. Uniformity in the cutoff directly leads to renormalisability.

These results can be extended to prove renormalisability for all sectors of the standard model. One aspect which still deserves attention, is to prove in a rigorous way the validity of the Ward identities in the standard model, to all orders, in the presence of chiral fermions. Apart from recovering in a simpler way results from traditional renormalisation theory, the method has also allowed to prove statements which are beyond the traditional methods, for an older review see [Müller, 2003]. We mention the convergence of the perturbative operator product expansion [Hollands and Kopper, 2012], bounds on theories containing irrelevant bare interaction terms, large momentum bounds and bounds on high orders in perturbation theory, temperature independent renormalisation for $T > 0$, renormalisation on manifolds and on spaces with boundary. Recently it could be proven with the aid of RGEs in the mean field approximation, but without recourse to perturbation theory, that there exist asymptotically free mean field scalar field theories. These have nonpolynomial bare lagrangians L_V^0 [Kopper, 2022].

4.1 The Callan-Symanzik equations

At the end of this short overview we would like to come back to the early days of the RGEs in perturbative quantum field theory. As we said the Wilson RG comprises the nowadays socalled old RG. Among the RGEs in the old sense, the Callan-Symanzik equations probably take the most prominent place. We want to deduce them here from (36), (37), (38), staying in the euclidean framework for a scalar field. As we said the solutions of the flow equations (63) are determined once we have fixed the renormalisation conditions. To align ourselves with the notation in perturbative quantum field theory we write the bare (inter)action in a slightly different parametrisation as

$$\hat{L}_b^0(z^{1/2}\phi) = \int_V \big[\, \frac{1}{2}(z-1)(\partial_\mu\phi)^2 + \frac{1}{2}(m_0^2 z - m^2)\,\phi^2 + \frac{g_0}{4!}\, z^2\,\phi^4 \,\big] \,. \tag{66}$$

The BPHZ renormalisation conditions corresponding to (64) are then

$$\mathscr{L}_{2,l}^{0,\Lambda_0}(\vec{0}) = \frac{1}{2}\,\delta_{l,0}\,m^2 \,, \quad \partial_{p^2}\mathscr{L}_{2,l}^{0,\Lambda_0}(\vec{0}) = \frac{1}{2}\,\delta_{l,0} \,, \quad \mathscr{L}_{4,l}^{0,\Lambda_0}(\vec{0}) = \delta_{l,0}\,\frac{g}{4!} \,, \tag{67}$$

where $\hat{\mathscr{L}}_{n,l}^{0,\Lambda_0}(\vec{p}) = \mathscr{L}_{n,l}^{0,\Lambda_0}(\vec{p}) + \frac{1}{2}\,\delta_{l,0}\,\delta_{n,2}\,(p^2+m^2)$. Using (66) we can write (59) for $\Lambda = 0$ as [11]

$$e^{-L^{0,\Lambda_0}(\varphi)} = \frac{1}{Z^{0,\Lambda_0}(0)} \int e^{-L^b(z^{1/2}(\phi+\varphi))}\, d\mu_{C^{0,\Lambda_0}}(\phi) \,. \tag{68}$$

Setting $\tilde{\phi} = z^{1/2}\,\phi$ and $\tilde{C} = zC$ [12], we can then write

$$e^{-L^{0,\Lambda_0}(\varphi)} = \frac{1}{\tilde{Z}^{0,\Lambda_0}(0)} \int e^{-L^b(\tilde{\phi}+z^{1/2}\,\varphi)}\, d\mu_{\tilde{C}^{0,\Lambda_0}}(\tilde{\phi}) \,. \tag{69}$$

[11] In this section we again set to one the factors of $\hbar$ for readibility. But one should note that all quantities are supposed to be expanded in the formal loop expansion. This applies also to the coefficient functions $\beta(g)$ and $\gamma(g)$ in (70).

[12] This change of covariance is associated with a normalisation factor which is finite only for Λ_0 finite and in finite volume, see (32). Again we do not dwell on this issue because the volume drops out, when considering moments of the log of the partition function.

