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Stéphane Balac

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Modeling optical Kerr effect in Fabry-Perot resonators

Stéphane BALAC

UNIV. RENNES, CNRS, IRMAR-UMR 6625, F-35000 Rennes, France

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Abstract

In a recent paper, see reference [3] in the bibliography Section, the authors have proposed a model for light-wave propagation in a fiber Fabry-Perot resonator containing a Kerr medium. They provide a set of two coupled non-linear partial differential equations for the forward and backward propagating electric field by exploiting connections between the physics in a medium with general Kerr non-linearity and the physics of a two-level atomic medium. Exploiting such an analogy has the advantage of the conciseness but it hides the underlying assumptions of the model. In this document, we propound a detailed justification of the way the coupled equations that describe light-wave propagation in a fiber Fabry-Perot resonator containing a Kerr medium can be deduced from Maxwell's equations. The main advantage of this presentation is to spotlight the various assumptions that must be made to reach this two coupled equations. In a second part of the document, following [3] again, we deduce from this set of two coupled equations an other non-linear partial differential equation, referred to as the Lugiato-Lefever equation for Fabry-Perot resonators, that allows to compute the forward and backward propagating electric field in the Kerr cavity by solving a single equation for a scalar unknown function rather than a set of two coupled non-linear equations.

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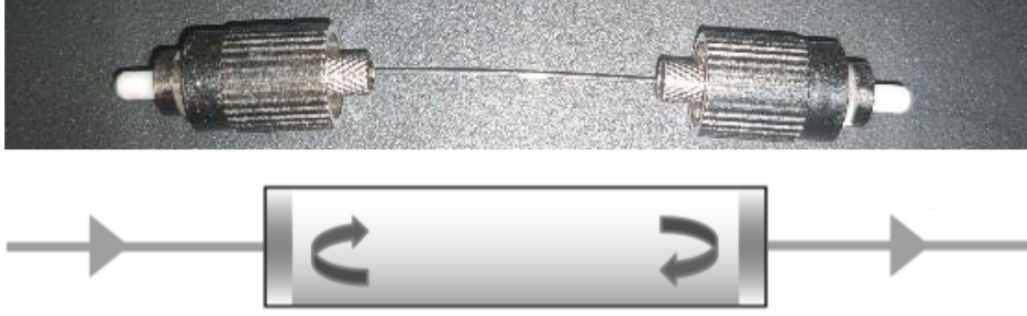


Figure 1: Top : A 7 cm long Fiber Fabry-Perot resonator. Mirrors are applied to the surface of fiber optic ferrules, which serve as terminations for the cord. Bottom : Schematic representation of a Fabry-Perot cavity. (Images courtesy of G. Bourcier and A. Fernandez, LAAS-CNRS.)

1 Introduction

Our concern in this work are fiber Fabry-Perot (FP) resonators that are optical devices formed by an optical fiber bounded at each end by a multi-layer dielectric mirror, see Fig. 1. Their ease of design and manipulation together with their compactness make them a very valuable alternative to whispering-gallery mode resonators, fiber ring resonators or integrated ring-resonators of high non-linearity, as source of Kerr frequency combs in all-fiber optical systems.

An approach for the theoretical understanding of Kerr comb generation process in whispering-gallery mode resonators and integrated ring-resonators relies on a spatiotemporal partial differential equation referred to as the Lugiato–Lefever equation (LLE), which is a nonlinear Schrödinger equation with damping, detuning, and driving [9] that have been initially introduced in the late 80’s by Luigi Lugiato and René Lefever as a model for spontaneous pattern formation in nonlinear optical systems. More recently, this model has been extended to describe Kerr comb generation process in FP resonators in [3] by using an analogy between Kerr effect in a fiber FP resonator and the physics of a two-level atomic medium.

Exploiting such an analogy has the advantage of the conciseness but it hides the underlying assumptions of the model for non-specialists of the topic. In this document we conduct a comprehensive modeling of light-wave propagation in a fiber Fabry-Perot resonator containing a Kerr medium starting from the general set of Maxwell’s equations and leading to the Lugiato–Lefever equation for a fiber Fabry-Perot resonator.

We proceed in two steps. In Section 2, from Maxwell’s equations we show how to obtain a set of two coupled non-linear partial differential equations for the forward and backward propagating electric field in the fiber. Then in a second step detailed in Section 3 we show how the Lugiato–Lefever equation can be deduced from this set of two partial differential equations. Throughout the document, we will take care to highlight all the mathematical assumptions and transformations, and as far as possible have related these assumptions to the physical setting of fiber Fabry-Perot resonators.

2 Wave propagation in a Fabry-Perot resonator

2.1 Maxwell’s equations in optics

Propagation of light into an optical fiber can be described by Maxwell’s equations [7, 13]:

$$\mathbf{curl} \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = \mathbf{0} \quad (1a)$$

$$\mathbf{curl} \mathbf{H} - \frac{\partial}{\partial t} \mathbf{D} = \mathbf{j} \quad (1b)$$

$$\mathbf{div} \mathbf{D} = \rho \quad (1c)$$

$$\mathbf{div} \mathbf{B} = 0 \quad (1d)$$

where \mathbf{E} and \mathbf{H} denote electric and magnetic field vectors, respectively, and \mathbf{D} and \mathbf{B} are corresponding electric and magnetic flux densities. The current density vector \mathbf{j} and the charge density ρ represent the sources for the electromagnetic field. In the absence of free charges in a medium such as in optical fibers, we have $\mathbf{j} = \mathbf{0}$ and $\rho = 0$. The flux densities \mathbf{D} and \mathbf{B} arise in response to the electric and magnetic fields \mathbf{E} and \mathbf{H} propagating inside the medium. In non-magnetic materials such as the ones involved in an optical fiber, the flux densities \mathbf{D} and \mathbf{B} are related to the fields \mathbf{E} and \mathbf{H} through the relations

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (2)$$

$$\mathbf{B} = \mu_0 \mathbf{H} \quad (3)$$

where ε_0 is the vacuum permittivity, μ_0 is the vacuum permeability, and \mathbf{P} is the induced electric polarization. Considering the curl of equation (1a) and combining it with equation (1b), we obtain after having taken into account relations (2) and (3) the following propagation equation for the electric field \mathbf{E}

$$\mathbf{curl\,curl\,E} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} - \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{P} \quad (4)$$

where c is the speed of light in vacuum and the relation $\mu_0 \varepsilon_0 = \frac{1}{c^2}$ was used. To complete the description, a constitutive relation between the induced polarization \mathbf{P} and the electric field \mathbf{E} is needed. In many cases in optics the electric polarization vector \mathbf{P} can be assumed independent of the electromagnetic field strength. However if the field strength is sufficiently strong as it is the case in this study, the dependence of this quantity on \mathbf{E} must be considered. It is convenient to decompose the polarization \mathbf{P} into a linear and a non-linear term as

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL} \quad (5)$$

where \mathbf{P}_L is the part of the electric polarization linear in the field amplitude and \mathbf{P}_{NL} is the non-linear part of this polarization. The mathematical expressions for these two electric polarization components will be discussed in the next section.

In the document, we adopt the following definition of the Fourier Transform of a function $f \in \mathbb{L}^2(\mathbb{R})$:

$$\forall \omega \in \mathbb{R} \quad \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{i\omega t} dt. \quad (6)$$

This choice for the definition of the Fourier Transform is made in accordance with the complex representation of harmonic fields (the so-called phasor notation) where we adopt the $e^{-i\omega t}$ convention, see (15a) below. The inverse Fourier Transform is then

$$\forall t \in \mathbb{R} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{-i\omega t} d\omega. \quad (7)$$

Throughout the document we will use the integral notation for the Fourier Transform although most often the Fourier Transform will have to be understood in the sense of distributions. The Fourier Transform of a function $f \in \mathbb{L}^2(\mathbb{R})$ will be denoted either \hat{f} or $\mathcal{F}(f)$. The properties of the Fourier Transform that will be used in the document are the following: $\widehat{f^{(k)}}(\omega) = (-i\omega)^k \hat{f}(\omega)$, $\mathcal{F}(t \mapsto f(t)e^{-i\omega_0 t})(\omega) = \hat{f}(\omega - \omega_0)$ and $\mathcal{F}(f^*)(\omega) = \hat{f}^*(-\omega)$ where $*$ refers to the complex conjugate.

2.2 Linear and non-linear polarization

The linear polarization governs linear-optical phenomena, i.e., it corresponds to the regime where the optical properties of a medium are independent of the field intensity. The relation between \mathbf{P}_L and the electric field \mathbf{E} can be expressed as

$$\mathbf{P}_L(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{+\infty} \bar{R}^{(1)}(t - t_1) \mathbf{E}(\mathbf{r}, t_1) dt_1 \quad (8)$$

where the scalar valued function $\bar{R}^{(1)}$ is the *linear response function*, which gives the contribution to the polarization produced at time t by an electric field applied at the earlier time t_1 . The total polarization is obtained by integrating these contributions over all previous times t_1 . In writing equation (8) as shown, it is assumed that $R^{(1)}$ obeys the causality condition $R^{(1)}(t) = 0$ for $t < 0$. This condition expresses the fact that \mathbf{P}_L depends only on past and not on future values of \mathbf{E} . Moreover, since the polarization \mathbf{P}_L and the electric field \mathbf{E} both are real-valued quantities, it is also required the polarization response function $\bar{R}^{(1)}$ to be a real-valued function.

Taking the Fourier transform of (8) considered as a convolution product, we find

$$\widehat{\mathbf{P}}_L(\mathbf{r}, \omega) = \varepsilon_0 \bar{\chi}^{(1)}(\omega) \widehat{\mathbf{E}}(\mathbf{r}, \omega) \quad (9)$$

where $\bar{\chi}^{(1)}(\omega) = \int_{-\infty}^{+\infty} \bar{R}^{(1)}(\tau) e^{-i\omega\tau} d\tau$. The scalar function $\bar{\chi}^{(1)}$ is the Fourier Transform of the linear response function $\bar{R}^{(1)}$ and it is known as the *linear optical susceptibility first order tensor*. The linear optical susceptibility $\bar{\chi}^{(1)}$ represents the dominant contribution to the polarization vector \mathbf{P} .

In general, the evaluation of \mathbf{P}_{NL} requires a quantum-mechanical approach since the origin of non-linear response is related to an-harmonic motion of bound electrons under the influence of an applied field. Although such an approach is often necessary when the optical frequency is near one of the medium resonance frequency, a phenomenological relation can be used to relate \mathbf{P} and \mathbf{E} far from medium resonances. This is the case for optical fibers in the wavelength range from 0.5 to 2 μm that is of interest here. Moreover, in the regime of “weak” fields, the non-linear part \mathbf{P}_{NL} of the polarization can be represented as a truncated power-series expansion in the field \mathbf{E} as [4, 12]:

$$\mathbf{P}_{NL}(\mathbf{r}, t) = \mathbf{P}^{(2)}(\mathbf{r}, t) + \mathbf{P}^{(3)}(\mathbf{r}, t) + \dots \quad (10)$$

where only the first two terms in the expansion are usually retained. These two terms can be expressed in a general form as

$$\mathbf{P}^{(2)}(\mathbf{r}, t) = \varepsilon_0 \int_0^{+\infty} \int_0^{+\infty} \bar{\bar{R}}^{(2)}(t - t_1, t - t_2) : \mathbf{E}(\mathbf{r}, t_1) \mathbf{E}(\mathbf{r}, t_2) dt_1 dt_2 \quad (11)$$

and

$$\mathbf{P}^{(3)}(\mathbf{r}, t) = \varepsilon_0 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \bar{\bar{R}}^{(3)}(t - t_1, t - t_2, t - t_3) \vdots \mathbf{E}(\mathbf{r}, t_1) \mathbf{E}(\mathbf{r}, t_2) \mathbf{E}(\mathbf{r}, t_3) dt_1 dt_2 dt_3 \quad (12)$$

where $\bar{\bar{R}}^{(2)}$ and $\bar{\bar{R}}^{(3)}$ are the second- and third-order *non-linear electric polarization response functions*. They are respectively second and third rank tensors. The causality condition requires that $\bar{\bar{R}}^{(2)}(\tau_1, \tau_2) = 0$ if either τ_1 or τ_2 is negative and that $\bar{\bar{R}}^{(3)}(\tau_1, \tau_2, \tau_3) = 0$ if either τ_1 , τ_2 or τ_3 is negative. Moreover, since the polarization \mathbf{P}_{NL} and the electric field \mathbf{E} both are real-valued quantities, we also require the polarization response functions $\bar{\bar{R}}^{(2)}$ and $\bar{\bar{R}}^{(3)}$ to be a real-valued function.

The second-order non-linear electric polarization response tensor $\bar{\bar{R}}^{(2)}$ is responsible for such non-linear effects as second-harmonic generation and sum-frequency generation. However, it is nonzero only for media that lack an inversion symmetry at the molecular level. As SiO_2 is a symmetric molecule, $\bar{\bar{R}}^{(2)}$ vanishes for silica glasses. As a result, optical fibers do not normally exhibit second-order non-linear effects. More generally as a consequence of this inversion symmetry all the even-order non-linear electric polarization response tensor vanish.

The lowest-order non-linear effects in optical fibers originate from the third order non-linear electric polarization response $\bar{\bar{R}}^{(3)}$, which is responsible for phenomena such as third harmonic generation, four-wave mixing, optical Kerr effects, \dots , see [11, chap. 5]. The optical Kerr effect we are interested in here is associated with the contributions to the third-order polarization $\mathbf{P}^{(3)}$.

