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The Surface counter-terms of the ϕ_4^4 theory on the half space $\mathbb{R}^+ \times \mathbb{R}^3$

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In a previous work, we established perturbative renormalizability to all orders of the massive ϕ_4^4 -theory on a half-space also called the semi-infinite massive ϕ_4^4 -theory. Five counter-terms which are functions depending on the position in the space, were needed to make the theory finite. The aim of the present paper is to establish that for a particular choice of the renormalization conditions the effective action consists of a part which is independent of the boundary conditions (Dirichlet, Neumann and Robin) plus a boundary term in the case of the Robin and Neumann boundary conditions. The key idea of our method is the decomposition of the correlators into a bulk part, which is defined as the scalar field model on the full space \mathbb{R}^4 with a quartic interaction restricted to the half-space, plus a remainder which we call "the surface part". We analyse this surface part and establish perturbatively that the ϕ_4^4 theory in $\mathbb{R}^+ \times \mathbb{R}^3$ is made finite by adding the bulk counter-terms and two additional counter-terms to the bare interaction in the case of Robin and Neumann boundary conditions. These surface counter-terms are position independent and are proportional to $\int_S \phi^2$ and $\int_S \phi \partial_n \phi$. For Dirichlet boundary conditions, we prove that no surface counter-terms are needed and the bulk counter-terms are sufficient to renormalize the connected amputated (Dirichlet) Schwinger functions. A key technical novelty as compared to our previous work is a proof that the power counting of the surface part of the correlators is better by one scaling dimension than their bulk counterparts.

I. INTRODUCTION

Renormalization group methods in the presence of boundaries have been developed in theoretical physics in the context of the semi-infinite scalar field model in^{13,14}. Besides the physical significance of this model as a prototype for systems with spatial inhomogeneities, it serves to illustrate the characteristic new features of the RG when translation invariance is broken by the presence of surfaces. Recently, we established a rigorous proof of perturbative renormalizability of this model in⁶ using the Polchinski flow equations. Initiated by Polchinski²⁸, this approach to renormalization has been adapted to various interesting quantum field theories. We analyzed the generating functional L^{Λ, Λ_0} of connected amputated Schwinger functions by proving inductively uniform upper bounds in the UV cutoff Λ_0 , wherefrom the renormalizability of the model is deduced. For theories that break translation invariance, the whole procedure is carried out in position space²³ or in the case of the semi-infinite model in a mixed position-momentum space. Hence, the correlation functions are distributions which imposes that the method of proof must be adapted to distributions. We mention a previous work in this direction²³ on the perturbative renormalization of the massive scalar field with a ϕ^4 interaction on Riemannian manifolds in which the proof of renormalization was completely performed in position space. Our work⁵ on the renormalization of the semi-infinite model adapts the proof of²³ to the case of a mixed position-momentum representation. The main particularity of working in position space within the flow equation framework is the appearance of weight factors associated to some combinatorial objects called "trees" in the bounds on the perturbative correlation functions. These weight factors imply tree decay between the position arguments of the correlation functions.

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The loss of translation invariance implies that the relevant parameters are not constants but are rather functions, which can possibly depend on the position $x \in V$, the domain on which the system is defined. In⁶, we considered BPHZ renormalization conditions, and we found that the semi-infinite model is renormalized by adding five position dependent counter-terms to the bare interaction given by

$$\begin{aligned} L_{\star}^{\Lambda_0, \Lambda_0}(\phi) = & \frac{\lambda}{4!} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \phi^4(z, x) \\ & + \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left(a_{\star}^{\Lambda_0}(z) \phi^2(z, x) - b_{\star}^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) \right. \\ & \left. - d_{\star}^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) - s_{\star}^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c_{\star}^{\Lambda_0}(z) \phi^4(z, x) \right), \end{aligned}$$

with \star denoting either Dirichlet, Neumann or Robin boundary conditions. The method developed in⁶ is of a pedagogical nature in the sense that it provides the detailed steps to establish the renormalizability of scalar field theories in the mixed position-momentum space. However, the approach does not highlight the effect of the presence of surfaces, in the sense that the correlation functions of the semi-infinite model and its translationally invariant counterpart can both be bounded with similar upper bounds using the method of proof of⁶. This also means that the steps of the proof in⁶ do not show the differences between the renormalization of the semi-infinite model and of the usual translationally invariant ϕ_4^4 theory in the mixed position-momentum space. A further issue is the dependence of the counter-terms terms in⁶ on the position. While mathematically well defined, the renormalization problem considered in⁶ differs from physical applications where the effective action is specified at Λ_0 , and is supposed to be independent of the position in space, as in the case in boundary critical phenomena^{12,14}. For these reasons, we investigate in this paper in more detail the structure of the counter-terms and their dependence on the surface by separating the surface and bulk effects. The results of this paper are also useful for a future publication, in which we prove a much stronger result with respect to the independence of the counter-terms of the position in space by establishing that these are given by the usual translationally invariant counter-terms plus some counter-terms supported on the surface.

In the mixed position momentum space, the possible propagators can be written as the sum of two terms. The first term is the propagator of the translationally invariant theory C_B and the second term denoted by $C_{S, \star}$ corresponds to the part that breaks translation invariance. By definition, $C_{S, \star}$ depends on the considered boundary condition with $\star \in \{D, N, R\}$. Inserting this decomposition in the Feynman graph expansion, we obtain graphs involving exclusively C_B (i.e. bulk graphs), and others involving $C_{S, \star}$ or $C_{S, \star}$ and C_B (i.e. surface graphs). In¹⁵ and in a different context in³¹, the authors argued that the power counting of the surface graphs is better by one scaling dimension than its bulk counterpart. We also mention the recent results^{1,2} on the 2D Ising model in cylindrical domains, in which the authors proceeded similarly by decomposing the generating function of correlations of observables located inside the volume into a bulk part (which in this case denotes the infinite plane counterpart of the considered model), plus a remainder denoted as the "edge" that accounts for the boundary effects. Their approach led to bounds on the "edge" part which are dimensionally better by one scaling dimension as compared to the bulk. A similar situation is expected in the context of the semi-infinite model¹⁵, by decomposing the correlation distributions into bulk and surface parts. The bulk part consists of all Feynman bulk integrals, which are identical to those of the translationally invariant theory, up to the restriction $z \geq 0$ on z -integrations. This implies that these graphs can be renormalized using the same counter-terms as for ϕ_4^4 in \mathbb{R}^4 with an interaction supported on the half-space which will be called in this paper the bulk theory. The remaining surface part contains surface Feynman graphs and these turn out, as we establish in the sequel, to be renormalized by adding position independent counter-terms supported on the surface. One of the important results of the present paper in agreement with the findings of^{1,2}, is that we obtain that the surface part admits a power counting dimensionally better by one scaling dimension as compared to its bulk counterpart. This modified scaling dimension appears in Theorem 1.

Our technique of proof is based on constructing a solution to the flow equation of the semi-infinite

model derived in⁶ which has the following form

$$\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n))$$

where $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ are the correlation distributions of the bulk theory. The surface correlation distributions $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ are defined inductively by a specific flow equation obtained from the respective flow equations of the semi-infinite model and the bulk theory, by imposing that $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0} + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ is a solution of the semi-infinite flow equation. These surface correlation distributions are the object of interest of this paper since they contain all surface singularities (at $\Lambda = \Lambda_0$) that can arise from the propagator $C_{S,\star}$ and hence are renormalized by the so-called surface counter-terms or equivalently counter-terms supported on the surface. In this paper, we will not look in detail into the bulk correlation distributions $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$, but we rather use their uniform boundedness w.r.t. the UV cutoff Λ_0 which can be obtained in the case of BPHZ renormalization conditions analogously to the bounds in⁶. As we mentioned before, this raises the possibility that the corresponding counter-terms depend on the position. The renormalization of $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ implies that at $\Lambda = \Lambda_0$ the effective action has the following form

$$\begin{aligned} L_R^{\Lambda_0,\Lambda_0}(\phi) = & \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left(\frac{\lambda}{4!} \phi^4(z,x) + \frac{1}{2} a_B^{\Lambda_0}(z) \phi^2(z,x) - \frac{1}{2} s_B^{\Lambda_0}(z) \phi(z,x) \partial_z \phi(z,x) \right. \\ & \left. - \frac{1}{2} b_B^{\Lambda_0}(z) \phi(z,x) \Delta_x \phi(z,x) - \frac{1}{2} d_B^{\Lambda_0}(z) \phi(z,x) \partial_z^2 \phi(z,x) + \frac{1}{4!} c_B^{\Lambda_0}(z) \phi^4(z,x) \right) \\ & + \int_{\mathbb{R}^3} d^3x \left(\frac{1}{2} s_R^{\Lambda_0} + c e^{\Lambda_0} \right) \phi^2(0,x), \quad (1) \end{aligned}$$

where c denotes the Robin parameter associated to Robin boundary conditions, and the position dependent counter-terms are those renormalizing the bulk theory.

The exposition is organized as follows. In Section II, we review the basic setting and recall some basic properties of the Robin, Neumann, Dirichlet and bulk heat kernels. Section III is devoted to define the scalar field theory in \mathbb{R}^4 with a quartic self-interaction restricted to the half-space $\mathbb{R}^+ \times \mathbb{R}^3$. The CAS of this theory obey the standard flow equations of the ϕ_4^4 -theory in \mathbb{R}^4 , with the exception that the z , z' integrations appearing on the RHS of the flow equations are restricted to \mathbb{R}^+ instead of \mathbb{R} . The bare interaction corresponding to this theory reads

$$\begin{aligned} L_B^{\Lambda_0,\Lambda_0}(\phi) = & \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left(\frac{\lambda}{4!} \phi^4(z,x) + \frac{1}{2} a_B^{\Lambda_0}(z) \phi^2(z,x) - \frac{1}{2} s_B^{\Lambda_0}(z) \phi(z,x) \partial_z \phi(z,x) \right. \\ & \left. - \frac{1}{2} b_B^{\Lambda_0}(z) \phi(z,x) \Delta_x \phi(z,x) - \frac{1}{2} d_B^{\Lambda_0}(z) \phi(z,x) \partial_z^2 \phi(z,x) + \frac{1}{4!} c_B^{\Lambda_0}(z) \phi^4(z,x) \right), \end{aligned}$$

where $a_B^{\Lambda_0}(z)$, $s_B^{\Lambda_0}(z)$, $b_B^{\Lambda_0}(z)$, $d_B^{\Lambda_0}(z)$ and $c_B^{\Lambda_0}(z)$ are the bulk counter-terms which can depend (smoothly) on z since the interaction breaks translation invariance. In Section IV, we construct the surface correlation distributions $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ associated to the boundary condition \star . In Section V, we introduce the trees and forest structures together with their corresponding weight factors which we need later in stating Theorem 1. Section VI is the central part of this paper. We present Theorem 1 which contains the power counting for the connected amputated Schwinger distributions (CAS) $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ as well as their boundedness w.r.t. to Λ_0 with \star denoting either Robin or Neumann boundary conditions. Then, Proposition 4 proves that the Dirichlet surface correlation distributions can be viewed as the limit of Robin surface correlation distributions when the Robin parameter c is taken to infinity. Theorem 1 together with Proposition 4 imply Corollary 1 which states that the Dirichlet surface correlation distributions when smeared with Dirichlet heat kernels are irrelevant. First order calculations in perturbation theory^{9,10} suggest that the amputated theory is renormalized differently as compared to the non-amputated one in the sense that the tadpole needs more counter-terms depending on whether one of its external points is on the surface or not. We confirm this statement to all orders of perturbation theory in Section VII. In the Appendices, we collect technical lemmas which we use in the proof of Theorem 1.

II. THE HEAT KERNELS AND THE PROPAGATORS

A. Some notations and the heat kernels

In the sequel, we will be using the following notations

$$\begin{aligned} \int_z &:= \int_0^\infty dz, \quad \vec{p}_n := (p_1, \dots, p_n), \quad (\vec{z}_n, \vec{p}_n) := ((z_1, p_1), \dots, (z_n, p_n)), \\ z_{i,j} &= (z_i, \dots, z_j), \quad \vec{p}_{i,j} = (p_i, \dots, p_j), \quad 1 \leq i \leq j \leq n, \\ \|\vec{p}_n\| &:= \max_{1 \leq i \leq n} |p_i|. \end{aligned}$$

We will also use the mixed position-momentum space representation which consists in taking the partial Fourier transform w.r.t. the variable $x \in \mathbb{R}^3$. We recall that in this representation, the Dirichlet, Neumann and Robin propagators simply read¹⁶

$$C_D(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[e^{-\sqrt{p^2 + m^2}|z-z'|} - e^{-\sqrt{p^2 + m^2}|z+z'|} \right], \quad (2)$$

$$C_N(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[e^{-\sqrt{p^2 + m^2}|z-z'|} + e^{-\sqrt{p^2 + m^2}|z+z'|} \right], \quad (3)$$

$$C_R(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[e^{-\sqrt{p^2 + m^2}|z-z'|} + \frac{\sqrt{p^2 + m^2} - c}{\sqrt{p^2 + m^2} + c} e^{-\sqrt{p^2 + m^2}|z+z'|} \right], \quad c > 0. \quad (4)$$

Note that the Dirichlet boundary condition corresponds to $c \rightarrow +\infty$ and the Neumann boundary condition to $c = 0$. For $\star \in \{D, R, N\}$, the propagator C_\star can also be written as

$$C_\star(p; z, z') = \int_0^\infty d\lambda e^{-\lambda(p^2 + m^2)} p_\star(\lambda; z, z'), \quad (5)$$

where the Dirichlet, Neumann and Robin heat kernels read for $z, z' \geq 0$

$$p_D\left(\frac{1}{\Lambda^2}; z, z'\right) = p_B\left(\frac{1}{\Lambda^2}; z, z'\right) - p_B\left(\frac{1}{\Lambda^2}; z, -z'\right), \quad (6)$$

$$p_N\left(\frac{1}{\Lambda^2}; z, z'\right) = p_B\left(\frac{1}{\Lambda^2}; z, z'\right) + p_B\left(\frac{1}{\Lambda^2}; z, -z'\right), \quad (7)$$

$$p_R\left(\frac{1}{\Lambda^2}; z, z'\right) = p_B\left(\frac{1}{\Lambda^2}; z, z'\right) + p_B\left(\frac{1}{\Lambda^2}; z, -z'\right) - 2 \int_0^\infty dw e^{-w} p_B\left(\frac{1}{\Lambda^2}; z, -\frac{w}{c} - z'\right), \quad (8)$$

and the bulk heat kernel p_B is given by

$$p_B(\tau; z, z') = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(z-z')^2}{2\tau}}, \quad \tau > 0. \quad (9)$$

It verifies the following basic properties:

- (The bulk semi-group property) For z_1 and z_2 in \mathbb{R}

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2) = p_B(\tau_1 + \tau_2; z_1, z_2). \quad (10)$$

- (The \star semi-group property) For z_1 and z_2 in \mathbb{R}^+ and $\star \in \{D, N, R\}$, we have

$$\int_{\mathbb{R}^+} du p_\star(\tau_1; z_1, u) p_\star(\tau_2; u, z_2) = p_\star(\tau_1 + \tau_2; z_1, z_2). \quad (11)$$

- (Completeness) For z_1 in \mathbb{R} , we have

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) = 1. \quad (12)$$

- For z_1 and z_2 in \mathbb{R}^+ , we have

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2) \leq 2 \int_{\mathbb{R}^+} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2). \quad (13)$$

- For $\delta \geq 0$, $\tau_\delta = (1 + \delta)\tau$ and $z_1, z_2 \in \mathbb{R}^+$, we have

$$p_B(\tau; z_1, z_2) \leq \sqrt{1 + \delta} p_B(\tau_\delta; z_1, z_2) \quad (14)$$

and for $\delta' > \delta$

$$|z_1 - z_2|^r p_B(\tau_\delta; z_1, z_2) \leq C_{\delta, \delta'} \tau^{\frac{r}{2}} p_B(\tau_{\delta'}; z_1, z_2), \quad (15)$$

where

$$C_{\delta, \delta'} = \sqrt{\frac{1 + \delta'}{1 + \delta}} \|x' e^{-\frac{x'^2}{2} \frac{\delta' - \delta}{(1 + \delta)(1 + \delta')}}\|_\infty \leq O(1) |\delta - \delta'|^{-\frac{r}{2}}. \quad (16)$$

- For $z, z' \in \mathbb{R}^+$, $\tau > 0$ and $c \geq 0$, we have

$$p_B(\tau; z, -z') \leq p_B(\tau; z, z'), \quad \int_{\mathbb{R}^+} dw e^{-w} p_B\left(\tau; z, -z' - \frac{w}{c}\right) \leq p_B(\tau; z, z'). \quad (17)$$

B. The regularized propagators

We denote by \star the type of boundary conditions considered. For $0 \leq \Lambda \leq \Lambda_0$, we define the regularized flowing propagator associated to the boundary condition $\star \in \{D, N, R\}$ as follows:

$$C_{\star}^{\Lambda, \Lambda_0}(p; z, z') := \int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\lambda p_{\star}(\lambda; z, z') e^{-\lambda(p^2 + m^2)}. \quad (18)$$

This can also be rewritten as

$$C_{\star}^{\Lambda, \Lambda_0}(p; z, z') = C_B^{\Lambda, \Lambda_0}(p; z, z') + C_{S, \star}^{\Lambda, \Lambda_0}(p; z, z'), \quad (19)$$

where

$$C_B^{\Lambda, \Lambda_0}(p; z, z') := \int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\lambda p_B(\lambda; z, z') e^{-\lambda(p^2 + m^2)} \quad (20)$$

and

$$C_{S, \star}^{\Lambda, \Lambda_0}(p; z, z') := \int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\lambda p_{S, \star}(\lambda; z, z') e^{-\lambda(p^2 + m^2)}, \quad (21)$$

with the surface heat kernel $p_{S, \star}$ defined as $p_{\star} - p_B$. In the case of Robin boundary conditions, the surface Robin heat kernel is given by

$$p_{S, R}\left(\frac{1}{\Lambda^2}; z, z'\right) := p_B\left(\frac{1}{\Lambda^2}; z, -z'\right) - 2 \int_0^\infty dw e^{-w} p_B\left(\frac{1}{\Lambda^2}; z, -\frac{w}{c} - z'\right). \quad (22)$$

Note that the Robin heat kernel and $p_{S,R}$ are uniformly bounded w.r.t. the Robin parameter c . Namely, we have using (17)

$$p_R(\tau; z, z') \leq 4 p_B(\tau; z, z'), \quad p_{S,R}(\tau; z, z') \leq 3 p_B(\tau; z, -z'), \quad (23)$$

for all $z, z' \geq 0$, $\tau > 0$ and $c \geq 0$. Similarly, we also have

$$p_D(\tau; z, z') \leq p_N(\tau; z, z') \leq 2 p_B(\tau; z, z'). \quad (24)$$

In the sequel, we denote the derivative of the flowing propagators w.r.t. Λ by

$$\dot{C}_\bullet^\Lambda(p; z, z') = \frac{\partial}{\partial \Lambda} C_{\bullet}^{\Lambda, \Lambda_0}(p; z, z') = \dot{C}^\Lambda(p) p_\bullet \left(\frac{1}{\Lambda^2}; z, z' \right), \quad (25)$$

where $\dot{C}^\Lambda(p) = -\frac{2}{\Lambda^3} e^{-\frac{p^2+m^2}{\Lambda^2}}$ and $\bullet \in \{\star, \{S, \star\}, B\}$ with $\star \in \{D, N, R\}$.

Given a polynomial \mathcal{P} and $w \in \mathbb{N}^3$, we have the following estimate on the 3-dimensional covariance

$$\left| \partial^w \dot{C}^\Lambda(p) \mathcal{P} \left(\frac{p}{\Lambda} \right) \right| \leq (\Lambda + m)^{-3-|w|} \tilde{\mathcal{P}} \left(\frac{|p|}{\Lambda + m} \right), \quad \forall p \in \mathbb{R}^3, \quad (26)$$

where $\tilde{\mathcal{P}}$ is a polynomial with positive coefficients. We refer to (34) for the multi-index w notation.

III. THE BULK THEORY ON THE HALF-SPACE $\mathbb{R}^+ \times \mathbb{R}^3$

A. The Action and the Flow Equations

We consider the theory of a real scalar field ϕ with mass m on the four dimensional Euclidean space-time \mathbb{R}^4 within the framework of functional integration. The point of departure to define this theory is to write the associated regularized path integral which is uniquely defined by the corresponding gaussian measure. The regularized flowing propagator is given by (20). Note that for $\Lambda \rightarrow 0$ and $\Lambda_0 \rightarrow \infty$ we recover the unregularized propagator. For finite Λ_0 and in finite volume the positivity and the regularity properties of C_B^{Λ, Λ_0} permit to define the theory rigorously from the functional integral

$$e^{-\frac{1}{\hbar} (L_B^{\Lambda, \Lambda_0}(\phi) + I^{\Lambda, \Lambda_0})} := \int d\mu_B^{\Lambda, \Lambda_0}(\Phi) e^{-\frac{1}{\hbar} L_B^{\Lambda_0, \Lambda_0}(\Phi + \phi)}, \quad (27)$$

$$L_B^{\Lambda, \Lambda_0}(0) = 0,$$

where the factors of \hbar have been introduced with regard to a systematic loop expansion considered later. Here, the Gaussian measure $d\mu_B^{\Lambda, \Lambda_0}$ is of mean zero and covariance $\hbar C_B^{\Lambda, \Lambda_0}$. The test function ϕ is in the support of $d\mu_B^{\Lambda, \Lambda_0}$ which in particular implies that it is in $\mathcal{C}^\infty(\mathbb{R}^4)$. This regularity stems from the UV-regularization determined by the cutoff Λ_0 which is imperative to have a well-defined functional integral. The normalization factor $e^{-\frac{1}{\hbar} I^{\Lambda, \Lambda_0}}$ is due to vacuum contributions. It diverges in infinite volume so that we can take the infinite volume limit only when it has been eliminated²⁰. We do not make the finite volume explicit here since it plays no role in the sequel.

The functional $L_B^{\Lambda_0, \Lambda_0}(\phi)$ is the bare interaction of a renormalizable theory including counter-terms, viewed as a formal power series in \hbar . It contains the tree order interaction and the related counter-terms. The interaction is supported only on the half-space $\mathbb{R}^+ \times \mathbb{R}^3$ which implies that translation invariance is broken in the z -direction (the semi-line). This raises the possibility that the counter-terms may be z -dependent. In general, the constraints on the bare action result from the

symmetry properties of the theory which are imposed, on its field content and on the form of the propagator. It is therefore natural to consider the general bare interaction

$$L_B^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_z \int_{\mathbb{R}^3} d^3x \phi^4(z, x) + \frac{1}{2} \int_z \int_{\mathbb{R}^3} d^3x \left(a_B^{\Lambda_0}(z) \phi^2(z, x) - b_B^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) \right. \\ \left. - d_B^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) - s_B^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c_B^{\Lambda_0}(z) \phi^4(z, x) \right). \quad (28)$$

Here we supposed the theory to be symmetric under $\phi \rightarrow -\phi$, and we included in (28) only relevant terms in the sense of the renormalization group. The functions $a_B^{\Lambda_0}(z)$, $b_B^{\Lambda_0}(z)$, $c_B^{\Lambda_0}(z)$, $d_B^{\Lambda_0}(z)$ and $s_B^{\Lambda_0}(z)$ are supposed to be smooth.

The flow equation (FE) is obtained from (27) on differentiating w.r.t. Λ . For the steps of the computation, we refer the reader to^{5,20,27}. It is a differential equation for the functional L_B^{Λ, Λ_0} :

$$\partial_\Lambda (L_B^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0}) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \phi}, \dot{C}_B^\Lambda \frac{\delta}{\delta \phi} \right\rangle L_B^{\Lambda, \Lambda_0} - \frac{1}{2} \left\langle \frac{\delta}{\delta \phi} L_B^{\Lambda, \Lambda_0}, \dot{C}_B^\Lambda \frac{\delta}{\delta \phi} L_B^{\Lambda, \Lambda_0} \right\rangle. \quad (29)$$

By $\langle \cdot, \cdot \rangle$ we denote the standard inner product in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$.

We expand the functional $L_B^{\Lambda, \Lambda_0}(\phi)$ in a formal power series w.r.t. \hbar ,

$$L_B^{\Lambda, \Lambda_0}(\phi) = \sum_{l=0}^{\infty} \hbar^l L_{l,B}^{\Lambda, \Lambda_0}(\phi).$$

Corresponding expansions for $a_B^{\Lambda_0}(z)$, $b_B^{\Lambda_0}(z)$, ..., are $a_B^{\Lambda_0}(z) = \sum_{l=1}^{\infty} \hbar^l a_{l,B}^{\Lambda_0}(z)$, etc. From $L_{l,B}^{\Lambda, \Lambda_0}(\phi)$ we obtain the CAS distributions of loop order l as

$$\mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, x_1), \dots, (z_n, x_n)) := \delta_{\phi(z_1, x_1)} \cdots \delta_{\phi(z_n, x_n)} L_{l,B}^{\Lambda, \Lambda_0} |_{\phi=0},$$

where we used the notation $\delta_{\phi(z,x)} = \delta / \delta \phi(z, x)$.

In the pz -representation, we set for r , r_1 and $r_2 \in \mathbb{N}^*$ and

$$\Phi_n(z_2, \dots, z_n) := \prod_{i=2}^n \phi_i(z_i), \quad \phi_i \in \mathcal{S}(\mathbb{R}^+)$$

$$\mathcal{D}_{l,n,r}^{\Lambda, \Lambda_0; (i)}(z_1; \vec{p}_n; \Phi_n) := \int_{(\mathbb{R}^+)^{n-1}} dz_{2,n} (z_1 - z_i)^r \mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi_2(z_2) \cdots \phi_n(z_n), \quad (30)$$

$$\mathcal{D}_{l,n,r_1,r_2}^{\Lambda, \Lambda_0; (i,j)}(z_1; \vec{p}_n; \Phi_n) := \int_{(\mathbb{R}^+)^{n-1}} dz_{2,n} (z_1 - z_i)^{r_1} (z_1 - z_j)^{r_2} \mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ \times \phi_2(z_2) \cdots \phi_n(z_n), \quad r_1 + r_2 = r, \quad (31)$$

and for $r = 0$

$$\mathcal{D}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \Phi_n) := \int_{(\mathbb{R}^+)^{n-1}} dz_{2,n} \mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi_2(z_2) \cdots \phi_n(z_n). \quad (32)$$

Here we denote

$$\delta^{(3)}(p_1 + \cdots + p_n) \mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = (2\pi)^{3(n-1)} \frac{\delta^n}{\delta \phi(z_1, p_1) \cdots \delta \phi(z_n, p_n)} L_{l,B}^{\Lambda, \Lambda_0}(\phi) |_{\phi=0}.$$

The distribution $\delta^{(3)}(p_1 + \dots + p_n)$ appears because of translation invariance in the x directions. The FE for the CAS distributions derived from (29) are

$$\begin{aligned} & \partial_\Lambda \partial^w \mathcal{D}_{l,n}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ &= \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \int_k \partial^w \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_B^\Lambda(k; z, z') \\ & - \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^+} dz' \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_{w_i} \left[\partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_B^\Lambda(p; z, z') \right. \\ & \quad \left. \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym}, \\ & p = -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n. \end{aligned} \quad (33)$$

The number $|w|$ of momentum derivatives, is characterized by a multi-index. We use the shorthand notation

$$\partial^w := \prod_{i=1}^n \prod_{\mu=0}^3 \left(\frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}} \quad \text{with } w = (w_{1,0}, \dots, w_{n,3}), \quad |w| = \sum_{i,\mu} w_{i,\mu}, \quad w_{i,\mu} \in \mathbb{N}^*. \quad (34)$$

The symbol "rsym" means summation over those permutations of the momenta $(z_1, p_1), \dots, (z_n, p_n)$, which do not leave invariant the (unordered) subsets $((z_1, p_1), \dots, (z_{n_1}, p_{n_1}))$ and $((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n))$, and therefore, produce mutually different pairs of (unordered) image subsets, and the primes restrict the summations to $n_1 + n_2 = n$, $l_1 + l_2 = l$, $w_1 + w_2 + w_3 = w$, respectively. The combinatorial factor $c_{\{w_i\}} = w!(w_1!w_2!w_3!)^{-1}$ stems from Leibniz's rule.

