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A TOPOLOGICAL ALGORITHM FOR THE FOURIER TRANSFORM OF STOKES DATA AT INFINITY

JEAN DOUÇOT AND ANDREAS HOHL

Abstract. We reinterpret a result of T. Mochizuki about the Fourier transform of Stokes data of irregular connections on the Riemann sphere in the language of Stokes local systems due to P. Boalch. We thus obtain a clean topological description of the Stokes matrices of the Fourier transform from infinity to infinity in a large number of cases of one level. In particular, this induces explicit isomorphisms between wild character varieties, in a much larger range of examples than those for which such isomorphisms have previously been written down. We conjecture that these isomorphisms are compatible with the quasi-Hamiltonian structure of the wild character varieties.

1. Introduction

Stokes data are generalised monodromy data associated to meromorphic connections with irregular singularities. Considering them is necessary to obtain an irregular analogue of the famous Riemann–Hilbert correspondence, which gives an equivalence between connections with regular singularities (the de Rham side) and monodromy data (the Betti side). The irregular version, known as the Riemann–Hilbert–Birkhoff correspondence, provides an equivalence between possibly irregular connections and generalised monodromy data. The latter can be encoded in several different ways, one of them being the notion of a *Stokes local system* ([Boa01, Boa21]), which is closely related to the wild fundamental group of [MR91]. Via this correspondence, moduli spaces of meromorphic connections with prescribed behaviour at the singularities give rise to the so-called wild character varieties, the irregular analogues of the usual character varieties. These moduli spaces can also be seen, via the nonabelian Hodge correspondence, as moduli spaces of irregular Higgs bundles (the Dolbeault side), and possess very rich geometric (symplectic [Boa01], hyperkähler [BB04]) structures, leading to the notion of (wild) nonabelian Hodge space [Boa18b].

The Fourier–Laplace transform is an operation acting on meromorphic connections on the affine line. It plays an important role in many contexts: for example, it underlies many symmetries of isomonodromy systems such as the existence of different Lax pairs for several Painlevé-type equations [Boa05, Boa08, Boa12, Yam16, SW04] and is an essential ingredient of Arinkin's extension [Ari10] of the Katz algorithm to rigid irregular connections.

In turn, the question of describing directly the action of the Fourier transform on the Betti side, i.e. on Stokes data, via the Riemann–Hilbert–Birkhoff correspondence, is natural and of great interest. In particular, a completely intrinsic topological description of the transformation of Stokes data via Fourier transform would be desirable.

This problem already has quite a long history. Explicit expressions for the Stokes data of the Fourier transform have been obtained mainly in two cases: the case of a Fourier transform of a regular singular system (see e.g. [BJL81], [Mal91, Chap. XII], [Boa15, §2] and [DHMS20]) and the case of quadratic monomial exponential factors at infinity (also called the case of *pure Gaussian type*, see e.g. [Sab16], [Ho22]). Beyond these cases, it has proved difficult to determine the Fourier transform of Stokes data outside of some specific situations. For example, in [Boa15], a symplectic isomorphism was found, in the so-called

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simply-laced case featuring one irregular singularity at infinity of order less than three together with regular singularities at finite distance, between two wild character varieties related by the Fourier transform, but it is unclear whether this isomorphism is the one induced by the Fourier transform.

On the other hand, there have been several studies aiming at tackling the general case, beginning with Malgrange's book [Mal91], which provides a complete answer, albeit hardly explicit. The problem has been considered more recently by T. Mochizuki, starting with the article [Moc10], which considers the general case with a different approach. In a second recent article [Moc21], T. Mochizuki develops another framework to obtain a more explicit transformation rule for Stokes data under Fourier transform in full generality. This is achieved by introducing a new description of Stokes data, which the author of loc. cit. calls *Stokes shells*. Although the result obtained there is valid in great generality, understanding the implications of its statement and proof in terms of explicit Stokes matrices is not an obvious task.

In this article, we will – in a large class of examples – reformulate the result of [Moc21] as a topological transformation rule for Stokes local systems, by making the connection between the latter and the notion of Stokes shells. Basically, what we do is the following: In his article, T. Mochizuki starts from the notions of Stokes filtrations and Stokes local systems (see [Moc21, Chapter 2] and in particular §2.4.2 for the reference to P. Boalch's work). He modifies the presentation of Stokes data and introduces his notion of "Stokes shells", which is more adapted to the formulation of his final result. It is our aim in this paper to "translate back" the latter to the world of Stokes local systems, to draw some conclusions and to give explicit examples.

In particular, our work is not at all to be seen as a replacement or a competing work with [Moc21]. On the contrary, the present article deeply relies on the results proved by T. Mochizuki and reinterprets them in certain cases in a language which we can directly use for explicit computations involving Stokes matrices.

The setting. We will consider a large class of examples, going much beyond the cases for which explicit formulas for the Stokes data of the Fourier transform have been known. The situation we will be considering is the case of meromorphic connections on the Riemann sphere where both the initial connection and its Fourier transform only have one singularity at infinity. This happens when all the exponential factors at ∞ of the initial connection are of slope > 1 (and, by the stationary phase formula, this is then the case for its Fourier transform as well).

In order to have a cleaner description of the algorithm, we will make the following further simplifying assumptions:

- The set of exponential factors at ∞ of the connection is of the form $\{az^k \mid a \in A\}$ for some fixed $k \in \mathbb{Q}_{>1}$ and some finite set $A \subset \mathbb{C}\backslash\{0\}$ (where z is a local coordinate at 0),
- $|a| = |b|$ for any $a, b \in A$, i.e. all parameters lie on the same circle centred at the origin.

Note that these assumptions seem to entail no true loss of generality from the weaker assumption that all exponential factors have pure level k, indeed up to admissible deformations it is always possible to move the coefficients of the exponential factors so as to satisfy these conditions. With some caution, we could thus probably consider the case of pure level k. We try, however, to keep it simple here and will not make this generalisation, since our aim is to give a description in the clearest possible way, which is already sufficiently involved in this slightly more restricted situation.

Main results. The main result of this work is an algorithm translating Mochizuki's description of the Fourier transform of Stokes data into the language of Stokes local systems,

in the class of cases covered by our main assumption: If (E, ∇) is a connection on the affine line satisfying this assumption, and $(\widehat{E}, \widehat{\nabla})$ is its Fourier transform, the algorithm allows us to determine the Stokes local system of $(\widehat{E}, \widehat{\nabla})$ as a function of the one of (E, ∇). While we refer the reader to the main body of the article for the fully detailed statements, let us sketch the main ideas underlying the algorithm and its structure.

It is well-known that the formal data, i.e. the Levelt–Turrittin normal form, of $(\widehat{E}, \widehat{\nabla})$ are determined by those of (E, ∇): This is expressed by the stationary phase formula [Mal91, Sab08], which says that the exponential factors of $(\widehat{E}, \widehat{\nabla})$ are related to those of $(\widehat{E}, \widehat{\nabla})$ by a Legendre transform.

Usually, the Legendre transform is viewed as providing a bijection between the sets of exponential factors before and after the Fourier transform. The crucial idea behind the algorithm is that it can actually be interpreted in a stronger sense. This relies on describing the formal data of the connections à la Deligne–Malgrange, in terms of the exponential local system $\mathcal{I}$. In brief, $\mathcal{I}$ is an infinite collection of *Stokes circles* lying over the circle ∂ of directions around infinity, and the exponential factors of the connection each correspond to one Stokes circle. The stronger version of the Legendre transform is the following

Crucial idea: The Legendre transform provides a self-homeomorphism ℓ of the collection $\mathcal{I}_{>1}$ of all Stokes circles of slope > 1 at infinity (see [Mal91, Dou23]).

Now, the idea of the algorithm is, roughly speaking, that it is possible to use ℓ to transport the Stokes data of (E, ∇) and obtain those of $(\widehat{E}, \widehat{\nabla})$.

To give a precise formulation of this idea, one needs to describe the formal data and the Stokes data of connections in a suitable way. The description we use is as follows:

- Formal data consist of a formal local system $V^0 \to \mathcal{I}$. The irregular class Θ of the formal local system determines an auxiliary surface $\widetilde{\Sigma}(\Theta)$, as well as a *fission surface* $\underline{\widetilde{\Sigma}}(\Theta)$. There is a finite collection of distinguished paths $\gamma_{i\to j}$ on the fission surface, whose endpoints i, j are points on some Stokes circle of Θ.
- For any formal local system V^0 with irregular class Θ, Stokes data compatible with these formal data, if they exist, consist of the data of a *Stokes local system* $\mathbb{V}$ on $\widetilde{\Sigma}(\Theta)$, or equivalently of a *fission local system* $\underline{\mathbb{V}}$ on the fission surface $\underline{\widetilde{\Sigma}}(\Theta)$.
- A fission local system $\underline{\mathbb{V}}$ can be reconstructed up to isomorphism from the data of the parallel transport $\rho_{i\to j}$ along a subset of the set of distinguished paths $\gamma_{i\to j}$ in the fission surface, the set of *Stokes paths*.

If (E, ∇) is an irregular connection on the affine line, with one singularity at infinity, its Levelt–Turrittin normal form determines a formal local system V^0, and the Stokes data of (E, ∇) determine a Stokes local system $\mathbb{V}$, and a fission local system $\underline{\mathbb{V}}$. The linear maps $\rho_{i\to j}$ basically correspond to the nontrivial entries of the Stokes matrices. In this language, Mochizuki's description of the Fourier transform of Stokes data can be quite simply formulated as follows:

Theorem 1.1 (see Theorem 5.3 for the fully detailed statement)**.** *Let (E, ∇) be an irreducible connection on the affine line satisfying our assumption, Θ its irregular class, and $\underline{\mathbb{V}}$ its fission local system. Let $(\widehat{E}, \widehat{\nabla})$ be the Fourier transform of (E, ∇), and $\widehat{V}^0$, $\widehat{\Theta}$, $\underline{\mathbb{V}}$ its formal local system, irregular class and fission local system respectively. We have:*

- *The formal local systems on both sides are related by the Legendre transform, up to a possible sign for the formal monodromy, i.e.*

$$\widehat{V}^0 \cong \ell_* V^0 \otimes W,$$

where W is a rank one local system on ∂ with monodromy $(-1)^{\mathrm{Irr}(I)}$ around any Stokes circle I.

- *The fission local system $\underline{\widehat{\mathbb{V}}}$ is determined as follows: For any Stokes path $\widehat{\gamma}_{\hat{i}\to\hat{j}}$ in the fission surface $\underline{\widetilde{\Sigma}}(\widehat{\Theta})$, the corresponding parallel transport $\widehat{\rho}_{\hat{i}\to\hat{j}}$ is obtained, up to a sign (which is explicitly determined), by pulling back the endpoints $\hat{i},\hat{j}$ to the initial fission surface $\underline{\widetilde{\Sigma}}(\Theta)$ using the inverse of the Legendre transform and taking the parallel transport in the fission local system $\underline{\mathbb{V}}$ along the path $\gamma_{\ell^{-1}(\hat{i})\to\ell^{-1}(\hat{j})}$, i.e.*

$$\widehat{\rho}_{\hat{i}\to\hat{j}} = \pm\rho_{\ell^{-1}(\hat{i})\to\ell^{-1}(\hat{j})}.$$

Let us note that one has to carefully keep track of the signs in the formal local system here in order to correctly determine $\underline{\widehat{\mathbb{V}}}$ from the parallel transports along the Stokes paths. In particular, the sign $(-1)^{\mathrm{Irr}(I)}$ comes by gluing the graded pieces of the formal local system from several intervals and introducing multiple signs in this gluing process, which is made precise in Section 5.

In particular, this allows us to compute explicitly the Stokes matrices of $(\widehat{E},\widehat{\nabla})$ as a function of the Stokes matrices of (E,∇). In turn, the algorithm induces an explicit isomorphism between the wild character varieties on both sides of the Fourier transform:

Theorem 1.2 (see Theorem 5.4)**.** *Let Θ be an irregular class satisfying the above assumptions and let $\widehat{\Theta}$ be its formal Fourier transform. The algorithm induces an algebraic isomorphism between the reduced representation varieties $\mathcal{E}_\Theta$ and $\mathcal{E}_{\widehat{\Theta}}$, provided that they are nonempty.*

Remarkably, the Fourier transform is thus algebraic both on the de Rham side and on the Betti side.

We conjecture that these isomorphisms are compatible with the quasi-Hamiltonian structures of the wild character varieties on both sides of the Fourier transform. This symplectic nature of the Fourier transform is already known in a few particular cases [Boa15, Sza15], and this is consistent with the broader perspective of 'global Lie theory' taken in [Boa18b], where one views the moduli spaces on both sides of the Fourier transform as two different 'representations' of the same abstract nonabelian Hodge space.

Finally, we discuss in detail several examples of explicit computations of the Stokes matrices of the Fourier transform, in all of which the conjecture is satisfied.

The isomorphisms provided by the algorithm are entirely topological and involve some very nontrivial combinatorics. While they give a very explicit way of computing Stokes data for the Fourier transform, it would be desirable to understand directly (without passing through the Riemann–Hilbert–Birkhoff correspondence and the result of [Moc21]) why they are well-defined, i.e. why they produce an element in the correct moduli space of Stokes local systems. It seems possible that some of the combinatorial structures which appear in the study of related spaces, such as braid varieties, and their cluster and symplectic structures [STZ19, CGGS20, GK22] might be used to shed some light on this question, or to understand why the isomorphisms are quasi-Hamiltonian.

Structure of the article. The article is organised as follows. We start by recalling a few facts about the formal data of irregular connections in Section 2, as well as about the description of Stokes data in terms of Stokes local systems, leading to the construction of wild character varieties, in Section 3.

We then review the description of Stokes data in terms of deformation data used by T. Mochizuki in [Moc21] and relate it to Stokes local systems in Section 4 in Section 4, via the observation that to each Stokes arrow in the Stokes diagram corresponds one deformation datum. The most technical work happens in §4.3, but it is not necessary to read these details in order to understand the final statement. In Section 5, we formulate the result of Mochizuki as an algorithm that yields the Stokes local system of the Fourier transform starting from the initial Stokes local system, thus inducing an isomorphism between the corresponding wild character varieties.

Finally, in Section 6, we use this algorithm to compute explicitly in several examples the Stokes matrices of the Fourier transform and determine the corresponding isomorphisms, which in all cases are symplectic, strongly suggesting that this will be true in the general case.

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2. Formal data of irregular connections

The classification of irregular connections up to formal gauge transformations is provided by the well-known Levelt–Turrittin theorem. It will be more convenient for us to use a formulation of the formal classification in terms of local systems, that we briefly review now, following mainly [Boa21, BY15].

2.1. The exponential local system.

The idea is to view the exponential factors as sections of a local system on the circle of directions around the singularity. Let $\Sigma = \mathbb{P}^1$. Let $\varpi\colon \widehat{\Sigma} \to \Sigma$ the real oriented blow-up at ∞ of Σ. The preimage $\partial := \varpi^{-1}(\infty)$ is a circle whose points correspond to the directions around ∞. An open subset of ∂ corresponds to a germ of open sector at infinity.

Let z be the standard complex coordinate on $\mathbb{C}$, so that $1/z$ provides a local coordinate $\mathbb{P}^1$ at infinity. The exponential local system $\mathcal{I}$ is a local system of sets (i.e. a covering space) on ∂ whose sections on open intervals in ∂ are holomorphic functions of the form

$$\sum_i a_i z^{k_i}$$

on the corresponding germs of open sectors, where $k_i \in \mathbb{Q}_{>0}$, and $a_i \in \mathbb{C}$. Each connected component of such a local section is a finite order cover of the circle ∂. More precisely, let r be the smallest integer such that the expression $q = \sum_i a_i z^{k_i}$ is a polynomial in $z^{1/r}$. The corresponding holomorphic function is multivalued, and becomes single-valued when passing to a finite covering $t^r = z$. Therefore, the corresponding connected component of $\mathcal{I}$, which we denote by $\langle q \rangle$, is an r-sheeted covering of ∂. As a topological space, it is homeomorphic to a circle, and $\mathcal{I}$ is thus a disjoint union of (an infinite number of) circles.

