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# Distributed Event-Triggered Leader-Follower Consensus of Nonlinear Input-Affine Multi-Agent Systems

M. Marchand<sup>1</sup>, V. Andrieu<sup>2</sup>, S. Bertrand<sup>1</sup> and H. Piet-Lahanier<sup>1</sup>

**Abstract—Achieving cooperative mission with a Multi-Agents System (MAS) in a distributed way require agents to communicate. Hence communication laws need to be defined. In this article, we proposed two new event-triggered communication laws to make a MAS converges to a Leader-Follower consensus in a distributed way.**

## I. INTRODUCTION

A Multi-Agent System (MAS) is a system composed of multiple autonomous agents that interact with each other to achieve a common goal. These agents are capable of acting independently and can have different sub-objectives, knowledge, and decision-making capabilities. The interactions between the agents of a MAS can result in complex and dynamic behaviours that are difficult to achieve with a single agent. They are used in various fields such as robotics [1], distributed systems [2]–[5] and artificial intelligence [6]–[8].

Centralised control and distributed control are two different approaches to control the dynamics of the agents of a MAS. Centralised control refers to a system where a single agent, often referred to a central coordinator, is responsible for coordinating the other agents. In this strategy of control, agents may have limited autonomy and decisions are made based on a global view of the system. On the other hand, distributed control refers to a system where decision-making is distributed among the agents in the MAS. In this strategy of control, each agent has its own objectives, knowledge and decision making capabilities. This approach can lead to more flexibility and robustness with respect to the loss of an agent, as the control computation does not rely on a single agent. Recently, there exists a growing interest in developing distributed law to make MAS reach a consensus. Consensus mission consists of synchronising the agents in MAS to a unique state. There exists two different strategies to achieve a consensus: a leader-followers consensus [9]–[12] in which an agent of the graph is designed as a leader and the other agents synchronise their states with the state of the leader, and a leaderless consensus [2], [13]. In these work, control laws are based on either continuous communication or periodic exchanges of the state values by agents within the same neighbourhood.

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Continuous communication refers to a constant exchange of messages between the agents without any specific trigger or event. This method, even if it is easily implemented, tends to saturate the transmission channel, and tends to be energy consuming. On the other hand, event-triggered communication only initiates a communication when a specific Communication Triggering Condition (CTC) holds, usually defined from a function of the current states and some information error, is verified. This results in a more efficient use of communication resources as messages are only sent when necessary.

In the linear case, firstly, control laws with event-triggered communication have been proposed to solve the consensus problem when agents are modelled by single and double integrator, see e.g. [3], [14]. Afterwards, the general linear dynamics case has been addressed, see e.g. [4]. Several extensions to improve the CTC have been suggested, see e.g. [15] resulting in reduced communications, and in handled perturbations.

Regarding nonlinear systems, firstly various distributed control laws with event-triggered communication has been proposed with dynamic class of Euler-Lagrange nonlinear dynamics, see e.g. [16]–[18]. Recently, event-triggered schemes dealing with packet losses have also been proposed for this type of dynamic, see e.g. [19]. Secondly, solutions for reaching the consensus with event-triggered communication in a MAS with nonlinear fully actuated input-affine dynamics have been suggested [20], [21]. In [22], the finite time consensus problem is considered with distributed event-triggered control law for this class of dynamics.

An interesting approach has also been proposed in [23] for minimum phase dynamics MAS. The consensus problem is split into a linear consensus problem and a nonlinear tracking problem. Indeed each agent in the MAS virtually computes a virtual consensus of a linear MAS with the same graph topology as the real one. Once the virtual consensus solved, the computed trajectories can be used a reference to track.

In this paper; we consider the leader-follower consensus problem for general nonlinear input-affine MAS. This article is an extension of [24]. Contrary to the article [24], the distributed control input is not applied through a constant matrix  $B$  but through a state dependant matrix  $g(x_i)$ , where  $x_i$  is the state of the agent  $i$ . The considered dynamics are neither fully actuated nor described by a minimum phase system.

The paper is organised as follows. In section II, the considered problem is exhibited. In section III, main results are provided. And finally, in section V illustration of the

proposed distributed event-triggered laws is shown

**Notations:** The set of positive real number is denoted by  $\mathbb{R}_+$ . Let  $A \in \mathbb{R}^{n \times n}$  be a matrix with real eigenvalues. We denote by  $\lambda_i(A)$  the  $i$ -th eigenvalue of the eigenvalues of  $A$  sorted in ascending order. The notation  $I_n$  denotes for the identity matrix of size  $n \times n$ . The Kronecker product is denoted as  $\otimes$ . Given a  $\mathcal{C}^1$  function  $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  and a  $\mathcal{C}^1$  vector field  $\xi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we define the Lie derivative of  $P$  along  $\xi$  as

$$\mathcal{L}_\xi P(x) = \mathfrak{d}_\xi P(x) + P(x) \frac{\partial \xi}{\partial x}(x) + \frac{\partial \xi}{\partial x}(x)^\top P(x) \quad (1)$$

where

$$\mathfrak{d}_\xi P(x) = \lim_{h \rightarrow 0} \frac{P(x + h\xi(x)) - P(x)}{h}, \quad (2)$$

for all  $x \in \mathbb{R}^n$ .

## II. PROBLEM STATEMENT

### A. System Description

Consider a MAS composed by a leader agent with state  $x_1 \in \mathbb{R}^n$ , and  $N - 1$  followers, with states  $x_2, \dots, x_N$  also in  $\mathbb{R}^n$ . Each agents' dynamics are identical and described by the following dynamics

$$\dot{x}_i(t) = f(x_i(t)) + g(x_i(t)) u_i(t), \quad \forall i \in \{1, \dots, N\}, \quad (3)$$

where  $x_i \in \mathbb{R}^n$  is the state of the agent  $i$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  a  $\mathcal{C}^1$  vector field,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $u_i \in \mathbb{R}^m$  is the control input applied to the agent for the MAS to perform its cooperative mission. By convention, the leader agent is not controlled, i.e.  $u_1(t) = 0$ . We define  $\mathbb{T}$  to be the time domain of the existence of the solution  $\mathbf{x} = (x_1^\top, \dots, x_N^\top)^\top$ .

The communication graph between the agents is represented by a triplet  $\{\mathcal{V}, \mathcal{E}, A\}$  in which  $\mathcal{V} = \{1, \dots, N\}$  denotes for the set of vertices or agents,  $\mathcal{E} \subset \{1, \dots, N\}^2$  is the set of edges which represents the communication possibility between agents and the matrix  $A = (a_{ij})_{(i,j) \in \{1, \dots, N\}^2} \in \mathbb{R}^{N \times N}$  denotes for the weight adjacency matrix. The entry  $a_{ij} > 0$  if  $(i, j) \in \mathcal{E}$ , meaning the agent  $j$  can transmit information to the agent  $i$ , i.e. the agent  $j$  is a neighbour of the agent  $i$ , while  $a_{ij} = 0$  if  $(i, j) \notin \mathcal{E}$ . From this framework, the Laplacian matrix associated to the communication graph,  $L = (l_{ij})_{(i,j) \in \{1, \dots, N\}^2} \in \mathbb{R}^{n \times N}$  can be defined as

$$l_{ij} = -a_{ij}, \quad \text{for } i \neq j, \quad l_{ij} = \sum_{k=1}^N a_{ik}, \quad \text{for } i = j. \quad (4)$$

The considered cooperative mission is the consensus, i.e. to have consensus manifold defined as

$$\mathcal{D} = \{\mathbf{x} = (x_1^\top, \dots, x_N^\top)^\top \in \mathbb{R}^{nN} \mid x_1 = \dots = x_N\} \quad (5)$$

asymptotically stable along the solution of the complete dynamical system (3). We denote for all  $\mathbf{x} \in \mathbb{R}^{nN}$  the Euclidean distance to the set  $\mathcal{D}$  by  $|\mathbf{x}|_{\mathcal{D}}$ . The main objective of this article is to define event-triggered distributed control laws to make the MAS described with the equation (3) converges asymptotically a consensus.

### B. Reducing communications

To achieve the consensus mission, the agents must either know or estimate the state of their neighbour. Due to the costs of communication, such as energy consumption or the risk of saturating the transmission channel, the number of state exchanges within the graph must be limited. A metric for evaluating the need to communicate, called CTC, must be defined so that agents know when to send their current state. Therefore, for all  $j \in \mathcal{V}$  there exists a discrete sequence  $\{t_{j,p}\}_{p \in \mathbb{N}}$ , where  $t_{j,p} \in \mathbb{R}_+$  is the  $p$ -th instant of communication from agent  $j$  to its neighbours.

### C. State estimation

As the neighboured agents do not send their states continuously, each agent has to estimate the state of its neighbours. The estimate of the agent  $j$  performed by the agent  $i$  is denoted as  $\hat{x}_j^i$ , for all  $(i, j) \in \mathcal{E}$ .

$$\begin{cases} \dot{\hat{x}}_j^i(t) = f(\hat{x}_j^i(t)), & \forall t \in [t_{j,p}, t_{j,(p+1)}) \\ \hat{x}_j^i(t_{j,p}) = x_j(t_{j,p}), \end{cases} \quad (6a)$$

$$\hat{x}_j^i(t_{j,p}) = x_j(t_{j,p}), \quad (6b)$$

where  $t_{j,p} \in \mathbb{R}_+$  is the  $p$ -th communication instant of the agent  $j$ .

*Remark 1:* The input of an agent is usually unknown by its neighbours, therefore to limit the complexity of the estimators and the amount of information exchanged, it was decided not to use the input  $u_i$  in the estimators (6). This corresponds to the open-loop estimation approach described for example in [4], [25]. Some articles, see e.g. [15], suggest more complicated structures of estimators which contains the input and tends to reduce the number of communication but may require exchanging more data in the messages.

In addition, in order to know whether its neighbours have a good representation of its state or not, each agent performs an estimation of its own state, using the same dynamics as the estimations performed by its neighbours, for all  $i \in \mathcal{V}$

$$\begin{cases} \dot{\hat{x}}_i^i(t) = f(\hat{x}_i^i(t)), & \forall t \in [t_{i,p}, t_{i,(p+1)}) \\ \hat{x}_i^i(t_{i,p}) = x_i^i(t_{i,p}). \end{cases} \quad (7a)$$

$$\hat{x}_i^i(t_{i,p}) = x_i^i(t_{i,p}). \quad (7b)$$

Note that if there are no communication delays or packet losses, then  $\hat{x}_i^j(t) = \hat{x}_i^i(t)$ , for all  $(i, j) \in \mathcal{E}$ . In this article, we do not consider packet losses nor communication delay. More work could be done on these phenomena.

## III. MAIN RESULTS

### A. Assumption

Before stating the theorem, we introduce two assumptions: one must assume connectivity of the communication graph and a stabilisability property (see for example [2]).

