



**HAL**  
open science

# Unusual functorialities for weakly constructible sheaves

Andreas Hohl, Pierre Schapira

► **To cite this version:**

Andreas Hohl, Pierre Schapira. Unusual functorialities for weakly constructible sheaves. 2023. hal-04483359

**HAL Id: hal-04483359**

**<https://hal.science/hal-04483359>**

Preprint submitted on 29 Feb 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Unusual functorialities for weakly constructible sheaves

Andreas Hohl\* and Pierre Schapira

## Abstract

We prove that various morphisms related to the six Grothendieck operations on sheaves become isomorphisms when restricted to (weakly) constructible sheaves.

## 1 Introduction

Let  $\mathbf{k}$  be a commutative unital ring of finite global dimension and denote by  $D^b(\mathbf{k}_X)$  the bounded derived category of sheaves on a good topological space  $X$ . There are some classical morphisms which are, in general, not isomorphisms, such as

$$\begin{aligned} R\mathcal{H}om(F_1, F_2) \overset{L}{\otimes} F_3 &\rightarrow R\mathcal{H}om(F_1, F_2 \overset{L}{\otimes} F_3), \\ f^! G_1 \overset{L}{\otimes} f^{-1} G_2 &\rightarrow f^!(G_1 \overset{L}{\otimes} G_2), \end{aligned}$$

for  $F_1, F_2, F_3 \in D^b(\mathbf{k}_X)$ ,  $G_1, G_2 \in D^b(\mathbf{k}_Y)$  and  $f: X \rightarrow Y$  a continuous map. We will prove here that, under suitable hypotheses of (weak) constructibility, these morphisms become isomorphisms.

Assuming that the duality functor is conservative in  $D^b(\mathbf{k})$  (see (3.3)), we prove the following results in the categories of real analytic manifolds and weakly  $\mathbb{R}$ -constructible sheaves. (See Theorems 4.2, 4.4 and 4.7.)

Let  $f: X \rightarrow Y$  be a morphism of real analytic manifolds, and let  $F_1, F_2, L_X \in D_{w\mathbb{R}c}^b(\mathbf{k}_X)$ ,  $G, L_Y \in D_{w\mathbb{R}c}^b(\mathbf{k}_Y)$  with moreover  $F_1$  being  $\mathbb{R}$ -constructible, and  $L_X, L_Y$  locally constant. Then we have the isomorphisms

$$\begin{aligned} f^! G \otimes f^{-1} L_Y &\simeq f^!(G \otimes L_Y), \\ f^{-1} R\mathcal{H}om(L_Y, G) &\simeq R\mathcal{H}om(f^{-1} L_Y, f^{-1} G), \\ R\mathcal{H}om(F_1, F_2) \otimes L_X &\simeq R\mathcal{H}om(F_1, F_2 \otimes L_X), \\ R\mathcal{H}om(L_X, F_2) \otimes F_1 &\simeq R\mathcal{H}om(L_X, F_2 \otimes F_1). \end{aligned}$$

---

\*The research of A.H. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Projektnummer 465657531.

For direct images we need slightly stronger assumptions: consider a morphism  $f: X_\infty \rightarrow Y_\infty$  of  $b$ -analytic manifolds (see [Sch23] for this notion) and let  $F \in D_{w\mathbb{R}c}^b(\mathbf{k}_{X_\infty})$  be weakly  $\mathbb{R}$ -constructible up to infinity and  $L_Y$  be locally constant. We prove the isomorphisms

$$\begin{aligned} Rf_*F \otimes L_Y &\simeq Rf_*(F \otimes f^{-1}L_Y), \\ Rf_!R\mathcal{H}om(f^{-1}L_Y, F) &\simeq R\mathcal{H}om(L_Y, Rf_!F). \end{aligned}$$

We start by introducing the notion (implicitly already defined in [KS90, § 3.4]) of weakly cohomologically constructible sheaves (wcc-sheaves, for short) and the full subcategory  $D_{wcc}^b(\mathbf{k}_X)$  of  $D^b(\mathbf{k}_X)$  consisting of such sheaves. On a real analytic manifold, the category of  $D_{wcc}^b(\mathbf{k}_X)$  contains the category  $D_{w\mathbb{R}c}^b(\mathbf{k}_X)$  of weakly  $\mathbb{R}$ -constructible sheaves.

We prove first that  $D_{wcc}^b(\mathbf{k}_X)$  is triangulated. Then our main tool is that for an object  $F$  of this category, for  $x \in X$  and  $L \in D^b(\mathbf{k})$ , one has functorial isomorphisms

$$R\mathcal{H}om(L_X, F)_x \simeq R\mathcal{H}om(L, F_x), \quad R\Gamma_x F \otimes L \simeq R\Gamma_x(F \otimes L_X),$$

where  $L_X$  denotes the constant sheaf associated with  $L$ .

The motivation for this note came through the preprint [Hoh23], where field extensions for sheaves are considered, and many of the desired functorialities of loc. cit. indeed follow from the more general set-up developed here.

We make the conjecture that the above results hold without Hypothesis (3.3).

## 2 Preliminaries

In all this paper, we work in a given universe  $\mathcal{U}$ . All limits and colimits (in particular, products and direct sums) are assumed to be small. Recall that  $\mathbf{k}$  is a commutative unital ring of finite global dimension. We assume that all topological spaces are “good”, that is, Hausdorff, locally compact, countable at infinity and of finite flabby dimension.

For a topological space  $X$  as above, one denotes by  $\text{Mod}(\mathbf{k}_X)$  the Grothendieck abelian category of sheaves of  $\mathbf{k}$ -modules and by  $D^b(\mathbf{k}_X)$  its bounded derived category. We need a slight modification of the notion of cohomologically constructible sheaves (see [KS90, Def. 3.4.1]).

We mainly follow the notations of [KS90]. In particular,

- $\omega_X$  denotes the dualizing complex and  $D_X$  the duality functor  $R\mathcal{H}om(\cdot, \omega_X)$ ,
- $D$  denotes the duality functor on  $D^b(\mathbf{k})$ ,
- $a_X: X \rightarrow \{\text{pt}\}$  denotes by the unique map from  $X$  to a one-point space. Hence, for  $F \in D^b(\mathbf{k}_X)$ ,  $R\Gamma(X; F) \simeq Ra_{X*}F$ .
- For  $L \in D^b(\mathbf{k})$ ,  $L_X$  denotes the constant sheaf on  $X$  with stalk  $L$ . More generally, for  $Z$  locally closed in  $X$ , one denotes by  $L_{XZ}$  the constant sheaf  $L_Z$  on  $Z$  extended by 0 on  $X \setminus Z$ . When  $Z$  is closed, we shall simply denote by  $L_Z$  the sheaf  $L_{XZ}$ .
- For  $x \in X$ , denoting by  $i_x: \{x\} \hookrightarrow X$  the embedding, and for  $F \in D^b(\mathbf{k}_X)$ , one denotes as usual by  $F_x = i_x^{-1}F$  its stalk at  $x$ . One also sets  $R\Gamma_x F = i_x^!F$ .