In [6, 12] expressions for the third order non-linear electric polarization are propound under the assumption that the Born-Oppenheimer approximation is valid for the states of the material, *i.e.* that, for the “optical” frequencies of interest, the electrons adiabatically follow both the nuclear motions and

the changing optical fields. Under this assumption, the third-order polarization $\mathbf{P}^{(3)}$ can be expressed as

$$\begin{aligned} \mathbf{P}^{(3)}(\mathbf{r}, t) &= \varepsilon_0 \bar{\bar{\bar{\chi}}}^{(3)}: \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \\ &+ \varepsilon_0 \mathbf{E}(\mathbf{r}, t) \int_{-\infty}^{+\infty} a(t-t_1) (\mathbf{E}(\mathbf{r}, t_1) \cdot \mathbf{E}(\mathbf{r}, t_1)) dt_1 \\ &+ \varepsilon_0 \int_{-\infty}^{+\infty} b(t-t_1) (\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t_1)) \mathbf{E}(\mathbf{r}, t_1) dt_1 \end{aligned} \quad (13)$$

where $\bar{\bar{\bar{\chi}}}^{(3)}$ denotes the third order optical susceptibility tensor and the two scalar functions a and b are causal ($a(t) = b(t) = 0$ when $t < 0$). The two distinct physical contributions to $\mathbf{P}^{(3)}$ taken into account in (13) are the following.

- The "electronic" contribution from the non-linear distortion of the electron orbits around the average positions of the nuclei. This polarization responds rapidly to field changes, within a few electronic cycles ($\approx 10^{-16}$ s.). In isotropic materials it contributes a term to $\mathbf{P}^{(3)}$ well approximated by the instantaneous form $\varepsilon_0 \bar{\bar{\bar{\chi}}}^{(3)}: \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t)$ for frequencies well below the electronic band gap. That is to say, the nonlinear response is assumed to be instantaneous so that the time dependence of $\bar{\bar{\bar{R}}}^{(3)}$ is given by the product of three delta functions of the form $\bar{\bar{\bar{R}}}^{(3)}(t-t_1, t-t_2, t-t_3) = \chi^{(3)} \delta(t-t_1)\delta(t-t_2)\delta(t-t_3)$ where $\bar{\bar{\bar{\chi}}}^{(3)}$ is a third order tensor time independent.
- A second "nuclear" contribution arises from change in the motions of the nuclei induced by an electromagnetic field. After the sudden impression of a field this contribution can be observed only following a time lapse of the order of the time required for a nucleus to execute a vibrational or rotational cycle ($\approx 10^{-12}$ s.). In isotropic materials this contribution to $\mathbf{P}^{(3)}$ corresponds to the two integral terms in (13) for frequencies well below the electronic band gap. In particular, contributions of this second kind can be associated with:
 - molecular reorientation mechanisms (unlikely, however, to be of particular relevance in solids);
 - Raman type nonlinearities resulting from the modulation of the electronic polarizability by the (Raman) vibrational modes driven by the incident field.

Let us neglect, as a first approximation, the non-instantaneous response of the medium represented by the last two terms in (13) (they turn out to be important only for very short pulses). In this case, relation (13) takes the simple form

$$\mathbf{P}^{(3)}(\mathbf{r}, t) = \varepsilon_0 \chi^{(3)}: \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \quad (14a)$$

that is to say, the components $P_i^{(3)}$, $i = 1, 2, 3$ are given by

$$P_i^{(3)}(\mathbf{r}, t) = \varepsilon_0 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{\ell=1}^3 \chi_{ijkl}^{(3)} E_j(\mathbf{r}, t) E_k(\mathbf{r}, t) E_\ell(\mathbf{r}, t). \quad (14b)$$

2.3 Wave equation in the spectral domain

Optical waves in the visible and infrared spectral regions generally have a spectral width much narrower than their carrier frequency ω_0 . Thus, we can write the electric field as

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} (\underline{\mathbf{E}}(\mathbf{r}, t) \exp(-i\omega_0 t) + c.c.) = \text{Re}(\underline{\mathbf{E}}(\mathbf{r}, t) \exp(-i\omega_0 t)) \quad (15a)$$

where the complex vector function $\underline{\mathbf{E}}$ referred as the *slowly time varying amplitude of the electric field* varies slowly in time compared to the period defined by ω_0 , that is to say

$$\left| \frac{\partial^2 \underline{\mathbf{E}}(\mathbf{r}, t)}{\partial t^2} \right| \ll \omega_0 \left| \frac{\partial \underline{\mathbf{E}}(\mathbf{r}, t)}{\partial t} \right| \quad \text{and} \quad \left| \frac{\partial \underline{\mathbf{E}}(\mathbf{r}, t)}{\partial t} \right| \ll \omega_0 |\underline{\mathbf{E}}(\mathbf{r}, t)|. \quad (15b)$$

In (15a), “c.c.” stands for “complex conjugate”. Similarly, without loss of generality we can assume that

$$\mathbf{P}_L(\mathbf{r}, t) = \frac{1}{2} (\underline{\mathbf{P}}_L(\mathbf{r}, t) \exp(-i\omega_0 t) + c.c.) \quad (15c)$$

and

$$\mathbf{P}^{(3)}(\mathbf{r}, t) = \frac{1}{2} (\underline{\mathbf{P}}^{(3)}(\mathbf{r}, t) \exp(-i\omega_0 t) + c.c.) \quad (15d)$$

Remark 1. *It has been shown, see [5] that the slowly varying approximation is actually redundant and its (artificial) necessity arises because one has deduced from Maxwell’s equations, which are inherently first-order, a second-order equation (the wave equation). Some authors thus use an alternative approach, the coupled-mode theory, which deals directly with first-order equations without need for the slowly varying approximation.*

From the propagation equation (4), we have

$$\mathbf{curl} \mathbf{curl} \mathbf{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = -\mu_0 \frac{\partial^2}{\partial t^2} \mathbf{P}_L - \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{P}^{(3)}. \quad (16)$$

Using the representation (15a), we obtain

$$\frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \left(\left(\frac{\partial^2}{\partial t^2} \underline{\mathbf{E}}(\mathbf{r}, t) - 2i\omega_0 \frac{\partial}{\partial t} \underline{\mathbf{E}}(\mathbf{r}, t) - \omega_0^2 \underline{\mathbf{E}}(\mathbf{r}, t) \right) e^{-i\omega_0 t} + c.c. \right) \quad (17)$$

with analogous expressions for \mathbf{P}_L and $\mathbf{P}^{(3)}$. Therefore, we deduce from (16) the following wave equation for $\underline{\mathbf{E}}$

$$\begin{aligned} \mathbf{curl} \mathbf{curl} \underline{\mathbf{E}} + \frac{1}{c^2} \left(\frac{\partial^2}{\partial t^2} \underline{\mathbf{E}} - 2i\omega_0 \frac{\partial}{\partial t} \underline{\mathbf{E}} - \omega_0^2 \underline{\mathbf{E}} \right) = \\ -\mu_0 \left(\frac{\partial^2}{\partial t^2} \underline{\mathbf{P}}_L - 2i\omega_0 \frac{\partial}{\partial t} \underline{\mathbf{P}}_L - \omega_0^2 \underline{\mathbf{P}}_L + \frac{\partial^2}{\partial t^2} \underline{\mathbf{P}}^{(3)} - 2i\omega_0 \frac{\partial}{\partial t} \underline{\mathbf{P}}^{(3)} - \omega_0^2 \underline{\mathbf{P}}^{(3)} \right). \end{aligned} \quad (18)$$

Using the properties of the Fourier Transform (namely $\mathcal{F} \left(\frac{\partial^n}{\partial t^n} \underline{\mathbf{E}} \right) = (-i\omega)^n \widehat{\underline{\mathbf{E}}}$), we obtain

$$\begin{aligned} \mathcal{F} \left(\frac{\partial^2}{\partial t^2} \underline{\mathbf{E}} - 2i\omega_0 \frac{\partial}{\partial t} \underline{\mathbf{E}} - \omega_0^2 \underline{\mathbf{E}} \right) (\mathbf{r}, \delta\omega) &= -\delta\omega^2 \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega) - 2\omega_0 \delta\omega \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega) - \omega_0^2 \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega) \\ &= -(\omega_0 + \delta\omega)^2 \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega), \end{aligned}$$

with similar relations for \mathbf{P}_L and $\mathbf{P}^{(3)}$. Thus, the wave equation (16) in the frequency domain reads

$$\mathbf{curl} \mathbf{curl} \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega) - \frac{(\omega_0 + \delta\omega)^2}{c^2} \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega) = \mu_0 (\omega_0 + \delta\omega)^2 \widehat{\underline{\mathbf{P}}}_L(\mathbf{r}, \delta\omega) + \mu_0 (\omega_0 + \delta\omega)^2 \widehat{\underline{\mathbf{P}}}^{(3)}(\mathbf{r}, \delta\omega). \quad (19)$$

From (8), we have $\mathbf{P}_L(\mathbf{r}, t) = \varepsilon_0 (\bar{R}^{(1)} * \mathbf{E})(\mathbf{r}, t)$ where $*$ refers to the convolution product of functions. From (15a) and (15c), we deduce that

$$\underline{\mathbf{P}}_L(\mathbf{r}, t) e^{-i\omega_0 t} = \varepsilon_0 (\bar{R}^{(1)} * e^{-i\omega_0 t} \underline{\mathbf{E}})(\mathbf{r}, t)$$

and then using the properties of the Fourier Transform, that

$$\widehat{\underline{\mathbf{P}}}_L(\mathbf{r}, \omega - \omega_0) = \varepsilon_0 \bar{\chi}^{(1)}(\omega) \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \omega - \omega_0). \quad (20)$$

Thus, the wave equation (19) can also be written as

$$\mathbf{curl} \mathbf{curl} \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega) - \frac{(\omega_0 + \delta\omega)^2}{c^2} \left(1 + \bar{\chi}^{(1)}(\omega_0 + \delta\omega) \right) \widehat{\underline{\mathbf{E}}}(\mathbf{r}, \delta\omega) = \mu_0 (\omega_0 + \delta\omega)^2 \widehat{\underline{\mathbf{P}}}^{(3)}(\mathbf{r}, \delta\omega) \quad (21)$$

where we have used the relation $\varepsilon_0 \mu_0 = 1/c^2$.

We introduce the frequency-dependent relative electric permittivity as

$$\varepsilon(\omega) = 1 + \bar{\chi}^{(1)}(\omega). \quad (22)$$

As $\bar{\chi}^{(1)}(\omega)$ is in general complex, so is $\varepsilon(\omega)$. Its real and imaginary parts are related to the so-called *refractive index* $n(\omega)$ and *absorption coefficient* (also termed *attenuation coefficient*) $\alpha(\omega)$ by [1]

$$n(\omega) = \sqrt{1 + \text{Re}(\bar{\chi}^{(1)}(\omega))} \approx 1 + \frac{1}{2} \text{Re}(\bar{\chi}^{(1)}(\omega)) \quad (23a)$$

$$\alpha(\omega) = \frac{\omega}{n(\omega)c} \text{Im}(\bar{\chi}^{(1)}(\omega)) \quad (23b)$$

so that

$$\varepsilon(\omega) = n^2(\omega) + i \frac{n(\omega)c}{\omega} \alpha(\omega) \approx \left(n(\omega) + i \frac{c}{2\omega} \alpha(\omega) \right)^2. \quad (24)$$

Finally, we conclude that the wave equation (16) in the frequency domain reads

$$\mathbf{curl} \mathbf{curl} \hat{\mathbf{E}}(\mathbf{r}, \delta\omega) - \frac{(\omega_0 + \delta\omega)^2}{c^2} \varepsilon(\omega_0 + \delta\omega) \hat{\mathbf{E}}(\mathbf{r}, \delta\omega) = \mu_0(\omega_0 + \delta\omega)^2 \hat{\mathbf{P}}^{(3)}(\mathbf{r}, \delta\omega). \quad (25)$$

Equation (25) can be re-written by using the calculus relation $\mathbf{curl} \mathbf{curl} \hat{\mathbf{E}} = \nabla(\text{div} \hat{\mathbf{E}}) - \Delta \hat{\mathbf{E}}$. However, on the contrary to linear optics for isotropic source-free media where the first term is vanishing because the divergence free electric flux density equation $\text{div} \mathbf{D} = 0$ implies $\text{div} \mathbf{E} = 0$, in nonlinear optics it is generally not the case. The term $\nabla(\text{div} \mathbf{E})$ is non-vanishing even for isotropic materials, as a consequence of the more general relation (2) between \mathbf{E} and \mathbf{D} . Fortunately, this term can often be shown to be small, even when it does not vanish identically, see [2, p. 71]. Let us explain the approximation. From Maxwell's equation (1c), we have

$$0 = \text{div}(\mathbf{D}) = \varepsilon_0 \text{div}(\mathbf{E}) + \text{div}(\mathbf{P}) = \varepsilon_0 \text{div}(\mathbf{E}) + \text{div}(\mathbf{P}_L) + \text{div}(\mathbf{P}^{(3)}).$$

Since the linear polarization represents the dominant contribution to the polarization, we may consider the approximation

$$\varepsilon_0 \text{div}(\mathbf{E}) + \text{div}(\mathbf{P}_L) \approx 0$$

that is to say, in the frequency domain

$$\varepsilon_0 \text{div}(\hat{\mathbf{E}}) + \text{div}(\hat{\mathbf{P}}_L) \approx 0.$$

As $\varepsilon(\omega)$ and $\bar{\chi}^{(1)}(\omega)$ are often independent of the spatial coordinates in both the core and the cladding of step-index fibers [1], using relation (20) the above approximation can be reformulated as

$$\begin{aligned} \varepsilon_0 \text{div}(\hat{\mathbf{E}}(\mathbf{r}, \delta\omega)) + \text{div}(\hat{\mathbf{P}}_L(\mathbf{r}, \delta\omega)) &= \varepsilon_0 (1 + \bar{\chi}^{(1)}(\omega_0 + \delta\omega)) \text{div}(\hat{\mathbf{E}}(\mathbf{r}, \delta\omega)) \\ &= \varepsilon_0 \varepsilon(\omega_0 + \delta\omega) \text{div}(\hat{\mathbf{E}}(\mathbf{r}, \delta\omega)) \approx 0 \end{aligned}$$

so that $\text{div}(\hat{\mathbf{E}}) \approx 0$.