*Assumption 1:* The communication graph is assumed to be connected, i.e. from any agent there exists a sequence of edges that connects it to any other agent, and undirected, i.e. if  $(i, j) \in \mathcal{E}$  then  $(j, i) \in \mathcal{E}$ .

As a consequence, the Laplacian matrix can be written as

$$L = \begin{pmatrix} l_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \quad (8)$$

where  $l_{11} \in \mathbb{R}_+$ ,  $L_{21} = L_{12}^\top \in \mathbb{R}^{(N-1)}$  and  $L_{22} = L_{22}^\top \in \mathbb{R}^{(N-1)^2}$ . We recall the separation theorem, also known as the Cauchy's interlacing theorem for a Hermitian matrix, see e.g. [26],

*Theorem 1 (Cauchy):* Let  $E_{22} \in \mathbb{C}^{(N-1) \times (N-1)}$  be Hermitian, let  $E_{21} \in \mathbb{C}^{(N-1)}$  and  $e_{11} \in \mathbb{R}$  and let  $E \in \mathbb{C}^{N \times N}$  such that

$$E = \begin{pmatrix} e_{11} & E_{21}^* \\ E_{21} & E_{22} \end{pmatrix}, \quad (9)$$

where  $*$  is used for noting the conjugate transpose operator (only here). Then

$$\begin{aligned} \lambda_1(E) &\leq \lambda_1(E_{22}) \leq \lambda_2(E) \leq \dots \leq \lambda_{N-1}(E) \\ &\leq \lambda_{N-1}(E_{22}) \leq \lambda_N(E). \end{aligned} \quad (10)$$

Using this theorem it can be noticed that  $L_{22}$  is a positive matrix, since  $\lambda_1(L) = 0$ . Furthermore, since  $L_{22}$  is full rank,  $L_{22}$  is a positive definite matrix satisfying

$$\varepsilon^{-1} I_{N-1} \leq L_{22}, \quad (11)$$

with  $\varepsilon \geq \lambda_2(L)^{-1}$  where  $I_{N-1}$  stands for the identity matrix of size  $N-1$ . Moreover, since the graph is connected, the Laplacian matrix  $L$  has  $N-1$  strictly positive eigenvalues and  $\lambda_1(L) = 0$ . Thus, for any  $\mu \leq \lambda_N(L)^{-1}$  one has

$$\mu L \leq I_N, \quad (12)$$

where  $I_N$  stands for the identity matrix of size  $N$ .

*Assumption 2:* There exists a  $\mathcal{C}^1$  function  $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ , such that for all  $x \in \mathbb{R}^n$ ,  $P(x)$  is a definite, positive, and symmetric matrix such that the following conditions hold

- The Control Matrix Function (CMF) condition holds

$$\begin{aligned} \mathcal{L}_f P(x) - \rho P(x) g(x) g(x)^\top P(x) &\preceq q P(x), \\ \underline{p} I_n \leq P(x) \leq \bar{p} I_n \end{aligned} \quad (13)$$

for any  $x \in \mathbb{R}^n$ , and  $\rho, q, \underline{p}, \bar{p} \in \mathbb{R}_+^*$ , where

$$\mathcal{L}_f P(x) = \dot{P}(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P(x). \quad (14)$$

- The function  $g$  is a killing vector field for  $P$ , i.e.

$$\mathcal{L}_g P(x) = 0, \quad \forall x \in \mathbb{R}^n. \quad (15)$$

- There exists  $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\Psi(x) = \frac{\partial \alpha}{\partial x}(x) = g(x)^\top P(x), \quad \forall x \in \mathbb{R}^n, \quad (16)$$

and  $\Psi(x)$  is Lipschitz, i.e. there exists  $k_\Psi \in \mathbb{R}_+$  such that

$$|\Psi(x) - \Psi(y)| \leq k_\Psi |x - y|, \quad \forall x, y \in \mathbb{R}^n, \quad (17)$$

and  $\Psi(x)$  is bounded, i.e. there exists  $\bar{m} \in \mathbb{R}_+$  such that

$$|\Psi(x)| \leq \sqrt{\bar{m}}, \quad \forall x \in \mathbb{R}^n. \quad (18)$$

Finding a matrix function  $P$  that satisfies all the conditions of the assumption 2 is not an easy task. Indeed, it depends on the coordinates in which the dynamics of the system is written. Using deep learning to learn a local  $P$  satisfying the conditions of the assumption 2 can be an interesting approach. It will be particularly powerful, for example, for

systems where trajectories are bounded, typically vehicles moving in a closed area. Also, in [27], the authors propose several methods to write the equation (13) as a linear matrix inequality (LMI), which could be easily checked. It is based on the decomposition of  $f$  as a linear system to which an incremental sector-bounded nonlinearity is added. This type of assumption was also used in [28].

## B. Asymptotic consensus

Consider the following CTC defining the sequence of communication instants  $t_{i,p}$  for the followers, i.e. for all  $i \in \{2, \dots, N\}$

$$\begin{aligned} t_{i,p+1} = \inf \left\{ s > t_{i,p} \mid 2T_i |e_i|^3 |w_i| + \varepsilon T_i |e_i|^2 |w_i|^2 \right. \\ \left. + S_i |e_i|^2 + Q_i |e_i|^4 - \sigma_i R_i |w_i|^2 > 0 \right\}, \end{aligned} \quad (19)$$

with  $e_i = x_i - \hat{x}_i^i$  the estimation error between  $x_i$  and  $\hat{x}_i^i$ , and  $w_i = \sum_{j=1}^N l_{ij} \hat{x}_j^i$ ,  $R_i, S_i, Q_i$  and  $T_i$  are positive number and their expression are detailed in (114)-(117) in section VII.

For the leader agent, the sequence of communication instants  $t_{1,p}$  are defined according to the following CTC

$$\begin{aligned} t_{1,p+1} = \inf \left\{ s > t_{1,p} \mid S_1 |e_1(s)|^2 + Q_1 |e_1|^4 \right. \\ \left. - \sigma_1 R_1 |w_1(s)|^2 > 0 \right\}, \end{aligned} \quad (20)$$

with constant  $R_1, S_1$  and  $Q_1$  defined in (118)-(120) in the section VII.

*Theorem 2:* If the assumption 1 and 2 hold. Then with the distributed control law  $u_i(t)$  for  $i = 2, \dots, N$  defined as

$$u_i = -\kappa \sum_{j=1}^N l_{ij} \alpha(\hat{x}_j^i(t)), \quad (21)$$

with  $\alpha$  defined in (16) and  $\kappa > \rho\varepsilon$ , where  $\rho$  and  $\varepsilon$  are respectively defined in (13) and (11), and the communication instant  $t_{i,p}$  defined according to the CTC (19), for  $i = 2, \dots, N$  and (20) for  $i = 1$ , there exists positive constants  $c_1, c_2$  such that for any  $\mathbf{x}(0) \in \mathbb{R}^{Nn}$  and for all  $t \in \mathbb{T}$

$$|\mathbf{x}(t)|_{\mathcal{D}} \leq c_1 \exp(-c_2 t) |\mathbf{x}(0)|_{\mathcal{D}}, \quad (22)$$

where  $\mathbf{x}(t)$  denotes the solution of the system (3) and  $\mathbb{T}$  is the time domain of the existence of the solution.

## C. Proof

*Proof:* The first steps of this proof is inspired by the proof of the theorem 1 in [12] in which authors are considering the consensus problem for a MAS described by (3) with continuous communications. For clarity reasons, the time dependency of variables will not be written. Define the error coordinates  $\mathbf{r} = [r_1^\top, \dots, r_N^\top]^\top$  with  $r_i = x_i - x_1$ , and the estimation error  $\mathbf{e} = [e_1^\top, \dots, e_N^\top]^\top$ . By convention,  $r_1 = 0$ , for all  $t$ . Since there are no communication delay or packet losses, one can notice that

$$\hat{x}_j^i = \hat{x}_j^j = x_1 + r_j - e_j. \quad (23)$$

The dynamics of  $r_i$  are given by

$$\dot{r}_i = f(r_i + x_1) - f(x_1) - \kappa g(r_i + x_1) \sum_{j=1}^N l_{ij} \alpha(x_1 + r_j - e_j). \quad (24)$$

Using the fact that

$$\sum_{j=1}^N l_{ij} g(x_1 + r_i) \alpha(x_1) = g(x_1 + r_i) \alpha(x_1) \sum_{j=1}^N l_{ij} = 0, \quad (25)$$

we can write

$$\begin{aligned} \dot{r}_i &= f(r_i + x_1) - f(x_1) \\ &\quad - \kappa g(r_i + x_1) \sum_{j=1}^N l_{ij} (\alpha(\hat{x}_1 + r_j - e_j) - \alpha(x_1)). \end{aligned} \quad (26)$$

Consider the  $\mathcal{C}^1$  functions  $\Phi_i(s, t) : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  defined as

$$\Phi_i(1, t_0) = e_i, \quad \Phi_i(0, t_0) = 0, \quad \Phi_i(s, t_0) = e_i s, \quad (27)$$

for all  $i \in \{1, \dots, N\}$  and  $\Gamma_i(s, t) : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  defined as

$$\Gamma_i(1, t_0) = r_i, \quad \Gamma_i(0, t_0) = 0, \quad \Gamma_i(s, t_0) = r_i s, \quad (28)$$

for all  $i \in \{1, \dots, N\}$ .  $\Gamma_i$  defined as solution of the following ordinary differential equation

$$\begin{aligned} \frac{\partial \Gamma_i}{\partial t} &= f(\Gamma_i + x_1) - f(x_1) \\ &\quad - \kappa g(\Gamma_i + x_1) \sum_{j=1}^N l_{ij} (\alpha(x_1 + \Gamma_j - \Phi_j) - \alpha(x_1)). \end{aligned} \quad (29)$$

We introduce  $\Gamma : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}^{Nn}$  defined as  $\Gamma = [\Gamma_1^\top, \dots, \Gamma_N^\top]^\top$ . In equation (29) and in the following we have skipped the dependency of the variables to  $t$  and  $s$  for clarity reasons. By convention  $\Gamma_1 = 0$ . Consider the following function as a Lyapunov candidate function

$$V_i(t) = \int_0^1 \frac{\partial \Gamma_i^\top}{\partial s} P(x_1 + \Gamma_i) \frac{\partial \Gamma_i}{\partial s} ds. \quad (30)$$

One can notice that, for all vector  $v \in \mathbb{R}^n$

$$\begin{aligned} \frac{\partial}{\partial t} v^\top P(x_1 + \Gamma_i) v &= v^\top \mathfrak{d}_f P(x_1 + \Gamma_i) v \\ &\quad - \kappa (\alpha(x_1 + \Gamma_i) - \alpha(x_1)) \sum_{j=1}^N l_{ij} v^\top \mathfrak{d}_g P(x_1 + \Gamma_i) v. \end{aligned} \quad (31)$$