## Ind-objects

We shall make use of ind-objects. For a short exposition see [KS90, § 1.11]. For a more detailed study, including new results that we shall use here, see [KS06, Ch. 6, § 8.6, Ch. 15]. Let us recall a few facts that we need, skipping some delicate questions of universes.

If  $\mathcal{C}$  is a category, one denotes by  $\text{Ind}(\mathcal{C})$  the category of ind-objects of  $\mathcal{C}$ , a full subcategory of the category  $\mathcal{C}^\wedge$  of functors from  $\mathcal{C}^{\text{op}}$  to  $\mathbf{Set}$ . Recall [KS06, § 6.1]

- The natural functor  $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$  is fully faithful.
- The category  $\text{Ind}(\mathcal{C})$  admits small filtrant colimits, denoted “colim”.
- Let  $\mathcal{I}$  be a small and filtrant category and  $\alpha: \mathcal{I} \rightarrow \mathcal{C}$  a functor. Let  $T: \mathcal{C} \rightarrow \mathcal{C}'$  be a functor. Then  $T(\text{“colim” } \alpha) \simeq \text{“colim” } (T \circ \alpha)$ .

Now we assume that  $\mathcal{C}$  is abelian. Recall [KS06, Th. 8.6.5] that

- The category  $\text{Ind}(\mathcal{C})$  is abelian and the fully faithful functor  $\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})$  is exact.
- Small filtrant colimits are exact in  $\text{Ind}(\mathcal{C})$ .

One should be aware that even if  $\mathcal{C}$  is a Grothendieck category,  $\text{Ind}(\mathcal{C})$  does not admit enough injectives in general.

Let  $\mathcal{I}$  be a small category and  $\alpha: \mathcal{I} \rightarrow \mathcal{C}$  be a functor. As already mentioned, one denotes by “colim”  $\alpha$  its colimit in  $\text{Ind}(\mathcal{C})$ . Note that if  $\mathcal{C}$  admits colimits, denoted  $\text{colim}$ , there is a natural morphism in  $\text{Ind}(\mathcal{C})$ :

$$\text{“colim” } \alpha \rightarrow \text{colim } \alpha$$

but this morphism is not an isomorphism in general. However:

- If “colim”  $\alpha$  belongs to  $\mathcal{C}$ , then “colim”  $\alpha \xrightarrow{\sim} \text{colim } \alpha$  (see [KS90, Cor. 1.11.7]). In this case, if  $T: \mathcal{C} \rightarrow \mathcal{C}'$  is a functor, then “colim”  $(T \circ \alpha) \xrightarrow{\sim} T(\text{colim } \alpha)$ . Therefore, “colim”  $(T \circ \alpha)$  belongs to  $\mathcal{C}$  and hence is isomorphic to  $\text{colim } (T \circ \alpha)$ .

## 3 Weakly cohomologically constructible sheaves

**Definition 3.1.** Let  $F \in D^b(\mathbf{k}_X)$ . We say that  $F$  is *weakly cohomologically constructible* (wcc for short) if for all  $x \in X$ , one has the isomorphisms

$$\text{“colim”}_{x \in U} \text{R}\Gamma(U; F) \xrightarrow{\sim} F_x, \quad \text{R}\Gamma_x F \xrightarrow{\sim} \text{“lim”}_{x \in U} \text{R}\Gamma_c(U; F).$$

We denote by  $D_{\text{wcc}}^b(\mathbf{k}_X)$  the full subcategory of  $D^b(\mathbf{k}_X)$  consisting of weakly cohomologically constructible sheaves,

**Remark 3.2.** As explained in [KS90, Rem. 4.3.2], the isomorphisms in Definition 3.1 hold as soon as the objects “colim”  $\text{R}\Gamma(U; F)$  and “lim”  $\text{R}\Gamma_c(U; F)$  are representable.

**Proposition 3.3.** *The category  $D_{\text{wcc}}^b(\mathbf{k}_X)$  is triangulated.*

*Proof.* (i) Remark first that for  $F \in D_{\text{wcc}}^b(\mathbf{k}_X)$  and  $j \in \mathbb{Z}$ , one has

$$\text{“colim”}_{x \in U} H^j(U; F) \xrightarrow{\simeq} H^j(F)_x.$$

(ii) Clearly, if  $F \in D_{\text{wcc}}^b(\mathbf{k}_X)$ , then so does the shifted sheaf  $F[j]$  for  $j \in \mathbb{Z}$ .

(iii) Consider a distinguished triangle  $F' \rightarrow F \rightarrow F'' \xrightarrow{+1}$  in  $D^b(\mathbf{k}_X)$  and assume that  $F', F'' \in D_{\text{wcc}}^b(\mathbf{k}_X)$ . We get the morphism of long exact sequence in the abelian category  $\text{Mod}(\mathbf{k})$

$$(3.1) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & H^j(U; F') & \longrightarrow & H^j(U; F) & \longrightarrow & H^j(U; F'') & \longrightarrow & H^{j+1}(U; F') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^j(F')_x & \longrightarrow & H^j(F)_x & \longrightarrow & H^j(F'')_x & \longrightarrow & H^{j+1}(F')_x & \longrightarrow & \cdots \end{array}$$

Applying the functor “colim” <sub>$x \in U$</sub>  and using [KS06, Th. 8.6.5], the first line gives rise to the long exact sequence in the abelian category  $\text{Ind}(\text{Mod}(\mathbf{k}))$ :

$$(3.2) \quad \cdots \rightarrow H^j(F')_x \rightarrow \text{“colim”}_{x \in U} H^j(U; F) \rightarrow H^j(F'')_x \rightarrow H^{j+1}(F')_x \rightarrow \cdots$$

We shall apply [KS06, Lem. 15.4.6], following its notations, to the category  $\mathcal{C} = \text{Mod}(\mathbf{k})$ . Consider the morphism

$$\varphi: \text{“colim”}_{x \in U} \text{R}\Gamma(U; F) \rightarrow F_x.$$

It follows from (3.1) that  $IH^j(\varphi)$  is an isomorphism for all  $j \in \mathbb{Z}$  and therefore  $\varphi$  is an isomorphism by loc. cit.

