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Small degree Salem numbers with trace -3

J.-M. Sac-Épée*

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Abstract

The contribution of this work is to provide tables of Salem numbers with trace -3 and small degrees, namely degrees $2d = 34, 36, 38$, and 40 . The implemented method also generates a list of totally positive polynomials of degrees $d = 17, 18, 19$, and 20 , with trace $2d - 3$, a single root strictly greater than 4 , and all other roots strictly between 0 and 4 .

Keywords— Salem numbers of small trace, Totally positive polynomials
MSC: 11K16

1 Introduction

A Salem number is a real algebraic integer strictly greater than 1 , whose conjugates all have a modulus less than or equal to 1 , and at least one conjugate has a modulus equal to 1 . The minimal polynomial P of a Salem number (referred to as a *Salem polynomial*) is self-reciprocal, i.e. $x^d P(1/x) = P(x)$, where d is the degree of P . A direct consequence of this is that a Salem number is conjugate to its inverse, and the rest of its conjugates are on the unit circle.

By definition, the degree of a Salem number is simply that of its minimal polynomial. This degree must be even and at least equal to 4 . Similarly, the trace of a Salem number is that of its minimal polynomial, i.e. the sum of its roots. With

$$P(x) = \sum_{k=0}^d a_k x^k \quad (a_d = 1),$$
 the trace of P is equal to $-a_{d-1}$.

A very comprehensive survey devoted to Salem numbers is [1], where the essential results and properties proved on the subject over the last few decades are presented and explained in detail.

As stated in [2], examples of Salem numbers of any strictly positive trace are available since the largest real root of the polynomial $x^4 - nx^3 - (2n + 1)x^2 - nx + 1$ is a Salem number (of trace n) for all $n \geq 1$. It is also well known that the polynomial $x^6 - x^4 - 2x^3 - x^2 + 1$ is the minimal polynomial of a Salem number of zero trace.

Things get much more complicated when it comes to finding Salem numbers with negative traces, and a great deal of work has been devoted to this.

In 2004, J. McKee and C.J. Smyth provided Salem numbers with trace -2 [6]. The same two authors proved in 2005 that there are Salem numbers of all negative traces [2]. In 2011, J. McKee [5] provided an explicit example of a Salem number of trace -3 (its minimal polynomial was of degree 54). Four years later, S. El Otmani,

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G. Rhin and J.-M. Sac-Épée provided a Salem number with degree 34 and trace -3 [8], expressing the conviction, motivated by the results of [4], that there is presumably no Salem number with trace -3 and degree 32. In 2024, G. Cherubini and P. Yatsyna proved [9] that there exist Salem numbers with trace -3 and every even degree ≥ 34 , notably providing a table containing an example of a Salem number of trace -3 for each of the degrees from 34 to 52, as well as an example at degree 58.

In the present work, our aim is to provide tables giving numerous examples of Salem numbers of trace -3 and small degrees, i.e. degrees between 34 and 40. The method on which our approach is based, namely integer linear optimization (since the variables of the problem are the *integer* coefficients of the polynomials to be found), has already enabled us to provide a list of totally positive polynomials of degree $d = 16$ and trace $2d - 3 = 29$ [4], as well as the Salem number of trace -3 and degree 34 previously mentioned [8]. This method is explained in detail in the two articles cited above, so we will just summarize the main points for the reader's convenience.

2 Outline of the method used to find Salem numbers

2.1 Relationship between Salem numbers and totally positive polynomials

Consider a Salem polynomial P with degree $2d$. By the change of variable $z = x + \frac{1}{x} + 2$, we obtain a monic integer polynomial Q of degree d . This polynomial Q is characterized by all its roots being real and positive: $d - 1$ roots are in $(0, 4)$, and only one root exceeds 4. Indeed, the Salem number τ for which P is the minimal polynomial is a real number greater than 1, so the corresponding root $\tau + 1/\tau + 2$ of Q is greater than 4. On the other hand, it is well known that the minimal polynomial of a Salem number is reciprocal, which is therefore the case for P . Knowing this, we can easily prove that the other roots of P actually have modulus 1, so that the corresponding roots of Q lie within the interval $(0, 4)$. This is thoroughly detailed in [11]. With this in mind, we then set ourselves the goal of finding totally positive monic polynomials of degree $d = 17, 18, 19, 20$ and trace $2d - 3 = 31, 33, 35, 37$, with $d - 1$ roots in $(0, 4)$ and only one root greater than 4. In addition to its own interest, each of the polynomials Q of degree d found will immediately provide us with a Salem number of degree $2d$ via an inverse change of variable.