More concretely, for $q = \sum_i a_i z^{k_i}$ as above, a point on the circle $\langle q \rangle$ is given by a pair $(\theta, [\tilde{q}])$, where $\theta \in \partial$ is a point and $[\tilde{q}]$ is the germ on a small sector around the direction θ of a holomorphic function $\tilde{q} = \sum_i a_i z^{k_i}$, given by a choice of determination of $z^{1/r}$. We denote by $\pi_q\colon \langle q \rangle \to \partial$ the corresponding covering (the restriction of π to $\langle q \rangle$). It is given by $(\theta, [\tilde{q}]) \mapsto \theta$.

The integer $r =: \mathrm{ram}(q)$ is the *ramification order* of the circle $\langle q \rangle$. If $r = 1$ we say that the circle $\langle q \rangle$ is unramified. The degree s of q as a polynomial in $z^{1/r}$ is the *irregularity* $s =: \mathrm{Irr}(\langle q \rangle)$ of $\langle q \rangle$ and the quotient $s/r =: \mathrm{slope}(I)$ is the *slope* of $\langle q \rangle$.

If $d \in \partial$, we denote by $\mathcal{I}_d$ the fibre of $\mathcal{I}$ over the direction d. Taking d as a basis point of ∂, the monodromy of $\mathcal{I}$ is the automorphism

$$\rho\colon \mathcal{I}_d \to \mathcal{I}_d$$

of the fibre $\mathcal{I}_d$ obtained when going once around ∂ in the positive direction.

2.2. Formal local systems.

Definition 2.1. *An $\mathcal{I}$-graded local system on ∂ is a local system $V^0 \to \partial$ of finite-dimensional vector spaces, together with a grading of each fibre V^0_d, with $d \in \partial$, by the set $\mathcal{I}_d$. This means that for any $d \in \mathcal{I}_d$, there is a direct sum decomposition*

$$V^0_d = \bigoplus_{i \in \mathcal{I}_d} V^0_d(i)$$

such that for any path γ from d to d' in ∂ (which naturally induces actions $\rho_\gamma\colon V_d \to V_{d'}$ and $\rho_\gamma\colon \mathcal{I}_d \to \mathcal{I}_{d'}$), one has

$$\rho_\gamma(V_d(i)) = V_q(\rho_\gamma(i))$$

For a fixed $d \in \partial$, let us set $n_i := \dim V_d^0(i)$ for $i \in \mathcal{I}_d$. Since V^0 is finite-dimensional, there is only a finite number of elements $i \in \mathcal{I}_d$ for which n_i is nonzero. Furthermore, since the grading is locally constant, n_i only depends on the connected component $\langle q\rangle$ of $\mathcal{I}$ containing i, and we set $n_{\langle q\rangle} := n_i$.

Definition 2.2. *An irregular class is a function $\Theta\colon \pi_0(\mathcal{I}) \to \mathbb{N}$ with finite support.*

The dimensions n_i of the graded pieces thus define an irregular class $\Theta(V^0)$ associated to V^0. The connected components $\langle q\rangle \in \pi_0(\mathcal{I})$ such that $n_{\langle q\rangle} \neq 0$ are called the *active circles*.

Let us again fix a basis direction $d \in \partial$ and denote also by $\rho \in \mathrm{Aut}(V_d)$ the monodromy of V:

$$\rho\colon V_d^0 \to V_d^0.$$

Since the grading is locally constant, the monodromy of V has to be compatible with the monodromy of $\mathcal{I}$, that is

$$\rho(V_d^0(i)) = V_d^0(\rho(i)),$$

for any $i \in \mathcal{I}_d$. In particular, the $\mathcal{I}$-graded local system V^0 globally splits as

$$V^0 = \bigoplus_{\langle q\rangle \in \pi_0(\mathcal{I})} V^0_{\langle q\rangle},$$

where $V^0_{\langle q\rangle}$ is a $\langle q\rangle$-graded local system and $V^0_{\langle q\rangle} \neq 0$ if and only if $\langle q\rangle$ is an active circle.

More explicitly, let $\langle q\rangle$ be a circle in $\mathcal{I}$ with ramification r. Fix $d \in \partial$ and choose a numbering on the fibre $\langle q\rangle_d = \{i_0, \dots, i_{r-1}\}$ such that $\rho(i_k) = i_{k+1}$ for $i = 0, \dots, r-1$ (with $i_r := i_0$). Then the monodromy of the piece $V^0_{\langle q\rangle,d} := \bigoplus_{k=0}^{r-1} V_d^0(i_k)$ has the form

$$\rho_{\langle q\rangle} = \begin{pmatrix} 0 & \dots & \dots & 0 & \rho_{r-1,0} \\ \rho_{0,1} & \ddots & & \vdots & 0 \\ 0 & \rho_{1,2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \rho_{r-2,r-1} & 0 \end{pmatrix} \in \mathrm{Aut}(V^0_{\langle q\rangle}) \tag{1}$$

with $\rho_{i_k,i_{k+1}}\colon V_d^0(i_k) \to V_d^0(i_{k+1})$.

The group $\mathrm{GrAut}(V^0)$ of graded automorphisms of V_d^0 is

$$\mathrm{GrAut}(V_d^0) = \prod_{i \in \mathcal{I}_d} \mathrm{GL}(V_d^0(i)).$$

An $\mathcal{I}$-graded local system V^0 is entirely determined up to isomorphism by the data of its irregular class Θ and the equivalence class $\mathcal{C}$ of its monodromy under graded automorphisms (acting by conjugation). We will refer to the pair $(\Theta, \mathcal{C})$ as the formal data of V^0.

This language enables us to have a more geometric formulation of the Levelt–Turrittin decomposition:

Theorem 2.3 (see [Mal91, Thm 2.3])**.** *The category of connections on the formal punctured disk is equivalent to the category of $\mathcal{I}$-graded local systems.*

In turn, to any algebraic connection (E, ∇) on the affine line is associated (by considering the corresponding connection on the formal punctured disk) a *formal local system* $V^0 \to \partial$ (which is $\mathcal{I}$-graded). The active circles correspond to the exponential factors of the connection in the Levelt–Turrittin normal form, and the monodromy of V^0 is related to

the exponent of formal monodromy. In particular, a connection is regular if and only if its only active circle is the tame circle $\langle 0\rangle$.

A useful observation is that it is equivalent to view an $\mathcal{I}$-graded local system $V^0 \to \partial$ as a local system on $\mathcal{I}$, such that the following diagram commutes:

$$\begin{tikzcd} V^0 \arrow[r] \arrow[dr] & \mathcal{I} \arrow[d, "\pi"] \\ & \partial \end{tikzcd}$$

If $i \in \mathcal{I}$, the fibre of V^0 over i is $V^0_i := V^0_d(i)$, where $d = \pi(i) \in \partial$. If $\langle q\rangle$ is a circle of ramification r, when going once around $\langle q\rangle$ we go r times around ∂. As a consequence, the monodromy $\rho_{\langle q\rangle}$ of the piece $V^0_{\langle q\rangle}$, now seen as a local system over $\langle q\rangle$, is related to the monodromy of $V^0_{\langle q\rangle}$ seen as a $\langle q\rangle$-graded local system by

$$\rho_{\langle q\rangle} = \rho_{0,r-1} \circ \rho_{r-1,r-2} \circ \cdots \circ \rho_{1,0} \colon V^0_d(i) \to V^0_d(i),$$

keeping the previous notations, with $i_0 = i$. We thus have (see [BY20] for this viewpoint):

Theorem 2.4. *The category of connections on the formal punctured disk is equivalent to the category of local systems on $\mathcal{I}$ with compact support.*

Finally, notice that the notion of local system on ∂ that we will be using here to describe the formal data of connections is closely related to the one of $2\pi\mathbb{Z}$-equivariant local system on $\mathbb{R}$ which is used in [Moc21]:

Lemma 2.5. *A local system on S^1 is the same as a $2\pi\mathbb{Z}$-equivariant local system on $\mathbb{R}$.*

Proof. If V^0 is a local system on S^1 and $\tau \colon \mathbb{R} \to S^1, x \mapsto e^{ix}$ is the universal covering, then $\tau^{-1}V^0$ is a $2\pi\mathbb{Z}$-equivariant local system on $\mathbb{R}$. The inverse construction is also easily described. □

Let $\langle q\rangle \subset \mathcal{I}$ be a circle in the the exponential local system, and denote by $\pi_q \colon \langle f\rangle \to \partial$ the projection as above. Then, under the correspondence just explained, an $\langle q\rangle$-graded local system V^0 on ∂ corresponds to a $2\pi\mathbb{Z}$-equivariant local system L on $\mathbb{R}$ such that each stalk is graded as

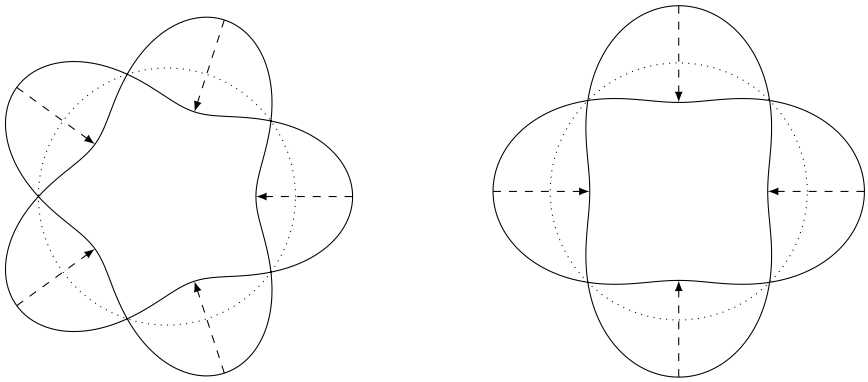
$$L_x = \bigoplus_{i \in \pi_q^{-1}(\tau(x))} V^0_x(i)$$

and the isomorphism $L_x \simeq L_{x+2\pi}$ given by the $2\pi\mathbb{Z}$-invariance is a graded isomorphism induced by the permutation that comes from the monodromy of V^0 at $\pi_q^{-1}(\tau(x))$.

3. Stokes local systems and wild character varieties

The essence of the Stokes phenomenon is that given an irregular connection (E, ∇), there exist isomorphisms between the local system V of its analytic solutions and the corresponding formal local system V^0, but these isomorphisms are only valid on sectors with limited angular width around the singularities. To have an analogue of the Riemann–Hilbert correspondence, which for connections with regular singularities provides an equivalence between connections and their monodromy data, in the case of irregular singularities one has to add extra data which account for passing from one sector to the next, known as Stokes data. In the literature, one finds many different ways of describing Stokes data. In this work, we will mostly use the formulation of [Boa21] in terms of Stokes local systems, which has the advantage of providing intrinsic explicit presentations of moduli spaces of Stokes data, which we briefly review in the next section. We will later relate it to another description used in [Moc21] in terms of deformation data.

We will present everything in our case of interest, where $\Sigma = \mathbb{P}^1(\mathbb{C})$ is the Riemann sphere, and the only marked point is at ∞. We write $\Sigma^\circ := \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$.



(A) One active circle $\langle z^{5/2} \rangle$ (B) Two active circles: $\langle z^2 \rangle$ and $\langle -z^2 \rangle$

FIGURE 1. Stokes diagrams for two examples of irregular classes. The diagram represents the growth rate of the exponential factors as a function of the direction around the singularity. The dotted circle separates the regions where the exponential is growing or decreasing near ∞. The Stokes arrows are also represented.

3.1. **Singular directions, Stokes arrows and Stokes groups.** Let (E, ∇) be a connection on the affine line and $V^0 \to \mathcal{I}$ the corresponding formal local system, with irregular class Θ. Let $\mathbb{I} \subset \mathcal{I}$ be the finite subcover of ∂ consisting of the active circles of V^0.

Definition 3.1. *Let $d \in \partial$. We define a partial order $\prec_d$ on $\mathbb{I}_d$ in the following way. Let $i, j \in \mathbb{I}_d$, and let q_i, q_j be the corresponding germs of holomorphic functions. We say that $i \prec_d j$ if and only if the exponential $e^{q_i - q_j}$ decays fastest in the direction d when z tends to ∞. This happens if the leading term of the difference $(q_i - q_j)(z)$ is in $\mathbb{R}_{<0}$ when z is in the direction d and $|z| \gg 1$.*

In this case, we will say there is a Stokes arrow from j to i over d, and we also write $j \to_d i$ instead of $i \prec_d j$.

Definition 3.2. *Let $d \in \partial$. If there exist $i, j \in \mathbb{I}_d$ such that $i \prec_d j$, then d is a* singular direction*, or* anti-Stokes direction *for (E, ∇). We denote by $\mathbb{A} \subset \partial$ the set of singular directions.*

When the exponential factors consist of only one monomial, the relative dominance order of the exponential factors can be visualised in the *Stokes diagram* of the connection (see Figure 1), which is obtained by plotting $|e^q| = e^{\mathrm{Re}(q)}$ as a function of $d \in \partial$ (for $|z| \gg 1$) for each active circle $\langle q \rangle$. The Stokes directions are the directions where some of the strands of the diagram cross.

Definition 3.3. *Let $d \in \mathbb{A}$ a singular direction. The Stokes group associated to d is the unipotent subgroup $\mathrm{Sto}_d \subset \mathrm{GL}(V_d^0)$ whose Lie algebra is given by*

$$\mathrm{sto}_d := \bigoplus_{i \prec_d j} \mathrm{Hom}(V_d^0(j), V_d^0(i)).$$

More concretely, an element of sto_d is a block matrix, with the blocks corresponding to the pieces $V_d^0(i)$, such that its diagonal blocks are identity matrices, for any Stokes arrow $i \prec_d j$ the corresponding block is an arbitrary matrix, and all other blocks are zero.

3.2. **Stokes local systems and fission local systems.** One possible way to describe Stokes data is given by the notion of *Stokes local systems* [BY15, Boa21], which we now describe. Given an irregular class Θ, the idea is to define a slightly modified surface by introducing a tangential puncture to the real blow-up of the Riemann surface at each singular direction. A Stokes local system is then a local system on this new surface which near ∂ is $\mathcal{I}$-graded (with irregular class Θ) and such that the monodromies around the tangential punctures are elements in the corresponding Stokes groups.

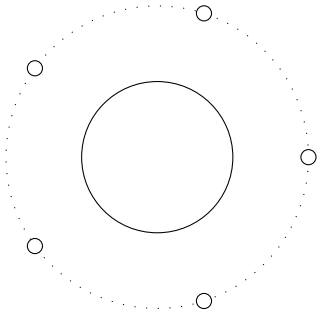


FIGURE 2. Local picture: The halo and the tangential punctures at a singularity.

More precisely, we define the new surface $\widetilde{\Sigma}(\Theta)$ as follows. We define the halo $\mathbb{H} \subset \widehat{\Sigma}$ as a tubular neighbourhood of ∂. One of the boundaries of $\mathbb{H}$ is the circle ∂. Let ∂' be the other boundary of $\mathbb{H}$. Let $e\colon \partial \to \partial'$ a homeomorphism preserving the orientation.

The irregular class Θ defines a set $\mathbb{A} \subset \partial$ of singular directions. We define a new surface as

$$\widetilde{\Sigma} := \widetilde{\Sigma}(\Theta) := \widehat{\Sigma} \setminus e(\mathbb{A}),$$

that is for each singular direction $d \in \mathbb{A}$ we remove from $\widehat{\Sigma}$ the corresponding *tangential puncture* $e(d)$ (see Figure 2). Let us denote by γ_d a small positive loop in $\widetilde{\Sigma}$ starting from d, going around the tangential puncture $e(d)$ (and no other puncture) in a positive sense and going back to d.