(iv) The proof for  $\text{R}\Gamma_x F$  is the same and we do not repeat it.  $\square$

**Proposition 3.4.** *Let  $F \in D_{\text{wcc}}^b(\mathbf{k}_X)$ . Then  $D_X F \in D_{\text{wcc}}^b(\mathbf{k}_X)$ . Moreover, one has the isomorphisms  $\text{R}\Gamma_x D_X F \simeq D(F_x)$  and  $(D_X F)_x \simeq D(\text{R}\Gamma_x F)$ .*

We shall adapt the proof of [KS90, Prop. 4.3.4 (iii)].

*Proof.* (i) The first isomorphism holds without any hypothesis, see [KS90, Exe. viii 3].

(ii) Recall first the isomorphism for  $U$  open and  $F \in D^b(\mathbf{k}_X)$ :

$$\text{R}\Gamma(U; D_X F) \simeq \text{RHom}(\text{R}\Gamma_c(U; F), \mathbf{k}).$$

Now assume that  $F \in D_{\text{wcc}}^b(\mathbf{k}_X)$ . Applying the functor “colim” <sub>$x \in U$</sub> , we get the isomorphisms

$$\begin{aligned} \text{“colim”}_{x \in U} \text{R}\Gamma(U; D_X F) &\simeq \text{RHom}(\text{“lim”}_{x \in U} \text{R}\Gamma_c(U; F), \mathbf{k}) \\ &\simeq \text{RHom}(\text{R}\Gamma_x F, \mathbf{k}) = D(\text{R}\Gamma_x F). \end{aligned}$$

This proves that “colim”  $\mathrm{R}\Gamma(U; D_X F)$  is representable, hence isomorphic to  $(D_X F)_x$ . This also proves the  $x \in U$  second isomorphism.

(iii) Let  $K$  be a compact neighborhood of  $x$ . One has

$$\begin{aligned} \mathrm{R}\Gamma_K(X, D_X F) &\simeq \mathrm{RHom}(\mathbf{k}_{XK}, D_X F) \\ &\simeq \mathrm{RHom}(F_K, \omega_X) \simeq \mathrm{RHom}(\mathrm{R}\Gamma(X; F_K), \mathbf{k}). \end{aligned}$$

Now, denote by  $\overset{\circ}{K}$  the interior of  $K$ . We have

$$\begin{aligned} \text{“lim”}_{x \in U} \mathrm{R}\Gamma_c(U; D_X F) &\simeq \text{“lim”}_{x \in \overset{\circ}{K}} \mathrm{R}\Gamma_K(X; D_X F) \\ &\simeq \mathrm{RHom}(\text{“colim”}_{x \in \overset{\circ}{K}} \mathrm{R}\Gamma(X; F_K), \mathbf{k}) \simeq \mathrm{RHom}(\text{“colim”}_{x \in U} \mathrm{R}\Gamma(U; F), \mathbf{k}) \\ &\simeq D(F_x). \end{aligned}$$

This proves that  $\text{“lim”}_{x \in U} \mathrm{R}\Gamma_c(U; D_X F)$  is representable, hence isomorphic to  $\mathrm{R}\Gamma_x D_X F$ .  $\square$

**Proposition 3.5.** *Let  $F \in D_{\mathrm{wcc}}^b(\mathbf{k}_X)$ , let  $L \in D^b(\mathbf{k})$  and let  $x \in X$ . Then  $F \overset{L}{\otimes} L$  and  $\mathrm{R}\mathcal{H}om(L_X, F)$  belong to  $D_{\mathrm{wcc}}^b(\mathbf{k}_X)$ . Moreover one has the isomorphisms*

$$\mathrm{R}\mathcal{H}om(L_X, F)_x \xrightarrow{\simeq} \mathrm{RHom}(L, F_x), \quad (\mathrm{R}\Gamma_x F) \overset{L}{\otimes} L \xrightarrow{\simeq} \mathrm{R}\Gamma_x(F \otimes L_X),$$

*Proof.* (i) One has

$$\begin{aligned} \mathrm{RHom}(L, F_x) &\simeq \mathrm{RHom}(L, \text{“colim”}_{x \in U} \mathrm{R}\Gamma(U; F)) \simeq \text{“colim”}_{x \in U} \mathrm{RHom}(L, \mathrm{R}\Gamma(U; F)) \\ &\simeq \text{“colim”}_{x \in U} \mathrm{RHom}(L_{XU}, F) \simeq \text{“colim”}_{x \in U} \mathrm{R}\Gamma(U; \mathrm{R}\mathcal{H}om(L_X, F)). \end{aligned}$$

This proves that  $\text{“colim”}_{x \in U} \mathrm{R}\Gamma(U; \mathrm{R}\mathcal{H}om(L_X, F))$  is representable as well as the first isomorphism.

(ii) One has

$$\begin{aligned} (\mathrm{R}\Gamma_x F) \overset{L}{\otimes} L &\simeq (\text{“lim”}_{x \in U} \mathrm{R}\Gamma_c(U; F)) \overset{L}{\otimes} L \simeq \text{“lim”}_{x \in U} (\mathrm{R}\Gamma_c(U; F) \overset{L}{\otimes} L) \\ &\simeq \text{“lim”}_{x \in U} \mathrm{R}\Gamma_c(U; F \overset{L}{\otimes} L_X). \end{aligned}$$

This proves that  $\text{“lim”}_{x \in U} \mathrm{R}\Gamma_c(U; F \otimes L_X)$  is representable as well as the second isomorphism.  $\square$

All along this paper, we shall consider the hypothesis

(3.3) For any  $M \in D^b(\mathbf{k})$ , if  $DM \simeq 0$ , then  $M \simeq 0$ .

Since  $D^b(\mathbf{k})$  is triangulated, this is equivalent to saying that the functor  $D: D^b(\mathbf{k})^{\mathrm{op}} \rightarrow D^b(\mathbf{k})$  is conservative.

**Example 3.6.** (i) Hypothesis (3.3) is obviously satisfied if  $\mathbf{k}$  is a field. Indeed, in this case,  $M \simeq \bigoplus_j H^j(M)[-j]$  and we are reduced to the case where  $M$  is a vector space. The result then follows since the map  $M \rightarrow DDM$  is injective.

(ii) This property is satisfied when  $\mathbf{k} = \mathbb{Z}$ . See [KS90, Exe. I.31].