Using the killing vector field (15), the relation (16) and (31), the time-derivative of  $V_i$  can be computed as

$$\begin{aligned} \dot{V}_i &= \int_0^1 \frac{\partial \Gamma_i^\top}{\partial s} \mathcal{L}_f P(x_1 + \Gamma_i) \frac{\partial \Gamma_i}{\partial s} - 2\kappa \frac{\partial \Gamma_i^\top}{\partial s} \Psi(x_1 + \Gamma_i)^\top \\ &\quad \times \sum_{j=1}^N l_{ij} \Psi(x_1 + \Gamma_j - \Phi_j) \left( \frac{\partial \Gamma_j}{\partial s} - \frac{\partial \Phi_j}{\partial s} \right) ds. \end{aligned} \quad (32)$$

Note that, according to the definition of  $\Phi_i$  and  $\Gamma_i$  in (27) and (28), we have

$$\frac{\partial \Gamma_i}{\partial s} = r_i, \quad \frac{\partial \Phi_i}{\partial s} = e_i, \quad \forall (s, t) \in [0, 1] \times \mathbb{R}_+. \quad (33)$$

Define  $V = \sum_{i=1}^N V_i$  and a function  $\Delta \Psi_i : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times n}$  as  $\Delta \Psi_i(-\Phi_j) = \Psi(x_1 + \Gamma_i - \Phi_j) - \Psi(x_1 + \Gamma_i)$ , for all  $i, j$ . One has

$$\begin{aligned} \dot{V} &= \int_0^1 \sum_{i=1}^N r_i^\top \mathcal{L}_f P(x_1 + \Gamma_i) r_i - \sum_{i=1}^N 2\kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \\ &\quad \times \sum_{j=1}^N l_{ij} (\Delta \Psi(-\Phi_j) + \Psi(x_1 + \Gamma_j)) (r_j - e_j) ds. \end{aligned} \quad (34)$$

By introducing the vector  $v_{\Psi_r} \in \mathbb{R}^{(N-1)n}$  such that

$$v_{\Psi_r} = \begin{pmatrix} \Psi(x_1 + \Gamma_2) r_2 \\ \vdots \\ \Psi(x_1 + \Gamma_N) r_N \end{pmatrix}, \quad (35)$$

one can notice that, since  $r_1 = 0$

$$\begin{aligned} \sum_{i=1}^N 2\kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \sum_{j=1}^N l_{ij} \Psi(x_1 + \Gamma_j) r_j \\ = \sum_{i=2}^N 2\kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \sum_{j=2}^N l_{ij} \Psi(x_1 + \Gamma_j) r_j, \\ = 2\kappa v_{\Psi_r}^\top (L_{22} \otimes I_{N-1}) v_{\Psi_r}, \end{aligned} \quad (36)$$

and according to the assumption 1

$$\begin{aligned} -2\kappa v_{\Psi_r}^\top (L_{22} \otimes I_{N-1}) v_{\Psi_r} \\ \leq -2\varepsilon^{-1} \kappa v_{\Psi_r}^\top v_{\Psi_r}. \end{aligned} \quad (37)$$

Using the relation (37) into (34), we obtain

$$\begin{aligned} \dot{V} &\leq \int_0^1 \sum_{i=1}^N r_i^\top \mathcal{L}_f P(x_1 + \Gamma_i) r_i - \varepsilon^{-1} \kappa v_{\Psi_r}^\top v_{\Psi_r} \\ &\quad - \sum_{i=1}^N 2\kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \sum_{j=1}^N l_{ij} \Delta \Psi_j(-\Phi_j) (r_j - e_j) \\ &\quad + \sum_{i=1}^N 2\kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \sum_{j=1}^N l_{ij} \Psi(x_1 + \Gamma_j) e_j ds. \end{aligned} \quad (38)$$

According to the assumption 2, since  $\kappa > \rho\varepsilon$ , we have

$$\begin{aligned} \dot{V} &\leq -qV + \int_0^1 \sum_{i=1}^N -2\kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \\ &\quad \times \sum_{j=1}^N l_{ij} \Delta \Psi_j(-\Phi_j) (r_j + e_j) \\ &\quad + \sum_{i=1}^N 2\kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \sum_{j=1}^N l_{ij} \Psi(x_1 + \Gamma_j) e_j ds. \end{aligned} \quad (39)$$

Note that with continuous communication the proof ends here. Due to event-triggered communication, a term depending on the estimation error remains and needs to be bounded. We define the function  $M : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  such that  $M(\Gamma_i) = \Psi(x_1 + \Gamma_i)^\top \Psi(x_1 + \Gamma_i)$ , and  $\Delta M_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  such that  $\Delta M_i(-\Phi_j) = \Delta \Psi_i(-\Phi_j)^\top \Delta \Psi_i(-\Phi_j)$ . In order to prove the convergence of the MAS to the consensus, we need to prove that the two final terms in (39) can be bounded by the first term. We recall the following relation

$$|x^\top y| \leq \frac{b}{2} x^\top x + \frac{1}{b} y^\top y, \quad \forall x, y \in \mathbb{R}^n, \quad (40)$$

for any constant  $b \in \mathbb{R}_+$ . Thus, we have

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N 2 l_{ij} \kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \Delta \Psi_j(-\Phi_j) (r_j + e_j) \\ & \leq \sum_{i=1}^N \sum_{j=1}^N \frac{|l_{ij}| \kappa}{b_1} (r_j + e_j)^\top \Delta M_j(-\Phi_j) (r_j + e_j) \\ & \quad + \sum_{i=1}^N \sum_{j=1}^N b_1 |l_{ij}| \kappa r_i^\top M(\Gamma_i) r_i, \end{aligned} \quad (41)$$

for any  $b_1 \in \mathbb{R}_+$ . According to the assumption 1, the graph is assumed to be undirected, thus  $l_{ij} = l_{ji}$ , thus by changing the index  $j$  into  $i$

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N 2 l_{ij} \kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \Delta \Psi_j(-\Phi_j) (r_j + e_j) \\ & \leq \sum_{i=1}^N \sum_{j=1}^N \frac{|l_{ij}| \kappa}{b_1} (r_i + e_i)^\top \Delta M_i(-\Phi_i) (r_i + e_i) \\ & \quad + \sum_{i=1}^N \sum_{j=1}^N b_1 |l_{ij}| \kappa r_i^\top M(\Gamma_i) r_i. \end{aligned} \quad (42)$$

Due to the properties of the Laplacian matrix, we have  $\sum_{j=1}^N |l_{ij}| = 2 l_{ii}$ , consequently

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N 2 l_{ij} \kappa r_i^\top \Psi(x_1 + \Gamma_i)^\top \Delta \Psi_j(-\Phi_j) (r_j + e_j) \\ & \leq \sum_{i=1}^N 2 \frac{l_{ii} \kappa}{b_1} (r_i + e_i)^\top \Delta M_i(-\Phi_i) (r_i + e_i) \\ & \quad + \sum_{i=1}^N 2 b_1 l_{ii} \kappa r_i^\top M(\Gamma_i) r_i. \end{aligned} \quad (43)$$

Similarly, we obtain

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1}^N 2 l_{ij} \kappa r_i^\top \Psi(Z + \Gamma_i)^\top \Psi(Z + \Gamma_j) e_j \\ & \leq \sum_{i=1}^N 2 b_1 l_{ii} \kappa r_i^\top M(\Gamma_i) r_i + 2 \frac{l_{ii} \kappa}{b_1} e_i^\top M(\Gamma_i) e_i. \end{aligned} \quad (44)$$

Using the relation (43) and (44) into (39), we have

$$\begin{aligned} \dot{V} & \leq -q V + \int_0^1 \sum_{i=1}^N \left[ 2 b_1 l_{ii} \kappa r_i^\top M(\Gamma_i) r_i \right. \\ & \quad + 2 b_1 l_{ii} \kappa r_i^\top M(\Gamma_i) r_i + 2 \frac{l_{ii}}{b_1} \kappa e_i^\top M(\Gamma_i) e_i \\ & \quad \left. + 2 \frac{l_{ii} \kappa}{b_1} (r_i + e_i)^\top \Delta M_i(-\Phi_i) (r_i + e_i) \right] ds. \end{aligned} \quad (45)$$

Introducing  $\eta \in (0, 1)$ , one has

$$\begin{aligned} \dot{V} & \leq -q(1 - \eta) V + \int_0^1 \sum_{i=1}^N \left[ -\eta q r_i^\top P(x_1 + \Gamma_i) r_i \right. \\ & \quad + 4 b_1 l_{ii} \kappa r_i^\top M(\Gamma_i) r_i + 2 \frac{l_{ii} \kappa}{b_1} e_i^\top M(\Gamma_i) e_i \\ & \quad + 2 \frac{l_{ii} \kappa}{b_1} (e_i^\top \Delta M_i(-\Phi_i) e_i + r_i^\top \Delta M_i(-\Phi_i) r_i) \\ & \quad \left. + 4 \frac{l_{ii} \kappa}{b_1} r_i^\top \Delta M_i(-\Phi_i) e_i \right] ds. \end{aligned} \quad (46)$$

We can define  $\beta = \max\{l_{11}, \dots, l_{NN}\}$ , thus the following upper bound can be made

$$\begin{aligned} \dot{V} & \leq -q(1 - \eta) V + \int_0^1 \sum_{i=1}^N \left[ -\eta q r_i^\top P(x_1 + \Gamma_i) r_i \right. \\ & \quad + 4 b_1 \beta \kappa r_i^\top M(\Gamma_i) r_i + 2 \frac{\beta \kappa}{b_1} e_i^\top M(\Gamma_i) e_i \\ & \quad + 2 \frac{\beta \kappa}{b_1} (e_i^\top \Delta M_i(-\Phi_i) e_i + r_i^\top \Delta M_i(-\Phi_i) r_i) \\ & \quad \left. + 4 \frac{\beta \kappa}{b_1} r_i^\top \Delta M_i(-\Phi_i) e_i \right] ds. \end{aligned} \quad (47)$$

From here, one can notice that the quantity  $r_i$  is not known by the agent  $i$  since it does not know the state of the leader agent. Therefore, in order to define a CTC, we replace the  $r_i$  in (47) by the quantity  $w_i = \sum_{j=1}^N l_{ij} x_j^i$  which is known by the agent  $i$ . One can notice that since  $r_1 = 0$  and  $\sum_{j=1}^N l_{ij} = 0$

$$\begin{aligned} \sum_{j=2}^N l_{ij} r_j & = \sum_{j=1}^N l_{ij} (x_j - x_1) \\ & = \sum_{j=1}^N l_{ij} x_j, \\ & = \sum_{j=1}^N l_{ij} \hat{x}_j^i + \sum_{j=1}^N l_{ij} e_j, \\ & = w_i + \xi_i, \end{aligned} \quad (48)$$

where  $\xi_i = \sum_{j=1}^N l_{ij} e_j$ . Since there are four term which depends on  $r_i$  in the expression (47), there will be four parts in the following of the proof.