2.2 Algorithmic approach

The initial phase consists of locating the zeros of such a polynomial Q of degree d using explicit auxiliary functions, a technique introduced in number theory by C.J. Smyth to study the absolute trace of a totally positive algebraic integer [7]. To bound the roots of the polynomials for each of the degrees $d = 17, 18, 19$ and 20 , we use a version of the auxiliary function due to V. Flammang [10], using which we establish that

Degrees d	Trace $2d - 3$	Intervals
17	31	(0,6.69)
18	33	(0,7.11)
19	35	(0,7.50)
20	37	(0,7.86)

Table 1: Intervals containing the roots of Q for degrees d

The second step is to propose bounds for the coefficients of the polynomials we are looking for. As explained in [4] and [8], this is achieved by uniformly drawing, in each interval, $d - 1$ real numbers in $(0, 4)$ and one real number greater than 4. With these real numbers, we reconstruct the polynomial (with real coefficients) of which they are the roots.. This is repeated many times, enabling us to propose reliable bounds for the integer coefficients of the polynomials of interest, based on the location of the real coefficients of the polynomials obtained via our random draws.

The objective is then to identify monic integer polynomials $Q = \sum_{i=0}^d a_i x^i$ ($a_d = 1$),

with all roots in the various intervals listed in Table 1, where the a_i 's are the unknown variables of the system we aim to solve. As mentioned above, polynomials Q are required to have a single root greater than 4, and all other roots in $(0, 4)$. For an interval in Table 1 corresponding to degree d , we first note that a polynomial Q that vanishes d times within this interval must alternate between strictly positive and strictly negative sign at points separating its roots. We leverage this observation as follows: we uniformly draw $d - 2$ real numbers β_i in interval $(0, 4)$, and we add to these $d - 2$ values the real numbers 0, 4, and the right-hand bound of the considered interval. We order these real numbers in ascending order, and we require the polynomial Q to change sign at each of these $d + 1$ real values, seen as roots separators. We must have $Q(0) < 0$ if d is odd, $Q(0) > 0$ if d is even, and the sign of Q must change when we move from one separator to the next. This gives us a list of linear constraints that form the first part of our optimization problem. The system is completed by adding lower and upper bounds on the a_i 's. In [4] and [8], the libraries used required the minimization of an artificially introduced linear function, subject to a set of constraints, and the result of the minimization itself was of no interest, since discovering the a_i 's that satisfied the constraints was the sole true objective. With the newly experimented tools (the JuMP modeling language and the Gurobi optimization library), we were able to develop a much more powerful algorithm, focused on feasibility (that is, specific compliance with constraints) without the need to carry out the actual minimization of a linear function simply introduced for practical reasons.

3 Implementation and results

The improved algorithm has been fully implemented in the Julia programming language. Polynomial manipulations are performed using the *Polynomials* package. For irreducibility tests, the *Nemo* package is used. The optimization problem is formulated within the framework of the *JuMP* modeling language, while the search for a solution that respects the constraints imposed is carried out using the *Gurobi* library via a wrapper maintained by the *JuMP* community.

The algorithm launches a loop with the following three steps:

- Randomly draw separators,
- Look for a vector of integers respecting the constraints,
- Test the irreducibility of the polynomial whose coefficients are the integers possibly found in the previous step.

The tables mentioned below can be viewed or downloaded in PDF format at the following address: <https://plmbox.math.cnrs.fr/d/5f3825ddf147481aa7e9/>

Table 2 contains two polynomials of degree 17 and trace 31, the second of which already appeared in [8]. It also includes four polynomials of degree 18 and trace 33, and fourteen polynomials of degree 19 and trace 35. Table 3 contains forty-five polynomials of degree 20 and trace 37.

