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Optimal growth models under Rawlsian criteria*

Stefano Bosi[†] and Thai Ha-Huy[‡]

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Abstract

This article revisits the optimal growth with three distinct Rawlsian approaches: (1) the original Rawlsian criterion, popular in economic literature; (2) the maximin with multiple discount factors, introduced by [Chambers and Echenique \(2018\)](#); (3) the rank-dependent criterion, applied by [Zuber and Asheim \(2012\)](#) to growth theory.

Finally, we develop a new approach based on a convex combination of the most common criteria: Ramsey-Cass-Koopmans and Rawls.

Keywords: Rawls criterion, maximin principle, multiple discount rates, Ramsey-Cass-Koopmans models.

JEL classification numbers: C61, D11, D90.

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1 Introduction

Fundamental questions in economic growth theory are how to define a criterion reconciling efficiency and equality, and how to compute the economy behavior under this criterion.

The most popular utilitarian approach to growth theory is the Ramsey-Cass-Koopmans criterion (hereinafter RCK). The main criticism addressed to this criterion is the unequal treatment between close and distant future. In fact, given an intertemporal utility stream, its value is computed weighting less the future generations. This leads to a situation where the current generation does not leave enough resource to the future ones. In his seminal contribution, [Ramsey \(1928\)](#) considers that discounting the welfare of future generations is "ethically indefensible and arises merely from the weakness of the imagination".¹

[Rawls \(1971\)](#) argues that, behind the "veil of ignorance", the economic agent must adopt the criterion which maximizes the welfare of the least generation. How the economy behaves under this criterion is an important question. [Arrow \(1973\)](#), [Calvo \(1977\)](#) and [Phelps and Riley \(1978\)](#) address this issue in different contexts such as constant productivity or uncertain technology and come to the pessimistic conclusion that, if the economy starts with a low stock of capital, in order to preserve intergenerational equality, any generation must remain at this poor level.

[Ha-Huy \(2022\)](#) extends the *maximin* criterion with multiple discounting in [Chambers and Echenique \(2018\)](#) to defines a new Rawls-like criterion. He argues that, behind the "veil of ignorance", the economic agent faces the ambiguity about the appropriate discount factor to evaluate the intertemporal utility stream. In this case,

¹See also [Zuber and Asheim \(2012\)](#) for a review.

given an intertemporal utility stream (u_0, u_1, \dots) , the criterion becomes

$$\inf_{\delta \in (0,1)} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u_t$$

Ha-Huy (2022) shows that, under this criterion, the economy behaves as under the Rawlsian one.

Criteria preserving equality present another difficulty. Diamond (1965) and Basu and Mitra (2003) proved that there is no function representing a criterion which satisfies both *Pareto*² and *anonymity*³ properties. In another words, it is difficult to reconcile efficiency and equality.

The *overtaking* criterion by Gale (1967) seems to overcome this difficulty, since it is both Paretian and anonymous. The future generations are not discounted. However, preferences represented by the *overtaking* criterion are not complete. Moreover, when the economy starts with a low initial stock of capital, the criteria that don't discount the future, ask the first generations to sacrifice.⁴

Zuber and Asheim (2012) applies the *rank-dependent* method of social welfare theory to growth theory. They interpret the discount factor as a way to avoid sacrifices of poor generations too harsh. The key idea is weighting more the worst generations in the intertemporal evaluation. They first provide an axiomatic foundation to this new criterion by applying the axioms introduced by Koopmans (1960, 1972), such as separability between present and future, stationarity and the Paretian property, to the set of utility streams either increasing or *rearrangeable as increasing* instead of to the whole set of streams. In this respect, the evaluation of a stream

²Increasing the welfare of one generation in keeping unchanged the others increases the evaluation of intertemporal utility sequence.

³Permuting the utility level of different generations does not change the evaluation of the intertemporal utility stream.

⁴Mirlees (1967) shows that the first generations' saving rate may go up to more than 50%.

(u_0, u_1, \dots) can be reformulated as:

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u_{\pi(t)}$$

where Π represents the set of all possible permutations of natural numbers $N \equiv \{0, 1, \dots\}$ and $\delta \in (0, 1)$ is a discount factor.

Zuber and Asheim (2012) apply this criterion to growth theory and prove that, in this case, the solution coincides with the optimal path of the Rawlsian criterion when the productivity is low, and with the optimal path of the Ramsey-Cass-Koopmans model when the productivity is high.⁵

An added value of our article is the global analysis of growth trajectories under the Zuber and Asheim (2012) criterion.⁶ Starting with a capital stock below the *modified golden rule*, the economy follows the same trajectory as in the RCK model: the consumption increases over time and converges to the level of *modified golden rule*. If the initial capital stock is between the *modified golden rule* and the *golden rule*, then it remains constant forever and equal to its initial value. Finally, if the initial capital stock is larger than the *golden rule*, then there are infinitely many optimal paths, all converging to the *golden rule*.