**Proposition 3.7.** *Assume (3.3). Then the functor  $\prod_{x \in X} \mathrm{R}\Gamma_x(\bullet): \mathrm{D}_{\mathrm{wcc}}^b(\mathbf{k}_X) \rightarrow \mathrm{D}^b(\mathbf{k})$  is conservative.*

*Proof.* Let  $F \in \mathrm{D}_{\mathrm{wcc}}^b(\mathbf{k}_X)$  and assume that  $\mathrm{R}\Gamma_x F \simeq 0$  for all  $x \in X$ . By Proposition 3.4, we get that  $D_X F \simeq 0$ . Hence  $\mathrm{R}\Gamma_x D_X F \simeq 0$  and by the same proposition, we get that  $D(F_x) \simeq 0$ . Using the hypothesis (3.3), we get  $F \simeq 0$ .  $\square$

## 4 Weakly $\mathbb{R}$ -constructible sheaves

The property of being weakly cohomologically constructible is not stable by the six operations. That is why we shall consider instead weakly  $\mathbb{R}$ -constructible sheaves. Hence, from now on, all manifolds and morphisms of manifolds will be real analytic.

Recall (see [KS90, Exe. I.30]) that  $M \in \mathrm{D}^b(\mathbf{k})$  is *perfect* if it is isomorphic to a bounded complex of finitely generated projective  $\mathbf{k}$ -modules. If  $M$  is perfect, then so is  $D(M)$  and the morphism  $M \rightarrow DD(M)$  is an isomorphism. We shall denote by  $\mathrm{D}_f^b(\mathbf{k})$  the full triangulated category of  $\mathrm{D}^b(\mathbf{k})$  consisting of perfect objects.

Let  $X$  be a real analytic manifold. As already mentioned, we denote by  $\mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$  (resp.  $\mathrm{D}_{\mathrm{Rc}}^b(\mathbf{k}_X)$ ) the full triangulated subcategory of  $\mathrm{D}^b(\mathbf{k}_X)$  consisting of weakly  $\mathbb{R}$ -constructible (resp.  $\mathbb{R}$ -constructible) sheaves on  $X$ .

Recall that  $F \in \mathrm{D}^b(\mathbf{k}_X)$  belongs to  $\mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$  if and only if its micro-support  $\mathrm{SS}(F)$  is contained in a conic subanalytic isotropic subset of  $T^*X$  and this is equivalent to the fact that  $\mathrm{SS}(F)$  is a conic subanalytic Lagrangian subset.

If  $F_1, F_2 \in \mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$ , then  $F_1 \overset{\mathrm{L}}{\otimes} F_2$  and  $\mathrm{R}\mathcal{H}om(F_1, F_2)$  belong to  $\mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$ . Moreover, if  $f: X \rightarrow Y$  is a morphism of real analytic manifolds and  $G \in \mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_Y)$ , then  $f^{-1}G$  and  $f^!G$  belong to  $\mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$ . If  $F \in \mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$  and  $f$  is proper on  $\mathrm{supp}(F)$ , then  $\mathrm{R}f_! F \simeq \mathrm{R}f_* F$  belongs to  $\mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_Y)$ . This follows from [KS90, Prop. 8.4.6].

Finally recall ([KS90, § 8.4] that  $F \in \mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$  is  $\mathbb{R}$ -constructible if for any  $x \in X$ ,  $F_x$  is perfect.

**Lemma 4.1.** *Weakly  $\mathbb{R}$ -constructible sheaves are weakly cohomologically constructible. In other words, the category  $\mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$  is a full triangulated subcategory of  $\mathrm{D}_{\mathrm{wcc}}^b(\mathbf{k}_X)$ .*

This result is implicitly proved in [KS90, Pro. 8.4.9]. For the reader's convenience, we repeat the proof (this is basically a slightly more detailed version of the proof of [KS90, Lemma 8.4.7]).

*Proof.* Let  $F \in \mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_X)$ . We want to prove that  $F$  is weakly cohomologically constructible, and this is a local problem, so we can assume  $X = \mathbb{R}^n$ .

Let  $x \in X$  and consider the real analytic function  $\varphi: X \rightarrow \mathbb{R}, y \mapsto |y - x|$ . Since  $F$  is weakly  $\mathbb{R}$ -constructible, its micro-support  $\mathrm{SS}(F)$  is a closed conic subanalytic isotropic subset of  $T^*X$  (see [KS90, Th. 8.4.2]). We can therefore apply the microlocal Bertini-Sard theorem (see [KS90, Prop. 8.3.12]), which shows that there exists  $b \in \mathbb{R}$  such that for all  $y \in X$  with  $0 < \varphi(y) < b$  we have  $d\varphi(y) \notin \mathrm{SS}(F)$ .

Now, it follows from the microlocal Morse lemma (see [KS90, Cor. 5.4.19]) that for any  $a \in \mathbb{R}$  with  $0 < a < b$ , the natural morphisms

$$\mathrm{R}\Gamma(B_b(x); F) \rightarrow \mathrm{R}\Gamma(B_a(x); F) \rightarrow F_x$$

are isomorphisms. (The second one is not directly part of the lemma, but is easily deduced, cf. e.g [KS90, Remark 2.6.9]).

Similarly (using  $-\varphi$  instead), we get that for a suitable  $b \in \mathbb{R}$  and  $0 < a < b$  the natural morphisms

$$\mathrm{R}\Gamma_x F \rightarrow \mathrm{R}\Gamma_{\overline{B_a(x)}}(X; F) \rightarrow \mathrm{R}\Gamma_{\overline{B_b(x)}}(X; F)$$

and

$$\mathrm{R}\Gamma_c(B_a(x); F) \rightarrow \mathrm{R}\Gamma_c(B_b(x); F)$$

are isomorphisms.

Since the balls  $B_a(x)$  make up a fundamental system of open neighborhoods of  $x$ , we obtain

$$\text{“colim”}_{x \in U} \mathrm{R}\Gamma(U; F) \simeq \text{“colim”}_{a \rightarrow 0} \mathrm{R}\Gamma(B_a(x); F) \simeq F_x,$$

$$\text{“lim”}_{x \in U} \mathrm{R}\Gamma_c(U; F) \simeq \text{“lim”}_{a \rightarrow 0} \mathrm{R}\Gamma_c(B_a(x); F) \simeq \text{“lim”}_{a \rightarrow 0} \mathrm{R}\Gamma_{\overline{B_a(x)}}(X; F) \simeq \mathrm{R}\Gamma_x F.$$

This completes the proof.  $\square$

## Inverse images

**Theorem 4.2.** *Let  $f: X \rightarrow Y$  be a morphism of real analytic manifolds. Let  $L_Y, G \in \mathrm{D}_{\mathrm{wRc}}^b(\mathbf{k}_Y)$  and assume that  $L_Y$  is locally constant. Then*

(a)  $f^{-1}\mathrm{R}\mathcal{H}om(L_Y, G) \xrightarrow{\simeq} \mathrm{R}\mathcal{H}om(f^{-1}L_Y, f^{-1}G).$

(b) *Assume (3.3). Then  $f^!G \otimes^L f^{-1}L_Y \xrightarrow{\simeq} f^!(G \otimes^L L_Y).$*

*Proof.* Since the problem is local on  $Y$ , we may assume that  $L_Y$  is the constant sheaf associated with for some  $L \in \mathrm{D}^b(\mathbf{k})$ . Hence  $f^{-1}L_Y \simeq L_X$ .