Definition 3.4. *A Stokes local system is a pair $(\mathbb{V}, \Theta)$ where Θ is an irregular class at infinity and $\mathbb{V}$ is a local system of vector spaces on $\widetilde{\Sigma}(\Theta)$ equipped with an I-grading (of dimension Θ) over the halo $\mathbb{H}$, where $\mathbb{I} \subset \mathcal{I}$ denotes the finite subcover corresponding to the active circles of Θ, such that the monodromy $\mathbb{S}_d := \rho(\gamma_d)$ is in $\mathrm{Sto}_d \subset \mathrm{GL}(V_d)$ for each $d \in \mathbb{A}$.*

Stokes local systems yield a topological description of the category of algebraic connections on the affine line $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ (cf. e.g. [Boa21]):

Theorem 3.5. *The category of algebraic connections on $\mathbb{C}$ is equivalent to the category of Stokes local systems $(\mathbb{V}, \Theta)$.*

Remark 3.6. There is a different way to view a Stokes local system, namely as a "local system on a fission surface". Let us briefly reflect on this notion here (as already alluded to in [Boa09, §4], see also [Boa14, §3.1] and the picture on the title page of [Boa21]).

Consider the following way to construct the so-called *fission surface* $\underline{\widetilde{\Sigma}}(\Theta)$:

Let Σ be a small disk around the origin in the complex plane, and fix an irregular class Θ at 0. (This is the local situation, and the generalisation to an arbitrary Riemann surface with more than one marked point is not difficult).

Let $\widehat{\Sigma}$ be the real blow-up of Σ at 0, with boundary circle ∂.

Each active exponent in Θ (i.e. each connected component of the covering $\mathbb{I} \to \partial$ determined by Θ) gives a covering circle $\langle q \rangle$ of ∂. We can consider a small strip $\langle q \rangle \times [0, \epsilon]$ (for some $\epsilon \ll 1$), and glue it to $\widehat{\Sigma}$ via the projection $p_q\colon \langle q \rangle \times \{0\} \simeq \langle q \rangle \to \partial$. In other words, we construct the pushout

$$\begin{tikzcd}
\langle q \rangle \arrow[r, "p_q"] \arrow[d, equal] & \partial \arrow[r, hook] & \widehat{\Sigma} \arrow[dd] \\
\langle q \rangle \times \{0\} \arrow[d, hook] & & \\
\langle q \rangle \times [0, \epsilon] \arrow[rr] & & \widehat{\Sigma} \cup_{p_q} (\langle q \rangle \times [0, \epsilon])
\end{tikzcd}$$

Doing this for all active exponents and removing all the points $d \in \mathbb{A} \subset \partial$ yields the fission surface that we denote by $\underline{\widetilde{\Sigma}} = \underline{\widetilde{\Sigma}}(\Theta)$, i.e. concretely we have

$$\underline{\widetilde{\Sigma}} = \left(\widehat{\Sigma} \cup_{p_{q_1}} (\langle q_1 \rangle \times [0,\epsilon]) \cup_{p_{q_2}} \dots \cup_{p_{q_n}} (\langle q_n \rangle \times [0,\epsilon])\right) \setminus \mathbb{A},$$

where $\langle q_1 \rangle, \dots, \langle q_n \rangle$ are the connected components of I.

A *fission local system* $\underline{\mathbb{V}}$ on the fission surface consists of a local system $\underline{\mathbb{V}}$ on $\widehat{\Sigma}$, a local system $\underline{\mathbb{V}}_i$ on any $\langle q_i \rangle \times [0,\epsilon]$, and for any connected component δ of $\partial \setminus \mathbb{A}$ an isomorphism $\oplus_{i=1}^n \underline{\mathbb{V}}_i|_\delta \simeq \underline{\mathbb{V}}|_\delta$.

Since an $\mathbb{I}$-graded local system on ∂ is the same as a local system on $\mathbb{I}$, it is clear that a fission local system is nothing but a local system on $\widetilde{\Sigma}(\Theta)$ graded by $\mathbb{I}$ on the halo $\mathbb{H}$. In particular, any Stokes local system can naturally be regarded as a fission local system. (Note that the notion of a fission local system associated to a Stokes filtered local system is indeed exactly what has been defined in [Boa21, Definition 8.1].)

The advantage of this viewpoint is that the maps induced by the so-called deformation data later are naturally captured by the notion of a path in the fission surface, which we will explain in Remark 4.8.

3.3. **Wild character varieties.** In the same way as moduli spaces of local systems are character varieties, moduli spaces of Stokes local systems give rise to wild character varieties.

Keeping previous notations, let $(\mathbb{V}, \Theta)$ a Stokes local system. Let us choose a base point $b \in \partial$. Let us also fix a framing of $\mathbb{V}$ at b, that is, an isomorphism of vector spaces

$$\mathbb{F} \cong \mathbb{V}_b$$

respecting the $\mathcal{I}_b$-grading on both sides, where

$$\mathbb{F} = \mathbb{C}^\Theta := \bigoplus_{j \in \mathcal{I}_b} \mathbb{C}^{\Theta(j)}.$$

Let $\Pi := \Pi_1(\widetilde{\Sigma}, b)$, the fundamental group of $\widetilde{\Sigma}$ with base point b. Let $\mathrm{Hom}(\Pi, G)$ be the set of isomorphism classes of representations of this group into $G := \mathrm{GL}(\mathbb{F})$. A framed local system on $\widetilde{\Sigma}$ determines via its monodromy an element in $\mathrm{Hom}(\Pi, G)$, see Figure 3. Actually, because of the conditions in the definition of a Stokes local system, the representation associated to a framed Stokes local system lives in a subset $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ of *Stokes representations*, that we now describe.

Let us view the boundary circle ∂ as a loop based at b. (The orientation is given by that of $\mathbb{P}^1$, i.e. counterclockwise in the local chart $t = z^{-1}$.) Let $V^0 \to \partial$ be the $\mathcal{I}$-graded local system on ∂ with irregular class Θ associated to $\mathbb{V}$. Let us denote by $\langle q_1 \rangle, \dots, \langle q_k \rangle$ its active circles, by n_l the multiplicity of $\langle q_l \rangle$, and by r_l its ramification order, for $l = 1, \dots, k$. In the direct sum decomposition $V_b^0 = \bigoplus_{l=1}^k (V^0_{\langle q \rangle_l})_b$, with $(V^0_{\langle q_l \rangle})_b = \bigoplus_{j \in (\langle q_l \rangle)_b} V_b^0(j)$, its monodromy (taking b as basis point) is of the form

$$\rho(\partial) = \begin{pmatrix} \rho_{\langle q_1 \rangle} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \rho_{\langle q_k \rangle} \end{pmatrix} \subset \mathrm{GL}(V_b^0) \cong \mathrm{GL}(\mathbb{F}),$$

with each $\rho_{\langle q_l \rangle}$ of the form (1). The set of such matrices (seen as elements in $\mathrm{GL}(\mathbb{F})$) is a twist of the group

$$H := \mathrm{GrAut}(\mathbb{F}) = \prod_{j \in (\mathcal{I})_b} \mathrm{GL}_{\Theta(j)}(\mathbb{C}),$$

that we denote by $H(\partial)$.

If $d \in \mathbb{A}$ is a singular direction, let $\lambda_d \subset \partial$ be an arc from b to d. By parallel translation along λ_d, we may identify the Stokes group $\mathrm{Sto}_d \subset \mathrm{GL}(\mathbb{V}_d)$ to a subgroup of $\mathrm{GL}(\mathbb{F}) = \mathrm{GL}_n(\mathbb{C})$, with $n = \sum_l n_l r_l$. Let us also define

$$\widehat{\gamma}_d := \lambda_d^{-1} \circ \gamma_d \circ \lambda_d \in \Pi$$

to be the simple loop around the tangential puncture $e(d)$ based on b where γ_d designates as before the simple loop around $e(d)$ based on d. From the definition of a Stokes local system we immediately get that the monodromy $\rho(\widehat{\gamma}_d)$ of $\mathbb{V}$ around $\widehat{\gamma}_d$ belongs to $\rho(\lambda_d^{-1}).\mathrm{Sto}_d.\rho(\lambda_d^{-1})$, which via the framing $\mathbb{F} \cong \mathbb{V}_b$ corresponds to a unipotent subgroup of G that we also denote Sto_d by a slight abuse of notation. In summary:

Lemma 3.7. *Let $\rho \in \mathrm{Hom}(\Pi, G)$ the representation associated to a Stokes local system $(\mathbb{V}, \Theta)$. Then ρ satisfies the following conditions:*

(1) $\rho(\partial) \in H(\partial)$,
(2) $\rho(\widehat{\gamma}_d) \in \mathrm{Sto}_d$ for any singular direction $d \in \mathbb{A}$.

Definition 3.8. *A representation $\rho \in \mathrm{Hom}(\Pi, G)$ is a Stokes representation if it satisfies these two conditions.*

We will call the space $\mathcal{R}_\Theta := \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ of Stokes representations with irregular class Θ the representation variety of Θ. It has the following explicit description:

$$\mathcal{R}_\Theta = \{(h, S_1, \dots, S_s) \in H(\partial) \times \mathrm{Sto}_{d_1} \times \cdots \times \mathrm{Sto}_{d_s} \mid hS_s \dots S_1 = 1\},$$

where the set of singular directions $\mathbb{A} = \{d_1, \dots, d_s\}$ is ordered in a positive sense.

There is a natural action of the group H on $\mathcal{R}_\Theta$, which amounts to changing the framing. The representation variety actually has a richer structure:

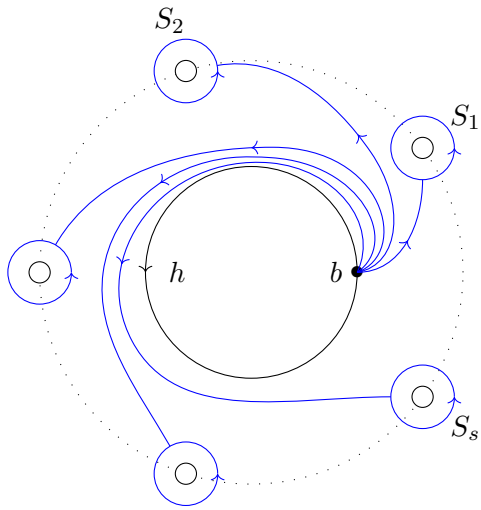


FIGURE 3. The paths for expressing the monodromies of a Stokes local system.

Theorem 3.9 ([Boa14, BY15])**.** *The representation variety $\mathcal{R}_\Theta$ is a smooth affine complex algebraic variety, and is a twisted quasi-Hamiltonian H-space, with moment map given by*

$$\mu \colon \mathcal{R}_\Theta \to H(\bar{\partial}),$$
$$\rho \mapsto \{\rho(\bar{\partial})\},$$

with $\bar{\partial}$ referring as above to the loop around ∂ in the negative sense.

Finally, the (symplectic) wild character variety $\mathcal{M}_B(\Sigma, \Theta)$ is defined as the affine GIT quotient of $\mathcal{R}_\Theta$ by H. Concretely, the points of the wild character variety are the closed orbits, and they correspond to irreducible Stokes representations [BY23]).

Theorem 3.10 ([Boa14, BY15])**.** *The wild character variety $\mathcal{M}_B(\Sigma, \Theta)$ has an algebraic Poisson structure. Its symplectic leaves, the symplectic wild character character varieties are the multiplicative symplectic quotients*

$$\mathcal{M}_B(\Sigma, \Theta, \mathcal{C}) = \mathcal{R}_\Theta /\!\!/_{\mathcal{C}} H = \mu^{-1}(\mathcal{C})/H,$$

where $\mathcal{C} \in H(\partial)$ is a twisted conjugacy class.

If (E,∇) is an algebraic connection on $\Sigma^\circ \subset \Sigma$, it determines an irregular class $\Theta\colon \pi_0(\mathcal{I}) \to \mathbb{N}$ and a formal local system $V^0 \to \mathcal{I}$ with irregular class Θ. The isomorphism class of the inverse of the monodromy of $V^0 \to \partial$ is a twisted conjugacy class $\mathcal{C} \subset H(\bar{\partial})$. The connection (E,∇) therefore determines a symplectic wild character variety $\mathcal{M}_B(E,\nabla) = \mathcal{M}_B(\Sigma,\Theta,\mathcal{C})$.

It will also be useful to consider a partial reduction of the representation variety $\mathcal{R}_\Theta$, obtained in the following way (see the last paragraph of [BY20]): Instead of considering a framing of the full fibre $\mathbb{V}_b$, for each active circle $\langle q_l\rangle$, let us just choose a point $i_l \in V^0_{\langle q_l\rangle,b}$ in the graded piece $V^0_{\langle q_l\rangle,b}$ and a framing of $\mathbb{V}_{b,i_l}$, which we will refer to as a *minimal framing*. Forgetting the framings at all other points in the fibre $\mathbb{V}_b$ amounts to acting on $\mathcal{R}_\Theta$ with a subgroup $H^\perp \cong \prod_{l=1}^k \mathrm{GL}_{n_l}(\mathbb{C})^{r_l-1}$ of H. Concretely, this action can be used to bring the piece $\rho_{\langle q_l\rangle}(\partial)$ of the formal monodromy $\rho(\partial)$ to the form

$$\rho_{\langle q_l\rangle}(\partial) = \begin{pmatrix} 0 & \dots & \dots & 0 & \rho_{r_l-1,0} \\ 1 & \ddots & & \vdots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{Aut}(V^0_{\langle q_l\rangle,b}), \tag{2}$$

where the first index corresponds to the point i_l (cf. equation (1)).

Definition 3.11. *The reduced representation variety of Θ is the quotient $\mathcal{E}_\Theta := \mathcal{R}_\Theta/H^\perp$.*

Furthermore, changing the minimal framing at i_l amounts to acting with the residual group $\breve{H} := \prod_{l=1}^k \mathrm{GL}_{n_i}(\mathbb{C})$. In turn, $\mathcal{E}_\Theta$ is a quasi-Hamiltonian $\breve{H}$-space, with moment map given by the monodromies around the $\langle q_l\rangle$ in the negative sense, and the symplectic wild character variety $\mathcal{M}_B(\Sigma,\Theta,\mathcal{C})$ is its (usual) quasi-Hamiltonian reduction $\mathcal{M}_B(\Sigma,\Theta,\mathcal{C}) = \mathcal{E}_\Theta /\!\!/_{\breve{\mathcal{C}}} \breve{H}$, where $\breve{\mathcal{C}} \subset \breve{H}$ is the (usual) conjugacy class in $\breve{H}$ induced by $\mathcal{C}$.

4. Deformation data

We now discuss a different description of Stokes data used in [Moc21], formulated in terms of 'deformation data'. We will slightly reformulate this description in order to relate it to Stokes local systems.

Throughout this section and the rest of this article, we will make the following assumption:

(3) There is a fixed $k \in \mathbb{Q}_{>1}$ such that any active circle $\langle f\rangle$ of the irregular class Θ is of the form $\langle f\rangle = \langle az^k\rangle$ for some $a \in \mathbb{C}\setminus\{0\}$, and for any two active circles $\langle az^k\rangle, \langle bz^k\rangle \subset \mathcal{I}$, one has $|a| = |b|$.

(We remark that it is not necessary in §4.1 and §5.1–5.2.)

4.1. **Distinguished intervals.** One of the main ingredients in the deformation data description consists in considering on each active circle distinguished intervals along which the associated exponential is either growing or decreasing, which we define in this paragraph. Recall the notation from §2.1.