Table 4, Table 5, Table 6 and Table 7 present the coefficients of the Salem polynomials of degree $2d$ obtained from polynomials of Table 2 and Table 3 by inverse change

of variable. Only the coefficients necessary for the complete knowledge of each of the polynomials are shown in the tables, as we recall that these polynomials are reciprocal. In the last column of each table, the Salem numbers whose minimal polynomial appears on the same line are listed.

4 Conclusion

The refinement of our optimization algorithm has enabled us to achieve results that were out of reach with previous versions. We hope that the lists of Salem numbers with trace -3 are complete for degrees 34, 36, and possibly 38, considering the number of times the algorithm has rediscovered each of them.

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x^{19}	x^{18}	x^{17}	x^{16}	x^{15}	x^{14}	x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
		1	-31	432	-3583	19736	-76285	213173	-437297	662123	-738327	600855	-351222	143998	-40022	7179	-772	44	-1
		1	-31	433	-3608	20013	-78079	220717	-458940	705459	-799257	660596	-391294	161786	-44982	7979	-837	46	-1
1	-33	493	-4418	26527	-112896	351385	-813619	1412249	-1838442	1784606	-1276751	661168	-241606	60150	60150	-9713	946	-49	1
1	-33	493	-4418	26526	-112873	351153	-812266	1407186	-1825688	1702537	-1250469	639916	-230279	56361	56361	-8974	871	-46	1
1	-33	494	-4445	26854	-115243	362498	-850194	1498144	-1983921	1962162	-1431230	754969	-280049	70262	70262	-11292	1074	-53	1
1	-33	494	-4445	26854	-115243	362499	-850212	1498282	-1984511	1963710	-1433817	757739	-281916	71023	71023	-11468	1095	-54	1
1	-35	557	-5344	34553	-159439	542423	-1386300	2687875	-3966530	4445822	-3758294	2367533	-1091721	359274	-81407	12063	-1083	52	-1
1	-35	557	-5344	34551	-159390	541892	-1382942	2674104	-3928030	4370711	-3655340	2268985	-1027046	331107	-73678	10832	-984	49	-1
1	-35	557	-5344	34551	-159391	541914	-1383154	2675283	-3932228	4380750	-3671796	2287527	-1041244	338312	-76006	11283	-1031	51	-1
1	-35	558	-5374	34956	-162647	559300	-1448266	2851355	-4280967	4888760	-4213778	2705933	-1270169	424300	-97142	14439	-1283	59	-1
1	-35	558	-5374	34955	-162625	559086	-1447051	2846876	-4269675	4868764	-4188594	2683303	-1255726	417842	-95176	14054	-1240	57	-1
1	-35	558	-5373	34930	-162344	557208	-1438734	2821126	-4212363	4775880	-4078746	2589161	-1198202	393469	-88329	12870	-1130	53	-1
1	-35	558	-5373	34931	-162369	557485	-1440530	2828706	-4234284	4820423	-4142921	2654512	-1244549	415716	-95214	14134	-1248	57	-1
1	-35	558	-5372	34904	-162040	555095	-1429011	2789911	-4140583	4656415	-3935052	2465780	-1124387	363890	-80862	11790	-1053	51	-1
1	-35	558	-5372	34904	-162041	555117	-1429223	2791089	-4144766	4666361	-3951199	2483725	-1137903	370645	-83028	12212	-1098	53	-1
1	-35	558	-5372	34904	-162042	555138	-1429415	2792093	-4148084	4673591	-3961718	2493842	-1144106	372869	-83387	12197	-1086	52	-1
1	-35	558	-5372	34905	-162064	555350	-1430594	2796295	-4158182	4690409	-3981464	2510414	-1154167	377281	-84729	12453	-1112	53	-1
1	-35	559	-5402	35310	-165319	572714	-1495491	2971138	-4502372	5187028	-4503873	2905886	-1365213	454403	-103264	15211	-1340	61	-1
1	-35	560	-5431	35691	-168321	588542	-1554487	3131239	-4823342	5664341	-5027914	3325004	-1603764	547583	-126895	18777	-1616	69	-1
1	-35	560	-5430	35664	-167995	586218	-1543608	3096048	-4742710	5532312	-4873839	3198484	-1532318	520824	-120639	17968	-1574	69	-1