a) **Part 1.** Concerning  $\sum_{i=1}^N r_i^\top M(\Gamma_i) r_i$ : We define the following vectors

$$\begin{aligned} \bar{\mathbf{r}} &= \begin{pmatrix} r_2 \\ \vdots \\ r_N \end{pmatrix}, \quad v_{me} = \begin{pmatrix} M(\Gamma_2) \xi_2 \\ \vdots \\ M(\Gamma_N) \xi_N \end{pmatrix}, \\ v_{mr} &= \begin{pmatrix} M(\Gamma_2) r_2 \\ \vdots \\ M(\Gamma_N) r_N \end{pmatrix}, \quad v_{mw} = \begin{pmatrix} M(\Gamma_2) w_2 \\ \vdots \\ M(\Gamma_N) w_N \end{pmatrix}. \end{aligned} \quad (49)$$

Since  $r_1 = 0$ , we have

$$\begin{aligned} \sum_{i=1}^N r_i^\top M(\Gamma_i) r_i &= \sum_{i=2}^N r_i^\top M(\Gamma_i) r_i, \\ &= \bar{\mathbf{r}}^\top (I_{N-1} \otimes I_n) v_{mr}. \end{aligned} \quad (50)$$

The function  $M(\cdot)$  is positive, therefore  $\sum_{i=1}^N r_i^\top M(\Gamma_i) r_i \geq 0$ . Due to the properties of  $L_{22}$  exhibit in (11), one has

$$\bar{\mathbf{r}}^\top (I_{N-1} \otimes I_n) v_{mr} \leq \varepsilon \bar{\mathbf{r}}^\top (L_{22} \otimes I_n) v_{mr}, \quad (51)$$

moreover since  $L_{22}$  is symmetric one has

$$\begin{aligned} \varepsilon \bar{\mathbf{r}}^\top (L_{22} \otimes I_n) v_{mr} &= \varepsilon ((L_{22} \otimes I_n) \bar{\mathbf{r}})^\top (I_{N-1} \otimes I_n) v_{mr}, \\ &= \varepsilon \sum_{i=2}^N \left( \sum_{j=2}^N l_{ij} r_j \right)^\top M(\Gamma_i) r_i, \\ &= \varepsilon \sum_{i=2}^N (w_i + \xi_i)^\top M(\Gamma_i) r_i, \\ &= \varepsilon (v_{mw}^\top (I_{N-1} \otimes I_n) \bar{\mathbf{r}} + v_{me}^\top (I_{N-1} \otimes I_n) \bar{\mathbf{r}}). \end{aligned} \quad (52)$$

One has

$$v_{me}^\top (I_{N-1} \otimes I_n) \bar{\mathbf{r}} \leq |\varepsilon v_{me}^\top (L_{22} \otimes I_n) \bar{\mathbf{r}}|, \quad (53)$$

$$v_{mw}^\top (I_{N-1} \otimes I_n) \bar{\mathbf{r}} \leq |\varepsilon v_{mw}^\top (L_{22} \otimes I_n) \bar{\mathbf{r}}|. \quad (54)$$

We can obtain

$$\begin{aligned} &|\varepsilon v_{mw}^\top (L_{22} \otimes I_n) \bar{\mathbf{r}}| \\ &= \left| \varepsilon \sum_{i=2}^N w_i^\top M(\Gamma_i) \left( \sum_{j=1}^N l_{ij} r_j \right) \right| \\ &= \left| \varepsilon \sum_{i=2}^N w_i^\top M(\Gamma_i) w_i \right| + \left| \varepsilon \sum_{i=2}^N w_i^\top M(\Gamma_i) \xi_i \right|, \end{aligned} \quad (55)$$

and

$$\begin{aligned} &|\varepsilon v_{me}^\top (L_{22} \otimes I_n) \bar{\mathbf{r}}| \\ &= \left| \varepsilon \sum_{i=2}^N \xi_i^\top M(\Gamma_i) \left( \sum_{j=1}^N l_{ij} r_j \right) \right| \\ &= \left| \varepsilon \sum_{i=2}^N \xi_i^\top M(\Gamma_i) w_i \right| + \left| \varepsilon \sum_{i=2}^N \xi_i^\top M(\Gamma_i) \xi_i \right|. \end{aligned} \quad (56)$$

Thus the following upper bound can be found

$$\begin{aligned} \sum_{i=1}^N r_i^\top M(\Gamma_i) r_i &\leq \varepsilon^2 \left( \sum_{i=2}^N |\xi_i^\top M(\Gamma_i) \xi_i| \right. \\ &\quad \left. + \sum_{i=2}^N |w_i^\top M(\Gamma_i) w_i| + 2 \sum_{i=2}^N |w_i^\top M(\Gamma_i) \xi_i| \right). \end{aligned} \quad (57)$$

The quantity  $\xi_i$  can not be computed by the agent  $i$  since it require the knowledge on  $e_i$  for all  $i \in \{1, \dots, N\}$ , therefore we need to replace it by quantity that agent  $i$  knows. Using the conditions (13) and (17), there exists  $\bar{m} \in \mathbb{R}_+$  such that

$$0 \leq M(x) \leq \bar{m} I_n, \quad \forall x \in \mathbb{R}^n. \quad (58)$$

One can notice that

$$\begin{aligned} |\xi_i^\top M(\Gamma_i) \xi_i| &\leq \sum_{i=2}^N \bar{m} \left| \sum_{j=1}^N l_{ij} e_j \right|^2, \\ &\leq \sum_{i=2}^N \bar{m} \left| \left( \sum_{j=2}^N l_{ij} e_j \right) + l_{i1} e_1 \right|^2, \\ &\leq \bar{m} \sum_{i=2}^N \left[ \left| \sum_{j=2}^N l_{ij} e_j \right|^2 + l_{i1}^2 |e_1|^2 + \sum_{j=2}^N |l_{ij}| |l_{i1}| |e_j| |e_1| \right]. \end{aligned} \quad (59)$$

On one hand, note that, using the relation (40),

$$\begin{aligned} \sum_{i=2}^N \left| \sum_{j=1}^N l_{ij} e_j \right|^2 &= \sum_{i=2}^N \sum_{j=2}^N \sum_{k=2}^N |l_{ij}| |l_{ik}| |e_j| |e_k|, \\ &\leq \sum_{i=2}^N \sum_{j=2}^N \sum_{k=2}^N \left( \frac{b_1}{2} l_{ij}^2 |e_j|^2 + \frac{1}{2b_1} l_{ik}^2 |e_k|^2 \right), \\ &\leq \sum_{i=2}^N \sum_{j=2}^N (N-1) \frac{b_1}{2} l_{ij}^2 |e_j|^2 \\ &\quad + \sum_{i=2}^N \sum_{j=2}^N (N-1) \frac{1}{2b_1} l_{ij}^2 |e_j|^2, \\ &\leq \sum_{i=2}^N \sum_{j=2}^N (N-1) l_{ij}^2 \left( \frac{b_1}{2} + \frac{1}{2b_1} \right) |e_j|^2, \end{aligned} \quad (60)$$

for any  $b_1 \in \mathbb{R}_+$ . We define the constant  $\bar{\zeta}_i = \sum_{j=2}^N l_{ij}^2$ . Since  $L_{22}$  is symmetric, we have

$$\sum_{i=2}^N \left| \sum_{j=1}^N l_{ij} e_j \right|^2 \leq \sum_{i=2}^N (N-1) \bar{\zeta}_i \left( \frac{b_1}{2} + \frac{1}{2b_1} \right) |e_i|^2. \quad (61)$$

On another hand, due to the properties of the Laplacian matrix, we have  $\sum_{j=2}^N |l_{ij}| = 2l_{ii} - |l_{i1}|$ , thus

$$\begin{aligned} \sum_{i=2}^N \sum_{j=2}^N |l_{ij}| |l_{i1}| |e_j| |e_1| &= \sum_{i=2}^N (2l_{ii} - |l_{i1}|) |l_{i1}| |e_i| |e_1|, \\ &\leq \sum_{i=2}^N \delta_i |e_i| |e_1|, \\ &\leq \sum_{i=2}^N \delta_i \left( \frac{b_1}{2} |e_1|^2 + \frac{1}{2b_1} |e_i|^2 \right), \end{aligned} \quad (62)$$

with  $\delta_i = (2l_{ii} - |l_{i1}|) |l_{i1}|$ . Finally, we have

$$\begin{aligned} \left| \sum_{i=2}^N \xi_i^\top M(\Gamma_i) \xi_i \right| &\leq \sum_{i=2}^N \bar{m} \left[ \left( (N-1) \bar{\zeta}_i \left( \frac{b_1}{2} + \frac{1}{2b_1} \right) \right. \right. \\ &\quad \left. \left. + \frac{\delta_i}{2b_1} \right) |e_i|^2 + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} |e_1|^2 \right]. \end{aligned} \quad (63)$$

Moreover

$$\begin{aligned} \left| \sum_{i=2}^N w_i^\top M(\Gamma_i) \xi_i \right| &\leq \bar{m} \sum_{i=2}^N |w_i| \left| \left( \sum_{j=1}^N l_{ij} e_j \right) \right|, \\ &\leq \bar{m} \sum_{i=2}^N \sum_{j=1}^N |w_i| |l_{ij} e_j| \\ &\leq \bar{m} \sum_{i=2}^N \sum_{j=1}^N \left( \frac{b_1}{2} |w_i|^2 + \frac{1}{2b_1} |l_{ij} e_j|^2 \right), \\ &\leq \sum_{i=2}^N \bar{m} \left( \frac{b_1 N}{2} |w_i|^2 + \frac{1}{2b_1} \zeta_i |e_i|^2 \right), \end{aligned} \quad (64)$$

with  $\zeta_i = \sum_{j=1}^N l_{ij}^2$ . Thus, using the relation (63) and (64) in (57), we can write

$$\begin{aligned} \sum_{i=1}^N r_i^\top M(\Gamma_i) r_i &\leq \sum_{i=2}^N \varepsilon^2 \bar{m} \left[ \left( (N-1) \bar{\zeta}_i \frac{b_1^2 + 1}{2b_1} + \frac{\zeta_i}{b_1} \right. \right. \\ &\quad \left. \left. + \frac{\delta_i}{2b_1} \right) |e_i|^2 + (1 + b_1 N) |w_i|^2 + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} |e_1|^2 \right], \end{aligned} \quad (65)$$

which concludes this part of the proof.