An additional added value, is the definition of new criteria balancing efficiency and equality. Alvarez-Cuadrado and Van Long (2009) in continuous time and Ha-Huy and Nguyen (2022) in discrete time study a convex combination of RCK and Rawlsian criteria.⁷ Both these contributions share the same idea: a Rawlsian cri-

⁵See Proposition 10 in Zuber and Asheim (2012).

⁶The *golden rule* capital stock maximizes the stationary consumption level. The *modified golden rule* is the steady state under the utilitarian criterion in the RCK model.

⁷In a purpose of balancing close and distant futures, Chichilnisky (1996, 1997) proposes *non-dictatorship* properties. These criteria have inspired many works thereafter. However, as pointed out by Heal (1998), under the assumption *non-dictatorship* there is no optimal path in models with

terion, stressing the equality between generations, tends to promote forever a state of the economy with low capital. Could we weaken the equality degree to ensure more future development? To this purpose, to what extent we have to sacrifice the equality principle? These contributions prove that, starting with a low level of capital, the economy follows the RCK optimal path in the long run, while with a high level, it converges to a RCK-like steady state which is a modified *modified golden rule* corresponding to a higher discount factor.

In our paper, we also consider a convex combination of criteria RCK and [Zuber and Asheim \(2012\)](#). If the initial capital stock is low, then the optimal path coincides with the optimal RCK trajectory. Otherwise, the optimal path coincides with the trajectory under the criterion in [Ha-Huy \(2022\)](#) and it converges to a higher steady state.

The article is organized as follows. In Section 2, we present three Rawlsian criteria and an equivalence result between the first two. Section 3 provides a global analysis of trajectories under the criterion in [Zuber and Asheim \(2012\)](#) and a convex combination of RCK and [Zuber and Asheim \(2012\)](#).

2 Three Rawlsian criteria

2.1 Fundamentals

Time is discrete: $t = 0, 1, \dots$. At time t , given the stock of capital $x_t \geq 0$, the production level is $f(x_t)$. The economic agent divides this output into consumption

renewable resources. A similar result holds in an optimal growth context (see [Ayong Le Kama et al. \(2013\)](#)). [Asheim and Ekeland \(2016\)](#) and [Figuières and Tidball \(2012\)](#) try to find more satisfactory solutions with alternative approaches.

c_t and capital investment for tomorrow, $x_{t+1} \geq 0$. Given the consumption c_t , the utility level at time t is $u(c_t)$. We assume that the production function f is concave and satisfies the Inada conditions: $f'(0) = \infty$ and $f'(\infty) < 1$. The utility function u is strictly increasing and strictly concave.

Let $x_0 > 0$ be a given capital stock and $\Phi(x_0)$ the set of feasible sequences $\chi = (x_t)_{t=0}^{\infty}$ with $0 \leq x_{t+1} \leq f(x_t)$ for any $t \geq 0$. Denote by Π the set of every possible permutations of natural numbers $\mathbb{N} \equiv \{\{0, 1, \dots\}\}$.

Let \bar{x} be the *golden rule*, maximizing the constant consumption $f(\bar{x}) - \bar{x}$, solution to $f'(x) = 1$. Given the discount factor $\delta \in (0, 1)$, let x^* be the *modified golden rule*, the steady state of the RCK problem, solution to $\delta f'(x) = 1$.

The economic agent maximizes the intertemporal utility $U(c_0, c_1, \dots)$ to smooth her consumption over time. More precisely, given the initial capital stock x_0 , she solves the following program:

$$\begin{aligned} v(x_0) &= \max U(c_0, c_1, \dots) \\ c_t + x_{t+1} &\leq f(x_t) \\ c_t, x_{t+1} &\geq 0 \end{aligned}$$

for any $t \geq 0$, given $x_0 > 0$. The function v is called the value of the optimization program.

(1) The criterion of [Rawls \(1971\)](#) focuses on the welfare of the worst generation. More precisely, given a consumption sequence (c_0, c_1, \dots) , the evaluation of the intertemporal utility stream is given by

$$U(c_0, c_1, \dots) = \inf_{t \geq 0} u(c_t)$$

(2) [Ha-Huy \(2022\)](#) considers the alternative maximin criterion with multiple discounting introduced by [Chambers and Echenique \(2018\)](#). Behind the Rawlsian "veil

of ignorance”, the economic agent does not know the right discount factor to apply in order to evaluate the utility stream. In this respect, she is ambiguous.

The maximin approach leads to a second Rawlsian criterion:

$$U(c_0, c_1, \dots) = \inf_{\delta \in (0,1)} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t)$$

where the argument in the RHS is an arithmetic average with geometric weights $(1 - \delta) \delta^t$. The second Rawlsian criterion can be also reinterpreted as an application of the first one to a population with heterogenous discounting $\delta_i \in (0, 1)$ and a planner who maximizes the intertemporal utility of the worst agent as a welfare function.