Table 2: Totally positive polynomials with degree $d = 17, 18, 19$ and trace $2d - 3 = 31, 33, 35$, a single root > 4 , and all other roots in $(0, 4)$.

x^{20}	x^{19}	x^{18}	x^{17}	x^{16}	x^{15}	x^{14}	x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
1	-37	625	-6399	44446	-222060	825522	-2329872	5048500	-8439635	10881501	-10768511	8104198	-4574262	1899487	-565429	116474	-15800	1309	-58	1
1	-37	625	-6398	44419	-221730	823105	-2318036	5007562	-8336641	10690269	-10505421	7837311	-4377327	1796212	-528252	107731	-14551	1213	-55	1
1	-37	625	-6398	44420	-221758	823452	-2320556	5019522	-8375790	10781149	-10656896	8018803	-4532351	1888945	-560022	117745	-16168	1354	-60	1
1	-37	625	-6398	44420	-221758	823406	-2320087	5016723	-8364912	10752164	-10602585	7946553	-4464268	1844094	-545867	111840	-15112	1252	-56	1
1	-37	625	-6398	44420	-221754	823362	-2319658	5014275	-8355817	10729005	-10561142	7893952	-4417035	1814480	-533222	108311	-14511	1197	-54	1
1	-37	625	-6397	44393	-221427	821013	-2306505	4977350	-8268158	10577853	-10371401	7721682	-4305574	1764264	-517945	105277	-14129	1168	-53	1
1	-37	626	-6429	44858	-225492	844888	-2408155	5282259	-8963730	11769570	-11904755	9193589	-5345760	2294145	-706514	150019	-20708	1695	-70	1
1	-37	626	-6428	44829	-225112	841911	-2392603	5225038	-8811017	11469514	-11468950	8726271	-4980804	2090616	-628026	130099	-17632	1443	-62	1
1	-37	626	-6428	44830	-225137	842191	-2394462	5233198	-8836024	11524592	-11557030	8829982	-5069161	2144406	-650788	136508	-18750	1549	-66	1
1	-37	626	-6428	44830	-225136	842168	-2394226	5231770	-8830351	11509007	-11526662	8787629	-5027014	2114896	-636621	132034	-17881	1457	-62	1
1	-37	626	-6428	44830	-225136	842168	-2394226	5231771	-8830369	11509014	-11527266	8789257	-5029844	2118095	-638940	133075	-18153	1494	-64	1
1	-37	626	-6428	44831	-225164	842493	-2396532	5244157	-8871910	11608670	-11699839	9006274	-5226294	2243942	-694391	149148	-21004	1765	-74	1
1	-37	626	-6428	44831	-225162	842470	-2396297	5241107	-8859521	11573861	-11630406	8906932	-5124480	2170088	-666355	139149	-18798	1503	-62	1
1	-37	626	-6427	44802	-224783	839517	-2381002	5185207	-8713504	11292862	-11232573	8494427	-4815249	2006263	-598222	123099	-16607	1359	-59	1
1	-37	626	-6427	44803	-224810	839841	-2383289	5196099	-8747502	11370755	-11361866	8650639	-4952078	1995204	-635760	134120	-18639	1568	-68	1
1	-37	626	-6427	44803	-224809	839817	-2383031	5194458	-8740630	11350803	-11320664	8589526	-4887098	2043125	-610360	125384	-16782	1353	-58	1
1	-37	626	-6427	44803	-224807	839772	-2382584	5191872	-8733094	11326157	-11277021	8535488	-4840735	2016215	-600243	123006	-16912	1357	-58	1
1	-37	627	-6457	45212	-228165	858322	-2455549	5402809	-9187016	12069406	-12190247	9375898	-5412124	2297027	-696834	145282	-19678	1589	-66	1
1	-37	627	-6457	45213	-228191	858625	-2457641	5412339	-9217208	12137715	-12301797	9507190	-5522069	2360827	-721344	151042	-20390	1621	-66	1