*b) Part 2.* Concerning  $\sum_{i=1}^N r_i^\top \Delta M_i(-\Phi_i) r_i$ : One can notice that this term is close to the term concerned by the first part. This only changes is the function  $\Delta M_i$  instead of  $M$ . In this first part, we have used the relation (58) to bound the function  $M$ . In this second part, using the relation (17), we propose the following bound on  $\Delta M_i$

$$\begin{aligned} |\Delta M_i(-\Phi_i)| &= |\Delta \Psi_i(-\Phi_i)^\top \Delta \Psi_i(-\Phi_i)|, \\ &= |\Delta \Psi_i(\Phi_i)|^2, \\ &\leq k_\Psi^2 |\Phi_i|^2. \end{aligned} \quad (66)$$

Now we can use the result of the previous part of the proof and we have

$$\begin{aligned} \sum_{i=1}^N r_i^\top \Delta M_i(-\Phi_i) r_i &\leq \sum_{i=2}^N \varepsilon^2 k_\Psi^2 |\Phi_i|^2 \left[ \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2\zeta_i}{2b_1} |e_i|^2 \right. \\ &\quad \left. + (1 + b_1 N) |w_i|^2 + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} |e_1|^2 \right], \end{aligned} \quad (67)$$

with  $\delta_i = (2l_{ii} - |l_{i1}|) |l_{i1}|$ ,  $\bar{\zeta}_i = \sum_{j=2}^N l_{ij}^2$  and  $\zeta_i = \sum_{j=1}^N l_{ij}^2$ . One can notice that the quantity  $|\Phi_i|$  is not known by the agent 1, therefore we need to separate the product  $|\Phi_i|^2 |e_1|^2$  into a sum, we have

$$\begin{aligned} \sum_{i=1}^N r_i^\top \Delta M_i(-\Phi_i) r_i &\leq \sum_{i=2}^N \varepsilon^2 k_\Psi^2 \left[ (1 + b_1 N) |w_i|^2 |\Phi_i|^2 \right. \\ &\quad \left. + \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2\zeta_i}{2b_1} |e_i|^2 |\Phi_i|^2 \right. \\ &\quad \left. + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} \left( \frac{b_1}{2} |e_1|^4 + \frac{1}{2b_1} |\Phi_i|^4 \right) \right], \end{aligned} \quad (68)$$

which conclude this part of the proof.

*c) Part 3.* Concerning  $\sum_{i=1}^N e_i^\top \Delta M_i(-\Phi_i) r_i$ : We define the following vector  $v_{\Delta e} \in \mathbb{R}^{(N-1)n}$

$$v_{\Delta e} = \begin{pmatrix} \Delta M_2(-\Phi_2) e_2 \\ \vdots \\ \Delta M_N(-\Phi_N) e_N \end{pmatrix}. \quad (69)$$

We have

$$\begin{aligned} \sum_{i=1}^N e_i^\top \Delta M_i(-\Phi_i) r_i &= \sum_{i=2}^N e_i^\top \Delta M_i(-\Phi_i) r_i, \\ &= v_{\Delta e}^\top (I_{N-1} \otimes I_n) \bar{\mathbf{r}}, \\ &\leq \varepsilon |v_{\Delta e}^\top (L_{22} \otimes I_n) \bar{\mathbf{r}}|, \\ &\leq \varepsilon \left| \sum_{i=2}^N e_i^\top \Delta M(-\Phi_i) \sum_{j=2}^N l_{ij} r_j \right|. \end{aligned} \quad (70)$$

Moreover using (48), we have

$$\begin{aligned} \sum_{i=1}^N e_i^\top \Delta M_i(-\Phi_i) r_i &\leq \varepsilon \left| \sum_{i=2}^N e_i^\top \Delta M(-\Phi_i) (w_i + \xi_i) \right|, \\ &\leq \sum_{i=2}^N \varepsilon k_\Psi^2 |\Phi_i|^2 |e_i| |w_i| + \sum_{i=2}^N \sum_{j=1}^N |l_{ij}| \varepsilon k_\Psi^2 |\Phi_i|^2 |e_i| |e_j|, \end{aligned} \quad (71)$$



And using the laplacian property  $\sum_{j=1}^N |l_{ij}| = 2l_{ii}$ , we have and

$$\begin{aligned} & \sum_{i=2}^N \sum_{j=1}^N |l_{ij}| |\Phi_i|^2 |e_i| |e_j| \\ & \leq \sum_{i=2}^N \sum_{j=1}^N |l_{ij}| |\Phi_i|^2 \left( \frac{b_1}{2} |e_i|^2 + \frac{1}{2b_1} |e_j|^2 \right) \\ & \leq \sum_{i=2}^N l_{ii} b_1 |\Phi_i|^2 |e_i|^2 + \sum_{i=2}^N \sum_{j=1}^N \frac{|l_{ij}|}{2b_1} |\Phi_i|^2 |e_j|^2, \end{aligned} \quad (72)$$

for any  $b_1 \in \mathbb{R}_+$ . Moreover,

$$\begin{aligned} & \sum_{i=2}^N \sum_{j=1}^N \frac{|l_{ij}|}{2b_1} |\Phi_i|^2 |e_j|^2 \\ & \leq \sum_{i=2}^N \sum_{j=1}^N \frac{|l_{ij}|}{2b_1} \left( \frac{b_1}{2} |\Phi_i|^4 + \frac{1}{2b_1} |e_j|^4 \right), \\ & \leq \sum_{i=2}^N \frac{l_{ii}}{2} |\Phi_i|^4 + \sum_{i=2}^N \sum_{j=1}^N \frac{|l_{ij}|}{4b_1^2} |e_j|^4, \end{aligned} \quad (73)$$

and

$$\begin{aligned} \sum_{i=2}^N \sum_{j=1}^N \frac{|l_{ij}|}{4b_1^2} |e_j|^4 &= \sum_{i=2}^N \frac{|l_{i1}|}{4b_1^2} |e_1|^4 + \sum_{i=2}^N \sum_{j=2}^N \frac{|l_{ij}|}{4b_1^2} |e_j|^4, \\ &= \sum_{i=2}^N \left( \frac{|l_{i1}|}{4b_1^2} |e_1|^4 + \frac{2l_{ii} - |l_{i1}|}{4b_1^2} |e_i|^4 \right). \end{aligned} \quad (74)$$

Finally, using (72)-(74) in (71), we obtained

$$\begin{aligned} \sum_{i=1}^N e_i^\top \Delta M_i (-\Phi_i) r_i &\leq \sum_{i=2}^N \varepsilon k_\Psi^2 \left( |\Phi_i|^2 |e_i| |w_i| \right. \\ & \quad \left. + l_{ii} b_1 |\Phi_i|^2 |e_i|^2 + \frac{l_{ii}}{2} |\Phi_i|^4 + \frac{|l_{i1}|}{4b_1^2} |e_1|^4 \right. \\ & \quad \left. + \frac{2l_{ii} - |l_{i1}|}{4b_1^2} |e_i|^4 \right), \end{aligned} \quad (75)$$

which completes this part of the proof.

*d) Part 4.* Concerning  $\sum_{i=1}^N -r_i^\top P(x_1 + \Gamma_i) r_i$ : Due to the relation (12), we have

$$-(I_N \otimes P(\cdot)) \leq -(\mu^2 L^2 \otimes P(\cdot)). \quad (76)$$

We define the vectors  $v_{pr} \in \mathbb{R}^{N \times n}$  defined as

$$v_{pr} = \begin{pmatrix} P(x_1 + \Gamma_1) r_1 \\ \vdots \\ P(x_1 + \Gamma_N) r_N \end{pmatrix}, \quad (77)$$

since  $P$  is positive, we notice that

$$\begin{aligned} \sum_{i=1}^N -r_i^\top P(x_1 + \Gamma_i) r_i &= -v_{pr}^\top (I_N \otimes I_n) \mathbf{r}, \\ &\leq -\mu^2 v_{pr}^\top (L^2 \otimes I_n) \mathbf{r}, \end{aligned} \quad (78)$$

$$\begin{aligned} & -\mu^2 v_{pr}^\top (L^2 \otimes I_n) \mathbf{r} \\ &= \sum_{i=1}^N -\mu^2 \left( \sum_{j=1}^N l_{ij} r_j \right)^\top P(x_1 + \Gamma_i) \left( \sum_{j=1}^N l_{ij} r_j \right) \\ &= \sum_{i=1}^N -\mu^2 (w_i + \xi_i)^\top P(x_1 + \Gamma_i) (w_i + \xi_i) \\ &= \sum_{i=1}^N -\mu^2 (w_i^\top P(x_1 + \Gamma_i) w_i + 2\xi_i^\top P(x_1 + \Gamma_i) w_i \\ & \quad + \xi_i^\top P(x_1 + \Gamma_i) \xi_i). \end{aligned} \quad (79)$$

Similarly as before, we want to replace the quantity  $\xi_i$  by quantities the agent  $i$  know. According to the assumption 2, one has

$$\begin{aligned} \sum_{i=1}^N \xi_i^\top P(x_1 + \Gamma_i) \xi_i &\leq \sum_{i=1}^N \bar{p} |\xi_i|^2, \\ &\leq \sum_{i=1}^N \bar{p} \left| \sum_{j=1}^N l_{ij} e_j \right|^2, \\ &\leq \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \bar{p} |l_{ij}| |e_j| |l_{ik}| |e_k|, \\ &\leq \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \bar{p} \left( \frac{b_2}{2} |l_{ij}|^2 |e_j|^2 + \frac{1}{2b_2} |l_{ik}|^2 |e_k|^2 \right), \\ &\leq \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \bar{p} \left( \frac{b_2}{2} |l_{ij}|^2 |e_i|^2 + \frac{1}{2b_2} |l_{ik}|^2 |e_i|^2 \right), \\ &\leq \sum_{i=1}^N \bar{p} N \zeta_i \frac{b_2^2 + 1}{2b_2} |e_i|^2, \end{aligned} \quad (80)$$

for any  $b_2 > 0$  and with  $\zeta_i = \sum_{j=1}^2 l_{ij}^2$ . And

$$\begin{aligned} \sum_{i=1}^N \xi_i^\top P(x_1 + \Gamma_i) w_i &\leq \sum_{i=1}^N \bar{p} |\xi_i| |w_i|, \\ &\leq \sum_{i=1}^N \bar{p} \left( \frac{1}{2b_2} |\xi_i|^2 + \frac{b_2}{2} |w_i|^2 \right). \end{aligned} \quad (81)$$

Using the relation (80), we obtain

$$\begin{aligned} \sum_{i=1}^N \xi_i^\top P(x_1 + \Gamma_i) w_i &\leq \sum_{i=1}^N \bar{p} N \zeta_i \frac{b_2^2 + 1}{4b_2^2} |e_i|^2 \\ & \quad + \sum_{i=1}^N \bar{p} \frac{b_2}{2} |w_i|^2. \end{aligned} \quad (82)$$

