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Asymptotically free solutions of the scalar mean field flow equations

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Abstract

The flow equations of the renormalisation group permit to analyse rigorously the perturbative n -point functions of renormalisable quantum field theories including gauge theories. In this paper we want to do a step towards a rigorous nonperturbative analysis of the flow equations (FEs). We restrict to massive scalar (one-component) fields and analyse a *mean field limit* where the Schwinger functions are considered to be momentum independent and thus are replaced by their zero momentum values. We analyse smooth solutions of the system of FEs for the n -point functions for different sets of boundary conditions. We will realise that allowing for nonvanishing irrelevant terms permits to construct *asymptotically free nontrivial smooth solutions of the scalar field mean field FEs*.

1 Introduction

Quantum field theory, originally developed to implement the principles of quantum mechanics in relativistic systems, has become the general theoretical framework to study physical systems with an infinite (or large) number of degrees of freedom. Relativistic quantum systems are described by relativistic quantum field theory, euclidean field theory gives access to critical systems in statistical mechanics, systems from solid state physics

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can be modeled by field theories at finite density and temperature. These systems have different kinematics which is reflected in particular by the form of the (free) propagator or two-point function. Interactions are introduced via the path integral formalism.

Aiming at mathematical rigour one is faced with the problem that path integrals describing interacting systems in field theory are generally not defined a priori. Whereas there is a complete theory of Gaussian measures applying to the noninteracting case, a mathematically oriented study of interacting field theories generally starts from regularised versions of the theory, where the number of degrees of freedom in space and momentum has been made (essentially) finite by hand, through the introduction of regulators like finite volume and large momentum cutoffs. One then studies correlation functions and proves that these have uniform limits in the cutoffs. For a general introduction to these methods see [13], euclidean scalar field theories are analysed in [31, 33].

The functional flow equation is a differential equation for the effective action functional of the field theory considered. The basic ideas go back to Wilson [35]. In its differential form it seems to appear first in [34]. When expanded in moments it becomes an infinite system of differential equations for the connected amputated Schwinger functions of the theory. In a seminal paper [30] Polchinski observed that when expanding these functions order by order in the number of loops, there is an airtight inductive scheme which permits to sufficiently control the perturbative functions such that renormalisability follows. In subsequent papers Polchinski's result was implemented in a rigorous path integral framework [16], extended to physical renormalisation conditions and sharpened such that cutoff independence became immediate [18, 17]. The scheme has also been extended to Minkowski space [19]. For reviews see [20, 27]. As a consequence ultraviolet renormalisability can be largely reduced to power counting on the basis of a sufficiently sharp induction hypothesis. The complicated combinatoric aspects of the problem, which had found their deep solution in Zimmermann's forest formula [37], turned out not to be intrinsic to the renormalisation problem, but rather to stem from the fact that the perturbative contributions had been split up in too fine a way, namely into Feynman diagram amplitudes ¹.

In contrast, methods originally stemming from statistical physics like cluster and Mayer expansions [26, 3, 13, 31, 4], permit to analyse regularised path integrals non-perturbatively, but are relatively straightforward only in theories which do not have to be renormalised in an essential way, like massive φ_2^4 and φ_3^4 or other superrenormalisable models. They are technically very hard to apply in strictly renormalisable theories. The

¹This remark does of course not put into question the value of Feynman diagrams. It only says that they are not optimally adapted for a mathematical analysis of the UV divergences (and related problems).

review [4] shows the state of the art and reveals important progress made in this respect over the last decades. In this context we cite a beautiful paper [32] on a construction of planar “wrong sign” φ_4^4 -theory which is intermediate between constructive and perturbative continuous renormalisation group methods. In quantum field theory with hindsight to particle physics the relevance of the constructive path integral method is also limited by the fact that the physically interesting theories are either plagued by the triviality statement [9, 1, 2], or in the case of quantum chromodynamics, by infrared problems which are presently beyond the scope of mathematical physics. As a consequence the work performed in constructive field theory has not entered typical text books on quantum field theory, in spite of the fact that the nonperturbative analysis of field theory is generally recognised to be an important problem.

Regarding on the other hand the differential FEs, one realises that they have not been used with much success in the rigorous analysis of quantum field theory beyond perturbation theory. A notable exception is [5] where the FEs are integrated over a finite interval for bounded and analytic initial data. Whereas the renormalisation problem becomes transparent in Polchinski’s framework, by being related immediately to power counting, the mathematical structure of the FEs for the n -point functions generally hinders a rigorous nonperturbative analysis, even in the absence of renormalisation, due to their dependence on the index n . This then even applies to φ_1^4 or φ_0^4 theory. We will come back to this point in the first remark after (29).

In perturbative or constructive quantum field theory, one typically starts from a bare action which contains only a few local monomials of low degree in the fields. In a theory like quantum electrodynamics this leads, after perturbative renormalisation, to results which are in extremely good agreement with experiment. The simple form of the bare action is also justified by the fact that higher order monomials in the fields, if not multiplied by a corresponding negative power of the renormalisation scale, are nonrenormalisable in perturbation theory and do not lead to physically predictive theories.

From the point of view of the Wilson renormalisation group such low order monomial lagrangians are related to an impressive fine tuning procedure - unless one would attribute a very distinguished physical role to the UV cutoff where the flow is started. To illustrate this point consider quantum electrodynamics. Starting from a bare lagrangian containing the monomials

$$\bar{\psi} \psi , \quad \bar{\psi} \partial^\mu \psi , \quad \bar{\psi} \gamma_\mu A^\mu \psi , \quad F_{\mu\nu} F^{\mu\nu} , \quad (\partial_\mu A^\mu)^2 ,$$

and integrating out degrees of freedom (in whatever small a momentum range), one immediately obtains a nonpolynomial effective lagrangian containing monomials of any degree

in the fields and their derivatives (as far as they are allowed by the symmetries of the theory). The above mentioned fine tuning thus consists in arranging the renormalisation group trajectory such that *the infinite number of all (connected amputated) higher n -point functions are forced to pass through 0 each, and for **all** momentum or position arguments, at exactly the same value of the renormalisation scale*, thus leaving only the small number of local terms we wrote above for the bare lagrangian of QED.

In perturbation theory the infinite number of terms generated by the renormalisation group evolution can be shown to contain each a negative power of the renormalisation group scale, this power corresponding to its mass dimension, times a suitable function bounded uniformly in the scale, up to logarithms. All these terms are uniformly bounded in the UV cutoff [27]. Therefore their contributions do not produce any new ultraviolet divergences, as compared to those stemming from the initial bare lagrangian. Thus the argument of nonrenormalisability disappears due to the aforementioned inverse powers of the renormalisation group scale. Nevertheless we should note that the the highly successful perturbative calculations in physical models like QED are based on such monomial bare lagrangians, for which also the perturbative calculations are easier to carry out.

In this paper we shall consider FEs for scalar field theories in four dimensions in the mean field limit. These have the same power counting and scaling behaviour as the full four dimensional theory and seem to capture well some of the basic features of four dimensional scalar field theories. Our point is that different, generally nonpolynomial bare actions, scaling with the cutoff as indicated before, may lead to essentially different solutions of the FEs. We will show that, depending on the choice of the bare lagrangian, one may in particular obtain asymptotically free solutions of the mean field scalar FEs which escape the so-called triviality statement [9, 1] which has been sharpened recently [2]. Our results are not in contradiction with this statement, since we will verify that for the bare Lagrangian containing only local terms of the type φ^4 and φ^2 , the trivial solution emerges indeed. We will characterise this solution quite explicitly. It will turn out that enforcing the fine tuned boundary conditions generates much larger values for the derivatives of the n -point functions w.r.t. the renormalisation group scale than for the asymptotically free solutions.

The mean field or local potential approximation to the FEs of scalar field theory has been analysed previously in the literature. One of the earliest references is [29] in which renormalisation group trajectories are calculated. In [38] critical exponents are evaluated in this approximation for $O(n)$ models with convincing results. The paper [7] gives a comprehensive rigorous analysis of fixed points of scalar mean field theory below and up to four dimensions. The mean field approximation is closely related to the so-called

hierarchical models for which a rigorous fixed point analysis has been performed in [10]. These fixed points are the nontrivial infrared fixed points relevant to the analysis of critical phenomena [36]. We are mainly interested in the existence of a (trivial) ultraviolet fixed point characteristic of UV asymptotically free field theories. We will comment in more detail on the paper [7] in section 4.1.

This paper is organized as follows: In section 2 we introduce the flow equations. In section 3 we perform the mean field limit. Section 4 is at the heart of our argument. We study various types of smooth solutions of mean field FEs. In 4.1 we study solutions for which we impose certain smallness and smoothness conditions at the UV boundary. In 4.2 we show that there are asymptotically free solutions among those studied in 4.1. Then in 4.3 we study a class of strictly positive asymptotically free solutions which are of similar type as those studied previously. In 4.4 we also impose bounds on the initial conditions which are sufficient to make the starting regularised path integral - or the starting mean field action in this approximation - well defined. In 4.5 we study the trivial solution corresponding to boundary data containing only the monomials φ^2 and φ^4 . This paper has strong overlap with an earlier preprint [22]. We have tried to clarify and present in a clearer way certain points and we have added important references. Many of these changes are due to remarks of David Brydges. In the present presentation we have not included section 5 from [22], which extends certain results to one-particle irreducible functions. These results are technically more complicated to obtain and (presently) do not seem to open new perspectives.

To describe our main result we introduce some notation (which is developed in more detail in the main text). We study the renormalisation group flow in terms of a logarithmic scale μ which varies between 0 for the bare action and $\mu_{\max} \gg 1$ for the fully integrated or renormalised action. Taking away the UV cutoff corresponds to the limit $\mu_{\max} \rightarrow \infty$. The field variable ϕ of the mean field approximation does not depend on position or momentum. Our statement is then:

Theorem: *There exist bare mean field actions $L_0(\phi)$ which are uniformly bounded from below for $\phi \in \mathbb{R}$ and locally analytic in the (constant) mean field variable ϕ such that the moments $A_n(\mu)$, $\mu \in [0, \mu_{\max}]$, of the effective mean field actions $L_\mu(\phi)$ exist as smooth solutions of the system of FEs. Choosing the bare action $L_0(\phi)$ suitably as a function of μ_{\max} , the ultraviolet limits $\mu_{\max} \rightarrow \infty$ of the moments of $L_{\mu_{\max}}(\phi)$ exist and are nontrivial. The bare action vanishes in the limit $\mu_{\max} \rightarrow \infty$ (asymptotic freedom).*

2 The flow equations

We consider a (one-component) self-interacting scalar field on four dimensional euclidean space. We adopt the renormalisation group flow equation framework [35, 34, 30]. In the following we will give a brief review of the general formalism and define the objects of interest for the purpose of this paper. See [27, 20, 15] for more comprehensive reviews of the flow equation approach within our context.

2.1 The flow equations for the effective action

We start formulating our theory with ultraviolet (UV) cutoff and infrared (IR) cutoff in the standard path integral formalism. This requires two main ingredients:

1. We define the regularised momentum space propagator as

$$C^{\alpha_0, \alpha}(p; m) = \frac{1}{p^2 + m^2} [\exp(-\alpha_0(p^2 + m^2)) - \exp(-\alpha(p^2 + m^2))] . \quad (1)$$

Upon removal of the cutoffs, i.e. in the limit $\alpha_0 \rightarrow 0$ (UV), $\alpha \rightarrow \infty$ (IR), we indeed recover the free propagator $\frac{1}{p^2 + m^2}$. For the Fourier transform, we use the convention

$$f(x) = \int_p \hat{f}(p) e^{ipx} \quad \text{using the shorthand} \quad \int_p := \int \frac{d^4p}{(2\pi)^4} \quad (2)$$

so that in position space

$$C^{\alpha_0, \alpha}(x - y; m) = \int_p e^{ip(x-y)} C^{\alpha_0, \alpha}(p; m) . \quad (3)$$

2. The bare interaction Lagrangian is supposed to be of the form ²

$$L_0(\varphi) = \int d^4x \left[b_0(\alpha_0) (\partial\varphi(x))^2 + \sum_{n \in 2\mathbb{N}} c_{0,n}(\alpha_0) \varphi^n(x) \right] , \quad (4)$$

where the bare couplings $b_0(\alpha_0)$, $c_{0,n}(\alpha_0)$ should be such that

$$-\infty < K < L_0(\varphi) < \infty \quad \forall \varphi \in \text{supp } \mu^{\alpha, \alpha_0} . \quad (5)$$

Here K is some finite real constant and μ^{α, α_0} is the normalised Gaussian measure with covariance (1) ³. The *basic field* φ is assumed to be in the support of μ^{α, α_0} .

In order to obtain a well defined limit of the quantities of interest for $\alpha_0 \rightarrow 0$, the constants $c_{0,n}$ generally need to be chosen as appropriate functions of the ultraviolet cutoff α_0 .

²In the mean field limit the term $b_0(\alpha_0) (\partial\varphi(x))^2$ vanishes.

³See the Appendix to Part I of [13] for mathematical details about Gaussian functional integrals.

The correlation (= Schwinger = n -point) functions of n basic fields with cutoff are defined by the expectation values

$$\begin{aligned} \langle \varphi(x_1) \cdots \varphi(x_n) \rangle &\equiv \mathbb{E}_{\mu^{\alpha, \alpha_0}} \left[\exp \left(-L_0 \right) \varphi(x_1) \cdots \varphi(x_n) \right] / Z^{\alpha, \alpha_0} \\ &= \int d\mu^{\alpha, \alpha_0} \exp \left(-L_0 \right) \varphi(x_1) \cdots \varphi(x_n) / Z^{\alpha, \alpha_0} . \end{aligned} \quad (6)$$

This expression is simply the standard euclidean path-integral, where the free part of the Lagrangian has been absorbed into $d\mu^{\alpha, \alpha_0}$. The normalisation factor Z^{α, α_0} is chosen so that $\langle 1 \rangle = 1$. For finite values of the cutoffs $0 < \alpha_0 < \alpha < \infty$ and on imposing a finite (space) volume, the functional integral (6) exists in the nonperturbative sense, if L_0 is bounded from below. In the perturbative theory it has been shown that one can remove the cutoffs, $\alpha_0 \rightarrow 0$ and $\alpha \rightarrow \infty$, for a suitable choice of the bare running couplings $b_0(\alpha_0)$, $c_{0,n}(\alpha_0)$ at each given but fixed order in the number of loops. The correct behaviour of these couplings (in terms of bounds) is determined from the FEs which are a system of differential equations in the parameter α for the (amputated) Schwinger functions.