B. Test functions and boundary conditions

The n -point correlation "functions" $\mathcal{D}_{l,n}^{\Lambda, \Lambda_0}$ when considered in the pz -representation are tempered distributions which belong for fixed \vec{p}_n to the space $\mathcal{S}'(\mathbb{R}^{+n})$ w.r.t. the semi-norms

$$\prod_{i=1}^n \mathcal{N}_2(\phi_i),$$

where $\mathcal{N}_2(\phi) := \sup_{0 \leq \alpha, \beta \leq 2} \|(1+z^\beta) \partial_z^\alpha \phi(z)\|_\infty$ and $\partial_z \phi|_{z=0} = \lim_{z \rightarrow 0^+} \partial_z \phi$. For additional informations on the topological construction of $\mathcal{S}'(\mathbb{R}^+)$, we refer the reader to²⁹. Our method of proof relies on inductive bounds deduced from the flow equations (33). The induction restricts our choice of the test functions. To proceed inductively we cannot admit any arbitrary test function in $\mathcal{S}'(\mathbb{R}^{+n})$. Let us give the set of test functions we will be using in the sequel: For $2 \leq s \leq n$, we define

$$\tau := \inf \tau_{2,s}, \quad \text{where } \tau_{2,s} = (\tau_2, \dots, \tau_s) \text{ with } \tau_i > 0,$$

and similarly $z_{2,s} = (z_2, \dots, z_s)$. Given $(y_2, \dots, y_s) \in \mathbb{R}^{s-1}$, we define

$$\phi_{\tau_{2,s}, y_{2,s}}(z_{2,s}) := \prod_{i=2}^s p_B(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i), \quad (35)$$

where χ^+ is the characteristic function of the semi-line \mathbb{R}^+ . This definition can be generalized by choosing any other subset of $s-1$ coordinates among z_2, \dots, z_n . The characteristic functions χ^+ are introduced in order to be able to extract the relevant terms in the sense of the renormalization group from the full n -point distributions and to get inductive control of the local counter terms. To go further one could either prove (in a more functional analysis type of approach) that our test functions are dense in the set of smooth rapidly decaying functions on \mathbb{R}^+ w.r.t. a suitable norm and

that $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \Phi_n)$ are continuous w.r.t. this semi-norm. Given the test functions ϕ_2, ϕ_3 and ϕ_4 in $\mathcal{S}(\mathbb{R}^+)$, we define

$$\Phi_4(z_2, z_3, z_4) = \prod_{i=2}^4 \phi_i(z_i).$$

The relevant terms contained in

$$\mathcal{D}_{l,2}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \phi_2) := \int_{z_2} \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \phi_2(z_2) \quad (36)$$

and

$$\mathcal{D}_{l,4}^{\Lambda,\Lambda_0}(z_1; \vec{0}; \Phi_4) := \int_{z_{2,4}} \mathcal{D}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) \Phi_4(z_2, z_3, z_4), \quad (37)$$

are then extracted using a Taylor expansion of the test functions ϕ_2 and Φ_4 , which gives

$$\begin{aligned} \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) &= a_{l,B}^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) - s_{l,B}^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) \\ &\quad - d_{l,B}^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1) + l_{l,2,B}^{\Lambda,\Lambda_0}(z_1; \phi_2), \end{aligned} \quad (38)$$

$$(\partial_{p^2} \mathcal{D}_{l,2}^{\Lambda,\Lambda_0})(z_1; 0, 0; \phi_2) = b_{l,B}^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) + (\partial_{p^2} l_{l,2}^{\Lambda,\Lambda_0})(z_1; \phi_2), \quad (39)$$

$$\mathcal{D}_{l,4}^{\Lambda,\Lambda_0}(z_1; 0, \dots, 0; \Phi_4) = c_{l,B}^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) \phi_3(z_1) \phi_4(z_1) + l_{l,4,B}^{\Lambda,\Lambda_0}(z_1; \Phi_4). \quad (40)$$

Then the relevant terms appear as

$$a_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (41)$$

$$s_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 (z_1 - z_2) \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (42)$$

$$d_{l,B}^{\Lambda,\Lambda_0}(z_1) = -\frac{1}{2} \int_0^\infty dz_2 (z_1 - z_2)^2 \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (43)$$

$$b_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \partial_{p^2} \left(\mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}, \quad (44)$$

$$c_{l,B}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 dz_3 dz_4 \mathcal{D}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)), \quad (45)$$

and the remainders $l_{l,2,B}^{\Lambda,\Lambda_0}(z_1; \phi_2)$, $(\partial_{p^2} l_{l,2,B}^{\Lambda,\Lambda_0})(z_1; \phi_2)$ and $l_{l,4,B}^{\Lambda,\Lambda_0}(z_1; \Phi_4)$ can be written as

$$l_{l,2,B}^{\Lambda,\Lambda_0}(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1; 0), (z_2; 0)), \quad (46)$$

$$(\partial_{p^2} l_{l,2,B}^{\Lambda,\Lambda_0})(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1) \partial_{p^2} \left(\mathcal{D}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}$$

and

$$\begin{aligned} l_{l,4,B}^{\Lambda,\Lambda_0}(z_1; \Phi_4) &= \int_{(\mathbb{R}^+)^3} dz_{2,4} \mathcal{D}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) \left[\int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1) \phi_3(z_3) \phi_4(z_4) \right. \\ &\quad \left. + \phi_2(z_1) \int_0^1 dt \partial_t \phi_3(tz_3 + (1-t)z_1) \phi_4(z_4) + \phi_2(z_1) \phi_3(z_1) \int_0^1 dt \partial_t \phi_4(tz_4 + (1-t)z_1) \right]. \end{aligned} \quad (47)$$

Boundary conditions at $\Lambda = \Lambda_0$:

The bare interaction implies that at $\Lambda = \Lambda_0$

$$\begin{aligned} \mathcal{D}_{l,2}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) &= \left(a_{l;B}^{\Lambda_0}(z_1) + b_{l;B}^{\Lambda_0}(z_1)p^2 - s_{l;B}^{\Lambda_0}(z_1)\partial_{z_1} - d_{l;B}^{\Lambda_0}(z_1)\partial_{z_1}^2 \right) \delta(z_1 - z_2), \\ \mathcal{D}_{l,4}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) &= \left(\lambda \delta_{l,0} + c_{l;B}^{\Lambda_0}(z_1)(1 - \delta_{l,0}) \right) \prod_{i=2}^4 \delta(z_1 - z_i), \\ \mathcal{D}_{l,n}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) &= 0, \quad \forall n \geq 5. \end{aligned} \quad (48)$$

Renormalization conditions at $\Lambda = 0$ (BPHZ renormalization conditions):

The renormalization conditions are fixed at $\Lambda = 0$ by imposing for all $z_1 \geq 0$

$$a_{l;B}^{0,\Lambda_0}(z_1) \equiv 0, \quad s_{l;B}^{0,\Lambda_0}(z_1) \equiv 0, \quad d_{l;B}^{0,\Lambda_0}(z_1) \equiv 0, \quad b_{l;B}^{0,\Lambda_0}(z_1) \equiv 0, \quad c_{l;B}^{0,\Lambda_0}(z_1) \equiv 0. \quad (49)$$

These will be adopted in the following. Note that the boundary conditions are invariant under $O(3)$ -symmetry. In the sequel, we need the following result:

Proposition 1. *For $0 \leq \Lambda \leq \Lambda_0 < \infty$, $1 \leq s \leq n$, $2 \leq i \leq n$ and $0 \leq r \leq 3$, we consider test functions of the form $\phi_{\tau_{2,s}, \nu_{2,s}}(z_{2,s})$, which are also denoted in shorthand as $\phi_{\tau_{2,s}, \nu_{2,s}}$. Adopting (48)-(49) we claim*

$$\begin{aligned} & \left| \partial^w \mathcal{D}_{l,n;r}^{\Lambda,\Lambda_0;(i)}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, \nu_{2,s}}) \right| \\ & \leq (\Lambda + m)^{4-n-|w|-r} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}), \end{aligned} \quad (50)$$

$$\begin{aligned} & \left| \partial^w \mathcal{D}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0;(i,j)}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, \nu_{2,s}}) \right| \\ & \leq (\Lambda + m)^{4-n-|w|-r_1-r_2} \mathcal{P}'_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q}'_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}'_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}), \end{aligned} \quad (51)$$

and $\partial^w \mathcal{D}_{l,n;r}^{\Lambda,\Lambda_0;(i)}(z_1; \vec{p}_n; \phi_{\tau_{2,s}, \nu_{2,s}})$ is continuous w.r.t. z_1 . Here, \mathcal{P}_i and \mathcal{Q}_i denote polynomials with non-negative coefficients which depend on $l, n, |w|, r$, but not on $\{p_i\}$, Λ , Λ_0 and z_1 . The polynomials \mathcal{Q}_i are reduced to a constant if $s = 1$, and for $l = 0$ all polynomials \mathcal{P}_i and \mathcal{Q}_i reduce to constants.

Since we considered the bulk theory in the mixed position-momentum space together with imposing BPHZ renormalization conditions, the proof of proposition 1 follows step by step the proof of Theorem 1 in⁶. For the sake of simplicity, we do not reproduce here the full proof and we refer the reader to⁶ for the details.

IV. THE SURFACE CORRELATION DISTRIBUTIONS

A. The semi-infinite theory

In this subsection, we recall the flow equations of the semi-infinite massive scalar field model presented in⁶:

$$\begin{aligned} \partial_\Lambda \partial^w \mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) &= \frac{1}{2} \int_z \int_{z'} \int_k \partial^w \mathcal{L}_{l-1,n+2;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_\star^\Lambda(k; z, z') \\ &\quad - \frac{1}{2} \int_z \int_{z'} \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_{w_i} \left[\partial^{w_1} \mathcal{L}_{l_1, n_1+1;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_\star^\Lambda(p; z, z') \right. \\ &\quad \left. \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1;\star}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym}, \\ p &= -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n. \end{aligned} \quad (52)$$

where $\dot{C}_\star^{\Lambda,\Lambda_0}(p; z, z')$ is defined in (19) and $\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n))$ denote the semi-infinite correlation distributions at loop order l and with n external points. The \star index refers to the type of considered boundary conditions, namely Dirichlet, Neumann and Robin. In⁶, we imposed the following mixed boundary conditions:

- **At $\Lambda = \Lambda_0$:**

$$\begin{aligned} \mathcal{L}_{l,2;\star}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) &= \left(a_{l;\star}^{\Lambda_0}(z_1) + b_{l;\star}^{\Lambda_0}(z_1)p^2 - s_{l;\star}^{\Lambda_0}(z_1)\partial_{z_1} - d_{l;\star}^{\Lambda_0}(z_1)\partial_{z_1}^2 \right) \delta(z_1 - z_2), \\ \mathcal{L}_{l,4;\star}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) &= \lambda \delta_{l,0} + c_{l;\star}^{\Lambda_0}(z_1)(1 - \delta_{l,0}) \prod_{i=2}^4 \delta(z_1 - z_i), \\ \mathcal{L}_{l,n;\star}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) &= 0, \quad \forall n \geq 5. \end{aligned} \quad (53)$$

- **At $\Lambda = 0$:** We impose BPHZ type renormalization conditions. Namely, for all $z_1 \geq 0$ we set

$$a_{l;\star}^{0,\Lambda_0}(z_1) \equiv 0, \quad s_{l;\star}^{0,\Lambda_0}(z_1) \equiv 0, \quad d_{l;\star}^{0,\Lambda_0}(z_1) \equiv 0, \quad b_{l;\star}^{0,\Lambda_0}(z_1) \equiv 0, \quad c_{l;\star}^{0,\Lambda_0}(z_1) \equiv 0, \quad (54)$$

where

$$a_{l;\star}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \mathcal{L}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (55)$$

$$s_{l;\star}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 (z_1 - z_2) \mathcal{L}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (56)$$

$$d_{l;\star}^{\Lambda,\Lambda_0}(z_1) = -\frac{1}{2} \int_0^\infty dz_2 (z_1 - z_2)^2 \mathcal{L}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (57)$$

$$b_{l;\star}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \partial_{p^2} \left(\mathcal{L}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}, \quad (58)$$

$$c_{l;\star}^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 dz_3 dz_4 \mathcal{L}_{l,4;\star}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)). \quad (59)$$

This yielded five position dependent counter-terms which appear in the bare interaction of our semi-infinite model

$$\begin{aligned} L_\star^{\Lambda_0,\Lambda_0}(\phi) &= \frac{\lambda}{4!} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \phi^4(z, x) + \frac{1}{2} \int_{\mathbb{R}^+} dz \int_{\mathbb{R}^3} d^3x \left(a_\star^{\Lambda_0}(z) \phi^2(z, x) - b_\star^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) \right. \\ &\quad \left. - d_\star^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) - s_\star^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c_\star^{\Lambda_0}(z) \phi^4(z, x) \right). \end{aligned} \quad (60)$$

In our previous work⁶, we saw that imposing constant renormalization conditions w.r.t. the position z at the scale $\Lambda = 0$ is at the expense of obtaining position dependent counter-terms and bounds which do not separate the bulk from the surface effects. In this work, we aim to differentiate the bulk and surface effects. Our strategy is based on extracting the surface counter-terms from the semi-infinite counter-terms by separating the bulk and the surface effects in the definition of the semi-infinite correlation distributions. Concretely, we proceed by subtracting the bulk correlation distributions defined in Section III from the semi-infinite correlation distributions and study the behaviour of the difference to which we refer as the surface correlation distributions. Namely, we write

$$\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) := \mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)) - \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}((\vec{z}_n, \vec{p}_n)). \quad (61)$$

The definition (61) allows to write the FEs verified by $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$, which we give explicitly in the next subsection.

B. The surface correlation distributions

Before getting to the mathematical definition of the surface correlation distributions, let us give a brief motivation based on a diagrammatic approach to the renormalization problem of the semi-infinite scalar field model. The propagator associated to the b.c. \star can be decomposed into a sum of the two contributions given in (19), where C_B^{Λ,Λ_0} is the regularized bulk propagator which is responsible for the singularities arising from coalescing of points and $C_{S,\star}^{\Lambda,\Lambda_0}$ is the part which is responsible of singularities arising when a point approaches the surface. Therefore, an arbitrary Feynman diagram of the semi-infinite model can be written as the sum of a diagram which contains only bulk internal lines consisting of propagators C_B^{Λ,Λ_0} only, and other diagrams which contain at least one surface internal line given by the propagator $C_{S,\star}^{\Lambda,\Lambda_0}$. Renormalizing the massive semi-infinite model then amounts to renormalizing the diagrams with only bulk internal lines and those with at least a surface internal line. This approach has the advantage to disentangle the surface divergences from the bulk divergences. From the renormalization group point of view, we proceed similarly by writing (61) with $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ (resp. $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$) consisting of all connected amputated diagrams with n external legs and l loops involving exclusively C_B^{Λ,Λ_0} (resp. C_B^{Λ,Λ_0} and at least one $C_{S,\star}^{\Lambda,\Lambda_0}$). Using the flow equations (33) and (52), we obtain the flow equations verified by the surface correlation distributions $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n))$

$$\begin{aligned} & \partial_\Lambda \partial^w \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \\ &= \frac{1}{2} \int_{z,z'} \int_k \partial^w \mathcal{S}_{l-1,n+2;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_\star^\Lambda(k; z, z') \\ &+ \frac{1}{2} \int_{z,z'} \int_k \partial^w \mathcal{D}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_{S,\star}^\Lambda(k; z, z') \\ &- \frac{1}{2} \int_{z,z'} \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_{w_i} \\ &\left[\partial^{w_1} \mathcal{S}_{l_1, n_1+1;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_\star^\Lambda(p; z, z') \partial^{w_2} \mathcal{S}_{l_2, n_2+1;\star}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \right. \\ &+ \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_\star^\Lambda(p; z, z') \partial^{w_2} \mathcal{S}_{l_2, n_2+1;\star}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \\ &+ \partial^{w_1} \mathcal{S}_{l_1, n_1+1;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_\star^\Lambda(p; z, z') \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \\ &\left. + \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_{S,\star}^\Lambda(p; z, z') \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{rsym}, \\ & \quad p = -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n. \end{aligned} \quad (62)$$

For the tree order $l = 0$ we have

$$\mathcal{S}_{0,4;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = 0. \quad (63)$$

The existence of $\mathcal{S}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n))$ is ensured by (63) and by the flow equations (62) through induction in $n + 2l$ and in l for fixed $n + 2l$.

C. Boundary and renormalization conditions

For ϕ_1 and ϕ_2 in $\mathcal{S}(\mathbb{R}^+)$, the relevant terms are contained in

$$\mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}(0,0) := \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \phi_1(z_1) \phi_2(z_2). \quad (64)$$

They are extracted from (64) by performing a Taylor expansion of the test functions ϕ_1 and ϕ_2 around 0 which gives

$$\begin{aligned} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}(0,0) = s_{l;\star}^{\Lambda,\Lambda_0} \phi_1(0) \phi_2(0) + e_{l;\star}^{\Lambda,\Lambda_0} \phi_1(0) (\partial_n \phi_2)(0) + h_{l;\star}^{\Lambda,\Lambda_0} \phi_2(0) (\partial_n \phi_1)(0) \\ + l_{l,2;\star}^{\Lambda,\Lambda_0}(\phi_1, \phi_2), \end{aligned} \quad (65)$$

with $(\partial_n \phi)(0) := \lim_{z \rightarrow 0} (\partial_z \phi)(z)$. Then the relevant terms $s_{l;\star}^{\Lambda,\Lambda_0}$, $e_{l;\star}^{\Lambda,\Lambda_0}$ and $h_{l;\star}^{\Lambda,\Lambda_0}$ are obtained as

$$\begin{aligned} s_{l;\star}^{\Lambda,\Lambda_0} &:= \int_{z_1, z_2} \mathcal{S}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad e_{l;\star}^{\Lambda,\Lambda_0} := \int_{z_1, z_2} z_2 \mathcal{S}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \\ h_{l;\star}^{\Lambda,\Lambda_0} &:= \int_{z_1, z_2} z_1 \mathcal{S}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)). \end{aligned} \quad (66)$$

Bose symmetry implies that

$$\int_{z_1, z_2} z_2 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) = \int_{z_1, z_2} z_1 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (67)$$

so that the counter-terms $e_{l;\star}^{\Lambda,\Lambda_0}$ and $h_{l;\star}^{\Lambda,\Lambda_0}$ are equal to all orders of perturbation theory. The remainder $l_{l,2;\star}^{\Lambda,\Lambda_0}(\phi_1, \phi_2)$ has the form

$$\begin{aligned} l_{l,2;\star}^{\Lambda,\Lambda_0}(\phi_1, \phi_2) = & \left(\int_{z_1, z_2} z_1 z_2 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \\ & + \phi_1(0) \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \\ & + \phi_2(0) \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ & + (\partial_n \phi_1)(0) \int_{z_1, z_2} z_1 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \\ & + (\partial_n \phi_2)(0) \int_{z_1, z_2} z_2 \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ & + \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \left(\int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \right) \times \left(\int_0^1 dt' (1-t') (\partial_{t'}^2 \phi_2)(t'z_2) \right). \end{aligned} \quad (68)$$

In the sequel, we use the following notations. For $\star \in \{R, N, D\}$, we write

$$\partial^w \mathcal{S}_{l,n;\star; r_1, r_2}^{\Lambda_0, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) := \int_{z_1, \dots, z_n} z_1^{r_1} z_2^{r_2} \partial^w \mathcal{S}_{l,n;\star}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i),$$

$$\partial^w \mathcal{S}_{l,n;\star; r_1, r_2}^{\Lambda_0, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}^*) := \int_{z_1, \dots, z_n} z_1^{r_1} z_2^{r_2} \partial^w \mathcal{S}_{l,n;\star}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_\star(\tau_i; z_i, y_i).$$

The boundary conditions imposed on $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ are the following:

- At $\Lambda = \Lambda_0$, we impose for $\star \in \{R, N\}$

$$\mathcal{S}_{l,2;\star}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = s_{l;\star}^{\Lambda_0,\Lambda_0} \delta_{z_1} \delta_{z_2} + e_{l;\star}^{\Lambda_0,\Lambda_0} (\delta_{z_1} \delta'_{z_2} + \delta'_{z_1} \delta_{z_2}), \quad \forall l \geq 1, \quad (69)$$

$$\mathcal{S}_{0,2;\star}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = 0,$$

$$\mathcal{S}_{l,n;\star}^{\Lambda_0,\Lambda_0}((\vec{z}_n, \vec{p}_n)) = 0, \quad \forall n \geq 4, \forall l \geq 0. \quad (70)$$

- At $\Lambda = 0$, we fix the renormalization conditions for $\star \in \{R, N\}$ as

$$s_{l;\star}^{0,\Lambda_0} = 0, \quad e_{l;\star}^{0,\Lambda_0} = 0. \quad (71)$$

- For Dirichlet boundary conditions we impose

$$\mathcal{S}_{l,n;D}^{\Lambda_0,\Lambda_0}((\vec{z}_n, \vec{p}_n)) = 0, \quad \forall n \geq 2, \forall l \geq 0. \quad (72)$$

Remarks 1. - The boundary conditions (69)-(72) together with the flow equations (62) and the tree order (63) define uniquely the surface correlation distributions

$$\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)), \quad \star \in \{D, R, N\}.$$

This can be verified inductively by taking the difference of two solutions of the flow equations which obey the same boundary conditions (69)-(72) and by proving to all orders of perturbation theory that this difference vanishes.

- We would like to emphasize w.r.t. (61) that we do not require any a priori knowledge on the semi-infinite correlation distributions $\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}$ to give a meaning to $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$. The flow equations (62) together with the bulk correlation distributions defined in Section III, the tree order (63) and the boundary conditions (69)-(72) are sufficient to define uniquely the surface correlation distributions. The relation (61) implies the flow equations to be verified by $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ such that the sum

$$\mathcal{D}_{l,n}^{\Lambda,\Lambda_0} + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$$

is a solution to the FEs (52).

V. TREES, FORESTS AND WEIGHT FACTORS

The bounds on the surface and bulk correlation distributions are specified in terms of weighted trees and forests, which we define in the following, and for which we also derive some properties that will be important later. Our trees basically represent tree level Feynman graphs, though this analogy should not be taken too literally; the trees and the incidence numbers of vertices are independent of the detailed form of the n -point interactions in the theory, the loop order controls the number of vertices of incidence number 2 of the trees and forests via a bound, but there is no one-to-one correspondence between the loop order and the number of these vertices. To each tree/forest, we assign a weight factor which appears in the bounds. These weight factors consist of products of bulk heat kernels $p_B(\cdot; z, z')$ with z and z' vertices of the tree/forest. The main idea behind introducing these trees is the fact that in perturbation theory, the scaling behaviour of correlation functions including loop corrections is modified only logarithmically with respect to the tree level. The tree structure is compatible with the flow equations, in particular with the quadratic term of the bulk flow equations which we recall

$$\mathcal{D}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((\vec{z}_{n_1}, \vec{p}_{n_1}), (z, p)) p_B\left(\frac{1}{\Lambda^2}; z, z'\right) \mathcal{D}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((\vec{z}_{n_1+1, n}, \vec{p}_{n_1+1, n}), (z', -p)) \dot{C}^\Lambda(p).$$

The heat kernel $p_B(1/\Lambda^2; z, z')$ corresponds to an internal line which merges the trees corresponding to the weight factors that bound each of the terms $\mathcal{D}_{l_i, n_i+1}^{\Lambda, \Lambda_0}$.

Since the aim of this paper is to understand the surface effects on the renormalizability of the semi-infinite model, we need to introduce additional combinatorial structures which we denote as surface trees. These have an external point on the surface. The appearance of forests, which are a collection of surface trees is motivated by the surface part of the propagator $C_{S, \star}^{\Lambda, \Lambda_0}$. The FE (62) contains terms with a factor of $\dot{C}_{S, \star}$

$$\dot{C}_{S, \star}(k; z, z') = p_{S, \star} \left(\frac{1}{\Lambda^2}; z, z' \right) \dot{C}^\Lambda(k). \quad (73)$$

Using (6)-(8) together with (17), we have

$$\left| p_{S, \star} \left(\frac{1}{\Lambda^2}; z, z' \right) \right| \leq O(1) p_B \left(\frac{1}{\Lambda^2}; z, -z' \right). \quad (74)$$

Furthermore, for $z, z' \geq 0$ we also have

$$p_B \left(\frac{1}{\Lambda^2}; z, -z' \right) \leq \sqrt{2\pi} \Lambda^{-1} p_B \left(\frac{1}{\Lambda^2}; z, 0 \right) p_B \left(\frac{1}{\Lambda^2}; z', 0 \right). \quad (75)$$

The bound (75) will play an important role in the sequel in lowering the power counting of the surface correlation distributions $\mathcal{S}_{l, n; \star}^{\Lambda, \Lambda_0}$. Furthermore, the heat kernels $p_B(1/\Lambda^2; z, 0)$ and $p_B(1/\Lambda^2; z', 0)$ imply that the trees corresponding to the weight factors bounding $\mathcal{D}_{l_i, n_i+1}^{\Lambda, \Lambda_0}$ in the flow equation (62) have new external legs $(z, 0)$ and $(z', 0)$. This motivates our definition of surface trees as those trees with an external leg $(z, 0)$ (i.e. surface external leg). Now the definition of forests which consist of a collection of trees is motivated by the fact that the bound (75) produces two surface external legs and hence two surface trees. The inductive bound of Theorem 1 used for the surface correlation distributions is then given in terms of weight factors associated with a product of surface trees, i.e. a forest.

First, we start with some notations that we will use in the sequel:

- For $s \geq 1$, we denote by σ_s the set $\{1, \dots, s\}$ and for $i \leq j$ we denote by $\sigma_{i,j}$ the set $\{i, \dots, j\}$.
- Let \mathcal{P}_s be the set of all the partitions of σ_s . For a partition $\Pi \in \mathcal{P}_s$, we write $\Pi = (\pi_i)_{1 \leq i \leq l_\Pi}$ with π_i denoting an element of the partition Π and l_Π the cardinality of Π .
- For $\Pi \in \mathcal{P}_s$ such that $r \in \pi_i$, we define

$$\pi_i^r := \pi_i \setminus \{r\}, \quad \Pi^r := \left\{ \{ \pi_j \}_{1 \leq j \leq l_\Pi, j \neq i}, \pi_i^r \right\}. \quad (76)$$

- Given $\Pi \in \mathcal{P}_{s+2}$ such that $\{s+1, s+2\} \in \pi_i$, we define the reduced sub-partition

$$\pi_i^{s+1, s+2} := \pi_i \setminus \{s+1, s+2\}. \quad (77)$$

- We denote by \mathcal{P}_s^1 the set of partitions which contain at least one sub-partition of length 1 (i.e. $\exists \pi_i \subset \Pi, |\pi_i| = 1$) and $\mathcal{P}_s^{1;c}$ its complementary set.
- We denote by $\tilde{\mathcal{P}}_{2,s}$ the set of partitions of length 2 of the set σ_s . Note that $\tilde{\mathcal{P}}_{2,s}$ is a subset of \mathcal{P}_s .

A. Bulk trees, surface trees and forests

- A tree is an undirected graph in which any two vertices are connected by exactly one path. We define the incidence number of a vertex z of the tree by the number of lines of the tree

that have z as an edge and we denote it by $c(z)$. Given a tree T , we denote by $\mathcal{V}(T)$ the set of vertices of the tree. The set of external vertices $\mathcal{V}_e(T)$ of the tree T is defined as follows

$$\mathcal{V}_e(T) := \{z \in \mathcal{V}(T) \mid c(z) = 1\}. \quad (78)$$

The set of internal vertices $\mathcal{V}_i(T)$ is then defined as $\mathcal{V}(T) \setminus \mathcal{V}_e(T)$.

- For $s \geq 2$, we denote by \mathcal{T}^s the set of all trees that have a root vertex and $s - 1$ external vertices. For a tree $T^s \in \mathcal{T}^s$ we will call $z_1 \in \mathbb{R}^+$ its root vertex. Denoting by $\mathcal{V}(T)$ the set of vertices of T^s , the set of external vertices of the tree T^s is defined as

$$\mathcal{V}_e(T^s) := \{z \in \mathcal{V}(T^s) \setminus \{z_1\} \mid c(z) = 1\}. \quad (79)$$

The set of internal vertices is the relative complement of $\mathcal{V}_e(T^s)$ in $\mathcal{V}(T^s) \setminus \{z_1\}$. Note that the root vertex z_1 is a vertex which is neither internal nor external. For simplicity, we use in the sequel the set of points $Y = \{y_2, \dots, y_s\}$ in \mathbb{R}^{s-1} to be identified with the external vertices of T^s . Likewise we call $Z = \{z_2, \dots, z_{r+1}\}$ the set of internal vertices of T^s where $z_i \in \mathbb{R}^+$ and r is the cardinality of $\mathcal{V}_i(T^s)$.