(a) Let  $x \in X$ , and set  $y = f(x)$ . Applying Proposition 3.5, one gets

$$\begin{aligned} \mathrm{R}\mathcal{H}om(L_X, f^{-1}G)_x &\simeq \mathrm{R}\mathcal{H}om(L, (f^{-1}G)_x) \\ &\simeq \mathrm{R}\mathcal{H}om(L, G_y) \simeq \mathrm{R}\mathcal{H}om(L_Y, G)_y \simeq (f^{-1}\mathrm{R}\mathcal{H}om(L_Y, G))_x. \end{aligned}$$

(b) Remark first that for any sheaf  $H$  on  $Y$ , one has  $\mathrm{R}\Gamma_x(f^!H) \simeq \mathrm{R}\Gamma_y H$ . Then using Proposition 3.5, one has

$$\begin{aligned} \mathrm{R}\Gamma_x(f^!G \otimes^L L_X) &\simeq (\mathrm{R}\Gamma_x f^!G) \otimes^L L \simeq (\mathrm{R}\Gamma_y G) \otimes^L L \\ \mathrm{R}\Gamma_x f^!(G \otimes^L L_Y) &\simeq \mathrm{R}\Gamma_y(G \otimes^L L_Y) \simeq (\mathrm{R}\Gamma_y G) \otimes^L L. \end{aligned}$$

Set  $A = f^!G \otimes^L L_X$  and  $B = f^!(G \otimes^L L_Y)$ . We have proved that the morphism  $A \rightarrow B$  induces for all  $x \in X$  an isomorphism  $\mathrm{R}\Gamma_x A \simeq \mathrm{R}\Gamma_x B$ . Then  $A \simeq B$  by Proposition 3.7.  $\square$



## Tensor product and hom

We shall make use of the following result, well-known among specialists. However, we shall give a proof for the reader's convenience.

**Lemma 4.3.** *Let  $L, M \in D^b(\mathbf{k})$  and let  $N \in D_f^b(\mathbf{k})$ . Then*

$$(4.1) \quad \mathrm{RHom}(L, M) \otimes^{\mathbf{L}} N \xrightarrow{\simeq} \mathrm{RHom}(L, M \otimes^{\mathbf{L}} N).$$

*Proof.* By Hypothesis, we may represent  $N$  with a bounded complex of projective modules of finite rank.

(i) Assume first that  $N = P$  is concentrated in a single degree. If  $P$  is of finite rank, there exists an integer  $n$  and an epimorphism  $\mathbf{k}^n \rightarrow P$ . If moreover  $P$  is projective, then this epimorphism has a retract and we get  $\mathbf{k}^n \simeq P \oplus Q$ . This proves the result in this case.

(ii) Now assume that  $N$  is represented by the complex  $0 \rightarrow P^0 \rightarrow \dots \rightarrow P^m \rightarrow 0$  with all  $P^j$ 's projective of finite rank. Here we assume for simplicity in the notations that  $P^0$  is in degree 0. Assume that the result is proved. Let us use the so-called "stupid truncation". Denote by  $N_0$  the complex  $0 \rightarrow P^0 \rightarrow \dots \rightarrow P^{m-1} \rightarrow 0$  and by  $u: N \rightarrow N_0$  the natural morphism. We have an exact sequence of complexes  $0 \rightarrow P^m[-m] \rightarrow N \xrightarrow{u} N_0 \rightarrow 0$  and it follows that the triangle  $P^m[-m] \rightarrow N \xrightarrow{u} N_0 \xrightarrow{+1}$  is distinguished. Arguing by induction on  $m$  the proof is complete.  $\square$

Let  $X$  and  $Y$  be real analytic manifolds. As usual, one denotes by  $q_1$  and  $q_2$  the projections from  $X \times Y$  to  $X$  and  $Y$ , respectively. One denotes  $\delta: X \rightarrow X \times X$  the diagonal morphism. One denotes by  $\boxtimes^{\mathbf{L}}$  the external product

$$F \boxtimes^{\mathbf{L}} G := q_1^{-1} F \otimes^{\mathbf{L}} q_2^{-1} G.$$

Recall [KS90, Prop. 3.4.4] that for  $F \in D_{\mathrm{Rc}}^b(\mathbf{k}_X)$  and  $G \in D^b(\mathbf{k}_Y)$ , one has the isomorphism

$$(4.2) \quad D_X F \boxtimes^{\mathbf{L}} G \xrightarrow{\simeq} \mathrm{R}\mathcal{H}om(q_1^{-1} F, q_2^! G).$$

Also note the isomorphism, for  $F_1, F_2 \in D^b(\mathbf{k}_X)$  and  $G_1, G_2 \in D^b(\mathbf{k}_Y)$ :

$$(4.3) \quad (F_1 \boxtimes^{\mathbf{L}} F_2) \otimes^{\mathbf{L}} (G_1 \boxtimes^{\mathbf{L}} G_2) \simeq (F_1 \otimes^{\mathbf{L}} G_1) \boxtimes^{\mathbf{L}} (F_2 \otimes^{\mathbf{L}} G_2).$$

**Theorem 4.4.** *Let  $L_X, F_1 \in D_{\mathrm{wRc}}^b(\mathbf{k}_X)$ , with  $L_X$  locally constant and let  $F_2 \in D_{\mathrm{Rc}}^b(\mathbf{k}_X)$ . Then*

$$(a) \quad \mathrm{R}\mathcal{H}om(L_X, F_1) \otimes^{\mathbf{L}} F_2 \xrightarrow{\simeq} \mathrm{R}\mathcal{H}om(L_X, F_1 \otimes^{\mathbf{L}} F_2).$$

$$(b) \quad \text{Assume (3.3). Then } \mathrm{R}\mathcal{H}om(F_2, F_1) \otimes^{\mathbf{L}} L_X \xrightarrow{\simeq} \mathrm{R}\mathcal{H}om(F_2, F_1 \otimes^{\mathbf{L}} L_X).$$

*Proof.* We may assume that  $L_X$  is the constant sheaf associated with  $L \in D^b(\mathbf{k})$ . The fact that  $F_1^{\mathbb{L}} \otimes F_2$  and  $R\mathcal{H}om(L_X, F_1^{\mathbb{L}} \otimes F_2)$  belong to  $D_{\text{w}\mathbb{R}\mathbb{C}}^b(\mathbf{k}_X)$  follows from [KS90, Prop. 8.4.6].