These differential equations are written most conveniently in terms of the hierarchy of “connected, amputated Schwinger functions” (CAS functions), whose generating functional is given by the following convolution⁴ of the Gaussian measure with the exponentiated interaction,

$$-L^{\alpha_0, \alpha} := \log \left[\mu^{\alpha_0, \alpha} \star \exp \left(-L_0 \right) \right] - \log Z^{\alpha_0, \alpha} . \quad (7)$$

The full Schwinger functions can be recovered from the CAS functions in the end. One can expand the functionals $L^{\alpha_0, \alpha}$ as formal power series in terms of Feynman diagrams with ℓ loops, n external legs and propagator $C^{\alpha_0, \alpha}(p)$. Then one can show that, indeed, only connected diagrams with an even number of external legs contribute, and that the (free) propagators on the external legs are removed. While we will not use diagrammatic decompositions in terms of Feynman diagrams, we start from analysing the functional (7) in momentum space, expanded in moments, i.e. powers of φ

$$L^{\alpha_0, \alpha}(\varphi) := \sum_{n \in 2\mathbb{N}} \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} \bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) \hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n) . \quad (8)$$

Here no statement is made about the convergence of this series. By performing the Fourier transformation in (4) we find the relation

$$(2\pi)^4 \delta^4 \left(\sum_{i=1}^n p_i \right) c_{0,n}(\alpha_0) = \bar{\mathcal{L}}_n^{\alpha_0, \alpha_0}(p_1, \dots, p_n) , \quad n \geq 4 , \quad (9)$$

⁴The convolution is defined in general by $(\mu^{\alpha_0, \alpha} \star F)(\varphi) = \int d\mu^{\alpha_0, \alpha}(\varphi') F(\varphi + \varphi')$.

$$(2\pi)^4 \delta^4(p_1 + p_2) [c_{0,2}(\alpha_0) + b_0(\alpha_0)p_1^2] = \bar{\mathcal{L}}_2^{\alpha_0, \alpha_0}(p_1, p_2) .$$

Translation invariance of the CAS functions in position space implies that $\bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$ is supported at $p_1 + \dots + p_n = 0$ (momentum conservation), and thus only depends on $n - 1$ independent four momenta. We write

$$\bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) = \delta^4\left(\sum_{i=1}^n p_i\right) \mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) \quad (10)$$

so that

$$c_{0,n}(\alpha_0) + \delta_{n,2} b_0(\alpha_0) p_1^2 = (2\pi)^{-4} \bar{\mathcal{L}}_n^{\alpha_0, \alpha_0}(p_1, \dots, p_n) . \quad (11)$$

We use the convention that the variable p_n is determined in terms of the remaining $n - 1$ four vectors by momentum conservation, i.e. $p_n = -p_1 - \dots - p_{n-1}$. One should keep in mind, however, that the functions $\bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$ are in fact fully symmetric under permutation of *all* p_1, \dots, p_n . Note also that the previous definitions imply that the free CAS two-point function is not included in the function $\mathcal{L}_2(p, -p)$ so that $\mathcal{L}_2(p, -p)$ vanishes, if the bare interaction does so.

To obtain the flow equations for the CAS functions, we take the α -derivative⁵ of (7):

$$\partial_\alpha L^{\alpha_0, \alpha} = \frac{1}{2} \left\langle \frac{\delta}{\delta\varphi}, \dot{C}^\alpha \star \frac{\delta}{\delta\varphi} \right\rangle L^{\alpha_0, \alpha} - \frac{1}{2} \left\langle \frac{\delta}{\delta\varphi} L^{\alpha_0, \alpha}, \dot{C}^\alpha \star \frac{\delta}{\delta\varphi} L^{\alpha_0, \alpha} \right\rangle + \partial_\alpha \log Z^{\alpha_0, \alpha} . \quad (12)$$

Here we use the following notation: We write \dot{C}^α for the derivative $\partial_\alpha C^{\alpha_0, \alpha}$, which, as we note, does not depend on α_0 . Further, by $\langle \cdot, \cdot \rangle$ we denote the standard scalar product in $L^2(\mathbb{R}^4, d^4x)$, and \star stands for convolution in \mathbb{R}^4 . As an example,

$$\left\langle \frac{\delta}{\delta\varphi}, \dot{C}^\alpha \star \frac{\delta}{\delta\varphi} \right\rangle = \int d^4x d^4y \dot{C}^\alpha(x - y; m) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} \quad (13)$$

is sometimes called the ‘‘functional Laplace operator’’. We can now write the flow equation (12) in an expanded version as

$$\begin{aligned} \partial_\alpha \bar{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) &= \binom{n+2}{2} \int_k \dot{C}^\alpha(k; m) \bar{\mathcal{L}}_{n+2}^{\alpha_0, \alpha}(k, -k, p_1, \dots, p_n) \\ &- \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 \mathbb{S} \left[\bar{\mathcal{L}}_{n_1}^{\alpha_0, \alpha}(p_1, \dots, p_{n_1-1}, q) \dot{C}^\alpha(q; m) \bar{\mathcal{L}}_{n_2}^{\alpha_0, \alpha}(-q, p_{n_1}, \dots, p_n) \right] , \end{aligned} \quad (14)$$

with $q = p_{n_1} + \dots + p_n = -p_1 - \dots - p_{n_1-1}$, and where \mathbb{S} is the symmetrisation operator acting on functions of the momenta (p_1, \dots, p_n) by taking the *mean value* over all permutations π of $1, \dots, n$ satisfying $\pi(1) < \pi(2) < \dots < \pi(n_1 - 1)$ and $\pi(n_1) < \dots < \pi(n)$.

⁵See for example [13] for the derivatives of Gaussian measures depending differentiably on a parameter.

Note again that for the theory proposed through (4), only even moments (i.e. even in n, n_1, n_2) will be nonvanishing due to the symmetry $\varphi \rightarrow -\varphi$. The infinite system of equations (14) then constitutes an infinite dimensional nonlinear dynamical system.

The CAS functions are defined uniquely as a solution to these differential equations on imposing suitable boundary conditions. Noting that $L^{\alpha_0, \alpha_0} = L_0$, these are fixed through the choice of the constants $c_{0,n}$ in L_0 (4). The CAS functions are then obtained by integrating the flow equations subject to the boundary conditions. For an existence and uniqueness proof in the context of perturbation theory see e.g. [18, 20, 27]. There also exist farther reaching results like e.g. large momentum bounds [23], bounds on large orders in perturbation theory [21], applications to finite temperature field theory [24], application to nonabelian gauge theories [6], or a proof of convergence of the operator product expansion [15]. The transition to Minkowski space is analysed in [19].

We have already mentioned the so-called triviality theorems which say that the continuum limit of lattice regularised scalar φ^4 theory in four dimensions [9, 1, 2]⁶ is (generalised) Gaussian. This statement has its counter part in perturbation theory in the so-called Landau pole which is related to the positive sign of the perturbative β -function of the theory. This sign leads to a growth of the effective coupling with the energy scale so that perturbation theory becomes unreliable. Using naively the Callan-Symanzik equations leads to a pole of the two-point function at high energies which violates unitarity.

The main point of this paper is to prove in the mean field context that irrelevant terms in the bare action can modify this behaviour and lead to UV asymptotically free theories. We shortly illustrate the mechanism which is behind our result by a lowest order perturbative calculation. To adapt the flow equations to this case we have to expand them also w.r.t. the number of loops $\ell \geq 0$. We use BPHZ type renormalization conditions at zero external momentum. Also, to simplify a bit, we set $m = 0$ and write $C(q) \equiv C(q; 0)$ for the massless propagator. In turn we impose the renormalisation conditions at $\alpha = \frac{1}{m^2}$ (in suitable units) and not at $\alpha = \infty$. This corresponds to a slight rearrangement on the infrared side without affecting the small α , i.e. ultraviolet properties. We also set

$$\hat{\mathcal{L}}_n = n! \mathcal{L}_n \quad (15)$$

which is the normalisation used in perturbation theory, The loop-expanded FEs then read

$$\begin{aligned} \partial_\alpha \hat{\mathcal{L}}_{n,\ell}^{\alpha_0, \alpha}(p_1, \dots, p_n) &= \frac{1}{2} \int_k \dot{C}^\alpha(k) \hat{\mathcal{L}}_{n+2,\ell-1}^{\alpha_0, \alpha}(k, -k, p_1, \dots, p_n) \\ &- \frac{1}{2} \sum_{\substack{n_1+n_2=n+2 \\ \ell_1+\ell_2=\ell}} \hat{\mathbb{S}} \left[\hat{\mathcal{L}}_{n_1,\ell_1}^{\alpha_0, \alpha}(p_1, \dots, p_{n_1-1}, q) \dot{C}^\alpha(q) \hat{\mathcal{L}}_{n_2,\ell_2}^{\alpha_0, \alpha}(-q, p_{n_1}, \dots, p_n) \right]. \end{aligned} \quad (16)$$

⁶in the symmetric phase and without additional irrelevant terms in the bare action

Here $\hat{\mathbb{S}}$ indicates not the mean value, but the *sum* over the rearrangements defined after (14), due to our change of normalization (15).

For the one-loop four-point function at zero external momentum this gives

$$\begin{aligned} \partial_\alpha \hat{\mathcal{L}}_{4,1}^{\alpha_0,\alpha}(0, \dots, 0) &= \frac{1}{2} \int_k \dot{C}^\alpha(k) \hat{\mathcal{L}}_{6,0}^{\alpha_0,\alpha}(k, -k, 0, \dots, 0) \\ &\quad - \frac{1}{2} \cdot 4 \hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(0, \dots, 0) \dot{C}^\alpha(0) \hat{\mathcal{L}}_{2,1}^{\alpha_0,\alpha}(0, 0) . \end{aligned} \quad (17)$$

First consider pure φ_4^4 -theory where $\hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(0, \dots, 0) = g$, $\hat{\mathcal{L}}_{6,0}^{\alpha_0,\alpha_0}(0, \dots, 0) = 0$. The terms on the r.h.s. of (17) are obtained from

$$\begin{aligned} \partial_\alpha \hat{\mathcal{L}}_{6,0}^{\alpha_0,\alpha}(k, -k, 0, \dots, 0) &= -\frac{1}{2} \cdot 8 \hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(k, -k, 0, 0) \dot{C}^\alpha(0) \hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(0, \dots, 0) \\ &\quad - \frac{1}{2} \cdot 12 \hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(0, 0, 0, k) \dot{C}^\alpha(k) \hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(-k, 0, 0, 0) , \\ \partial_\alpha \hat{\mathcal{L}}_{2,1}^{\alpha_0,\alpha}(0, 0) &= \frac{1}{2} \int_k \dot{C}^\alpha(k) \hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(k, -k, 0, 0) \end{aligned}$$

so that we find for pure φ_4^4 -theory

$$\begin{aligned} \partial_\alpha \hat{\mathcal{L}}_{4,1}^{\alpha_0,\alpha}(0, \dots, 0) &= -2g^2 \int_k \dot{C}^\alpha(k) \int_{\alpha_0}^\alpha d\alpha' \dot{C}^{\alpha'}(0) - 3g^2 \int_k \dot{C}^\alpha(k) \int_{\alpha_0}^\alpha d\alpha' \dot{C}^{\alpha'}(k) \\ &\quad - 2g^2 \dot{C}^\alpha(0) \left(- \int_\alpha^{\frac{1}{m^2}} d\alpha' \int_k \dot{C}^{\alpha'}(k) \right) - 4g \dot{C}^\alpha(0) \hat{\mathcal{L}}_{2,1}^{\alpha_0, \frac{1}{m^2}}(0, 0) . \end{aligned}$$

For the last two terms remember that we impose BPHZ renormalisation conditions at $\alpha = \frac{1}{m^2}$. Those conditions imply $\hat{\mathcal{L}}_{2,1}^{\alpha_0, \frac{1}{m^2}}(0, 0) = 0$. Performing the integrals then gives

$$\begin{aligned} \alpha \partial_\alpha \hat{\mathcal{L}}_{4,1}^{\alpha_0,\alpha}(0, \dots, 0) &= -2g^2 \frac{1}{16\pi^2} \left(1 - \frac{\alpha_0}{\alpha}\right) - 3g^2 \frac{1}{16\pi^2} \left(\frac{1}{2} \frac{\alpha - \alpha_0}{\alpha + \alpha_0}\right) + 2g^2 \frac{1}{16\pi^2} (1 - \alpha m^2) \\ &= -\frac{g^2}{16\pi^2} \left(\frac{3}{2} + 2\alpha m^2 + \mathcal{O}\left(\frac{\alpha_0}{\alpha}\right)\right) \xrightarrow{\alpha m^2 \rightarrow 0, \alpha_0 \rightarrow 0} -\frac{3}{2} \frac{g^2}{16\pi^2} , \end{aligned} \quad (18)$$

which is the standard value for the one-loop β -function of φ_4^4 -theory. As a consequence the effective coupling $\hat{\mathcal{L}}_{4,1}^{\alpha_0,\alpha}(0, \dots, 0)$ grows logarithmically with the energy scale (squared) $\frac{1}{\alpha}$ for large energies.