- We denote by $c_1 = c(z_1)$ the incidence number of the root vertex. We call a line p an external line of the tree if one of its edges is in Y . The set of external lines is denoted \mathcal{J} . The remaining lines are called internal lines of the tree and are denoted by \mathcal{I} .
- By T_l^s we denote a tree $T^s \in \mathcal{T}^s$ satisfying $v_2 + \delta_{c_1,1} \leq 3l - 2 + s/2$ for $l \geq 1$ and satisfying $v_2 = 0$ for $l = 0$, where v_n is the number of vertices having incidence number n . Then \mathcal{T}_l^s denotes the set of all trees T_l^s . We indicate the external vertices and internal vertices of the tree by writing $T_l^s(z_1, y_{2,s}, \vec{z})$ with $y_{2,s} = (y_2, \dots, y_s)$ and $\vec{z} = (z_2, \dots, z_{r+1})$.
- For $s \geq 1$, we define the set of bulk trees $\hat{\mathcal{T}}_l^s$ as the set of all trees with s external vertices which satisfy $v_2 \leq 3l - 2 + \frac{s}{2}$ for all $l \geq 1$, or $v_2 = 0$ for $l = 0$.
- Let $s \geq 1$. For $y_{1,s} := (y_1, \dots, y_s) \in \mathbb{R}^s$, we define the set of surface trees $\mathcal{T}^{s,0}$ to be the set consisting of all trees of $s + 1$ external vertices $\{y_1, \dots, y_s, 0\}$. In the sequel, we refer to the external vertex 0 as the surface external vertex to distinguish it from the other external vertices.
- By $T_l^{s,0}$ we denote a surface tree $T^{s,0} \in \mathcal{T}^{s,0}$ satisfying $v_2 \leq 3l - 2 + \frac{s+1}{2}$ for $l \geq 1$ and satisfying $v_2 = 0$ for $l = 0$. Then $\mathcal{T}_l^{s,0}$ denotes the set of all surface trees $T_l^{s,0}$. For a tree $T_l^{s,0} \in \mathcal{T}_l^{s,0}$, the set $\{y_1, \dots, y_s, 0\}$ of points in \mathbb{R} is identified with its external vertices, and $\vec{z} = (z_1, \dots, z_r)$ with the set of its internal vertices such that $r := |\mathcal{V}_i(T_l^{s,0})|$. We indicate the external vertices and the internal vertices of the tree by writing $T_l^{s,0}(Y_{\sigma_s}, 0, \vec{z})$. Note that this definition implies for all $l' \leq l$, $\mathcal{T}_{l'}^{s,0} \subset \mathcal{T}_l^{s,0}$. The definition we use here of internal/external vertices is (78).
- For $s = 1$, the set of surface trees $\mathcal{T}_1^{1,0}$ includes as well the surface tree $T_1^{1,0}(y_1, 0)$ with no internal vertices and one surface external vertex attached to the external vertex y_1 .
- Given a partition $\Pi \in \mathcal{P}_s$ and trees $T_l^{s\pi_i,0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \in \mathcal{T}_l^{s\pi_i,0}$, we define the forest $W_l^s(\Pi)$ as follows,

$$W_l^s(\Pi, y_{1,s}, \vec{z}) = \cup_{i=1}^{l_\Pi} T_l^{s\pi_i,0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \text{ where } s_{\pi_i} := |\pi_i|, \sum_{i=1}^{l_\Pi} s_{\pi_i} = s \text{ and } Y_{\pi_j} = \{y_i \mid i \in \pi_j\}. \quad (80)$$

We write shortly $W_l^s(\Pi, y_{1,s}, \vec{z}) \equiv W_l^s(\Pi)$, but one should keep in mind that the forest depends on $y_{1,s}$ and \vec{z} . There is a slight abusive of notation in (80), in the sense that $T_l^{s\pi_i,0}$ are not sets but rather surface trees, however we use the union symbol $\cup_{i=1}^{l_\Pi}$ to emphasize the fact that the

forest is a collection of surface trees. Then the set of all forests $W_l^s(\Pi)$ denoted by $\mathcal{W}_l^s(\Pi)$, is defined as:

$$\mathcal{W}_l^s(\Pi) := \left\{ \bigcup_{i=1}^{l_\Pi} T_i^{s\pi_i, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \mid T_i^{s\pi_i, 0} \in \mathcal{T}_i^{s\pi_i, 0}, \forall 1 \leq i \leq l_\Pi \right\}. \quad (81)$$

Note that for the trivial partition $\Pi_0 = \sigma_s$, the length of the partition is equal to one. Therefore, the set $W_l^s(\sigma_s)$ reduces to surface trees $\mathcal{T}_l^{s, 0}$. We write

$$\mathcal{W}_l^s(\sigma_s) = \mathcal{T}_l^{s, 0}. \quad (82)$$

This implies that each tree $T_l^{s\pi_i, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i})$ can be identified with a forest in $\mathcal{W}_l^{s\pi_i}(\sigma_{s\pi_i})$, where $\sigma_{s\pi_i} := \bigcup_{k \in \pi_i} \{k\}$.

- We define the global set of forests \mathcal{W}_l^s by

$$\mathcal{W}_l^s := \{ \bigcup_{\Pi \in \mathcal{P}_s} W_l^s(\Pi) \mid W_l^s(\Pi) \in \mathcal{W}_l^s(\Pi) \}.$$

To illustrate these concepts, we give some examples of trees and forests for $s = 3$ and $l = 2$. The set of partitions is in this case

$$\mathcal{P}_3 = \{ \{ \{1\}, \{2\}, \{3\} \}, \{ \{1\}, \{2, 3\} \}, \{ \{2\}, \{1, 3\} \}, \{ \{3\}, \{1, 2\} \}, \sigma_3 \}. \quad (83)$$

- For the trivial partition $\Pi_0 = \sigma_3$, the partition length l_Π is equal to one and therefore the elements of the set $\mathcal{W}_2^3(\Pi_0)$ are the trees $T_2^{3, 0} \in \mathcal{T}_2^{3, 0}$ such that $v_2 \leq 5$. For $v_2 = 3$, Figure 1 is an example of a surface tree in $\mathcal{W}_2^3(\Pi_0)$.

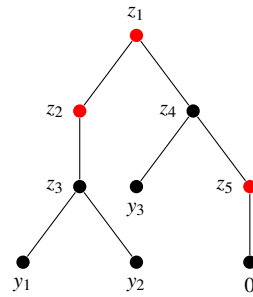


FIG. 1: Example of a forest $W_2^3(\Pi_0)$ with $v_2 = 3$ and $\vec{z} = (z_1, \dots, z_5)$. The red color is used for the internal vertices of incidence number 2.

- For the partition $\Pi_1 = \{ \{1\}, \{2\}, \{3\} \}$, an element of $\mathcal{W}_2^3(\Pi_1)$ (i.e. the set of forests of the partition Π_1) is given by the forest in Figure 2.

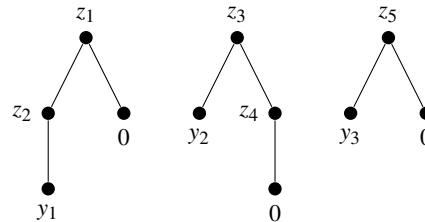


FIG. 2: Example of a forest $W_2^3(\Pi_1)$ with $v_2 = 5$ and $\vec{z} = (z_1, \dots, z_5)$.

This forest is composed of three trees. Each tree has two external vertices. The external

vertex y_i has an index which belongs to the sub-partition $\{i\}$. Note that the total number of vertices of incidence number 2 does not exceed 5 (in this case it is equal to 5). Note also that all the internal vertices of a surface tree with only two external vertices are of incidence number 2.

- For the partition $\Pi_2 = \{\{1, 2\}, \{3\}\}$, Figure 3 is an example of a forest in $\mathcal{W}_2^3(\Pi_2)$ with a total number of vertices of incidence number 2 equal to 4.

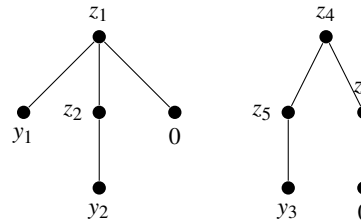


FIG. 3: Example of a forest $W_2^3(\Pi_2) \in \mathcal{W}_2^3(\Pi_2)$ with $v_2 = 4$.

Similar examples for the forest $W_2^3(\Pi_3)$ (respectively $W_2^3(\Pi_4)$) for $\Pi_3 = \{1, 3\}, \{2\}$ (resp. $\Pi_4 = \{\{2, 3\}, \{1\}\}$) can be constructed by replacing in Figure 2 the vertices $\{y_1, y_2\}$ by $\{y_1, y_3\}$ and the vertex y_3 by y_2 (resp. $\{y_1, y_2\}$ by $\{y_2, y_3\}$ and y_3 by y_1). An example of a forest in the global set of forests \mathcal{W}_2^3 is $\cup_{i=0}^4 W_2^3(\Pi_i)$.

B. Some operations on Forests and Trees

1. Reduction

Let $W_{l-1}^{s+2}(\Pi)$ be a forest in $\mathcal{W}_{l-1}^{s+2}(\Pi)$. In this part, we define and explain the process of reducing the forest $W_{l-1}^{s+2}(\Pi)$ to a forest in \mathcal{W}_l^s .

Definition 1. (Reduced partition) Let $s \geq 1$ and Π be in \mathcal{P}_{s+2} . We denote by π_i and π_j the sub-partitions of Π such that $s+1 \in \pi_i$ and $s+2 \in \pi_j$. The reduced partition $\Pi^{s+1, s+2}$ is defined as follows,

$$\Pi^{s+1, s+2} = \begin{cases} \{\pi_k\}_{1 \leq k \leq l_{\Pi}, k \notin \{i, j\}}, \pi_i^{s+1}, \pi_j^{s+2} & \text{if } i \neq j \\ \{\pi_k\}_{1 \leq k \leq l_{\Pi}, k \neq i}, \pi_i^{s+1, s+2} & \text{otherwise,} \end{cases}$$

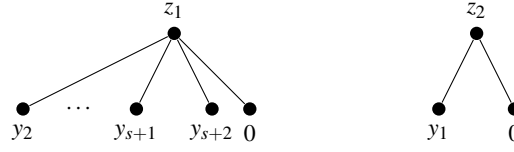
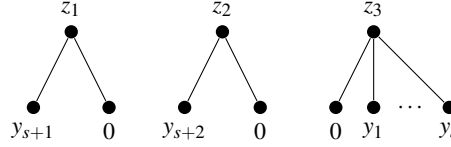
where we used the notations (76)-(77).

Proposition 2. (Reduction process) Let $s \geq 1$. For $\Pi \in \mathcal{P}_{s+2}$, we define $C_{y_{s+1}, y_{s+2}}$ to be the operator which acts on a forest $W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_{l-1}^{s+2}(\Pi)$ by removing the two external legs attached to y_{s+1} and y_{s+2} . If this operation produces an internal vertex of incidence number one, it is removed until an internal vertex of incidence number $c(z) \geq 2$ is reached. We have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2}). \quad (84)$$

Proof. The set \mathcal{P}_{s+2} can be separated into two subsets $\tilde{\mathcal{P}}_{s+2}$ and $\tilde{\mathcal{P}}_{s+2}^c$ defined as follows:

- $\tilde{\mathcal{P}}_{s+2}$ is defined as a subset of \mathcal{P}_{s+2} which contains all the partitions Π that admit a sub-partition π_i such that $\{s+1, s+2\} \in \pi_i$.
- $\tilde{\mathcal{P}}_{s+2}^c$ is the complementary set of $\tilde{\mathcal{P}}_{s+2}$.

FIG. 4: Example of a forest $W_{l-1}^{s+2}(\Pi)$ where $\Pi \in \tilde{\mathcal{P}}_{s+2}$ and $l_{\Pi} = 2$.FIG. 5: Example of a forest $W_{l-1}^{s+2}(\Pi^c)$ where $\Pi^c \in \tilde{\mathcal{P}}_{s+2}^c$ and $l_{\Pi} = 3$.

Diagrammatically, the global set of forests \mathcal{W}_{l-1}^{s+2} is partitioned into two subsets: the subset of forests for which y_{s+1} and y_{s+2} both belong to the same surface tree and the subset of forests in which y_{s+1} and y_{s+2} belong to different surface trees.

The proof of the statement (84) follows directly from establishing that

$$\forall \Pi \in \tilde{\mathcal{P}}_{s+2} : C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2}) \quad (85)$$

and

$$\forall \Pi \in \tilde{\mathcal{P}}_{s+2}^c : C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2}). \quad (86)$$

- First, we prove (85). Given a partition Π in $\tilde{\mathcal{P}}_{s+2}$, there exists a sub-partition $\pi_i \in \Pi$ such that $\{s+1, s+2\} \in \pi_i$. Therefore, we can write in slightly abusive notation

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i}^{l_{\pi}} T_{l-1}^{s_k, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}) \cup C_{y_{s+1}, y_{s+2}} T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}), \quad (87)$$

where the tree $T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i})$ can be identified with a forest in $\mathcal{W}_{l-1}^{s_{\pi_i}}(\sigma_{s_{\pi_i}})$. Deducing (85) amounts to prove for $s_{\pi_i} > 2$

$$C_{y_{s+1}, y_{s+2}} T_{l-1}^{s_{\pi_i}, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \in \mathcal{F}_l^{s_{\pi_i}-2, 0}. \quad (88)$$

For $s_{\pi_i} = 2$

$$C_{y_{s+1}, y_{s+2}} T_{l-1}^{2, 0}(y_{s+1}, y_{s+2}, 0, \vec{z}_{\pi_i}) = \emptyset, \quad (89)$$

and we have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i}^{l_{\pi}} T_{l-1}^{s_k, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}), \quad (90)$$

which is clearly in $W_l^s(\Pi^s)$. To treat the case $s_{\pi_i} > 2$, the discussion is simplified by considering the case of the trivial partition $\Pi = \sigma_{s+2}$ s.t. $s \geq 1$. In this case, the set of forest $\mathcal{W}_{l-1}^{s+2}(\sigma_{s+2})$ is given by all the surface trees $T_{l-1}^{s+2, 0}$ (see (82)). Let J_i (resp. J_j) be the external line which attaches the internal vertex z_i (resp. z_j) to the external vertex y_{s+1} (resp. y_{s+2}). The operator $C_{y_{s+1}, y_{s+2}}$ removes the external legs J_i and J_j from the forest $W_{l-1}^{s+2}(\sigma_{s+2})$, and if one of the internal vertices z_i and z_j becomes of incidence number one, it is removed

and the process continues until an internal vertex z of incidence number $c(z) \geq 2$ is reached. This implies that v'_2 (i.e. the number of vertices of incidence number 2 of the new forest $C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\sigma_{s+2})$) is at most $v_2 + 2$, with v_2 the number of vertices of incidence number 2 of $W_{l-1}^{s+2}(\sigma_{s+2})$. Therefore,

$$v'_2 \leq v_2 + 2 \leq 3(l-1) - 2 + \frac{s+3}{2} + 2 \leq 3l - 2 + \frac{s+1}{2}.$$

The last point to verify is that the reduction process converges for $s \geq 1$ in the sense that we have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\sigma_{s+2}) \neq \emptyset. \quad (91)$$

In order to obtain (91), we need to prove that there exists at least one internal vertex \tilde{z} such that $c(\tilde{z}) \geq 2$. If $W_{l-1}^{s+2}(\sigma_{s+2})$ has at least one internal vertex such that $c(z) \geq 4$, then eventually (91) holds. If all the internal vertices are of incidence number less than or equal to 3, then since $s \geq 1$, the tree $W_{l-1}^{s+2}(\sigma_{s+2})$ has a number of external vertices greater than or equal to 4 (taking into account the surface external vertex 0 as well). This implies that it has at least two internal vertices z and z' such that $c(z) = c(z') = 3$ which leads directly to (91). This proves (85).

- Now, we prove (86). Take $\Pi \in \tilde{\mathcal{F}}_{s+2}^c$, there exist $\pi_i, \pi_j \in \Pi$ such that $i \neq j$, $\{s+1\} \in \pi_i$ and $\{s+2\} \in \pi_j$. Therefore, we can write

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i, k \neq j}^{l_\Pi} T_{l-1}^{s\pi_k, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}) \cup C_{y_{s+1}} T_{l-1}^{s\pi_i, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \cup C_{y_{s+2}} T_{l-1}^{s\pi_j, 0}(Y_{\pi_j}, 0, \vec{z}_{\pi_j}), \quad (92)$$

where the operator $C_{y_{s+1}}$ acts on the tree $T_{l-1}^{s\pi_i, 0}$ by removing the external leg to which y_{s+1} is attached and by removing all the internal vertices which through this process become of incidence number one. Following the same steps of the discussion above, we deduce that $v'_{2,i}$ the number of vertices of incidence number 2 of the tree $C_{y_{s+1}} T_{l-1}^{s\pi_i, 0}$ is at most $v_{2,i} + 1$, with $v_{2,i}$ the number of vertices of incidence number 2 of $T_{l-1}^{s\pi_i+1, 0}$. We have

$$v'_{2,i} \leq v_{2,i} + 1 \leq 3(l-1) - 2 + \frac{s\pi_i + 1}{2} \leq 3l - 2 + \frac{s\pi_i}{2},$$

which implies that $C_{y_{s+1}} T_{l-1}^{s\pi_i, 0} \in \mathcal{F}_l^{s\pi_i-1, 0}$.

Here again, we need to verify that the reduction process of the forest W_{l-1}^{s+2} converges in the sense of (91). If $|\pi_i| = |\pi_j| = 1$, then we have

$$C_{y_{s+1}, y_{s+2}} W_{l-1}^{s+2}(\Pi) = \bigcup_{k=1, k \neq i, k \neq j}^{l_\Pi} T_{l-1}^{s\pi_k, 0}(Y_{\pi_k}, 0, \vec{z}_{\pi_k}). \quad (93)$$

If $|\pi_i| \geq 2$, we have

$$C_{y_{s+1}} T_{l-1}^{s\pi_i, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i}) \neq \emptyset.$$

This holds since the tree $T_{l-1}^{s\pi_i, 0}(Y_{\pi_i}, 0, \vec{z}_{\pi_i})$ has at least three external vertices which implies that there exists at least one internal vertex such that $c(z) \geq 3$, and removing at most one external leg at each step of the reduction process implies that the incidence number of z is strictly greater than 1 at the end of the process.

□

2. Fusion

In this part, we define and explain the merging process of a bulk tree with a forest.

Proposition 3. For $s \geq 2$ and $l \geq 0$, we consider the partition $(\tilde{\pi}_1, \tilde{\pi}_2)$ in $\tilde{\mathcal{P}}_{2,s}$ such that $|\tilde{\pi}_i| = s_i$ and $s_1 + s_2 = s$. Given a partition Π of the set $\tilde{\pi}_2 \cup \{s_2 + 1\}$, we define the a -merging operator $M_{y_{s_1+1}, y_{s_2+1}}^a$ acting on the forest $W_{l_2}^{s_2+1}(\Pi)$ and the bulk tree $\hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s_1+1}; \vec{z})$ at the external vertices y_{s_1+1} and y_{s_2+1} following the steps below:

- (a) Let $J_{s_1+1} = (z, y_{s_1+1})$ and $J_{s_2+1} = (z', y_{s_2+1})$ be the external legs which attach respectively y_{s_1+1} to the internal vertex $z \in \hat{T}_{l_1}^{s_1+1}$ and y_{s_2+1} to the internal vertex $z' \in W_{l_2}^{s_2+1}(\Pi)$. In the first step of the merging process J_{s_1+1} and J_{s_2+1} are removed.
- (b) A new internal line (z, z') is added.

Similarly, we define the b -merging operator $M_{y_{s_1+1}, y_{s_2+1}}^b$ acting on $W_{l_2}^{s_2+1}(\Pi)$ and $\hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s_1+1}, \vec{z})$ following the same steps above except for adding an internal vertex of incidence number 2, which replaces the internal line (z, z') in step (b) by the two internal lines (z, u) and (u, z') . Then we claim

$$M_{y_{s_1+1}, y_{s_2+1}}^i \left(\hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s_1+1}; \vec{z}), W_{l_2}^{s_2+1}(\Pi) \right) \in \mathcal{W}_i^s(\Pi'), \quad i \in \{a, b\} \quad (94)$$

where $\Pi' := \tilde{\pi}_1 \cup \Pi^{s_2+1}$ (we used the notation (76)) and $l := l_1 + l_2$.

Proof. Let π_i be the sub-partition of Π such that $s_2 + 1 \in \pi_i$. The merging operators (a) and (b) act only on the tree $T_{l_2}^{s_{\pi_i}, 0}$ since all the surface trees corresponding to the remaining sub-partitions do not have external vertices on which the merging operators act. Therefore, without loss of generality, we simplify the discussion by considering the case of a partition Π of length one.

The first and second step of the two merging processes create a tree with $s + 1$ external vertices given by the set

$$\{Y_{\tilde{\pi}_1}\} \cup \{Y_{\tilde{\pi}_2}, 0\}.$$

The only difference between the two cases is related to the set of internal vertices, which in case of (a) is given by the union of the internal vertices of the bulk tree $\hat{T}_{l_1}^{s_1+1}(Y_{\tilde{\pi}_1}, y_{s_1+1}, \vec{z})$ and the surface tree $T_{l_2}^{s_{\pi_i}, 0}$. For (b), a new vertex of incidence number 2 is added, which implies

$$v_{2,a} = v_{2,1} + v_{2,2}, \quad v_{2,b} = v_{2,1} + v_{2,2} + 1, \quad (95)$$

where $v_{2,i}$ denotes the number of vertices of incidence number 2 of the surface tree obtained through the merging process (i). Therefore, we obtain

$$v_{2,i} \leq 3(l_1 + l_2) - 4 + \frac{s_1 + s_2 + 3}{2} + 1 = 3l - 2 + \frac{s+1}{2}, \quad i \in \{a, b\}. \quad (96)$$

This concludes that the surface trees obtained through the merging processes (a) and (b) are indeed in $\mathcal{W}_i^s(\sigma_s)$. \square

C. Weight factors

1. The bulk weight factors

Let $0 < \delta < 1$. Given a set $\tau_{2,s} := \{\tau_2, \dots, \tau_s\}$ with $\tau := \inf_{2 \leq i \leq s} \tau_i$, a set of external vertices $y_{2,s} = \{y_2, \dots, y_s\} \in \mathbb{R}^{s-1}$ and a set of internal vertices $\vec{z} = (z_2, \dots, z_{r+1}) \in (\mathbb{R}^+)^r$, and attributing positive parameters $\Lambda_{\mathcal{J}} = \{\Lambda_I | I \in \mathcal{J}\}$ to the internal lines, the weight factor $\mathcal{F}_{\delta}(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_l^s(z_1, y_{2,s}, \vec{z}))$ of

a tree $T_I^s(z_1, y_{2,s}, \vec{z})$ at scales Λ_I is defined as a product of heat kernels associated with the internal and external lines of the tree. We set

$$\mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_I^s(z_1, y_{2,s}, \vec{z})) := \prod_{I \in \mathcal{J}} p_B\left(\frac{1+\delta}{\Lambda_I^2}; I\right) \prod_{J \in \mathcal{J}} p_B(\tau_{J,\delta}; J), \quad (97)$$

where $\tau_{J,\delta}$ denotes the entry $\tau_{i,\delta}$ in τ carrying the index of the external coordinate y_i in which the external line J ends, and $\tau_{i,\delta} := (1+\delta)\tau_i$. For $I = \{a, b\}$ the notation $p_B(\frac{1+\delta}{\Lambda_I^2}; I)$ stands for $p_B(\frac{1+\delta}{\Lambda_I^2}; a, b)$. We also define the integrated weight factor

$$\mathcal{F}_\delta(\Lambda, \tau_{2,s}; T_I^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_I \leq \Lambda_0} \int_{\vec{z}} \mathcal{F}_\delta(\Lambda_{\mathcal{J}}, \tau_{2,s}; T_I^s(z_1, y_{2,s}, \vec{z})). \quad (98)$$

It depends on Λ_0 , but note that its limit for $\Lambda_0 \rightarrow \infty$ exists, and that typically the sup is expected to be taken for the minimal values of Λ admitted. Therefore we suppress the dependence on Λ_0 in the notation. The definitions (97)-(98) can be generalized to a bulk tree \hat{T}_I^s . Finally we introduce the global weight factor $\mathcal{F}(\Lambda, \tau_{2,s}, z_1, y_{2,s})$, which is defined through

$$\mathcal{F}_{s,l;\delta}(\Lambda, \tau_{2,s}, z_1, y_{2,s}) := \sum_{T_I^s \in \mathcal{T}_I^s} \mathcal{F}_\delta(\Lambda, \tau_{2,s}; T_I^s; z_1, y_{2,s}). \quad (99)$$

Similarly, we define the global bulk weight factor

$$\hat{\mathcal{F}}_{s,l;\delta}(\Lambda, \tau_{2,s}, y_{2,s}) := \sum_{\hat{T}_I^s \in \hat{\mathcal{T}}_I^s} \mathcal{F}_\delta(\Lambda, \tau_{2,s}; \hat{T}_I^s; y_{2,s}). \quad (100)$$

If this does not lead to ambiguity we write shortly

$$\mathcal{F}_{s,l;\delta}^\Lambda(\tau_{2,s}) \equiv \mathcal{F}_{s,l;\delta}(\Lambda, \tau_{2,s}, z_1, y_{2,s}). \quad (101)$$

For $s = 1$ we set $\mathcal{F}_{1,l;\delta}^\Lambda \equiv 1$.

2. The surface weight factors

- In the sequel, we will use the following notations:

$$\tau_{\pi_i} := \{\tau_k | k \in \pi_i\}, \quad Y_{\pi_i} := \{y_k | k \in \pi_i\}, \quad \tau_{\pi_i,\delta} := \{(1+\delta)\tau_k | k \in \pi_i\}, \quad \tau := \inf_{1 \leq i \leq s} \tau_i. \quad (102)$$

- Let $0 < \delta < 1$ and $\tau_{1,s} := \{\tau_1, \dots, \tau_s\}$ such that $\tau > 0$ and let $Y_{\sigma_s} \in \mathbb{R}^s$ be the set of the external vertices. Given a partition $\Pi \in \mathcal{P}_s$, let $\vec{z}_\Pi = (\vec{z}_{\pi_1}, \dots, \vec{z}_{\pi_\Pi}) \in (\mathbb{R}^+)^p$, where each vector \vec{z}_{π_i} consists of the internal vertices of the tree $T_I^{s\pi_i,0}$ in the forest $W_I^s(\Pi)$ and p is the number of internal vertices of $W_I^s(\Pi)$. We denote by $\mathcal{J} = \cup_{i=1}^{l_\Pi} \mathcal{J}_k$ the set of the internal lines of the trees of $W_I^s(\Pi)$ and by $\mathcal{J} = \cup_{k=1}^{l_\Pi} \mathcal{J}_k$ the set of the external lines which link an internal vertex to an external vertex belonging to the set Y_{σ_s} . Each set \mathcal{J}_k (resp. \mathcal{J}_k) denotes the internal lines (resp. the external lines) of the tree $T_I^{s\pi_k,0}$. We also use the notation $\mathcal{J}_k^0 = \{\mathcal{J}_k^0 | 1 \leq k \leq l_\Pi\}$ to denote the set of surface external lines which link an internal vertex to 0.
- Attributing positive parameters $\Lambda_{\mathcal{J}} = \{\Lambda_I | I \in \mathcal{J}\}$ to the internal lines and $\tilde{\Lambda} = \{\tilde{\Lambda}_k | k \in \mathcal{J}_k^0\}$ to the surface external lines, the weight factor $\mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{1,s}; W_I^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s})$ of the forest $W_I^s(\Pi)$ at scales Λ_I and $\tilde{\Lambda}_k$ is defined as the product of heat kernels associated to the internal

and external lines of each tree of the forest. For a sub-partition $\pi_k \in \Pi$, we define the weight factor of the tree $T_l^{s\pi_k,0}$ as follows:

$$\begin{aligned} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}_k}, \tilde{\Lambda}_k; \tau_{\pi_k}; T_l^{s\pi_k,0}; \vec{z}_{\pi_k}; Y_{\pi_k} \right) \\ := \prod_{I \in \mathcal{J}_k} p_B \left(\frac{1+\delta}{\Lambda_I^2}; I \right) \prod_{J \in \mathcal{J}_k} p_B((1+\delta)\tau_J; J) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_k^2}; J_0^k \right), \end{aligned} \quad (103)$$

where we used the same notations as in (97) and J_0^k denotes the line which links an internal vertex to the external vertex 0 with an attributed positive parameter $\tilde{\Lambda}_k$. For a surface tree $T_l^{1,0}$ with no internal vertices, the surface weight factor reads

$$\mathcal{F}_\delta^0 \left(\tau_{\pi_k}; T_l^{1,0}; y_{\pi_k} \right) := p_B((1+\delta)\tau_{\pi_k}; y_{\pi_k}, 0). \quad (104)$$

The weight factor of the forest $W_l^s(\Pi)$ is defined for $\Pi \in \mathcal{P}_s^{1,c}$ as follows:

$$\mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s} \right) := \prod_{\pi_k \in \Pi} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}_k}, \tilde{\Lambda}_k; \tau_{\pi_k}; T_l^{s\pi_k,0}; \vec{z}_{\pi_k}; Y_{\pi_k} \right). \quad (105)$$

For $\Pi \in \mathcal{P}_s^1$, it is given by

$$\begin{aligned} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s} \right) := \prod_{\pi_k} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}_k}, \tilde{\Lambda}_k; \tau_{\pi_k}; T_l^{s\pi_k,0}; \vec{z}_{\pi_k}; Y_{\pi_k} \right) \\ \times \prod_{\tilde{\pi}_k} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}_k}, \tilde{\Lambda}_k; 2\tau_{\tilde{\pi}_k}; T_l^{s\tilde{\pi}_k,0}; \vec{z}_{\tilde{\pi}_k}; Y_{\tilde{\pi}_k} \right), \end{aligned} \quad (106)$$

where the product $\prod_{\tilde{\pi}_k}$ runs over all sub-partitions in Π of length equal to 1.

- We also define the integrated surface weight factor

$$\mathcal{F}_\delta^0 \left(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s} \right) := \sup_{\Lambda \leq \Lambda_I, \tilde{\Lambda}_k \leq \Lambda_0} \int_{\vec{z}_\Pi} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{1,s}; W_l^s(\Pi); \vec{z}_\Pi; Y_{\sigma_s} \right), \quad (107)$$

where $\int_{\vec{z}_\Pi} := \prod_{i=1}^p \int_0^\infty dz_i$. The weight factor associated to a global forest W_l^s is defined as

$$\mathcal{F}_\delta^0 \left(\Lambda, \tau_{1,s}; W_l^s; Y_{\sigma_s} \right) := \sum_{\Pi \in \mathcal{P}_s} \mathcal{F}_\delta^0 \left(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s} \right). \quad (108)$$

- We define the global surface weight factor as follows,

$$\mathcal{F}_{s,l;\delta}^0 \left(\Lambda, \tau_{1,s}; Y_{\sigma_s} \right) := \sum_{W_l^s \in \mathcal{W}_l^s} \mathcal{F}_\delta^0 \left(\Lambda, \tau_{1,s}; W_l^s; Y_{\sigma_s} \right). \quad (109)$$

If it does not lead to ambiguity we write shortly

$$\mathcal{F}_{s,l;\delta}^{\Lambda,0} \left(\tau_{1,s} \right) \equiv \mathcal{F}_{s,l;\delta}^0 \left(\Lambda, \tau_{1,s}; Y_{\sigma_s} \right). \quad (110)$$

For $s = 0$ we set $\mathcal{F}_{0,l;\delta}^{\Lambda,0} \equiv 1$.