(3) In [Zuber and Asheim \(2012\)](#), the discount factor is necessary only to prevent hard sacrifices of the worst generations. In this respect, they reconsider the *rank-dependent* method well-known in social welfare literature from a dynamic perspective. More precisely, they apply the axiomatic system elaborated by [Koopmans \(1960, 1972\)](#) to increasing utility streams or to streams that can be rearranged as increasing. In doing so, they introduce a third Rawlsian criterion:

$$U(c_0, c_1, \dots) = \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})$$

or, equivalently,

$$U(c_0, c_1, \dots) = \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(c_{\pi(t)}) \tag{1}$$

where Π represents the set of all permutations in the set of natural numbers \mathbb{N} and δ is a given discount factor in $(0, 1)$. The third Rawlsian criterion weights more the worst generations.

Preliminarily, we are interested in sufficient conditions ensuring that, for each Rawlsian criterion, an optimal path exists.

To keep things as simple as possible, we assume a set of sufficient conditions provided by [Le Van and Morhaim \(2002\)](#), that is Hypothesis H1 to H6. Among them, *tail-insensitivity* plays the key role by entailing the continuity of the intertemporal utility function for the product topology and, hence, the continuity of the value function in the initial capital stock. Under these assumptions, for each Rawlsian criterion, an optimal capital accumulation path always exists.

2.2 The first two Rawlsian criteria

The first Rawlsian criterion is studied in a series of contributions, such as [Arrow \(1970, 1973\)](#), [Calvo \(1977\)](#), [Phelps and Riley \(1978\)](#) among others. The second one is applied in the same context of our model by [Ha-Huy \(2022\)](#).

(1) Let us present shortly the dynamic properties of the economy under the first Rawlsian criterion.

(1.1) If $0 \leq x_0 \leq \bar{x}$, then for any feasible path $\chi \in \Phi(x_0)$, we have

$$\inf_{t \geq 0} u(f(x_t) - x_{t+1}) \leq u(f(x_0) - x_0)$$

The equality happens if and only if $x_t = x_0$ for every $t \geq 0$. Then the optimal sequence is (x_0, x_0, \dots) .

(1.2) In the case $x_0 > \bar{x}$, we have

$$\inf_{t \geq 0} u(f(x_t) - x_{t+1}) \leq u(f(\bar{x}) - \bar{x}) \tag{2}$$

There is an infinite number of optimal sequences (x_0, x_1, \dots) such that equality may be obtained. They are all decreasing and converge to \bar{x} .

(2) Under the second Rawlsian criterion, the optimal path has the same qualitative properties.

(2.1) If $0 \leq x_0 \leq \bar{x}$, the unique optimal path is the initial condition forever: (x_0, x_0, \dots) (see Corollary 2.1 in Ha-Huy (2022)). The value function becomes $v(x_0) = u(f(x_0) - x_0)$.

(2.2) If $x_0 > \bar{x}$, then any optimal path under the first Rawlsian criterion is also an optimal path under the second one. Any optimal path is decreasing and converge to \bar{x} (see Corollary 2.1 in Ha-Huy (2022)). The value function is $v(x_0) = u(f(\bar{x}) - \bar{x})$.

We observe that, though these two criteria are different, they lead to the same optimal path and the same value function. Starting from with a low capital stock, the economy remains poor forever.

2.3 The third Rawlsian criterion

Focus on the third Rawlsian criterion (1). Consider the following optimization problem:

$$v(x_0) = \max_{\chi \in \Phi(x_0)} \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}) - x_{\pi(t)+1})$$

The next lemma states some fundamental properties. If the consumption sequence is increasing, the third criterion and the utilitarian criterion (RCK) result in the same evaluation with larger weights on poorer generations. In the general case, the evaluation under the third criterion is bounded from above by the limit infimum of the utility sequence, normalized by $1 - \delta$.

Lemma 1. *Consider a consumption sequence (c_0, c_1, \dots) .*

(1) *If the sequence (c_0, c_1, \dots) is non-decreasing, then*

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(c_{\pi(t)}) = \sum_{t=0}^{\infty} \delta^t u(c_t) \quad (3)$$

(2) *If, for some utility level w and any $\varepsilon > 0$, there exist infinitely many t such*

that $u(c_t) \leq w + (1 - \delta)\varepsilon$, then

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(c_{\pi(t)}) \leq \frac{w}{1 - \delta}$$

(3) If there exist infinitely many t such that $u(c_t) \leq w + (1 - \delta)\varepsilon$ for any $\varepsilon > 0$ and there exists τ such $u(c_\tau) < w$, then the inequality in part (2) becomes strict:

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(c_{\pi(t)}) < \frac{w}{1 - \delta}$$

Using results in Lemma 1, we can prove the next proposition 1 describing the optimal path under the third criterion. When the economy starts with a low capital stock, $x_0 \leq x^*$, where x^* is the solution to $\delta f'(x) = 1$ (*modified golden rule*), then it follows the same path we obtain under the RCK criterion. When the economy starts from an intermediate level, $x^* \leq x_0 \leq \bar{x}$, then its capital level remains constant forever. If the initial capital stock is large enough, $x_0 \geq \bar{x}$, any solution is the same we obtain under the first criterion, that is a decreasing sequence that converges to the *golden rule*.