And

$$-w_i^\top P(x_1 + \Gamma_i) w_i \leq -\underline{p} |w_i|^2 \quad (83)$$

Using the relation (80), (82) and (83) in (79), we finally have

$$\sum_{i=1}^N -r_i^\top P(x_1 + \Gamma_i) r_i \leq \sum_{i=1}^N \mu^2 \left( -(\underline{p} - \bar{p} b_2) |w_i|^2 + \bar{p} N \zeta_i \frac{b_2^2 + 1}{2 b_2} \left( \frac{1}{b_2} + 1 \right) |e_i|^2 \right), \quad (84)$$

which concludes this part of the proof. The unknown quantities by the agent  $i$  has been replaced by ones which are known. One can notice that

$$e_i^\top M_i(\Gamma_i) e_i \leq \bar{m} |e_i|^2, \quad (85)$$

$$e_i^\top \Delta M_i(-\Phi_i) e_i \leq k_\phi^2 |\Phi_i|^2 |e_i|^2. \quad (86)$$

Now, we can return to the Lyapunov function. Using (65), (68), (75) and (84)-(86) in (47) we obtain the following inequality. The terms have not yet be reorganised to facilitate the reviewing process.

$$\begin{aligned} \dot{V} \leq & -q(1-\eta) V + \sum_{i=1}^N \eta q \mu^2 \left( -(\underline{p} - \bar{p} b_2) |w_i|^2 \right. \\ & \left. + \bar{p} N \zeta_i \frac{b_2^2 + 1}{2 b_2} \left( \frac{1}{b_2} + 1 \right) |e_i|^2 \right) + \sum_{i=2}^N 4 b_1 \beta \kappa \varepsilon^2 \bar{m} \\ & \times \left[ \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i}{2 b_1} |e_i|^2 \right. \\ & \left. + (1 + b_1 N) |w_i|^2 + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} |e_1|^2 \right] \\ & + \sum_{i=1}^N 2 \frac{\beta \kappa}{b_1} \bar{m} |e_i|^2 + \sum_{i=1}^N 2 \frac{\beta \kappa}{b_1} k_\phi^2 |e_i|^2 \int_0^1 |\Phi_i|^2 ds \\ & + \sum_{i=2}^N 2 \frac{\beta \kappa}{b_1} \varepsilon^2 k_\Psi^2 \left[ (1 + b_1 N) |w_i|^2 \int_0^1 |\Phi_i|^2 ds \right. \\ & \left. + \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i}{2 b_1} |e_i|^2 \int_0^1 |\Phi_i|^2 ds \right. \\ & \left. + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} \left( \frac{b_1}{2} |e_1|^4 + \frac{1}{2 b_1} \int_0^1 |\Phi_i|^4 ds \right) \right] \\ & + \sum_{i=2}^N 4 \frac{\beta \kappa}{b_1} \varepsilon k_\Psi^2 \left( |e_i| |w_i| \int_0^1 |\Phi_i|^2 ds \right. \\ & \left. + l_{ii} b_1 |e_i|^2 \int_0^1 |\Phi_i|^2 ds + \frac{l_{ii}}{2} \int_0^1 |\Phi_i|^4 ds + \frac{|l_{i1}|}{4 b_1^2} |e_i|^4 \right. \\ & \left. + \frac{2 l_{ii} - |l_{i1}|}{4 b_1^2} |e_i|^4 \right). \quad (87) \end{aligned}$$

One can notice that

$$\int_0^1 |\Phi_i(s)|^k ds = \int_0^1 |e_i|^k s^k ds = \frac{1}{k} |e_i|^k, \quad (88)$$

for all  $k \in \mathbb{N}$ . Using the relation (88), we have

$$\begin{aligned} \dot{V} \leq & -q(1-\eta) V + \sum_{i=1}^N \eta q \mu^2 \left( -(\underline{p} - \bar{p} b_2) |w_i|^2 \right. \\ & \left. + \bar{p} N \zeta_i \frac{b_2^2 + 1}{2 b_2} \left( \frac{1}{b_2} + 1 \right) |e_i|^2 \right) + \sum_{i=2}^N 4 b_1 \beta \kappa \varepsilon^2 \bar{m} \\ & \times \left[ \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i}{2 b_1} |e_i|^2 + (1 + b_1 N) |w_i|^2 \right. \\ & \left. + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} |e_1|^2 \right] + \sum_{i=1}^N 2 \frac{\beta \kappa}{b_1} \bar{m} |e_i|^2 \\ & + \sum_{i=1}^N \frac{\beta \kappa}{b_1} k_\phi^2 |e_i|^4 + \sum_{i=2}^N 2 \frac{\beta \kappa}{b_1} \varepsilon^2 k_\Psi^2 \left[ \frac{1 + b_1 N}{2} |w_i|^2 |e_i|^2 \right. \\ & \left. + \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i}{4 b_1} |e_i|^4 \right. \\ & \left. + \frac{l_{i1}^2 (1 + \delta_i b_1)}{2} \left( \frac{b_1}{2} |e_1|^4 + \frac{1}{8 b_1} |e_i|^4 \right) \right] \\ & + \sum_{i=2}^N 4 \frac{\beta \kappa}{b_1} \varepsilon k_\Psi^2 \left[ \frac{1}{2} |e_i|^3 |w_i| + \frac{|l_{i1}|}{4 b_1^2} |e_i|^4 \right. \\ & \left. + \left( \frac{l_{ii} (4 b_1 + 1)}{8} + \frac{2 l_{ii} - |l_{i1}|}{4 b_1^2} \right) |e_i|^4 \right]. \quad (89) \end{aligned}$$

Now we reorganise the term to exhibit the CTCs (19) and (20)

$$\begin{aligned} \dot{V} \leq & -q(1-\eta) V - \eta q \mu^2 (\underline{p} - \bar{p} b_2) |w_1|^2 \\ & + \left( \frac{2 \beta \kappa k_\phi^2 + \beta \kappa \varepsilon^2 k_\Psi^2 \omega_1 b_1}{2 b_1} + \frac{\beta \kappa \varepsilon k_\Psi^2 |l_{11}|}{b_1^3} \right) |e_1|^4 \\ & + \left( 2 b_1 \beta \kappa \varepsilon^2 \bar{m} \omega_1 + \frac{\eta q \mu^2 \bar{p} N \zeta_i (b_2^2 + 1) (1 + b_2)}{2 b_2^2} \right. \\ & \left. + 2 \frac{\beta \kappa \bar{m}}{b_1} \right) |e_1|^2 + \sum_{i=2}^N \left[ 2 \frac{\beta \kappa \varepsilon k_\Psi^2}{b_1} |e_i|^3 |w_i| \right. \\ & \left. - (\eta q \mu^2 (\underline{p} - \bar{p} b_2) - 4 b_1 \beta \kappa \varepsilon^2 \bar{m} (1 + b_1 N)) |w_i|^2 \right. \\ & \left. + \left( 2 \frac{\beta \kappa \bar{m}}{b_1} + \frac{\eta q \mu^2 \bar{p} N \zeta_i (b_2^2 + 1) (1 + b_2)}{2 b_2^2} \right) \right. \\ & \left. + 2 b_1 \beta \kappa \varepsilon^2 \bar{m} \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i}{b_1} |e_i|^2 \right. \\ & \left. + \frac{\beta \kappa}{b_1} k_\Psi^2 \left( 1 + \varepsilon \frac{l_{ii} b_1^2 (4 b_1 + 1) + 2 (2 l_{ii} - |l_{i1}|)}{2 b_1^2} \right) \right. \\ & \left. + \varepsilon^2 \frac{4 ((N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i)}{8 b_1} \right. \\ & \left. + \frac{\varepsilon^2 l_{i1}^2 (1 + b_1 \delta_i)}{8 b_1} \right) |e_i|^4 \\ & \left. + \frac{\beta \kappa \varepsilon^2 k_\Psi^2 (1 + b_1 N)}{b_1} |w_i|^2 |e_i|^2 \right]. \quad (90) \end{aligned}$$

with  $\omega_1 = \sum_{i=2}^N l_{i1}^2 (1 + b_1 |l_{i1}| (2\beta - |l_{i1}|))$ . We recognise the constant  $Q_i, R_i, S_i, T_i$  and  $Q_1, R_1$  and  $S_1$ , thus we

have

$$\begin{aligned} \dot{V} &\leq -q(1-\eta)V - R_1 |w_1|^2 + Q_1 |e_1|^4 + S_1 |e_1|^2 \\ &+ \sum_{i=2}^N \left[ 2T_i |e_i|^3 |w_i| - R_i |w_i|^2 + S_i |e_i|^2 + Q_i |e_i|^4 \right. \\ &\left. + T_i (1 + b_1 N) |w_i|^2 |e_i|^2 \right]. \end{aligned} \quad (91)$$

According to the CTC (19), if  $2T_i |e_i|^3 |w_i| + S_i |e_i|^2 + Q_i |e_i|^4 + T_i (1 + b_1 N) |w_i|^2 |e_i|^2 > \sigma_i R_i |w_i|^2$  then a communication occurs, and the estimation error is reset to 0. Thus we have

$$\begin{aligned} 2T_i |e_i|^3 |w_i| + S_i |e_i|^2 + Q_i |e_i|^4 + T_i (1 + b_1 N) |w_i|^2 |e_i|^2 \\ \leq \sigma_i R_i |w_i|^2 \end{aligned} \quad (92)$$

Similarly for the CTC (20) for the leader agent. We obtain

$$\dot{V} \leq -q(1-\eta)V - \sum_{i=1}^N (1-\sigma_i) R_i |w_i|^2. \quad (93)$$

If we choose  $0 < b_2 < \frac{p}{\bar{p}N} - \frac{4b_1\beta\kappa\varepsilon^2\bar{m}(1+b_1N)}{\eta q \mu^2 \bar{p}N}$  where  $0 < b_1 < \frac{1}{2N} \left( \sqrt{1 + \frac{N\eta q p \mu^2}{\beta\kappa\varepsilon^2\bar{m}}} - 1 \right)$  then the  $R_i$  are positive number for  $i = 1, \dots, N$ . Since  $R_1$  and  $R_i$  are positive number and  $\sigma_1 < 1$  and  $\sigma_i < 1$ , we obtain

$$\dot{V} \leq -q(1-\eta)V, \quad (94)$$

which conclude the proof.  $\blacksquare$

#### IV. ZENO

The CTCs (19)-(20) do not exclude any Zeno behaviour. In this section, we proposed new CTCs which exclude Zeno behaviour. Before stating the theorem, we introduce two assumptions

*Assumption 3:* The input applies to the agents is bounded, i.e. there exists  $\bar{\alpha}$  such that

$$|\alpha(x)| < \bar{\alpha}, \quad \forall x. \quad (95)$$

In practice, this assumption is always verified, since the value of the input is proportional to the energy given to the agent. As the agents have a limited amount of energy, their input is always limited. So it is not a strong assumption.