Remarks 2. • The definitions (98) and (107) imply for $0 \leq \Lambda' \leq \Lambda$

$$\mathcal{F}_{s,l;\delta}^{\Lambda,0} \left(\tau_{1,s} \right) \leq \mathcal{F}_{s,l;\delta}^{\Lambda',0} \left(\tau_{1,s} \right), \quad \mathcal{F}_{s,l;\delta}^\Lambda \left(\tau_{1,s} \right) \leq \mathcal{F}_{s,l;\delta}^{\Lambda'} \left(\tau_{1,s} \right). \quad (111)$$

- Combining the bound (15) together with the definitions (97) and (103), the following bounds hold for all $0 < \delta < \delta'$ and $0 \leq \Lambda \leq \Lambda_0$

$$\mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) \leq C(\delta, \delta') \mathcal{F}_{s,l;\delta'}^{\Lambda,0}(\tau_{1,s}), \quad \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) \leq C'(\delta, \delta') \mathcal{F}_{s,l;\delta'}^{\Lambda}(\tau_{1,s}). \quad (112)$$

The constants C and C' are explicitly given by

$$C(\delta, \delta') := \sup_{(\mathcal{I}, \mathcal{J}) \in T_l^{s,0}, T_l^{s,0} \in \mathcal{T}_l^{s,0}} C_{\delta, \delta'}^{|\mathcal{I}|+|\mathcal{J}|}, \quad C'(\delta, \delta') := \sup_{(\mathcal{I}, \mathcal{J}) \in T_l^s, T_l^s \in \mathcal{T}_l^s} C_{\delta, \delta'}^{|\mathcal{I}|+|\mathcal{J}|},$$

where \mathcal{I} and \mathcal{J} are respectively the set of internal and external lines of the tree T_l^s (resp. surface tree $T_l^{s,0}$) and $|\cdot|$ denotes their cardinality. The constant $C_{\delta, \delta'}$ is given by (16) for $r = 0$.

D. Inequalities

For the proof in Sec VI, we need to bound the tree/forest weight factors for reduced forests and for merged trees and forests.

Lemma 1. (*Reduction*) Let $\tau, \delta > 0, 0 \leq \Lambda \leq \Lambda_0$ and $Y_{\sigma_s} \in \mathbb{R}^s$, we have

$$\int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta}^0 \left(\Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right) \leq O(1) \Lambda \mathcal{F}_{s,l;\delta}^0(\Lambda; \tau_{1,s}; Y_{\sigma_s}), \quad (113)$$

where the constant $O(1)$ depends only on s and l .

Proof. Let us recall the definition of the surface weight factor, which in this case is given by

$$\begin{aligned} \mathcal{F}_{s+2,l-1;\delta}^0 \left(\Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s+2}} \right) \\ = \sum_{W_{l-1}^{s+2} \in \mathcal{W}_{l-1}^{s+2}} \sum_{\Pi \in \mathcal{P}_{s+2}} \mathcal{F}_{\delta}^0 \left(\Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; W_{l-1}^{s+2}(\Pi); Y_{\sigma_{s+2}} \right), \end{aligned} \quad (114)$$

where $y_{s+1} := u$ and $y_{s+2} = u$. The weight factor

$$\mathcal{F}_{\delta}^0 \left(\Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; W_{l-1}^{s+2}(\Pi); Y_{\sigma_{s+2}} \right)$$

is given by (103)-(106). Let (z_i, u) and (z_j, u) be the external lines which attach respectively the internal vertices z_i and z_j to the external vertices y_{s+1} and y_{s+2} . Using (10), we obtain

$$\int_{\mathbb{R}} du p_B \left(\frac{\alpha_1(1+\delta)}{2\Lambda^2}; z_i, u \right) p_B \left(\frac{\alpha_2(1+\delta)}{2\Lambda^2}; z_j, u \right) = p_B \left(\frac{(\alpha_1 + \alpha_2)(1+\delta)}{2\Lambda^2}; z_i, z_j \right) \leq \Lambda. \quad (115)$$

We recall that a tree of two external vertices (including the surface external vertex 0) corresponds to a sub-partition of length 1 and the surface weight factor associated to these trees differs from a surface tree of three or more external vertices by a factor 2 multiplying the parameter τ_i of the corresponding external vertex, as it appears in (106). Therefore, the constants α_1 and α_2 take either the value 2 or 1 depending on whether the two external vertices at u belong to a surface tree of only two external vertices or more. The bound (115) removes the external legs (z_i, u) and (z_j, u) from the forest $W_{l-1}^{s+2}(\Pi)$ by bounding their contribution in the surface weight factor by Λ . Furthermore, the property

$$\int_0^{\infty} dz p_B \left(\frac{1+\delta}{\Lambda_i^2}; z, z' \right) \leq 1 \quad (116)$$

implies that all internal vertices which after removing (z_i, u) and (z_j, u) , their incidence number is equal to one are removed. These two steps correspond to reducing the forest $W_{l-1}^{s+2}(\Pi)$ at the external vertices (u, u) . Therefore, we have

$$\int_{\mathbb{R}} du \mathcal{F}_{\delta}^0 \left(\Lambda; \tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; W_{l-1}^{s+2}(\Pi); Y_{\sigma_s}, u, u \right) \leq \Lambda \mathcal{F}_{\delta}^0 \left(\Lambda; \tau_{1,s}; C_{u,u} W_{l-1}^{s+2}(\Pi); Y_{\sigma_s} \right). \quad (117)$$

Proposition 1 gives that $C_{u,u} W_{l-1}^{s+2}(\Pi) \in \mathcal{W}_l^s(\Pi^{s+1, s+2})$, where $\Pi^{s+1, s+2} \in \mathcal{P}_s$ is the reduced partition obtained from Π . Hence, we obtain

$$\begin{aligned} \sum_{W_{l-1}^{s+2} \in \mathcal{W}_{l-1}^{s+2}} \sum_{\Pi \in \mathcal{P}_{s+2}} \mathcal{F}_{\delta}^0 \left(\Lambda, \tau_{1,s}; C_{u,u} W_{l-1}^{s+2}(\Pi); Y_{\sigma_s} \right) \\ \leq O(1) \sum_{W_l^s \in \mathcal{W}_l^s} \sum_{\Pi \in \mathcal{P}_s} \mathcal{F}_{\delta}^0 \left(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s} \right). \end{aligned} \quad (118)$$

The constant $O(1)$ takes into account that the reduction operator is not a one-to-one map, in the sense that the same forest can be obtained from the reduction of different forests, which implies that some weight factors $\mathcal{F}_{\delta}^0 \left(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s} \right)$ are possibly summed more than once in (117). Combining (117) and (118) gives the final bound (113).

If the external vertex y_{s+1} belong to the sub-surface tree $T_l^{1,0}(y_{s+1}, 0)$ with no internal vertex, (115) reads in this case

$$\int_{\mathbb{R}} du p_B \left(\frac{(1+\delta)}{\Lambda^2}; 0, u \right) p_B \left(\frac{(1+\delta)}{\Lambda^2}; z_j, u \right) = p_B \left(\frac{2(1+\delta)}{\Lambda^2}; 0, z_j \right) \leq \Lambda. \quad (119)$$

If y_{s+2} is also an external vertex of $T_l^{1,0}(y_{s+2}, 0)$, then z_j is replaced by 0 in (119). This corresponds to removing the sub-surface trees from the forest $W_{l-1}^{s+2}(\Pi)$. The bound (118) is obtained in these two cases following the same line of reasoning as before. \square

Lemma 2. (Forest-Forest Fusion) Let $\delta, \delta' > 0$ and $1 \leq l_1, l_2 \leq l-1$ such that $l_1 + l_2 = l$. Given $(\tilde{\pi}_1, \tilde{\pi}_2) \in \tilde{\mathcal{P}}_{2,s}$, we have

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^0 \left(\Lambda; \tau_{\tilde{\pi}_1}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_1}, u \right) \mathcal{F}_{s_2+1, l_2; \delta'}^0 \left(\Lambda; \tau_{\tilde{\pi}_2}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_2}, u \right) \\ \leq \Lambda \mathcal{F}_{s, l; \delta''}^0 \left(\Lambda; \tau_{1,s}; Y_{\sigma_s} \right), \end{aligned} \quad (120)$$

where $s_i := |\pi_i|$ and $\delta'' = \max(\delta, \delta')$.

Proof. Without loss of generality, we consider the ordered sub-partitions $\tilde{\pi}_1 := \sigma_{s_1}$ and $\tilde{\pi}_2 := \sigma_{s_1+1, s}$. To establish (120), it is sufficient to bound

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^0 \left(\Lambda; \tau_{1, s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \\ \times \mathcal{F}_{s_2+1, l_2; \delta'}^0 \left(\Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; W_{l_2}^{s_2+1}(\Pi_2); Y_{\sigma_{s_1+1, s}}, u \right) \end{aligned} \quad (121)$$

where $\Pi_1 \in \tilde{\mathcal{P}}_{s_1+1}$ and $\Pi_2 \in \tilde{\mathcal{P}}_{s_2+1}$. The sets $\tilde{\mathcal{P}}_{s_1+1}$ and $\tilde{\mathcal{P}}_{s_2+1}$ denote respectively the set of partitions of $\sigma_{s_1} \cup \{s+1\}$ and $\sigma_{s_1+1, s} \cup \{s+2\}$. Using (112), we can bound (121) by

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta''}^0 \left(\Lambda; \tau_{1, s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \\ \times \mathcal{F}_{s_2+1, l_2; \delta''}^0 \left(\Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; W_{l_2}^{s_2+1}(\Pi_2); Y_{\sigma_{s_1+1, s}}, u \right), \end{aligned} \quad (122)$$

where $\delta'' = \max(\delta, \delta')$.

Let π_i and π_j be respectively the sub-partitions in Π_1 and Π_2 such that $\{s+1\} \in \pi_i$ and $\{s+2\} \in \pi_j$. We denote by z_i and z_j the internal vertices in the sub-surface trees $T_{l_1}^{s\pi_i,0}(Y_{\pi_i}, u, 0)$ and $T_{l_2}^{s\pi_j,0}(Y_{\pi_j}, u, 0)$ in the forests $W_{l_1}^{s_1+1}(\Pi_1)$ and $W_{l_2}^{s_2+1}(\Pi_2)$, which are attached to u . As we mentioned previously, the bound (115) amputates the external legs (z_i, u) and (z_j, u) and bounds their contribution in (122) by Λ . Furthermore, (116) implies that all internal vertices of incidence number 1 are removed. The amputation can possibly create in each tree at most one internal vertex of incidence number 2. Denoting by $T_{l_1}^{s\pi_i-1,0}(Y_{\pi_i}, 0)$ the surface tree obtained by amputating the external leg (z, u) from $T_{l_1}^{s\pi_i,0}(Y_{\pi_i}, u, 0)$, we deduce

$$\tilde{v}_{2,i}^{s_1} \leq v_{2,i}^{s_1} + 1 \leq 3l_1 - 1 + \frac{s\pi_i}{2} + \frac{1}{2},$$

where $\tilde{v}_{2,i}^{s_1}$ and $v_{2,i}^{s_1}$ denote respectively the number of vertices of incidence number 2 of the surface trees $T_{l_1}^{s\pi_i-1,0}(Y_{\pi_i}, 0)$ and $T_{l_1}^{s\pi_i,0}(Y_{\pi_i}, u, 0)$. Since $1 \leq l_1 \leq l-1$, we obtain

$$\tilde{v}_{2,i}^{s_1} \leq 3l - 3 - 1 + \frac{s\pi_i}{2} + \frac{1}{2} \leq 3l - 2 + \frac{s\pi_i}{2}. \quad (123)$$

Proceeding similarly with $T_{l_2}^{s\pi_j,0}(Y_{\pi_j}, u, 0)$, we deduce that the number of vertices of the amputated tree obeys

$$\tilde{v}_{2,j}^{s_2} \leq 3l - 2 + \frac{s\pi_j}{2}. \quad (124)$$

From (123) and (124), we deduce that $T_{l_i}^{s\pi_i-1,0}(Y_{\pi_i}, 0) \in \mathcal{F}_l^{s\pi_i-1,0}$. Therefore, we obtain that

$$\int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta''}^0 \left(\Lambda; \tau_{\pi_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\pi_1}, u \right) \times \mathcal{F}_{s_2+1, l_2; \delta''}^0 \left(\Lambda; \tau_{\pi_2}, \frac{1}{2\Lambda^2}; W_{l_2}^{s_2+1}(\Pi_2); Y_{\pi_2}, u \right) \quad (125)$$

is bounded by

$$\Lambda \mathcal{F}_{\delta''}^0 \left(\Lambda; \tau_{1, s_1}; W_{l_1}^{s_1}(\Pi_1^{s_1+1}); Y_{\sigma_{s_1}} \right) \mathcal{F}_{\delta''}^0 \left(\Lambda; \tau_{s_1+1, s}; W_{l_2}^{s_2}(\Pi_2^{s_2+1}); Y_{\sigma_{s_1+1, s}} \right), \quad (126)$$

where we used the notation (76). Note that $\Pi_1^{s_1+1} \cup \Pi_2^{s_2+1} \in \mathcal{P}_s$, which together with (126) gives the integrated surface weight factor of the forest $W_{l_1}^{s_1}(\Pi_1^{s_1+1}) \cup W_{l_2}^{s_2}(\Pi_2^{s_2+1})$. Hence (126) is bounded by

$$\Lambda \mathcal{F}_{\delta''}^0 \left(\Lambda; \tau_{1, s}; W_l^s(\Pi); Y_{\sigma_s} \right),$$

where $\Pi = \Pi_1^{s_1+1} \cup \Pi_2^{s_2+1}$ belongs to \mathcal{P}_s and we deduce

$$\int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^{\Lambda; 0} \left(\tau_{1, s_1}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{s_2+1, l_2; \delta'}^{\Lambda; 0} \left(\tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1+1, s}}, u \right) \leq \Lambda \mathcal{F}_{s, l; \delta''}^{\Lambda; 0}(\tau_{1, s}). \quad (127)$$

□

Lemma 3. (*Bulk tree-Forest Fusion*) Let $\delta, \delta' > 0$ and $l_1, l_2 \geq 0$ such that $l_1 + l_2 = l$. Given $(\tilde{\pi}_1, \tilde{\pi}_2) \in \tilde{\mathcal{P}}_{2, s}$, we have

$$\int_{\mathbb{R}} du \mathcal{F}_{s_1+1, l_1; \delta}^0 \left(\Lambda; \tau_{\tilde{\pi}_1}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_1}, u \right) \mathcal{F}_{s_2+1, l_2; \delta'}^0 \left(\Lambda; \tau_{\tilde{\pi}_2}, \frac{1}{2\Lambda^2}; Y_{\tilde{\pi}_2}, u \right) \leq O(1) \mathcal{F}_{s, l; \delta''}^0 \left(\Lambda; \tau_{1, s}; Y_{\sigma_s} \right) \quad (128)$$

where $\delta'' = \max(\delta, \delta')$.

Proof. Without loss of generality, we again consider the ordered sub-partitions σ_{s_1} and $\sigma_{s_1+1:s}$. In order to obtain the bound (128), it is sufficient to bound for a given $\Pi_1 \in \tilde{\mathcal{F}}_{s_1}$

$$\int_{\mathbb{R}} du \mathcal{F}_{\delta}^0 \left(\Lambda; \tau_{1,s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{\delta'} \left(\Lambda; \tau_{s_1+1,s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u \right). \quad (129)$$

Using the bound (112), we bound (129) by

$$\int_{\mathbb{R}} du \mathcal{F}_{\delta''}^0 \left(\Lambda; \tau_{1,s_1}, \frac{1}{2\Lambda^2}; W_{l_1}^{s_1+1}(\Pi_1); Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{\delta''} \left(\Lambda; \tau_{s_1+1,s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u \right), \quad (130)$$

where $\delta'' := \max(\delta, \delta')$. Let z_i and z_j be respectively the internal vertices attached to u in $W_{l_1}^{s_1+1}(\Pi_1)$ and to u in $\hat{T}_{l_2}^{s_2+1}$. Interchanging the integral over u with the integral over the internal vertices of the forest $W_{l_1}^{s_1+1}(\Pi_1)$ and the bulk tree $\hat{T}_{l_2}^{s_2+1}$ in their respective weight factors and using (10) we deduce

$$\int_{\mathbb{R}} du p_B \left(\frac{\alpha(1+\delta)}{2\Lambda^2}; z_i, u \right) p_B \left(\frac{1+\delta}{2\Lambda^2}; z_j, u \right) = p_B \left(\frac{(\alpha+1)(1+\delta)}{2\Lambda^2}; z_i, z_j \right) \quad (131)$$

with $\alpha \in \{1, 2\}$. Here, we proceed similarly to (115) to differentiate the surface trees with two external vertices from other surface trees with more than two external vertices. For $\alpha = 2$, we keep the integration over u and write

$$\begin{aligned} \int_{\mathbb{R}} du p_B \left(\frac{1+\delta}{\Lambda^2}; z_i, u \right) p_B \left(\frac{1+\delta}{2\Lambda^2}; z_j, u \right) \\ \leq 2 \int_0^{\infty} du p_B \left(\frac{1+\delta}{\Lambda^2}; z_i, u \right) p_B \left(\frac{1+\delta}{2\Lambda^2}; z_j, u \right). \end{aligned} \quad (132)$$

Therefore, (131) and (132) correspond to the fact that the two external legs attached to (z_i, u) and (z_j, u) are removed. If $\alpha = 1$, the external lines are replaced by the internal line (z_i, z_j) and for $\alpha = 2$ the vertex u becomes internal with incidence number 2. The first case corresponds to the steps of merging the forest $W_{l_1}^{s_1+1}(\Pi_1)$ and the bulk tree $\hat{T}_{l_2}^{s_2+1}$ at the external points (u, u) through the process a). In the second case, the forest and the tree are merged following the merging process b). From Proposition 3 we have

$$M_{u,u}^i \left(W_{l_1}^{s_1+1}(\Pi_1), \hat{T}_{l_2}^{s_2+1} \right) \in \mathcal{W}_i^s(\Pi'), \quad i \in \{a, b\}$$

where $\Pi' = \Pi_1^{s_1+1} \cup \sigma_{s_1+1:s}$. This implies that (130) is bounded by

$$2 \mathcal{F}_{\delta''}^0 \left(\Lambda; \tau_{1,s}; W_i^s(\Pi'); Y_{\sigma_s} \right).$$

Therefore we deduce

$$\begin{aligned} \int_{\mathbb{R}} du \mathcal{F}_{s_1+1,l_1;\delta}^0 \left(\Lambda; \tau_{\pi_1}, \frac{1}{2\Lambda^2}; Y_{\pi_1}, u \right) \hat{\mathcal{F}}_{s_2+1,l_2;\delta'} \left(\Lambda; \tau_{\pi_2}, \frac{1}{2\Lambda^2}; Y_{\pi_2}, u \right) \\ \leq O(1) \mathcal{F}_{s,l;\delta''}^0 \left(\Lambda; \tau_{1,s}; Y_{\sigma_s} \right), \end{aligned}$$

where $O(1)$ is a constant which depends on s and l that takes into account that the same forest can be obtained through the fusion of different forests and bulk trees. \square

VI. RESULTS AND PROOFS

Before stating the results of this paper, we recall some notations that are used frequently in the sequel:

$$\tau_{1,s} := (\tau_1, \dots, \tau_s), \quad y_{1,s} := (y_1, \dots, y_s). \quad (133)$$

A. Results

The central result of this paper is summarized below in Theorem 1, which provides a uniform bound in the UV cutoff Λ_0 of the surface correlation "functions" in the case of Robin and Neumann boundary conditions. As we mentioned before in section IV-B, the surface correlation "functions" $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ are tempered distributions when considered in the mixed position-momentum space. Hence, in order to bound the n -point surface correlation distributions, one should first **smear** them with an appropriate set of test functions. Since the renormalization proof by the method of flow equations is inductive, the choice of test functions is restricted by the flow equations. For simplicity, we choose the bulk heat kernels

$$\phi_{\tau_{1,s},y_{1,s}}(\vec{z}_n) = \prod_{i=1}^s p_B(\tau_i; z_i, y_i),$$

but one should keep in mind that this choice is not unique.

The bound (135) consists of familiar factors which are also present in the inductive bounds that establishes the perturbative renormalization of the scalar field theory in momentum space²⁰ in \mathbb{R}^4 , in particular the power counting factor in terms of $\Lambda + m$ as well as the polynomials \mathcal{P}_1 and \mathcal{P}_2 .

There is also the additional factor (see (103)-(107))

$$\mathcal{F}_{s,l;\delta}^0(\Lambda, \tau_{1,s}; Y_{\sigma_s}) := \sum_{W_l^s \in \mathcal{W}_l^s} \sum_{\Pi \in \mathcal{P}_s} \mathcal{F}_{\delta}^0(\Lambda, \tau_{1,s}; W_l^s(\Pi); Y_{\sigma_s}), \quad (134)$$

denoted by the surface weight factor. On one hand, this factor appears because we work in position space in the z -direction, but on the other hand it comes from the contribution of the factor $\hat{C}_{S,\star}$ in the flow equation (62) through the bounds (73)-(75). Note that the behaviour w.r.t. the flow parameter Λ in the surface weight factor (134) is traced by controlling the number of vertices of incidence number 2 of the forests through the following bound

$$v_2 \leq 3l - 2 + \frac{s+1}{2}.$$

This bound is compatible with the inductive scheme but is not optimal in the sense that sharper upper bounds rendering the proof even more complicated could have been achieved.

The definition of the surface correlation "distributions" introduced in section IV-B implies that $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ contain all the contributions from the surface part $C_{S,\star}^{\Lambda,\Lambda_0}$ of the propagator $C_{\star}^{\Lambda,\Lambda_0}$. Hence, the right-hand side of the bound (135) contains elements that account for the effect of the presence of the surface on the renormalization of $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ which are:

- A dimensionally better power counting (i.e. $(\Lambda + m)^{3-n}$) as compared to the bound on the bulk correlation distributions given in proposition 1.
- The surface weight factor.

Theorem 1. (Boundedness) *Let $0 \leq \Lambda \leq \Lambda_0 < \infty$ and $(r_i, s) \in \mathbb{N}^2$ such that $0 \leq r_i \leq 4$ and $0 \leq s \leq n$. For $Y_{\sigma_s} \in \mathbb{R}^s$, $\star \in \{R, N\}$ and adopting (69)-(71) we claim*

$$\begin{aligned} \left| \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| &\leq (\Lambda + m)^{3-n-r_1-r_2-|w|} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ &\times \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^0(\Lambda, \tau_{1,s}, Y_{\sigma_s}), \quad \forall n \geq 2. \quad (135) \end{aligned}$$

Here and subsequently \mathcal{P}_i and \mathcal{Q}_i denote polynomials with non-negative coefficients which depend on $l, n, |w|, \delta, r_1, r_2$, but not on $\{p_i\}$, Λ , Λ_0 and c . The polynomial \mathcal{Q}_i is reduced to a constant for $s = 1$, and for $l = 0$ all polynomials \mathcal{P}_i reduce to constants. The parameter δ depends on the loop order l and verifies $0 < \delta_l \leq \delta_{l+1} < 1$.

As a consequence of Theorem 1, we have:

Proposition 4. For fixed $0 \leq \Lambda \leq \Lambda_0 < \infty$, $\tau > 0$ and $(y_1, \dots, y_n) \in (\mathbb{R}^+)^n$, we have:

$$\mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \prod_{i=1}^n p_D(\tau_i; \cdot, y_i) \right) = \lim_{c \rightarrow +\infty} \mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \prod_{i=1}^n p_R(\tau_i; \cdot, y_i) \right), \quad (136)$$

where the parameter c denotes the Robin parameter.

Corollary 1. For Dirichlet boundary conditions, adopting (69)-(72) and the assumptions of Proposition 4 we have

$$\begin{aligned} & \left| \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \prod_{i=1}^n p_D(\tau_i; \cdot, y_i) \right) \right| \\ & \leq (\Lambda + m)^{3-n} \mathcal{P}_3 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_4 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_2 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{n,l;\delta}^0(\Lambda, \tau_{1,n}, Y_{\sigma_n}) \quad \forall n \geq 4, \end{aligned} \quad (137)$$

and for $n = 2$ we have

$$\begin{aligned} & \left| \mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0} \left(p, -p; \prod_{i=1}^2 p_D(\tau_i; \cdot, y_i) \right) \right| \\ & \leq (\Lambda + m)^{-1} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P}_5 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_6 \left(\frac{|p|}{\Lambda + m} \right) \mathcal{Q}_3 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^0(\Lambda, \tau_{1,2}, Y_{\sigma_2}). \end{aligned} \quad (138)$$

Theorem 2. (Convergence) Let $0 \leq \Lambda \leq \Lambda_0 < \infty$. Using the same notations, conventions and adopting the same renormalization conditions (69)-(72) as in Theorem 1 and Proposition 4, we have the following bounds

$$\begin{aligned} (A) \quad & \left| \partial_{\Lambda_0} \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) \right| \leq \frac{(\Lambda + m)^{4-n-|w|-r_1-r_2}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_1 \left(\log \frac{\Lambda_0 + m}{m} \right) \tilde{\mathcal{P}}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ & \quad \times \tilde{\mathcal{Q}}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}), \quad \forall n + |w| + r_1 + r_2 \geq 2, \star \in \{N, R\}. \end{aligned} \quad (139)$$

$$\begin{aligned} (B) \quad & \left| \partial_{\Lambda_0} \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) \right| \\ & \leq \frac{(\Lambda + m)^{4-n}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_1 \left(\log \frac{\Lambda_0 + m}{m} \right) \tilde{\mathcal{P}}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathcal{Q}}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{n,l;\delta}^{\Lambda,0}(\tau_{1,n}), \quad \forall n \geq 4. \end{aligned} \quad (140)$$

$$\begin{aligned} (C) \quad & \left| \partial_{\Lambda_0} \mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) \right| \\ & \leq \tau^{-1} (\Lambda_0 + m)^{-2} \tilde{\mathcal{P}}_1 \left(\log \frac{\Lambda_0 + m}{m} \right) \tilde{\mathcal{P}}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathcal{Q}}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \end{aligned} \quad (141)$$

Remarks 3. • There are two differences between the Robin/Neumann case (135) and the Dirichlet case (137)-(138): The boundary conditions (72) for $\mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0}$ are imposed at scale $\Lambda = \Lambda_0$ only, whereas for $\mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}$ we imposed mixed boundary conditions (69)-(71). The second difference concerns the type of test functions considered, which in the case of Dirichlet are product of Dirichlet heat kernels (i.e. $\prod_{i=1}^n p_D(\tau_i; z_i, y_i)$), whereas in the case of Robin and Neumann b.c. the test functions are product of bulk heat kernels and characteristic functions of the semi-lines (i.e. $\prod_{i=1}^s p_B(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i)$).

- The bounds (135) and (137)-(138) can be established by induction separately using the associated flow equations. For the Dirichlet boundary conditions, the associated flow equations are integrated from Λ to Λ_0 . For the Robin/Neumann cases, the flow equations are integrated from 0 to Λ for the relevant terms using the boundary condition (71) and from Λ to Λ_0 for the irrelevant terms using the boundary condition (69).
- Adopting the boundary conditions (48)-(49) together with (69)-(71), the distributions $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ and $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ are uniquely defined as the solutions of the flow equations (33) and (62). Furthermore, their sum

$$\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0} = \mathcal{D}_{l,n}^{\Lambda,\Lambda_0} + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}, \quad (142)$$

is the unique solution of the flow equations (52) such that $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$ and $\mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$ obey respectively (48)-(49) and (69)-(71). Theorem 1 together with Proposition 1 gives for $s \geq 1$

$$\begin{aligned} & \left| \partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \prod_{i=1}^s p_B(\tau_i; \cdot, y_i) \right) \right| \\ & \leq \left\{ (\Lambda + m)^{4-n-|w|-r} \hat{\mathcal{F}}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) + (\Lambda + m)^{3-n-|w|-r} \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) \right\} \\ & \quad \times \mathcal{P}_5 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_6 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_3 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right), \quad (143) \end{aligned}$$

where

$$\begin{aligned} & \partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \prod_{i=1}^s p_B(\tau_i; \cdot, y_i) \right) \\ & := \int_{z_1, \dots, z_n} (z_1 - z_2)^r \partial^w \mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) \end{aligned}$$

$$\text{and } \hat{\mathcal{F}}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) := \int_{z_1} p_B(\tau_1; z_1, y_1) \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{2,s}).$$

The bound (143) implies that $\partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0}(\vec{p}_n; \prod_{i=1}^s p_B(\tau_i; \cdot, y_i))$ are bounded uniformly w.r.t. Λ_0 . It is also possible to deduce a convergence Theorem which implies the existence of the limit $\Lambda \rightarrow 0$ and $\Lambda_0 \rightarrow \infty$ for $\partial^w \mathcal{L}_{l,n;r,\star}^{\Lambda,\Lambda_0}$ which we do not explicit here.

- We do not prove Theorem 2 since there is no novelty in the proof, which is mainly based on combining arguments from the proof of Theorem 1 with the steps of the proof of the convergence theorem in⁶.
- The difference between $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0} + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}$, and $\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}$ studied in⁶, is their distributional structure, in the sense that one can prove inductively using the FEs (62) and the boundary conditions (69)-(70) that

$$\begin{aligned} \mathcal{D}_{l,n}^{\Lambda,\Lambda_0}(z_1; \phi_{\tau_{2,s}, y_{2,s}}) + \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}(z_1; \phi_{\tau_{2,s}, y_{2,s}}) &= a_{l,n;\star}^{\Lambda,\Lambda_0}(z_1, y_{2,s}, \tau_{2,s}) + b_{l,n;\star}^{\Lambda,\Lambda_0}(y_{2,s}, \tau_{2,s}) \delta_{z_1} \\ & \quad + c_{l,n;\star}^{\Lambda,\Lambda_0}(y_{2,s}, \tau_{2,s}) \delta'_{z_1}, \quad (144) \end{aligned}$$

where $a_{l,n;\star}^{\Lambda,\Lambda_0}$ is smooth w.r.t. z_1 and $a_{l,n;\star}^{\Lambda,\Lambda_0}$, $b_{l,n;\star}^{\Lambda,\Lambda_0}$ and $c_{l,n;\star}^{\Lambda,\Lambda_0}$ are smooth w.r.t. $y_{2,s}$ and $\tau_{2,s}$. However, the semi-infinite correlation distributions $\mathcal{L}_{l,n;\star}^{\Lambda,\Lambda_0}$ considered in⁶ are smooth w.r.t. z_1 which is a consequence of the type of mixed b.c.s imposed on the semi-infinite correlation distributions.