(a) Let  $x \in X$ . One has

$$\begin{aligned} (R\mathcal{H}om(L_X, F_1^{\mathbb{L}} \otimes F_2)_x) &\simeq R\mathcal{H}om(L_X, F_1)_x \otimes^{\mathbb{L}} F_{2x} \simeq \text{RHom}(L, F_{1x}) \otimes^{\mathbb{L}} F_{2x} \\ &\simeq \text{RHom}(L, F_{1x} \otimes^{\mathbb{L}} F_{2x}) \simeq (R\mathcal{H}om(L_X, F_1^{\mathbb{L}} \otimes F_2))_x. \end{aligned}$$

The second and fourth isomorphisms follow from Proposition 3.5 and the third one from Lemma 4.3.

(b) One has

$$\begin{aligned} R\mathcal{H}om(F_2, F_1^{\mathbb{L}}) \otimes^{\mathbb{L}} L_X &\simeq \delta^!(D_X F_2 \boxtimes^{\mathbb{L}} F_1) \otimes^{\mathbb{L}} \delta^{-1}(\mathbf{k}_X \boxtimes^{\mathbb{L}} L_X) \simeq \delta^!((D_X F_2 \boxtimes^{\mathbb{L}} F_1) \otimes^{\mathbb{L}} (\mathbf{k}_X \boxtimes^{\mathbb{L}} L_X)) \\ &\simeq \delta^!((D_X F_2 \otimes^{\mathbb{L}} \mathbf{k}_X) \boxtimes^{\mathbb{L}} (F_1 \otimes^{\mathbb{L}} L_X)) \simeq R\mathcal{H}om(F_2, F_1^{\mathbb{L}}) \otimes^{\mathbb{L}} L_X. \end{aligned}$$

Here, the second isomorphism follows from Theorem 4.2 (b). The other ones follow from (4.2) and (4.3).  $\square$

## Direct images

Let  $f: X \rightarrow Y$  be a morphism of real analytic manifolds. Let  $F \in D_{\text{w}\mathbb{R}\mathbb{C}}^b(\mathbf{k}_X)$  and  $L_Y \in D^b(\mathbf{k}_Y)$  being locally constant. One can ask if the morphism

$$Rf_* F \otimes^{\mathbb{L}} L_Y \rightarrow Rf_*(F \otimes^{\mathbb{L}} f^{-1} L_Y)$$

is an isomorphism. The answer is negative in general, even if we require  $F$  to be constructible, thanks to an example of [Hoh23, Rem. 4.6].

However, there is a positive answer when considering sheaves *constructible up to infinity*. Before proving the result for general direct images, let us establish it in the particular case of open embeddings.

**Lemma 4.5.** *Let  $j: U \hookrightarrow X$  be the open embedding of a subanalytic relatively compact open subset  $U$  of  $X$ . Let  $F \in D_{\text{w}\mathbb{R}\mathbb{C}}^b(\mathbf{k}_U)$  and assume that there exists  $G \in D_{\text{w}\mathbb{R}\mathbb{C}}^b(\mathbf{k}_X)$  with  $j^{-1}G \simeq F$ . Let  $L_X \in D^b(\mathbf{k}_X)$  be locally constant. Then*

(a)  $Rj_! R\mathcal{H}om(j^{-1}L_X, F) \xrightarrow{\simeq} R\mathcal{H}om(L_X, Rj_! F)$ .

(b) *Assume (3.3). Then  $Rj_* F \otimes^{\mathbb{L}} L_X \xrightarrow{\simeq} Rj_*(F \otimes^{\mathbb{L}} j^{-1}L_X)$ .*

*Proof.* As above, we may assume that  $L_X$  is the constant sheaf associated with  $L \in D^b(\mathbf{k})$ . Let  $G \in D_{\text{w}\mathbb{R}\mathbb{C}}^b(\mathbf{k}_X)$  be such that  $j^{-1}G \simeq F$ . Then  $Rj_* F \simeq R\Gamma_U G \simeq R\mathcal{H}om(\mathbf{k}_{XU}, G)$  and  $Rj_! F \simeq G_U \simeq \mathbf{k}_{XU} \otimes G$ .

(a) Note that  $j^{-1}\mathbf{R}\mathcal{H}om(L_X, G) \simeq \mathbf{R}\mathcal{H}om(j^{-1}L_X, F)$ . Using Theorem 4.4 (a), we get

$$\begin{aligned} \mathbf{R}j_!\mathbf{R}\mathcal{H}om(j^{-1}L_X, F) &\simeq \mathbf{k}_{XU} \otimes \mathbf{R}\mathcal{H}om(L_X, G) \simeq \mathbf{R}\mathcal{H}om(L_X, G \otimes \mathbf{k}_{XU}) \\ &\simeq \mathbf{R}\mathcal{H}om(L_X, \mathbf{R}j_!F). \end{aligned}$$

(b) Using Theorem 4.4 (b), we have

$$\begin{aligned} \mathbf{R}j_*F^{\mathbb{L}} \otimes L_X &\simeq \mathbf{R}\mathcal{H}om(\mathbf{k}_{XU}, G)^{\mathbb{L}} \otimes L_X \simeq \mathbf{R}\mathcal{H}om(\mathbf{k}_{XU}, G^{\mathbb{L}} \otimes L_X) \\ &\simeq \mathbf{R}j_*j^{-1}(G \otimes L_X) \simeq \mathbf{R}j_*(F^{\mathbb{L}} \otimes j^{-1}L_X). \end{aligned}$$

□

Recall the following notions extracted from [Sch23].

**Definition 4.6.** A b-analytic manifold  $X_\infty$  is a pair  $(X, \widehat{X})$  with  $X \subset \widehat{X}$  an open embedding of real analytic manifolds such that  $X$  is relatively compact and subanalytic in  $\widehat{X}$ .

A morphism  $f: X_\infty = (X, \widehat{X}) \rightarrow Y_\infty = (Y, \widehat{Y})$  of b-analytic manifolds is a morphism of real analytic manifolds  $f: X \rightarrow Y$  such that the graph  $\Gamma_f$  of  $f$  in  $X \times Y$  is subanalytic in  $\widehat{X} \times \widehat{Y}$ .