1	-37	627	-6457	45213	-228191	858625	-2457641	5412339	-9217208	12137715	-12301797	9507190	-5522069	2360827	-721344	151042	-20390	1621	-66	1
1	-37	627	-6457	45213	-228191	858625	-2457641	5412339	-9217208	12137715	-12301797	9507190	-5522069	2360827	-721344	151042	-20390	1621	-66	1
1	-37	627	-6456	45184	-227810	855625	-2441857	5353767	-9059466	11824414	-11842438	9016474	-5154098	2152040	-643776	132678	-17884	1455	-62	1
1	-37	627	-6456	45184	-227810	855625	-2441856	5353748	-9059310	11824430	-11842546	9012408	-5135825	2147726	-641477	131945	-17762	1447	-62	1
1	-37	627	-6456	45184	-227810	855625	-2441856	5353748	-9059309	11824413	-11842425	9011936	-5134715	2146096	-639982	131115	-17503	1408	-60	1
1	-37	627	-6456	45184	-227810	855625	-2441856	5353748	-9059309	11824414	-11842438	9012005	-5134910	2146416	-640275	131289	-17543	1414	-60	1
1	-37	627	-6456	45184	-227810	855626	-2441878	5353960	-9060487	11828596	-11852371	9028084	-5152669	2159643	-646788	133322	-17940	1455	-62	1
1	-37	627	-6456	45185	-227838	855974	-2444417	5366077	-9100380	11921723	-12008396	9215903	-5313756	2256284	-686103	143642	-19552	1585	-66	1
1	-37	627	-6456	45185	-227838	855974	-2444438	5366272	-9101438	11925456	-12017410	9231150	-5331914	2271354	-694588	146725	-20212	1654	-66	1
1	-37	627	-6456	45185	-227837	855951	-2444183	5364688	-9095036	11907718	-11982813	9183119	-5284481	2238431	-678944	141878	-19318	1573	-66	1
1	-37	627	-6456	45185	-227837	855951	-2444181	5364649	-9094706	11906123	-11977932	9173209	-5270879	2225814	-671175	138825	-18606	1487	-62	1
1	-37	627	-6456	45185	-227837	855952	-2444206	5364922	-9096425	11913062	-11996815	9208666	-5317092	2267276	-696202	148578	-20895	1772	-76	1
1	-37	627	-6456	45185	-227835	855904	-2443694	5361716	-9083310	11876144	-11923421	9104529	-5295224	2245421	-681799	142556	-19400	1577	-66	1
1	-37	627	-6456	45185	-227835	855904	-2443692	5361676	-9082964	11874441	-11918141	9093749	-5196994	2178655	-651191	133485	-17773	1420	-60	1
1	-37	628	-6486	45592	-231142	873873	-2512751	5555367	-9486362	12503537	-12654056	9737224	-5614086	2376431	-718438	149359	-20224	1638	-68	1
1	-37	628	-6486	45592	-231142	873873	-2512750	5555346	-9486170	12502532	-12650724	9729912	-5603303	2365794	-711572	146576	-19562	1556	-64	1
1	-37	628	-6486	45592	-231142	873873	-2512750	5555346	-9486170	12502532	-12650724	9729912	-5603303	2365794	-711572	146576	-19562	1556	-64	1
1	-37	628	-6486	45593	-231169	874201	-2515119	5566671	-9523924	12592623	-12806213	9923750	-5775760	2472712	-755964	158148	-21258	1669	-66	1
1	-37	628	-6486	45593	-231168	874176	-2514842	5564877	-9516380	12570979	-12762865	9862764	-5715893	2432511	-738161	153272	-20528	1625	-66	1
1	-37	628	-6486	45593	-231167	874152	-2514587	5563295	-9510015	12553535	-12729583	9818381	-5674984	2407163	-728125	150950	-20263	1615	-66	1
1	-37	628	-6485	45564	-230788	871199	-2499292	5507696	-9364014	12272639	-12332084	9406401	-5365891	2242486	-667539	136173	-18026	1430	-60	1