If on the other hand we set $\hat{\mathcal{L}}_{4,0}^{\alpha_0,\alpha}(0, \dots, 0) = 0$ but impose a nonzero boundary value for the six-point function setting

$$\hat{\mathcal{L}}_{6,0}^{\alpha_0,\alpha_0}(p_1, \dots, p_6) = g_6 \alpha_0$$

then (17) gives

$$\alpha \partial_\alpha \hat{\mathcal{L}}_{4,1}^{\alpha_0, \alpha}(0, \dots, 0) = \frac{1}{2} \frac{g_6}{16\pi^2} \frac{\alpha_0}{\alpha} > 0, \quad \hat{\mathcal{L}}_{4,1}^{\alpha_0, \alpha}(0, \dots, 0) = \frac{1}{2} \frac{g_6}{16\pi^2} \left(1 - \frac{\alpha_0}{\alpha}\right) \xrightarrow{\alpha_0 \rightarrow 0} \frac{1}{2} \frac{g_6}{16\pi^2}.$$

For $g_6 > 0$ we thus get to lowest order an effective four-point coupling decreasing with energy. This lowest order perturbative argument is evidently not conclusive. In fact we now have to analyse the flow of the six-point function which will be negative unless we add also a bare term to the eight-point function etc.

Our subsequent analysis is devoted to unravel necessary conditions on the bare action to obtain an asymptotically free theory. It is nonperturbative but limited to the mean field case.

3 The mean field limit of the flow equations

The flow equations constitute an infinite dimensional nonlinear dynamical system. The system of functions $\mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$ is defined on configuration spaces whose dimension also goes to infinity for $n \rightarrow \infty$. The mean field limit implies a drastic simplification of this system. It is obtained by replacing the functions $\mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$ by constants $A_n^{\alpha_0, \alpha}$ corresponding to their zero momentum values. So one sets all external momenta equal to zero in (14). In particular one replaces $\mathcal{L}_{n+2}^{\alpha_0, \alpha}(k, -k, p_1, \dots, p_n)$ in the integral by $\mathcal{L}_{n+2}^{\alpha_0, \alpha}(0, \dots, 0)$. We hope that this simplification nevertheless captures essential aspects of the behaviour of the full dynamical system. This hope is in particular based on the fact that the simplification amounts to replacing the derived propagator in the second term on the r.h.s of (14) by 1. In fact we have

$$0 < \dot{C}^\alpha(q; m) = e^{-\alpha(q^2 + m^2)} \leq e^{-\alpha m^2} \leq 1. \quad (19)$$

So the full system is obtained from the simplified one by contracting the second term on the r.h.s. in a momentum dependent manner. Controlling this contraction is maybe not completely out of range. In this respect we also remind the fact that the critical behaviour in statistical physics is exactly described by the mean field approximation in $d > 4$ dimensions [9, 1], as was first pointed out by Ginzburg [12]. The hard technical problems one is faced with when going beyond mean field, are in particular due to the fact that the n -point functions we want to construct have to respect Bose symmetry and euclidean invariance.

A rather mild second simplification which we will adopt, consists in choosing $m = 0$ and in restricting in turn our analysis to the interval $\alpha \in [\alpha_0, 1]$ ⁷ to avoid infrared

⁷on choosing units such that the original mass satisfies $m^2 = 1$

problems. It should then be a straightforward extension of the present analysis to take the limit $\alpha \rightarrow \infty$ while keeping $m > 0$.

The system of mean field FEs (21) below is obtained from (14) by setting all external momenta and m equal to zero writing

$$A_n^{\alpha_0, \alpha} := \mathcal{L}_n^{\alpha_0, \alpha}(0, \dots, 0) . \quad (20)$$

The mean field effective action $L_{\text{mf}}^{\alpha_0, \alpha}$ action then takes the form (as a formal power series)

$$L_{\text{mf}}^{\alpha_0, \alpha}(\phi) = \sum_n A_n^{\alpha_0, \alpha} \phi^n ,$$

where $\phi \in \mathbb{R}$. The k -integral in (14) can now be carried out explicitly, and we get for $n \in 2\mathbb{N}$

$$\partial_\alpha A_n^{\alpha_0, \alpha} = \binom{n+2}{2} c_\alpha A_{n+2}^{\alpha_0, \alpha} - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 A_{n_1}^{\alpha_0, \alpha} A_{n_2}^{\alpha_0, \alpha} , \quad (21)$$

where the sum, here and subsequently, can always be restricted to even values of n_1, n_2 . Furthermore

$$c_\alpha := \frac{c}{\alpha^2} \quad \text{with} \quad c := \frac{1}{16\pi^2} \quad (22)$$

so that c_α is the value at $m = 0$ of

$$c_\alpha(m) := \int_k \dot{C}^\alpha(k; m) = \frac{1}{16\pi^2} \frac{1}{\alpha^2} e^{-\alpha m^2} . \quad (23)$$

It is useful to factor out the basic scaling behaviour w.r.t. α and a combinatoric factor, setting

$$A_n^{\alpha_0, \alpha} =: \alpha^{n/2-2} \frac{1}{n} a_n(\alpha) , \quad (24)$$

where we suppressed the variable α_0 on the r.h.s. for shortness. In terms of the functions $a_n(\alpha)$, our dynamical system can be rewritten as

$$a_{n+2}(\alpha) = \frac{1}{(n+1)c} \sum_{n_1+n_2=n+2} a_{n_1}(\alpha) a_{n_2}(\alpha) + \frac{n-4}{n(n+1)c} a_n(\alpha) + \frac{2}{n(n+1)c} \alpha \partial_\alpha a_n(\alpha) . \quad (25)$$

This system permits to construct the functions $a_n(\alpha)$ inductively in n if the function $a_2(\alpha)$ is known. We make another change of variables in order to also factor out the $1/c$ factors

$$f_n(\mu) := c^{n/2-1} a_n(\alpha) = \alpha^{2-n/2} c^{n/2-1} n A_n^{\alpha_0, \alpha} , \quad \mu := \ln\left(\frac{\alpha}{\alpha_0}\right) . \quad (26)$$

The system (25) can then be rewritten

$$f_{n+2} = \frac{1}{n+1} \sum_{n_1+n_2=n+2} f_{n_1} f_{n_2} + \frac{n-4}{n(n+1)} f_n + \frac{2}{n(n+1)} \partial_\mu f_n, \quad \mu \in [0, \ln \frac{1}{\alpha_0}]. \quad (27)$$

We have again suppressed the dependence on α_0 in the notation, and in fact the solutions which we are going to construct, will depend on μ only.

Making the functions f_2 and f_4 more explicit, this gives equivalently

$$f_4 = \frac{1}{3} f_2 (f_2 - 1) + \frac{1}{3} \partial_\mu f_2, \quad (28)$$

$$f_{n+2} = \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} f_{n_1} f_{n_2} + \frac{1}{n+1} f_n [2f_2 + 1 - \frac{4}{n}] + \frac{2}{n(n+1)} \partial_\mu f_n, \quad n \geq 4. \quad (29)$$

Smooth solutions of the dynamical system (28), (29) are fixed if we choose a smooth function $f_2(\mu)$. In perturbative quantum field theory one primarily considers the flow of the four-point function which is represented by $f_4(\mu)$. From (28) we realise that we may first fix $f_4(\mu)$ and then solve the differential equation (28) for $f_2(\mu)$ to obtain a solution for $f_2(\mu)$.

We add a few general remarks in relation with the structure of the system (28), (29).

- The first remark concerns what one might call the *combinatorial instability* of the system. By this we mean that trying to solve the system by iterated integration starting from a first educated guess on the f_n , will not define a convergent procedure. In fact the prefactor $\frac{2}{n(n+1)}$ in front of $\partial_\mu f_n$, will lead to a blow-up w.r.t. n on iteration, unless one were able take into account cancellations of terms of opposite sign. But this is typically beyond scope. The remark applies also when considering scalar (mean) field theory in lower dimensions.
- As a consequence of the previous statement we rather proceed in a different way: We start by fixing $f_2(\mu)$ and construct the higher n -point functions from the two-point function. This will permit us to find smooth solutions of the system (29). When trying to apply this procedure to the full system (14), one is faced with the problem of how to define a function $\mathcal{L}_{n+2}^{\alpha_0, \alpha}(p_1, \dots, p_{n+2})$ on prescribing its integrals

$$\int_k \dot{C}^\alpha(k; m) \mathcal{L}_{n+2}^{\alpha_0, \alpha}(k, -k, p_1, \dots, p_n).$$

The function $\mathcal{L}_{n+2}^{\alpha_0, \alpha}(p_1, \dots, p_{n+2})$ has to be Bose symmetric and symmetric under the euclidean group, in particular translation invariant. It also should have good

analyticity properties as required by a full-fledged quantum field theory, which can be analytically continued to Minkowski space. A hard challenge is to identify the conditions which determine these functions uniquely in agreement with the axioms of quantum field theory.

- We also mention in this context the so-called hierarchy problem of scalar field theory. It consists in the observation that in perturbative scalar field theory the two-point function diverges quadratically with the UV cutoff $\Lambda_0 = \alpha_0^{-1/2}$, as suggested by (26). In fact it is the only term diverging stronger than logarithmically in perturbation theory, even when inspecting the whole of the standard model of particle physics. It is then argued that this divergence leads to a fine-tuning problem when viewing Λ_0 as a very high energy scale (“the Planck mass”) since fixing the mass of the Higgs particle associated to the scalar field at its much lower physical value requires fine-tuning of the corresponding counter terms. Consequently this quadratic divergence is often cited as a motivation for supersymmetric (or other) extensions of the standard model where the perturbative divergences are only logarithmic. Once we look at the rescaled system (28), (29) - the same rescaling can be performed for the full system (14) - this quadratic divergence disappears. In the scale free system there is no particular instability linked to the two-point function. To some degree it seems that the precedent fine-tuning problem is only palpable if one attributes direct physical significance to the mass counter terms. On the other hand supersymmetric cancellations appear to be due to a subtle fine-tuning procedure from the point of view of the corresponding dynamical FE system.

4 Solutions of the mean field equations

We will consider solutions of (28), (29) which are smooth functions of the renormalisation group scale μ in the interval $[0, \ln \frac{1}{\alpha_0}]$. The existence of the ultraviolet limit means that the system of solutions has a finite limit for $\alpha = 1 \Leftrightarrow \mu = \ln \frac{1}{\alpha_0}$ (“where all degrees of freedom have been integrated out”) when the UV cutoff $1/\alpha_0$ is sent to infinity. In other words claiming the existence of a mean field solution of the FEs in the UV limit is tantamount to prove that

$$\text{the limits } \lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) \text{ exist for all } n, \text{ where } \mu_{max} = \ln \frac{1}{\alpha_0}. \quad (30)$$

The solutions studied in 4.1 and in 4.2 are the simplest to obtain. We will show that there exist smooth solutions $f_n(\mu)$, uniformly bounded w.r.t. μ , which vanish at $\mu = 0$

when taking the UV limit $\alpha_0 \rightarrow 0$. They are thus asymptotically free in the ultraviolet. For these solutions we however do not control the signs of the n -point functions, not even at $\mu = 0$, i.e. for the bare action. We find a bare action which is nonpolynomial, and its moments are not necessarily positive. From the functional integral point of view the existence of the bare action for an arbitrary field configuration in the support of the Gaussian measure is therefore not assured. And for the (mean) field configurations for which the bare action exists, we do not know whether it is uniformly bounded from below.

In section 4.3 we will then study solutions with strictly positive boundary conditions at $\mu = 0$ for all n -point functions. So the bare action is nonpolynomial, and all of its moments have positive coefficients. The bare action is bounded from below (by 0) whenever it is well-defined. The solutions we obtain are again ultraviolet asymptotically free. Still the bare action (restricted to finite volume) may be not well-defined for all admissible field configurations - or for all values of ϕ in the mean field limit - since it may diverge due to its nonpolynomial character.

We therefore study in section 4.4 solutions, the boundary conditions of which, while being again nonpolynomial, can be resummed into bounded functions of the field variable and lead to well-defined bare actions in the (finite volume) path integral. These actions are also bounded from below. So the (regularised) path integral exists, and the mean field bare action is globally well-defined. The mean field solutions from 4.4 constitute subclasses of those considered in 4.1 and 4.2. We show in particular that there exist UV asymptotically free mean field solutions with well-defined path integral. The proof requires much sharper restrictions on the couplings than those needed in 4.1 and 4.2. Section 4.4 presents the main result of this paper.

Finally we study the boundary conditions of pure φ^4 theory in section 4.5. The mean field solutions corresponding to these boundary conditions have alternating signs (at least for small μ) and large μ -derivatives which is related to the aforementioned fine-tuning of the boundary conditions required for polynomial bare actions.

We shall find that with the exception of 4.4, the upper bounds on the coupling constants required in the existence proofs of the solutions are quite moderate when compared to constructive field theory upper bounds which typically are “astronomically small” (like exponentials of a very big negative number) due to the high complexity of the contributions from iterated cluster expansions. The upper bounds from 4.4 are however much smaller and not really made explicit. This is because the proof of Theorem 1 is delicate. We did not try to optimise the bounds w.r.t. the size of the couplings, also for the sake of readability. More realistic upper bounds should be attainable with reasonable effort.

4.1 Bounded mean field solutions

The simplest solutions of (28), (29) are scale-invariant solutions for which

$$\partial_\mu f_2 \equiv 0 . \quad (31)$$

By induction it then follows directly from (28), (29) that

$$\partial_\mu f_n \equiv 0 \quad \forall n \quad (32)$$

so that we obtain the μ independent system

$$f_4 = \frac{1}{3} f_2 (f_2 - 1) , \quad f_{n+2} = \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} f_{n_1} f_{n_2} + \frac{1}{n+1} f_n [2f_2 + 1 - \frac{4}{n}] , \quad n \geq 4 . \quad (33)$$

The solutions of (33) are fully determined on imposing the value of f_2 .

In [7] Felder has analysed rigorously and in great generality global solutions of the differential equation

$$\frac{1}{2} u_{xx} - \frac{d-2}{2} x u_x + du - \frac{1}{2} u_x^2 = 0 \quad (34)$$

in dimensions $2 < d < 4$. The system (33) is obtained when taking $d \rightarrow 4$, by expanding a solution $u(x)$ in moments according to ⁸

$$u(x) = \sum_{n \geq 0} \frac{f_n}{n} x^n . \quad (35)$$

Felder's conclusion is that for $d \rightarrow 4 - 0$ the nontrivial fixed point solution u_4 found in dimensions $3 < d < 4$ tends to zero. Felder's analysis does not exclude the existence of fixed points other than those he found explicitly. If other fixed points can be excluded, then the conclusion to be drawn from [7] and our results is that the moment expansion (35) is not valid for arbitrarily large $x \in \mathbb{R}$ or that the solution u obtained from the momentum expansion is not sufficiently regular to satisfy (34).