- If the bulk correlation distributions obey the bound

$$\begin{aligned} & \left| \partial^w \mathcal{D}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}) \right| \\ & \leq \left\{ (\Lambda+m)^{3-n-r_1-r_2-|w|} \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}) + (\Lambda+m)^{4-n-r_1-r_2-|w|} \mathcal{F}_{s,l;\delta}^{\Lambda}(\tau_{1,s}) \right\} \\ & \quad \times \mathcal{P}_1 \left(\log \frac{\Lambda+m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda+m} \right) \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right), \quad \forall n \geq 2, \quad \forall s \geq 1 \end{aligned} \quad (145)$$

instead of (50)-(51), the bound (135) still holds.

- The bound (135) holds also for the surface correlation distributions folded with \star heat kernels (i.e. $\star \in \{N, R\}$), that is

$$\begin{aligned} & \left| \partial^w \mathcal{S}_{l,n;\star;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s},y_{1,s}}^\star) \right| \leq (\Lambda+m)^{3-n-r_1-r_2-|w|} \mathcal{P}_1 \left(\log \frac{\Lambda+m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda+m} \right) \\ & \quad \times \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{F}_{s,l;\delta}^{\Lambda,0}(\tau_{1,s}), \quad \forall n \geq 2, \end{aligned} \quad (146)$$

where the external points $y_{1,s}$ belong to $(\mathbb{R}^+)^s$. This is a direct consequence of (7)-(8) together with the bounds (17). In particular, the bound (146) implies that $\mathcal{S}_{l,n;R;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau,y_{1,s}}^R)$ is uniformly bounded w.r.t. the Robin parameter c .

B. Proof of Theorem 1

Outline of the proof: The bounds (135) and (137)-(138) are proven inductively using the standard inductive scheme which proceeds upwards in l , for given l upwards in n , and for given (n, l) downwards in $|w|$ starting from some arbitrary $|w_{\max}| \geq 3$. The bounds (135) and (137)-(138) can be proven separately. Let us explain the general steps in establishing (135). First, we verify that the bound (135) holds at the tree order. The terms on the RHS of the FE are prior to the one in the LHS in the inductive order, therefore we use the induction hypothesis for the terms in the RHS to bound the term on the LHS. In this part of the proof, the key ingredient is the stability of the reduction and fusion operations of the trees and forests under the flow equation, which is guaranteed by Lemmas 1-3. Afterwards, we integrate this bound from Λ to Λ_0 for the irrelevant terms using the boundary conditions (70). The relevant part requires a careful analysis. We proceed by performing a Taylor expansion of the test functions ϕ_{τ_1,y_1} and ϕ_{τ_2,y_2} around $z_1 = 0$ and $z_2 = 0$ in

$$\partial_{\Lambda} \mathcal{S}_{l,2;\star}^{\Lambda,\Lambda_0}(\vec{0}; \phi_{\tau_{1,2},y_{1,2}}).$$

This step allows to separate the relevant and irrelevant parts in the 2-point surface correlator at zero momenta. Then, the relevant terms are integrated from 0 to Λ using the renormalization (71). The remainder of the Taylor expansion is irrelevant and it is integrated from Λ to Λ_0 using again (70) together with the properties of the surface weight factor for $s = 2$, which are gathered in the Appendix.

Proof. We establish the proof in the case of the Robin boundary conditions. For the Neumann boundary conditions, we proceed similarly. In the sequel, we omit the subscript R from $\mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0}$.

The induction starts at the tree order for which we have

$$\mathcal{S}_{0,4}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = 0,$$

and the bound (135) obviously holds.

1. The right-hand side of the FEs

The bounds that we want to obtain for the RHS of the flow equations (62) are of the form

$$\left| \partial_\Lambda \partial^w \mathcal{S}_{l,n;r_1,r_2}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,s}, y_{1,s}}) \right| \leq (\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \times \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta}^0(\Lambda, \tau_{1,s}, y_{1,s}), \quad (147)$$

for all $n \geq 2$, $0 \leq s \leq n$ and $0 \leq r_1, r_2 \leq 4$. In the sequel, we drop the lower indices from the polynomials \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{Q}_1 . But one should keep in mind that these polynomials, whenever they appear, may have different positive coefficients which depend on $l, n, |w|, \delta_l$ only and not on $\{p_i\}$, $\Lambda, \Lambda_0, \tau_{1,s}$ and the Robin parameter c .

The bound (147) is established by bounding each of the terms on the RHS of the FE (62). We consider first the case $r_1 = r_2 = 0$.

- We start by treating the linear terms R_1^D and R_1^S given by

$$R_1^S := \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_R \left(\frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) \quad (148)$$

and

$$R_1^D := \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_{S,R} \left(\frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i), \quad (149)$$

where p_R and $p_{S,R}$ are given by (8) and (22). First, we bound R_1^S . Using the decomposition of the Robin heat kernel (8), we obtain that R_1^S can be written as the sum of three contributions such that for each contribution the Robin heat kernel p_R in R_1^S is replaced by a term from the decomposition (8). We analyze first the term

$$\tilde{R}_1^S := \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_B \left(\frac{1}{\Lambda^2}; z, z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i).$$

Using the semi-group property for the bulk heat kernel (10), \tilde{R}_1^S can be rewritten as

$$\int_{\mathbb{R}} du \int_{z,z'} \int_{z_{1,n}} \int_k \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \\ \times \dot{C}^\Lambda(k) p_B \left(\frac{1}{2\Lambda^2}; z, u \right) p_B \left(\frac{1}{2\Lambda^2}; z', u \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i).$$

We now insert the induction hypothesis to obtain that \tilde{R}_1^S is bounded by

$$(\Lambda + m)^{1-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_k |\dot{C}^\Lambda(k)| \mathcal{P} \left(\frac{|k|}{\Lambda + m}, \frac{\|\vec{p}_n\|}{\Lambda + m} \right) \\ \int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_1}^{\Lambda,0} \left(\tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right). \quad (150)$$

Using

$$\int_k |\dot{C}^\Lambda(k)| \left(\frac{|k|}{\Lambda+m} \right)^\alpha = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{2}{\Lambda^3} e^{-\frac{k^2+m^2}{\Lambda^2}} \left(\frac{|k|}{\Lambda+m} \right)^\alpha \leq O(1), \quad \forall \alpha \in \mathbb{N}, \quad (151)$$

we have

$$|\tilde{R}_1^S| \leq (\Lambda+m)^{1-n-|w|} \mathcal{P} \left(\log \frac{\Lambda+m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda+m} \right) \int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_1}^{\Lambda,0} \left(\tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right). \quad (152)$$

Applying Lemma 1, we obtain the bound

$$|\tilde{R}_1^S| \leq (\Lambda+m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda+m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda+m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{F}_{s,l;\delta_1}^{\Lambda,0}(\tau_{1,s}). \quad (153)$$

The other contributions to R_1^S are

$$\int_{z,z'} \int_{z_1,n} \int_k \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \times \dot{C}^\Lambda(k) p_B \left(\frac{1}{\Lambda^2}; z, -z' \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) \quad (154)$$

and

$$-2 \int_{z,z'} \int_{z_1,n} \int_k \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \times \dot{C}^\Lambda(k) \int_v e^{-v} p_B \left(\frac{1}{\Lambda^2}; z, -z' - \frac{v}{c} \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i). \quad (155)$$

These terms can be rewritten using (10) as

$$\int_{\mathbb{R}} du \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_{1,s}, y_{1,s}} \times p_B \left(\frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left(\frac{1}{2\Lambda^2}; \cdot, -u \right)) \quad (156)$$

and

$$\int_{\mathbb{R}} du \int_v e^{-v} \partial^w \mathcal{S}_{l-1,n+2}^{\Lambda,\Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_{1,s}, y_{1,s}} \times p_B \left(\frac{1}{2\Lambda^2}; \cdot, u \right) p_B \left(\frac{1}{2\Lambda^2}; \cdot, -\frac{v}{c} - u \right)). \quad (157)$$

Applying the induction hypothesis, we obtain that (156) is bounded by

$$(\Lambda+m)^{1-n-|w|} \mathcal{P} \left(\log \frac{\Lambda+m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda+m} \right) \int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_1}^{\Lambda,0} \left(\tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u \right). \quad (158)$$

Similarly, we have the following bound for (157)

$$(\Lambda+m)^{1-n-|w|} \mathcal{P} \left(\log \frac{\Lambda+m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \int_k \dot{C}(k) \mathcal{P} \left(\frac{|k|}{\Lambda+m}, \frac{\|\vec{p}_n\|}{\Lambda+m} \right) \int_{\mathbb{R}} du \int_v e^{-v} \mathcal{F}_{s+2,l-1;\delta_1}^{\Lambda,0} \left(\tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u - \frac{v}{c} \right). \quad (159)$$

Using the bounds (17) and remembering the definition (103) of the surface weight factor, we deduce that

$$\int_{\mathbb{R}} du \int_v e^{-v} \mathcal{F}_{s+2,l-1;\delta_1}^{\Lambda,0} \left(\tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u - \frac{v}{c} \right)$$

and

$$\int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_1}^{\Lambda,0} \left(\tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, -u \right)$$

are bounded by

$$\int_{\mathbb{R}} du \mathcal{F}_{s+2,l-1;\delta_1}^{\Lambda,0} \left(\tau_{1,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2}; Y_{\sigma_s}, u, u \right).$$

The rest of the proof follows the steps used to obtain the final bound for \tilde{R}_1^S , which gives

$$|R_1^S| \leq (\Lambda + m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_1}^{\Lambda,0} (\tau_{1,s}). \quad (160)$$

Now we analyse R_1^D . This term is independent of the induction hypothesis and will be bounded using only the bound (50) for $\mathcal{D}_{l,n}^{\Lambda,\Lambda_0}$. Using (10), R_1^D can be rewritten as

$$\int_{\mathbb{R}} du \int_{z,z'} \int_{z_1,n} \int_k \partial^w \mathcal{D}_{l-1,n+2}^{\Lambda,\Lambda_0} ((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \times \dot{C}^\Lambda(k) p_B \left(\frac{1}{2\Lambda^2}; z, u \right) p_B \left(\frac{1}{2\Lambda^2}; z', -u \right) \prod_{i=1}^s p_B(\tau_i; z_i, y_i).$$

The bound (50) implies that R_1^D is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \int_{\mathbb{R}} du \int_{z_1} p_B(\tau_1; z_1, y_1) \times \sum_{T_{l-1}^{s+2}(z_1, y_{2,s}, u, -u, \vec{z})} \int_{\vec{z}} \mathcal{F}_{\delta_2} \left(\Lambda; \left\{ \tau_{2,s}, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2} \right\}; T_{l-1}^{s+2}(z_1, y_{2,s}, u, -u, \vec{z}) \right). \quad (161)$$

For any contribution to (161) we denote by z', z'' the vertices in the tree $T_{l-1}^{s+2}(z_1, y_{2,s}, u, -u)$ to which the test functions $p_B \left(\frac{1+\delta_2}{2\Lambda^2}; u, \cdot \right)$ and $p_B \left(\frac{1+\delta_2}{2\Lambda^2}; \cdot, -u \right)$ are attached. Performing the integral over u we obtain using (10)

$$\int_{\mathbb{R}} du p_B \left(\frac{1+\delta_2}{2\Lambda^2}; z', u \right) p_B \left(\frac{1+\delta_2}{2\Lambda^2}; -u, z'' \right) = p_B \left(\frac{1+\delta_2}{\Lambda^2}; z', -z'' \right) \leq p_B \left(\frac{1+\delta_2}{\Lambda^2}; z', 0 \right). \quad (162)$$

The bound (162) implies that the legs (z, u) and (z', u) are amputated from the tree T_{l-1}^{s+2} and (z', u) is replaced by the surface external leg $(z', 0)$ with the parameter Λ . If z'' is of incidence number one, it is removed using $\int_{z''} p_B \left((1+\delta_2)/\Lambda^2; z, z'' \right) \leq 1$, and this operation is iterated until a vertex \tilde{z} such that $c(\tilde{z}) \geq 2$ is reached. This iteration process converges to a non-empty tree since for $s \geq 1$, there exists at least one internal vertex of incidence number greater or equal to 3 in the tree T_{l-1}^{s+2} . The integration over z_1 in (161) implies that z_1 becomes an internal vertex attached to y_1 . Therefore, the reduction process produces a tree which belongs to $\mathcal{T}^{s,0}$.

Furthermore, v'_2 which denotes the number of vertices of incidence number 2 of the new tree, is increased at most by 2. This stems from the reduction process which can produce one additional internal vertex such that $c(z) = 2$ when the vertex z'' is removed, but also from the vertex z_1 which was initially a root vertex. If z_1 had an incidence number equal to one then after introducing the test function $p_B(\tau_1; z_1, y_1)$, it becomes internal of incidence number 2. If v_2 is the number of vertices of incidence number 2 of T_{l-1}^{s+2} , then $v'_2 \leq v_2 + \delta_{c_1,1} + 1$ which implies

$$v'_2 \leq v_2 + \delta_{c_1,1} + 1 \leq 3(l-1) - 2 + \frac{s+2}{2} + 1 \leq 3l - 2 + \frac{s+1}{2}.$$

This also means that the obtained tree is a surface tree in $\mathcal{F}_l^{s,0} \equiv \mathcal{W}_l^s(\sigma_s)$, which can also be seen as the set of forests corresponding to the trivial partition. Therefore, R_1^D is bounded by

$$(\Lambda+m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda+m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda+m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{F}_{\delta_2}^0(\Lambda; \tau_{1,s}; W_i^s(\sigma_s); Y_{\sigma_s}) \quad (163)$$

which implies (see (108))

$$|R_1^D| \leq (\Lambda+m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda+m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda+m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m} \right) \mathcal{F}_{s,l;\delta_2}^{\Lambda,0}(\tau_{1,s}). \quad (164)$$

- In this part, we treat the quadratic terms on the RHS of the flow equations. It is enough to analyse the terms from the symmetrized sum in which the arguments (z_i, p_i) appear ordered in $(\mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}, \mathcal{S}_{l_2, n_2+1}^{\Lambda, \Lambda_0}), (\mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}, \mathcal{S}_{l_2, n_2+1}^{\Lambda, \Lambda_0})$ and $(\mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}, \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0})$. These terms are given by

$$R_2^{SS} := \int_{z_{1,n}} \int_{z, z'} z_1^{r_1} z_2^{r_2} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) p_R \left(\frac{1}{\Lambda^2}; z, z' \right) \\ \times \partial^{w_2} \mathcal{S}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i),$$

$$R_2^{DS} := \int_{z_{1,n}} \int_{z, z'} z_1^{r_1} z_2^{r_2} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) p_R \left(\frac{1}{\Lambda^2}; z, z' \right) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i),$$

and

$$R_2^{DD} := \int_{z_{1,n}} \int_{z, z'} z_1^{r_1} z_2^{r_2} \partial^{w_1} \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) p_{S,R} \left(\frac{1}{\Lambda^2}; z, z' \right) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i).$$

First, we treat the case $(r_1, r_2) = (0, 0)$.

- We start with the term R_2^{DS} . The property (11) implies that R_2^{DS} can be rewritten as

$$\int_{\mathbb{R}^+} du \int_{z_{1,n}} \int_{z, z'} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) \\ \times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) p_R \left(\frac{1}{2\Lambda^2}; z, u \right) p_R \left(\frac{1}{2\Lambda^2}; z', u \right).$$

Using the decomposition of the Robin heat kernel (8), we restrict our analysis to the following term only

$$\begin{aligned} \tilde{R}_2^{DS} &:= \int_{\mathbb{R}^+} du \int_{z_1, n} \int_{z, z'} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s P_B(\tau_i; z_i, y_i) P_B\left(\frac{1}{2\Lambda^2}; z, u\right) P_B\left(\frac{1}{2\Lambda^2}; z', u\right). \end{aligned}$$

The line of reasoning in treating the remaining contributions in R_2^{DS} is similar to the one used in bounding R_1^S . We define

$$\phi'_{s_1}(z_{1, n_1}) = \prod_{r=1}^{n_1} \phi_i(z_i), \quad \phi''_{s_2}(z_{n_1+1, n}) = \prod_{r=n_1+1}^n \phi_i(z_i), \quad (165)$$

where

$$\phi_i(z_i) = \begin{cases} P_B(\tau_i; z_i, y_i) & \text{if } i \leq s \\ \chi^+(z_i) & \text{otherwise.} \end{cases} \quad (166)$$

Note that $s_1 = n_1$ if $n_1 \leq s$ and $s_2 = s - n_1$. Otherwise, we have $s_1 = s$ and $s_2 = 0$. Without loss we consider the case $n_1 < s$. Therefore, \tilde{R}_2^{DS} can be rewritten as

$$\begin{aligned} \tilde{R}_2^{DS} &= \int_{\mathbb{R}^+} du \int_{z'} \partial^{w_1} \mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(\vec{p}_{n_1}, p; \phi'_{s_1} \times P_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(z'; -p, \vec{p}_{n_1+1, n}; \phi''_{s_2}) \times P_B\left(\frac{1}{2\Lambda^2}; u, z'\right). \quad (167) \end{aligned}$$

Applying the induction hypothesis to $\mathcal{S}_{l_1, n_1+1}^{\Lambda, \Lambda_0}$ and using the bound (50) for $\mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}$, we obtain that \tilde{R}_2^{DS} is bounded by

$$\begin{aligned} &(\Lambda + m)^{2-n-|w|} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \\ &\times \int_{\mathbb{R}^+} du \mathcal{F}_{s_1+1, l_1; \delta_3}^0\left(\Lambda; \tau_{1, s_1}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1}}, u\right) \\ &\times \int_{z'} \mathcal{F}_{s_2, l_2; \delta_3'}\left(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1:s}}\right) P_B\left(\frac{1}{2\Lambda^2}; z', u\right). \end{aligned}$$

Since the global weight factor $\mathcal{F}_{s_2, l_2; \delta_3'}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1:s}})$ is a sum of the weight factors of all trees $T_{l_2}^{s_2}(z'; z''; Y_{\sigma_{s_1+1:s}})$ in $\mathcal{T}_{l_2}^{s_2}$, we deduce that the integration over z' of the corresponding weight factor gives the global weight factor of the bulk tree $\hat{T}_{l_2}^{s_2+1}(z', z''; Y_{\sigma_{s_1+1:s}}, u)$ obtained from $T_{l_2}^{s_2}$ by converting z' into an internal vertex and attaching it to the external vertex u . Therefore we can write

$$\begin{aligned} &\hat{\mathcal{F}}_{s_2+1, l_2; \delta_3'}\left(\Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1+1:s}}, u\right) \\ &= \sum_{\hat{T}_{l_2}^{s_2+1} \in \hat{\mathcal{T}}_{l_2}^{s_2+1}} \mathcal{F}_{\delta_3'}\left(\Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u\right), \quad (168) \end{aligned}$$

with

$$\begin{aligned} &\mathcal{F}_{\delta_3'}\left(\Lambda; \tau_{s_1+1, s}, \frac{1}{2\Lambda^2}; \hat{T}_{l_2}^{s_2+1}; Y_{\sigma_{s_1+1:s}}, u\right) \\ &= \int_{z'} \mathcal{F}_{\delta_3'}\left(\Lambda; \tau_{s_1+1, s}; T_{l_2}^{s_2}; z'; Y_{\sigma_{s_1+1:s}}\right) P_B\left(\frac{1}{2\Lambda^2}; z', u\right). \quad (169) \end{aligned}$$

Applying Lemma 3, we deduce that \tilde{R}_2^{DS} is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_3''}^{\Lambda,0}(\tau_{1,s}),$$

where $\delta_3'' := \max(\delta_3, \delta_3')$.

– In this part we bound the term R_2^{SS} . As for R_2^{DS} , we only treat the term

$$\begin{aligned} \tilde{R}_2^{SS} &:= \int_{\mathbb{R}^+} du \int_{z_1,n} \int_{z,z'} \partial^{w_1} \mathcal{S}_{l_1,n_1+1}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{S}_{l_2,n_2+1}^{\Lambda,\Lambda_0}((z', -p), \dots, (z_n, p_n)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i) p_B \left(\frac{1}{2\Lambda^2}; z, u \right) p_B \left(\frac{1}{2\Lambda^2}; z', u \right). \end{aligned}$$

Using the same notations (165)-(166), we rewrite \tilde{R}_2^{SS} as follows,

$$\begin{aligned} \tilde{R}_2^{SS} &= \int_{\mathbb{R}^+} du \partial^{w_1} \mathcal{S}_{l_1,n_1+1}^{\Lambda,\Lambda_0}(\vec{p}_{1,n_1}, p; \phi'_{s_1} \times p_B \left(\frac{1}{2\Lambda^2}; \cdot, u \right)) \partial^{w_3} \dot{C}^\Lambda(p) \\ &\times \partial^{w_2} \mathcal{S}_{l_2,n_2+1}^{\Lambda,\Lambda_0}(-p, \vec{p}_{n_1+1,n}; \phi''_{s_2} \times p_B \left(\frac{1}{2\Lambda^2}; \cdot, u \right)). \end{aligned} \quad (170)$$

Note that $\tilde{R}_{2,SS} = 0$ for $l_i \in \{0, l\}$. Using the induction hypothesis, we obtain

$$\begin{aligned} |\tilde{R}_2^{SS}| &\leq (\Lambda + m)^{1-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ &\times \int_{\mathbb{R}} du \mathcal{F}_{s_1+1,l_1;\delta_3}^0 \left(\Lambda; \tau_{1,s_1}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1}}, u \right) \mathcal{F}_{s_2+1,l_2;\delta_4}^0 \left(\Lambda; \tau_{s_1+1,s}, \frac{1}{2\Lambda^2}; Y_{\sigma_{s_1+1,s}}, u \right). \end{aligned}$$

Applying Lemma 2, we deduce that \tilde{R}_2^{SS} is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l;\delta_5}^{\Lambda,0}(\tau_{1,s}),$$

where $\delta_5 := \max(\delta_3, \delta_4)$.

– In this part, we bound the term R_2^{DD} , which we rewrite using (165)-(166) as follows,

$$\begin{aligned} \int_{z,z'} \partial^{w_1} \mathcal{S}_{l_1,n_1+1}^{\Lambda,\Lambda_0}(z; \vec{p}_{1,n_1}, p; \phi'_{s_1}) \partial^{w_3} \dot{C}^\Lambda(p) p_{S,R} \left(\frac{1}{\Lambda^2}; z, -z' \right) \\ \times \partial^{w_2} \mathcal{S}_{l_2,n_2+1}^{\Lambda,\Lambda_0}(z'; \vec{p}_{n_1+1,n}, -p; \phi''_{s_2}). \end{aligned}$$

Using the bounds (23) and (50), we obtain that R_2^{DD} is bounded by

$$\begin{aligned} (\Lambda + m)^{3-n-|w|} e^{-\frac{m^2}{2\Lambda^2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ \times \int_{z,z'} \mathcal{F}_{s_1+1,l_1;\delta_4'} \left(\Lambda; \tau_{1,s_1}; z; Y_{\sigma_{s_1}} \right) \mathcal{F}_{s_2+1,l_2;\delta_3'} \left(\Lambda; \tau_{s_1+1,s}; z'; Y_{\sigma_{s_1+1,s}} \right) p_B \left(\frac{1}{\Lambda^2}; z', -z \right). \end{aligned} \quad (171)$$

The bound

$$p_B \left(\frac{1}{\Lambda^2}; z', -z \right) \leq \sqrt{2\pi} \Lambda^{-1} p_B \left(\frac{1}{\Lambda^2}; z', 0 \right) p_B \left(\frac{1}{\Lambda^2}; z, 0 \right) \quad (172)$$

together with

$$\Lambda^{-\alpha} e^{-\frac{m^2}{2\Lambda^2}} \leq O(1) (\Lambda + m)^{-\alpha} \quad \text{for } \alpha \in \mathbb{N}, \quad (173)$$

gives

$$\begin{aligned} & e^{-\frac{m^2}{2\Lambda^2}} \int_{z,z'} \mathcal{F}_{s_1+1,l_1;\delta'_4} \left(\Lambda; \tau_{1,s_1}; z; Y_{\sigma_{s_1}} \right) \mathcal{F}_{s_2+1,l_2;\delta'_3} \left(\Lambda; \tau_{s_1+1,s}; z'; Y_{\sigma_{s_1+1,s}} \right) p_B \left(\frac{1}{\Lambda^2}; z', -z \right) \\ & \leq O(1) (\Lambda + m)^{-1} \int_z \mathcal{F}_{s_1+1,l_1;\delta'_4} \left(\Lambda; \tau_{1,s_1}; z; Y_{\sigma_{s_1}} \right) p_B \left(\frac{1}{\Lambda^2}; z, 0 \right) \\ & \quad \times \int_{z'} \mathcal{F}_{s_2+1,l_2;\delta'_3} \left(\Lambda; \tau_{s_1+1,s}; z'; Y_{\sigma_{s_1+1,s}} \right) p_B \left(\frac{1}{\Lambda^2}; z', 0 \right). \end{aligned}$$

From the definition (99), we have

$$\begin{aligned} & \int_z \mathcal{F}_{s_1+1,l_1;\delta'_4} \left(\Lambda; \tau_{1,s_1}; z; Y_{\sigma_{s_1}} \right) p_B \left(\frac{1}{\Lambda^2}; z, 0 \right) \\ & = \sum_{T_{l_1}^{s_1+1} \in \mathcal{T}_{l_1}^{s_1+1}} \int_z \mathcal{F}_{s_1+1,l_1;\delta'_4} \left(\Lambda; \tau_{1,s_1}; T_{l_1}^{s_1+1}; z; Y_{\sigma_{s_1}} \right) p_B \left(\frac{1}{\Lambda^2}; z, 0 \right). \quad (174) \end{aligned}$$

We define the tree $T_{l_1}^{s_1,0}$ obtained from $T_{l_1}^{s_1+1}$ by converting the root vertex z into an internal vertex and attaching z to the surface external vertex 0. Clearly, $T_{l_1}^{s_1,0}$ belongs to $\mathcal{T}_{l_1}^{s_1,0}$. Denoting by v'_2 the number of vertices of incidence number 2 of $T_{l_1}^{s_1,0}$, we clearly have

$$v'_2 = v_2 + \delta_{c(z),1}$$

with v_2 denoting the number of vertices of incidence number 2 of $T_{l_1}^{s_1+1}$. Remembering that $T_{l_1}^{s_1+1}$ is in $\mathcal{T}_{l_1}^{s_1+1}$, we have

$$v'_2 = v_2 + \delta_{c(z),1} \leq 3l - 2 + \frac{s_1 + 1}{2}.$$

This implies that $T_{l_1}^{s_1,0} \in \mathcal{T}_{l_1}^{s_1,0}$. Using the definitions (97), (98) and (103), we obtain

$$\begin{aligned} & \sum_{T_{l_1}^{s_1+1} \in \mathcal{T}_{l_1}^{s_1+1}} \int_z \mathcal{F}_{s_1+1,l_1;\delta'_4} \left(\Lambda; \tau_{1,s_1}; T_{l_1}^{s_1+1}; z; Y_{\sigma_{s_1}} \right) p_B \left(\frac{1}{\Lambda^2}; z, 0 \right) \\ & \leq \sum_{T_{l_1}^{s_1,0} \in \mathcal{T}_{l_1}^{s_1,0}} \mathcal{F}_{s_1,l_1;\delta'_4}^0 \left(\Lambda; \tau_{1,s_1}; T_{l_1}^{s_1,0}; Y_{\sigma_{s_1}}, 0 \right). \quad (175) \end{aligned}$$

A similar reasoning gives

$$\begin{aligned} & \sum_{T_{l_2}^{s_2+1} \in \mathcal{T}_{l_2}^{s_2+1}} \int_{z'} \mathcal{F}_{s_2+1,l_2;\delta'_3} \left(\Lambda; \tau_{s_1+1,s}; T_{l_2}^{s_2+1}; z'; Y_{\sigma_{s_1+1,s}} \right) p_B \left(\frac{1}{\Lambda^2}; z', 0 \right) \\ & \leq \sum_{T_{l_2}^{s_2,0} \in \mathcal{T}_{l_2}^{s_2,0}} \mathcal{F}_{s_2,l_2;\delta'_3}^0 \left(\Lambda; \tau_{s_1+1,s}; T_{l_2}^{s_2,0}; Y_{\sigma_{s_1+1,s}}, 0 \right). \quad (176) \end{aligned}$$

Hence, we deduce

$$\begin{aligned} & \int_z \mathcal{F}_{s_1+1, l_1; \delta_4'}(\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) p_B \left(\frac{1}{\Lambda^2}; z, 0 \right) \\ & \quad \times \int_{z'} \mathcal{F}_{s_2+1, l_2; \delta_3'}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1, s}}) p_B \left(\frac{1}{\Lambda^2}; z', 0 \right) \\ & \leq O(1) \left(\sum_{W_l^s \in \mathcal{W}_l^s} \sum_{\{\Pi \in \mathcal{P}_s, l_{\Pi}=2\}} \mathcal{F}_{\delta_6}^0(\Lambda; \tau_{1, s}; W_l^s(\Pi); Y_{\sigma_s}) \right), \quad (177) \end{aligned}$$

where $\Pi := \{\Pi_1, \Pi_2\}$, $\Pi_1 := \{1, \dots, s_1\}$ and $\Pi_2 := \{s_1 + 1, \dots, s\}$. In (177), we also used the bound (112) and $\delta_6 := \max(\delta_3', \delta_4')$. Therefore, we find that R_2^{DD} is bounded by

$$\begin{aligned} & (\Lambda + m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \quad \times \left(\sum_{W_l^s \in \mathcal{W}_l^s} \sum_{\{\Pi \in \mathcal{P}_s, l_{\Pi}=2\}} \mathcal{F}_{\delta_6}^0(\Lambda; \tau_{1, s}; W_l^s(\Pi); Y_{\sigma_s}) \right). \quad (178) \end{aligned}$$