Any circle $\langle f\rangle \subset \mathcal{I}$ associated to an exponent of slope > 1 has a set $S_0(f)$ of distinguished points defined by

$$S_0(f) := \{(\theta, [\tilde{f}]) \mid \lim_{r\to\infty} \mathrm{Re}\, \tilde{f}(re^{i\theta}) = 0\}.$$

It is not difficult to see that $\#S_0(f) = 2m$ if $f = ax^{m/n}$ for m,n coprime. The points of $S_0(f)$ correspond to the directions $d \in \partial$ such that $|e^{f(z)}|$ (for z near ∞) passes from increasing to decreasing (or vice versa) when crossing the direction d. In a Stokes diagram as in Figure 1, these are exactly the points where the circle $\langle f\rangle$ intersects the dashed

one; indeed, this dashed circle corresponds to $f = 0$, for which $e^{f(z)}$ is constant (notice, however, that in this picture each of the three intersection points of $\langle f \rangle$ with the dashed circle defines two elements of $S_0(x^{3/2}) \subset \langle x^{3/2} \rangle$.) We also observe that the set of points $\pi_f(S_0(f)) \subset \partial$ consists exactly of the Stokes directions of the pair $(0, \langle f \rangle)$.

The distinguished intervals of a circle $\langle f \rangle$ are then defined as the intervals between the points of $S_0(f)$:

Definition 4.1. *Let $\langle f \rangle \subset \mathcal{I}$ be a circle. We define $A(f)$ to be the set of connected components of $\langle f \rangle \setminus S_0(f)$. We denote by $A_+(f) \subset A(f)$ (resp. $A_-(f) \subset A(f)$) the subset of those intervals such that for (one, hence any) $(\theta, [\tilde{f}]) \in \langle f \rangle$, we have*

$$\operatorname{Re} \tilde{f}(re^{i\theta}) > 0 \qquad \textit{(resp. } < 0\textit{)}$$

for $r \gg 0$.

Clearly, the intervals contained in $A(f)$ have length $\frac{\pi \cdot n}{m}$ for $f = ax^{m/n}$.

An observation that will be essential in relating deformation data to Stokes local system is that pairwise intersections of distinguished intervals are in one-to-one correspondence with Stokes arrows:

Lemma 4.2. *Let $k \in \mathbb{Q}_{>1}$, let $f = ax^k$ and $g = bx^k$ be exponents with $|a| = |b|$, and let $I \in A_+(f)$, $J \in A_-(g)$ be two sectors such that their projections to ∂ intersect nontrivially, i.e. $\pi_f(I) \cap \pi_g(J) \neq \emptyset$. Then the midpoint $d \in \partial$ of the intersection $\pi_f(I) \cap \pi_g(J)$ is a singular direction for f and g and there is a Stokes arrow $d_I \to_d d_J$ over d, where d_I, d_J are the unique points of I and J lying over d.*

Conversely, for every Stokes arrow $d_f \to_d d_g$ there is a pair of sectors $I \in A_+(f)$, $J \in A_-(g)$ with $d_f \in I$, $d_g \in J$ such that d is the midpoint of $\pi_f(I) \cap \pi_g(J)$.

Proof. This is clear from the definition of a Stokes arrow: having a Stokes arrow means that the real parts of the corresponding branches of the exponents in question have a maximal difference (their difference is a point of maximal decay). This certainly happens at a point where the two real parts differ in sign, and since the absolute values of the coefficients are equal, the maximal difference must appear in the middle of the intersection for symmetry reasons. □

4.2. **Preferred splittings on sectors.** We now briefly recall a few more facts that come into play in the definition of deformation data. These are constructed in [Moc21] starting from the data a Stokes-filtered local system, which is still another way of describing Stokes data (the local system of analytic solutions of an algebraic connection on the affine line comes naturally equipped with the structure of a Stokes-filtered local system.) While we refer the reader, for example, to [Mal91, Sab13, Boa21] for the full definition and further details about Stokes-filtered local systems, we will only use here the main fact allowing to pass from Stokes-filtered local systems to deformation data or Stokes local systems, which is the following property.

Let (V, Θ, F) be a Stokes-filtered local system on $(\mathbb{P}^1, \infty)$ with irregular class Θ, and let $V^0 \to \partial$ be the corresponding formal local system (obtained by taking the associated graded). Then any direction $\theta \in \partial$ that is not a singular direction determines a preferred isomorphism

$$V|_S \simeq V^0|_S \tag{4}$$

on a sufficiently small neighbourhood $S \subset \partial$ of θ. In particular, one gets a splitting of the Stokes filtration

$$V|_S \simeq \bigoplus_{I \in \pi_0(S)} V^0|_I.$$

(Here, the right-hand side is to be read as a local system on S by the natural homeomorphisms $I \simeq S$ for any $I \in \pi_0(S)$.) The latter can be made more precise: The decomposition (4) is valid on the full open interval bounded by the singular directions between which θ lies, the so-called *singular sector* associated to θ. Even more, it is valid on the associated

(open) *supersector*: We can choose S to be the singular sector enlarged by $\frac{\pi}{2k}$ on both sides.

If an interval $S \subset \partial$ is large enough, there is a unique such splitting.

Lemma 4.3. *Let $S = (\theta_0, \theta_0 + \frac{\pi}{k} + \varepsilon) \subset \partial$ for some $\varepsilon \ll 1$. Then there is a unique supersector containing S and the midpoint of S lies in the associated singular sector.*

Proof. The overlap of two consecutive supersectors is an open interval of angle $\frac{\pi}{k}$. This shows that S cannot be contained in two supersectors. On the other hand, it is not difficult to see that S is certainly contained in one supersector, and that the midpoint of S will lie in the central part of the supersector, which is the associated singular sector. □

The lemma implies in particular the following: If $S = \pi_f(I_+)$ for $\langle f \rangle \in \mathcal{I}$ and $I \in A(f)$, where I_+ denotes the interval I closed on the right (at its endpoint), then there is a unique splitting on S and it is the splitting induced by that on the singular sector containing the midpoint of I or, if the midpoint is a singular direction, the singular sector starting there. This splitting is called the *canonical splitting* on I_+. A similar statement obviously holds for I_-.

4.3. **Deformation data and comparison of canonical splittings.** We are now in the position to reformulate some statements of [Moc21] in the language we introduced above. In what follows, we tacitly make the following identification: If I is a contractible set, $p \in I$ a point and V_1, V_2 local systems on I, then it is equivalent to give one of the following: a morphism $V_1 \to V_2$; a morphism $(V_1)_p \to (V_2)_p$; a morphism $V_1(I) \to V_2(I)$. Moreover, if $I \in A(f)$, we will write $I_\partial := \pi_f(I) \subset \partial$. For each active circle $\langle f \rangle$, we denote by $V^0_{\langle f \rangle} \subset V^0$ the corresponding graded piece of a formal local system V^0.

Let V be a Stokes-filtered local system on $(\mathbb{P}^1, \infty)$ with irregular class Θ, and let V^0 be the associated formal local system on ∂. In [Moc21], a notion equivalent to that of a Stokes filtered local system on S^1 is introduced: the notion of a Stokes shell. We note that, given our assumptions, the local systems $\mathcal{K}_\lambda$ (for an exponent λ) from loc. cit. correspond to the formal pieces of the formal local system in our terminology.

In our case, such a Stokes shell consists of the data of the corresponding formal local system together with so-called *deformation data*. The latter are linear maps of the following form: Let $f, g \in \mathcal{I}$ and $I \in A_+(f)$, $J \in A_-(g)$ such that $I_\partial \cap J_\partial \neq \emptyset$, and denote by m_I, m_J their midpoints, then a deformation datum is a linear map

$$\mathcal{R}^I_J \colon (V^0_{\langle f \rangle})_{m_I} \to (V^0_{\langle g \rangle})_{m_J}.$$

By Lemma 4.2, this means that there will be exactly one deformation datum for every Stokes arrow. Here, we will relate the deformation data morphism $\mathcal{R}^I_J$ to the comparison of the canonical splittings on I and J.

Proposition 4.4. *Let $\langle f \rangle$ and $\langle g \rangle$ be active exponents with respect to the irregular class Θ. Let $I \in A_+(f)$ and $J \in A_-(g)$ be two intervals such that $I_\partial \cap J_\partial \neq \emptyset$. Then there are splittings*

$$V|_{I_\partial} \xrightarrow[\sim]{\varphi} \bigoplus_{S \in \pi_0(\pi^{-1}(I_\partial))} V^0|_S \qquad \text{and} \qquad V|_{J_\partial} \xrightarrow[\sim]{\psi} \bigoplus_{S \in \pi_0(\pi^{-1}(J_\partial))} V^0|_S$$

coming from the canonical splittings on $(I_\partial)_-$ and $(J_\partial)_+$, respectively, and we can consider the associated morphism

$$\begin{aligned} V^0_{\langle f \rangle}|_{I \cap \pi^{-1}(J_\partial)} &\hookrightarrow \bigoplus_{U \in \pi_0(\pi^{-1}(I_\partial \cap J_\partial))} V^0|_U \\ &\xrightarrow[\sim]{\psi \circ \varphi^{-1}} \bigoplus_{U \in \pi_0(\pi^{-1}(I_\partial \cap J_\partial))} V^0|_U \twoheadrightarrow V^0_{\langle g \rangle}|_{J \cap \pi^{-1}(I_\partial)}. \end{aligned}$$

This is equivalent to Mochizuki's deformation datum $\mathcal{R}^I_J$ associated to this situation.

Proof. If $I_\partial = J_\partial$, the statement is clear from the definition in [Moc21].

In the case where $I_\partial \neq J_\partial$, the author of loc. cit. does not directly define deformation data in terms of $\psi \circ \varphi^{-1}$. In fact, he gives two different constructions, depending on if J_∂ "comes after" I_∂ (we write $J_\partial > I_\partial$, by which we mean that J_∂ contains the endpoint of I_∂, thinking in terms of the given orientation of ∂) or not. We will therefore need to distinguish two cases, and we will start with the one where $J_\partial > I_\partial$. In this case, the construction in [Moc21] is the following (this is a slight reformulation of p. 22 in loc. cit. in our situation and language):

One considers the situation at the point θ_0, the endpoint of I (which is contained in the interior of J). For any sector $U \in A$ such that $\theta_0 \in (U_\partial)_-$ there is a splitting

$$V|_{U_\partial} \xrightarrow[\sim]{\varphi_U} \bigoplus_{S \in \pi_0(\pi^{-1}(U_\partial))} V^0|_S \tag{5}$$

coming from that on $(U_\partial)_-$. Then in a small neighbourhood W of θ_0, we have a decomposition

$$\nu \colon V|_W \xrightarrow{\sim} \bigoplus_{U \in A, \theta_0 \in (U_\partial)_-} V^0|_U|_W$$

defined by combining all those from (5) in the natural way: Take the natural total order $\leq$ on the sectors $U \in A$ with $\theta_0 \in (U_\partial)_-$ given by comparing the real parts of the corresponding exponents at θ_0. Start with a maximal U_1 and the associated decomposition

$$V|_{(U_1)_\partial} \xrightarrow[\sim]{\varphi_{U_1}} \bigoplus_{S \in \pi_0(\pi^{-1}((U_1)_\partial))} V^0|_S.$$

Now take a maximal element U_2 of the remaining sectors. Then $\varphi_{U_2} \circ \varphi_{U_1}^{-1}$ induces on $(U_1)_\partial \cap (U_2)_\partial$ an isomorphism (since it has to be compatible with the filtration induced by the ordering)

$$\bigoplus_{S \in \pi_0(\pi^{-1}((U_1)_\partial)) \setminus \{U_1\}} V^0|_S \simeq \bigoplus_{S \in \pi_0(\pi^{-1}((U_2)_\partial)) \setminus \{U_1\}} V^0|_S.$$

Composing this isomorphism (taking it to be the identity on the summand corresponding to $S = U_1$) with φ_{U_1}, we find

$$V|_{(U_1)_\partial \cap (U_2)_\partial} \xrightarrow{\sim} V^0|_{U_1}|_{(U_1)_\partial \cap (U_2)_\partial} \oplus \bigoplus_{S \in \pi_0(\pi^{-1}((U_2)_\partial))} V^0|_S|_{(U_1)_\partial \cap (U_2)_\partial}.$$

Continuing this procedure yields the desired isomorphism.

Now, the deformation datum is defined in [Moc21] as

$$V^0_{\langle f \rangle}|_I|_W \hookrightarrow \bigoplus_{S \in \pi_0(\pi^{-1}(I_\partial))} V^0|_S|_W \xrightarrow[\sim]{\nu \circ \varphi^{-1}} \bigoplus_{U \in A, \theta_0 \in (U_\partial)_-} V^0|_U|_W \twoheadrightarrow V^0_{\langle g \rangle}|_J|_W$$

It is not too difficult to see that, by the construction of ν given above, this is the desired morphism, since the other components of $\varphi_U \circ \varphi^{-1}(V^0_{\langle f \rangle}|_I|_U)$ give no extra contribution to the component $V^0_{\langle g \rangle}|_J|_W$ due to the fact that the overlap of any such U with J contains a Stokes direction of the pair (f_U, g). This completes the proof for the case $J_\partial > I_\partial$.

Let us now indicate the proof for the other case, when $I_\partial > J_\partial$. The construction of Mochizuki goes differently here, namely through the interval $I - \frac{\pi}{k}$ as follows: Denote the splitting on this shifted sector induced by the one on $((I - \frac{\pi}{k})_\partial)_+$ by $\widetilde{\varphi}$. Let θ_0 be the startpoint of I. Then we can construct a splitting ν as in the proof above. By the definition in [Moc21], the deformation datum is (-1) times the morphism

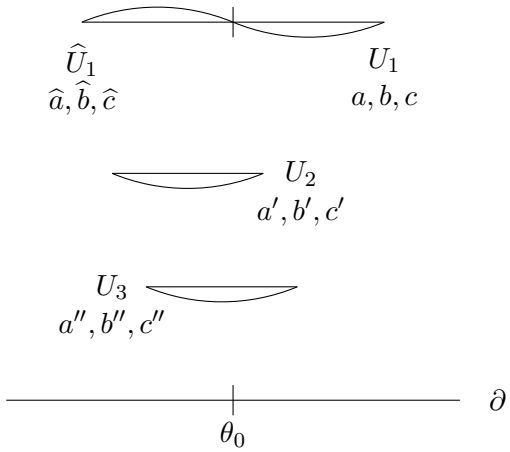
$$V^0_{\langle f \rangle}|_I|_W \simeq V^0_{\langle f \rangle}|_{I - \frac{\pi}{k}}|_W \hookrightarrow \bigoplus_{S \in \pi_0(\pi^{-1}((I - \frac{\pi}{k})_\partial))} V^0|_S|_W \xrightarrow{\nu \circ \widetilde{\varphi}^{-1}} \bigoplus_{U \in A, \theta_0 \in (U_\partial)_-} V^0|_U|_W \twoheadrightarrow V^0_{\langle g \rangle}|_J|_W.$$

It now remains to see that the image of $V^0_{\langle f\rangle}|_{I-\frac{\pi}{k}}|_W$ under $\nu\circ\widetilde{\varphi}^{-1}$ contributes a summand of $-\lambda(V^0_{\langle f\rangle}|_I|_W)$ to the J-th graded piece, where $\lambda\colon V^0_{\langle f\rangle}|_I|_W \to V^0_{\langle g\rangle}|_J|_W$ is the matrix entry of $\psi\circ\varphi^{-1}$ that we want to get.