It is instructive to look at the fixed point solutions of (33), i.e. on performing the moment expansion. We consider different cases as regards the value of f_2 .

a) $0 < |f_2| \leq \varepsilon \ll 1$

In this regime we find that $f_4 = \mathcal{O}(\varepsilon)$, with sign opposite to that of f_2 , $f_6 = \mathcal{O}(\varepsilon^2)$ with negative sign, and $f_n > 0$, $f_n = \mathcal{O}(\varepsilon)$ for $n \geq 8$. So we have an action bounded from below. This regime is not perturbative, in the sense that f_n is not of increasing order in ε for increasing

⁸There is a slight difference of normalisation since Felder's flow parameter t satisfies $2t = \mu$. As a consequence, the second term on the r.h.s. of (27) is multiplied by 2.

n . The $|f_n|$ for $n \geq 6$ are bounded by ε^2 times a numerical coefficient becoming small ($\ll 1$) rapidly for increasing n .

b) $0 < f_2 < 1$

In this case $f_4 < 0$. Generally the f_n may have either sign, depending on n .

c) $f_2 = 1$

We find $f_n = 0 \quad \forall n \geq 4$, i.e. a “free theory”. This is the so-called high temperature fixed point.

d) $f_2 > 1$

By induction on n one finds that the coefficients f_n of the system (33) satisfy

- $f_n > 0$, so the action is bounded from below.
- f_n are strictly increasing when viewed as functions of f_2 and geometrically bounded by a constant to the power n .

In particular for $f_2 = 1 + \varepsilon$ with $0 < \varepsilon \ll 1$ we find $f_4 = \frac{1}{3}(1 + \varepsilon)\varepsilon$, $f_6 = \frac{2}{15}(1 + \varepsilon)^2\varepsilon$, $f_n = O(\varepsilon) \quad \forall n \geq 4$. The $|f_n|$ for $n \geq 6$ are bounded by ε times a numerical coefficient becoming small ($\ll 1$) rapidly for increasing n .

e) $f_2 < 0$

In this case we do not control the signs of the f_n . The $|f_n|$ may become large in modulus for large f_2 .

It is relatively straightforward to see (and follows from the subsequent bounds) that the moment expansion (35) has (at least) a finite radius of convergence for f_2 close to 0 or close to 1.

We now study more general solutions for which all $|f_n(\mu)|$ are uniformly bounded⁹. We consider a smooth two-point function satisfying

$$-K_1 \delta \leq f_2(0) \leq -\delta < 0, \quad |\partial_\mu^l f_2(\mu)| \leq \frac{K_1^l \delta^{l+1}}{(l+1)^2} l! \quad \forall \mu \in [0, \ln \frac{1}{\alpha_0}] \quad \text{and} \quad \forall l \geq 0 \quad (36)$$

for some fixed $0 < \delta \leq 1$, $K_1 > 1$ being a positive constant. We restrict ourselves for simplicity and definiteness to the interval $0 < \delta \leq 1$, but larger values could be analysed similarly. We note that the sign of $f_2(0)$ in (36) is in agreement with the sign of the mass counter term in perturbative φ^4 -theory at lowest order.

Proposition 4.1 *For suitable $K \geq \sup(K_1, 4)$ and $f_2(\mu)$ satisfying (36), the functions $f_n(\mu)$ solving (29) are smooth and satisfy for $\mu \in [0, \ln \frac{1}{\alpha_0}]$, $n \in 2\mathbb{N} + 2$, $l \geq 0$,*

$$|\partial_\mu^l f_n(\mu)| \leq \frac{K^{n+l-2} \delta^{l+1}}{(l+1)^2} \frac{(n+l-2)!}{(n-2)!}. \quad (37)$$

⁹It is possible to choose the initial conditions such that $|f_n(\mu)| \leq 1 \quad \forall n, \mu$.

Remark.

Assuming that $f_2(\mu)$, and then also all $f_n(\mu)$ are defined for complex values of μ , the bound (37) implies that the $f_n(\mu)$ are analytic in a strip of width $\frac{2}{\delta K}$ around the real μ -axis for $0 \leq \mu \leq \ln \frac{1}{\alpha_0}$. The mean field action $L_{\text{mf}}^{\alpha_0, \alpha}(\phi)$ is then analytic w.r.t. the field variable ϕ in a disk of radius $(c/\alpha K^2)^{1/2}$ uniformly in α_0 , see (41) below.

Proof. The proof is by induction in $n + l \geq 2$.

The bounds hold for the two-point function by assumption (36). Verification of the bounds on $\partial_\mu^l f_4(\mu)$ using (28) is straightforward and simpler than the general case $n \geq 4$. So we leave this case to the reader. For $n \geq 4$ we insert the induction hypothesis on the r.h.s. of (29), derived l times w.r.t. μ . This gives the bound

$$\begin{aligned}
& \left| \partial_\mu^l f_{n+2}(\mu) \right| \leq \\
& \frac{\delta^{l+2}}{n+1} \sum_{\substack{n_1+n_2=n+2, n_i \geq 4 \\ l_1+l_2=l}} \binom{l}{l_1} \frac{K^{n+l-2}}{(l_1+1)^2 (l_2+1)^2} \frac{(n_1+l_1-2)!}{(n_1-2)!} \frac{(n_2+l_2-2)!}{(n_2-2)!} \\
& + \frac{\delta^{l+1}}{n+1} \sum_{l_1+l_2=l} \binom{l}{l_1} \frac{K^{n+l_1-2}}{(l_1+1)^2} \frac{(n+l_1-2)!}{(n-2)!} \left[\delta \frac{K^{2+l_2-2}}{(l_2+1)^2} 2(2+l_2-2)! + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \\
& + \frac{2\delta^{l+2}}{n(n+1)} \frac{K^{n+l+1-2}}{(l+2)^2} \frac{(n+l+1-2)!}{(n-2)!}.
\end{aligned} \tag{38}$$

From the standard bound (all entries are supposed to be nonnegative integers)

$$\binom{l}{l_1} \binom{n-2}{n_1-2} \leq \binom{n-2+l}{n_1-2+l_1} \tag{39}$$

which follows from Vandermonde's identity, we obtain

$$\binom{l}{l_1} \frac{(n_1+l_1-2)!}{(n_1-2)!} \frac{(n_2+l_2-2)!}{(n_2-2)!} = \binom{l}{l_1} \binom{n-2}{n_1-2} \frac{(n_1+l_1-2)!(n_2+l_2-2)!}{(n-2)!} \leq \frac{(n+l-2)!}{(n-2)!}.$$

This then allows to deduce from (38)

$$\begin{aligned}
& \left| \partial_\mu^l f_{n+2}(\mu) \right| \leq \frac{\delta^{l+2}}{n+1} \sum_{\substack{n_1+n_2=n+2, n_i \geq 4 \\ l_1+l_2=l}} \frac{K^{n+l-2}}{(l_1+1)^2 (l_2+1)^2} \frac{(n+l-2)!}{(n-2)!} \\
& + \frac{\delta^{l+1}}{n+1} \sum_{l_1+l_2=l} \frac{K^{n+l-2}}{(l_1+1)^2} \frac{(n+l-2)!}{(n-2)!} \left[\frac{2\delta}{(l_2+1)^2} + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \\
& + \frac{2\delta^{l+2}}{n(n+1)} \frac{K^{n+l+1-2}}{(l+2)^2} \frac{(n+l+1-2)!}{(n-2)!}.
\end{aligned} \tag{40}$$

We may use the bounds

$$\frac{1}{(l_1 + 1)^2 (l_2 + 1)^2} \leq \frac{2}{(l + 1)^2} \left(\frac{1}{(l_1 + 1)^2} + \frac{1}{(l_2 + 1)^2} \right), \quad \sum_{l \geq 1} \frac{1}{l^2} \leq 2$$

to get

$$\sum_{\substack{n_1 + n_2 = n + 2, n_i \geq 4 \\ l_1 + l_2 = l}} \frac{1}{(l_1 + 1)^2 (l_2 + 1)^2} \leq \frac{8n}{(l + 1)^2}.$$

Choosing K sufficiently large such that for $n \geq 6$

$$\frac{\delta}{n + 1} \sum_{\substack{n_1 + n_2 = n + 2, n_i \geq 4 \\ l_1 + l_2 = l}} \frac{1}{(l_1 + 1)^2 (l_2 + 1)^2} \leq \frac{1}{3} \frac{K^2}{(l + 1)^2} \frac{(n + l)(n + l - 1)}{n(n - 1)}$$

and such that

$$\frac{1}{n + 1} \sum_{l_1 + l_2 = l} \frac{1}{(l_1 + 1)^2 (l_2 + 1)^2} [2\delta + \delta_{l_2, 0} (1 - \frac{4}{n})] \leq \frac{K^2}{2(l + 1)^2} \frac{(n + l)(n + l - 1)}{n(n - 1)}$$

and such that

$$\frac{2\delta}{n(n + 1)} \frac{1}{(l + 2)^2} \leq \frac{1}{6} K \frac{1}{(l + 1)^2} \frac{n + l}{n(n - 1)}$$

we find that (38) is bounded by

$$\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{6} \right) \frac{\delta^{l+1} K^{n+l}}{(l + 1)^2} \frac{(n + l)!}{n!}.$$

One can straightforwardly convince oneself that $K = 4$ is admissible for $\delta = 1$, $K_1 \leq 4$. Smaller values of K are allowed if $\delta < 1$, $K_1 < 4$. \square

Going back to the dynamical system (21) we obtain from the set of smooth functions $f_n(\mu)$, the system of smooth functions $A_n^{\alpha_0, \alpha}$. If the functions $f_n(\mu)$ satisfy the bounds from Proposition 4.1, then the $A_n^{\alpha_0, \alpha}$ satisfy uniformly in α_0 the bounds

$$|A_n^{\alpha_0, \alpha}| \leq \delta \left(\frac{\alpha K^2}{c} \right)^{\frac{n-2}{2}} \frac{1}{\alpha n} \quad \text{for } 0 < \alpha_0 \leq \alpha \leq 1. \quad (41)$$

So the mean field action

$$L_{\text{mf}}^{\alpha_0, \alpha}(\phi) = \sum_{n \in 2\mathbb{N}} A_n^{\alpha_0, \alpha} \phi^n \quad (42)$$

is analytic in a disk of radius $[\frac{c}{\alpha K^2}]^{\frac{1}{2}}$.

4.2 Asymptotically free mean field solutions

Generally ultra-violet asymptotically free theories are those for which the effective couplings, as a function of the energy scale, tend to zero if the energy scale tends to infinity. In the perturbative analysis of the field theories from particle physics this expresses itself in the fact that the so-called β -function has a negative leading term when expanded in a (formal) power series with respect to the renormalised couplings. In ϕ^4 scalar theory this renormalised coupling, generally called g , is related to the four-point function, the flow of which is then viewed as a function of the energy scale, see also the discussion after (17). The β -function controls this flow of the effective coupling $g(\lambda)$ as a function of the logarithmic energy scale λ via

$$\frac{dg}{d\lambda} = \beta(g(\lambda)) .$$

If β is negative, the coupling vanishes logarithmically for $\lambda \rightarrow \infty$,

$$g(\lambda) \rightarrow 0 \quad \text{for} \quad \lambda \rightarrow \infty : \quad \text{asymptotic freedom} .$$

Here $g(0)$ is the finite (positive) renormalised coupling. Since in renormalised perturbation theory, all higher n -point functions are expanded with respect to the renormalised coupling, asymptotic freedom in the perturbative context implies that all these functions asymptotically vanish at high energies.

In the mean field context the logarithmic energy scale is given by the variable μ . The ultraviolet limit corresponds to $\mu = 0$, the relation between μ and λ is

$$\lambda = \mu_{\max} - \mu .$$

To stay close to previous notations we may set

$$\tilde{f}_n(\lambda) = f_n(\mu)|_{\mu=\mu_{\max}-\lambda} .$$

Note that the $\tilde{f}_n(\lambda)$ also depend on the parameter μ_{\max} which is the ultra-violet cutoff. This cutoff is to be sent to infinity while keeping the physical couplings at $\lambda = 0$ fixed. Asymptotically free solutions fulfill

$$\lim_{\lambda \rightarrow \infty} \tilde{f}_n(\lambda) = 0 \quad \forall n \in 2\mathbb{N} , \tag{43}$$

while the physical couplings $\tilde{f}_n(0)$ are kept fixed. In our nonperturbative context the previous statement (43) directly includes all n -point functions since we do not (necessarily) expand the higher n -point functions in terms of the four-point function.

On the contrary, solutions for which

$$\lim_{\mu_{\max} \rightarrow \infty} \tilde{f}_n(0) = 0 \quad \forall n \in 2\mathbb{N} , \tag{44}$$

while the $\tilde{f}_n(\mu_{max})$ are kept fixed, are called trivial. We come back to the trivial solution in section 4.5.

In our analysis it is more natural to stay with the variable μ , since we are posing our boundary conditions at $\mu = 0$. We want to show now that there is a subclass of asymptotically free solutions among those from Proposition 4.1. These solutions describe a mean field theory in the sense that the solutions $f_n(\mu_{max})$ have a well-defined limit for $\mu_{max} \rightarrow \infty \Leftrightarrow \alpha_0 \rightarrow 0$. We choose

$$f_2(\mu) = -\delta(\mu), \quad \delta(\mu_{max}) = \delta, \quad \partial_\mu \delta(\mu) = \beta \delta^2(\mu), \quad 0 < \delta, \quad \beta < \frac{1}{2}, \quad \mu \in [0, \ln \frac{1}{\alpha_0}] . \quad (45)$$

Here $\beta \delta^2(\mu)$ plays the role of the β -function from the previous discussion.