Therefore, in the sequel we will use the following simplified form of the wave equation (25)

$$\Delta \hat{\mathbf{E}}(\mathbf{r}, \delta\omega) + \frac{(\omega_0 + \delta\omega)^2}{c^2} \varepsilon(\omega_0 + \delta\omega) \hat{\mathbf{E}}(\mathbf{r}, \delta\omega) = -\mu_0(\omega_0 + \delta\omega)^2 \hat{\mathbf{P}}^{(3)}(\mathbf{r}, \delta\omega). \quad (26)$$

2.4 A simplified framework

We now introduce some additional assumptions in the model.

In optical fibers, the longitudinal component E_z of the electric field \mathbf{E} corresponding to the fundamental fiber mode HE_{11} is generally much smaller than the transverse one and the latter dominates the non-linear effects. A major simplification consists in assuming that the polarization state of the

incident light is preserved during its propagation inside an optical fiber and to fix things we assume in the sequel that the electric field is polarized along the \mathbf{e}_x direction. Even if this is not really the case in practice, this assumption is quite relevant when the fiber length is small as it is the case for the Fabry-Perot resonator investigated in the Rollmops project.¹

Thus, from now on, we assume the electric field polarized in the \mathbf{e}_x direction orthogonal to the propagation direction \mathbf{e}_z defined by the fiber axis. Moreover, we neglect the variation of the slowly varying electric field envelope \underline{E} with respect to the transverse variables x and y . This approximation is referred to as the *paraxial approximation*. Thus, we assume that the electric field has the following expression deduced from the more general expression (15a)

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} \left(\underline{E}(z, t) \exp(-i\omega_0 t) \mathbf{e}_x + c.c. \right) \quad (27)$$

where the scalar function \underline{E} is referred to as the *slowly varying electric field envelope*. We recall that in (27), it is assumed that the envelope of the electric field wave varies slowly in time compared to a period $T = \frac{2\pi}{\omega_0}$, see (15b).

From (14), we have

$$\begin{aligned} \mathbf{P}^{(3)}(z, t) \cdot \mathbf{e}_x &= \frac{1}{8} \varepsilon_0 \chi_{1111}^{(3)} \left(\underline{E}(z, t) \exp(-i\omega_0 t) + c.c. \right)^3 \\ &= \frac{1}{8} \varepsilon_0 \chi_{1111}^{(3)} \left(\underline{E}^3(z, t) \exp(-3i\omega_0 t) + 3\underline{E}^2(z, t)\underline{E}^*(z, t) \exp(-i\omega_0 t) \right) + c.c. \end{aligned}$$

In this expression of the non-linear polarization, the quantity $\underline{E}^3(z, t) \exp(-3i\omega_0 t)$ is related to third harmonic generation [11], a phenomenon not relevant in the context of this study whereas the quantity $\underline{E}^2(z, t)\underline{E}^*(z, t) \exp(-i\omega_0 t)$ models the Kerr effect [11]. Thus, neglecting third harmonic generation and taking into account the Kerr effect only, we obtain

$$P_1^{(3)}(z, t) = \mathbf{P}^{(3)}(z, t) \cdot \mathbf{e}_x = \frac{3}{8} \varepsilon_0 \chi_{1111}^{(3)} \underline{E}^2(z, t)\underline{E}^*(z, t) \exp(-i\omega_0 t) + c.c. \quad (28)$$

In accordance with (15d), we deduce that $\underline{\mathbf{P}}^{(3)}(z, t) = \underline{P}^{(3)}(z, t) \mathbf{e}_x$ where

$$\underline{P}^{(3)}(z, t) = \frac{3}{4} \varepsilon_0 \chi_{1111}^{(3)} \underline{E}^2(z, t)\underline{E}^*(z, t) \quad (29)$$

From (26), we deduce that the scalar slowly varying electric field envelope \underline{E} is such that in the spectral domain

$$\frac{\partial^2}{\partial z^2} \widehat{\underline{E}}(z, \delta\omega) + \frac{(\omega_0 + \delta\omega)^2}{c^2} \varepsilon(\omega_0 + \delta\omega) \widehat{\underline{E}}(z, \delta\omega) = -\mu_0(\omega_0 + \delta\omega)^2 \widehat{\underline{P}}^{(3)}(z, \delta\omega). \quad (30)$$

We define $k(\omega) \stackrel{\text{def}}{=} \frac{\omega}{c} \sqrt{\varepsilon(\omega)}$ and we consider the Taylor expansion of $k(\omega)$ around the frequency ω_0 :

$$k(\omega) = k(\omega_0) + (\omega - \omega_0)k'(\omega_0) + \frac{1}{2}(\omega - \omega_0)^2 k''(\omega_0) + \mathcal{O}_{\omega_0}((\omega - \omega_0)^3)$$

and

$$k(\omega)^2 = k(\omega_0)^2 + 2(\omega - \omega_0)k(\omega_0)k_1 + (\omega - \omega_0)^2(k_1^2 + k(\omega_0)k_2) + \mathcal{O}_{\omega_0}((\omega - \omega_0)^3)$$

where we have set $k_1 = k'(\omega_0)$ and $k_2 = k''(\omega_0)$. Truncating the Taylor expansion to its first three terms, we obtain from the propagation equation (30) that

$$\frac{\partial^2}{\partial z^2} \widehat{\underline{E}}(z, \delta\omega) + (k(\omega_0)^2 + 2k_1 k(\omega_0) \delta\omega + (k_1^2 + k(\omega_0)k_2) \delta\omega^2) \widehat{\underline{E}}(z, \delta\omega) = -\mu_0 \omega_0^2 \widehat{\underline{P}}^{(3)}(z, \delta\omega) \quad (31)$$

where we have made the approximation $(\omega_0 + \delta\omega)^2 \approx \omega_0^2$ in the RHS of equation (30), see remark 2 below.

¹For an overview of the way of modeling light-wave propagation without such an assumption, we refer to [8, chap. 1].

We have taken into account the constitutive relation (9) for the linear polarization in the spectral domain. Now, in order to take into account the constitutive relation (14a) for the non-linear polarization, we must write (30) back into the temporal domain. To obtain the wave equation for $A(z, t)$ in the time domain, we multiply each side of (31) by $\frac{1}{2\pi} e^{-i\delta\omega t}$ and integrate for $\delta\omega$ along the real line. By inverse Fourier Transform, we obtain

$$\begin{aligned} & \bullet \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\partial^2}{\partial z^2} \widehat{E}(z, \delta\omega) e^{-i\delta\omega t} d\delta\omega = \frac{\partial^2}{\partial z^2} E(z, t) \\ & \bullet \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta\omega^\ell \widehat{E}(z, \delta\omega) e^{-i\delta\omega t} d\delta\omega = i^\ell \frac{\partial^\ell}{\partial t^\ell} E(z, t) \\ & \bullet \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{P}^{(3)}(z, \delta\omega) e^{-i\delta\omega t} d\delta\omega = \underline{P}^{(3)}(z, t) \end{aligned}$$

It follows from (31) that $\underline{E}(z, t)$ is such that

$$\frac{\partial^2 \underline{E}}{\partial z^2}(z, t) + k(\omega_0)^2 \underline{E}(z, t) + 2ik(\omega_0)k_1 \frac{\partial \underline{E}}{\partial t}(z, t) - (k_1^2 + k(\omega_0)k_2) \frac{\partial^2 \underline{E}}{\partial t^2}(z, t) = -\mu_0 \omega_0^2 \underline{P}^{(3)}(z, t) \quad (32)$$

and taking into account relation (29), we obtain the following propagation equation for $\underline{E}(z, t)$

$$\begin{aligned} \frac{\partial^2 \underline{E}}{\partial z^2}(z, t) + k(\omega_0)^2 \underline{E}(z, t) + 2ik(\omega_0)k_1 \frac{\partial \underline{E}}{\partial t}(z, t) - (k_1^2 + k(\omega_0)k_2) \frac{\partial^2 \underline{E}}{\partial t^2}(z, t) \\ = -\frac{3}{4} \frac{\omega_0^2}{c^2} \chi_{1111}^{(3)} \underline{E}^2(z, t) \underline{E}^*(z, t) \end{aligned} \quad (33)$$

Remark 2. The expression in the RHS of equation (32) obtained by the approximation $(\omega_0 + \delta\omega)^2 \approx \omega_0^2$ we have made in the RHS of equation (30) to obtain (31). If we had skipped this approximation, the RHS of equation (32) would be

$$-\mu_0 \left(\omega_0^2 \underline{P}^{(3)}(z, t) + 2i\omega_0 \frac{\partial}{\partial t} \underline{P}^{(3)}(z, t) - \frac{\partial^2}{\partial t^2} \underline{P}^{(3)}(z, t) \right)$$

with $\underline{P}^{(3)}$ given by (29). Note that we have

$$\frac{\partial}{\partial t} \underline{P}^{(3)}(z, t) = \frac{3}{4} \varepsilon_0 \chi_{1111}^{(3)} \left(2\underline{E}(z, t) \underline{E}^*(z, t) \frac{\partial}{\partial t} \underline{E}(z, t) + \underline{E}^2(z, t) \frac{\partial}{\partial t} \underline{E}^*(z, t) \right)$$

so that

$$\left| \frac{\partial}{\partial t} \underline{P}^{(3)}(z, t) \right| \leq \frac{9}{4} \varepsilon_0 \chi_{1111}^{(3)} |\underline{E}(z, t)|^2 \left| \frac{\partial}{\partial t} \underline{E}(z, t) \right|.$$

Under the slowly time varying envelope approximation, see (15b), we have $\left| \frac{\partial}{\partial t} \underline{E}(z, t) \right| \ll \omega_0 |\underline{E}(z, t)|$ and therefore

$$\left| \frac{\partial}{\partial t} \underline{P}^{(3)}(z, t) \right| \ll \frac{9}{4} \varepsilon_0 \chi_{1111}^{(3)} \omega_0 |\underline{E}(z, t)|^3 = 3\omega_0 |\underline{P}^{(3)}(z, t)|.$$

A similar relation can be obtained for the second order derivative $\left| \frac{\partial^2}{\partial t^2} \underline{P}^{(3)}(z, t) \right|$. Thus, the approximation $(\omega_0 + \delta\omega)^2 \approx \omega_0^2$ we have made in the RHS of equation (30) to obtain (31) amounts to using the slowly time varying envelope approximation and does not constitute an additional assumption.

From (24), we have

$$k(\omega) = \frac{\omega}{c} \sqrt{\varepsilon(\omega)} = \frac{\omega}{c} \left(n(\omega) + i \frac{c}{2\omega} \alpha(\omega) \right). \quad (34)$$

We assume that the attenuation coefficient α does not vary significantly with respect to the frequency ω , that is to say in the sequel we make the approximation $\alpha(\omega) \approx \alpha(\omega_0)$ for all $\omega \in \mathbb{R}$. Moreover,

$$k(\omega_0) = \frac{\omega_0}{c} n(\omega_0) + \frac{i}{2} \alpha(\omega_0) = k_0 + \frac{i}{2} \alpha(\omega_0) \quad (35a)$$

where $k_0 = \frac{\omega_0}{c} n(\omega_0) = \text{Re}(k(\omega_0))$ and

$$k_1 = k'(\omega_0) = \frac{1}{c} \left(n(\omega_0) + \omega_0 \frac{dn}{d\omega}(\omega_0) \right) = \frac{n_g}{c} \quad (35b)$$

$$k_2 = k''(\omega_0) = \frac{1}{c} \left(2 \frac{dn}{d\omega}(\omega_0) + \omega_0 \frac{d^2n}{d\omega^2}(\omega_0) \right) \quad (35c)$$

where n_g is the so-called the *group index*.

Physically speaking, the envelope of an optical pulse moves at the group velocity $v_g \stackrel{\text{def}}{=} \frac{c}{n_g}$ while the parameter k_2 represents dispersion of the group velocity and is responsible for pulse broadening. This phenomenon is known as the group-velocity dispersion (GVD) and k_2 is the GVD parameter.