It suffices to see that during this procedure no other, "unwanted" contributions are made to the J-factor of the target. □

Remark 4.5. Since the proof of the previous proposition might be hard to understand, let us illustrate the idea in a concrete example:

Consider a Stokes-filtered local system of rank 3 with an irregular class consisting of 3 exponents f, g and h, such that the picture looks as follows:



For each of these intervals U_i, one has a canonical splitting for the local system V on $((U_i)_\partial)_-$, and one also has a canonical splitting on $(\widehat{U}_\partial)_+$. Each splitting corresponds to choosing a basis of the local system on this interval downstairs. We denote these bases by a,b,c etc., as shown in the picture. On the other hand, Mochizuki's splitting, denoted ν in the above proof, corresponds to considering the basis a,b',c''. The statement of the first proposition is then the following: If we express $\widehat{a}$ in the basis a'',b'',c'' and in the basis a,b',c'', then the coefficient of c'' is the same in both expressions.

Let us also explain the second proposition: Let us assume that, if we express a in terms of $\widehat{a},\widehat{b},\widehat{c}$, the coefficient of $\widehat{a}$ is 1 (this can always be achieved simply by rescaling one of these bases). The statement of the second proposition then says the following in this case: Express $\widehat{a}$ in terms of the basis a,b',c'', and express a in terms of the basis a'',b'',c'', then the coefficients of c'' in both expressions only differ in sign.

Both statements can be checked by an easy computation, and it is tedious but not difficult to generalise these arguments to more complicated cases.

Let us note that, if $I_\partial \neq J_\partial$, then the deformation datum in the above proposition does not change if we replace the splitting coming from $(I_\partial)_-$ by the one from $(I_\partial)_+$, and similarly for the splitting on J_∂. However, in the case $I_\partial = J_\partial$, the deformation datum really depends on the particular choice of splitting.

4.4. **Stokes local systems and deformation data.** Having understood the deformation data as in the previous subsection, we can also associate similar data to Stokes local systems. First of all, in view of the above observations, let us give the following definition:

Definition 4.6. *Let $\mathbb{V}$ be a Stokes local system with irregular class Θ, and let V^0 be the corresponding formal local system. Let $\langle f\rangle$ be an active circle and let $I\in A(f)$ be a distinguished interval.*

If $I\in A(f)$, we define the map

$$\iota_I\colon (V^0_{\langle f\rangle})_{m_I} \hookrightarrow \mathbb{V}_{\pi_f(m_I)}$$

induced by the splitting on the singular sector just to the left of $\pi_f(m_I)$ (or at $\pi_f(m_I)$ if it is not a singular direction). (Here, "to the left" means "going backwards wth respect to the given orientation on ∂".)

If $I \in A(f)$, we define the map

$$\pi_I \colon \mathbb{V}_{\pi_f(m_I)} \to (V^0_{\langle f\rangle})_{m_I}$$

induced by the splitting on the singular sector just to the right of $\pi_f(m_I)$ (or at $\pi_f(m_I)$ if it is not a singular direction).

In view of Proposition 4.4, we define as follows the deformation datum associated to a Stokes local system:

Definition 4.7. *Let d be a singular direction and let $d_f \to_d d_g$ be a Stokes arrow over it. Then there exist sectors I and J as in Lemma 4.2. Denote their midpoints by $p := m_I$ and $q := m_J$, respectively. Then the associated deformation datum is the map*

$$R^p_q \colon (V^0_{\langle f\rangle})_{m_I} \longrightarrow (V^0_{\langle g\rangle})_{m_J}$$

given by the composition

$$(V^0_{\langle f\rangle})_{m_I} \overset{\iota_I}{\hookrightarrow} \mathbb{V}_{m_I} \underset{\simeq}{\longrightarrow} \mathbb{V}_{m_J} \overset{\pi_J}{\twoheadrightarrow} (V^0_{\langle g\rangle})_{m_J},$$

where the middle arrow is parallel transport in the local system in the interior (not in the halo) of $\widetilde{\Sigma}(\Theta)$.

Proposition 4.4 implies that the deformation data in the sense of Mochizuki associated to a Stokes-filtered local system V coincides with the deformation data associated by the definition above to the Stokes local system $\mathbb{V}$ corresponding to V.

Remark 4.8. We can relate the deformation data defined above to the parallel transport in the Stokes local system along distinguished paths on $\widetilde{\Sigma}(\Theta)$ as follows: Given a Stokes arrow, it determines two sectors $\pi_f(I)$ and $\pi_g(J)$ on ∂, where the corresponding real parts of the exponents are positive and negative, respectively. We can connect their midpoints by a path in $\widetilde{\Sigma}(\Theta)$. Parallel transport along this path in the local system gives an isomorphism stalks

$$\mathbb{V}^0_{\pi_f(m_I)} \xrightarrow{\sim} \mathbb{V}^0_{\pi_g(m_J)}.$$

Pre- and post-composing this with the inclusion and projection associated to the $\langle f\rangle$- and $\langle g\rangle$-graded pieces, respectively, respectively, yields the deformation datum.

In what follows, we will illustrate this by a path as drawn in Figure 4.

All this can be formulated in an even cleaner way if we view a Stokes local system as a fission local system as explained in Remark 3.6: One can draw a *Stokes path* on the fission surface as follows: Recall that the fission surface $\underline{\widetilde{\Sigma}}(\Theta)$ looks like the real blow-up $\widehat{\Sigma}$ glued to a small strip for each exponent at the boundary ∂, and punctured at the singular directions. Start at the point $(m_I, \epsilon) \in \langle f\rangle \times [0, \epsilon]$. Then connect it by a straight line with $(m_I, 0) \in \langle f\rangle \times [0, \epsilon]$, which identified with a point in ∂. We can draw a path in $\widehat{\Sigma}$ from this point to $(m_J, 0) \in \langle g\rangle \times [0, \epsilon]$ and connect it again by a straight line with (m_J, ϵ). This is a path in the surface $\underline{\widetilde{\Sigma}}(\Theta)$, and if we interpret every transition of the circle ∂ (where the surface "splits") as the obvious inclusion or projection, the parallel transport in the fission local system along this path naturally represents the deformation data.

A Stokes local system can be reconstructed from its deformation data together with its formal local system (i.e. the data referred to as a Stokes shell by Mochizuki).

Proposition 4.9. *A Stokes local system $\mathbb{V}$ is determined by the following data:*

- *For each active exponent $\langle f\rangle$ the (twisted, i.e. $\langle f\rangle$-graded) local system $V^0_{\langle f\rangle}$ on ∂, viewed as a local system on $\langle f\rangle$, and hence given by the following data:*
  for any $I \in A(f)$, the vector space $K^f_I := (V^0_{\langle f\rangle})_{m_I} \simeq \Gamma(I, V^0_{\langle f\rangle})$, and for any two adjacent intervals $I, J \in A(f)$, J following I with respect to the given orientation, an isomorphism $K^f_I \xrightarrow{\sim} K^f_J$,

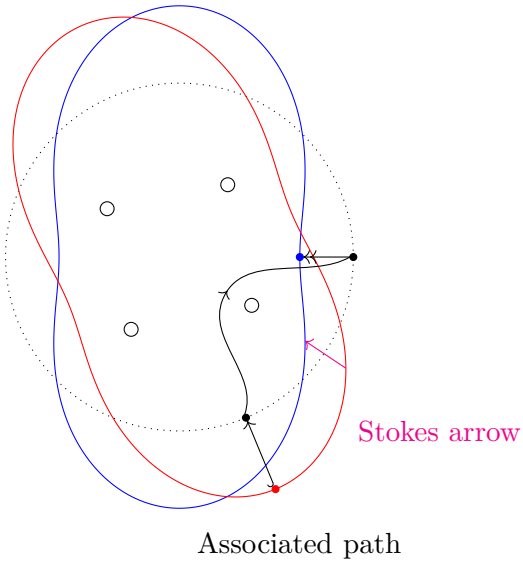


FIGURE 4. A deformation datum associated to a Stokes arrow over a singular direction can be pictured as a path from a leaf to another. Note that at the boundary the surface is equipped with a covering by the exponential factors, so it is best to picture the blue and red circles above the black one (in the third dimension).

- *for each Stokes arrow, the corresponding deformation datum R^p_q, as in Definition 4.7.*

Proof. This observation follows, for example, from the fact that the notion of Stokes shell is equivalent to that of a Stokes-filtered local system and hence to that of a Stokes local system (combining [Moc21] and [Boa21]). □

5. The algorithm

In this section, we describe the algorithm of [Moc21] for the Fourier transform of Stokes data in the class of cases we are interested in here, and reformulate it in the language of Stokes local systems, yielding isomorphisms between wild character varieties.

5.1. **The Legendre transform as a homeomorphism of circles.** A well-known fact about the Fourier transform on irregular connections on the affine line is that there exists an explicit way of determining the irregular class $\widehat{\Theta}$ of the Fourier transform of a connection in terms of its irregular class Θ, given by the so-called stationary phase formula [Mal91, Sab08]. The basic statement is that the active circles of $\widehat{\Theta}$ are obtained from those of Θ by a Legendre transform.

In the case which interests us here, for exponential factors at infinity of slope > 1 (i.e. with leading term az^k, $k \in \mathbb{Q}_{>1}$), the Legendre transform is given as follows: If $f(z)$ is such an exponent, its Legendre transform $g(w)$ is determined by the equations

$$w = f'(z) \quad \text{and} \quad g(w) = f(z) - zw.$$

The Legendre transform provides a permutation of the set of active circles of slope > 1, and this is the way it is often presented in the literature. The crucial idea underlying the description of [Moc21] of the Fourier transform of Stokes data is that the Legendre transform can be seen in a stronger way, not only as a bijection between the sets of active circles of Θ and $\widehat{\Theta}$, but for any active circle $\langle f \rangle$ of Θ, as a homeomorphism between $\langle f \rangle$ and the corresponding circle $\langle \widehat{f} \rangle$ of $\widehat{\Theta}$.

This can be observed as follows (this is similar to the presentation in [Dou23, §3.1], to which we refer for details): Let us denote the circle of directions at ∞ before Fourier

transform by ∂ and the one after Fourier transform by ∂'. As explained above, a point of $\langle f\rangle$ is a pair $(\theta, [\tilde{f}])$ with $\theta \in \partial$ and $\tilde{f}$ a branch of f near θ (so $\tilde{f}(z)$ is a holomorphic function on a small sector $S(U)$ defined by a small neighbourhood $U \subset \partial$ of p; we shrink U and $S(U)$ if necessary).

Now we consider the function $\phi(z) := \tilde{f}'(z) = \frac{\mathrm{d}\tilde{f}}{\mathrm{d}z}(z)$. If defines a biholomorphic map between $S(U)$ and some small sector $S(U')$ for $U' \subset \partial'$. In particular, it sends $\theta \in \partial$ to a unique point $\theta' \in \partial'$. Concretely, we can set $\theta' = \lim_{r\to\infty} \arg \tilde{f}'(r \cdot e^{i\theta})$.

Next, define a function on $S(U')$ by $\tilde{g}(w) := \tilde{f}(\phi^{-1}(w)) - \phi^{-1}(w)w$. By continuing this function continuously, it defines a circle $\langle g\rangle$ on which $(\theta', [\tilde{g}])$ is a point. It can be checked that this defines a homeomorphism of circles $\ell_f \colon \langle f\rangle \xrightarrow{\sim} \langle g\rangle$ preserving the orientation.

5.2. **Correspondence between distinguished intervals.** We now show that the Legendre transform behaves well with respect to the distinguished intervals. This appears to be a main reason why T. Mochizuki prefers them to the actual Stokes or singular directions and builds his theory around them (see [Moc21, §1.6.1]).

Lemma 5.1. *If $\langle g\rangle$ is the Legendre transform of and exponent $\langle f\rangle$ of slope > 1, then the homeomorphism $\ell_f \colon \langle f\rangle \xrightarrow{\sim} \langle g\rangle$ induces a bijection between $S_0(f)$ and $S_0(g)$.*

Proof. First of all, note that the statement makes sense since $\#S_0(f) = \#S_0(g)$: If the highest power appearing in f is $z^{m/n}$, then the highest power appearing in g is $w^{m/(m-n)}$. Since the Legendre transform ℓ_f is a homeomorphism between two circles, it suffices to show that any point of $S_0(f)$ is mapped to a point of $S_0(g)$.

Let $(\theta, [\tilde{f}]) \in S_0(f)$, i.e. $\lim_{r\to\infty} \operatorname{Re} \tilde{f}(re^{i\theta}) = 0$, and denote by $(\theta', [\tilde{g}])$ the corresponding point in the Legendre transform. Note that for $f = z^{m/n}$, we have $z\tilde{f}'(z) = \frac{m}{n}\tilde{f}(z)$, and hence in general one has

$$\lim_{|z|\to\infty} z\tilde{f}'(z) = \lim_{|z|\to\infty} \frac{m}{n}\tilde{f}(z),$$

where the limit means approaching ∞ along a fixed path (the same on both sides of the equality). Hence, we can conclude

$$\begin{aligned}\lim_{r'\to\infty} \operatorname{Re} \tilde{g}(r'e^{i\theta'}) &= \lim_{r\to\infty} \operatorname{Re} \tilde{g}(\tilde{f}'(re^{i\theta})) \\ &= \lim_{r\to\infty} \left(\operatorname{Re} \tilde{f}(re^{i\theta}) - \frac{m}{n} \operatorname{Re} \tilde{f}(re^{i\theta})\right) = 0.\end{aligned}$$

The first equality holds since $\tilde{g}$ is a continuous function (with values in $\mathbb{C}\sqcup\{\infty\}$) defined up to the boundary of the real blow-up space, and the sequences $r'e^{i\theta'}$ and $\tilde{f}'(re^{i\theta})$ approach the same point on the boundary for $r' \to \infty$ and $r \to \infty$, respectively. □

The following follows directly from Lemma 5.1 and its proof.

Corollary 5.2. *If $\langle g\rangle$ is the Legendre transform of an exponent $\langle f\rangle$ of slope > 1, then the Legendre transform induces a bijection between $A(f)$ and $A(g)$. Moreover, this bijection sends $A_+(f)$ to $A_-(g)$ and $A_-(f)$ to $A_+(g)$.*

It is not difficult to see that this corresponds to the maps $\nu_0^{\pm}$ from [Moc21, §1.8.2, §5.2.4]. To be more precise, if $J \in A_-(g)$ and $I \in A_+(f)$ is the corresponding sector via Legendre transform, we have $\pi_f(I) = \nu_0^+(\pi_g(J))$ as sectors on ∂.

5.3. **The algorithm for computing the Fourier transform.** We are now ready to translate Mochizuki's description of the Stokes data of the Fourier transform to the language of Stokes local systems.

Let (E, ∇) be an algebraic connection on the affine line and $(\widehat{E}, \widehat{\nabla})$ its Fourier transform[1]. Let $\mathbb{V}$ be the Stokes local system associated to (E, ∇) via the Riemann–Hilbert–Birkhoff correspondence (describing flat sections of (E, ∇)), and by $\widehat{\mathbb{V}}$ the one of $(\widehat{E}, \widehat{\nabla})$. We assume that the irregular class Θ of $\mathbb{V}$ satisfies the assumption (3).

[1]To be consistent with [Moc21], we consider the Fourier transform given by the automorphism of the Weyl algebra $\mathbb{C}[z]\langle\partial_z\rangle \simeq \mathbb{C}[w]\langle\partial_w\rangle$ where by $z \mapsto \partial_w, \partial_z \mapsto -w$.

If $\mathbb{V}$ has irregular class Θ and associated formal local system V^0 with graded pieces $V^0_{\langle f\rangle}$ for any active circle $\langle f\rangle$, then the $V^0_{\langle f\rangle}$ can be viewed as local systems on the circles $\langle f\rangle$. Therefore, they are determined by vector spaces $K^f_I := \Gamma(I, V^0_{\langle f\rangle})$ for any $I \in A(f)$ and identifications $\kappa_I \colon K^f_I \xrightarrow{\sim} K^f_J$ for any neighbouring sectors $I, J \in A(f)$, where J follows I with respect to the given orientation. Note that K^f_I is naturally identified with $(V^0_{\langle f\rangle})_{m_I}$ for the midpoint m_I of I.