Let  $F \in \mathbf{D}_{\mathbf{wRc}}^b(\mathbf{k}_X)$ . One says that  $F$  is “weakly constructible up to infinity” or simply weakly b-constructible, if  $j_{X!}F$  (or, equivalently,  $\mathbf{R}j_*F$ ) belongs to  $\mathbf{D}_{\mathbf{wRc}}^b(\mathbf{k}_{\widehat{X}})$ . One denotes by  $\mathbf{D}_{\mathbf{wRc}}^b(\mathbf{k}_{X_\infty})$  the full triangulated subcategory of  $\mathbf{D}_{\mathbf{wRc}}^b(\mathbf{k}_X)$  consisting of weakly b-constructible sheaves.

**Theorem 4.7.** *Let  $f: X_\infty \rightarrow Y_\infty$  be a morphism of b-analytic manifolds. Let  $F \in \mathbf{D}_{\mathbf{wRc}}^b(\mathbf{k}_{X_\infty})$  and let  $L_Y \in \mathbf{D}^b(\mathbf{k}_Y)$  be locally constant. Then  $\mathbf{R}f_!F$  and  $\mathbf{R}f_*F$  belong to  $\mathbf{D}_{\mathbf{wRc}}^b(\mathbf{k}_{X_\infty})$  and*

$$(a) \quad \mathbf{R}f_!\mathbf{R}\mathcal{H}om(f^{-1}L_Y, F) \xrightarrow{\simeq} \mathbf{R}\mathcal{H}om(L_Y, \mathbf{R}f_!F).$$

$$(b) \quad \text{Assume (3.3). Then } (\mathbf{R}f_*F)^{\mathbb{L}} \otimes L_Y \xrightarrow{\simeq} \mathbf{R}f_*(F^{\mathbb{L}} \otimes f^{-1}L_Y).$$

*Proof.* Here again we may assume that  $L_Y$  is the constant sheaf associated with some  $L \in \mathbf{D}^b(\mathbf{k})$ . Set for short  $Z = \widehat{X} \times \widehat{Y}$  and denote by  $q_1$  and  $q_2$  the first and second projection from  $Z$  to  $\widehat{X}$  and  $\widehat{Y}$ . Denote by  $\Gamma_f \subset Z$  the graph of  $f$ . Note that,  $\Gamma_f$  is subanalytic in  $Z$  by definition, and relatively compact in  $Z$  since it is contained in the relatively compact subset  $X \times Y$ .

One has

$$\begin{aligned} \mathbf{R}f_!F &\simeq j_Y^{-1}\mathbf{R}q_{2!}(q_1^{-1}j_{X!}F^{\mathbb{L}} \otimes \mathbf{k}_{\Gamma_f}), \\ \mathbf{R}f_*F &\simeq j_Y^!\mathbf{R}q_{2*}\mathbf{R}\mathcal{H}om(\mathbf{k}_{\Gamma_f}, q_1^!\mathbf{R}j_{X*}F). \end{aligned}$$

Note that the supports of  $q_1^{-1}j_{X!}F \otimes \mathbf{k}_{\Gamma_f}$  and  $\mathbf{R}\mathcal{H}om(\mathbf{k}_{\Gamma_f}, q_1^!\mathbf{R}j_{X*}F)$  are contained in  $\Gamma_f$  and hence compact in  $Z$ .

(a) Let us apply the functor  $R\mathcal{H}om(L_Y, \bullet)$  to the first isomorphism. We get

$$\begin{aligned}
R\mathcal{H}om(L_Y, Rf_!F) &\simeq R\mathcal{H}om(j_Y^{-1}L_{\widehat{Y}}, j_Y^{-1}Rq_{2*}(q_1^{-1}j_{X!}F^{\mathbb{L}} \otimes \mathbf{k}_{\Gamma_f})) \\
&\simeq j_Y^{-1}R\mathcal{H}om(L_{\widehat{Y}}, Rq_{2*}(q_1^{-1}j_{X!}F^{\mathbb{L}} \otimes \mathbf{k}_{\Gamma_f})) \\
&\simeq j_Y^{-1}Rq_{2*}R\mathcal{H}om(L_{\widehat{X} \times \widehat{Y}}, q_1^{-1}j_{X!}F^{\mathbb{L}} \otimes \mathbf{k}_{\Gamma_f}) \\
&\simeq j_Y^{-1}Rq_{2*}(R\mathcal{H}om(q_1^{-1}L_{\widehat{X}}, q_1^{-1}j_{X!}F^{\mathbb{L}}) \otimes \mathbf{k}_{\Gamma_f}) \\
&\simeq j_Y^{-1}Rq_{2*}(q_1^{-1}R\mathcal{H}om(L_{\widehat{X}}, j_{X!}F^{\mathbb{L}}) \otimes \mathbf{k}_{\Gamma_f}) \\
&\simeq j_Y^{-1}Rq_{2*}(q_1^{-1}j_{X!}R\mathcal{H}om(L_X, F)^{\mathbb{L}} \otimes \mathbf{k}_{\Gamma_f}) \\
&\simeq Rf_!R\mathcal{H}om(f^{-1}L_Y, F).
\end{aligned}$$

The second and fifth isomorphism use Theorem 4.2 (a), the fourth uses Theorem 4.4 (a), and the sixth isomorphism uses Lemma 4.5 (a). On the other hand, the third isomorphism is classical (see [KS90, (2.6.15)]).

(b) The proof is completely analogous, using parts (b) of the statements mentioned above instead.  $\square$

Recall the isomorphisms which hold for any  $F \in D^b(\mathbf{k}_X)$  and  $L \in D^b(\mathbf{k})$ . One has

$$(4.4) \quad \begin{aligned}
R\Gamma_c(X; F^{\mathbb{L}} \otimes L_X) &\simeq R\Gamma_c(X; F)^{\mathbb{L}} \otimes L, \\
R\Gamma(X; R\mathcal{H}om(L_X, F)) &\simeq R\mathrm{Hom}(L, R\Gamma(X; F)).
\end{aligned}$$

**Corollary 4.8.** *Let  $F \in D_{\mathrm{wRc}}^b(\mathbf{k}_X)$  and let  $L \in D^b(\mathbf{k})$ . Let  $U$  be an open relatively compact subanalytic subset of  $X$ . Then*

$$(a) \quad R\Gamma_c(U; R\mathcal{H}om(L_X, F)) \simeq R\mathrm{Hom}(L, R\Gamma_c(U; F)).$$

$$(b) \quad \text{Assume (3.3). Then } R\Gamma(U, F^{\mathbb{L}} \otimes L_X) \simeq R\Gamma(U; F)^{\mathbb{L}} \otimes L.$$

*Proof.* Apply Theorem 4.7 to the sheaf  $F|_U$  with  $X_\infty = (U, X)$ ,  $Y = \mathrm{pt}$  and  $f = a_U$ .  $\square$

Note that this corollary applies in particular when  $X_\infty = (X, \widehat{X})$  is  $b$ -analytic and  $F \in D_{\mathrm{wRc}}^b(\mathbf{k}_{X_\infty})$ . In this case, one gets

$$\begin{aligned}
R\Gamma_c(X; R\mathcal{H}om(L_X, F)) &\simeq R\mathrm{Hom}(L, R\Gamma_c(X; F)), \\
R\Gamma(X, F^{\mathbb{L}} \otimes L_X) &\simeq R\Gamma(X; F)^{\mathbb{L}} \otimes L.
\end{aligned}$$

## Duality

From our above result, we obtain slight generalisations of [KS90, Exe. VIII.3].