Since the attenuation coefficient in (35a) can be considered as negligible with respect to $k_0 = \frac{\omega_0}{c} n(\omega_0)$, we make the additional assumptions

$$k(\omega_0)^2 = \left(k_0 + \frac{i}{2} \alpha(\omega_0) \right)^2 = k_0^2 + ik_0 \alpha(\omega_0) - \frac{1}{4} \alpha(\omega_0)^2 \approx k_0^2 + ik_0 \alpha(\omega_0) \quad (36a)$$

$$k(\omega_0)k_1 \approx k_0k_1 = \frac{\omega_0}{c^2} n(\omega_0)n_g = \frac{\omega_0}{v_g^2} \frac{n(\omega_0)}{n_g}. \quad (36b)$$

We also assume that $k_1^2 \ll k(\omega_0)k_2$ and we use the approximation $k(\omega_0)k_2 \approx k_0k_2$. The propagation equation (33) then reads

$$\frac{\partial^2 \underline{E}}{\partial z^2}(z, t) + (k_0^2 + ik_0 \alpha_0) \underline{E}(z, t) + 2ik_0k_1 \frac{\partial \underline{E}}{\partial t}(z, t) - k_0k_2 \frac{\partial^2 \underline{E}}{\partial t^2}(z, t) = -\frac{3\omega_0^2}{4c^2} \chi_{1111}^{(3)} \underline{E}^2(z, t) \underline{E}^*(z, t) \quad (37)$$

where $\alpha_0 = \alpha(\omega_0)$.

2.5 Bidirectional propagation in the Fabry-Perot resonator

When neglecting the non-linear phenomenon in an optical fiber, we show that the electric field reads $E(z, t) = E_0 e^{-i(\omega_0 t - \tilde{\gamma} z)} \mathbf{e}_x + \text{c.c.}$ where the so-called *mode propagation constant* $\tilde{\gamma}$ is such that

$$\tilde{\gamma} = \frac{\omega_0}{c} n_{\text{eff}}$$

and n_{eff} refers to the *effective index of the mode*. Taking into account this information and the fact that the non-linear polarization effects are small compared to the effects induced by the linear polarization, we assume that the slowly varying electric field amplitude in the Fabry-Perot cavity can be expressed in the form

$$\underline{E}(z, t) = E_F(z, t) e^{i\gamma z} + E_B(z, t) e^{-i\gamma z} + \text{c.c.} \quad (38)$$

where E_F and E_B denote respectively the time and position slowly varying electric field amplitude in the forward and backward direction and γ is such that

$$\gamma \stackrel{\text{def}}{=} \frac{\omega_0}{c} n(\omega_0) = k_0. \quad (39)$$

Thus, in addition to the slowly *time* varying envelope approximation, see (15b), we will make use of the slowly *space* varying envelope approximation that can be expressed as follows:

$$\left| \frac{\partial^2}{\partial z^2} E_F(z, t) \right| \ll 2\gamma \left| \frac{\partial}{\partial z} E_F(z, t) \right|, \quad \left| \frac{\partial^2}{\partial z^2} E_B(z, t) \right| \ll 2\gamma \left| \frac{\partial}{\partial z} E_B(z, t) \right|. \quad (40)$$

Let us now deduce from (37) the propagation equations satisfied by E_F and E_B . From (38), we deduce that

$$\begin{aligned} \underline{E}^2(z, t) \underline{E}^*(z, t) &= \left(E_F(z, t) e^{i\gamma z} + E_B(z, t) e^{-i\gamma z} \right)^2 \left(E_F^*(z, t) e^{-i\gamma z} + E_B^*(z, t) e^{i\gamma z} \right) \\ &= \left(|E_F(z, t)|^2 + 2|E_B(z, t)|^2 \right) E_F(z, t) e^{i\gamma z} \\ &\quad + \left(|E_B(z, t)|^2 + 2|E_F(z, t)|^2 \right) E_B(z, t) e^{-i\gamma z} \\ &\quad + E_F^2(z, t) E_B^*(z, t) e^{3i\gamma z} + E_B^2(z, t) E_F^*(z, t) e^{-3i\gamma z} \end{aligned} \quad (41)$$

We also have

$$\begin{aligned} \frac{\partial^2}{\partial z^2} \underline{E}(z, t) = & \left(\frac{\partial^2}{\partial z^2} E_F(z, t) + 2i\gamma \frac{\partial}{\partial z} E_F(z, t) - \gamma^2 E_F(z, t) \right) e^{i\gamma z} \\ & + \left(\frac{\partial^2}{\partial z^2} E_B(z, t) - 2i\gamma \frac{\partial}{\partial z} E_B(z, t) - \gamma^2 E_B(z, t) \right) e^{-i\gamma z} \end{aligned} \quad (42)$$

Using the slowly space varying envelope approximation (40), we have

$$\frac{\partial^2}{\partial z^2} \underline{E}(z, t) \approx \left(2i\gamma \frac{\partial}{\partial z} E_F(z, t) - \gamma^2 E_F(z, t) \right) e^{i\gamma z} - \left(2i\gamma \frac{\partial}{\partial z} E_B(z, t) + \gamma^2 E_B(z, t) \right) e^{-i\gamma z}. \quad (43)$$

It follows that the propagation equation (37) reads

$$\begin{aligned} & \left(2i\gamma \frac{\partial}{\partial z} E_F(z, t) - (\gamma^2 - k_0^2 - i\alpha_0 k_0) E_F(z, t) + \frac{2i\gamma^2}{\omega_0} \frac{\partial}{\partial t} E_F(z, t) - k_0 k_2 \frac{\partial^2}{\partial t^2} E_F(z, t) \right) e^{i\gamma z} \\ & + \left(-2i\gamma \frac{\partial}{\partial z} E_B(z, t) - (\gamma^2 - k_0^2 - i\alpha_0 k_0) E_B(z, t) + \frac{2i\gamma^2}{\omega_0} \frac{\partial}{\partial t} E_B(z, t) - k_0 k_2 \frac{\partial^2}{\partial t^2} E_B(z, t) \right) e^{-i\gamma z} \\ = & -\frac{3\omega_0^2}{4c^2} \chi_{1111}^{(3)} \left(|E_F(z, t)|^2 + 2|E_B(z, t)|^2 \right) E_F(z, t) e^{i\gamma z} \\ & - \frac{3\omega_0^2}{4c^2} \chi_{1111}^{(3)} \left(|E_B(z, t)|^2 + 2|E_F(z, t)|^2 \right) E_B(z, t) e^{-i\gamma z} \\ & - \frac{3\omega_0^2}{4c^2} \chi_{1111}^{(3)} \left(E_F^2(z, t) E_B^*(z, t) e^{3i\gamma z} + E_B^2(z, t) E_F^*(z, t) e^{-3i\gamma z} \right) \end{aligned} \quad (44)$$

In (44), the terms $e^{\pm 3i\gamma z}$ are connected to the terms $e^{\pm 3i\omega_0 t}$ in the full expression of the electric field

$$E(z, t) = E_F(z, t) e^{i(\gamma z - \omega_0 t)} + E_B(z, t) e^{-i(\gamma z + \omega_0 t)} + \text{c.c.}$$

and as explain on page 8, they are related to third harmonic generation [11], a phenomenon not relevant in the context of this study. Thus, from now on, we will drop these two terms. Moreover, using relation $\gamma^2 = k_0^2$, see (39), equation (44) reads

$$\begin{aligned} & \left(2i\gamma \frac{\partial}{\partial z} E_F(z, t) + i\alpha_0 k_0 E_F(z, t) + \frac{2i\gamma^2}{\omega_0} \frac{\partial}{\partial t} E_F(z, t) - k_0 k_2 \frac{\partial^2}{\partial t^2} E_F(z, t) \right) e^{i\gamma z} \\ & + \left(-2i\gamma \frac{\partial}{\partial z} E_B(z, t) + i\alpha_0 k_0 E_B(z, t) + \frac{2i\gamma^2}{\omega_0} \frac{\partial}{\partial t} E_B(z, t) - k_0 k_2 \frac{\partial^2}{\partial t^2} E_B(z, t) \right) e^{-i\gamma z} \\ = & -\frac{3\omega_0^2}{4c^2} \chi_{1111}^{(3)} \left(|E_F(z, t)|^2 + 2|E_B(z, t)|^2 \right) E_F(z, t) e^{i\gamma z} \\ & - \frac{3\omega_0^2}{4c^2} \chi_{1111}^{(3)} \left(|E_B(z, t)|^2 + 2|E_F(z, t)|^2 \right) E_B(z, t) e^{-i\gamma z}. \end{aligned} \quad (45)$$

We now can “separate” the different terms involved in equation (45) according to the power of $e^{i\gamma z}$ they are related to. The underlying idea, see [10, sect. 14.3], is that two scales are involved in equation (45) for the space variable. The first one is the scale characterized by the cavity length L around 10^{-2} m. The related variable is $z \in [0, L]$. The second one is the wavelength spatial scale encountered in the terms $e^{\pm i\gamma z}$. The variable

$$\varphi \stackrel{\text{def}}{=} \gamma z = \frac{\omega_0 n_0}{c} z = \frac{2\pi n_0}{\lambda_0} z$$

with λ_0 typically around $1 \mu\text{m}$ for light-wave can be considered as a “fast” variable compared to z , the “slow” variable, since $\frac{2\pi n_0}{\lambda_0} \approx 6\pi \cdot 10^6 \text{ m}^{-1}$.

Equation (45) can be expressed in the form

$$\sum_{p \in \{\pm 1\}} A_p(z, t) e^{ip\gamma z} = 0 \quad (46)$$

where the functions A_p , that can readily be identified in (45), inherit from the slowly space varying envelope approximation assumption for E_B and E_F , see (40), the same property. In particular, we have

$$\left| \frac{\partial}{\partial z^2} A_p(z, t) \right| \ll \gamma \left| \frac{\partial}{\partial z} A_p(z, t) \right|, \quad \left| \frac{\partial}{\partial z} A_p(z, t) \right| \ll |\gamma A_p(z, t)|. \quad (47)$$

For $q \in \mathbb{Z}$, let's multiply (46) by $e^{iq\gamma z}$ and integrate over $[0, \tilde{L}]$ with $\tilde{L} \leq L$ to be specified latter; we obtain

$$\sum_{p \in \{\pm 1\}} \int_0^{\tilde{L}} A_p(z, t) e^{ip\gamma z} e^{iq\gamma z} dz = 0.$$

Then by the change of variables $\varphi = \gamma z$, we get

$$\frac{1}{\gamma} \sum_{p \in \{\pm 1\}} \int_0^{\gamma \tilde{L}} A_p\left(\frac{\varphi}{\gamma}, t\right) e^{ip\varphi} e^{iq\varphi} d\varphi = 0. \quad (48)$$

According to [10, sect. 8.3] in cavities, the phase plays a dominant role because the mirrors create a feedback loop that gives rise to multiple interference of light. The interference is fully constructive when the input light has a frequency ω_0 such that $\omega_0 = \frac{\pi c j}{L}$ with $j \in \mathbb{N}^*$. In such a case, we have $\gamma \tilde{L} = \pi j n(\omega_0) \frac{\tilde{L}}{L}$. We choose \tilde{L} such that $n(\omega_0) \frac{\tilde{L}}{L} = 2$, that is to say $\tilde{L} = \frac{2}{n(\omega_0)} L$. Then, equation (48) reads

$$\sum_{p \in \{\pm 1\}} \int_0^{2\pi} \tilde{A}_p(\varphi, t) e^{i(p+q)\varphi} d\varphi = 0 \quad (49)$$

where we have set $\tilde{A}_p(\varphi, t) \stackrel{\text{def}}{=} A_p\left(\frac{\varphi}{\gamma}, t\right)$. We have

$$\forall \ell \in \mathbb{N}^* \quad \frac{\partial^\ell}{\partial \varphi^\ell} \tilde{A}_p(\varphi, t) = \gamma^\ell \frac{\partial^\ell}{\partial z^\ell} A_p(z, t)$$

and we deduce from the slowly space varying envelope approximation assumption for A , see (50), that

$$\forall \ell \in \mathbb{N}^* \quad \left| \frac{\partial^\ell}{\partial \varphi^\ell} \tilde{A}_p(\varphi, t) \right| \ll |\tilde{A}_p(\varphi, t)|. \quad (50)$$

Then, by Taylor expansion

$$\forall \varphi \in [0, 2\pi] \quad \tilde{A}_p(\varphi, t) = \tilde{A}_p(0, t) + \varphi \frac{\partial}{\partial \varphi} \tilde{A}_p(0, t) + \frac{1}{2} \varphi^2 \frac{\partial^2}{\partial \varphi^2} \tilde{A}_p(0, t) + \dots \quad (51)$$

we can deduce that $\tilde{A}_p(\varphi, t)$ is almost constant over $[0, 2\pi]$. Thus, we can approximate relation (48) by

$$\sum_{p \in \{\pm 1\}} \tilde{A}_p(\varphi, t) \int_0^{2\pi} e^{i(p+q)\varphi} d\varphi = 0. \quad (52)$$

Since the family of functions $\varphi \mapsto e^{ip\varphi}$ is orthogonal for the scalar product in $\mathbb{L}^2([0, 2\pi[)$, we conclude that for all $q \in \{\pm 1\}$ we have $\tilde{A}_q(\varphi, t) = 0$.