Analogously, the formal local system associated with $\widehat{\mathbb{V}}$ has graded pieces $\widehat{V}^0_{\langle \widehat{f}\rangle}$ for any active circle $\langle \widehat{f}\rangle$ of $\widehat{\Theta}$, and they are determined by vector spaces $\widehat{K}^{\widehat{f}}_{\widehat{I}} := \Gamma(\widehat{I}, \widehat{V}^0_{\langle \widehat{f}\rangle})$ for any $\widehat{I} \in A(\widehat{f})$ and identifications $\widehat{\kappa}_{\widehat{I}} \colon \widehat{K}^{\widehat{f}}_{\widehat{I}} \xrightarrow{\sim} \widehat{K}^{\widehat{f}}_{\widehat{J}}$ for neighbouring sectors $\widehat{I}, \widehat{J} \in A(\widehat{f})$.

For $\langle f\rangle \in \mathcal{I}$ we will denote by $\langle \widehat{f}\rangle$ its Legendre transform and denote by $\ell_f \colon \langle f\rangle \xrightarrow{\sim} \langle \widehat{f}\rangle$ the homeomorphism of circles defined by the Legendre transform. Therefore, the active circles of $\widehat{\mathbb{V}}$ are the $\langle \widehat{f}\rangle$ for $\langle f\rangle \in \mathcal{I}$.

The following theorem is a reformulation of the main result of [Moc21] in our situation and language set up above. Given $\mathbb{V}$, we can express the data as in Proposition 4.9 for $\widehat{\mathbb{V}}$, which in turn determines $\widehat{\mathbb{V}}$.

Theorem 5.3. *In the above situation, the Stokes local system $\widehat{\mathbb{V}}$ is the Stokes local system determined by the following data:*

- *The data encoding the local system $\widehat{V}^0_{\langle \widehat{f}\rangle}$ on $\langle \widehat{f}\rangle$ are obtained as follows:*

  For any $\widehat{I} \in A(\widehat{f})$, we set
  $$\widehat{K}^{\widehat{f}}_{\widehat{I}} := K^f_{\ell_f^{-1}(\widehat{I})}$$
  and for neighbouring sectors $\widehat{I}, \widehat{J} \in A(\widehat{f})$, $\widehat{J}$ following $\widehat{I}$, the isomorphism
  $$\widehat{\kappa}_{\widehat{I}} \colon \widehat{K}^{\widehat{f}}_{\widehat{I}} \xrightarrow{\sim} \widehat{K}^{\widehat{f}}_{\widehat{J}}$$
  is given by
  (1) $\kappa_{\ell^{-1}(\widehat{I})} \colon K^f_{\ell_f^{-1}(\widehat{I})} \xrightarrow{\sim} K^f_{\ell_f^{-1}(\widehat{J})}$ *if $\widehat{I} \in A_-(\widehat{f})$ and $\widehat{J} \in A_+(\widehat{f})$,*
  (2) (-1) *times the identification $\kappa_{\ell^{-1}(\widehat{I})} \colon K^f_{\ell_f^{-1}(\widehat{I})} \xrightarrow{\sim} K^f_{\ell_f^{-1}(\widehat{J})}$ otherwise.*
- *For any Stokes arrow $\widehat{d}_{\widehat{f}} \to_{\widehat{d}} \widehat{d}_{\widehat{g}}$ over a singular direction $\widehat{d}$ of $\widehat{\Theta}$, denote by $\widehat{I} \in A(\widehat{f})$ and $\widehat{J} \in A(\widehat{g})$ the sectors whose intersection has $\widehat{d}$ as a midpoint. Then the deformation datum*
  $$\widehat{R}^{m_{\widehat{I}}}_{m_{\widehat{J}}} \colon (\widehat{V}^0_{\langle \widehat{f}\rangle})_{m_{\widehat{I}}} \to (\widehat{V}^0_{\langle \widehat{g}\rangle})_{m_{\widehat{J}}}$$
  is obtained by considering $I = \ell_f^{-1}(\widehat{I})$ and $J = \ell_g^{-1}(\widehat{J})$ and taking the linear map given by the composition
  $$(\widehat{V}^0_{\langle \widehat{f}\rangle})_{m_I} \overset{\iota_I}{\hookrightarrow} \mathbb{V}_{m_I} \xrightarrow{\sim} \mathbb{V}_{m_J} \overset{\pi_J}{\twoheadrightarrow} (\widehat{V}^0_{\langle \widehat{g}\rangle})_{m_J}$$
  if $\widehat{J}$ comes before $\widehat{I}$ and (-1) times this morphism otherwise (in particular if $\pi_f(\widehat{I}) = \pi_g(\widehat{J})$).

More intuitively, the second part of the above theorem means the following: To obtain a deformation datum of $\widehat{\mathbb{V}}$, map the corresponding path to the fission surface of $\mathbb{V}$ via the Legendre transform and read off the corresponding map (up to a sign). Note that the path from m_I to m_J in the picture before Fourier transform that we obtain is not a path associated to a Stokes arrow of $\mathbb{V}$. We will make this explicit in many examples later on.

Proof. With the relations developed in the previous subsections, this is the statement of [Moc21, Theorem 7.3.1] (see also [Moc21, §1.8] for a more explicit version) in our case.

As remarked above, one observes that the objects $\mathcal{K}_\lambda$ from loc. cit. correspond to the graded pieces of the formal local system and that the notion of deformation data from loc. cit. correspond to that for Stokes local systems introduced above, as shown in Proposition 4.4.

A major difference in our formulation is that we consider everything over the circle ∂, instead of $2\pi\mathbb{Z}$-equivariant objects on the universal covering space $\mathbb{R}$. This does not pose any problem in our case since the only singularity of our system is at the point ∞, hence the topological monodromy (the monodromy of the Stokes local system around the origin) is equal to the identity. Therefore, one can identify an interval on ∂ with its copy shifted by 2π without getting a contribution in the formulas for the deformation data in loc. cit.

Moreover, a priori, the formulae in loc. cit. describe the Fourier transform of a system localised (or co-localised) at 0, while there is no singularity at 0 in our case. However, the Fourier transform of the system without a singular point at 0 can be constructed from these systems. This is described in [Moc21] and amounts to "forgetting" (i.e. setting to zero) the graded piece for the exponent $\langle 0\rangle$ and all the associated deformation data. □

Note that Theorem 5.3 gives us a completely topological way to pass from the data of the Stokes local system $\mathbb{V}$ (given, for example, by monodromies as in §3.3) to the formal data and deformation data of $\widehat{\mathbb{V}}$. On the other hand, we know how these deformation data are extracted from $\widehat{\mathbb{V}}$ (see Definition 4.7), which gives us an explicit way of reconstructing a description in terms of monodromies as in §3.3 from these deformation data. Altogether, this leads to a completely explicit algorithmic way of computing the Stokes data of the Fourier transform in terms of the Stokes data of the original system, and we illustrate this in explicit examples in Section 6.

To arrive at this explicit description, notice that given a choice of reference direction $b \in \partial$, a point i_l above b in each active circle of Θ as in §3.3, and analogous choices of reference direction $\widehat{b}$, points $\widehat{i}_l$ for $\widehat{\Theta}$, via the algorithm, any choice of compatible minimal framing for Θ determines a compatible minimal framing for $\widehat{\Theta}$, in a way equivariant under the action of $\breve{H}$.

5.4. **Isomorphisms of wild character varieties.** The algorithm for the topological Fourier transform can be reformulated in terms of the explicit presentations of Stokes local systems. Let Θ be an irregular class at infinity, and $\widehat{\Theta}$ its formal Fourier transform. Let us fix choices of reference directions $b \in \partial$, points i_l, and $\widehat{b} \in \widehat{\partial}$, points $\widehat{i}_l$, a compatible minimal framing for Θ and the corresponding minimal framing for $\widehat{\Theta}$. As discussed in §3.3, this identifies the moduli spaces of (minimally framed) Stokes local systems with irregular class Θ, $\widehat{\Theta}$ with the reduced representation varieties $\mathcal{E}_\Theta$ and $\mathcal{E}_{\widehat{\Theta}}$. We have:

Theorem 5.4. *Let Θ be an irregular class at infinity satsifying our assumptions, and $\widehat{\Theta}$ its formal Fourier transform. The Fourier transform induces an algebraic isomorphism Φ between the representation varieties $\mathcal{E}_\Theta$ and $\mathcal{E}_{\widehat{\Theta}}$, provided that they are nonempty.*

Proof. The isomorphism is the one induced by the algorithm from Theorem 5.3. The highly nontrivial point is that this map is well-defined. This is a consequence of the general theory of [Moc21]: If $\rho \in \mathcal{E}_\Theta$, then there exists a connection ∇ on the affine line with irregular class Θ whose associated Stokes representation is isomorphic to ρ. Then the Fourier transform $\widehat{\nabla}$ of ∇ is a connection with irregular class $\widehat{\Theta}$, and [Moc21] shows that its Stokes data are obtained from those of ∇ by the deformation data description of the algorithm. The deformation data of $\widehat{\nabla}$ correspond to a Stokes representation $\widehat{\rho}$ with irregular class $\widehat{\Theta}$, which defines an element $\widehat{\rho} \in \mathcal{E}_{\widehat{\Theta}}$. That the map is a bijection follows immediately from the fact that the algorithm can be performed in reverse, using the Legendre transform in the other direction to transport paths. Finally, that the map is algebraic follows from the fact that the entries of the new Stokes matrices are (up to a sign) entries of products of the initial Stokes matrices, and in turn polynomial expressions in the entries of the initial Stokes matrices. □

Remark 5.5. Since the algorithm is entirely topological, a natural question which arises at this point is whether there is a simpler direct topological proof of the fact that $\Phi(\rho)$ is an element of $\mathcal{E}_{\widehat{\Theta}}$, which does not involve the Riemann–Hilbert–Birkhoff correspondence.

Recall that the representation varieties $\mathcal{E}_\Theta$ and $\mathcal{E}_{\widehat{\Theta}}$ possess a quasi-Hamiltonian structure. We conjecture that these structures are preserved by the isomorphism induced by the Fourier transform.

Conjecture 5.6. *The isomorphism Φ is compatible with the quasi-Hamiltonian structures on $\mathcal{E}_\Theta$ and $\mathcal{E}_{\widehat{\Theta}}$.*

This conjecture is quite natural: indeed it is known that the Fourier transform induces a symplectic isomorphism of wild character varieties in the case of connections on $\mathbb{P}^1$ with an untwisted irregular singularity at infinity of order 2 together with regular singularities at finite distance [Boa15]. Furthermore, in [Sza15] it is shown that in this case the Fourier transform preserves the full hyperkähler structure of the moduli space.

6. Examples

Is this section, we consider a few examples to illustrate the featuring some fully explicit computations with Stokes matrices. In particular, in all cases, the conjectures stated above are satisfied.

6.1. **The case of pure Gaussian type.** Consider first a system of pure Gaussian type, with irregular class

$$\Theta = \left\langle \frac{1}{2}z^2 \right\rangle + \left\langle \frac{1+i}{2\sqrt{2}}z^2 \right\rangle.$$

(This short notation shall mean that the two summands are the only active circles, both of multiplicity one, i.e. mapped to 1 by the irregular class Θ.)

There are four singular directions, and hence any Stokes local system with irregular class Θ is determined by four Stokes factors and the formal monodromy. Explicitly, if we choose a base point $b \in \partial$ and a framing at b, the Stokes data of any connection with irregular class Θ are of the form:

$$S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix} \quad S_4 = \begin{pmatrix} 1 & s_4 \\ 0 & 1 \end{pmatrix} \quad h = \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}$$

and must satisfy $hS_4S_3S_2S_1 = \mathrm{id}$. Let us denote by Sto_i, $i = 1, \dots, 4$ the Stokes groups corresponding to these forms of the Stokes matrices.

In turn, the representation variety associated to Θ is

$$\mathcal{R}_\Theta = \{(h, S_1, S_2, S_3, S_4) \in H \times \mathrm{Sto}_1 \times \mathrm{Sto}_2 \times \mathrm{Sto}_3 \times \mathrm{Sto}_4 \mid hS_4S_3S_2S_1 = \mathrm{id}\}.$$

Furthermore, denoting by $H \subset \mathrm{GL}_2(\mathbb{C})$ the subgroup of diagonal matrices, any conjugacy class $\mathcal{C} \subset H$ is just a singleton $\mathcal{C} = \{h\}$ for some h, so the wild character variety associated to the formal data $(\Theta, \mathcal{C})$ corresponds to the quotient

$$\mathcal{M}_B(\Theta, \mathcal{C}) = \{(S_1, S_2, S_3, S_4) \in \mathrm{Sto}_1 \times \mathrm{Sto}_2 \times \mathrm{Sto}_3 \times \mathrm{Sto}_4 \mid hS_4S_3S_2S_1 = \mathrm{id}\}/H.$$

Since all Stokes matrices have determinant 1, it follows that $\mathcal{M}_B(\Theta, \mathcal{C})$ can be possibly nonempty only for $\det h = 1$, i.e.

$$h = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix},$$

for $\tau \in \mathbb{C}^*$, which we assume to be satisfied in the rest of this paragraph.

The initial situation, before the Fourier transform, is summarised by the picture below, where each active circle is divided into four distinguished intervals (see Figure 5).

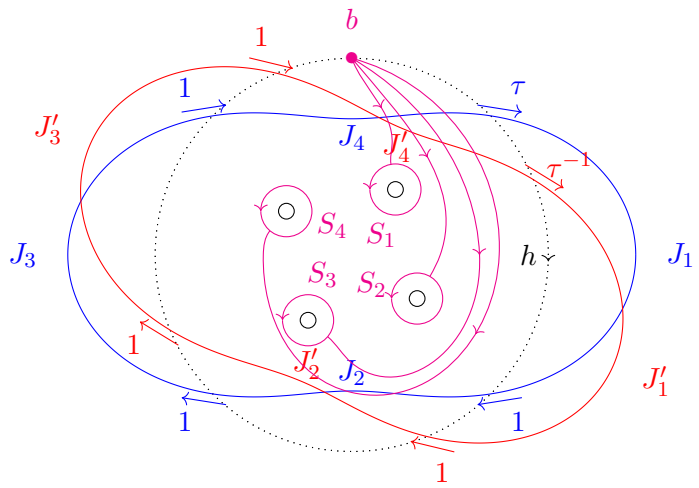


FIGURE 5. The Stokes local system before Fourier transform. If we choose a base point b at the boundary, it is determined by the monodromies around the punctures at the singular directions (given by the Stokes factors S_i) and the formal monodromy h along the boundary circle ∂ (represented here by the dotted circle). The corresponding graded local system can be encoded by dividing every exponent circle into four intervals J_i and J_i', and associating a vector space to each interval, with gluing maps for passing from one to the next. We can choose bases on these intervals such that the graded local systems there are glued by three identity maps and a single nontrivial map.
Note that – in contrast to Figure 3 – this picture shows the real blow-up of $\mathbb{P}^1$ at ∞ from a different perspective: here, the affine line is in the interior (i.e. the origin is at the center of this picture).

Now, we want to determine the similar picture after Fourier transform. The formal Fourier transform of Θ is given (via the Legendre transform) by

$$\widehat{\Theta} = \left\langle -\frac{1}{2}w^2 \right\rangle + \left\langle \frac{-1+i}{2\sqrt{2}}w^2 \right\rangle .$$

The space of Stokes representations and the wild character varieties with irregular class $\widehat{\Theta}$ have a similar form as those with irregular class Θ.