This shows that R_2^{DD} is bounded by

$$(\Lambda + m)^{2-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l; \delta_6}^{\Lambda, 0}(\tau_{1, s}).$$

- Case $(r_1, r_2) \neq (0, 0)$: The linear terms R_1^S and R_1^D together with the non-linear term R_2^{SS} are treated following the same steps as before. The only terms that require a careful analysis are R_2^{DS} and R_2^{DD} . To shorten the discussion, we restrict our analysis to the term R_2^{DD} . R_2^{DS} can be treated following the same line of reasoning. We write

$$z_1^{r_1} = \sum_{\alpha_1 + \beta_1 = r_1} \binom{r_1}{\alpha_1} (z_1 - z)^{\alpha_1} z^{\beta_1}, \quad z_2^{r_2} = \sum_{\alpha_2 + \beta_2 = r_2} \binom{r_2}{\alpha_2} (z_2 - z')^{\alpha_2} z'^{\beta_2}.$$

This allows to rewrite R_2^{DD} for all $n_1 \geq 2$ as follows

$$\begin{aligned} & \sum_{\alpha_1 + \beta_1 = r_1, \alpha_2 + \beta_2 = r_2} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} z^{\beta_1 + \beta_2} \partial^{w_1} \mathcal{D}_{l_1, n_1 + 1; \alpha_1, \alpha_2}^{\Lambda, \Lambda_0; (1, 2)}(z; \vec{p}_{1, n_1}, p; \phi'_{s_1}) \partial^{w_3} \dot{C}^{\Lambda}(p) \\ & \quad \times \partial^{w_2} \mathcal{D}_{l_2, n_2 + 1}^{\Lambda, \Lambda_0}(z'; \vec{p}_{n_1 + 1, n}, -p; \phi''_{s_2}) p_{S, R} \left(\frac{1}{\Lambda^2}; z, z' \right), \quad (179) \end{aligned}$$

and for $n_1 = 1$ we have

$$\begin{aligned} & \sum_{\alpha_1 + \beta_1 = r_1, \alpha_2 + \beta_2 = r_2} \binom{r_1}{\alpha_1} \binom{r_2}{\alpha_2} z^{\beta_1} z'^{\beta_2} \partial^{w_1} \mathcal{D}_{l_1, 2; \alpha_1}^{\Lambda, \Lambda_0; (1)}(z; p_1, p; \phi'_{s_1}) \partial^{w_3} \dot{C}^{\Lambda}(p) \\ & \quad \times \partial^{w_2} \mathcal{D}_{l_2, n; \alpha_2}^{\Lambda, \Lambda_0; (2)}(z'; \vec{p}_{2, n}, -p; \phi''_{s_2}) p_{S, R} \left(\frac{1}{\Lambda^2}; z, z' \right). \quad (180) \end{aligned}$$

Using the bounds (50), (51) and (23), we deduce that the summands in (179)-(180) are bounded by

$$\begin{aligned} & (\Lambda + m)^{3-n-|w| - \alpha_1 - \alpha_2} e^{-\frac{m^2}{2\Lambda^2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \quad \times \int_{z, z'} \mathcal{F}_{s_1, l_1; \delta_1''}(\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) \mathcal{F}_{s_2, l_2; \delta_2''}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1, s}}) \chi_{n_1}^{\Lambda}(z, z'), \quad (181) \end{aligned}$$

where

$$\chi_{n_1}^\Lambda(z, z') = \begin{cases} z^{\beta_1} z'^{\beta_2} p_B\left(\frac{1}{\Lambda^2}; z', -z\right) & \text{if } n_1 = 1, \\ z^{\beta_1 + \beta_2} p_B\left(\frac{1}{\Lambda^2}; z', -z\right) & \text{otherwise.} \end{cases}$$

Using the bound (172) together with (15) and (172), we deduce for all $\delta, \tilde{\delta} > 0$

$$\chi_{n_1}^\Lambda(z, z') \leq O(1) \Lambda^{-1-\beta_1-\beta_2} p_B\left(\frac{1+\delta}{\Lambda^2}; z, 0\right) p_B\left(\frac{1+\tilde{\delta}}{\Lambda^2}; z', 0\right), \quad (182)$$

which implies together with (173) that (181) is bounded by

$$\begin{aligned} & (\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \\ & \times \int_z \mathcal{F}_{s_1, l_1; \delta_1''}(\Lambda; \tau_{1, s_1}; z; Y_{\sigma_{s_1}}) p_B\left(\frac{1+\delta_1''}{\Lambda^2}; z, 0\right) \\ & \times \int_{z'} \mathcal{F}_{s_2, l_2; \delta_2''}(\Lambda; \tau_{s_1+1, s}; z'; Y_{\sigma_{s_1+1, s}}) p_B\left(\frac{1+\delta_2''}{\Lambda^2}; z', 0\right). \end{aligned} \quad (183)$$

This together with the bounds (177) and (178) imply the final bound for R_2^{DD} given by

$$(\Lambda + m)^{2-n-|w|-r_1-r_2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{s, l; \delta_7}^{\Lambda, 0}(\tau_{1, s}), \quad (184)$$

where $\delta_7 := \max\{\delta_1'', \delta_2''\}$.

Using the bound (112), we deduce (147) where $\delta := \max\{\delta_i, \delta_i', \delta_i'', 1 \leq i \leq 7\}$.

2. Integration of the FEs

- We start by integrating the irrelevant terms for which $n + |w| + r_1 + r_2 \geq 4$. In this case, (147) is integrated from Λ to Λ_0 using the boundary condition (69)-(70) together with (111) and we obtain

$$\begin{aligned} & \left| \partial^w \mathcal{S}_{l, n; r_1, r_2}^{\Lambda, \Lambda_0}(\vec{p}_n; \phi_{\tau_{1, s}, y_{1, s}}) \right| \\ & \leq (\Lambda + m)^{3-n-|w|-r_1-r_2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{s, l; \delta}^{\Lambda, 0}(\tau_{1, s}). \end{aligned} \quad (185)$$

- The relevant terms are those that correspond to $n + |w| + r_1 + r_2 \leq 3$. These read

$$\int_0^\infty dz_1 \int_0^\infty dz_2 z_1^{r_1} z_2^{r_2} \partial_\Lambda \mathcal{S}_{l, 2}^{\Lambda, \Lambda_0}((z_1, p), (z_2, -p)) \prod_{i=1}^s p_B(\tau_i; z_i, y_i), \quad (186)$$

where r_1, r_2 are integers such that $r_1 + r_2 \leq 1$, and $0 \leq s \leq 2$. We restrict our analysis to the case $s = 2$, the case $s = 1$ can be treated similarly and the case $s = 0$ will be integrated in the sequel. For $s = 2$, the relevant part is extracted from

$$\int_{z_1, z_2} \partial_\Lambda \mathcal{S}_{l, 2}^{\Lambda, \Lambda_0}((z_1, p), (z_2, -p)) \phi_1(z_1) \phi_2(z_2) \quad (187)$$

by performing a Taylor expansion of ϕ_1 and ϕ_2 around $z_i = 0$ at $p = 0$, where $\phi_i(z_i) := p_B(\tau_i; z_i, y_i)$, using (65) and (68). The bound (147) for $s = 0$ and $r_1 + r_2 \leq 1$ gives

$$\left| \partial_\Lambda s_l^{\Lambda, \Lambda_0} \right| \leq \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right), \quad \left| \partial_\Lambda e_l^{\Lambda, \Lambda_0} \right| \leq (\Lambda + m)^{-1} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right). \quad (188)$$

Integrating (188) from 0 to Λ and using the renormalization conditions (71), we have

$$|s_l^{\Lambda, \Lambda_0}| \leq (\Lambda + m) \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \quad |e_l^{\Lambda, \Lambda_0}| \leq \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right). \quad (189)$$

Applying Lemma 6 from the Appendix, we have

$$\begin{aligned} & \left| s_l^{\Lambda, \Lambda_0} \phi_1(0) \phi_2(0) + e_l^{\Lambda, \Lambda_0} \{ \phi_1(0) (\partial_n \phi_2)(0) + \phi_2(0) (\partial_n \phi_1)(0) \} \right| \\ & \leq (\Lambda + m) \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \end{aligned} \quad (190)$$

Now, we bound and integrate the remainder $\partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(\phi_1, \phi_2)$, which is irrelevant as we will see in the sequel, from Λ to Λ_0 . We distinguish between the two cases:

– $\underline{\Lambda \leq 3\sqrt{l}\tau^{-\frac{1}{2}}}$: Using (147) and (190), we deduce that $\partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(\phi_1, \phi_2)$ is bounded by

$$O(1) \max(m^2, \tau^{-1}) (\Lambda + m)^{-2} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \quad (191)$$

– $\underline{\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}}$: In the sequel, we restrict our analysis to the integration of the following terms, for which we need to proceed differently.

$$\mathcal{H}_1 := \left(\int_{z_1, z_2} z_1 z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0), \quad (192)$$

$$\mathcal{H}_2 := (\partial_n \phi_2)(0) \int_{z_1, z_2} z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1(tz_1)), \quad (193)$$

and

$$\begin{aligned} \mathcal{H}_3 := & \int_{z_1, z_2} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \left(\int_0^1 dt (1-t) \partial_t^2 \phi_1(tz_1) \right) \\ & \times \left(\int_0^1 dt' (1-t') \partial_{t'}^2 \phi_2(t'z_2) \right). \end{aligned} \quad (194)$$

The other terms which also contribute to $\partial_\Lambda l_{l,2}^{\Lambda, \Lambda_0}(\phi_1, \phi_2)$ can be treated similarly.

* We start first with \mathcal{H}_1 for which the bound (147) implies that

$$\left| \left(\int_{z_1, z_2} z_1 z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \right) \right| \leq (\Lambda + m)^{-2} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right). \quad (195)$$

Using Lemma 6, we obtain

$$\begin{aligned} & \left| \left(\int_{z_1, z_2} z_1 z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \right| \\ & \leq (\Lambda + m)^{-2} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \end{aligned} \quad (196)$$

* The term \mathcal{H}_2 : we have for $0 \leq t \leq 1$

$$\phi_i(tz_i) = \frac{1}{\sqrt{2\pi\tau_i}} e^{-\frac{(tz_i - y_i)^2}{2\tau_i}}. \quad (197)$$

Differentiating (197) twice w.r.t. t , we obtain

$$\partial_t^2 (\phi_i(tz_i)) = \frac{1}{t} \left[-\frac{z_i^2}{\tau_i} + \frac{z_i^2(tz_i - y_i)^2}{\tau_i^2} \right] p_B \left(\frac{\tau_i}{t^2}; z_i, \frac{y_i}{t} \right), \quad (198)$$

which implies that the term

$$\int_{z_1, z_2} z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) \partial_t^2 \phi_1(tz_1) \quad (199)$$

can be rewritten as

$$\sum_{(\alpha, \beta) \in \mathcal{S}_2} c_{\alpha\beta} \frac{y_1^\beta}{\tau_1^{1+\frac{\beta+\alpha}{2}}} \int_0^1 dt t^{\alpha-1} (1-t) \int_{z_1, z_2} z_2 z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) p_B \left(\frac{\tau_1}{t^2}; z_1, \frac{y_1}{t} \right), \quad (200)$$

where $\mathcal{S}_2 := \{(0, 0), (\alpha, \beta) \mid \alpha + \beta = 2, (\alpha, \beta) \in \mathbb{N}^2\}$, and the coefficients $c_{\alpha\beta} \in \mathbb{R}$ depend only on the exponents α and β . The bound (147) implies that the term

$$\int_{z_1, z_2} z_2 z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) p_B \left(\frac{\tau_1}{t^2}; z_1, \frac{y_1}{t} \right) \quad (201)$$

is bounded by

$$(\Lambda + m)^{-3-\alpha} e^{-\frac{m^2}{2\Lambda^2}} \mathcal{Q} \left(\frac{t\tau_1^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{1,l;\delta}^0 \left(\Lambda; \frac{\tau_1}{t^2}, \frac{y_1}{t} \right). \quad (202)$$

From Lemma 7, we obtain

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^\beta \mathcal{F}_{1,l;\delta}^0 \left(\Lambda; \frac{\tau_1}{t^2}, \frac{y_1}{t} \right) \leq O(1) t \left(1 + \frac{\sqrt{\tau_1}}{\Lambda} \right)^\beta \mathcal{F}_{1,l;\delta'}^{\Lambda, 0}(\tau_1), \quad (203)$$

where $0 < \delta < \delta'$. Using (173) together with (201) and (202), we deduce that (200) is bounded by

$$(\Lambda + m)^{-3} \tau_1^{-1} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{1,l;\delta'}^{\Lambda, 0}(\tau_1). \quad (204)$$

Lemma 6 together with (A3) imply that \mathcal{H}_2 is bounded by

$$(\Lambda + m)^{-3} \tau_1^{-1} \tau_2^{-\frac{1}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta'}^{\Lambda, 0}(\tau_{1,2}). \quad (205)$$

Following similar steps, we obtain

$$\begin{aligned} & \left| (\partial_n \phi_1)(0) \int_{z_1, z_2} z_2 \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2(tz_1)) \right| \\ & \leq (\Lambda + m)^{-3} \tau_1^{-\frac{1}{2}} \tau_2^{-1} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta'}^{\Lambda, 0}(\tau_{1,2}) \end{aligned} \quad (206)$$

and

$$\begin{aligned} & \left| \phi_i(0) \int_{z_1, z_2} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \int_0^1 dt (1-t) (\partial_t^2 \phi_j(tz_1)) \right| \\ & \leq (\Lambda + m)^{-2} \tau_j^{-1} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta'}^{\Lambda, 0}(\tau_{1,2}). \end{aligned} \quad (207)$$

* The term \mathcal{H}_3 : We start from (194) and (198) and write the term \mathcal{H}_3 as follows

$$\sum_{(\alpha, \beta) \in \mathcal{J}_2} \sum_{(\alpha', \beta') \in \mathcal{J}_2} c_{\alpha\beta} c_{\alpha'\beta'} \frac{y_1^\beta}{\tau_1^{1+\frac{\beta+\alpha}{2}}} \frac{y_2^{\beta'}}{\tau_2^{1+\frac{\beta'+\alpha'}{2}}} \int_0^1 dt dt' t^{\alpha-1} (1-t) t'^{\alpha'-1} (1-t') \int_{z_1, z_2} z_2^{2+\alpha'} z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) p_B\left(\frac{\tau_1}{t^2}; z_1, \frac{y_1}{t}\right) p_B\left(\frac{\tau_2}{t'^2}; z_2, \frac{y_2}{t'}\right). \quad (208)$$

The bound (147) implies that the term

$$\int_{z_1, z_2} z_2^{2+\alpha'} z_1^{2+\alpha} \partial_\Lambda \mathcal{S}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) p_B\left(\frac{\tau_1}{t^2}; z_1, \frac{y_1}{t}\right) p_B\left(\frac{\tau_2}{t'^2}; z_2, \frac{y_2}{t'}\right)$$

is bounded by

$$(\Lambda + m)^{-4-\alpha-\alpha'} e^{-\frac{m^2}{2\Lambda^2}} \mathcal{Q}\left(\frac{t\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta}^0\left(\Lambda; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'}\right).$$

From Lemma 8 and (173), we deduce that (208) is bounded by

$$\tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} (\Lambda + m)^{-2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0}(\tau_{1,2}), \quad (209)$$

where $0 < \delta < \delta' < 1$.

The boundary conditions (69) together with (68), (200) and (208) imply that

$$l_{l,2;R}^{\Lambda_0, \Lambda_0}(\phi_1, \phi_2) = 0.$$

Integrating from Λ to Λ_0 , we obtain for $\Lambda_0 \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$

$$l_{l,2}^{\Lambda, \Lambda_0}(\phi_1, \phi_2) = \int_\Lambda^{3\sqrt{l}\tau^{-\frac{1}{2}}} d\lambda \partial_\lambda l_{l,2}^{\lambda, \Lambda_0}(\phi_1, \phi_2) + \int_{3\sqrt{l}\tau^{-\frac{1}{2}}}^{\Lambda_0} d\lambda \partial_\lambda l_{l,2}^{\lambda, \Lambda_0}(\phi_1, \phi_2). \quad (210)$$

Using the bounds (191), (196), (205) and (209) together with (206) and (207), we obtain that the remainder $l_{l,2}^{\Lambda, \Lambda_0}$ is bounded by

$$(\Lambda + m) \max\left\{\frac{\tau^{-1}}{(\Lambda + m)^2}, \frac{m^2}{(\Lambda + m)^2}\right\} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{2,l;\delta'}^{\Lambda,0}(\tau_{1,2}). \quad (211)$$

For $\Lambda_0 \leq 3\sqrt{l}\tau^{-\frac{1}{2}}$, we conclude by integrating (191) from Λ to Λ_0 and we deduce

$$\left| \partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda, \Lambda_0}(\vec{0}; \phi_{\tau_{1,s}, y_{1,s}}) \right| \leq (\Lambda + m)^{1-|w|-r_1-r_2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{s,l;\delta'}^{\Lambda,0}(\tau_{1,s}), \quad (212)$$

where we used (112).

- Extension to general momenta: We now extend the bound (212) to general momenta using the Taylor formula with integral remainder, which reads

$$\partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda, \Lambda_0}(p, -p; \phi_{\tau_{1,s}, y_{1,s}}) = \partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda, \Lambda_0}(\vec{0}; \phi_{\tau_{1,s}, y_{1,s}}) + \int_0^1 dt \partial_t \partial^w \mathcal{S}_{l,2;r_1,r_2}^{\Lambda, \Lambda_0}(tp, -tp; \phi_{\tau_{1,s}, y_{1,s}}). \quad (213)$$

Applying this formula, the bound of the integrand (due to the derivative) yields an additional factor $(\Lambda + m)^{-1}$ which combines with the momentum produced by the t -derivative to give a bound as in (135).

This ends the proof of Theorem 1. \square

C. Proof of Proposition 4

Proof. The proof follows the same inductive scheme used in the proof of Theorem 1. For the tree order, we have

$$\mathcal{S}_{0,n;\star}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = 0, \quad \forall n \geq 2, \star \in \{D, R, N\}.$$

Clearly, the statement (136) holds.

A1) We start by verifying inductively the following statement

$$\partial_\Lambda \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) = \lim_{c \rightarrow +\infty} \partial_\Lambda \mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R). \quad (214)$$

In the sequel, we use the symbol \star to denote either Dirichlet or Robin boundary conditions. Given $\Pi = (\pi_1, \pi_2) \in \mathcal{P}_n$ such that $|\pi_i| = n_i$ and $n_1 + n_2 = n$, we introduce the following notations:

$$\begin{aligned} & \mathcal{S}_{l_i, n_i+1; \star}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; \star}; Y_{\pi_i}, u) \\ & := \int_{\vec{z}_{\pi_i}, z} \mathcal{S}_{l_i, n_i+1; \star}^{\Lambda, \Lambda_0}((\vec{z}_{\pi_i}, \vec{p}_{\pi_i}), (z, p)) \prod_{i \in \pi_i} p_\star(\tau_i; z_i, y_i) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right), \quad i \in \{1, 2\} \end{aligned} \quad (215)$$

and

$$\begin{aligned} \mathcal{S}_{l-1, n+2; \star}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; \star}; Y_{\sigma_n}, u, u) & := \int_{\vec{z}_n, z, z'} \mathcal{S}_{l-1, n+2; \star}^{\Lambda, \Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \\ & \times \phi_{\tau_{1,n}, y_{1,n}}^\star(z_{1,n}) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right) p_\star\left(\frac{1}{2\Lambda^2}; z', u\right), \end{aligned} \quad (216)$$

where

$$\Phi_{\pi_i}^{\Lambda; \star}(\vec{z}_{\pi_i}, z) = \phi_{\pi_i}^\star(\vec{z}_{\pi_i}) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right) \quad \text{with} \quad \phi_{\pi_i}^\star(\vec{z}_{\pi_i}) := \prod_{i \in \pi_i} p_\star(\tau_i; z_i, y_i),$$

and

$$\Phi_{n+2}^{\Lambda; \star}(z_{1,n}, z, z') = \phi_{\tau_{1,n}, y_{1,n}}^\star(z_{1,n}) p_\star\left(\frac{1}{2\Lambda^2}; z, u\right) p_\star\left(\frac{1}{2\Lambda^2}; z', u\right).$$

We consider the flows equations (62) smeared with the test functions $\phi_{\tau_{1,n}, y_{1,n}}^\star$ given by

$$\begin{aligned} & \partial_\Lambda \mathcal{S}_{l,n;\star}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^\star) \\ & = \frac{1}{2} \int_u \int_k \mathcal{S}_{l-1, n+2; \star}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; \star}; Y_{\sigma_n}, u, u) \dot{C}^\Lambda(k) \\ & + \frac{1}{2} \int_{z, z'} \int_k \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_{S, \star}^\Lambda(k; z, z') \phi_{\tau_{1,n}, y_{1,n}}^\star(z_{1,n}) \\ & - \frac{1}{2} \sum_{l_1, l_2} \sum_{\pi_1, \pi_2} \left[\int_u \left\{ \mathcal{S}_{l_1, n_1+1; \star}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_1}, -p; \Phi_{\pi_1}^{\Lambda; \star}; Y_{\pi_1}, u) \dot{C}^\Lambda(p) \right. \right. \\ & \quad \times \mathcal{S}_{l_2, n_2+1; \star}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_2}, p; \Phi_{\pi_2}^{\Lambda; \star}; Y_{\pi_2}, u) \\ & + \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_1}, -p; \Phi_{\pi_1}^{\Lambda; \star}; Y_{\pi_1}, u) \dot{C}^\Lambda(p) \mathcal{S}_{l_2, n_2+1; \star}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_2}, p; \Phi_{\pi_2}^{\Lambda; \star}; Y_{\pi_2}, u) \\ & \left. \left. + \mathcal{S}_{l_1, n_1+1; \star}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_1}, -p; \Phi_{\pi_1}^{\Lambda; \star}; Y_{\pi_1}, u) \dot{C}^\Lambda(p) \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_2}, p; \Phi_{\pi_2}^{\Lambda; \star}; Y_{\pi_2}, u) \right\} \right. \\ & \left. + \int_{z, z'} \mathcal{D}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(z; -p, \vec{p}_{1, n_1}; \phi_{\pi_1}^\star) \dot{C}_{S, \star}^\Lambda(p; z, z') \mathcal{D}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(z'; p, \vec{p}_{n_1+1, n}; \phi_{\pi_2}^\star) \right]_{rsym}, \\ & p = - \sum_{i \in \pi_1} p_i = \sum_{i \in \pi_2} p_i, \end{aligned} \quad (217)$$

where we used (115) and the notations (215)-(216). Here, the prime denotes all pairs (l_1, l_2) such that $l_1 + l_2 = l$, and the double prime refers to a summation over $(\pi_1, \pi_2) \in \tilde{\mathcal{P}}_{2;n}$ with $n_i := |\pi_i|$.

Using the induction hypothesis, we obtain

$$\mathcal{S}_{l-1, n+2; D}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; D}; Y_{\sigma_n}, u, u) = \lim_{c \rightarrow +\infty} \mathcal{S}_{l-1, n+2; R}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; R}; Y_{\sigma_n}, u, u), \quad (218)$$

$$\begin{aligned} \mathcal{S}_{l_i, n_i+1; D}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; D}; Y_{\pi_i}, u) \\ = \lim_{c \rightarrow +\infty} \mathcal{S}_{l_i, n_i+1; R}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; R}; Y_{\pi_i}, u), \quad i \in \{1, 2\}. \end{aligned} \quad (219)$$

For $\tau_i > 0$ and $y_i \in \mathbb{R}^+$, we have

$$\lim_{c \rightarrow +\infty} \mathcal{N}_p(p_D(\tau_i; \cdot, y_i) - p_R(\tau_i; \cdot, y_i)) = 0, \quad (220)$$

where for ϕ in $\mathcal{S}(\mathbb{R}^+)$ the semi-norm \mathcal{N}_p is given by

$$\mathcal{N}_p(\phi) = \sum_{\alpha, \beta \leq p} \sup_{x \in \mathbb{R}^+} |x^\alpha \partial^\beta \phi(x)|.$$

Remembering that $\mathcal{D}_{l, n}^{\Lambda, \Lambda_0} \in \mathcal{S}'(\mathbb{R}^{+n})$ and using (220), we deduce

$$\mathcal{D}_{l_i, n_i+1}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; D}; Y_{\pi_i}, u) = \lim_{c \rightarrow +\infty} \mathcal{D}_{l_i, n_i+1}^{\Lambda, \Lambda_0}(\vec{p}_{\pi_i}, (-1)^i p; \Phi_{\pi_i}^{\Lambda; R}; Y_{\pi_i}, u). \quad (221)$$

We rewrite the term

$$\int_k \int_{z, z'} \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}((\vec{z}_n, \vec{p}_n), (z, k), (z', -k)) \dot{C}_{S, R}^{\Lambda}(k; z, z') \phi_{\tau_1, n, y_1, n}^R(z_1, n) \quad (222)$$

as follows

$$\begin{aligned} \int_k \int_u \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; R}; Y_{\sigma_n}, u, u) \dot{C}^{\Lambda}(k) \\ - \int_k \int_{\mathbb{R}} du \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_1, n, y_1, n}^R \times p_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right) p_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right)) \dot{C}^{\Lambda}(k). \end{aligned} \quad (223)$$

Following the same steps that led to (221), we obtain

$$\mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; D}; Y_{\sigma_n}, u, u) = \lim_{c \rightarrow +\infty} \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \Phi_{n+2}^{\Lambda; R}; Y_{\sigma_n}, u, u) \quad (224)$$

and

$$\begin{aligned} \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_1, n, y_1, n}^D \times p_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right) p_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right)) \\ = \lim_{c \rightarrow +\infty} \mathcal{D}_{l-1, n+2}^{\Lambda, \Lambda_0}(\vec{p}_n, k, -k; \phi_{\tau_1, n, y_1, n}^R \times p_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right) p_B\left(\frac{1}{2\Lambda^2}; \cdot, u\right)). \end{aligned} \quad (225)$$

Part A1) in the proof of Theorem 1 implies that the integrands of each term on the RHS of the FEs (217), in the case of Robin boundary conditions are bounded independently of the Robin parameter c , and the Lemmas 1-3 show that these bounds are integrable w.r.t. u . We refer the reader to the proof of Theorem 1 for more details.