Finally, we deduce from equation (45) that

$$\begin{aligned} 2ik_0 \frac{\partial}{\partial z} E_F(z, t) + i\alpha_0 k_0 E_F(z, t) + 2ik_0 k_1 \frac{\partial}{\partial t} E_F(z, t) \\ - k_0 k_2 \frac{\partial^2}{\partial t^2} E_F(z, t) = -\frac{3}{4} \frac{\omega_0^2}{c^2} \chi_{1111}^{(3)} \left(|E_F(z, t)|^2 + 2|E_B(z, t)|^2 \right) E_F(z, t) \end{aligned} \quad (53a)$$

$$\begin{aligned} -2ik_0 \frac{\partial}{\partial z} E_B(z, t) + i\alpha_0 k_0 E_B(z, t) + 2ik_0 k_1 \frac{\partial}{\partial t} E_B(z, t) \\ - k_0 k_2 \frac{\partial^2}{\partial t^2} E_B(z, t) = -\frac{3}{4} \frac{\omega_0^2}{c^2} \chi_{1111}^{(3)} \left(|E_B(z, t)|^2 + 2|E_F(z, t)|^2 \right) E_B(z, t). \end{aligned} \quad (53b)$$

By multiplying equations (53a) and (53b) by $-i\frac{\omega_0}{2\gamma^2}$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} E_F(z, t) + v_g \frac{\partial}{\partial z} E_F(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} E_F(z, t) - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} E_F(z, t) \\ &\quad + i\frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} \left(|E_F(z, t)|^2 + 2|E_B(z, t)|^2 \right) E_F(z, t) \end{aligned} \quad (54a)$$

$$\begin{aligned} \frac{\partial}{\partial t} E_B(z, t) - v_g \frac{\partial}{\partial z} E_B(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} E_B(z, t) - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} E_B(z, t) \\ &\quad + i\frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} \left(|E_B(z, t)|^2 + 2|E_F(z, t)|^2 \right) E_B(z, t) \end{aligned} \quad (54b)$$

where we have use the relation $k_1 \stackrel{\text{def}}{=} k'(\omega_0) = \frac{n_g}{c} = \frac{1}{v_g}$, see (35b), and the approximation $n_g \approx n(\omega_0) \stackrel{\text{def}}{=} n_0$.

Finally, the electric field component of light-wave, assumed to be polarized along \mathbf{e}_x , in a Fabry-Perot resonator containing a non-linear Kerr medium can be obtained in the framework that has been detailed above as

$$\mathbf{E}(z, t) = \left(E_F(z, t) e^{i\gamma z} + E_B(z, t) e^{-i\gamma z} \right) e^{-i\omega_0 t} \mathbf{e}_x + c.c. \quad (55)$$

where E_F and E_B , the slowly varying envelope of the forward and backward waves, are solutions to the non-linear coupled partial differential equations (54a) and (54b). It remains to specify the boundary conditions at the Fabry-Perot cavity ends, what we are going to do in the next section.

2.6 Boundary conditions at the Fabry-Perot cavity ends

The two ends of the Fabry-Perot cavity are made of two Bragg mirrors. We denote by L the length of the Fabry-Perot cavity and by r_1 and r_2 the reflection coefficients of the mirrors (defined as the modulus of the ratio of the complex amplitude of the reflected wave to that of the incident wave) respectively at the position $z = 0$ (front end) and $z = L$ (back end). We have $r_1, r_2 \in [0, 1]$.

We assume that an incident field with a sinusoidal time variation at the frequency ω_0 is injected in the front end of the Fabry-Perot cavity and that a second incident field with the same sinusoidal time variation at the frequency ω_0 is injected in the back end. In accordance with (27) and (38), the injected electric field at $z = 0$ reads

$$\mathbf{E}_{in,F}(t) = \frac{1}{2} \left(E_F^{in} \exp(-i\omega_0 t) \mathbf{e}_x + c.c. \right) \quad (56a)$$

whereas the injected electric field at $z = L$ reads

$$\mathbf{E}_{in,B}(t) = \frac{1}{2} \left(E_B^{in} \exp(-i\omega_0 t) \mathbf{e}_x + c.c. \right) \quad (56b)$$

We denote by \mathbf{t}_1 the transmission coefficient across the bragg mirror located at $z = 0$ and by \mathbf{t}_2 the transmission coefficient across the bragg mirror located at $z = L$. (The transmission coefficient is defined as the modulus of the complex amplitude of the transmitted wave to that of the incident wave.) We have $\mathbf{t}_1, \mathbf{t}_2 \in [0, 1]$.

The boundary conditions at $z = 0$ and $z = L$ read for all $t \in \mathbb{R}$

$$\underline{E}_F(0, t) = r_1 \underline{E}_B(0, t) + \mathbf{t}_1 E_F^{in} \quad (57a)$$

$$\underline{E}_B(L, t) = r_2 \underline{E}_F(L, t) + \mathbf{t}_2 E_B^{in} \quad (57b)$$

where according to (38), $\underline{E}_F(z, t) = E_F(z, t) e^{i\gamma z}$ and $\underline{E}_B(z, t) = E_B(z, t) e^{-i\gamma z}$. These boundary conditions can also be expressed as

$$E_F(0, t) = r_1 E_B(0, t) + \mathbf{t}_1 E_F^{in} \quad (58a)$$

$$E_B(L, t) = r_2 E_F(L, t) e^{2i\gamma L} + \mathbf{t}_2 E_B^{in} e^{i\gamma L} \quad (58b)$$

Following [3] and [10, eq. (14.15)], we can express boundary condition (58b) in a slightly different way. By periodicity of the complex exponential, we have for all $j \in \mathbb{Z}$

$$\begin{aligned} E_B(L, t) &= r_2 E_F(L, t) e^{2i\gamma L - 2\pi j i} + t_2 E_B^{in} e^{i\gamma L} \\ &= r_2 E_F(L, t) e^{-id_0} + t_2 E_B^{in} e^{i\gamma L} \end{aligned} \quad (58c)$$

where $d_0 = 2\pi j - 2\gamma L$. Let j be chosen such that $\omega_c = \frac{\pi j c}{n(\omega_0)L}$ is the cavity frequency the closest to the input frequency ω_0 . Then,

$$d_0 = 2\pi j - 2\gamma L = 2Ln(\omega_0) \frac{\omega_c - \omega_0}{c} \quad (58d)$$

and the parameter d_0 can be interpreted as the phase deviation from the cavity resonance.

2.7 An equivalent problem with simple boundary conditions

Our goal in this section is to introduce two new unknown functions F_F and F_B in place of E_F and E_B such that F_F and F_B satisfy the following boundary conditions

$$\forall t \in \mathbb{R} \quad F_F(0, t) = F_B(0, t) \quad (59a)$$

$$F_F(L, t) = F_B(L, t) \quad (59b)$$

We look for F_F and F_B according to the following relations

$$E_F(z, t) = e^{a_F(L-z)} \left(b_F F_F(z, t) + (L-z) c_F + z d_F \right) \quad (60a)$$

$$E_B(z, t) = e^{a_B z} \left(b_B F_B(z, t) + (L-z) c_B + z d_B \right) \quad (60b)$$

Boundary conditions (58a)–(58c) give rise to the following relations

$$e^{a_F L} \left(b_F F_F(0, t) + L c_F \right) = r_1 \left(b_B F_B(0, t) + L c_B \right) + t_1 E_F^{in} \quad (61a)$$

$$e^{a_B L} \left(b_B F_B(L, t) + L d_B \right) = r_2 \left(b_F F_F(L, t) + L d_F \right) e^{-id_0} + t_2 E_B^{in} e^{i\gamma L} \quad (61b)$$

Boundary conditions (59a)–(59b) are satisfied if

$$e^{a_F L} b_F = r_1 b_B \quad (62a)$$

$$L c_F e^{a_F L} = r_1 L c_B + t_1 E_F^{in} \quad (62b)$$

$$e^{a_B L} b_B = r_2 b_F e^{-id_0} \quad (62c)$$

$$L d_B e^{a_B L} = r_2 L d_F e^{-id_0} + t_2 E_B^{in} e^{i\gamma L}. \quad (62d)$$

From (62a) and (62c) we deduce that

$$a_B + a_F = \frac{1}{L} \left(\ln(r_1 r_2) - id_0 \right). \quad (63)$$

This relation is satisfied if we set for instance

$$a_B = \frac{1}{2L} \left(\ln(r_1 r_2) - id_0 \right) \quad (64a)$$

$$a_F = \frac{1}{2L} \left(\ln(r_1 r_2) - id_0 \right). \quad (64b)$$

With this choice, we have from (62a)

$$b_B = \sqrt{\frac{r_2}{r_1}} e^{\frac{-id_0}{2}} b_F. \quad (65)$$

Then, from (62b), we deduce that

$$\sqrt{r_2} e^{-\frac{id_0}{2}} c_F = \sqrt{r_1} c_B + \frac{t_1}{L\sqrt{r_1}} E_F^{in} \quad (66)$$

and from (62d), we deduce that

$$\sqrt{r_2} e^{-id_0} d_F = \sqrt{r_1} e^{-\frac{id_0}{2}} d_B - \frac{t_2}{L\sqrt{r_2}} e^{i\gamma L} E_B^{in}. \quad (67)$$

The parameters b_F, c_F, d_F remains free and we have

$$E_F(z, t) = e^{a_F(L-z)} \left(b_F F_F(z, t) + (L-z) c_F + z d_F \right) \quad (68a)$$

$$E_B(z, t) = e^{a_F z} \sqrt{\frac{r_2}{r_1}} e^{-\frac{id_0}{2}} \left(b_F F_B(z, t) + (L-z) \left(c_F - \sqrt{\frac{r_1}{r_2}} e^{i\frac{d_0}{2}} \frac{t_1}{Lr_1} E_F^{in} \right) + z \left(d_F + e^{i\gamma L} e^{id_0} \frac{t_2}{Lr_2} E_B^{in} \right) \right). \quad (68b)$$

In order to have similar expressions for the forward and backward electric field, we choose the following values for the parameters b_F, c_F, d_F :

$$b_F = 1 \quad (69a)$$

$$c_F = \sqrt{\frac{r_1}{r_2}} e^{i\frac{d_0}{2}} \frac{t_1}{2Lr_1} E_F^{in} \quad (69b)$$

$$d_F = -e^{id_0} e^{i\gamma L} \frac{t_2}{2Lr_2} E_B^{in} \quad (69c)$$

and we finally have

$$E_F(z, t) = e^{a_F(L-z)} \left(F_F(z, t) + c_F (L-z) + z d_F \right) = \rho^{\frac{1}{4}} e^{a_F(L-z)} \left(\rho^{-\frac{1}{4}} F_F(z, t) + \rho^{-\frac{1}{4}} c_F (L-z) + z \rho^{-\frac{1}{4}} d_F \right) \quad (70a)$$

$$E_B(z, t) = \sqrt{\frac{r_2}{r_1}} e^{-\frac{id_0}{2}} e^{a_F z} \left(F_B(z, t) - c_F (L-z) - d_F z \right) = \rho^{-\frac{1}{4}} e^{-\frac{id_0}{2}} e^{a_F z} \left(\rho^{-\frac{1}{4}} F_B(z, t) - \rho^{-\frac{1}{4}} c_F (L-z) - z \rho^{-\frac{1}{4}} d_F \right) \quad (70b)$$

where we have set $\rho = r_1/r_2$.

Remark 3. When the source electric field is launched in the Fabry-Perot resonator at the front end only, that is to say when $E_B^{in} = 0$, we have from (69c) $d_F = 0$. Moreover, when the two end mirrors have the same reflexivity, i.e. $r_1 = r_2$, we have $\rho = 1$.

It remains to deduce from (54a)–(54b) the propagation equations for F_F and F_B . We have

$$\frac{\partial^\ell E_F}{\partial t^\ell}(z, t) = e^{a_F(L-z)} \frac{\partial^\ell F_F}{\partial t^\ell}(z, t) \quad (71a)$$

$$\frac{\partial^\ell E_B}{\partial t^\ell}(z, t) = \sqrt{\frac{r_2}{r_1}} e^{-\frac{id_0}{2}} e^{a_F z} \frac{\partial^\ell F_B}{\partial t^\ell}(z, t) \quad (71b)$$

and

$$\frac{\partial E_F}{\partial z}(z, t) = e^{a_F(L-z)} \left(-a_F F_F(z, t) - c_F a_F (L-z) - a_F d_F z + \frac{\partial F_F}{\partial z}(z, t) - c_F + d_F \right) \quad (71c)$$

$$\frac{\partial E_B}{\partial z}(z, t) = \sqrt{\frac{r_2}{r_1}} e^{-\frac{id_0}{2}} e^{a_F z} \left(a_F F_B(z, t) - c_F a_F (L-z) - a_F d_F z + \frac{\partial F_B}{\partial z}(z, t) + c_F - d_F \right). \quad (71d)$$

We deduce from the propagation equations (54a)–(54b) that F_F and F_B satisfy

$$\begin{aligned} \frac{\partial}{\partial t} F_F(z, t) + v_g \frac{\partial}{\partial z} F_F(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} F_F(z, t) - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} (F_F(z, t) + c_F(L - z) + d_F z) \\ &\quad + i \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} N_F(F_F, F_B) (F_F(z, t) + c_F(L - z) + d_F z) \\ &\quad + v_g (a_F F_F(z, t) + c_F a_F(L - z) + a_F d_F z + c_F - d_F) \end{aligned} \quad (72a)$$

$$\begin{aligned} \frac{\partial}{\partial t} F_B(z, t) - v_g \frac{\partial}{\partial z} F_B(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} F_B(z, t) - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} (F_B(z, t) - c_F(L - z) - d_F z) \\ &\quad + i \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} N_B(F_F, F_B) (F_B(z, t) - c_F(L - z) - d_F z) \\ &\quad + v_g (a_F F_B(z, t) - c_F a_F(L - z) - a_F d_F z + c_F - d_F) \end{aligned} \quad (72b)$$

where

$$\begin{aligned} N_F(F_F, F_B) &= |E_F(z, t)|^2 + 2|E_B(z, t)|^2 = |e^{2a_F(z-L)} |F_F(z, t) + c_F(L - z) + z d_F|^2 \\ &\quad + 2 \frac{r_2}{r_1} |e^{2a_F z}| |F_B(z, t) - c_F(L - z) - d_F z|^2 \end{aligned} \quad (73a)$$

$$\begin{aligned} N_B(F_F, F_B) &= |E_B(z, t)|^2 + 2|E_F(z, t)|^2 = \frac{r_2}{r_1} |e^{2a_F z}| |F_B(z, t) - c_F(L - z) - d_F z|^2 \\ &\quad + 2 |e^{2a_F(z-L)} |F_F(z, t) + c_F(L - z) + z d_F|^2 \end{aligned} \quad (73b)$$