Now the renormalised CASFs are defined to be the moments of $L^{0,\Lambda_0}(\varphi)$ (for $\Lambda_0 \to \infty$), and the bare CASFs (indicated by a subscript b) are defined to be the moments of

$$-\log\Big(\int e^{-L^b(\vec{\phi}+z^{1/2}\varphi)}\, d\mu_{\tilde{C}^{0,\Lambda_0}}(\vec{\phi})\Big)$$

so that we obtain the relation[13]

$$\hat{\mathscr{L}}_n^{0,\Lambda_0}(m^2;g;\vec{p}) = z^{n/2}(m_0^2;g_0)\,\hat{\mathscr{L}}_{b;n}^{\Lambda_0}(m_0^2;g_0;\vec{p})\ .$$

The bare functions generally do not have finite limits for $\Lambda_0 \to \infty$. If we now change the (renormalised) mass into $m^2(s) = (1+s)m^2$, keeping g_0 fixed, we get

$$\Big[\frac{d}{ds}\Big|_{g_0} \hat{\mathscr{L}}_n^{0,\Lambda_0}(m^2(s);g(s);\vec{p})\Big]\Big|_{s=0} = \frac{d}{ds}\Big|_{g_0}\Big[z^{n/2}(m_0^2(s);g_0)\,\hat{\mathscr{L}}_{b;n}^{\Lambda_0}(m_0^2(s);g_0;\vec{p})\Big]\Big|_{s=0}\ ,$$

which gives

$$\Big[m^2\frac{\partial}{\partial m^2} + \beta(g)\frac{\partial}{\partial g} - n\,\gamma(g)\Big]\hat{\mathscr{L}}_n^{0,\Lambda_0}(m^2;g;\vec{p}) = \hat{\mathscr{L}}_{1;n}^{\Lambda_0}(m^2;g;\vec{p})\ , \tag{70}$$

where

$$\beta(g) = \frac{dg}{ds}\Big|_{g_0,s=0}\ , \quad \gamma(g) = \frac{1}{2}\,\frac{d\log z}{ds}\Big|_{g_0,s=0}\ .$$

The functions

$$\hat{\mathscr{L}}_{1;n}^{\Lambda_0}(m^2;g;\vec{p}) = z^{n/2}(m_0^2(s);g_0)\,\frac{d}{ds}\Big|_{g_0}\hat{\mathscr{L}}_{b,n}^{\Lambda_0}(m_0^2(s);g_0;\vec{p})\,\Big|_{s=0}$$

are the moments obtained via the flow equations from the bare action

$$L_1^0(z^{1/2}\phi) = \int\Big[\frac{1}{2}(z-1)(\partial_\mu\phi)^2 + \frac{1}{2}(m_0^2 z - m^2)\,\phi^2 + \frac{g_0}{4!}\,z^2\,\phi^4 + \frac{1}{2}\,\frac{dm_0^2}{ds}\Big|_{g_0,s=0} z\,\phi^2\Big]\,d^4x \tag{71}$$

to first order in $\frac{z}{2}\frac{dm_0^2}{ds}\big|_{g_0,s=0}\int\phi^2(x)\,d^4x$. Renormalisation theory tells us that the $\hat{\mathscr{L}}_{1;n}^{0,\Lambda_0}(m;g;\vec{p})$ have finite limits for $\Lambda_0 \to \infty$ order by order in perturbation theory. Their precise renormalisation conditions can be read from (70) at $\Lambda = 0$, evaluated at the renormalisation points, i.e. zero momentum in the BPHZ case, using the renormalisation conditions (67). The equations (70) are the Callan-Symanzik equations [Callan, 1970], [Symanzik, 1970].

5 Conclusions

The renormalisation (semi)group is a general method to analyse systems with a large (or infinite) number of degrees of freedom [14]. Putting the RG into work requires to establish a

[13] It is customary in perturbative field theory to parametrise renormalised and bare functions in terms of the respective constants (m^2, g) and (m_0^2, g_0), understanding that these are mutually unique functions of each other.

[14] Basically all physical systems are of this type.

hierarchy among these degrees of freedom, mostly based on a one-dimensional ordering, e.g. according to a length or an energy scale. Then the effect of the degrees of freedom on physical observables such as correlation functions is taken into account successively, according to the chosen hierarchy. The RG has become an indispensable tool for analysing such systems in the general situation, where no explicit solutions, due to the particular symmetry structure of the system, are available. The importance and domains of application of the RG have been growing ever since its advent in the 1950ies and its new conception by Wilson around 1970. In our presentation we focussed on a few rigorous results in the theory of critical phenomena and in constructive field theory. The renormalisation group equations are continous versions of the RG, which may be given the form of an infinite dimensional dynamical system. Here we focussed on the reformulation and extension of rigorous perturbative renormalisation theory in this framework. In the future there will certainly be other fields of physics analysed with the RG method, like e.g. the analysis of coherent quantum systems and tensor networks.

Since the method is very general, it is probably not astonishing that its implementation is not system independent. Rigorous RG analysis often requires large effort, and results are obtained only for a very limited range of parameters. On the other hand it is often the only method at hand to obtain rigorous results on systems which are not strongly restricted by symmetries. So for the future we do not only hope for results on new systems, but also for technical and conceptual simplifications and for extensions to physically more realistic situations. One may safely predict that the RG and the RGEs will continue to be of primordial interest for mathematical physics in the foreseeable future.

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