Recall that for any  $G \in D^b(\mathbf{k}_Y)$ , one has  $D_X f^{-1}G \simeq f^! D_Y G$ .

**Corollary 4.9.** *Let  $f: X \rightarrow Y$  be a morphism of real analytic manifolds. Let  $G \in D_{\mathbb{R}c}^b(\mathbf{k}_Y)$  and assume that  $L_Y \in D^b(\mathbf{k}_Y)$  is locally constant. Then*

$$D_X f^!(G \otimes^L L_Y) \simeq f^{-1} D_Y(G \otimes^L L_Y).$$

*Proof.* One has the chain of isomorphisms

$$\begin{aligned} D_X f^!(G \otimes^L L_Y) &\simeq R\mathcal{H}om(f^! G \otimes^L f^{-1} L_Y, \omega_X) \\ &\simeq R\mathcal{H}om(f^{-1} L_Y, D_X(f^! G)) \simeq R\mathcal{H}om(f^{-1} L_Y, f^{-1} D_Y G) \\ &\simeq f^{-1} R\mathcal{H}om(L_Y, D_Y G) \simeq f^{-1} D_Y(G \otimes^L L_Y). \end{aligned}$$

□

Recall that for any  $F \in D^b(\mathbf{k}_X)$ , one has  $D_Y Rf_! F \simeq Rf_* D_X F$ .

**Corollary 4.10.** *Let  $f: X_\infty \rightarrow Y_\infty$  be a morphism of  $b$ -analytic manifolds. Let  $F \in D_{\mathbb{R}c}^b(\mathbf{k}_{X_\infty})$  and let  $L_Y \in D^b(\mathbf{k}_Y)$  be locally constant. Then*

$$Rf_! D_X(F \otimes^L f^{-1} L_Y) \simeq D_Y(Rf_* F \otimes^L L_Y).$$

*Proof.* One has the chain of isomorphisms

$$\begin{aligned} Rf_! D_X(F \otimes^L f^{-1} L_Y) &\simeq Rf_! R\mathcal{H}om(f^{-1} L_Y, D_X F) \\ &\simeq R\mathcal{H}om(L_Y, Rf_! D_X F) \simeq R\mathcal{H}om(L_Y, D_Y Rf_* F) \\ &\simeq D_Y(Rf_* F \otimes^L L_Y). \end{aligned}$$

□

## Micro-support

**Remark 4.11.** Consider a field extension  $\mathbf{l}$  of the field  $\mathbf{k}$ . Then of course all preceding results apply with  $L = \mathbf{1}_X$  (the case of interest in [Hoh23]). Moreover, note that [KS90, Rem. 5.1.5] asserts that if  $for$  denotes the forgetful functor

$$for: D^b(\mathbf{l}_X) \rightarrow D^b(\mathbf{k}_X),$$

and if  $F \in D^b(\mathbf{l}_X)$ , then the micro-support of  $F$  and that of  $for(F)$  are the same.

To conclude, let us consider the action of the functors  $L \otimes \cdot$  and  $R\mathcal{H}om(L, \cdot)$  on the micro-support. Remark first that for  $F, L_X \in D^b(\mathbf{k}_X)$  with  $L_X$  locally constant, one has

$$(4.5) \quad \text{SS}(L_X \otimes^L F) \subset \text{SS}(F), \quad \text{SS}(R\mathcal{H}om(L_X, F)) \subset \text{SS}(F).$$

Indeed, one has by [KS90, Prop. 5.4.14]

$$\text{SS}(L_X \otimes^L F) \subset T_X^* X + \text{SS}(F) = \text{SS}(F),$$

and similarly with  $R\mathcal{H}om(L_X, F)$ .

**Proposition 4.12.** *Assume that  $\mathbf{k}$  is a field. Let  $F, L_X \in D_{\text{wRc}}^b(\mathbf{k}_X)$  with  $L_X$  locally constant,  $L_X \neq 0$ . Then*

$$\text{SS}(L_X \otimes F) = \text{SS}(F), \quad \text{SS}(\mathcal{R}\mathcal{H}om(L_X, F)) = \text{SS}(F).$$

*Proof.* The problem is local and we may assume that  $L_X$  is the constant sheaf associated with  $L \in D^b(\mathbf{k})$ . Then  $L = \bigoplus_j H^j(L)[-j]$  and we may assume that  $L \in \text{Mod}(\mathbf{k})$ . In this case, there exists  $K \in \text{Mod}(\mathbf{k})$  such that  $L \simeq \mathbf{k} \oplus K$ , hence  $L_X \simeq \mathbf{k}_X \oplus K_X$  and the result follows (see e.g. [KS90, Prop. 5.1.3]).  $\square$

## References

- [Hoh23] Andreas Hohl, *Field extensions and Galois descent for sheaves of vector spaces* (2023), available at [arxiv:2302.14837v1](https://arxiv.org/abs/2302.14837v1).
- [KS90] Masaki Kashiwara and Pierre Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer-Verlag, Berlin, 1990.
- [KS06] ———, *Categories and Sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 332, Springer-Verlag, Berlin, 2006.
- [Sch23] Pierre Schapira, *Constructible sheaves and functions up to infinity*, Journal of Applied and Computational Topology (2023, to appear), available at [arXiv:2012.09652](https://arxiv.org/abs/2012.09652).

Andreas Hohl  
 Université Paris Cité et Sorbonne Université,  
 CNRS, IMJ-PRG, F-75013 Paris, France  
 e-mail: andreas.hohl@imj-prg.fr  
<https://www.andreashohl.eu>

Pierre Schapira  
 Sorbonne Université, CNRS IMJ-PRG  
 e-mail: pierre.schapira@imj-prg.fr  
<http://webusers.imj-prg.fr/~pierre.schapira/>