The well-known solution of (45) is

$$\delta(\mu) = \frac{\delta}{1 + (\mu_{max} - \mu)\beta \delta} . \quad (46)$$

Evidently (45) and (46) verify the assumptions of Proposition 4.1. We have in particular

$$\lim_{\mu_{max} \rightarrow \infty} f_2(\mu_{max}) = -\delta, \quad \lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_2(\mu_{max}) = -\beta^l l! \delta^{l+1} . \quad (47)$$

By straightforward induction in $n + l$, proceeding as in the proof Proposition 4.1, we then find that the limits $\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max})$, $n \geq 4$, also exist and obey the bounds of Proposition 4.1. We collect our findings in

Proposition 4.2 *Among the solutions from Proposition 4.1 there are nontrivial asymptotically free solutions, for which the following relations hold*

$$f_2(\mu_{max}) < 0, \quad f_4(\mu_{max}) > 0, \quad (48)$$

$$\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) \text{ exists } \quad \forall n \in 2\mathbb{N}, \quad (49)$$

$$\lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_n(0) = 0 \quad \forall n \in 2\mathbb{N}, \quad l \in \mathbb{N}_0 . \quad (50)$$

Proof. When choosing boundary conditions (45), the second inequality in (48) is true if $\frac{1}{3} \delta(\mu_{max})(\delta(\mu_{max}) + 1) - \partial_\mu \delta(\mu_{max}) > 0$, which is the case for δ, β bounded as in (45). For these solutions the last statement (50) follows by induction proceeding as in the proof of Proposition 4.1. In fact it suffices to replace in this proof the powers of δ appearing in the inductive bound (38) by the respective powers of $\delta(\mu)$, and to note that for $\mu_{max} \rightarrow \infty$ we have $\delta(0) \rightarrow 0$.

□

We also remark that solutions of the type (45), but with β negative, lead to trivial theories, namely we find that

$$\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) = 0 , \quad (51)$$

if $\delta(0)$ is fixed to be positive and not too large. In fact one obtains in this case

$$\delta(\mu) = \frac{\delta(0)}{1 - \mu \beta \delta(0)} , \quad (52)$$

which vanishes for $\mu = \mu_{max} \rightarrow \infty$. This implies the vanishing of all f_n in this limit, using again the inductive scheme of proof of Proposition 4.1. We do not work out this point further here. We will come back to the triviality question for pure φ_4^4 in 4.5.

4.3 Asymptotically free mean field solutions with positive bare values

The bare actions constructed from the solutions $f_n(0)$ in 4.1 are generally not bounded from below. In this subsection we look at solutions for which all f_n are positive and monotonic in μ :

$$f_n(\mu) \geq 0 , \quad \partial_\mu f_n(\mu) \geq 0 . \quad (53)$$

The first inequalities for $\mu = 0$ assure positivity - and thus in particular boundedness from below - of the bare action, whenever it is well-defined.

We obtain smooth solutions of (28), (29) satisfying the conditions (53) on considering functions $f_2(\mu)$ such that

$$f_2(\mu) = 1 + \delta(\mu) , \quad 0 < \delta(\mu) \leq 1 , \quad \partial_\mu \delta = \beta \delta^2(\mu) , \quad 0 < \beta \leq 1 , \quad \mu \in [0, \ln \frac{1}{\alpha_0}] . \quad (54)$$

The main difference between (54) and (45) is that f_2 in (54) is not of order δ . So the solutions studied are nonperturbative from the beginning.

Rewriting (28) as

$$3 f_4(\mu) = (1 + \delta(\mu)) \delta(\mu) + \partial_\mu \delta(\mu) = \left(1 + (1 + \beta) \delta(\mu)\right) \delta(\mu) , \quad (55)$$

we see that the relations (54), (55) imply

$$\partial_\mu^l f_4(\mu) \geq 0 \quad \forall l . \quad (56)$$

Proposition 4.3 For suitable $K > 1$ and $f_2(\mu) = 1 + \delta(\mu)$ smooth, satisfying (54), the functions $f_n(\mu)$ are smooth, and satisfy for $\mu \in [0, \ln \frac{1}{\alpha_0}]$, $n \in 2\mathbb{N} + 2$, $l \geq 0$

$$0 < \partial_\mu^l f_n(\mu) \leq \delta(\mu) \frac{K^{n+l-2}}{(l+1)^2} \frac{(n+l-2)!}{(n-2)!} . \quad (57)$$

Proof. The proof is by induction in $n+l \geq 2$. Positivity follows immediately by inspecting the r.h.s. of (29) using (54) and (56). The bound does not contain higher powers of $\delta(\mu)$ as in Proposition 4.1 since f_2 is no more of order δ . Otherwise the proof follows strictly the one of Proposition 4.1. So we do not rewrite it. We find again that $K = 4$ is an admissible value. \square

From (21) and Proposition 4.3 we find again bounds uniform in the UV cutoff α_0 for the $A_n^{\alpha_0, \alpha}$

$$0 < A_n^{\alpha_0, \alpha} < (\delta_{n,2} + \delta(\mu)) \left(\frac{\alpha K^2}{c} \right)^{\frac{n-2}{2}} \frac{1}{\alpha n} \quad \text{for } 0 < \alpha_0 \leq \alpha \leq 1 . \quad (58)$$

The initial data $f_n(0) \geq 0$ assure the positivity of all moments of the bare action, which obey (58). The bound assures again analyticity of the mean field effective action (42) in a disk of radius $(\frac{c}{\alpha K^2})^{1/2}$.

The solutions studied in Proposition 4.3 are again asymptotically free. When choosing $\delta = \delta(\mu_{max}) > 0$ fixed, we have statements analogous to (46, 47, 48, 49, 50):

$$\delta(\mu) = \frac{\delta}{1 + (\mu_{max} - \mu)\beta\delta} , \quad (59)$$

$$\lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_2(\mu_{max}) = \delta_{l,0} + \beta^l l! \delta^{l+1} , \quad \lim_{\mu_{max} \rightarrow \infty} f_4(\mu_{max}) = \frac{1}{3}(1 + \delta)\delta + \beta\delta^2 , \quad (60)$$

$$\lim_{\mu_{max} \rightarrow \infty} f_n(\mu_{max}) \quad \text{exists and is positive} \quad \forall n \in 2\mathbb{N} , \quad (61)$$

$$\lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_2(0) = \delta_{l,0} , \quad \lim_{\mu_{max} \rightarrow \infty} \partial_\mu^l f_n(0) = 0 \quad \forall n \in 2\mathbb{N} + 2, l \in \mathbb{N}_0 . \quad (62)$$

These asymptotically free solutions seem quite special. An interesting task is to analyse the different classes of solutions more systematically. As regards the UV limit, the basic possibilities are nontrivial asymptotically free or safe (i.e. scale independent) solutions, or trivial solutions which are free at $\mu = \mu_{max}$.

In perturbative quantum field theory one generally analyses in a first place the scaling behaviour of the four-point function, and not that of the two-point function. When

imposing $f_4(\mu)$ and then solving the differential equation for the two-point function (55), which is of Riccati type, we find as a particular solution

$$f_2(\mu) = -3f_4(\mu) . \quad (63)$$

This implies that we can find all solutions of the Riccati equation.

For example, the solution satisfying $f_2(0) = 0$ is given by

$$f_2(\mu) = \frac{3 f_4(0) e^{\int_0^\mu (6f_4(\mu') + 1) d\mu'}}{1 + 3 f_4(0) \int_0^\mu d\mu' e^{\int_0^{\mu'} (6f_4(\mu'') + 1) d\mu''}} - 3 f_4(\mu) . \quad (64)$$

It satisfies

$$f_2(\mu) \geq 0 , \quad \partial_\mu f_2(\mu) \geq 0 , \quad f_2(0) = 0 .$$

Higher order derivatives of this solution are not positive for all values of μ so that the corresponding solutions of (29) cannot be expected to satisfy Proposition 4.3 for $l = 0$. Since the solution (64) corresponds to a vanishing mass counter term, it may well be that the ultraviolet limit for this solution does not exist. It would be interesting to know whether it is possible to construct along these lines asymptotically free solutions, in particular for the four-point function, satisfying Proposition 4.3 and which are such that the bare action is well defined and bounded from below.

4.4 Solutions of bounded action

The mean field actions $L_{\text{mf}}^{\alpha_0, \alpha}(\phi)$ corresponding to the solutions of the mean field FEs constructed in the previous sections 4.1, 4.2, 4.3 are analytic w.r.t. ϕ in a disk of radius $(\frac{c}{\alpha K^2})^{1/2}$. For those constructed in 4.3 all coefficients $A_n^{\alpha_0, \alpha}$ of the moment expansion are positive. But $L_0(\phi) = L_{\text{mf}}^{\alpha_0, \alpha_0}(\phi)$ is not defined for arbitrary $\phi \in \mathbb{R}$. And if we want to obtain this bare action as the mean field limit of an action functional $L_0(\varphi(x))$ of a complete scalar field theory, then $L_0(\varphi(x))$ is generally not well-defined on the whole of the support of the Gaussian measure, even in the presence of regulators, due to its nonpolynomial character.

The solutions we will construct in this section satisfy sufficiently strong bounds in order to assure well-defined bare field action functionals or bare mean field action functions, bounded from below. Since the estimates become more delicate the upper bounds on the coupling constants required, are much more restrictive. The solutions from this section will in fact be subclasses of those from section 4.1, 4.2.

The bare functional (4) $L_0(\varphi(x)) = \sum_{n \in 2\mathbb{N}} \int d^4x c_{0,n} \varphi^n(x)$ is local. The constants $c_{0,n}$ are related to the $A_n^{\alpha_0, \alpha_0}$ via

$$c_{0,n} = \frac{1}{(2\pi)^4} A_n^{\alpha_0, \alpha_0}$$

as can be seen from (9), (10), (20). For the $A_n^{\alpha_0, \alpha_0}$ we have deduced the bounds (41) resp. (58). The functional $L_0(\varphi)$ is well-defined for all φ in

$$\mathcal{D}(\alpha_0) = \left\{ \varphi \mid \varphi \in \bigcap_{n \in 2\mathbb{N}} L^n(\mathbb{R}^4, d^4x), \quad L_0(\varphi) < \infty \right\}.$$

If the bounds (41) resp. (58) hold, the set $\mathcal{D}(\alpha_0)$ contains

$$\mathcal{M}_\varepsilon(\alpha_0) := \left\{ \varphi \mid \varphi \in \bigcap_{n \in 2\mathbb{N}} L^n(\mathbb{R}^4, d^4x), \quad \limsup \|\varphi\|_n < \left(\frac{c}{\alpha_0 K^2}\right)^{1/2} - \varepsilon \right\} \subset \mathcal{D}(\alpha_0)$$

for arbitrarily small positive ε . The sets $\mathcal{M}_\varepsilon(\alpha_0)$ do not exhaust the support of the measure $\mu(C^{\alpha_0, \alpha})$ for finite α_0 . One might then be tempted to introduce one more regularisation by setting

$$\begin{aligned} V(\varphi) &\equiv e^{-L_0(\varphi)}, & \text{if } \varphi \in \text{supp } \mu(C^{\alpha_0, \alpha}) \cap \mathcal{D}(\alpha_0), \\ V(\varphi) &\equiv 0, & \text{if } \varphi \in \text{supp } \mu(C^{\alpha_0, \alpha}) - \mathcal{D}(\alpha_0). \end{aligned}$$

But $V(\varphi)$ is not differentiable w.r.t. φ , and it is thus no more possible to derive the FEs from the path integral by partial integration. In fact boundary terms appear where the potential $V(\varphi)$ is cut off.

To impose boundedness from the beginning we now consider local bare actions (as functionals) of the form

$$L_0(\varphi(x)) = \sum_{n \in 2\mathbb{N}} \int d^4x \tilde{\mathcal{L}}_n^{\alpha_0, \alpha_0} \sin(\alpha_0^{n/2} \varphi^n(x)) \alpha_0^{-n/2}. \quad (65)$$

In the mean field approximation we obtain correspondingly ¹⁰

$$\boxed{L_0(\phi) = \sum_{n \in 2\mathbb{N}} \tilde{A}_n^{\alpha_0, \alpha_0} \sin(\alpha_0^{n/2} \phi^n) \alpha_0^{2-n/2}}. \quad (66)$$

We will show in the following that there exist solutions such that the $|\tilde{A}_n^{\alpha_0, \alpha_0}|$ are bounded by $n^{-5/4} \alpha_0^{n/2-2}$. Since the sine functions in (66) are bounded by 1, the series in (66) is in fact convergent. The boundary conditions we will impose are a subclass of

¹⁰We could then also try to analyse the functional $L^{\alpha_0, \alpha}(\varphi(x))$ and the function $L_{\text{mf}}^{\alpha_0, \alpha}(\phi)$ for general values of α in the same form

$$L_0(\varphi(x)) = \sum_{n \in 2\mathbb{N}} \int d^4x \tilde{\mathcal{L}}_n^{\alpha_0, \alpha} \sin(\alpha^{n/2} \varphi^n(x)) \alpha^{-n/2}, \quad L_{\text{mf}}^{\alpha_0, \alpha}(\phi) = \sum_{n \in 2\mathbb{N}} \tilde{A}_n^{\alpha_0, \alpha} \sin(\alpha^{n/2} \phi^n) \alpha^{2-n/2}$$

which would be more complicated. We do not do so here since our main point is to show that the starting functional integral and its mean field limit are well-defined.

those considered in 4.1. So for general α , the moments of the effective action $L_{\text{mf}}^{\alpha_0, \alpha}(\phi)$ can be written as in the previous sections, and the flow of the moments can be studied as before.