Table 3: Totally positive polynomials with degree $d = 20$ and trace $2d - 3 = 37$, a single root > 4 , and all other roots in $(0, 4)$.

x^{34}	x^{33}	x^{32}	x^{31}	x^{30}	x^{29}	x^{28}	x^{27}	x^{26}	x^{25}	x^{24}	x^{23}	x^{22}	x^{21}	x^{20}	x^{19}	x^{18}	x^{17}	Salem numbers
1	3	1	-15	-52	-107	-167	-220	-266	-317	-383	-462	-540	-603	-646	-673	-688	-693	2.9718375039
1	3	2	-10	-40	-89	-149	-208	-257	-293	-315	-322	-311	-281	-237	-191	-156	-143	2.7616448085

Table 4: Salem numbers of degree 34 and trace -3 .

x^{36}	x^{35}	x^{34}	x^{33}	x^{32}	x^{31}	x^{30}	x^{29}	x^{28}	x^{27}	x^{26}	x^{25}	x^{24}	x^{23}	x^{22}	x^{21}	x^{20}	x^{19}	x^{18}	Salem numbers
1	3	1	-15	-52	-106	-161	-201	-224	-245	-282	-340	-404	-452	-473	-474	-469	-467	-467	2.9585975515
1	3	1	-15	-53	-111	-173	-217	-228	-206	-161	-103	-34	47	136	221	289	331	345	2.9778824718
1	3	2	-10	-39	-82	-124	-146	-136	-96	-40	12	45	55	50	43	43	48	51	2.6852347917
1	3	2	-10	-39	-82	-123	-140	-118	-60	14	77	111	113	92	61	32	12	5	2.6779781031

Table 5: Salem numbers of degree 36 and trace -3 .

x^{38}	x^{37}	x^{36}	x^{35}	x^{34}	x^{33}	x^{32}	x^{31}	x^{30}	x^{29}	x^{28}	x^{27}	x^{26}	x^{25}	x^{24}	x^{23}	x^{22}	x^{21}	x^{20}	x^{19}	Salem numbers
1	3	0	-20	-63	-118	-161	-176	-173	-183	-228	-298	-357	-376	-360	-345	-364	-419	-479	-505	3.1314069583
1	3	0	-20	-65	-129	-190	-222	-213	-174	-128	-94	-77	-70	-61	-42	-11	26	57	69	3.1753281563
1	3	0	-20	-65	-130	-196	-240	-248	-221	-168	-100	-27	43	107	166	223	276	316	331	3.1837638764
1	3	1	-16	-59	-132	-227	-328	-421	-502	-575	-648	-726	-807	-882	-940	-972	-979	-973	-969	3.1073115409
1	3	1	-16	-60	-140	-260	-421	-623	-864	-1135	-1422	-1710	-1986	-2238	-2455	-2626	-2745	-2813	-2835	3.1627585984
1	3	1	-15	-53	-113	-185	-254	-308	-343	-360	-364	-363	-367	-384	-417	-460	-502	-532	-543	3.0026153431
1	3	1	-15	-52	-108	-173	-238	-302	-369	-436	-491	-522	-528	-518	-502	-482	-458	-436	-427	2.9830177489
1	3	1	-14	-47	-93	-136	-157	-146	-109	-62	-22	2	12	18	29	49	74	95	103	2.8626146783
1	3	1	-14	-47	-94	-142	-175	-182	-163	-126	-83	-45	-18	0	15	33	54	72	79	2.8769732691
1	3	1	-14	-47	-95	-149	-199	-237	-260	-270	-274	-280	-294	-318	-350	-382	-406	-419	-423	2.8963426610
1	3	1	-14	-46	-87	-118	-122	-98	-62	-31	-10	9	35	67	93	103	97	85	79	2.8205243729
1	3	2	-10	-40	-88	-143	-189	-214	-217	-208	-203	-214	-244	-285	-323	-346	-352	-348	-345	2.7432751935
1	3	3	-5	-26	-60	-100	-137	-164	-177	-173	-150	-107	-47	25	104	185	259	313	333	2.4399913855
1	3	3	-4	-21	-48	-81	-114	-142	-163	-176	-183	-188	-195	-206	-221	-236	-247	-253	-255	2.3319842011