At this stage, we make use of the low transmission approximation [3, 10] that consists in assuming that the transmitted fields into the cavity $t_1 E_F^{in}$ and $t_2 E_B^{in}$ are small compared to the electric field propagating inside the cavity. Thus, the coefficients c_F and d_F are small compared to F_F and F_B and we make the following approximations

$$\frac{\alpha_0}{2} \frac{c}{n(\omega_0)} (F_F(z, t) + c_F(L - z) + d_F z) \approx \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} F_F(z, t) \quad (74a)$$

$$\frac{\alpha_0}{2} \frac{c}{n(\omega_0)} (F_B(z, t) - c_F(L - z) - d_F z) \approx \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} F_B(z, t) \quad (74b)$$

$$\frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} N_F(F_F, F_B) (F_F(z, t) + c_F(L - z) + d_F z) \approx \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} N_F(F_F, F_B) F_F(z, t) \quad (74c)$$

$$\frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} N_B(F_F, F_B) (F_B(z, t) - c_F(L - z) - d_F z) \approx \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} N_B(F_F, F_B) F_B(z, t) \quad (74d)$$

$$v_g (a_F F_F(z, t) + c_F a_F(L - z) + a_F d_F z + c_F - d_F) \approx v_g (a_F F_F(z, t) + c_F - d_F) \quad (74e)$$

$$v_g (a_F F_B(z, t) - c_F a_F(L - z) - a_F d_F z + c_F - d_F) \approx v_g (a_F F_B(z, t) + c_F - d_F) \quad (74f)$$

and

$$N_F(F_F, F_B) \approx |e^{2a_F(z-L)} |F_F(z, t)|^2 + 2 \frac{r_2}{r_1} |e^{2a_F z}| |F_B(z, t)|^2 \quad (75a)$$

$$N_B(F_F, F_B) \approx \frac{r_2}{r_1} |e^{2a_F z}| |F_B(z, t)|^2 + 2 |e^{2a_F(z-L)} |F_F(z, t)|^2 \quad (75b)$$

The coefficient a_F is given by (64b). We have $a_F = \frac{1}{2L} (\ln(r_1 r_2) - id_0)$ and it follows that

$$|e^{2a_F(z-L)}| = |e^{\frac{z-L}{L} \ln(r_1 r_2)}| \quad \text{and} \quad |e^{2a_F z}| = |e^{\frac{z}{L} \ln(r_1 r_2)}|.$$

In the situation considered here, the reflection coefficients r_1 and r_2 are assumed to be close to the value 1. Thus we may consider the approximation $|e^{2a_F(z-L)}| \approx 1$ and $|e^{2a_F z}| \approx 1$ for all $z \in [0, L]$ in relations (75a) and (75b), that is to say assume that

$$N_F(F_F, F_B) \approx \frac{1}{r_1} \left(r_1 |F_F(z, t)|^2 + 2r_2 |F_B(z, t)|^2 \right) = |F_F(z, t)|^2 + 2\rho^{-1} |F_B(z, t)|^2 \quad (76a)$$

$$N_B(F_F, F_B) \approx \frac{1}{r_1} \left(r_2 |F_B(z, t)|^2 + 2r_1 |F_F(z, t)|^2 \right) = \rho^{-1} |F_B(z, t)|^2 + 2 |F_F(z, t)|^2. \quad (76b)$$

Remark 4. When the two end mirrors have distinct reflexivity, i.e. $r_1 \neq r_2$, the non-linear terms N_B and N_F are not “symmetric”. Whereas when $r_1 = r_2$, we have $\rho = 1$ and $N_F(F_F, F_B) = N(F_F, F_B)$, $N_B(F_F, F_B) = N(F_B, F_F)$ where $N(x, y) = |x|^2 + 2|y|^2$. It is clear from (60a)–(60b) that there is no possibility of choosing the parameter a_F, b_F, c_F and d_F to have such a relation when $r_1 \neq r_2$. Indeed, with the approximation introduced latter in the development, such a relation would require to have $|b_F| = |b_B|$. Unfortunately, this condition is not compatible with (66) when $r_1 \neq r_2$.

Finally, the propagation equations (72a)–(72b) are simplified into the following ones:

$$\begin{aligned} \frac{\partial}{\partial t} F_F(z, t) + v_g \frac{\partial}{\partial z} F_F(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} F_F(z, t) + \left(v_g a_F - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} \right) F_F(z, t) \\ &+ i \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2 r_1} \chi_{1111}^{(3)} \left(r_1 |F_F(z, t)|^2 + 2r_2 |F_B(z, t)|^2 \right) F_F(z, t) + v_g (c_F - d_F) \end{aligned} \quad (77a)$$

$$\begin{aligned} \frac{\partial}{\partial t} F_B(z, t) - v_g \frac{\partial}{\partial z} F_B(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} F_B(z, t) + \left(v_g a_F - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} \right) F_B(z, t) \\ &+ i \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2 r_1} \chi_{1111}^{(3)} \left(r_2 |F_B(z, t)|^2 + 2r_1 |F_F(z, t)|^2 \right) F_B(z, t) + v_g (c_F - d_F) \end{aligned} \quad (77b)$$

where from (69b)–(69c)

$$v_g (c_F - d_F) = \frac{v_g}{2L} e^{i \frac{d_0}{2}} \left(\frac{t_1}{\sqrt{r_1 r_2}} E_F^{in} + e^{\frac{1}{2} i (2k_0 L + d_0)} \frac{t_2}{r_2} E_B^{in} \right) = \frac{v_g}{2L} F^{in} \quad (78)$$

where

$$F^{in} = e^{i \frac{d_0}{2}} \left(\frac{t_1}{\sqrt{r_1 r_2}} E_F^{in} + e^{\frac{1}{2} i (2k_0 L + d_0)} \frac{t_2}{r_2} E_B^{in} \right) \quad (79)$$

and

$$v_g a_F - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} \approx v_g \left(\frac{1}{2L} (\ln(r_1 r_2) - id_0) - \frac{\alpha_0}{2} \right) \quad (80)$$

The boundary conditions at the two fiber ends are

$$F_F(0) = F_B(0) \quad \text{and} \quad F_F(L) = F_B(L). \quad (81)$$

Let us now consider the special case when $r_1 = r_2$, $t_1 = t_2$ under the assumption that there is no loss in reflection, nor in transmission at the two mirrors. Let $R = r_1^2 = r_2^2$ and $T = t_1^2 = t_2^2$. Under the assumption that there is no loss, we have $R + T = 1$. Let us consider the expression of the coefficient a_F defined in (64b). We have

$$a_F = \frac{1}{2L} (\ln(R) - id_0) = \frac{1}{2L} (\ln(1 - T) - id_0)$$

and when the transmission is much lower than the reflection, using the approximation $\ln(1 - T) \approx -T$, we deduce that

$$a_F = -\frac{T}{2L} - i \frac{d_0}{2L}. \quad (82)$$

It follows that relation (80) in this special case reads

$$v_g a_F - \frac{\alpha_0}{2} \frac{c}{n(\omega_0)} = -\frac{v_g}{2L} (T + id_0 + \alpha_0 L). \quad (83)$$

3 The Lugiato-Lefever equation for a Fabry-Perot resonator

3.1 Problem simplification for two identical Bragg mirrors

From now on, we assume that $r_1 = r_2$ and we denote by r their common value. We also assume that $t_1 = t_2$ and we denote by t their common value. We also have $T = t^2$, $R = r^2$ and $T + R = 1$. Problem

(77a)–(77b) with boundary conditions (81) reads

$$\begin{aligned} \frac{\partial}{\partial t} F_F(z, t) + v_g \frac{\partial}{\partial z} F_F(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} F_F(z, t) - \frac{v_g}{2L} (T + id_0 + \alpha_0 L) F_F(z, t) \\ &+ i \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} \left(|F_F(z, t)|^2 + 2|F_B(z, t)|^2 \right) F_F(z, t) + \frac{v_g}{2L} F^{in} \end{aligned} \quad (84a)$$

$$\begin{aligned} \frac{\partial}{\partial t} F_B(z, t) - v_g \frac{\partial}{\partial z} F_B(z, t) &= -\frac{ick_2}{2n(\omega_0)} \frac{\partial^2}{\partial t^2} F_B(z, t) - \frac{v_g}{2L} (T + id_0 + \alpha_0 L) F_B(z, t) \\ &+ i \frac{3}{8} \frac{\omega_0}{n(\omega_0)^2} \chi_{1111}^{(3)} \left(|F_B(z, t)|^2 + 2|F_F(z, t)|^2 \right) F_B(z, t) + \frac{v_g}{2L} F^{in} \end{aligned} \quad (84b)$$

where

$$F^{in} = e^{i\frac{d_0}{2} \frac{t}{r}} \left(E_F^{in} + e^{\frac{1}{2}i(2k_0L+d_0)} E_B^{in} \right). \quad (85)$$

Let us define as new unknowns \check{F}_F and \check{F}_B such that

$$F_F(z, t) = \sqrt{\frac{8n_0^2 \kappa}{3\omega_0 \chi_{1111}^{(3)}}} \check{F}_F(z, t) \quad (86a)$$

$$F_B(z, t) = \sqrt{\frac{8n_0^2 \kappa}{3\omega_0 \chi_{1111}^{(3)}}} \check{F}_B(z, t) \quad (86b)$$

where

$$\kappa = \frac{c(1-R)}{2n_0L}. \quad (87)$$

From (84a)–(84b), we deduce that \check{F}_F and \check{F}_B satisfy the following propagation equations

$$\begin{aligned} \frac{\partial}{\partial t} \check{F}_F(z, t) + v_g \frac{\partial}{\partial z} \check{F}_F(z, t) &= -i\kappa \tilde{a} \frac{\partial^2}{\partial t^2} \check{F}_F(z, t) - \kappa \bar{b} \check{F}_F(z, t) \\ &+ i\kappa \left(|\check{F}_F(z, t)|^2 + 2|\check{F}_B(z, t)|^2 \right) \check{F}_F(z, t) + \kappa \check{F}^{in} \end{aligned} \quad (88a)$$

$$\begin{aligned} \frac{\partial}{\partial t} \check{F}_B(z, t) - v_g \frac{\partial}{\partial z} \check{F}_B(z, t) &= -i\kappa \tilde{a} \frac{\partial^2}{\partial t^2} \check{F}_B(z, t) - \kappa \bar{b} \check{F}_B(z, t) \\ &+ i\kappa \left(|\check{F}_B(z, t)|^2 + 2|\check{F}_F(z, t)|^2 \right) \check{F}_B(z, t) + \kappa \check{F}^{in} \end{aligned} \quad (88b)$$

where

$$\check{F}^{in} = \sqrt{\frac{3\omega_0 \chi_{1111}^{(3)} n(\omega_0) L}{4n_g^2 c(1-R)^3}} F^{in} \approx \sqrt{\frac{3}{4} \frac{\omega_0}{n(\omega_0)} \frac{L}{c(1-R)^3}} F^{in} \quad (89)$$

(we have use the approximation $n_g \approx n(\omega_0)$) and

$$\tilde{a} = \frac{Lk_2}{1-R} \quad (90a)$$

$$\bar{b} = 1 + i\frac{d_0}{T} + \alpha_0 \frac{L}{T} = 1 + i\alpha + \alpha_0 \frac{L}{T} \quad (90b)$$

where $\alpha \stackrel{\text{def}}{=} d_0/T$. Note that the quantity $\alpha_0 \frac{L}{T}$ correspond to the bi-stability parameter introduced in [10, equ. (11.27) p.121].

The boundary conditions at the two fiber ends deduced from (81) are

$$\forall t \in \mathbb{R} \quad \check{F}_F(0, t) = \check{F}_B(0, t) \quad \text{and} \quad \check{F}_F(L, t) = \check{F}_B(L, t). \quad (91a)$$

The initial conditions are

$$\forall z \in [0, L] \quad \check{F}_F(z, 0) = 0 \quad \text{and} \quad \check{F}_B(z, 0) = 0. \quad (91b)$$

Remark 5. Equations (88a)–(88b) coincide with equations (11)–(12) in [3]. Note that in [3] the resonator is assumed to be non-absorbent so that $\alpha_0 = 0$ and $\bar{b} = 1 + i\alpha$.

Note also that the term v_g^2 is missing in the definition (90a) of \tilde{a} and the second order derivative is with respect to z here whereas it is for t in [3]. That is to say the second order derivative $\frac{\partial^2}{\partial t^2} \check{F}_B$ in (88a)–(88b) is changed for $v_g^2 \frac{\partial^2}{\partial z^2} \check{F}_B$ in [3]. The explanation of this difference is as follows: Since $R \approx 1$, we have $\kappa \approx 0$ and equation (88a) can be roughly approximated by

$$\frac{\partial}{\partial t} \check{F}_F(z, t) + v_g \frac{\partial}{\partial z} \check{F}_F(z, t) = 0. \quad (92)$$

If we differentiate (92) with respect to t , we obtain

$$\frac{\partial^2}{\partial t^2} \check{F}_F(z, t) = -v_g \frac{\partial^2}{\partial z \partial t} \check{F}_F(z, t)$$

and if we differentiate (92) with respect to z , we obtain

$$\frac{\partial^2}{\partial t \partial z} \check{F}_F(z, t) = -v_g \frac{\partial^2}{\partial z^2} \check{F}_F(z, t).$$

Combining these two relations, we get

$$\frac{\partial^2}{\partial t^2} \check{F}_F(z, t) = v_g^2 \frac{\partial^2}{\partial z^2} \check{F}_F(z, t). \quad (93)$$

Using relation (93) in (88a) we obtain the same equation as equation (11) in [3] (except however for the definition of \bar{b}).