We now analyse the FEs for the functions $\tilde{A}_n^{\alpha_0, \alpha}$ and their α -derivatives evaluated at $\alpha = \alpha_0$. We will show that there are convergent solutions for sufficiently small values of $\tilde{A}_2^{\alpha_0, \alpha_0}$ and its μ -derivatives. These FEs are obtained by expressing the moments $A_n^{\alpha_0, \alpha_0}$ in terms of the $\tilde{A}_{n'}^{\alpha_0, \alpha_0}$. Expanding (65) in moments as we did in (14) and taking the mean field limit, as in (21), the mean field FEs for the $\tilde{A}_n^{\alpha_0, \alpha_0}$ take the form ¹¹

$$\begin{aligned}
& \sum_{\substack{\nu \geq 0 \\ (1+2\nu)n' = n}} \alpha_0^{n'\nu} \frac{(-1)^\nu}{(2\nu+1)!} \partial_\alpha \tilde{A}_{n'} + \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} \alpha_0^{n'\nu-1} n'\nu \frac{(-1)^\nu}{(2\nu+1)!} \tilde{A}_{n'} \\
&= \frac{c}{2}(n+2)(n+1) \alpha_0^{-2} \tilde{A}_{n+2} + \frac{c}{2} \alpha_0^{-2} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} \alpha_0^{n'\nu} n'(n'-1) \frac{(-1)^\nu}{(2\nu)!} \tilde{A}_{n'} \\
&+ \frac{c}{2} \alpha_0^{-2} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} \alpha_0^{n'\nu} n'^2 \frac{(-1)^\nu}{(2\nu-1)!} \tilde{A}_{n'} \\
&- \frac{1}{2} \sum_{\substack{n_1+n_2=n+2, \nu_1, \nu_2 \geq 0 \\ (1+2\nu_1)n'_1 = n_1 \\ (1+2\nu_2)n''_2 = n_2}} \alpha_0^{n'\nu_1 + n''\nu_2} n' n'' \frac{(-1)^{\nu_1+\nu_2}}{(2\nu_1)!(2\nu_2)!} \tilde{A}_{n'_1} \tilde{A}_{n''_2} .
\end{aligned} \tag{67}$$

The additional terms appearing as compared to (14) and (21) carry $\nu \geq 1$ or $\nu_i \geq 1$ in the various sums. They stem from higher order terms on expanding the sine in (65). Note that

$$\tilde{A}_2 \equiv \tilde{A}_2^{\alpha_0, \alpha_0} = A_2^{\alpha_0, \alpha_0} . \tag{68}$$

As before, see (24), (25), we introduce dimensionless quantities $\tilde{a}(n)$ via the definition

$$\tilde{A}_n =: \frac{1}{n} \alpha_0^{n/2-2} \tilde{a}(n) . \tag{69}$$

Comparing to (24) we find again for $n = 2$

$$\tilde{a}(2) = a_2(\alpha_0) .$$

¹¹We suppress the upper index α_0, α_0 on \tilde{A}_n, \tilde{a}_n . Note that the second term in the first line of (67) stems from deriving the power $\alpha^{\nu n'}$ at $\alpha = \alpha_0$.

This gives the following FEs for the $\tilde{a}(n)$, evaluated at $\alpha = \alpha_0$ or equivalently at $\mu = 0$

$$\begin{aligned}
\tilde{a}(n+2) &= \frac{1}{n+1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} (n'-1) \frac{(-1)^{\nu-1}}{(2\nu)!} \tilde{a}(n') + \frac{1}{n+1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} n' \frac{(-1)^{\nu-1}}{(2\nu-1)!} \tilde{a}(n') \\
&+ \frac{1}{(n+1)c} \sum_{\substack{n_1+n_2=n+2, \nu_1, \nu_2 \geq 0 \\ (1+2\nu_1)n' = n_1 \\ (1+2\nu_2)n'' = n_2}} (-1)^{\nu_1+\nu_2} \frac{1}{(2\nu_1)!(2\nu_2)!} \tilde{a}(n') \tilde{a}(n'') \\
&+ \frac{2}{(n+1)c} \frac{n-4}{2n} \tilde{a}(n) + \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} \frac{n'-4}{2n'} \frac{(-1)^\nu}{(2\nu+1)!} \tilde{a}(n') \\
&+ \frac{2}{(n+1)c} \frac{1}{n} \partial_\mu \tilde{a}(n) + \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} \frac{1}{n'} \frac{(-1)^\nu}{(2\nu+1)!} \partial_\mu \tilde{a}(n') \\
&+ \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} (-1)^\nu \frac{\nu}{(2\nu+1)!} \tilde{a}(n') .
\end{aligned} \tag{70}$$

For $n = 2$ we obtain simply

$$\tilde{a}(4) = \frac{1}{3c} [\tilde{a}(2)(\tilde{a}(2) - 1) + \partial_\mu \tilde{a}(2)] \tag{71}$$

in agreement with (25) for $n = 2$.

For the μ -derivatives evaluated at $\mu = 0$ we write for shortness

$$\tilde{a}(n, l) := \partial_\mu^l \tilde{a}(n) . \tag{72}$$

Our bound will be expressed in terms of the decomposition of n into prime numbers. For any integer n we can write its prime number decomposition

$$n = \prod_{j \geq 2, j \text{ prime}} j^{p_j(n)} = 2^{p_2(n)} \cdot 3^{p_3(n)} \cdot 5^{p_5(n)} \cdot 7^{p_7(n)} \cdot 11^{p_{11}(n)} \dots , \tag{73}$$

where the nonnegative integer $p_j(n)$ is the (uniquely determined) multiplicity of the prime number j in the prime number decomposition of n . We then define

$$\begin{aligned}
B(n, 0) &:= B(n) := \left[2^{p_2(n)/4} \cdot 3^{p_3(n)/4} \cdot 5^{p_5(n)/2} \prod_{j \geq 7, j \text{ prime}} j^{\frac{9}{8}p_j(n)} \right]^{-1} , \\
B_\varepsilon(n) &:= B(n) \varepsilon , \quad B(n, l) := B(n) \frac{(n+l)!}{n!} , \quad B_\varepsilon(n, l) := B(n, l) \varepsilon^{l+1} .
\end{aligned} \tag{74}$$

It is straightforward to verify¹² that

$$\sum_{\nu \geq 1} \frac{1}{B(1+2\nu)(2\nu+1)!} \leq \frac{1}{4}, \quad (75)$$

$$\sum_{\nu \geq 1} \frac{2\nu}{B(1+2\nu)(2\nu+1)!} \leq \frac{11}{20}, \quad (76)$$

and as a consequence

$$\sum_{\nu \geq 0} \frac{1}{B(1+2\nu)(2\nu)!} \leq 2. \quad (77)$$

Lemma 4.1 *For fixed $\varepsilon > 0$ sufficiently small, there exists ε' , $0 < \varepsilon' \leq \varepsilon$ such that if*

$$|\tilde{a}(2, l)| \leq B_{\varepsilon'}(2, l) \quad \forall l \geq 0, \quad (78)$$

then

$$|\tilde{a}(n, l)| \leq B_{\varepsilon}(n, l) \quad \forall l \geq 0, n \geq 2. \quad (79)$$

Remark.

On inspecting (71), (70) it is obvious that the bound (79) holds for all (n, l) with $n + l \leq N_0$, for some fixed N_0 , if ε' is chosen sufficiently small depending on N_0 and ε . We will not derive an explicit upper bound on $\varepsilon'(\varepsilon, N_0) > 0$, satisfying ourselves with the existence statement, but comment on the size of N_0 in the proof. We are not ambitious on the size of $\varepsilon, \varepsilon'$. To show that one may allow for not too small values of $\varepsilon, \varepsilon'$, one has to consider small values of $N = n + l$ individually, and also to bound in a different way those particular cases, where $B(n + 2)$ is much bigger than $B(n)$. The most stringent case is $n + 2 = 2^k$ with k large. Then $B(n + 2) = (n + 2)^{-1/4}$, whereas $B(n)$ may equal $n^{-9/8}$.

Proof. We proceed by induction in $N = n + l \geq 2$, going up in l for fixed N . We consider (70) and bound successively the terms on the r.h.s. of (70) by induction. This then allows to verify the inductive bound for the l.h.s. The μ -derivatives of (70) will be treated afterwards.

- 1st term

The first term to bound is

$$\frac{1}{n+1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} (n' - 1) \frac{(-1)^{\nu-1}}{(2\nu)!} \tilde{a}(n'). \quad (80)$$

¹²The convergence statement is trivial, we give quite precise bounds though this does not really matter.

By induction $\tilde{a}(n') \leq B_\varepsilon(n')$. From the definition (74) it follows that

$$B_\varepsilon(n_1 n_2) = B_\varepsilon(n_1) B(n_2)$$

for integers n_1, n_2 . Therefore for $(1 + 2\nu)n' = n + 2$,

$$\tilde{a}(n') \leq \frac{B_\varepsilon(n + 2)}{B(1 + 2\nu)}.$$

Using $(n' - 1)(1 + 2\nu) \leq n + 1$ and (75) we get

$$\left| \frac{1}{n + 1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} (n' - 1) \frac{(-1)^{\nu-1}}{(2\nu)!} \tilde{a}(n') \right| \leq \frac{1}{4} B_\varepsilon(n + 2). \quad (81)$$

- 2nd term

For the second term we get similarly using (76)

$$\left| \frac{1}{n + 1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} n' \frac{(-1)^{\nu-1}}{(2\nu - 1)!} \tilde{a}(n') \right| \leq \frac{n + 2}{n + 1} \frac{11}{20} B_\varepsilon(n + 2). \quad (82)$$

- 3rd term

$$\begin{aligned} & \left| \frac{1}{(n + 1)c} \sum_{\substack{n_1 + n_2 = n + 2 \\ \nu_1, \nu_2 \geq 0 \\ (1+2\nu_1)n' = n_1 \\ (1+2\nu_2)n'' = n_2}} (-1)^{\nu_1 + \nu_2} \frac{1}{(2\nu_1)!(2\nu_2)!} \tilde{a}(n') \tilde{a}(n'') \right| \\ & \leq \frac{1}{(n + 1)c} \sum_{\substack{n_1 + n_2 = n + 2 \\ \nu_1, \nu_2 \geq 0 \\ (1+2\nu_1)n' = n_1 \\ (1+2\nu_2)n'' = n_2}} \frac{1}{B(1 + 2\nu_1)(2\nu_1)! B(1 + 2\nu_2)(2\nu_2)!} B_\varepsilon(n_1) B_\varepsilon(n_2) \quad (83) \\ & \leq \frac{4}{(n + 1)c} 2 \sum_{\substack{n_1 \leq n_2 \\ n_1 + n_2 = n + 2}} B_\varepsilon(n_1) B_\varepsilon(n_2), \end{aligned}$$

where we used (77). We have

$$\sum_{\substack{n_1 \leq n_2 \\ n_1 + n_2 = n + 2}} B(n_1) B(n_2) \leq \sum_{n_1 \leq (n+2)/2} B(n_1) \left[\frac{2}{n + 2} \right]^{\frac{1}{4}} \quad (84)$$

and ¹³

$$\sum_{n_1 \leq (n+2)/2} B(n_1) \leq \sum_{\mu, \nu, \rho \geq 0} 2^{-\frac{\mu}{4}} 3^{-\frac{\nu}{4}} 5^{-\frac{\rho}{2}} \sum_{n' \geq 2} \left(\frac{1}{n'} \right)^{\frac{9}{8}} \leq K \quad (85)$$

¹³We may choose $K = 400$.

so that if we choose ε sufficiently small to assure

$$\frac{8 \cdot 2^{\frac{1}{4}} K}{(n+1)c} \varepsilon \leq \frac{1}{30} \frac{1}{(n+2)^{7/8}}, \quad (86)$$

(83) is bounded by

$$\frac{1}{30} \frac{1}{(n+2)^{9/8}} \varepsilon \leq \frac{1}{30} B_\varepsilon(n+2). \quad (87)$$

• 4th and 5th term

$$\begin{aligned} & \left| \frac{2}{(n+1)c} \frac{n-4}{2n} \tilde{a}(n) + \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n'=n}} \frac{n'-4}{2n'} \frac{(-1)^\nu}{(2\nu+1)!} \tilde{a}(n') \right| \\ & \leq \frac{(n-4)}{n(n+1)c} \frac{5}{4} B_\varepsilon(n) \leq \frac{1}{20} B_\varepsilon(n+2) \quad \text{for } n \geq \left(\frac{25}{c}\right)^8. \end{aligned} \quad (88)$$

• 6th and 7th term

$$\begin{aligned} & \frac{2}{(n+1)c} \left| \sum_{\substack{\nu \geq 0 \\ (1+2\nu)n'=n}} \frac{1}{n'} \frac{(-1)^\nu}{(2\nu+1)!} \tilde{a}(n', 1) \right| \leq \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 0 \\ (1+2\nu)n'=n}} \frac{1}{B(1+2\nu)(2\nu+1)!} B_\varepsilon(n, 1) \\ & \leq \frac{2}{(n+1)c} \frac{5}{4} B_\varepsilon(n, 1) \leq \frac{1}{20} B_\varepsilon(n+2, 1) \quad \text{for } n \geq \left(\frac{25}{c}\right)^8. \end{aligned} \quad (89)$$

• 8th term

$$\begin{aligned} & \frac{2}{(n+1)c} \left| \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n'=n}} (-1)^\nu \frac{\nu}{(2\nu+1)!} \tilde{a}(n') \right| \leq \frac{1}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n'=n}} \frac{2\nu}{B(1+2\nu)(2\nu+1)!} B_\varepsilon(n) \\ & \leq \frac{11}{20(n+1)c} B_\varepsilon(n) \leq \frac{1}{30} B_\varepsilon(n+2) \quad \text{for } n \geq \left(\frac{33}{2c}\right)^8. \end{aligned} \quad (90)$$

Using our lower bound on n (88) we then obtain

$$\frac{1}{4} + \frac{11}{20} \frac{n+2}{n+1} + \frac{2}{20} + \frac{2}{30} < 1$$

and thus have inductively proven the assertion

$$\tilde{a}(n+2) \leq B_\varepsilon(n+2). \quad (91)$$

The lower bound on n (88) (which equals N_0 for $l = 0$) and the upper bound on ε (86) could be relaxed if treating individually a number of different cases.