Table 6: Salem numbers of degree 38 and trace -3 .

x^{40}	x^{39}	x^{38}	x^{37}	x^{36}	x^{35}	x^{34}	x^{33}	x^{32}	x^{31}	x^{30}	x^{29}	x^{28}	x^{27}	x^{26}	x^{25}	x^{24}	x^{23}	x^{22}	x^{21}	x^{20}	Salem numbers
1	3	-1	-30	-112	-274	-527	-857	-1227	-1587	-1883	-2068	-2110	-1998	-1744	-1382	-960	-535	-168	82	171	3.8007708538
1	3	-1	-29	-105	-247	-452	-691	-919	-1095	-1198	-1234	-1229	-1217	-1226	-1268	-1335	-1408	-1468	-1505	-1517	3.7206401214
1	3	-1	-29	-104	-243	-449	-715	-1033	-1396	-1787	-2172	-2507	-2759	-2920	-3003	-3025	-3001	-2949	-2897	-2875	3.7193813271
1	3	-1	-29	-104	-241	-435	-664	-906	-1155	-1418	-1699	-1983	-2239	-2440	-2584	-2693	-2794	-2894	-2974	-3005	3.7085516704
1	3	-1	-29	-104	-239	-419	-597	-712	-721	-623	-463	-308	-210	-177	-172	-138	-40	107	242	297	3.6933302502
1	3	-1	-28	-97	-215	-366	-515	-625	-674	-658	-586	-474	-346	-232	-157	-125	-120	-121	-119	-117	3.6222259269
1	3	0	-24	-90	-212	-384	-580	-766	-913	-998	-999	-891	-655	-293	163	658	1127	1510	1760	1847	3.5263915198
1	3	0	-23	-85	-199	-361	-548	-725	-856	-911	-872	-735	-512	-229	80	383	653	868	1008	1057	3.4775939743
1	3	0	-23	-84	-192	-335	-482	-599	-667	-685	-661	-598	-492	-343	-168	2	136	220	260	271	3.4505894081
1	3	0	-23	-84	-191	-328	-455	-525	-509	-411	-267	-125	-21	37	68	104	166	249	323	353	3.4416559110
1	3	0	-23	-84	-191	-328	-455	-524	-503	-392	-225	-53	79	151	173	176	189	221	257	273	3.4415063642
1	3	0	-23	-83	-187	-323	-467	-600	-720	-835	-949	-1054	-1139	-1203	-1257	-1311	-1365	-1410	-1438	-1447	3.4387135979
1	3	0	-23	-83	-186	-317	-446	-546	-607	-632	-624	-576	-477	-326	-141	48	211	330	400	423	3.4313736405
1	3	0	-23	-83	-185	-310	-420	-478	-468	-399	-297	-192	-113	-87	-135	-260	-440	-629	-772	-825	3.4226176184
1	3	0	-22	-78	-173	-293	-408	-486	-509	-477	-402	-297	-173	-42	80	178	243	277	290	293	3.3753589936
1	3	0	-22	-78	-171	-280	-364	-383	-324	-213	-106	-58	-91	-178	-255	-255	-151	23	185	251	3.3596154717
1	3	0	-22	-77	-168	-283	-408	-544	-709	-918	-1164	-1417	-1646	-1840	-2009	-2165	-2308	-2425	-2501	-2527	3.3686279815
1	3	0	-22	-77	-167	-277	-387	-489	-590	-696	-799	-880	-927	-945	-948	-942	-924	-893	-861	-847	3.3601726633
1	3	0	-22	-77	-167	-275	-372	-432	-442	-399	-306	-172	-17	130	242	309	339	349	352	353	3.3538404471
1	3	0	-22	-77	-165	-262	-330	-343	-308	-259	-229	-202	-142	-38	76	155	180	170	161	161	3.3390688183