In the sequel, taking into account Remark 5, we will consider (88a)–(88b) where the second order partial differential of \check{F}_F and \check{F}_B with respect to t is changed for the second order partial differential of F_F and F_B with respect to z thanks to the approximation (93). Thus, we look for $\check{F}_F : [0, L] \times \mathbb{R} \rightarrow \mathbb{C}$ and $\check{F}_B : [0, L] \times \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\begin{aligned} \frac{\partial}{\partial t} \check{F}_F(z, t) + v_g \frac{\partial}{\partial z} \check{F}_F(z, t) &= -i\kappa \bar{a} \frac{\partial^2}{\partial z^2} \check{F}_F(z, t) - \kappa \bar{b} \check{F}_F(z, t) \\ &+ i\kappa \left(|\check{F}_F(z, t)|^2 + 2|\check{F}_B(z, t)|^2 \right) \check{F}_F(z, t) + \kappa \check{F}^{in} \end{aligned} \quad (94a)$$

$$\begin{aligned} \frac{\partial}{\partial t} \check{F}_B(z, t) - v_g \frac{\partial}{\partial z} \check{F}_B(z, t) &= -i\kappa \bar{a} \frac{\partial^2}{\partial z^2} \check{F}_B(z, t) - \kappa \bar{b} \check{F}_B(z, t) \\ &+ i\kappa \left(|\check{F}_B(z, t)|^2 + 2|\check{F}_F(z, t)|^2 \right) \check{F}_B(z, t) + \kappa \check{F}^{in} \end{aligned} \quad (94b)$$

where

$$\bar{a} = \frac{L k_2 v_g^2}{1 - R}. \quad (95)$$

The boundary conditions are given by (91a) and we have the initial conditions given by (91b).

3.2 Fourier series decomposition

Let us introduce as unknowns the two functions \check{F}_F and \check{F}_B defined over $\mathbb{R} \times \mathbb{R}$ and related to \check{F}_F and \check{F}_B thanks to the following “periodisation” process:

1. for all $z \in [0, L]$, we set $\check{F}_F(z, t) = \check{F}_F(z, t)$ and $\check{F}_B(z, t) = \check{F}_B(z, t)$
2. for all $z \in [-L, 0]$, we set $\check{F}_F(z, t) = \check{F}_B(-z, t)$ and $\check{F}_B(z, t) = \check{F}_F(-z, t)$
3. for all $p \in \mathbb{Z}$ and for all $z \in [-L + 2pL, L + 2pL]$, we set $\check{F}_F(z, t) = \check{F}_F(z - 2pL, t)$ and $\check{F}_B(z, t) = \check{F}_B(z - 2pL, t)$

It follows from this definition of the two functions \check{F}_F and \check{F}_B that

$$\forall z \in \mathbb{R} \quad \forall t \in \mathbb{R} \quad \check{F}_B(z, t) = \check{F}_F(-z, t). \quad (96)$$

Moreover, these two functions are continuous over \mathbb{R} and have regularity C^∞ except perhaps at points $\pm pL$, $p \in \mathbb{Z}$.

Thanks to the above first property and the fact that equations (94a) and (94b) for \check{F}_F and \check{F}_B over $[0, L]$ only differs from the sign in front of the partial differential with respect to z but are otherwise identical through the permutation $F_B \leftrightarrow F_F$, we readily obtain that for all $z \in \mathbb{R}$ and for all $t \in \mathbb{R}$

$$\begin{aligned} \frac{\partial}{\partial t} \check{F}_F(z, t) + v_g \frac{\partial}{\partial z} \check{F}_F(z, t) &= -i\kappa \bar{a} \frac{\partial^2}{\partial z^2} \check{F}_F(z, t) - \kappa \bar{b} \check{F}_F(z, t) \\ &\quad + i\kappa \left(|\check{F}_F(z, t)|^2 + 2|\check{F}_B(z, t)|^2 \right) \check{F}_F(z, t) + \kappa \check{F}^{in} \end{aligned} \quad (97a)$$

$$\begin{aligned} \frac{\partial}{\partial t} \check{F}_B(z, t) - v_g \frac{\partial}{\partial z} \check{F}_B(z, t) &= -i\kappa \bar{a} \frac{\partial^2}{\partial z^2} \check{F}_B(z, t) - \kappa \bar{b} \check{F}_B(z, t) \\ &\quad + i\kappa \left(|\check{F}_B(z, t)|^2 + 2|\check{F}_F(z, t)|^2 \right) \check{F}_B(z, t) + \kappa \check{F}^{in} \end{aligned} \quad (97b)$$

By construction, the unknown function \check{F}_F is periodic with respect to the space variable z with period $2L$. It belongs to $C_M^1(\mathbb{R}) \cap C^0(\mathbb{R})$ so that from Dirichlet theorem it can be considered as the sum of a Fourier series:

$$\check{F}_F(z, t) = \sum_{p \in \mathbb{Z}} \check{f}_p(t) e^{i\pi p z / L} \quad (98a)$$

where

$$\forall p \in \mathbb{Z} \quad \check{f}_p(t) = \frac{1}{2L} \int_{-L}^L \check{F}_F(z, t) e^{i\pi p z / L} dz. \quad (98b)$$

(Note that this definition of the Fourier coefficients is coherent with the definition (6) of the Fourier Transform made earlier in the document.) We could also consider the Fourier series expansion of the second unknown function \check{F}_B , but we rather use relation (96) to express \check{F}_B as

$$\check{F}_B(z, t) = \sum_{p \in \mathbb{Z}} \check{f}_p(t) e^{-i\pi p z / L}. \quad (99)$$

Notation. For all $p \in \mathbb{Z}$, let $\mathbf{E}_p(z) \stackrel{\text{def}}{=} e^{i\pi p z / L}$. We have $\mathbf{E}_p^*(z) = e^{-i\pi p z / L} = \mathbf{E}_{-p}^*(z)$ and for all $p, q \in \mathbb{Z}$, $\mathbf{E}_p(z) \mathbf{E}_q(z) = e^{i\pi p z / L} e^{i\pi q z / L} = e^{i\pi(p+q)z / L} = \mathbf{E}_{p+q}(z)$.

Let us now look for the differential equations satisfied by the Fourier coefficients $f_p, p \in \mathbb{Z}$. To this end, we substitute (98a) into (97a). Let us note beforehand that

$$|\check{F}_F(z, t)|^2 = \check{F}_F(z, t) \check{F}_F^*(z, t) = \sum_{p \in \mathbb{Z}} \check{f}_p(t) \mathbf{E}_p(z) \sum_{q \in \mathbb{Z}} \check{f}_q^*(t) \mathbf{E}_{-q}(z) = \sum_{p, q \in \mathbb{Z}} \check{f}_p(t) \check{f}_q^*(t) \mathbf{E}_{p-q}(z).$$

Similarly, we get $|\check{F}_B(z, t)|^2 = \sum_{p, q \in \mathbb{Z}} \check{f}_p(t) \check{f}_q^*(t) \mathbf{E}_{-p+q}(z)$. Thus, from (97a), we have

$$\begin{aligned} \sum_{p \in \mathbb{Z}} \mathbf{E}_p(z) \left\{ \frac{d\check{f}_p}{dt}(t) + \left(i\pi p \frac{v_g}{L} - i\kappa \bar{a} \left(\frac{\pi p}{L} \right)^2 + \kappa \bar{b} \right) \check{f}_p(t) - \kappa \check{F}^{in} \right. \\ \left. - i\kappa \check{f}_p(t) \left(\sum_{\ell, q \in \mathbb{Z}} \check{f}_\ell(t) \check{f}_q^*(t) \mathbf{E}_{\ell-q}(z) + 2 \sum_{\ell, q \in \mathbb{Z}} \check{f}_\ell(t) \check{f}_q^*(t) \mathbf{E}_{-\ell+q}(z) \right) \right\} = 0. \end{aligned} \quad (100)$$

In a next step, we multiply (100) by $\frac{1}{2L} \mathbf{E}_{-n}(z)$ and we integrate over $[0, 2L]$ for the space variable z . Using the orthogonality of the family $(\mathbf{E}_p)_{p \in \mathbb{Z}}$ (namely, we have $\int_0^{2L} \mathbf{E}_p(z) \mathbf{E}_q(z) dz = 2L \delta_{p+q}$ where δ_n refers to Dirac symbol such that $\delta_n = 1$ if $n = 0$ and $\delta_n = 0$ otherwise), we obtain

$$\frac{d\check{f}_p}{dt}(t) + \left(i\pi p \frac{v_g}{L} - i\kappa \bar{a} \left(\frac{\pi p}{L} \right)^2 + \kappa \bar{b} \right) \check{f}_p(t) - \kappa \check{F}^{in} \delta_p - i\kappa \mathcal{N}(f_p(t)) = 0 \quad (101)$$

where

$$\mathcal{N}(f_p(t)) = \frac{1}{2L} \sum_{p \in \mathbb{Z}} \int_0^{2L} \check{f}_p(t) \mathbf{E}_p(z) \left(\sum_{\ell, q \in \mathbb{Z}} \check{f}_\ell(t) \check{f}_q^*(t) \mathbf{E}_{\ell-q}(z) + 2 \sum_{\ell, q \in \mathbb{Z}} \check{f}_\ell(t) \check{f}_q^*(t) \mathbf{E}_{-\ell+q}(z) \right) \mathbf{E}_{-n}(z) dz.$$

We have

$$\begin{aligned} \mathcal{N}(f_p(t)) &= \frac{1}{2L} \sum_{p, q, \ell \in \mathbb{Z}} \check{f}_p(t) \check{f}_\ell(t) \check{f}_q^*(t) \left(\int_0^{2L} \mathbf{E}_{\ell-q+p-n}(z) dz + 2 \int_0^{2L} \mathbf{E}_{-\ell+q+p-n}(z) dz \right) \\ &= \sum_{p, q, \ell \in \mathbb{Z}} \check{f}_p(t) \check{f}_\ell(t) \check{f}_q^*(t) \left(\delta_{\ell-q+p-n} + 2\delta_{-\ell+q+p-n} \right) \\ &= \sum_{p, q \in \mathbb{Z}} \check{f}_p(t) \check{f}_q^*(t) \left(\check{f}_{n+q-p}(t) + 2\check{f}_{q+p-n}(t) \right). \end{aligned}$$

Finally, we conclude that the sequence $(\check{f}_n)_{n \in \mathbb{Z}}$ of Fourier coefficients defined in (98b) satisfies the differential equations

$$\frac{d\check{f}_n}{dt}(t) + i\pi n \frac{v_g}{L} \check{f}_n(t) = \kappa \left\{ (ia_n - \bar{b}) \check{f}_n(t) + \check{F}^{in} \delta_n + i \sum_{p, q \in \mathbb{Z}} \check{f}_p(t) \check{f}_q^*(t) \left(\check{f}_{n+q-p}(t) + 2\check{f}_{q+p-n}(t) \right) \right\} \quad (102)$$

where

$$a_n \stackrel{\text{def}}{=} \bar{a} \left(\frac{\pi n}{L} \right)^2. \quad (103)$$

Let us now define for all $n \in \mathbb{N}$ the unknown

$$g_n : t \in \mathbb{R} \mapsto \check{f}_n(t) e^{i\pi n \frac{v_g}{L} t}. \quad (104)$$

We have $\check{f}_n(t) = \mathbf{E}_{-n}(v_g t) g_n(t)$ and $\frac{d\check{f}_n}{dt}(t) = \left(\frac{dg_n}{dt}(t) - i\pi n \frac{v_g}{L} g_n(t) \right) \mathbf{E}_{-n}(v_g t)$ and we deduce from (102) that for all $n \in \mathbb{Z}$, g_n satisfies the differential equation

$$\begin{aligned} \frac{dg_n}{dt}(t) &= \kappa \left\{ (ia_n - \bar{b}) g_n(t) + \check{F}^{in} \delta_n \mathbf{E}_n(v_g t) \right. \\ &\quad \left. + i \mathbf{E}_n(v_g t) \sum_{p, q \in \mathbb{Z}} g_p(t) g_q^*(t) \mathbf{E}_{-p+q}(v_g t) \left(g_{n+q-p}(t) \mathbf{E}_{-n-q+p}(v_g t) + 2g_{q+p-n}(t) \mathbf{E}_{-q-p+n}(v_g t) \right) \right\} \\ &= \kappa \left\{ (ia_n - \bar{b}) g_n(t) + \check{F}^{in} + i \sum_{p, q \in \mathbb{Z}} g_p(t) g_q^*(t) \left(g_{n+q-p}(t) + 2g_{q+p-n}(t) \mathbf{E}_{2n-2p}(v_g t) \right) \right\}. \end{aligned} \quad (105)$$

In equation (105) two time scales are involved:

1. The first time scale is related to the cavity round-trip time $T_{\text{RT}} = 2L/v_g$ and appears in the complex exponential terms $\mathbf{E}_{2n-2p}(v_g t)$;
2. The second time scale is related to the cavity decay time $\kappa^{-1} = \frac{2L}{v_g T}$ and appears in $g_n(t)$.