It is straightforward to verify the assertion for the $\tilde{a}(4, l)$ by bounding inductively the μ -derivatives of (71). When taking μ -derivatives of (70) we get

$$\begin{aligned}
& \tilde{a}(n+2, l) \\
&= \frac{1}{n+1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} (n'-1) \frac{(-1)^{\nu-1}}{(2\nu)!} \tilde{a}(n', l) + \frac{1}{n+1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} \frac{(n'-1)^2}{n'} \frac{(-1)^{\nu-1}}{(2\nu-1)!} \tilde{a}(n', l) \\
&+ \frac{1}{(n+1)c} \sum_{l'+l''=l} \binom{l}{l'} \sum_{\substack{n_1+n_2=n+2 \\ \nu_1, \nu_2 \geq 0 \\ (1+2\nu_1)n' = n_1 \\ (1+2\nu_2)n'' = n_2}} (-1)^{\nu_1+\nu_2} \frac{1}{(2\nu_1)!(2\nu_2)!} \tilde{a}(n', l') \tilde{a}(n'', l'') \\
&+ \frac{2}{(n+1)c} \frac{n-4}{2n} \tilde{a}(n, l) + \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} \frac{n'-4}{2n'} \frac{(-1)^\nu}{(2\nu+1)!} \tilde{a}(n', l) \\
&+ \frac{2}{(n+1)c} \frac{1}{n} \tilde{a}(n, l+1) + \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} \frac{1}{n'} \frac{(-1)^\nu}{(2\nu+1)!} \tilde{a}(n', l+1) \\
&+ \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} (-1)^\nu \frac{\nu}{(2\nu+1)!} \tilde{a}(n', l) .
\end{aligned} \tag{92}$$

Going from l to $l+1$, the inductive bound allowed for the l.h.s., i.e. for $a(n+2, l)$, is multiplied by a factor of $n+2+l+1$. The respective bounds on the linear terms on the r.h.s. take factors of

- 1) $n'+l+1$, 2) $n'+l+1$, 4) $n+l+1$, 5) $n'+l+1$, 6) $n+l+2$, 7) $n'+l+2$, 8) $n'+l+1$.

All these factors are strictly smaller than the one allowed for the l.h.s. so that the inductive verification of the bound for $l > 0$ follows directly from its verification for $l = 0$. For the quadratic term (the third term) we use the bound (39) which gives

$$\binom{l}{l'} \frac{(n'+l)!}{n'!} \frac{(n''+l'')!}{n''!} \leq l! \binom{n+2+l}{n+2} . \tag{93}$$

Here the factors $\frac{(n'+l)!}{n'!}$, $\frac{(n''+l'')!}{n''!}$ stem from the inductive bounds on the $\tilde{a}(n', l')$, $\tilde{a}(n'', l'')$. The expression on the r.h.s. of (93) corresponds to the factorials appearing in the definition of $B(n+2, l)$. The sum over l', l'' then gives a factor of $l+1$ which is again smaller than $n+2+l+1$. The remaining part of the bound is established as for the third item (83)-(87). \square

We note that for ε' sufficiently small, the assumptions (78) imply the assumptions (36) of Proposition 4.1 for $\mu = 0$. So we may choose $f_2(\mu)$ such that $f_2(0)$ satisfies (78)

and (36) and such that $f_2(\mu)$ satisfies (36)¹⁴. By choosing appropriately the signs of the $\tilde{a}(2, l)$ we can also verify the assumptions (45), sufficient for Proposition 4.2. As a consequence of Propositions 4.1 and 4.2 and of Lemma 4.1, we have therefore proven

Theorem 1 *For $\varepsilon' > 0$ sufficiently small, the solutions of the mean field FEs corresponding to the bounded action (66) obeying (78) verify the conditions of Proposition 4.1. For appropriate choices of the $\partial_\mu^l f_2(\mu)|_{\mu=0}$ they also verify Proposition 4.2. This implies the existence of nontrivial asymptotically free solutions of bounded mean field action for the scalar field mean field FEs. In fact we have proven that the bare mean field action for the boundary conditions considered satisfies*

$$\left| L_0(\phi) \right| \leq \varepsilon \sum_{n \in 2\mathbb{N}} n^{-5/4} \quad \forall \phi \in \mathbb{R} \quad , \quad (94)$$

where $\varepsilon > 0$ from Lemma 4.1 has to be chosen sufficiently small.

4.5 The trivial solution

It has been proven by Fröhlich [9] and Aizenman [1] under mild assumptions that the pure symmetric (one- and two-component) φ_4^4 -theory is interaction-free, i.e. (generalised) Gaussian. These results apply to the continuum limit of lattice regularised φ_4^4 -theory in dimension $d > 4$; in $d = 4$ under the additional assumption that the theory has infinite wave function renormalisation. Quite recently it could be proven by Aizenman and Duminil-Copin with the aid of multi-scale techniques [2] that the assumption on infinite wave function renormalisation in $d = 4$ can be dropped. These proofs do not require restrictions on the size of the φ_4^4 coupling. We note that beyond four dimensions it is also known that the critical behaviour of the theory is exactly described by the mean field approximation [12, 9, 1]. The fact that the continuum limit is interaction-free, is proven by showing that the truncated (i.e. connected) four-point function vanishes in this limit. By inequalities due to Glimm, Jaffe [14] and to Newman [28], the vanishing of the truncated four-point function implies the vanishing of the truncated higher n -point functions as well. The triviality result seems quite robust and has also been confirmed by a comprehensive analysis including numerical work [25].

The result we present in this section is much weaker than those from [9, 1, 2]. The section is mainly intended to show that the present approach is coherent with the triviality results in spite of the asymptotically free solutions presented in the previous sections. We will show that weakly coupled pure φ_4^4 -theory is trivial in the mean-field limit, when

¹⁴Remember that $\tilde{a}(2, l) = \partial_\mu^l f_2(\mu)|_{\mu=0}$.

treated with the continuous FEs. This result confirms the importance of irrelevant terms proven in the previous section.

In the mean field approximation we cannot analyse the role of the wave function renormalisation. Whereas the wave function renormalisation is important for the triviality issue, as mentioned above, there are indications from asymptotically free models which can be constructed nonperturbatively, that for such models the wave function renormalisation is finite, and thus need not be included in the bare action. This remark applies in particular to the $2d$ Gross-Neveu model where finiteness of the wave function renormalisation was shown in [11, 8]. The Gross-Neveu model is of similar diagrammatic structure as φ^4 -theory, apart from the modifications due to Fermi-Dirac statistics.

As stated before the boundary conditions of pure φ_4^4 -theory appear to be natural when aiming at a continuum description of Ising type lattice models. From the point of view of the renormalisation group FEs, they rather correspond to a fine-tuned problem since the infinite number of trajectories $f_n(\mu)$, $n > 4$, are supposed to pass all through 0 at the same value $\mu = 0$. In the full theory the boundary conditions even require that all these trajectories pass through 0 at $\mu = 0$ for *all* values of the momentum or position arguments. It will turn out that enforcing these conditions tends to make grow higher derivatives of the $f_n(\mu)$ more rapidly with increasing n than in the case of the solutions we have considered so far. In any case, since we want to argue that our considerations grasp important aspects of scalar field theory, it is important to look at pure φ_4^4 -theory in our context. It will turn out that we can construct the trivial solution quite explicitly for all values of the renormalisation group parameter and sufficiently small bare coupling, so that we are in agreement with the above cited results [9, 1, 2].

When starting with a bare action functional

$$L_0(\varphi(x)) = \int d^4x \left(c_{0,2} \varphi^2(x) + c_{0,4} \varphi^4(x) \right), \quad (95)$$

we obtain in the mean field limit from (95) using (11), (20) and (26),

$$f_2(0) = \alpha_0 2 (2\pi)^4 c_{0,2}, \quad f_4(0) = 4c (2\pi)^4 c_{0,4} = 4\pi^2 c_{0,4}, \quad f_n(0) = 0, \quad n > 4. \quad (96)$$

As a consequence of the pure φ_4^4 boundary conditions we have

Lemma 4.2 *For smooth solutions $f_n(\mu)$ of (28), (29) respecting the boundary conditions (96)*

$$\text{we have } \partial_\mu^l f_n(0) = 0 \quad \text{for } n \geq 6 \quad \text{and } 0 \leq l \leq \frac{n}{2} - 3. \quad (97)$$

Proof. We proceed as usual by induction in $N = n + 2l \in \mathbb{N}$, going up in l for fixed N and starting at $N = 6$. For $N = 6$ the assertion just corresponds to the boundary condition

$$f_6(0) = 0 .$$

For $N > 6$ and $l = 0$ the statement follows from (96). Then (29), derived l times at $\mu = 0$, and solved for $\partial_\mu^{l+1} f_n(0)$ implies with the help of the induction hypothesis for $l < \frac{n}{2} - 3$

$$\partial_\mu^{l+1} f_n(0) = 0 ,$$

since all other terms appearing in (27) derived l times w.r.t. μ vanish by induction. We note in particular that for the products

$$\partial_\mu^{l_1} f_{n_1}(0) \partial_\mu^{l_2} f_{n_2}(0)$$

with $l_1 + l_2 = l$ and $n_1 + n_2 = n + 2$, the condition $l < \frac{n}{2} - 3$ implies that either $l_1 \leq \frac{n_1}{2} - 3$ or $l_2 \leq \frac{n_2}{2} - 3$. \square

As a consequence of Lemma 4.2 we can write smooth solutions verifying (96) as

$$f_n(\mu) = \mu^{\frac{n}{2}-2} g_n(\mu) , \quad n \geq 4 , \quad (98)$$

where the $g_n(\mu)$ are smooth. The system (29) can then be rewritten as

$$\begin{aligned} \mu^2 g_{n+2} &= \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1} g_{n_2} \\ &+ \mu \frac{1}{n+1} g_n \left(2f_2 + 1 - \frac{4}{n} \right) + \frac{n-4}{n(n+1)} g_n + \frac{2}{n(n+1)} \mu \partial_\mu g_n , \quad n \geq 4 . \end{aligned} \quad (99)$$

Expanding the g_n and f_2 in a (for the moment) formal Taylor series around $\mu = 0$

$$g_n(\mu) = \sum_{k \geq 0} g_{n,k} \mu^k , \quad f_2(\mu) = \sum_{k \geq 0} f_{2,k} \mu^k , \quad (100)$$

we find for the coefficients from (28) and (29)

$$g_{4,k} = \frac{1}{3} \left[(k+1) f_{2,k+1} - f_{2,k} + \sum_{0 \leq \nu \leq k} f_{2,\nu} f_{2,k-\nu} \right] , \quad (101)$$

$$\begin{aligned} g_{n+2,k} &= \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{0 \leq \nu \leq k+2} g_{n_1,\nu} g_{n_2,k+2-\nu} + \frac{2}{n+1} \sum_{0 \leq \nu \leq k+1} g_{n,\nu} f_{2,k+1-\nu} \\ &+ \frac{1}{n+1} g_{n,k+1} \left(1 - \frac{4}{n} \right) + \frac{n+2k}{n(n+1)} g_{n,k+2} , \quad n \geq 4 . \end{aligned} \quad (102)$$

We can rewrite (101), (102) as

$$f_{2,k+1} = \frac{1}{k+1} \left[3g_{4,k} + f_{2,k} - \sum_{0 \leq \nu \leq k} f_{2,\nu} f_{2,k-\nu} \right], \quad (103)$$

$$\begin{aligned} g_{n,k+2} = & -\frac{n-4}{n+2k} g_{n,k+1} - \frac{2n}{n+2k} \sum_{0 \leq \nu \leq k+1} g_{n,\nu} f_{2,k+1-\nu} \\ & - \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{0 \leq \nu \leq k+2} g_{n_1,\nu} g_{n_2,k+2-\nu} + \frac{n(n+1)}{n+2k} g_{n+2,k}. \end{aligned} \quad (104)$$

Regularity of the system (99) at $\mu = 0$ also implies for $n \geq 4$

$$\frac{n-4}{n} g_{n,0} + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,0} = 0, \quad (105)$$

$$\frac{2}{n} g_{n,1} + \frac{n-4}{n} g_{n,1} + 2 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,1} + g_{n,0} \left(2f_{2,0} + 1 - \frac{4}{n} \right) = 0. \quad (106)$$

If we choose freely $f_{2,0}$, $g_{4,0}$, equations (105), (106) fix all other $g_{n,0}$, $g_{n,1}$. All terms $f_{2,k}$ with $k \geq 1$ and $g_{n,k}$ with $k \geq 2$ are then determined through (103), (104).