A2) Integration: Lebesgue's dominated convergence theorem together with (218)-(225) and the FEs (217) give

$$\partial_\Lambda \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) = \lim_{c \rightarrow +\infty} \partial_\Lambda \mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R). \quad (226)$$

This implies (again by the Lebesgue's dominated convergence theorem and the integrability of the bound (147) in the proof of Theorem 1)

$$\int_\Lambda^{\Lambda_0} d\lambda \partial_\lambda \mathcal{S}_{l,n;D}^{\lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) = \lim_{c \rightarrow +\infty} \int_\Lambda^{\Lambda_0} d\lambda \partial_\lambda \mathcal{S}_{l,n;R}^{\lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R). \quad (227)$$

– Irrelevant terms: These terms are characterized by $n \geq 4$. Using the boundary condition

$$\mathcal{S}_{l,n;\star}^{\Lambda_0,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^\star) = 0, \quad \star \in \{D, R\} \quad (228)$$

together with (227), we deduce

$$\mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) = \lim_{c \rightarrow +\infty} \mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R). \quad (229)$$

– Relevant terms ($n = 2$): We have

$$\begin{aligned} \int_\Lambda^{\Lambda_0} d\lambda \partial_\lambda \mathcal{S}_{l,2;R}^{\lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R) \\ = \mathcal{S}_{l,2;R}^{\Lambda_0,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R) - \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^R) \end{aligned} \quad (230)$$

and

$$\int_\Lambda^{\Lambda_0} d\lambda \partial_\lambda \mathcal{S}_{l,2;D}^{\lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D) = -\mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,n}, y_{1,n}}^D). \quad (231)$$

In (231), we used the boundary condition (72) for the Dirichlet case. The boundary condition (69) implies

$$\begin{aligned} \mathcal{S}_{l,2;R}^{\Lambda_0,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^R) &= s_{l;R}^{\Lambda_0,\Lambda_0} \prod_{i=1}^2 p_R(\tau_i; y_i; 0) \\ &+ e_{l;R}^{\Lambda_0,\Lambda_0} \{p_R(\tau_1; y_1; 0) \partial_n p_R(\tau_2; y_2; 0) + p_R(\tau_2; y_2; 0) \partial_n p_R(\tau_1; y_1; 0)\}. \end{aligned} \quad (232)$$

Using

$$|\partial_n p_R(\tau_1; y_1; 0)| \leq O(1) \tau_i^{-\frac{1}{2}} p_B(\tau_i, \delta; y_i; 0),$$

and the fact that $s_{l;R}^{\Lambda_0,\Lambda_0}$ and $e_{l;R}^{\Lambda_0,\Lambda_0}$ are uniformly bounded w.r.t. the Robin parameter c , which is implied by the bound given in Theorem 1 for $s = 1$, $r_1 \in \{0, 1\}$ and $r_2 = 0$, we obtain from $p_R \rightarrow_{c \rightarrow +\infty} p_D$

$$\lim_{c \rightarrow +\infty} \mathcal{S}_{l,2;R}^{\Lambda_0,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^R) = 0. \quad (233)$$

Therefore, we deduce

$$\mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^D) = \lim_{c \rightarrow +\infty} \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0}(\vec{p}_n; \phi_{\tau_{1,2}, y_{1,2}}^R). \quad (234)$$

This ends the proof of (136). □

D. Proof of Corollary 1

Proof. In this part, we prove the bounds (137) and (138). As a consequence of Theorem 1, we have for Robin boundary conditions

$$\begin{aligned} & \left| \mathcal{S}_{l,n;R}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \phi_{\tau_{1,n},y_{1,n}}^R \right) \right| \\ & \leq (\Lambda + m)^{3-n} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{n,l;\delta}^{\Lambda,0}(\tau_{1,n}), \quad \forall n \geq 2. \end{aligned} \quad (235)$$

Using Theorem 1 and taking the limit $c \rightarrow +\infty$, we deduce

$$\begin{aligned} & \left| \mathcal{S}_{l,n;D}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \phi_{\tau,y_{1,s}}^D \right) \right| \\ & \leq (\Lambda + m)^{3-n} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{n,l;\delta}^{\Lambda,0}(\tau_{1,n}), \quad \forall n \geq 2. \end{aligned} \quad (236)$$

For $n = 2$, it is possible to obtain a sharper bound by performing a Taylor expansion around 0 of the test functions $p_R(\tau_i; \cdot, y_i)$ as follows

$$\begin{aligned} \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \phi_{\tau_{1,2},y_{1,2}}^R \right) &= s_{l;R}^{\Lambda,\Lambda_0} p_R(\tau_1; y_1, 0) p_R(\tau_2; y_2, 0) + e_{l;R}^{\Lambda,\Lambda_0} \{ p_R(\tau_1; y_1, 0) (\partial_n p_R)(\tau_2; y_2, 0) \\ & \quad + p_R(\tau_2; y_2, 0) (\partial_n p_R)(\tau_1; y_1, 0) \} + l_{l,2;R}^{\Lambda,\Lambda_0} (p_R(\tau_1; \cdot, y_1), p_R(\tau_2; \cdot, y_2)). \end{aligned} \quad (237)$$

Taking the limit $c \rightarrow \infty$, we deduce

$$\mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \phi_{\tau_{1,2},y_{1,2}}^D \right) = \lim_{c \rightarrow +\infty} \tilde{l}_{l,2;R}^{\Lambda,\Lambda_0} (p_R(\tau_1; \cdot, y_1), p_R(\tau_2; \cdot, y_2)), \quad (238)$$

where the remainder $\tilde{l}_{l,2;R}^{\Lambda,\Lambda_0}$ is given by

$$\begin{aligned} & \left(\int_{z_1, z_2} z_1 z_2 \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) (\partial_n \phi_1)(0) (\partial_n \phi_2)(0) \\ & + \int_{z_1, z_2} \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \left(\int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \right) \times \left(\int_0^1 dt' (1-t') (\partial_{t'}^2 \phi_2)(t'z_2) \right) \\ & + (\partial_n \phi_2)(0) \int_{z_1, z_2} z_2 \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \int_0^1 dt (1-t) (\partial_t^2 \phi_1)(tz_1) \\ & + (\partial_n \phi_1)(0) \int_{z_1, z_2} z_1 \mathcal{S}_{l,2;R}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \int_0^1 dt (1-t) (\partial_t^2 \phi_2)(tz_2) \end{aligned} \quad (239)$$

and $\phi_i(z_i) := p_R(\tau_i; z_i, y_i)$. These terms can be bounded in a similar way as \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 in the proof of Theorem 1. One should keep in mind that the test functions considered within the proof of Theorem 1 were products of bulk heat kernels. However, the same bounds (192), (205) and (194) which are uniform in c , hold for Robin type test functions using the bounds (17). Therefore, we deduce

$$\left| \tilde{l}_{l,2;R}^{\Lambda,\Lambda_0}(\phi_1, \phi_2) \right| \leq (\Lambda + m)^{-2} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}), \quad (240)$$

which gives

$$\left| \mathcal{S}_{l,2;D}^{\Lambda,\Lambda_0} \left(\vec{p}_n; \phi_{\tau_{1,2},y_{1,2}}^R \right) \right| \leq (\Lambda + m)^{-2} \tau_1^{-\frac{1}{2}} \tau_2^{-\frac{1}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}), \quad (241)$$

and this ends the proof of Corollary 1. \square

E. The minimal form of the bare interaction

In this section, we show that the bare interaction (1) corresponds to the boundary conditions imposed in Theorem 1 for $L_\star^{\Lambda_0, \Lambda_0}$. Given $\phi \in \text{supp } \mu_\star^{\Lambda_0, \Lambda_0}$, we expand $L_\star^{\Lambda_0, \Lambda_0}(\phi)$ in powers of the field ϕ :

$$L_\star^{\Lambda_0, \Lambda_0}(\phi) = \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\vec{z}_n} \int_{\mathbb{R}^{3n}} \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \mathcal{S}_{l,n;\star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \times \delta^{(3)}(p_1 + \dots + p_n) \phi(z_1, p_1) \cdots \phi(z_n, p_n). \quad (242)$$

Using (142), we can write

$$L_\star^{\Lambda_0, \Lambda_0}(\phi) = D^{\Lambda_0, \Lambda_0}(\phi) + S_\star^{\Lambda_0, \Lambda_0}(\phi),$$

where

$$D^{\Lambda_0, \Lambda_0}(\phi) := \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\vec{z}_n} \int_{\mathbb{R}^{3n}} \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \mathcal{D}_{l,n}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \delta^{(3)}(p_1 + \dots + p_n) \times \phi(z_1, p_1) \cdots \phi(z_n, p_n) \quad (243)$$

and

$$S_\star^{\Lambda_0, \Lambda_0}(\phi) = \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\vec{z}_n} \int_{\mathbb{R}^{3n}} \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3} \mathcal{S}_{l,n;\star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \delta^{(3)}(p_1 + \dots + p_n) \times \phi(z_1, p_1) \cdots \phi(z_n, p_n). \quad (244)$$

Proposition 1 in⁶ implies that there exists f in $L^2(\mathbb{R}^+)$ such that

$$\phi(z, p) = \int_0^\infty dz' C_\star^{\Lambda_0, \Lambda_0}(p; z, z') f(p, z'),$$

which can be rewritten as

$$\phi(p, z) = \int_0^\infty dz' \int_{\frac{1}{\Lambda^2}}^{\frac{1}{\Lambda_0^2}} d\lambda e^{-\lambda(p^2 + m^2)} p_\star(\lambda; z, z') f(p, z').$$

Therefore, we can write

$$\begin{aligned} & \int_{z_1, \dots, z_n} \mathcal{S}_{l,n;\star}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi(z_1, p_1) \cdots \phi(z_n, p_n) \\ &= \int_{z'_n} \prod_{i=1}^n \int_{\frac{1}{\Lambda^2}}^{\frac{1}{\Lambda_0^2}} d\lambda_i e^{-\lambda_i(p_i^2 + m^2)} f(p_i, z'_i) \mathcal{S}_{l,n;\star}^{\Lambda_0, \Lambda_0}(\vec{p}_n; \phi_{\lambda_{1,n}, z'_{1,n}}^\star). \end{aligned} \quad (245)$$

The boundary conditions (70) and (72) imply that

$$S_\star^{\Lambda_0, \Lambda_0}(\phi) = \frac{1}{2} \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) \phi(z_1, p) \phi(z_2, -p), \quad (246)$$

where $\mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p))$ is given by (69) for Robin/Neumann boundary conditions and by (72) for Dirichlet boundary conditions:

- Robin/Neumann boundary conditions ($c \geq 0$): In this case, we obtain

$$S_\star^{\Lambda_0, \Lambda_0}(\phi) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2} s_\star^{\Lambda_0} + c e_\star^{\Lambda_0} \right) \phi(0, p) \phi(0, -p), \quad (247)$$

where

$$s_\star^{\Lambda_0} = \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)), \quad e_\star^{\Lambda_0} = \int_{z_1, z_2} z_1 \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) \quad (248)$$

and $\star \in \{R, N\}$.

- Dirichlet boundary conditions: For Dirichlet boundary conditions, we obtain

$$S_D^{\Lambda_0, \Lambda_0}(\phi) = 0.$$

VII. THE AMPUTATED VS THE NON-AMPUTATED THEORY

For quantum field theories on spaces without boundary, the renormalization problem of the amputated and the non-amputated theory is equivalent in the sense that the required counter-terms render finite the amputated and unamputated amplitudes, independently of the location of the external points of the unamputated diagrams^{20,23}. However, first order calculations¹⁰ give clear evidence that this is not the case when one considers the renormalization of the semi-infinite model. The tadpole diverges w.r.t. the UV cutoff, and its renormalization depends on the location of the external points (i.e. if they are on the surface or not). If the two external points are not on the surface, then in addition to the usual mass counter-term only one additional surface counter-term, which diverges linearly in the UV cutoff, is needed. This is not the case when one considers the tadpole with at least one external point on the surface. The latter needs one additional surface counter-term which diverges logarithmically w.r.t. the UV cutoff. This suggests that the amputated and non-amputated diagrams are renormalized differently for the semi-infinite model. In this section, we prove the following proposition which sheds some light on this finding to all loop orders in perturbation theory:

Proposition 5. *Let $\star \in \{R, N\}$. We denote by C_\star the unregularized propagator associated to the boundary condition \star . For nonvanishing $s_{l;\star}^{\Lambda_0}$ and $e_{l;\star}^{\Lambda_0}$, we have for $y_2 > 0$*

$$\begin{aligned} \lim_{y_1 \rightarrow 0^+} \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) \\ \neq \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, 0) C_R(p; z_2, y_2) \end{aligned} \quad (249)$$

and

$$\begin{aligned} \lim_{y_1 \rightarrow 0^+} \lim_{y_2 \rightarrow 0^+} \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) \\ \neq \int_{z_1, z_2} \mathcal{S}_{l,2;\star}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, 0) C_R(p; z_2, 0). \end{aligned} \quad (250)$$

Proof. We give the proof of Proposition 5 in the case of Robin boundary conditions. Neumann b.c. can be treated analogously.

We proved in Section 3 that for the particular choice of the boundary conditions (69)-(71), the bare interaction is of the form (1) and we have

$$\mathcal{S}_{l,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) = \left(s_R^{\Lambda_0} + e_R^{\Lambda_0} (\partial_{z_1} + \partial_{z_2}) \right) \delta_{z_1} \delta_{z_2}. \quad (251)$$

Hence, we obtain for $y_1, y_2 > 0$

$$\begin{aligned} \int_{z_1, z_2} \mathcal{S}_{l,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) = s_R^{\Lambda_0} C_R(p; 0, y_1) C_R(p; 0, y_2) \\ + e_R^{\Lambda_0} \{ C_R(p; 0, y_1) \partial_n C_R(p; 0, y_2) + C_R(p; 0, y_2) \partial_n C_R(p; 0, y_1) \}. \end{aligned} \quad (252)$$

Using

$$\lim_{z \rightarrow 0} \lim_{y \rightarrow 0} \partial_z C_R(p; z, y) = -\kappa_p C_R(p; 0, 0), \quad \lim_{y \rightarrow 0} \lim_{z \rightarrow 0} \partial_z C_R(p; z, y) = c C_R(p; 0, 0) \quad (253)$$

with $\kappa_p := \sqrt{p^2 + m^2}$, we deduce

$$\begin{aligned} \int_{z_1, z_2} \mathcal{S}_{l,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, y_2) \\ = \left(s_R^{\Lambda_0} + 2c e_R^{\Lambda_0} \right) C_R(p; 0, y_1) C_R(p; 0, y_2), \quad \forall y_1, y_2 > 0, \end{aligned} \quad (254)$$

$$\begin{aligned} & \int_{z_1, z_2} \mathcal{S}_{1,2;R}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, y_1) C_R(p; z_2, 0) \\ &= \left(s_R^{\Lambda_0} + 2c e_R^{\Lambda_0} \right) C_R(p; 0, y_1) C_R(p; 0, 0) - e_R^{\Lambda_0} C_R(p; 0, y_1) (\kappa_p + c) C_R(p; 0, 0), \end{aligned} \quad (255)$$

$$\begin{aligned} & \int_{z_1, z_2} \mathcal{S}_{1,2}^{\Lambda_0, \Lambda_0}((z_1, p), (z_2, -p)) C_R(p; z_1, 0) C_R(p; z_2, 0) \\ &= \left(s_R^{\Lambda_0} + 2c e_R^{\Lambda_0} \right) C_R(p; 0, 0) C_R(p; 0, 0) - 2 e_R^{\Lambda_0} C_R(p; 0, 0) (\kappa_p + c) C_R(p; 0, 0), \end{aligned} \quad (256)$$

from which (249) and (250) follow directly. \square

Remarks 4. Denoting $g_R^{\Lambda_0} = s_R^{\Lambda_0} + 2c e_R^{\Lambda_0}$, we deduce that (254) implies that the unamputated two-point function of the semi-infinite model which has two external points in the interior of the bulk, requires only the surface counter-term $g_R^{\Lambda_0}$ to be renormalized. This is not the case when at least one of the external points is on the surface. From (255) and (256), we deduce that $g_R^{\Lambda_0}$ is not sufficient and the additional surface counter-term $e_R^{\Lambda_0}$ is required to make the two-point function finite. This generalizes the remarks given in^{11,14} and¹⁰ concerning the tadpole to all loop orders.

Appendix A: Some properties of the surface weight factor for $s = 2$ and $l \geq 1$

In this Appendix, we prove several lemmas that we use in the proof of Theorem 1. These lemmas concern the case $s = 2$ for which the set of partitions \mathcal{P}_2 simply reads

$$\mathcal{P}_2 := \{\Pi_0, \Pi_1\},$$

where $\Pi_0 := \sigma_2$, $\Pi_1 = \{\pi_1, \pi_2\}$ and $\pi_i = \{i\}$.
From the definition (81), we have

$$\mathcal{W}_l^2(\sigma_2) = \left\{ T_l^{2,0}(Y_{\sigma_2}, 0; \vec{z}) \mid T_l^{2,0} \in \mathcal{T}^{2,0}, \quad v_2 \leq 3l - 1 \right\}$$

and

$$\mathcal{W}_l^2(\Pi_1) = \left\{ T_{l;1}^{1,0}(y_1, 0; \vec{z}) \cup T_{l;2}^{1,0}(y_2, 0; \vec{z}) \mid T_{l;1}^{1,0}, T_{l;2}^{1,0} \in \mathcal{T}_l^{1,0} \right\},$$

which implies that the global surface weight factor $\mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2})$ simply reads

$$\mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}) = \sum_{T_l^{2,0} \in \mathcal{W}_l^2(\sigma_2)} \mathcal{F}_\delta^0(\Lambda, \tau_1, \tau_2; T_l^{2,0}; Y_{\sigma_2}) + \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_2), \quad (\text{A1})$$

where

$$\mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i) := \sum_{T_l^{1,0} \in \mathcal{T}_l^{1,0}} \mathcal{F}_\delta^0(\Lambda, 2\tau_i; T_l^{1,0}; y_i). \quad (\text{A2})$$

Note that (A1) implies

$$\mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_2) \leq \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \quad (\text{A3})$$

Lemma 4. *Let v be the total number of vertices of incidence number 2 of the surface tree $T_l^{2,0}$. For $p \geq 1$, we denote by $\vec{z} = (z_1, \dots, z_p)$ the set of the internal vertices of $T_l^{2,0}$ and $(y_1, y_2) \in (\mathbb{R}^+)^2$ its external vertices. For $\Lambda_{\mathcal{J}} := \{\Lambda_i \mid 1 \leq i \leq v-1\}$, $\tilde{\Lambda} \in [\Lambda, \Lambda_0]$ and $\tau_1, \tau_2 > 0$, we have*

$$\int_{\vec{z}} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2}) \leq \int_0^\infty dz p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \quad (\text{A4})$$

and

$$\int_0^\infty dz p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \leq 2^v \int_{\vec{z}} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2}), \quad (\text{A5})$$

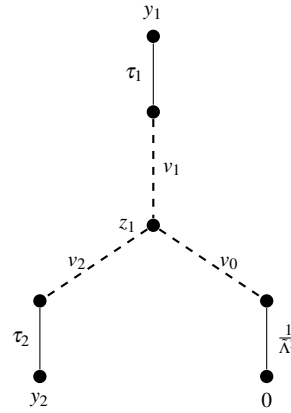
where $0 \leq v_1, v_2, v_0 \leq v$ such that $v_1 + v_2 + v_0 = v$, $c_{1,\delta} := (1+\delta)c_1$ and $c_{2,\delta} := (1+\delta)c_2$. The parameters c_1 , c_2 and $\tilde{\Lambda}_1$ are given by

$$c_1 = \tau_1 + \left(\sum_{i=1}^{v_1} \frac{1}{\Lambda_i^2} \right) (1 - \delta_{v_1,0}), \quad (\text{A6})$$

$$c_2 = \tau_2 + \left(\sum_{i=v_1+1}^{v_1+v_2-1} \frac{1}{\Lambda_i^2} \right) (1 - \delta_{v_2,0}), \quad (\text{A7})$$

$$\frac{1}{\tilde{\Lambda}_1^2} = \left(\sum_{i=v_1+v_2}^{v-1} \frac{1}{\Lambda_i^2} \right) (1 - \delta_{v,0}) + \frac{1}{\Lambda^2}. \quad (\text{A8})$$

Proof. • First, we prove the bound (A4). A tree $T_l^{2,0}$ in $\mathcal{W}_l^2(\sigma_2)$ is of the following form



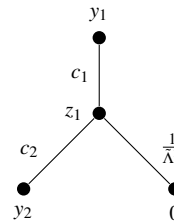
It contains one internal vertex of incidence number 3 and all the other internal vertices are of incidence number 2. We assume that each dashed line contains a number $v_i \geq 1$ of internal vertices of incidence number 2. The case $v_i = 0$ can be treated similarly. Remember that $v_0 + v_1 + v_2 = v$. Let $\{z_2, \dots, z_{v_1+1}\}$, $\{z_{v_1+2}, \dots, z_{v_1+v_2}\}$ and $\{z_{v_1+v_2+1}, \dots, z_v\}$ be respectively the internal vertices on the paths from z_1 to y_1 , z_1 to y_2 and z_1 to 0 . From (103), the integral surface weight factor of $T_l^{2,0}$ is then given by

$$\begin{aligned} \int_{\vec{z}} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2}) &= \int_{z_1, \dots, z_v} \prod_{j=2}^{v_1+1} p_B\left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j\right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1) \\ &\times p_B\left(\frac{1+\delta}{\Lambda_{v_1+1}^2}; z_1, z_{v_1+2}\right) \prod_{j=v_1+3}^{v_1+v_2} p_B\left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j\right) p_B(\tau_{2,\delta}; z_{v_1+v_2}, y_2) \\ &\times p_B\left(\frac{1+\delta}{\Lambda_{v_1+v_2}^2}; z_1, z_{v_1+v_2+1}\right) \prod_{j=v_1+v_2+2}^v p_B\left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j\right) p_B\left(\frac{1+\delta}{\tilde{\Lambda}^2}; z_v, 0\right). \quad (\text{A9}) \end{aligned}$$

Bounding the integral over \mathbb{R}^+ by the integral over \mathbb{R} and using (10), we obtain

$$\int_{z_2 \dots z_{v_1+1}} \prod_{j=2}^{v_1+1} p_B\left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j\right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1) \leq p_B(c_{1,\delta}; z_1, y_1), \quad (\text{A10})$$

where $c_{1,\delta} = (1+\delta) \left(\tau_1 + \sum_{i=1}^{v_1} \frac{1}{\Lambda_i^2} \right)$. Proceeding similarly on the paths from z_1 to y_2 and from z_1 to 0 , we obtain that the weight factor of a tree in $\mathcal{W}_l^2(\sigma_2)$ is bounded by the weight factor of the tree



where c_1 , c_2 and $\frac{1}{\Lambda_1^2}$ are the new parameters associated respectively to the edges (z_1, y_1) , (z_1, y_2) and $(z_1, 0)$. The relations between these new parameters and those of the tree $T_l^{2,0}$ are

given by

$$c_2 = \tau_2 + \left(\sum_{i=1}^{v_2} \frac{1}{\Lambda_{i+v_1}^2} \right) \quad \text{and} \quad \frac{1}{\tilde{\Lambda}_1^2} = \left(\sum_{i=v_1+v_2}^{v-1} \frac{1}{\Lambda_i^2} \right) + \frac{1}{\tilde{\Lambda}^2}.$$

This proves the statement (A4).

- To prove (A5), we assume without loss of generality that $v_1 \geq 1$. Using (13), we find

$$\begin{aligned} p_B(c_{1,\delta}; z_1, y_1) &= \int_{\mathbb{R}^{v_1}} dz_2 \cdots dz_{v_1+1} \prod_{j=2}^{v_1+1} p_B \left(\frac{1+\delta}{\Lambda_{j-2}^2}; z_j, z_{j-1} \right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1) \\ &\leq 2^{v_1} \int_{(\mathbb{R}^+)^{v_1}} dz_2 \cdots dz_{v_1+1} \prod_{j=2}^{v_1+1} p_B \left(\frac{1+\delta}{\Lambda_{j-2}^2}; z_j, z_{j-1} \right) p_B(\tau_{1,\delta}; z_{v_1+1}, y_1). \end{aligned} \quad (\text{A11})$$

Proceeding similarly for $p_B(c_{2,\delta}; z_1, y_2)$ and $p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right)$, we deduce

$$\begin{aligned} \int_0^\infty dz p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \\ \leq 2^v \int_{\vec{z}} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_1, \tau_2; T_l^{2,0}; \vec{z}; Y_{\sigma_2}). \end{aligned} \quad (\text{A12})$$

□

Lemma 5. Let $W_l^2(\Pi_1) := T_{l;1}^{1,0}(y_1, 0; \vec{z}) \cup T_{l;2}^{1,0}(y_2, 0; \vec{z}')$ be a forest in $\mathcal{W}_l^2(\Pi_1)$ with $(y_1, y_2) \in (\mathbb{R}^+)^2$ and $v_{2,1}$ (resp. $v_{2,2}$) the total number of vertices of incidence number 2 of the tree $T_{l;1}^{1,0}$ (resp. $T_{l;2}^{1,0}$). For $p, q \geq 1$, we denote by $\vec{z} = (z_1, \dots, z_p)$ (resp. $\vec{z}' = (z'_1, \dots, z'_q)$) the set of the internal vertices of $T_{l;1}^{1,0}$ (resp. $T_{l;2}^{1,0}$). For $\Lambda_{\mathcal{J}} := \{\Lambda_i, \Lambda'_j \mid 1 \leq i \leq v_{2,1} - 1, 1 \leq j \leq v_{2,2} - 1\}$, $\tilde{\Lambda}_1, \tilde{\Lambda}_2 \in [\Lambda, \Lambda_0]$ and $\tau_1, \tau_2 > 0$, we have

$$\int_{\vec{z}} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_1, \tau_2; W_l^2(\Pi_1); \vec{z}; Y_{\sigma_2}) \leq p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0) \quad (\text{A13})$$

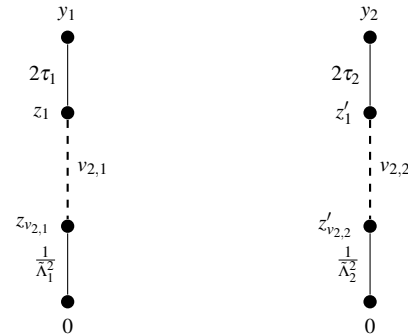
and

$$p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0) \leq 2^{v_{2,1}+v_{2,2}} \int_{\vec{z}} \mathcal{F}_\delta^0(\Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_1, \tau_2; W_l^2(\Pi_1); \vec{z}; Y_{\sigma_2}), \quad (\text{A14})$$

where \tilde{c}_1 and \tilde{c}_2 are given by

$$\begin{aligned} \tilde{c}_1 &= 2\tau_1 + \sum_{i=1}^{v_{2,1}-1} \frac{1}{\Lambda_{i,1}^2} + \frac{1}{\tilde{\Lambda}_1^2}, \\ \tilde{c}_2 &= 2\tau_2 + \sum_{i=1}^{v_{2,2}-1} \frac{1}{\Lambda_{i,2}^2} + \frac{1}{\tilde{\Lambda}_2^2}. \end{aligned}$$

Proof. The forest $W_l^2(\Pi_1)$ is of the following form



The integrated weight factor of this forest reads

$$\begin{aligned} & \int_{\vec{z}} \mathcal{F}_\delta^0 (\Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_{1,2}; W_l^2(\Pi_1); \vec{z}; y_1, y_2) \\ &= \int_0^\infty dz_1 \cdots dz_{v_{2,1}} \prod_{j=2}^{v_{2,1}} p_B \left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_j, z_{j-1} \right) p_B(2\tau_{1,\delta}; z_1, y_1) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z_{v_{2,1}}, 0 \right) \\ & \times \int_0^\infty dz'_1 \cdots dz'_{v_{2,2}} \prod_{j=2}^{v_{2,2}} p_B \left(\frac{1+\delta}{\Lambda_{j-1}^{\prime 2}}; z'_j, z'_{j-1} \right) p_B(2\tau_{2,\delta}; z'_1, y_2) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_2^2}; z'_{v_{2,2}}, 0 \right). \end{aligned}$$

Bounding the integral over \mathbb{R}^+ by the integral over \mathbb{R} and using (10), we obtain

$$\int_{\vec{z}} \mathcal{F}_\delta^0 (\Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_{1,2}; W_l^2(\Pi_1); \vec{z}; y_1, y_2) \leq p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0),$$

where for $v_{2,i} > 1$

$$\tilde{c}_1 = 2\tau_1 + \sum_{i=1}^{v_{2,1}-1} \frac{1}{\Lambda_{i,1}^2} + \frac{1}{\tilde{\Lambda}_1^2}, \quad (\text{A15})$$

$$\tilde{c}_2 = 2\tau_2 + \sum_{i=1}^{v_{2,2}-1} \frac{1}{\Lambda_{i,2}^2} + \frac{1}{\tilde{\Lambda}_2^2}. \quad (\text{A16})$$

If $v_{2,i} = 1$, then $\tilde{c}_i = 2\tau_i + \frac{1}{\tilde{\Lambda}_i^2}$.

Using again (13) and proceeding as in (A11), we deduce

$$p_B(\tilde{c}_{1,\delta}; y_1, 0) p_B(\tilde{c}_{2,\delta}; y_2, 0) \leq 2^{v_{2,1}+v_{2,2}} \int_{\vec{z}} \mathcal{F}_\delta^0 (\Lambda_{\mathcal{J}}, \tilde{\Lambda}_1, \tilde{\Lambda}_2; \tau_{1,2}; W_l^2(\Pi_1); \vec{z}; y_1, y_2). \quad (\text{A17})$$

□

Lemma 6. For $0 \leq \alpha \leq 1$ and $y_1, y_2 \in \mathbb{R}$, we have

$$|\partial_n^\alpha \phi_i(0)| \leq C_{0,\delta} \tau_i^{-\frac{\alpha}{2}} \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i), \quad \forall 0 < \delta < 1, \quad (\text{A18})$$

where $\partial_n^\alpha \phi_i(0) = \lim_{z_i \rightarrow 0^+} \partial_{z_i}^\alpha \phi_i(z_i)$ with $\phi_i(z_i) := p_B(\tau_i; z_i, y_i)$ and $C_{0,\delta}$ is defined in (15).

Proof. For $\alpha = 0$, we have

$$\phi_i(0) = \frac{1}{\sqrt{2\pi\tau_i}} e^{-\frac{y_i^2}{2\tau_i}} \leq \sqrt{2} p_B(2\tau_i; y_i, 0). \quad (\text{A19})$$

For $\alpha = 1$, we have $|\partial_n \phi_i(0)| = \frac{y_i}{\tau_i} \phi_i(0)$. Using the bound (15) for $r = 1$, we obtain

$$|\partial_n \phi_i(0)| \leq \sqrt{2} C_{0,\delta} \tau_i^{-\frac{1}{2}} p_B(2\tau_i; y_i, 0). \quad (\text{A20})$$

We consider the surface tree $T_{l;i}^{1,0}$ which consists of the external vertex y_i and the surface external vertex 0. We associate to the external line $(0, y_i)$ the parameter $2\tau_i$. The integrated surface weight factor of the tree $T_{l;i}^{1,0}$ then reads

$$\mathcal{F}_{1,l;\delta}^0(2\tau_i; T_{l;i}^{1,0}; y_i) = p_B(2\tau_i; y_i, 0). \quad (\text{A21})$$

Since $T_{l;i}^{1,0} \in \mathcal{W}_l^1(\pi_1)$, we obtain using (A2)

$$\mathcal{F}_{1,l;\delta}^0(2\tau_i; T_{l;i}^{1,0}; y_i) \leq \mathcal{F}_{1,l;\delta}^0(\tau_i). \quad (\text{A22})$$

Combining the bounds (A19), (A20) and (A22), we deduce

$$|\partial_n^\alpha \phi_i(0)| \leq \sqrt{2} C_{0,\delta} \tau_i^{-\frac{\alpha}{2}} \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i), \quad \alpha \in \{0,1\}. \quad (\text{A23})$$

Furthermore, we also obtain

$$\left| \partial_n^\alpha \phi_1(0) \partial_n^\beta \phi_2(0) \right| \leq 2 C_{0,\delta}^2 \tau_1^{-\frac{\alpha}{2}} \tau_2^{-\frac{\beta}{2}} \prod_{i=1}^2 \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_i), \quad \alpha, \beta \in \{0,1\}. \quad (\text{A24})$$

Recalling (A3), we deduce

$$\left| \partial_n^\alpha \phi_1(0) \partial_n^\beta \phi_2(0) \right| \leq 2 C_{0,\delta}^2 \tau_1^{-\frac{\alpha}{2}} \tau_2^{-\frac{\beta}{2}} \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}). \quad (\text{A25})$$

□

Lemma 7. For $0 < t \leq 1$, $\gamma \in \mathbb{N}$, $y_1 \in \mathbb{R}$ and $0 < \delta < \delta' < 1$, we have

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^\gamma \mathcal{F}_{1,l;\delta}^{\Lambda,0}(t, \tau_1) \leq O(1) t \left(1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^\gamma \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1), \quad (\text{A26})$$

where

$$\mathcal{F}_{1,l;\delta}^{\Lambda,0}(t, \tau_1) := \sum_{T_l^{1,0} \in \mathcal{T}_l^{1,0}} \mathcal{F}_\delta^0 \left(\Lambda, \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) = \sum_{v=0}^{3l-1} \mathcal{F}_\delta^0 \left(\Lambda, \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right), \quad (\text{A27})$$

and $O(1)$ is a constant which depends on δ , δ' , v , γ and the loop order l .