For a high-Q cavity, the transmission parameter T is much smaller than 1, so that the two scales are widely separated : $\kappa^{-1} \gg T_{\text{RT}}$. In the following, we consider the high-Q limit regime. Let S be a time much longer than the cavity round-trip time T_{RT} but much shorter than the cavity decay time κ^{-1} such that $S = \ell L/v_g$ where $\ell \in \mathbb{N}^*$. The integer ℓ is such that $\ell \gg 2$ and $\ell \ll 2/T$.

We integrate equation (105) over the interval $[0, S]$. We can notice that

$$\int_0^S \frac{dg_n}{dt}(t) dt = g_n(S) - g_n(0) = S \frac{g_n(S) - g_n(0)}{S}$$

and since S is much shorter than the time scale of the variation of g_n , we may use the approximation

$$\int_0^S \frac{dg_n}{dt}(t) dt \approx S g'_n(S).$$

We may also consider the following approximations since S is much shorter than the time scale of the variation of g_n

$$\int_0^S g_n(t) dt \approx S g_n(S)$$

and

$$\int_0^S g_p(t) g_q^*(t) g_{n+q-p}(t) dt \approx S g_p(S) g_q^*(S) g_{n+q-p}(S).$$

Moreover, taking into account that for all $n \in \mathbb{N}$, g_n varies on the time scale related to the cavity decay time $\kappa^{-1} = \frac{2L}{v_g T}$ whereas the complex exponential E_n varies on the time scale related to the cavity round-trip time T_{RT} and that in the high-Q limit approximation $T_{RT} \ll \kappa^{-1}$, we have

$$\begin{aligned} \int_0^S g_p(t) g_q^*(t) g_{q+p-n}(t) E_{2n-2p}(v_g t) dt &\approx g_p(S) g_q^*(S) g_{q+p-n}(S) \int_0^S E_{2n-2p}(v_g t) dt \\ &= g_p(S) g_q^*(S) g_{q+p-n}(S) \int_0^{\ell L/v_g} e^{i2\pi(n-p)\frac{v_g}{L}t} dt \\ &= \frac{L}{2\pi v_g} \int_0^{2\pi\ell} e^{i(n-p)\theta} d\theta = \frac{L}{v_g} \ell \delta_{n-p} = S \delta_{n-p}. \end{aligned}$$

Therefore after using the above relations and dividing both sides of the equation by S , we deduce from (105) the following approximate differential equation for g_n

$$\begin{aligned} \frac{dg_n}{dt}(S) &= \kappa \left\{ (ia_n - \bar{b}) g_n(S) + \check{F}^{in} + i \sum_{p,q \in \mathbb{Z}} g_p(S) g_q^*(S) (g_{n+q-p}(S) \right. \\ &\quad \left. + 2i \sum_{p,q \in \mathbb{Z}} g_p(S) g_q^*(S) g_{q+p-n}(S) \delta_{n-p} \right\} \\ &= \kappa \left\{ (ia_n - \bar{b}) g_n(S) + \check{F}^{in} + i \sum_{p,q \in \mathbb{Z}} g_p(S) g_q^*(S) g_{n+q-p}(S) \right. \\ &\quad \left. + 2i g_n(S) \sum_{q \in \mathbb{Z}} g_q(S) g_q^*(S) \right\}. \end{aligned} \quad (106)$$

This equation does not anymore involve the time scale related to the cavity round-trip time $T_{RT} = 2L/v_g$ that was appearing in the complex exponential terms $E_{2n-2p}(t)$ and it is valid for all $S = \ell L/v_g$ where the integer ℓ is such that $\ell \ll 2/T$. In the high-Q limit regime where T tends to zero, it can be considered valid for all time S .

In a last step, we introduce as new unknown the function $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ defined as the sum of the trigonometric series in z

$$\psi(z, t) = \sum_{n \in \mathbb{Z}} g_n(t) e^{i\pi n z/L}. \quad (107)$$

From (104), we have $g_n(t) = \check{f}_n(t) e^{i\pi n \frac{v_g}{L} t}$ and from the convergence of the Fourier series (98a) $\sum_{p \in \mathbb{Z}} \check{f}_p(t) e^{i\pi p z/L}$, ψ is defined on \mathbb{R} .

We have

$$\begin{aligned} \int_{-L}^L |\psi(z, t)|^2 dz &= \int_{-L}^L \psi(z, t) \psi^*(z, t) dz = \int_{-L}^L \sum_{p,q \in \mathbb{Z}} g_p(t) g_q^*(t) e^{i\pi(p-q)z/L} dz \\ &= \sum_{p,q \in \mathbb{Z}} g_p(t) g_q^*(t) \int_{-L}^L e^{i\pi(p-q)z/L} dz = 2L \sum_{p,q \in \mathbb{Z}} g_p(t) g_q^*(t) \delta_{p-q} \\ &= 2L \sum_{p \in \mathbb{Z}} |g_p(t)|^2. \end{aligned} \quad (108a)$$

We also have from (103)

$$\sum_{n \in \mathbb{Z}} a_n g_n(t) e^{i\pi n z/L} = \bar{a} \sum_{n \in \mathbb{Z}} \left(\frac{\pi n}{L}\right)^2 g_n(t) e^{i\pi n z/L} = -\bar{a} \frac{\partial^2}{\partial z^2} \sum_{n \in \mathbb{Z}} g_n(t) e^{i\pi n z/L} = -\bar{a} \frac{\partial^2 \psi}{\partial z^2}(z, t). \quad (108b)$$

Lastly, we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} g_p(S) e^{i\pi n z/L} \sum_{p, q \in \mathbb{Z}} g_p(S) g_q^*(S) g_{n+q-p}(S) &= \sum_{n, p, q \in \mathbb{Z}} g_p(S) E_p(z) g_q^*(S) E_{-q}(z) g_{n+q-p}(S) E_{n+q-p}(z) \\ &= \sum_{p, q \in \mathbb{Z}} g_p(S) E_p(z) g_q^*(S) E_{-q}(z) \sum_{n \in \mathbb{Z}} g_{n+q-p}(S) E_{n+q-p}(z) \\ &= \sum_{p \in \mathbb{Z}} g_p(S) E_p(z) \sum_{q \in \mathbb{Z}} g_q^*(S) E_{-q}(z) \sum_{m \in \mathbb{Z}} g_m(S) E_{-m}(z) \\ &= \psi(z, t) \psi^*(z, t) \psi(z, t) = \psi(z, t) |\psi(z, t)|^2. \end{aligned} \quad (108c)$$

Thus, by multiplying (106) by $E_n(z) \stackrel{\text{def}}{=} e^{i\pi n z/L}$ and by summing the resulting equations for all $n \in \mathbb{Z}$, we obtain when taking into account relations (108a)–(108c)

$$\frac{\partial \psi}{\partial t}(z, t) = \kappa \left(-i\bar{a} \frac{\partial^2 \psi}{\partial z^2}(z, t) - \bar{b} \psi(z, t) + \tilde{F}^{in} + i\psi(z, t) |\psi(z, t)|^2 + 2i\psi(z, t) \frac{1}{2L} \int_{-L}^L |\psi(\zeta, t)|^2 d\zeta \right). \quad (109)$$

This equation constitutes the Lugiato-Lefever equation (LLE) for a Fabry-Perot cavity as introduced in [3]. The non-linear term $i\kappa \psi(z, t) |\psi(z, t)|^2$ describes processes of self-phase-modulation and cross phase-modulation. The nonlinear integral term $2i\kappa \psi(z, t) \int_{-L}^L |\psi(\zeta, t)|^2 d\zeta$ is specific to the LLE for a Fabry-Perot cavity when compared to the temporal-longitudinal LLE in a ring cavity and it corrects the coefficients of self-phase-modulation and cross-phase-modulation [3].

Together with (109), the new unknown function ψ satisfies the periodic boundary condition

$$\psi(-L, t) = \psi(L, t) \quad \forall t \in \mathbb{R}^+. \quad (110)$$

For well posedness, we must also impose that

$$\frac{\partial \psi}{\partial z}(-L, t) = \frac{\partial \psi}{\partial z}(L, t) \quad \forall t \in \mathbb{R}^+. \quad (111)$$

Moreover, we have the initial condition at time $t = 0$ deduced from (107), (104) and (98a)

$$\forall z \in \mathbb{R} \quad \psi(z, 0) = \sum_{n \in \mathbb{Z}} g_n(0) e^{i\pi n z/L} = \sum_{n \in \mathbb{Z}} \check{f}_n(0) e^{i\pi n z/L} = \check{F}_F(z, 0) = \check{F}_B(-z, 0). \quad (112)$$

Since ψ is $2L$ -periodic, we actually have

$$\psi(z, 0) = \begin{cases} \check{F}_F(z, 0) = \check{F}_F(z, 0) = \sqrt{\frac{3\omega_0 \chi_{1111}^{(3)}}{8n_0^2 \kappa}} F_F(z, 0) & \forall z \in [0, L] \\ \check{F}_F(z, 0) = \check{F}_B(-z, 0) = \sqrt{\frac{3\omega_0 \chi_{1111}^{(3)}}{8n_0^2 \kappa}} F_B(z, 0) & \forall z \in [-L, 0] \end{cases}. \quad (113)$$

Let us consider the Fourier series decomposition of ψ with respect to the variable z , see (107),

$$\forall z \in [-L, L] \quad \forall t \in \mathbb{R}^+ \quad \psi(z, t) = \sum_{n \in \mathbb{Z}} g_n(t) e^{i\pi n \frac{z}{L}}. \quad (114)$$

From (98a) and (99), we have for all $z \in [0, L]$ and for all $t \in \mathbb{R}^+$

$$F_F(z, t) = \sum_{n \in \mathbb{Z}} \check{f}_n(t) e^{i\pi n z/L} \quad F_B(z, t) = \sum_{n \in \mathbb{Z}} \check{f}_n(t) e^{-i\pi n z/L} \quad (115)$$

where from (104)

$$\check{f}_n(t) = g_n(t) e^{i\pi n \frac{v_g}{L} t}. \quad (116)$$

We conclude that once equation (109) with boundary conditions (110) and (111) and initial condition (113) is solved, we deduce the forward and backward components E_F and E_B of the electric field thanks to relations (70a) and (70b) where F_F and F_B are obtain as the sum of the series (115).

In particular, we deduce from Parseval identity that for all $t \in \mathbb{R}^+$

$$\begin{aligned} \|F_B(\cdot, t)\|_{\mathbb{L}^2([-L, L])}^2 &= \|F_F(\cdot, t)\|_{\mathbb{L}^2([-L, L])}^2 = 2L \sum_{n \in \mathbb{Z}} |\check{f}_n(t)|^2 \\ &= 2L \sum_{n \in \mathbb{Z}} |g_n(t)|^2 = \|\psi(\cdot, t)\|_{\mathbb{L}^2([-L, L])}^2 \end{aligned} \quad (117)$$

It can be useful to introduce the normalized variables

$$\tau = \kappa t = \frac{v_g T}{2L} t \quad \text{and} \quad \theta = \pi \frac{z}{L} \quad (118)$$

and the corresponding new unknown $\tilde{\psi}$ such that $\tilde{\psi}(\theta, \tau) = \psi(z, t)$ The Lugiato-Lefever equation (109) for $\tilde{\psi}$ reads

$$\frac{\partial \tilde{\psi}}{\partial \tau}(\theta, \tau) = -i \frac{\beta}{2} \frac{\partial^2 \tilde{\psi}}{\partial \theta^2}(\theta, \tau) - \bar{b} \tilde{\psi}(\theta, \tau) + i \tilde{\psi}(\theta, \tau) \left(|\tilde{\psi}(\theta, \tau)|^2 + \frac{1}{\pi} \|\tilde{\psi}(\tau)\|_0^2 \right) + F(\theta, \tau) \quad \forall \theta \in [-\pi, \pi] \quad (119a)$$

where

$$\|\tilde{\psi}(\tau)\|_0^2 = \int_{-\pi}^{\pi} |\tilde{\psi}(\zeta, \tau)|^2 d\zeta \quad (119b)$$

and

$$\beta = \frac{2\pi^2}{L^2} \bar{a}. \quad (119c)$$

4 Conclusion

We have conducted a comprehensive modeling of light-wave propagation in a fiber Fabry-Perot resonator containing a Kerr medium starting from the general set of Maxwell's equations. We have highlighted all the mathematical assumptions and transformations, and as far as possible have related these assumptions to the physical setting of fiber Fabry-Perot resonators, that lead to the Lugiato-Lefever equation that have recently been proposed to describe in a as simple as practicably possible way time evolution of the electric field envelope in a fiber Fabry-Perot resonator.

Modeling was conducted in two steps. In a first step, we have shown how we can pass from Maxwell's equations to a set of two coupled non-linear partial differential equations for the forward and backward propagating electric field in the fiber. This set of two coupled equations is not easy to deal with because the two unknowns are two counter-propagating fields that are related to each other both by a non-linear term present in each of the two equations and by the boundary conditions at the two fiber ends. For this reason, a further simplification is introduced in a second step. The main idea is that assuming that the Bragg mirrors at the two ends of the fiber are identical, it is possible to expand the two counter-propagating fields in Fourier series expansions with respect to the propagation variable z with exactly the same Fourier coefficients but with opposite sign frequencies. By substituting these Fourier series expansions in the preceding set of two coupled non-linear partial differential equations, we obtain after some simplification in the crossed non-linear terms and after reforming an appropriate series expansion, the so-called Lugiato-Lefever equation.

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