Lemma 4.3 *We consider smooth solutions $f_n(\mu)$ of (28), (29) respecting the boundary conditions (96) and assume that for $\mu = 0$*

$$|f_{2,0}| \leq \frac{\varepsilon}{4}, \quad 0 \leq f_{4,0} = g_{4,0} \leq \frac{\varepsilon}{32} \quad (107)$$

with $0 < \varepsilon \leq 10^{-2}$. Then

$$|f_{2,1}| \leq \frac{\varepsilon}{2}, \quad |g_{4,1}| \leq \frac{\varepsilon^2}{32} \quad (108)$$

and for $n \geq 6$

$$|g_{n,0}| \leq \frac{\varepsilon^{\frac{n}{2}-1}}{2n^2}, \quad |g_{n,1}| \leq \frac{\varepsilon^{\frac{n}{2}-1}}{n^2}. \quad (109)$$

The constants $g_{n,0}$ are alternating in sign:

$$g_{n,0} = (-1)^{n/2} |g_{n,0}|. \quad (110)$$

Proof. For $f_{2,1}$ we find explicitly from (103)

$$f_{2,1} = 3f_{4,0} - f_{2,0}(f_{2,0} - 1) \leq \frac{\varepsilon}{2}.$$

Similarly from (106)

$$g_{4,1} = 4 g_{4,0} f_{2,0} \quad \text{so that} \quad |g_{4,1}| \leq \frac{\varepsilon^2}{32}.$$

We then proceed as usual by induction in n , treating first $g_{n,0}$. For $n \geq 6$ we find from (105)

$$|g_{n,0}| \leq \frac{n}{n-4} \frac{1}{4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{\varepsilon^{\frac{n}{2}-1}}{n_1^2 (n+2-n_1)^2} \leq \frac{1}{2} \frac{\varepsilon^{\frac{n}{2}-1}}{n^2}.$$

For $n \geq 6$ we also get from (106)

$$|g_{n,1}| \leq \frac{2n}{n-2} \frac{1}{2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{\varepsilon^{\frac{n}{2}-1}}{n_1^2 (n+2-n_1)^2} + \frac{n}{n-2} \frac{\varepsilon^{\frac{n}{2}-1}}{2n^2} \left(\frac{\varepsilon^2}{2} + 1 - \frac{4}{n} \right) \leq \frac{\varepsilon^{\frac{n}{2}-1}}{n^2}.$$

The two previous bounds on the sums over n_1 can be verified explicitly for $n \leq 10$. For $n \geq 12$ we use

$$\begin{aligned} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4, n_i \in 2\mathbb{N}}} \frac{1}{n_1^2 (n+2-n_1)^2} &\leq \frac{1}{16} \sum_{\substack{n_1+n_2=\frac{n}{2}+1 \\ n_i \geq 2, n_i \in \mathbb{N}}} \frac{1}{n_1^2 (\frac{n}{2}+1-n_1)^2} \\ &\leq \frac{1}{8} \sum_{\substack{n_1 \leq \frac{n}{2}+1 \\ n_1 \geq 3}} \frac{4}{n_1^2 (n+2)^2} + \frac{1}{8} \frac{4}{4(n-2)^2} \\ &\leq \frac{1}{2(n+2)^2} \underbrace{\left(\zeta(2) - \frac{5}{4} \right)}_{\leq 0.4} + \frac{1}{8(n-2)^2} \leq \frac{1}{5} \frac{1}{(n+2)^2} + \frac{1}{(n-2)^2}, \end{aligned}$$

and

$$\frac{n}{n-2} \left(\frac{1}{5} \frac{1}{(n+2)^2} + \frac{1}{8} \frac{1}{(n-2)^2} \right) \leq \frac{1}{2n^2} (1 - \varepsilon^2).$$

The statement on the signs follows from (105) by induction in n , using that $g_{4,0} > 0$. \square

Lemma 4.4 *Under the same assumptions as in Lemma 4.3 we have the bounds*

$$|g_{n,k}| \leq 2^{k-2} \varepsilon^{n/2-1} \left(k + \frac{n-4}{2} \right)!, \quad |f_{2,k}| \leq 2^k \varepsilon |k-1|. \quad (111)$$

Proof. We proceed by induction going up in $N = n + k$ using (104). For $g_{n,1}$, $g_{n,0}$ and $f_{2,1}$, $f_{n,0}$ we use the bounds from Lemma 4.3. We obtain from (104), (109) and (111)

$$\begin{aligned}
|g_{n,k+2}| &\leq 2^k \varepsilon^{\frac{n}{2}-1} \left[\frac{n-4}{2(n+2k)} \left(k+1 + \frac{n-4}{2}\right)! + \frac{\varepsilon n}{n+2k} \sum_{0 \leq \nu \leq k+1} \left(\nu + \frac{n-4}{2}\right)! |k-\nu|! \right. \\
&\quad + \frac{n}{4(n+2k)} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{0 \leq \nu \leq k+2} \left(\nu + \frac{n_1-4}{2}\right)! \left(k+2-\nu + \frac{n_2-4}{2}\right)! \\
&\quad \left. + \frac{n(n+1)\varepsilon}{4(n+2k)} \left(k + \frac{n-2}{2}\right)! \right] \tag{112}
\end{aligned}$$

$$\begin{aligned}
&\leq 2^k \varepsilon^{\frac{n}{2}-1} \left(k+2 + \frac{n-4}{2}\right)! \left[\frac{n-4}{2n} \frac{2}{n} + \frac{2\varepsilon n}{n} \frac{2}{n} + \frac{n}{4n} \frac{n}{2} \frac{2}{n} + \frac{n(n+1)\varepsilon}{4n} \frac{2}{n} \right] \tag{113} \\
&\leq 2^k \varepsilon^{\frac{n}{2}-1} \left(k+2 + \frac{n-4}{2}\right)! .
\end{aligned}$$

We used

$$\sum_{0 \leq \nu \leq n-a} (n-\nu)! \nu! \leq 2n! \quad \text{for } a \in \mathbb{N}, \quad a \leq n,$$

and

$$\sum_{0 \leq \nu \leq k} (a+\nu)! (b+k-\nu)! \leq \sum_{0 \leq \nu \leq k} (A+\nu)! (A+k-\nu)! \quad \text{with } A = \sup(a, b), \quad a, b \in \mathbb{N}.$$

For $n = 2$ the bound follows from (103) and Lemma 4.3. \square

We note that the bounds we derived are not sufficient to prove convergence of the Taylor expansion around $\mu = 0$, in contrast to the bounds (37). So (100) still stand as formal power series. We think the factorial behaviour of the bounds is not far from optimal and trace the large size of the derivatives back to the particular boundary conditions. We now show that there exist smooth solutions corresponding to the formal power series (100).

Proposition 4.4 *There exist smooth solutions $f_n(\mu)$ of (28), (29) respecting the boundary conditions (96). They vanish in the limit $\mu_{max} = \ln \frac{1}{\alpha_0} \rightarrow \infty$.*

Proof. We study two-point functions $f_2(\mu)$ of the form

$$f_2(\mu) = \sum_{n \geq 1} b_n \frac{x_n^{n-1}}{1+x_n^n}, \quad \text{where } x_n = n\mu \quad \text{and} \quad |b_n| < 1. \tag{114}$$

This ansatz is the most important ingredient in our construction of the trivial solution. If it is well-defined, then all the $f_n(\mu)$ and thus all the $g_n(\mu)$ are determined as functions of $f_2(\mu)$, as follows from (28), (29). Expanding as in (100)

$$f_2(\mu) = \sum_{k \geq 0} f_{2,k} \mu^k ,$$

we find for the Taylor coefficients

$$f_{2,k} = (k+1)^k \sum_{\rho=1}^{k+1} b_{\{\frac{k+1}{\rho}\}} (-1)^{\rho-1} \left(\frac{1}{\rho}\right)^k . \quad (115)$$

Here we set $b_0 := 0$, and for integers n, m

$$\left\{\frac{n}{m}\right\} := \begin{cases} \frac{n}{m} & , \text{ if } \frac{n}{m} \in \mathbb{N} \\ 0 & , \text{ otherwise} \end{cases} . \quad (116)$$

We have in particular

$$f_{2,0} = b_1 , \quad f_{2,1} = 2b_2 - b_1 . \quad (117)$$

Choosing $f_{2,0}$ and $f_{4,0}$ such that the assumptions of Lemma 4.3 are fulfilled, Lemma 4.4 implies for smooth solutions of (28), (29) respecting the boundary conditions (96)

$$\left|f_{2,k}\right| \leq 2^k \varepsilon |k-1|! .$$

We then claim that the coefficients b_n in (114) obey the bounds

$$|b_n| \leq 4 \left(\frac{3}{4}\right)^n \varepsilon . \quad (118)$$

The claim is easily verified for b_1 to b_3 using Lemmata 4.3 and 4.4. For $n \geq 3$ we obtain inductively from (115):

$$\begin{aligned} |b_{n+1}| &\leq 2^n \frac{(n-1)!}{(n+1)^n} \varepsilon + \sum_{\rho=2}^{n+1} \left|b_{\{\frac{n+1}{\rho}\}}\right| \left(\frac{1}{\rho}\right)^n \\ &\leq \left(2^n \frac{(n-1)!}{(n+1)^n} + 4 \sum_{\rho=2}^{n+1} \left(\frac{3}{4}\right)^{\frac{n+1}{\rho}} \left(\frac{1}{\rho}\right)^n\right) \varepsilon \leq 3 \left(\frac{3}{4}\right)^n \varepsilon \end{aligned} \quad (119)$$

using that for $n \geq 3$

$$2^n \frac{(n-1)!}{(n+1)^n} \leq \inf\left(\frac{1}{4}, \left(\frac{3}{4}\right)^n\right) \quad \text{and} \quad \sum_{\rho=2}^{n+1} \left(\frac{3}{4}\right)^{\frac{n+1}{\rho}} \left(\frac{1}{\rho}\right)^n \leq \sum_{\rho=2}^{n+1} \left(\frac{1}{\rho}\right)^n \leq \zeta(n) - 1 \leq 2^{1-n} .$$

The bound (118) implies absolute convergence of the series in (114), uniformly in μ so that $f_2(\mu)$ is smooth and well-defined for $0 \leq \mu \leq \ln \frac{1}{\alpha_0}$. The free choice of $f_{2,0}$ and $f_{4,0}$ fixes b_1 and b_2 . All b_n , $n \geq 3$, are uniquely determined by (103), (104) and (105) as a consequence of the boundary conditions (96) and the smoothness condition.

Uniform absolute convergence in $[0, \infty)$ of the series (114) and its derivatives

$$\sum_{n \geq 1} b_n \partial_\mu^l \frac{x_n^{n-1}}{1 + x_n^n}$$

and the fact that

$$\lim_{\mu \rightarrow \infty} \partial_\mu^l \frac{x_n^{n-1}}{1 + x_n^n} = 0$$

imply

$$\lim_{\mu \rightarrow \infty} \partial_\mu^l f_2(\mu) = 0 \quad \forall l \geq 0. \quad (120)$$

The functions $\partial_\mu^l f_n(\mu)$ for $n \geq 4$ are determined by $\partial_\mu^l f_2(\mu)$ through (28), (29). Proceeding by induction in $n \geq 4$ one finds straightforwardly

- The solutions $f_n(\mu)$ are smooth bounded functions of μ .
- Together with all their derivatives they have vanishing limits for $\mu \rightarrow \infty$:

$$\lim_{\mu \rightarrow \infty} \partial_\mu^l f_n(\mu) = 0. \quad (121)$$

□

We collect the previous findings in the following

Theorem 2 *Triviality of weakly coupled mean field pure φ_4^4 -theory :*

We consider smooth solutions of the mean field flow equations $f_n \in C^\infty[0, \mu_{\max}]$. For the boundary conditions (96), setting

$$0 \leq c_{0,4} \leq \frac{\varepsilon}{2^7 \pi^2}, \quad |c_{0,2}| \leq \Lambda_0^2 \frac{\varepsilon}{2^7 \pi^4}, \quad \varepsilon = 10^{-2},$$

these solutions vanish in the UV limit $\mu_{\max} = \ln \frac{1}{\alpha_0} \rightarrow \infty$, i.e. on removing the UV cutoff $\Lambda_0^{-2} = \alpha_0 \rightarrow 0$.

Proof. We have seen in Proposition 4.4 that there exist solutions with the required properties. For given boundary conditions these solutions are unique since the difference of two given systems of solutions satisfies a system of linear FEs with vanishing boundary conditions.

□

In contrast with the solutions considered in the previous sections the solutions respecting the pure φ_4^4 boundary conditions cannot be shown to be, and probably are not, analytic in a strip of finite width around $[0, \mu_{\max}]$. The obstruction is closely related to the result of Lemma 4.2 which says that the derivatives f_n have to vanish in increasing order with n at $\mu = 0$.

We note that the value of ε in Theorem 2 is certainly not optimal and can easily be improved. More ambitiously one may also try to reach values of ε exceeding 1.

We close this section with three **general remarks**:

- *The Landau pole*

In perturbative field theory the triviality of pure φ_4^4 -theory reflects itself in the divergence of the energy dependent coupling when going to high energies. As discussed in section 4.2, this means that when we fix the physical coupling $g(\lambda)$, $\lambda = \mu_{\max} - \mu$, at low energies, $\lambda = 0$ in our setting, via

$$g := g(0) := f_2(\mu_{\max}) , \quad (122)$$

then

$$g(\lambda) := f_2(\mu_{\max} - \mu)$$

diverges at a finite value of μ , unless we let $g(0) \rightarrow 0$ which implies triviality. This is indeed the case for the solution we found. If we truncate for simplicity the expression (114) at lowest order setting

$$f_2(\mu) = a_1 \frac{1}{1 + \mu} \quad (123)$$

we get

$$g(\lambda) = \frac{g(0)}{1 - \beta g(0) \lambda} \quad \text{with} \quad \beta = \frac{1}{a_1} .$$

The Landau pole is situated at $\lambda_L = \frac{1}{\beta g(0)}$. In physical perturbation theory one normally chooses $f_4(\mu_{\max})$ to define $g(0)$, but this does not change the reasoning since f_2 and f_4 can be expressed in terms of each other and behave in a similar way for large μ . Nor do the conclusions change when taking the full expression (114) instead of (123) since all entries in the absolutely convergent series in (114) behave similarly for $\mu \rightarrow \infty$. Since the perturbative truncations get out of control, way before the Landau divergence occurs, perturbation theory does not allow to make hard statements about triviality.

- *Perturbation theory*

The solutions we considered in the previous sections are not perturbative, which is reflected by the fact that the bounds from Propositions 4.1, 4.3 and from section 4.4 do not involve a power proportional to n of the small parameters δ or ε . For the trivial solution, the perturbative behaviour w.r.t. the bare coupling is revealed by the factor of $\varepsilon^{n/2-1}$ appearing in the bounds of Lemma 4.4. It should also be possible and would be interesting to reexpress the formal power series in ε as a formal power series in the renormalized coupling g (122) and to show that the coefficients of these series are termwise finite for $\mu_{max} \rightarrow \infty$. This would correspond to a perturbative renormalizability proof for mean field ϕ_4^4 -theory. Our nonperturbative proof implies finiteness and even triviality for $\mu_{max} \rightarrow \infty$, but we did not analyse the expansion in powers of g .

- *Relation to the triviality theorems*

A more explicit analysis of the trivial solutions studied here should also allow for comparison with the mean field solutions of the Ising model, and with the properties of the correlation functions studied in [9, 1, 2], on taking the mean field limit.

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