1	3	1	-16	-58	-125	-203	-273	-324	-358	-382	-400	-410	-410	-403	-396	-393	-394	-396	-397	-397	3.0674618430
1	3	1	-16	-57	-120	-189	-244	-274	-282	-276	-260	-231	-187	-134	-84	-44	-13	13	33	41	3.0404935832
1	3	1	-16	-57	-119	-184	-231	-252	-256	-256	-256	-248	-223	-181	-131	-81	-35	5	34	45	3.0329139322
1	3	1	-16	-57	-119	-184	-231	-251	-250	-236	-209	-162	-93	-11	70	143	208	265	307	323	3.0324565729
1	3	1	-15	-52	-105	-154	-176	-164	-137	-128	-161	-232	-312	-368	-386	-375	-356	-344	-341	-341	2.9406722181
1	3	1	-15	-52	-105	-154	-175	-157	-112	-67	-46	-52	-67	-69	-52	-29	-20	-32	-53	-63	2.9384433127
1	3	1	-15	-52	-105	-154	-175	-157	-111	-62	-34	-36	-61	-94	-122	-137	-138	-129	-118	-113	2.9383434641
1	3	1	-15	-52	-105	-154	-175	-157	-111	-61	-27	-11	-1	13	26	23	-7	-57	-104	-123	2.9382595870
1	3	1	-15	-52	-105	-154	-175	-157	-111	-61	-27	-11	0	20	51	84	107	116	116	115	2.9382561494
1	3	1	-15	-52	-105	-153	-169	-139	-75	-8	30	26	-7	-39	-42	-2	75	166	239	267	2.9339655654
1	3	1	-15	-51	-101	-149	-184	-212	-251	-311	-384	-453	-512	-571	-640	-709	-754	-761	-744	-733	2.9358570973
1	3	1	-15	-51	-101	-148	-177	-185	-178	-159	-126	-78	-20	41	100	153	192	211	214	213	2.9281342716
1	3	1	-15	-51	-100	-142	-159	-151	-136	-131	-138	-147	-153	-164	-191	-230	-263	-275	-270	-265	2.9161201862
1	3	1	-15	-51	-100	-142	-157	-138	-92	-30	37	97	136	144	120	72	10	-54	-104	-123	2.9120722906
1	3	1	-15	-51	-100	-141	-154	-137	-108	-86	-73	-52	-7	58	120	158	168	162	154	151	2.9121557127
1	3	1	-15	-51	-100	-141	-152	-127	-82	-41	-18	-8	4	28	61	94	120	138	149	153	2.9098555842
1	3	1	-15	-51	-98	-129	-116	-55	25	82	92	67	40	34	44	48	29	-9	-46	-61	2.8817787817
1	3	1	-15	-51	-98	-129	-114	-43	61	151	178	117	-18	-174	-278	-269	-135	73	260	335	2.8785434334
1	3	2	-9	-34	-71	-113	-154	-192	-227	-256	-274	-276	-260	-227	-181	-128	-77	-36	-10	-1	2.6316862080
1	3	2	-9	-34	-71	-113	-153	-187	-214	-233	-244	-248	-247	-243	-239	-238	-243	-253	-263	-267	2.6296111462
1	3	2	-9	-34	-71	-112	-147	-167	-165	-137	-89	-38	-4	2	-19	-58	-106	-154	-191	-205	2.6188371017
1	3	2	-9	-33	-66	-99	-123	-132	-124	-100	-66	-32	-8	2	0	-9	-22	-36	-47	-51	2.5752377703
1	3	2	-9	-33	-65	-94	-111	-114	-106	-89	-65	-38	-14	4	18	33	50	66	77	81	2.5602938026
1	3	2	-9	-33	-64	-88	-93	-78	-53	-29	-11	3	15	22	18	3	-16	-30	-36	-37	2.5347241894
1	3	2	-8	-28	-52	-71	-81	-85	-86	-80	-62	-33	-1	26	46	63	78	89	95	97	2.4352709937

Table 7: Salem numbers of degree 40 and trace -3 .