Let us first describe the new formal data. Each of the active circles of $\widehat{\Theta}$ is again subdivided into four distinguished intervals. Via the Legendre transform, these intervals are identified with the ones before Fourier transform, and by the first part of the transformation rule of Proposition 5.3, the new graded local systems is encoded by having the same vector spaces on corresponding intervals, but introducing some signs in the gluing maps (see Figure 6).

In particular, the conjugacy class of the new formal monodromy is $\widehat{\mathcal{C}} = \mathcal{C}$.

Next, let us compute the deformation data of the Fourier transform using the second part of Proposition 5.3. Therefore, we have to do the following: for any Stokes arrow, we draw the corresponding path in the irregular curve of the Fourier transform, we transform it (by applying the Legendre transform to its endpoints) to a path in the irregular curve of the original system, and we determine the map associated to it explicitly since the Stokes local system is explicitly given on the left-hand side.

We illustrate this with one Stokes arrow (see Figure 7). We draw the path associated to the Stokes arrow and pull it back to the picture before Fourier transform.

Now, comparing the picture on the left with Figure 5, we note that the parallel transport along this path in the initial Stokes local system (not taking into account the inclusion

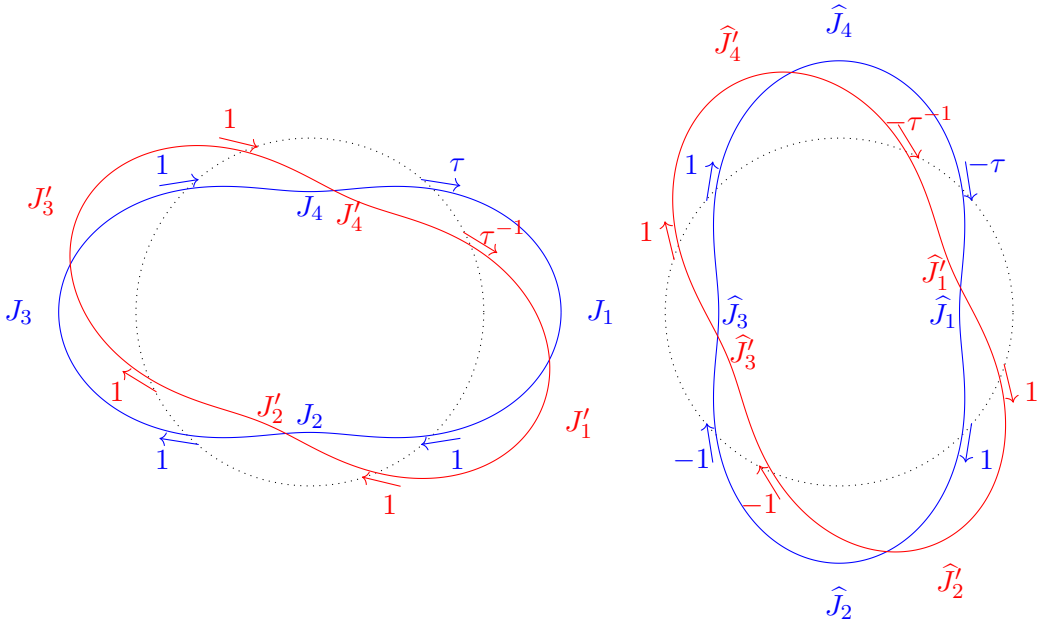


FIGURE 6. The transformation of the formal monodromy: for each exponent, the four intervals before and after Fourier transform are in bijection via the Legendre transform (as indicated by the numbering). The gluing maps either remain the same (for transitions from a sector with negative exponent to one with a positive exponent in the Fourier transform) or change sign (otherwise). Of course, here this means that the local systems on the circles remain unchanged globally since we introduce an even number of sign changes, but we keep the information about the local gluings here for the algorithm.

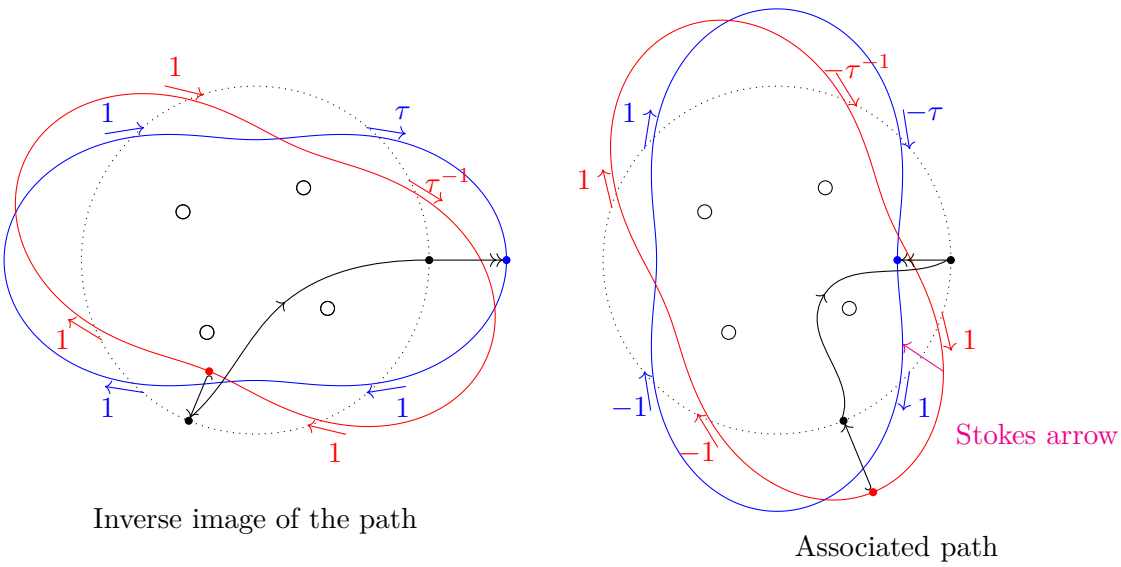


FIGURE 7. Transporting the distinguished path associated to a Stokes arrow to the initial Stokes diagram.

and projection for the moment) corresponds to

$$\begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} S_2^{-1} \begin{pmatrix} 1 \times \tau & 0 \\ 0 & 1 \times \tau^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -s_2\tau^2 \\ 0 & 1 \end{pmatrix}$$

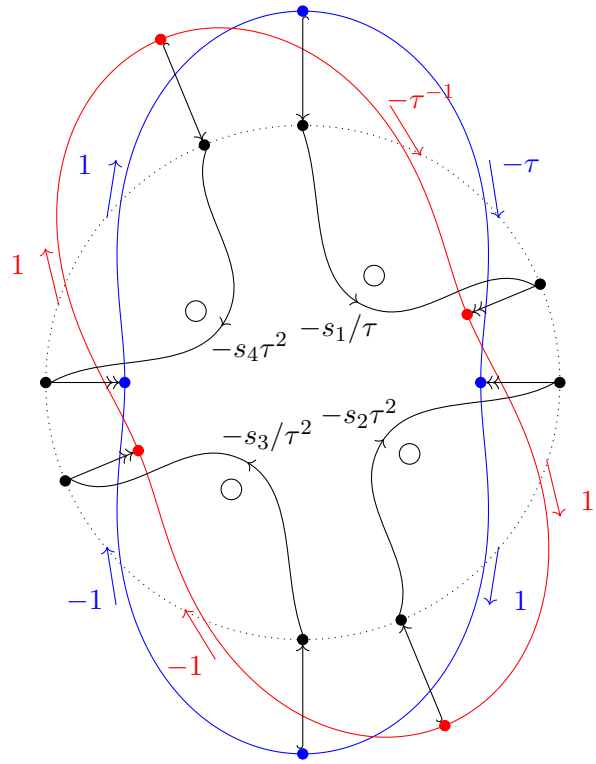


FIGURE 8. Deformation data of the Fourier transform

(The first matrix is the part of the formal monodromy connecting the endpoint of the path with our basepoint b, and the third matrix connects the startpoint of our path with b.)

Together with the inclusion of the red strand (the second exponent) at the beginning and the projection to the blue strand (the first exponent) at the end of the path, this means that the deformation datum associated to that this path corresponds to the $(1,2)$ entry of this matrix, which is $-s_2\tau^2$. If we do the same with the other deformation data (remembering that in cases where the orientation of the path is clockwise, we need to change the sign at the end), we get the full collection of deformation data for the Fourier transform, shown in Figure 8.

Finally, to obtain an explicit Stokes matrix description of the Stokes local system of the Fourier transform, let us choose a basepoint $\widehat{b}$ at the boundary in the picture for the Fourier transform. The Stokes matrices $\widehat{S}_i$ correspond to the paths in Figure 9.

We compare Figure 8 and Figure 9 to determine the matrices $\widehat{S}_i$ from the deformation data. For example, consider the deformation datum with value $-s_3/\tau^2$. The parallel transport along the corresponding path (without the inclusion and projection) is also given by the matrix

$$\begin{pmatrix} (-1)\times 1\times(-\tau) & 0 \\ 0 & (-1)\times 1\times(-\tau^{-1}) \end{pmatrix} \widehat{S}_3 \begin{pmatrix} 1\times(-\tau) & 0 \\ 0 & 1\times(-\tau^{-1}) \end{pmatrix}^{-1}.$$

The deformation datum itself, now taking the inclusion of the first component at the beginning and the projection to the second component at the end, corresponds to the $(2,1)$ component of this matrix.

Since we know a priori that $\widehat{S}_3$ is of the form $\begin{pmatrix} 1 & 0 \\ \widehat{s}_3 & 1 \end{pmatrix}$, it follows that $\widehat{s}_3 = s_3$. Doing this for all deformation data paths, we get:

$$\widehat{S}_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \quad \widehat{S}_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \quad \widehat{S}_3 = \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix} \quad \widehat{S}_4 = \begin{pmatrix} 1 & s_4 \\ 0 & 1 \end{pmatrix} \quad \widehat{h} = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}.$$

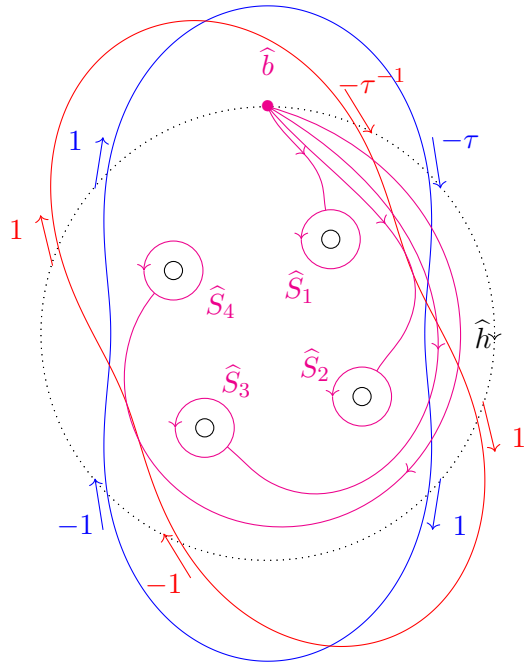


FIGURE 9. The monodromies of the Stokes local system of the Fourier transform with respect to a choice of basepoint $\widehat{b}$.

We thus obtain with this choice of parametrisation $\widehat{S}_i = S_i$ and $\widehat{h} = h$, that is, the Stokes matrices after Fourier transform are the same as the initial ones. In particular, we clearly have here a well-defined isomorphism between the spaces of Stokes representations on both sides, which in turn induces (through symplectic reduction by H) isomorphisms at the level of the wild character varieties. The Poisson/symplectic structures are obviously preserved.

To summarise, we have established the following.

Proposition 6.1. *Keeping the previous notations, let $\mathcal{C} = \{h\}$ be a conjugacy class for Θ and $\widehat{\mathcal{C}} = \mathcal{C}$. The Fourier transform induces the isomorphism $\Phi\colon \mathcal{M}_B(\Theta, \mathcal{C}) \to \mathcal{M}_B(\widehat{\Theta}, \widehat{\mathcal{C}})$ given by*

$$\widehat{S}_i = S_i, \qquad \widehat{h} = h$$

with respect to our choices of the base point and framing at this base point.

Remark 6.2. Let us remark that this result is consistent with previous studies of Fourier transforms of Stokes data for the Gaussian case [Sab16, Ho22]: Although we consider different conditions on the exponents here, we still get a result in the same spirit: With respect to suitable choices, the representations of Stokes data on both sides of the Fourier transform coincide.

6.2. **The Airy case.** Let us now discuss an example featuring a twisted irregular class. Consider the irregular class $\Theta = \langle z^3/3 \rangle$. Its Stokes structure is trivial (no Stokes arrows). In turn, a wild character variety with this irregular class is nonempty only if its formal monodromy is the identity, so there exists a unique rank one Stokes local system $\mathbb{V}$ with irregular class Θ. Let us determine its Fourier transform.

The formal Fourier transform of Θ is $\widehat{\Theta} = \langle \frac{2}{3} w^{3/2} \rangle$, which is closely related to the Airy equation.

The formal local system associated to $\mathbb{V}$ is the local system on $\langle z^3/3 \rangle$ with trivial monodromy, which we view again as glued from one-dimensional constant sheaves (with stalk $\mathbb{C}$) on 6 sectors. The Legendre transform maps the circle $\langle z^3/3 \rangle$ onto $\langle \frac{2}{3} w^{3/2} \rangle$. This identifies the sectors on both sides, and the algorithm gives us the induced transition maps defining the formal monodromy of the new formal local system, as indicated on the figure

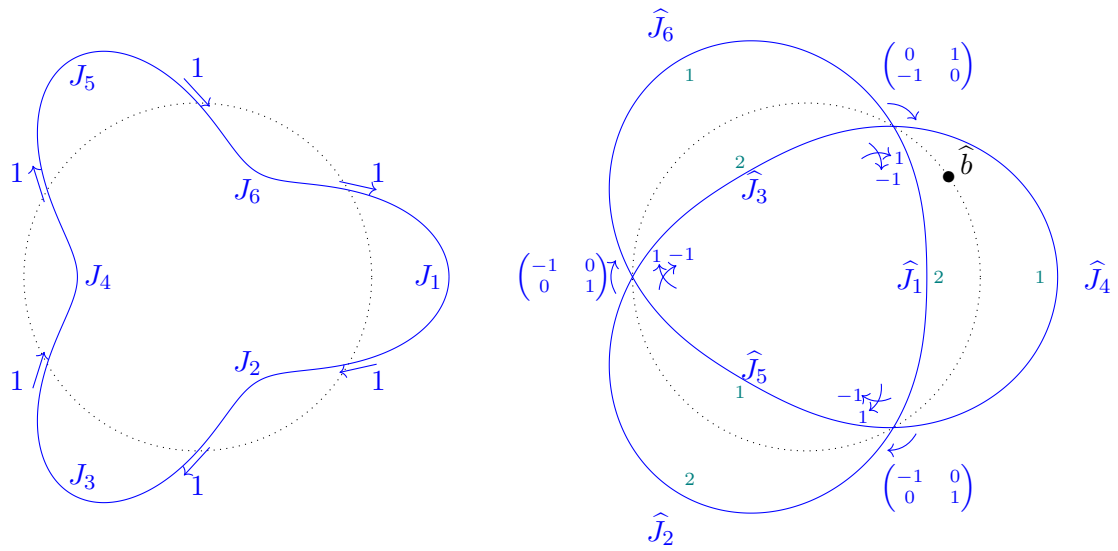


FIGURE 10. Left: the formal local system before Fourier transform, Right: the formal local system after Fourier transform. The numbers in the circles show the identification of sectors given by Legendre transform. The green numbers show the numbering of the strands, given by choosing a numbering near a basepoint $\widehat{b}$ and continuing it compatibly in clockwise direction. The matrices are the partial formal monodromies given by the transformation rule (they come from the formal monodromies on the left, but change signs whenever one transits from a positive to a negative strand).

below. If we choose a basepoint and an ordering of the strands, we get three transition matrices (in blue) for the now partial formal monodromies, the last one taking into account the twist (permutation of the strands).