Proof. We consider the surface tree $T_l^{1,0} \in \mathcal{T}_l^{1,0}$ with the external vertex $\frac{y_1}{t}$ and the internal vertices $\vec{z}_v = (z_1, \dots, z_v)$. Let v be the number of its vertices of incidence number 2 and $\Lambda_{\mathcal{J}} = (\Lambda_i)_{1 \leq i \leq v-1}$, Λ_v and τ_1/t^2 be respectively the parameters associated to the internal lines, the surface external line and the external line of $T_l^{1,0}$. Then the integrated surface weight factor of $T_l^{1,0}$ reads

$$\begin{aligned} & \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \\ &= \int_{\vec{z}_v} p_B \left(\frac{2\tau_{1,\delta}}{t^2}; \frac{y_1}{t}, z_1 \right) \prod_{i=1}^{v-1} p_B \left(\frac{1+\delta}{\Lambda_i^2}; z_i, z_{i+1} \right) p_B \left(\frac{1+\delta}{\Lambda_v^2}; z_v, 0 \right). \end{aligned} \quad (\text{A28})$$

If $y_1 \in \mathbb{R}^-$, (A28) is bounded by

$$p_B \left(\frac{2\tau_{1,\delta}}{t^2}; \frac{y_1}{t}, 0 \right) \int_{\vec{z}_v} \prod_{i=1}^{v-1} p_B \left(\frac{1+\delta}{\Lambda_i^2}; z_i, z_{i+1} \right) p_B \left(\frac{1+\delta}{\Lambda_v^2}; z_v, 0 \right). \quad (\text{A29})$$

Therefore, we obtain

$$\begin{aligned} & \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \\ & \leq O(1) t p_B(2\tau_{1,\delta}; y_1, 0) \leq O(1) t \mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_1), \quad \forall T_l^{1,0} \in \mathcal{T}_l^{1,0}. \end{aligned} \quad (\text{A30})$$

Using (15), we deduce

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^\gamma \mathcal{F}_{1,l;\delta}^0(t, \tau_1) \leq O(1) t \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1), \quad (\text{A31})$$

where $0 < \delta < \delta' < 1$.

Now, we treat the case in which $y_1 \in \mathbb{R}^+$. Bounding the integral over $(\mathbb{R}^+)^{\nu}$ by the integral over \mathbb{R}^{ν} in (A28) and using (10), we obtain

$$\begin{aligned} \mathcal{F}_{\delta}^0 \left(\Lambda_{\mathcal{J}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \\ \leq t p_B \left(2\tau_{1,\delta} + \sum_{i=1}^{\nu} \frac{t^2(1+\delta)}{\Lambda_i^2}; y_1, 0 \right) \leq t p_B \left(2\tau_{1,\delta} + \sum_{i=1}^{\nu} \frac{(1+\delta)}{\Lambda_i^2}; y_1, 0 \right). \end{aligned} \quad (\text{A32})$$

Furthermore, using (15) we obtain for all $0 \leq \delta < \delta' < 1$

$$\begin{aligned} \left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma} p_B \left(2\tau_{1,\delta} + \sum_{i=1}^{\nu} \frac{1+\delta}{\Lambda_i^2}; y_1, 0 \right) \\ \leq C_{\delta,\delta'} \left(1 + \sum_{i=1}^{\nu} \frac{\tau_1^{-\frac{1}{2}}}{\sqrt{2}\Lambda_i} \right)^{\gamma} p_B \left(2\tau_{1,\delta'} + \sum_{i=1}^{\nu} \frac{1+\delta'}{\Lambda_i^2}; y_1, 0 \right). \end{aligned} \quad (\text{A33})$$

Since $\Lambda_i, \Lambda_{\nu} \geq \Lambda$ for all $i \in \mathcal{J}$, we deduce that (A33) is bounded by

$$C_{\delta,\delta'} \max \left(\left(\frac{\nu}{\sqrt{2}} \right)^{\gamma}, 1 \right) \left(1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^{\gamma} p_B \left(2\tau_{1,\delta'} + \sum_{i=1}^{\nu} \frac{1+\delta'}{\Lambda_i^2}; y_1, 0 \right) \quad (\text{A34})$$

Proceeding similarly to (A11), we deduce

$$p_B \left(2\tau_{1,\delta'} + \sum_{i=1}^{\nu} \frac{1+\delta'}{\Lambda_i^2}; y_1, 0 \right) \leq 2^{\nu} \mathcal{F}_{\delta'}^0 \left(\Lambda_{\mathcal{J}}, \Lambda_{\nu}; 2\tau_1; T_l^{1,0}; y_1 \right), \quad (\text{A35})$$

which together with (A32) and (A33) imply

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma} \mathcal{F}_{\delta}^0 \left(\Lambda_{\mathcal{J}}; \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \leq C t \left(1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^{\gamma} \mathcal{F}_{\delta'}^0 \left(\Lambda_{\mathcal{J}}; 2\tau_1; T_l^{1,0}; y_1 \right), \quad (\text{A36})$$

where $C := 2^{\nu} C_{\delta,\delta'} \max \left(\left(\frac{\nu}{\sqrt{2}} \right)^{\gamma}, 1 \right)$. Using

$$\mathcal{F}_{1,l;\delta}^{\Lambda,0}(\tau_1) := \sum_{\nu=0}^{3l-1} \mathcal{F}_{\delta}^0 \left(\Lambda, 2\tau_1; T_l^{1,0}; y_1 \right), \quad (\text{A37})$$

we deduce

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma} \mathcal{F}_{1,l;\delta}^0(t, \tau_1) \leq O(1) t \left(1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^{\gamma} \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1). \quad (\text{A38})$$

□

Lemma 8. Let $\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$, $0 < \delta < \delta' \leq 1$ and $(y_1, y_2) \in \mathbb{R}^2$. For $0 < t, t' \leq 1$, $l \geq 1$ and $\gamma_1, \gamma_2 \in \mathbb{N}$, we have

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \mathcal{F}_{2,l;\delta}^0 \left(\Lambda; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \leq tt' \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathcal{F}_{2,l;\delta'}^0(\Lambda; \tau_{1,2}; y_{1,2}). \quad (\text{A39})$$

The polynomial \mathcal{Q} has nonnegative coefficients which are independent of τ_1, τ_2 and Λ but depend on $l, \delta, \delta', \gamma_1$ and γ_2 .

Proof. Using (A1), $\mathcal{F}_{2,t;\delta}^{\Lambda,0} \left(\frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; \frac{y_1}{t}, \frac{y_2}{t'} \right)$ can be written as follows

$$\begin{aligned} & \sum_{T_l^{2,0} \in \mathcal{W}_l^2(\sigma_2)} \mathcal{F}_\delta^0 \left(\Lambda; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & + \left(\sum_{T_l^{1,0} \in \mathcal{T}_l^{1,0}} \mathcal{F}_\delta^0 \left(\Lambda, \frac{2\tau_1}{t^2}; T_l^{1,0}; \frac{y_1}{t} \right) \right) \cdot \left(\sum_{\tilde{T}_l^{1,0} \in \mathcal{T}_l^{1,0}} \mathcal{F}_\delta^0 \left(\Lambda, \frac{2\tau_2}{t'^2}; \tilde{T}_l^{1,0}; \frac{y_2}{t'} \right) \right). \end{aligned} \quad (\text{A40})$$

• First, we prove

$$\begin{aligned} & \left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \sum_{T_l^{2,0} \in \mathcal{W}_l^2(\sigma_2)} \mathcal{F}^0 \left(\Lambda, \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq tt' \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathcal{F}_{2,t;\delta'}^0(\Lambda; \tau_{1,2}, y_{1,2}), \end{aligned} \quad (\text{A41})$$

where $0 < \delta < \delta' < 1$. Let us start first with the case $y_1, y_2 \geq 0$. Given a surface tree $T_l^{2,0}$ in $\mathcal{W}_l^2(\sigma_2)$, we have by Lemma 4

$$\begin{aligned} & \mathcal{F}_\delta^0 \left(\Lambda, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq \int_0^\infty dz p_B \left(c_{1,\delta}(t); z, \frac{y_1}{t} \right) p_B \left(c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right), \end{aligned} \quad (\text{A42})$$

where $c_{1,\delta}(t) = c_1(t)(1+\delta)$ and $c_{2,\delta}(t') = c_2(t')(1+\delta)$. The parameters $c_1(t)$, $c_2(t')$ and $\tilde{\Lambda}_1$ are given by (A6) with $\tau_1 \rightarrow \tau_1/t^2$ and $\tau_2 \rightarrow \tau_2/t'^2$. For $y_1 \leq y_2$, we write

$$\begin{aligned} & \int_0^\infty dz p_B \left(c_{1,\delta}(t); z, \frac{y_1}{t} \right) p_B \left(c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \\ & = \mathcal{I}(0, y_1) + \mathcal{I}(y_1, y_2) + \mathcal{I}(y_2, +\infty), \end{aligned} \quad (\text{A43})$$

where

$$\mathcal{I}(a, b) := \int_a^b dz p_B \left(c_{1,\delta}(t); z, \frac{y_1}{t} \right) p_B \left(c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right).$$

– First, we bound $\mathcal{I}(0, y_1)$. For $0 \leq t, t' \leq 1$, we have

$$p_B \left(c_{1,\delta}(t); z, \frac{y_1}{t} \right) \leq t p_B \left(c_{1,\delta}; tz, y_1 \right), \quad (\text{A44})$$

$$p_B \left(c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) \leq t' p_B \left(c_{2,\delta}; t'z, y_2 \right), \quad (\text{A45})$$

with $c_i := c_i(1)$. For $0 \leq z \leq y_1 \leq y_2$, we also have

$$p_B \left(c_{1,\delta}; tz, y_1 \right) \leq t p_B \left(c_{1,\delta}; z, y_1 \right), \quad (\text{A46})$$

$$p_B \left(c_{2,\delta}; t'z, y_2 \right) \leq t' p_B \left(c_{2,\delta}; z, y_2 \right). \quad (\text{A47})$$

This implies

$$\mathcal{I}(0, y_1) \leq t t' \int_0^\infty dz p_B \left(c_{1,\delta}; z, y_1 \right) p_B \left(c_{2,\delta}; z, y_2 \right) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right), \quad (\text{A48})$$

which again by Lemma 4 is bounded by

$$O(1) \int_{\vec{z}} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{\sigma_2}; T_l^{2,0}; \vec{z}; Y_{\sigma_2} \right) \leq O(1) \mathcal{F}_{2,l;\delta}^{\Lambda,0}(\tau_{1,2}).$$

For $(\gamma_1, \gamma_2) \neq (0, 0)$, we need to bound also the following term

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} \mathcal{S}(0, y_1). \quad (\text{A49})$$

Using (A48), (A49) is bounded by

$$t' t' \tau^{-\gamma_1 - \gamma_2} \sum_{k=0}^{\gamma_1} \sum_{k'=0}^{\gamma_2} \binom{\gamma_1}{k} \binom{\gamma_2}{k'} \int_0^\infty dz |y_1 - z|^k |y_2 - z|^{k'} z^{\gamma_1 + \gamma_2 - k - k'} \\ \times p_B(c_{1,\delta}; z, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right). \quad (\text{A50})$$

Using (15), we obtain

$$z^{\gamma_1 + \gamma_2 - k - k'} p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \leq C_{\delta,\delta'} \tilde{\Lambda}_1^{-\gamma_1 - \gamma_2 + k' + k} p_B\left(\frac{1+\delta'}{\tilde{\Lambda}_1^2}; z, 0\right). \quad (\text{A51})$$

Since $\Lambda_i, \tilde{\Lambda} \geq \Lambda$ for all $i \in \mathcal{J}$, we deduce that

$$\frac{1}{\tilde{\Lambda}_1^2} = \left(\sum_{i=v_1+v_2+1}^v \frac{1}{\tilde{\Lambda}_i^2} \right) (1 - \delta_{v,0}) + \frac{1}{\tilde{\Lambda}^2} \leq \frac{v_0+1}{\Lambda^2} \quad \text{and} \quad c_i \leq \tau_i + \frac{v_i}{\Lambda^2}. \quad (\text{A52})$$

This gives

$$z^{\gamma_1 + \gamma_2 - k - k'} p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right) \leq O(1) \Lambda^{-\gamma_1 - \gamma_2 + k' + k} p_B\left(\frac{1+\delta'}{\tilde{\Lambda}_1^2}; z, 0\right). \quad (\text{A53})$$

Similarly, we have

$$|y_i - z|^{n-k} p_B(c_{i,\delta}; y_i, z) \leq C'_{\delta,\delta'} c_i^{\frac{k}{2}} p_B(c_{i,\delta'}; y_i, z), \quad (\text{A54})$$

$$\leq O(1) \tau_i^{\frac{k}{2}} \left(1 + \frac{\tau_i^{-\frac{1}{2}}}{\Lambda} \right)^k p_B(c_{i,\delta'}; y_i, z), \quad (\text{A55})$$

where we used (A52). Whenever it appears, $O(1)$ denotes a constant which depends on δ, δ', l and v . Combining (A50), (A53) and (A55), we deduce that (A49) is bounded by

$$t' t' \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \int_0^\infty dz p_B(c_{1,\delta'}; z, y_1) p_B(c_{2,\delta'}; z, y_2) p_B\left(\frac{1+\delta'}{\tilde{\Lambda}_1^2}; z, 0\right). \quad (\text{A56})$$

By Lemma 4, we deduce that (A49) is bounded by

$$t' t' \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \int_{\vec{z}} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \tau_{\sigma_2}; T_l^{2,0}; \vec{z}; Y_{\sigma_2} \right). \quad (\text{A57})$$

– Using the bounds (A44), (A45) and (A47), the term $\mathcal{S}(y_1, y_2)$ is bounded by

$$t' t' \int_{y_1}^{y_2} dz p_B(c_{1,\delta}; tz, y_1) p_B(c_{2,\delta}; z, y_2) p_B\left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0\right). \quad (\text{A58})$$

For $z \geq y_1$, we have

$$p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \leq p_B \left(\frac{2(1+\delta)}{\tilde{\Lambda}_1^2}; z, 0 \right) \exp \left(-\frac{y_1^2 \tilde{\Lambda}_1^2}{4(1+\delta)} \right). \quad (\text{A59})$$

Knowing that $v_0 \leq 3l - 1$ together with

$$\forall \Lambda_i \in \Lambda_{\mathcal{J}}, \quad \Lambda_i \geq \Lambda, \quad \tilde{\Lambda} \geq \Lambda, \quad (\text{A60})$$

and recalling (A52), we obtain

$$\tilde{\Lambda}_1 \geq \frac{\Lambda}{\sqrt{3l}} \geq \sqrt{3}\tau^{-\frac{1}{2}}. \quad (\text{A61})$$

where we also used $\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$. This implies

$$\exp \left(-\frac{y_1^2 \tilde{\Lambda}_1^2}{4(1+\delta)} \right) \leq \exp \left(-\frac{y_1^2 \Lambda^2}{12(1+\delta)l} \right) \leq \exp \left(-\frac{y_1^2}{2(1+\delta)\tau_1} \right). \quad (\text{A62})$$

Furthermore, we have

$$p_B(c_{1,\delta}; tz, y_1) \leq \frac{1}{\sqrt{2\pi c_{1,\delta}}}. \quad (\text{A63})$$

Combining (A63) with the fact that $c_1 \geq \tau_1$, we deduce

$$\mathcal{J}(y_1, y_2) \leq \int_{\mathbb{R}} dz p_B(c_{2,\delta}; z, y_2) p_B \left(\frac{2(1+\delta)}{\tilde{\Lambda}_1^2}; z, 0 \right) p_B(c_{1,\delta}; y_1, 0),$$

and by (10) we deduce that $\mathcal{J}(y_1, y_2)$ is bounded by

$$p_B \left(c_{2,\delta} + \frac{2(1+\delta)}{\tilde{\Lambda}_1^2}; y_2, 0 \right) p_B(c_{1,\delta}; y_1, 0).$$

Using the property (15) of the bulk heat kernel together with (A60), we obtain

$$\begin{aligned} & \left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\gamma_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\gamma_2} p_B \left(c_{2,\delta} + \frac{2(1+\delta)}{\tilde{\Lambda}_1^2}; y_2, 0 \right) p_B(c_{1,\delta}; y_1, 0) \\ & \leq \mathcal{Q} \left(\frac{\tau_i^{-\frac{1}{2}}}{\Lambda} \right) p_B \left(c_{2,\delta'} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2}; y_2, 0 \right) p_B(c_{1,\delta'}; y_1, 0), \end{aligned} \quad (\text{A64})$$

where $0 < \delta < \delta' < 1$. For $\Lambda \geq 3\sqrt{l}\tau^{-\frac{1}{2}}$ and $l \geq 1$, we have

$$\forall \Lambda_i \in \Lambda_{\mathcal{J}}, \quad \Lambda_i \geq \Lambda \geq \sqrt{3l}\tau_2^{-\frac{1}{2}}, \quad (\text{A65})$$

and this implies

$$\begin{aligned} \frac{1}{\Lambda_{v_1+v_2}^2} & \leq \frac{\tau_2}{3}, \quad \frac{1}{\Lambda_{v_1+v_2-1}^2} \leq \frac{\tau_2}{3}, \\ \frac{1}{\tilde{\Lambda}_1^2} & = \left(\sum_{i=v_1+v_2+1}^v \frac{1}{\Lambda_i^2} \right) (1 - \delta_{v,0}) + \frac{1}{\Lambda^2} \leq \frac{v_0+1}{\Lambda^2} \leq \frac{\tau_2}{3}, \end{aligned} \quad (\text{A66})$$

where again we used that $v_0 \leq 3l - 1$. Hence, we have

$$c_{2,\delta'} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2} = \tau_{2,\delta'} + \sum_{i=1}^{v_2} \frac{1+\delta'}{\Lambda_{i+v_1}^2} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2} \leq 2\tau_{2,\delta'} + \sum_{i=1}^{v_2-2} \frac{1+\delta'}{\Lambda_{i+v_1}^2} + \frac{1+\delta'}{\tilde{\Lambda}_1^2}, \quad (\text{A67})$$

which gives

$$p_B \left(c_{2,\delta'} + \frac{2(1+\delta')}{\tilde{\Lambda}_1^2}; y_2, 0 \right) \leq \sqrt{2} p_B \left(2\tau_{2,\delta'} + \sum_{i=1}^{v_2-2} \frac{1+\delta'}{\Lambda_{i+v_1}^2} + \frac{1+\delta'}{\tilde{\Lambda}_1^2}; y_2, 0 \right). \quad (\text{A68})$$

(13) together with Lemma 4 gives

$$\begin{aligned} & p_B \left(2\tau_{2,\delta'} + \sum_{i=1}^{v_2-2} \frac{1+\delta'}{\Lambda_{i+v_1}^2} + \frac{1+\delta'}{\tilde{\Lambda}_1^2}; y_2, 0 \right) \\ & \leq O(1) \int_{z_1} \cdots \int_{z_{v_2+v_0-1}} p_B(2\tau_{2,\delta'}; z_1, y_2) \prod_{i=2}^{v_2-1} p_B \left(\frac{1+\delta'}{\Lambda_{v_1+i-1}^2}; z_i, z_{i-1} \right) \\ & \quad \times \prod_{i=0}^{v_0-1} p_B \left(\frac{1+\delta'}{\tilde{\Lambda}_{v_1+v_2+i}^2}; z_{v_2+i}, z_{v_2+i-1} \right) p_B \left(\frac{1+\delta'}{\tilde{\Lambda}^2}; z_{v_2+v_0-1}, 0 \right). \quad (\text{A69}) \end{aligned}$$

The RHS of (A69) corresponds to the integrated surface weight factor of a surface tree $T_l^{1,0}$ which has an external vertex y_2 and $v_2 + v_0 - 1$ internal vertices which all are of incidence number 2. This tree belongs to the set of forests $\mathscr{W}_l^1(\pi_2)$ if and only if

$$v_0 + v_2 - 1 \leq 3l - 1. \quad (\text{A70})$$

Since the tree $T_l^{2,0} \left(\frac{\tau_1}{t^2}, \frac{\tau_2}{t^2}; \frac{y_1}{t}, \frac{y_2}{t} \right)$ is in the forest $\mathscr{W}_l^2(\sigma_2)$, v_0 , v_1 and v_2 necessarily verify

$$v_0 + v_2 - 1 \leq 3l - 2 - v_1 + \frac{1}{2},$$

which implies (A70). Hence, $T_l^{1,0}$ belongs to the set $\mathscr{W}_l^1(\pi_2)$. From (A68) and (A69), we deduce

$$p_B \left(c_{2,\delta'} + \frac{1+\delta'}{\tilde{\Lambda}^2}; y_2, 0 \right) \leq O(1) \mathscr{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2). \quad (\text{A71})$$

Using (A11), we obtain

$$p_B(c_{1,\delta'}; y_1, 0) \leq O(1) \mathscr{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1). \quad (\text{A72})$$

(A58) and (A64) together with (A71) and (A75) give

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\eta_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\eta_2} \mathscr{I}(y_1, y_2) \leq t t' \mathscr{D} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathscr{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathscr{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2), \quad (\text{A73})$$

where all the constants were absorbed in the coefficients of the polynomial \mathscr{D} .

– The last term to bound is

$$\mathscr{I}(y_2, +\infty) := \int_{y_2}^{\infty} dz p_B(c_{1,\delta}(t); z, \frac{y_1}{t}) p_B(c_{2,\delta}(t'); z, \frac{y_2}{t'}) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right).$$

For $z \geq y_2 \geq y_1$ we have

$$p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \leq \frac{\tilde{\Lambda}_1}{\sqrt{2\pi}} \exp \left(-\frac{z^2 \tilde{\Lambda}_1^2}{6(1+\delta)} \right) \exp \left(-\frac{y_1^2}{2c_{1,\delta}} \right) \exp \left(-\frac{y_2^2}{2c_{2,\delta}} \right),$$

where we used (A61) and $c_i \geq \tau_i$.

Bounding respectively $p_B(c_{1,\delta}(t); z, \frac{y_1}{t})$ and $p_B(c_{2,\delta}(t'); z, \frac{y_2}{t'})$ by $\frac{t}{\sqrt{2\pi c_{1,\delta}}}$ and $\frac{t'}{\sqrt{2\pi c_{2,\delta}}}$ we deduce that $\mathcal{I}(y_2, +\infty)$ is bounded by

$$O(1) t t' p_B(c_{1,\delta}; y_1, 0) p_B(c_{2,\delta}; y_2, 0).$$

Using the bound (15), we find

$$\begin{aligned} \left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\eta_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\eta_2} p_B(c_{2,\delta}; y_2, 0) p_B(c_{1,\delta}; y_1, 0) \\ \leq \mathcal{Q} \left(\frac{\tau_i^{-\frac{1}{2}}}{\Lambda} \right) p_B(c_{2,\delta'}; y_2, 0) p_B(c_{1,\delta'}; y_1, 0), \end{aligned} \quad (\text{A74})$$

where $0 < \delta < \delta' < 1$. Using (A11), we deduce

$$p_B(c_{i,\delta'}; y_i, 0) \leq O(1) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_i), \quad (\text{A75})$$

which implies

$$\left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\eta_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\eta_2} \mathcal{I}(y_2, +\infty) \leq t t' \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2). \quad (\text{A76})$$

Combining (A57), (A73) and (A76) together with (A3) and (A42), we obtain (A41).

Now, we treat the case in which the external vertices are negative: Given a surface tree $T_l^{2,0}$ in $\mathcal{W}_{2,l}(\sigma_2)$, we recall

$$\begin{aligned} \int_{\tilde{z}} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{I}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \tilde{z}; \frac{y_1}{t}, \frac{y_2}{t'} \right) &= \int_{z_1, \dots, z_v} \prod_{j=2}^{v_1+1} p_B \left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B \left(\frac{\tau_{1,\delta}}{t}; z_{v_1+1}, \frac{y_1}{t} \right) \\ &\times p_B \left(\frac{1+\delta}{\Lambda_{v_1+1}^2}; z_1, z_{v_1+2} \right) \prod_{j=v_1+3}^{v_1+v_2} p_B \left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B \left(\frac{\tau_{2,\delta}}{t'^2}; z_{v_1+v_2}, \frac{y_2}{t'} \right) \\ &\times p_B \left(\frac{1+\delta}{\Lambda_{v_1+v_2}^2}; z_1, z_{v_1+v_2+1} \right) \prod_{j=v_1+v_2+2}^v p_B \left(\frac{1+\delta}{\Lambda_{j-1}^2}; z_{j-1}, z_j \right) p_B \left(\frac{1+\delta}{\tilde{\Lambda}^2}; z_v, 0 \right). \end{aligned} \quad (\text{A77})$$

For $y_1 \leq 0$ and $z_{v_1+1} \in \mathbb{R}^+$, we have

$$p_B \left(\frac{\tau_{1,\delta}}{t^2}; z, \frac{y_1}{t} \right) \leq t p_B(\tau_{1,\delta}; y_1, 0).$$

Hence, we obtain for $y_1, y_2 \leq 0$ using (15)

$$\begin{aligned} \left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\eta_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\eta_2} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{I}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}; \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ \leq O(1) t t' p_B(\tau_{1,\delta'}; y_1, 0) p_B(\tau_{2,\delta'}; y_2, 0), \end{aligned} \quad (\text{A78})$$

which is bounded by

$$O(1) t t' \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,l;\delta'}^{\Lambda,0}(\tau_2). \quad (\text{A79})$$

If $y_1 \leq 0$ and $y_2 \geq 0$, we have

$$\begin{aligned} & \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}, \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq O(1) t p_B(\tau_{1,\delta}; y_1, 0) \int_0^\infty dz p_B \left(c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right). \end{aligned} \quad (\text{A80})$$

Using (10) together with (A11) and (A2), we have

$$\int_0^\infty dz p_B \left(c_{2,\delta}(t'); z, \frac{y_2}{t'} \right) p_B \left(\frac{1+\delta}{\tilde{\Lambda}_1^2}; z, 0 \right) \leq \mathcal{F}_{1,t;\delta}^{\Lambda,0}(t', \tau_2). \quad (\text{A81})$$

Combining Lemma 7 with (A80) and (A81) gives

$$\begin{aligned} & \left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\eta_1} \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\eta_2} \mathcal{F}_\delta^0 \left(\Lambda_{\mathcal{J}}, \tilde{\Lambda}; \frac{\tau_1}{t^2}, \frac{\tau_2}{t'^2}; T_l^{2,0}, \frac{y_1}{t}, \frac{y_2}{t'} \right) \\ & \leq O(1) t t' \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda} \right) \mathcal{F}_{1,t;\delta'}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,t;\delta'}^{\Lambda,0}(\tau_2). \end{aligned} \quad (\text{A82})$$

Using (A3), we obtain the bound (A41).

- By definition, we have

$$\mathcal{F}_{1,t;\delta}^{\Lambda,0}(t, \tau_1) = \sum_{T_l^{1,0} \in \mathcal{T}_l^{1,0}} \mathcal{F}_\delta^0 \left(\Lambda, \frac{\tau_1}{t^2}; T_l^{1,0}, \frac{y_1}{t} \right). \quad (\text{A83})$$

Applying Lemma 7 to the global surface weight factors $\mathcal{F}_{1,t;\delta}^{\Lambda,0}(t, \tau_1)$ and $\mathcal{F}_{1,t;\delta}^{\Lambda,0}(t', \tau_2)$ we obtain

$$\begin{aligned} & \left(\frac{y_1}{\sqrt{\tau_1}} \right)^{\eta_1} \sum_{T_l^{1,0} \in \mathcal{T}_l^{1,0}} \mathcal{F}_\delta^0 \left(\Lambda, \frac{\tau_1}{t^2}; T_l^{1,0}, \frac{y_1}{t} \right) \\ & \quad \times \left(\frac{y_2}{\sqrt{\tau_2}} \right)^{\eta_2} \sum_{\tilde{T}_l^{1,0} \in \mathcal{T}_l^{1,0}} \mathcal{F}_\delta^0 \left(\Lambda, \frac{\tau_2}{t'^2}; \tilde{T}_l^{1,0}, \frac{y_2}{t'} \right) \\ & \leq O(1) t t' \left(1 + \frac{\tau_1^{-\frac{1}{2}}}{\Lambda} \right)^{\eta_1} \left(1 + \frac{\tau_2^{-\frac{1}{2}}}{\Lambda} \right)^{\eta_2} \mathcal{F}_{1,t;\delta'}^{\Lambda,0}(\tau_1) \mathcal{F}_{1,t;\delta'}^{\Lambda,0}(\tau_2), \end{aligned} \quad (\text{A84})$$

and this together with (A3), (A40) and (A41) conclude the proof of Lemma 8. □

DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

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