*Assumption 4:* The function  $f$  are Lipschitz, i.e. there exist  $k_f \in \mathbb{R}_+$  such that

$$|f(x) - f(y)| \leq k_f |x - y|, \quad \forall x, y \in \mathbb{R}^n, \quad (96)$$

A positive term  $\xi > 0$  is added in the CTC to avoid Zeno behaviour when  $w_i$  is small. Consider the following CTC defining the sequence of communication instants  $t_{i,p}$  for the followers, i.e. for all  $i \in \{2, \dots, N\}$

$$\begin{aligned} t_{i,p+1} = \inf \left\{ s > t_{i,p} \mid 2T_i |e_i|^3 |w_i| + \varepsilon T_i |e_i|^2 |w_i|^2 \right. \\ \left. + S_i |e_i|^2 + Q_i |e_i|^4 - \sigma_i R_i |w_i|^2 - \xi > 0 \right\}, \end{aligned} \quad (97)$$

with  $e_i = x_i - \hat{x}_i^i$  the estimation error between  $x_i$  and  $\hat{x}_i^i$ , and  $w_i = \sum_{j=1}^N l_{ij} \hat{x}_j^j$ ,  $R_i, S_i, Q_i$  and  $T_i$  are positive number

and their expression are detailed in (114)-(117) in Appendix VII.

For the leader agent, the sequence of communication instants  $t_{1,p}$  are defined according to the following CTC

$$\begin{aligned} t_{1,p+1} = \inf \left\{ s > t_{1,p} \mid S_1 |e_1(s)|^2 + Q_1 |e_1|^4 \right. \\ \left. - \sigma_1 R_1 |w_1(s)|^2 - \xi > 0 \right\}, \end{aligned} \quad (98)$$

with constant  $R_1, S_1$  and  $Q_1$  defined in (118)-(120) in Appendix VII.

*Theorem 3:* If the assumption 1, 2, 3 and 4 hold. Then with the distributed control law  $u_i(t)$  for  $i = 2, \dots, N$  defined in (21), and the communication instant  $t_{i,p}$  defined according to the CTC (97), for  $i = 2, \dots, N$  and (98) for  $i = 1$ , there exists positive constants  $c_3, c_4$  such that for any  $\mathbf{x}(0) \in \mathbb{R}^{Nn}$

$$\lim_{t \rightarrow \infty} |\mathbf{x}|_{\mathcal{D}}^2 \leq c_3 e^{-c_4 t} + \frac{N\xi}{pq(1-\eta)}. \quad (99)$$

with  $\eta \in (0, 1)$  and  $q, p$  satisfying (13) and where  $\mathbf{x}(t)$  denotes the solution of the system (3). Moreover, there is no Zeno behaviour.

*Proof:* Starting from (93), we have

$$\dot{V} \leq -q(1-\eta)V - \sum_{i=1}^N \left[ (1-\sigma_i) R_i |w_i|^2 - \xi \right]. \quad (100)$$

Since  $R_i > 0$  for all  $i$  and  $0 < \sigma_i < 1$ , we have

$$\dot{V} \leq -q(1-\eta)V + N\xi. \quad (101)$$

By integrating (101), we obtain

$$\begin{aligned} V(t) &\leq e^{-q(1-\eta)t} V(0) + N\xi \int_0^t e^{-q(1-\eta)(t-\tau)} d\tau, \\ &\leq e^{-q(1-\eta)t} \left( V(0) + \frac{N\xi}{q(1-\eta)} \right) + \frac{N\xi}{q(1-\eta)}. \end{aligned} \quad (102)$$

We recall that  $V = \int_0^1 \sum_{i=2}^N r_i^T P(x_1 + r_i s) r_i ds$ . According to the properties of the function  $P$ , we have

$$p |r_i(t)|^2 \leq V(t). \quad (103)$$

Hence

$$|r_i(t)|^2 \leq c_1 e^{-c_2 t} + \frac{N\xi}{pq(1-\eta)}. \quad (104)$$

with  $c_1 = \frac{1}{p} \left( V(0) + \frac{N\xi}{q(1-\eta)} \right)$  and  $c_2 = q(1-\eta)$ . And

$$|\mathbf{x}|_{\mathcal{D}}^2 = \min_{z \in \mathbb{R}^n} \sum_{i=1}^N |z - x_i|^2 \leq \sum_{i=1}^N |x_1 - x_i|^2 = \sum_{i=1}^N |r_i|^2. \quad (105)$$

Thus

$$\lim_{t \rightarrow \infty} |\mathbf{x}|_{\mathcal{D}}^2 \leq c_1 e^{-c_2 t} + \frac{N\xi}{pq(1-\eta)}. \quad (106)$$

In the second part, we want to show that with the CTC (97) the inter event time between two communications can be

bounded. Let's consider the error  $|\tilde{x}_i|$  dynamics, using the assumptions 2 and 4, we have,

$$\begin{aligned} \frac{d}{dt}|\tilde{x}_i| &= (\tilde{x}_i^\top \tilde{x}_i)^{-\frac{1}{2}} \tilde{x}_i^\top \frac{d}{dt} \tilde{x}_i, \\ &= \frac{\tilde{x}_i^\top}{(\tilde{x}_i^\top \tilde{x}_i)^{\frac{1}{2}}} \left( f(x_i) - f(\hat{x}_i^i) - \kappa g(x_i) \sum_{j=1}^N l_{ij} \alpha(\hat{x}_j^i) \right), \\ &\leq k_f |\tilde{x}_i| + \kappa \left| g(x_i) \sum_{j=1}^N l_{ij} \alpha(\hat{x}_j^i) \right|, \\ &\leq k_f |\tilde{x}_i| + \kappa |g(x_i)| \left| \sum_{j=1}^N l_{ij} \alpha(\hat{x}_j^i) \right|, \\ &\leq k_f |\tilde{x}_i| + \kappa \frac{\sqrt{\bar{m}}}{\bar{p}} \left| \sum_{j=1}^N l_{ij} \alpha(\hat{x}_j^i) \right|. \end{aligned} \quad (107)$$

Moreover according to the assumption 3, we have

$$\frac{d}{dt}|\tilde{x}_i| \leq k_f |\tilde{x}_i| + 2\kappa \frac{\sqrt{\bar{m}}}{\bar{p}} |l_{ii}| |\bar{\alpha}|. \quad (108)$$

From here, it can be noticed that the derivative of the estimation error is bounded. We denote by  $d_i$  in this proof only the quantity  $2T_i |\tilde{x}_i|^3 |w_i| + S_i |\tilde{x}_i|^2 + Q_i |\tilde{x}_i|^4 + T_i (1 + b_1 N) |w_i|^2 |\tilde{x}_i|^2$  for all  $i \in \mathcal{V} \setminus \{1\}$ . It can be noticed that the time it takes for  $d_i$  to grow from 0 to  $\sigma_i R_i |w_i|^2 + \xi$  is not smaller than the time needed for  $d_i$  to grow from 0 to  $\xi$ . Since the the derivative of the estimation error is bounded, there exists a positive time for  $d_i$  to grow from 0 to  $\xi$ . This reasoning is similar for the leader agent and its CTC. ■

## V. SIMULATION EXAMPLES

Consider a MAS composed by  $N = 5$  agents whose dynamics are described by the relation (3) with

$$f(x) = \begin{pmatrix} \varsigma + \theta \cos(\varkappa) \\ -\varkappa + \theta \cos(\varsigma) + \theta^2 \cos(\varkappa) \sin(\varkappa) \end{pmatrix}, \quad (109)$$

with  $x = [\varkappa, \varsigma]^\top$ . The function  $f$  is parameterised by  $\theta = 0.5$ . To assess the robustness of the proposed CTCs and the control laws, the estimators are parametrised by  $\hat{\theta}$  instead of  $\theta$ . The function  $g$  is defined as

$$g(x) = \begin{pmatrix} \cos(\varsigma) \\ -1 \end{pmatrix}. \quad (110)$$

The graph formed by the agents is illustrated in Figure 1.

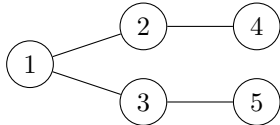


Fig. 1: Structure of the considered MAS

The agent 1 is the leader agent, thus,  $u_1(t) = 0$  for all  $t$ . A sampling period of  $T_s = 0.04$  s has been chosen for all the numerical simulations. The simulation duration is  $T_{sim} = 20$  s. Selecting  $\mu = 0.27$  and  $\varepsilon = 2.7$  meet the assumption 1 with the communication graph described in Figure 1.

As underlined previously, solving the relation (13) to obtain the matrix function  $P$  is a complex task. In this example we consider a deep learning method to approximate the matrix function  $P$  solution of the relation (13) using neural network  $\sigma_P$ , see for example [29], [30]. Deep learning is a numerical method that approximates a local optimal solution. Therefore, the database on which the neural network  $\sigma_P$  is trained must correctly represent the state space in which the neural network  $\sigma_P$  will operate. The algorithm proposed in [29] is illustrated in Algorithm 1.

---

### Algorithm 1 Finding a matrix function satisfying (13)

---

**Input:** number of training epochs  $N_e$ , number  $N_\tau$  of training points, the parameters  $p, \rho, q$  defined in (13).

Generate a training data set  $\mathcal{S}_\tau$  of  $N_\tau$  points  $\tau = \{x\}$  from multiple random samples.

Initialise the parameters of a neural network  $\sigma_P(\cdot)$ .

**for**  $\varepsilon = 1$  to  $N_e$  **do**

**for** each trajectory  $\tau$  in  $\mathcal{S}_\tau$  **do**

    Compute

$$M_1(x) = \mathcal{L}_f \sigma_P(x) - \rho \sigma_P(x) g(x) g(x)^\top \sigma_P(x) + q \sigma_P(x).$$

    Compute  $M_2(x) = \mathcal{L}_g \sigma_P(x)$ .

    Compute  $M_3(x) = -\sigma_P(x) + p I_n$ .

    Compute  $\lambda_i(M_1(x))$ ,  $\lambda_i(M_2(x))$  and  $\lambda_i(M_3(x))$  respectively the eigenvalues of  $M_1(x)$ ,  $M_2(x)$  and  $M_3(x)$ .

    Evaluate the loss

$$\mathcal{L}_\tau = \sum_{i=1}^N [\max(0, \lambda_i(M_1(x))) + \max(0, \lambda_i(M_2(x))) - \max(0, \lambda_i(M_2(x))) + \max(0, \lambda_i(M_3(x)))].$$

**end for**

  Update the parameters of the network  $\sigma_P(\cdot)$  by minimising  $\frac{1}{N_\tau} \sum_{\tau \in \mathcal{S}_\tau} \mathcal{L}_\tau$  using backward propagation.