The formal local systems thus look as shown in Figure 10.

There are three Stokes arrows in the Airy case. Each of them corresponds to a path on the Stokes diagram of $\widehat{\Theta}$. Via the Legendre transform, this path can be transported to a path in the initial Stokes diagram, the parallel transport along which defines a linear map (a number in this explicit case). Let us detail this process in the example drawn in Figure 11.

Notice that the path associated to the Stokes arrow is closed here, but its inverse image in the picture before Fourier transform is not. The path on the left corresponds to the identity map. The deformation data map associated to this Stokes arrow is therefore given by -1 (the transformation rule tells us to change the sign). The computation of the deformation data corresponding to other two Stokes arrows is similar. All three deformation data are indicated on the figure below.

To explicitly compute the Stokes matrices for Airy, we now need to relate them to these deformation data. To this end, we choose a basepoint $\widehat{b}$, and we get the picture shown in Figure 12.

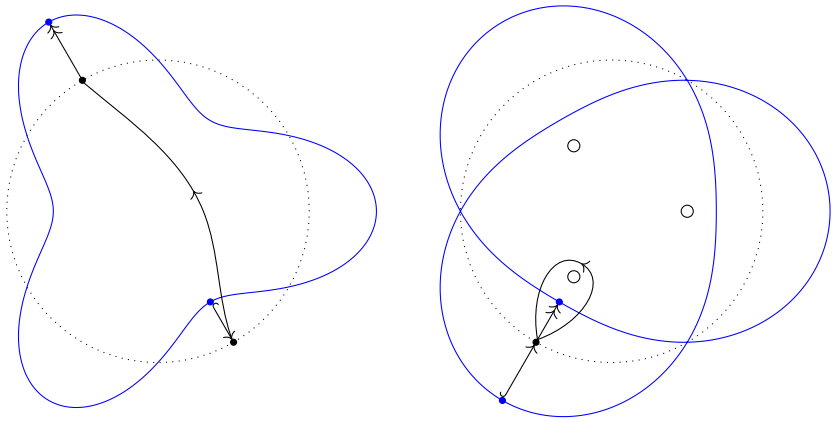
Let us, for instance, detail how to compute S_2. We know that it is of the form

$$\widehat{S}_2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Now, the parallel transport along the path we considered above corresponds to the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \widehat{S}_2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

(go back to the basepoint via a partial formal monodromy, apply the loop $\widehat{S}_2$ and go away from the basepoint again), and the number associated to its deformation datum should



Inverse image of path Path associated to one Stokes arrow

FIGURE 11. A Stokes arrow between a positive and a negative sector corresponds to a path connecting the midpoints of the two sectors, going around at least one singular direction (composed with the embedding of the positive strand and the projection to the negative strand). Via the Legendre transform it has a corresponding path on the left-hand side, in the picture before Fourier transform. Notice that this path does not need to correspond to a Stokes arrow.

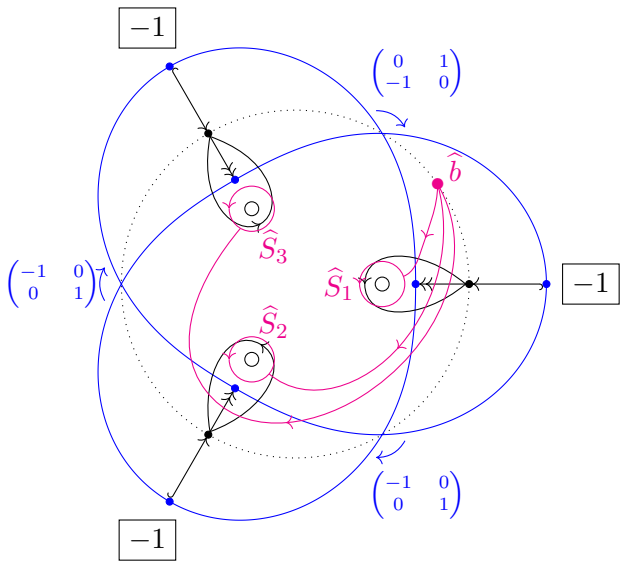


FIGURE 12. The monodromy data defining the Stokes local system with respect to a basepoint (in magenta) and the deformation data (in black) for the system after Fourier transform.

correspond to the $(1,2)$-entry of this (first row, second column), so we get the equation

$$-a = -1,$$

and hence

$$\widehat{S}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Analogously we get

$$\widehat{S}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \widehat{S}_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The new formal monodromy is the product of the partial formal monodromies

$$\widehat{h} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We check easily that

$$\widehat{h}\widehat{S}_3\widehat{S}_2\widehat{S}_1 = \mathrm{id},$$

so the algorithm provides as expected a valid set of Stokes data $(\widehat{h}, \widehat{S}_1, \widehat{S}_2, \widehat{S}_3) \in \mathcal{M}_B(\widehat{\Theta}, -1)$ for $\widehat{\Theta}$.

In this case, the Betti moduli spaces $\mathcal{M}_B(\Theta, 1)$ and $\mathcal{M}_B(\widehat{\Theta}, -1)$ both consist of just a point, so the algorithm induces an isomorphism as expected (and there is no symplectic form to be considered).

6.3. **The case of $\langle z^{5/3} \rangle$ and $\langle w^{5/2} \rangle$.** We now discuss an example where one obtains a more interesting isomorphism. Let us consider the irregular class $\Theta = \langle z^{5/3} \rangle$ with only one active circle, of slope $5/3$. Its image under Fourier transform is an irregular class with one active circle (up to a positive real coefficient) $\widehat{\Theta} = \langle w^{5/2} \rangle$. Let us once again describe the isomorphism between the two corresponding wild character varieties induced by the Fourier transform. This time we will give less details about the passage between the Stokes matrices and the deformation data descriptions.

The Stokes diagram corresponding to $\Theta = \langle z^{5/3} \rangle$ is drawn in Figure 13 below. There are ten singular directions and ten Stokes arrows. We choose a reference direction $b \in \partial$ as indicated on the figure. Up to choosing an appropriate framing, one may assume that the formal monodromy has the form

$$h = \begin{pmatrix} 0 & 0 & \tau \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

with $\tau \in \mathbb{C}^*$, with any conjugacy class $\mathcal{C}$ for the formal monodromy corresponding to one value of τ. With the choice of numbering of the strands at b indicated on the figure, the Stokes matrices are of the form

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_1 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s_2 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & s_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 1 & 0 & t_4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s_5 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_6 = \begin{pmatrix} 1 & 0 & 0 \\ s_6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_7 & 0 & 1 \end{pmatrix}, \quad S_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s_8 & 1 \end{pmatrix},$$
$$S_9 = \begin{pmatrix} 1 & s_9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_{10} = \begin{pmatrix} 1 & 0 & s_{10} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us denote by $\mathrm{Sto}_1, \dots, \mathrm{Sto}_5$ the corresponding Stokes groups. From the quasi-Hamiltonian description of wild character varieties, it follows that the Betti moduli space (taking into account that here $\breve{H} = \mathbb{C}^*$ acts trivially on $\mathcal{E}_\Theta$) is explicitly given by

$$\mathcal{M}_B(\langle z^{5/3} \rangle, \tau) = \left\{ (S_1, \dots, S_{10}) \in \prod_{i=1}^{10} \widehat{\mathrm{Sto}_i} \mid hS_{10} \dots S_1 = 1 \right\}.$$

The Stokes diagram for $\langle w^{5/2} \rangle$ is drawn in Figure 14 below. There are 10 Stokes arrows on 10 singular directions. We choose a reference direction $\widehat{b}$ as indicated on the figure. Again, if we choose an appropriate framing (which is indeed given by the Legendre transform), we can assume that the formal monodromy has the form

$$\widehat{h} = \begin{pmatrix} 0 & \widehat{\tau} \\ -1 & 0 \end{pmatrix}$$

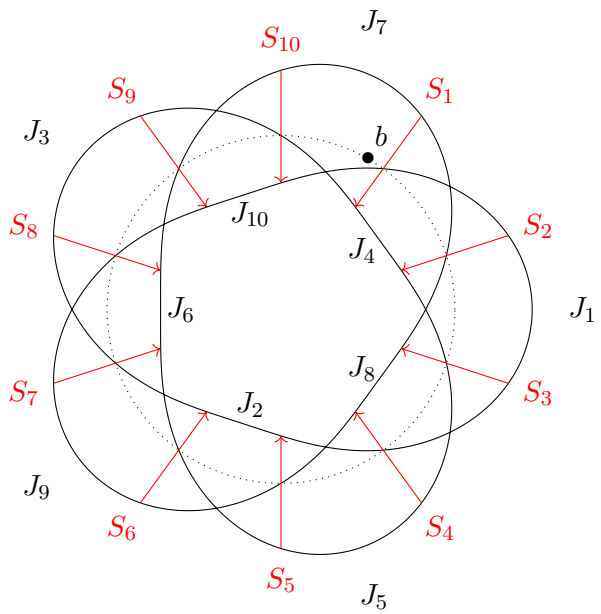


FIGURE 13. The Stokes diagram of $\langle z^{5/3}\rangle$, with the distinguished intervals and the Stokes arrows.

with a conjugcacy class for the formal monodromy corresponding to one value of $\widehat{\tau}$. The Stokes matrices have the form

$$\widehat{S}_1 = \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}, \quad \widehat{S}_2 = \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix}, \quad \widehat{S}_3 = \begin{pmatrix} 1 & 0 \\ t_3 & 1 \end{pmatrix}, \quad \widehat{S}_4 = \begin{pmatrix} 1 & t_4 \\ 0 & 1 \end{pmatrix}, \quad \widehat{S}_5 = \begin{pmatrix} 1 & 0 \\ t_5 & 1 \end{pmatrix}.$$

Let us denote by $\widehat{\mathrm{Sto}}_i$, $i = 1, \dots, 10$ the corresponding Stokes groups. The Betti moduli space associated to $\langle w^{5/3}\rangle$ and formal monodromy determined by $\widehat{\tau}$ is thus given by

$$\mathcal{M}_B(\langle w^{5/2}\rangle, \widehat{\tau}) = \left\{ (\widehat{S}_1, \dots, \widehat{S}_5) \in \prod_{i=1}^{5} \mathrm{Sto}_i \mid h\widehat{S}_5\widehat{S}_4\widehat{S}_3\widehat{S}_2\widehat{S}_1 = 1 \right\}.$$

A first observation is the following:

Lemma 6.3. *$\mathcal{M}_B(\langle z^{5/2}\rangle, \tau)$ is nonempty if and only if $\tau = 1$. $\mathcal{M}_B(\langle w^{5/3}\rangle, \widehat{\tau})$ is nonempty if and only if $\widehat{\tau} = 1$.*

Proof. For both moduli spaces, considering the determinant of the product of the Stokes matrices immediately implies that the condition $\tau = 1$ or $\widehat{\tau} = 1$ is necessary. It is then possible to check directly that in this case the moduli space is nonempty. □

Remark 6.4. Notice that this condition for the moduli spaces to be nonempty is consistent with the sign change of the formal monodromy in the formal Fourier transform.

Both circles $\langle z^{5/3}\rangle$ and $\langle w^{5/2}\rangle$ are divided into 10 distinguished intervals $J_1, \dots J_{10}$, and $\widehat{J}_1, \dots, \widehat{J}_{10}$, as indicated in Figures 13 and 14, respectively, with the Legendre transform sending J_i to $\widehat{J}_i$. Applying the algorithm, we obtain:

Proposition 6.5. *The Fourier transform induces between $\mathcal{M}_B(\langle z^{5/2}\rangle, 1)$ and $\mathcal{M}_B(\langle w^{5/3}\rangle, 1)$ the isomorphism Φ given by*

$$\begin{cases} t_1 &= -s_6 \\ t_2 &= -s_9 \\ t_3 &= -s_7 \\ t_4 &= -s_5 \\ t_5 &= -s_8 \end{cases}$$

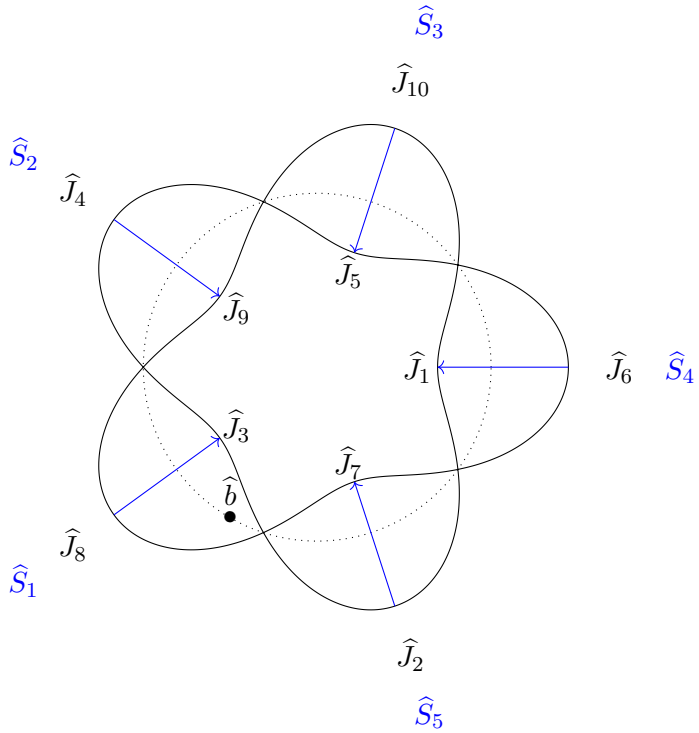


FIGURE 14. The Stokes diagram of $\langle w^{5/2} \rangle$, with the distinguished intervals and the Stokes arrows.

using the choices of basepoints and parametrisation of the Stokes matrices described above. Furthermore, this isomorphism is symplectic.

Proof. The table below summarises the main ingredients of the computation for each coefficient t_i: It indicates the Stokes arrow corresponding to the coefficient, the corresponding entry of a product of the initial Stokes matrices obtained by transporting back the Stokes arrows on the initial Stokes diagram, as well as the extra sign coming from the changes of signs in the formal monodromy.

coefficient	Stokes arrow	Entry of matrix product	extra sign
t_1	$8 \to 3$	s_6	$-$
t_2	$4 \to 9$	$-s_9$	$+$
t_3	$10 \to 5$	$-s_7$	$+$
t_4	$6 \to 1$	$-s_5$	$+$
t_5	$2 \to 7$	τs_8	$-$

The procedure is completely analogous to the one detailed in the previous examples. Let us just give some details on the computation of the first coefficient: For t_1, the corresponding Stokes arrow is $8 \to 3$. To go from J_8 to J_3 on the initial picture, one needs to consider the product of Stokes factors $S_8 \cdots S_4$. Then, taking into account the numbering of the strands (from high to low at b), one considers its entry at position $(2, 1)$, which is s_6. The "extra sign" $\varepsilon_{8\to 3} = +$ comes from the signs visible in Theorem 5.3: The corresponding deformation datum for the Fourier transform will be $-s_6$ in this case, and when reconstructing the Stokes matrix $\widehat{S}_1$ from this deformation datum, one takes into account the signs induced by the Legendre transform on the level of formal monodromy, which in total gives another negative sign. In the case of t_5, an additional factor of τ occurs in this last step, which cancels with the one in the matrix entry. The remaining coefficients are obtained in a similar way.

We can check explicitly that this indeed provides a well-defined isomorphism. The fact that this isomorphism preserves the symplectic structure can also be checked by direct computation. □

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