**end for**

---

A training database is created by randomly generating  $10^9$  different values of  $x$  according to a normal distribution  $\mathcal{N}(0, 100)$ . The parameters of the neural network are then optimised so that the neural network  $\sigma_P$  satisfies all the necessary conditions. Finally, the performance of the obtained solution is tested on a database composed of  $10^9$  random different values of  $x$ , not seen during training, generated according to a normal distribution  $\mathcal{N}(0, 100)$ . Once the model trained, we obtained  $q = 0.001$ ,  $\rho = 5.5$ ,  $\bar{p} = 0.019$ ,  $p = 0.0094$  and  $\bar{m} = 0.0006$  and  $k_\Psi = 10^{-7}$ . We have selected  $\kappa = 80$ . One can notice that  $\kappa > \rho\varepsilon$ . We also choose  $b_1 = 6.8e - 8$  and  $b_2 = 0.0185$ . To illustrate our results, we have selecting  $\sigma_i = 10^{12}$  which does not guarantee any stability but provides better results.

The initial condition  $\mathbf{x}(0)$  are chosen as

$$\mathbf{x}(0) = \begin{pmatrix} [0.8033, 0.1748]^\top \\ [0.0890, -0.6137]^\top \\ [0.0462, -1.3683]^\top \\ [0.3375, 1.0111]^\top \\ [-1.4352, 0.9774]^\top \end{pmatrix}. \quad (111)$$

The robustness of the CTCs and the control laws is assessed by considering several values of  $\hat{\theta}$ . The figure 2 presents the time evolution of the disagreement  $\Upsilon$  defined as

$$\Upsilon(k) = \sum_{i=1}^N |x_i(t) - x_m(t)|, \quad (112)$$

with  $x_m = \frac{1}{N} \sum_{i=1}^N x_i(t)$ , of the MAS and the communication instants for each agent for several values of  $\hat{\theta}$  using the CTC (19).

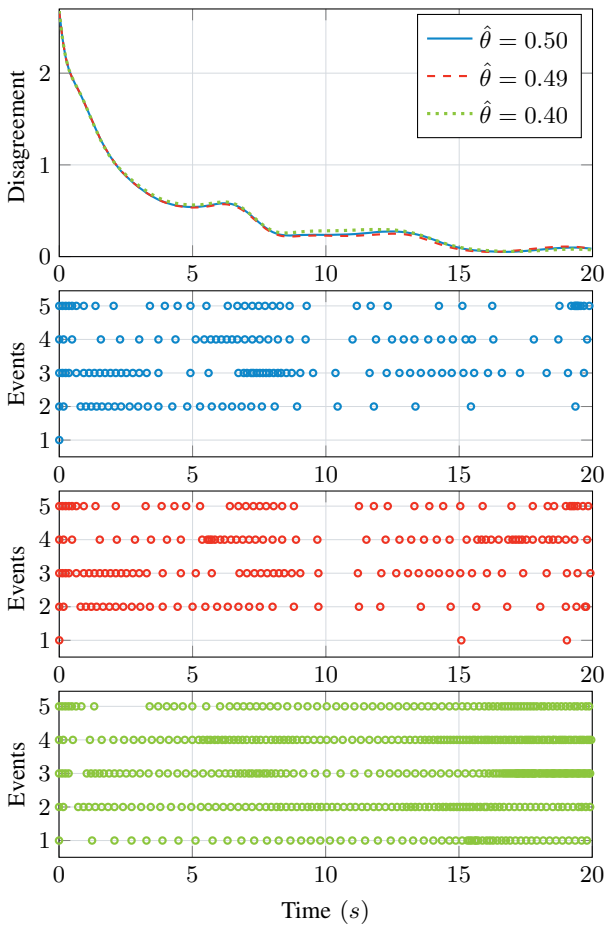


Fig. 2: Time evolution of the disagreement of the MAS and the communication events using the CTCs (19) and (20) for different values of  $\hat{\theta}$ .

It can be noticed that the MAS converges to the consensus for all the values of  $\hat{\theta}$ . Moreover, the communications are not triggered at each instant. The leader agent does not need to initiate a lot of communications since its estimation error is only due to the wrong representation of  $\theta$  and not of the input.

The figure 3 shows the time evolution of the disagreement of the MAS and the communication instants using the CTCs (97)-(98) which exclude any Zeno behaviour but does not guarantee an asymptotic consensus of the MAS. The parameters  $\xi$  is chosen as  $\xi = 5e7$ . The other parameters remain unchanged.

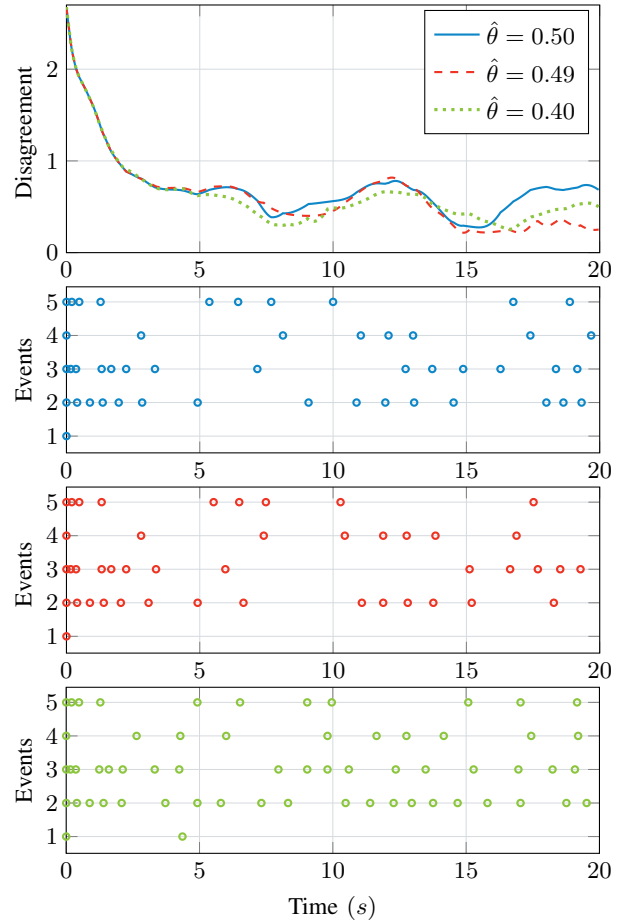


Fig. 3: Time evolution of the disagreement of the MAS and the communication events using the CTCs (97) and (98) for different values of  $\hat{\theta}$ .

It can be seen that the number of communications has decreased. However, the asymptotic convergence of the MAS to a consensus is not obtained.

In order to compare the resulting performances, we introduce a performance index  $\Gamma$  to qualify the evolution of the consensus error and the reduction of communications from the CTC.  $\Gamma$  is defined as

$$\Gamma = \frac{1}{N} \sum_{k=0}^T \left[ \sum_{i=1}^N |x_i(k T_s) - x_1(k T_s)| + \chi v_i(k T_s) \right],$$

where  $T = \frac{T_{sim}}{T_s}$  is the number of simulated instants,  $v_i(k)$  is defined as

$$v_i(k T_s) = \begin{cases} 1, & \text{if the CTC is verified for the agent } i, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\chi > 0$  is a tuning parameter that allows to balance between consensus accuracy and communication reduction.

For this simulation, we selected  $\chi = 0.3$ . The Table I present the value of  $\Gamma$  for each simulation.

Values of $\Gamma$						
	CTC (19)			CTC (97)		
$\hat{\theta}$	0.50	0.49	0.40	0.50	0.49	0.40
$\Gamma$	120	120	140	133	127	128

TABLE I: Comparison of number of communications in the MAS

## VI. CONCLUSION

In this article, firstly new distributed event-triggered control laws have been proposed for the leader-follower consensus problem of a MAS described by control-affine nonlinear dynamics. They are based on two novel Communication Triggering Conditions (CTCs): the first one allows to obtain an asymptotic consensus but without any guarantee of Zeno behaviour, and the second one excludes Zeno behaviour but a practical consensus property is ensured in this case. Simulations are presented to compare the performance of the two approaches. Robustness to noises is also illustrated. However, the obtained CTCs tends to be too conservative resulting in many unnecessary communications. Future works concerning the optimisation of the communication laws are investigated.

## VII. APPENDIX

The real constant  $R_i, S_i, Q_i, T_i$ , for all  $i \in \{2, \dots, N\}$  are defined as

$$R_i = \eta q \mu^2 (p - \bar{p} b_2) - 4 b_1 \beta \kappa \varepsilon^2 \bar{m} (1 + b_1 N), \quad (114)$$

$$S_i = 2 \frac{\beta \kappa \bar{m}}{b_1} + \frac{\eta q \mu^2 \bar{p} N \zeta_i (b_2^2 + 1) (1 + b_2)}{2 b_2^2} + 2 b_1 \beta \kappa \varepsilon^2 \bar{m} \frac{(N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i}{b_1}, \quad (115)$$

$$Q_i = \frac{\beta \kappa}{b_1} k_{\Psi}^2 \left( 1 + \varepsilon \frac{l_{ii} b_1^2 (4 b_1 + 1) + 2 (2 l_{ii} - |l_{i1}|)}{2 b_1^2} + \varepsilon^2 \frac{4((N-1) \bar{\zeta}_i (b_1^2 + 1) + \delta_i + 2 \zeta_i) + l_{i1}^2 (1 + b_1 \delta_i)}{8 b_1} \right), \quad (116)$$

$$T_i = \frac{\beta \kappa \varepsilon k_{\Psi}^2}{b_1}, \quad (117)$$

where  $0 < \eta < 1$ ,  $0 < b_1 < \frac{1}{2N} \left( \sqrt{1 + \frac{N \eta q p \mu^2}{\beta \kappa \varepsilon^2 \bar{m}}} - 1 \right)$ ,  $0 < b_2 < \frac{p}{\bar{p} N} - \frac{4 b_1 \beta \kappa \varepsilon^2 \bar{m} (1 + b_1 N)}{\eta q \mu^2 \bar{p} N}$ ,  $\delta_i = (2 l_{ii} - |l_{i1}|) |l_{i1}|$ ,  $\bar{\zeta}_i = \sum_{j=2}^N l_{ij}^2$  and  $\zeta_i = \sum_{j=1}^N l_{ij}^2$ . The real constant

$R_1, S_1, Q_1$  are defined as

$$R_1 = \eta q \mu^2 (p - \bar{p} b_2), \quad (118)$$

$$S_1 = 2 b_1 \beta \kappa \varepsilon^2 \bar{m} \omega_1 b_1 + \frac{\eta q \mu^2 \bar{p} N \zeta_1 (b_2^2 + 1) (1 + b_2)}{2 b_2^2} + 2 \frac{\beta \kappa \bar{m}}{b_1}, \quad (119)$$

$$Q_1 = \frac{2 \beta \kappa k_{\Phi}^2 + \beta \kappa \varepsilon^2 k_{\Psi}^2 \omega_1 b_1^2}{2 b_1} + \frac{\beta \kappa \varepsilon k_{\Psi}^2 |l_{11}|}{b_1 b_1^2} \quad (120)$$

where  $\omega_1 = \sum_{i=2}^N l_{i1}^2 (1 + b_1 |l_{i1}| (2\beta - |l_{i1}|))$

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