Proposition 1. (1) If $x_0 \leq x^*$, then the optimal path χ^* is the solution under RCK criterion. This is an increasing sequence converging to x^* . The value of this problem is the same of the RCK program.

(2) If $x^* < x_0 \leq \bar{x}$, the optimal path is $\chi^* = (x_0, x_0, \dots)$ and the value of (1) is given by

$$v(x_0) = \frac{u(f(x_0) - x_0)}{1 - \delta}$$

(3) If $x_0 > \bar{x}$, there exists an infinite number of solutions, all converging to \bar{x} , and the value is given by

$$v(x_0) = \frac{u(f(\bar{x}) - \bar{x})}{1 - \delta}$$

3 The efficiency-equality balance

Any criterion treating all generations equally satisfies the *anonymity* property, ensuring that the evaluation of an intertemporal utility stream does not change after a permutation of welfare level of different generations. However, [Diamond \(1965\)](#) and [Basu and Mitra \(2003\)](#) have shown that these criteria can not be represented by a function satisfying also the *Pareto* property.

[Alvarez-Cuadrado and Van Long \(2009\)](#) in continuous time and [Ha-Huy and Nguyen \(2022\)](#) in discrete time consider a convex combination of two criteria: RCK and the first Rawls. In our notation, we have

$$U(c_0, c_1, \dots) = \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + a \inf_{t \geq 0} u(c_t) \quad (4)$$

Parameter $a \geq 0$ represents the equality degree in this hybrid criterion. As limit cases, if $a = 0$, we recover the standard RCK criterion and, if a converges to infinity, we have the first Rawlsian one.

The key idea to solve the optimization problem under this criterion is the following. If we renounce a little to intergenerational equality, we can accumulate more capital and promote future development.

There exists an optimal sacrifice level of equality. The results about qualitative properties of the optimal path can be divided into two parts.

(1) When the initial capital is low, $x_0 \leq x^*$, where x^* is the *modified golden rule*, then its stock slowly increases in the period of life of the first generations. However, in the long run, the trajectory of the capital stock will approach and, from a critical date on, coincides with the RCK optimal path.⁸ The equality parameter a in the hybrid criterion reduces the distance in terms of utility between the best and the

⁸See Proposition 2.1 in [Ha-Huy and Nguyen \(2022\)](#).

worst generations.

(2) The case $x_0 > x^*$ is more complicated to handle. Two critical levels matter: $0 \leq a_1 \leq a_2$ with $a_2 > 0$.

(2.1) If $0 \leq a \leq a_1$, the equality part plays no role and the optimal path coincides the one under RCK criterion (see Proposition 2.2, part (iii), in [Ha-Huy and Nguyen \(2022\)](#)).

(2.2) When $a_1 < a < a_2$, the economy converges to *modified golden rule* as in the RCK model, but with a larger discount factor $\tilde{\delta} > \delta$ (see Proposition 2.2, part (i), in [Ha-Huy and Nguyen \(2022\)](#)).

(2.3) If $a \geq a_2$, the equality part fully dominates and we recover the optimal path of the first Rawlsian criterion (see Proposition 2.2, part (ii), in [Ha-Huy and Nguyen \(2022\)](#)).

We see that the equality degree a increases the welfare of generations living in a far future with respect to their level under the standard utilitarian criterion. It is easy to see that the criteria represented by (4) and

$$U(c_0, c_1, \dots) = (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{t \geq 0} u(c_t)$$

with $\lambda = a / (1 + a)$ are equivalent.

Along similar lines, we define a new hybrid criterion which combines a standard utilitarian part with an equality part, and we study the optimization problem under the convex combination of RCK and [Zuber and Asheim \(2012\)](#). More precisely, we introduce the following criterion:

$$U(c_0, c_1, \dots) = (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})$$

where the parameter $\lambda \in [0, 1]$ represents the equality degree in the new criterion.

When $\lambda = 0$, we recover the standard RCK criterion, while, when $\lambda = 1$, the third Rawlsian criterion, considered in subsection 2.3.

There are two cases.

(1) When the initial capital is low, $x_0 \leq x^*$, because of the high productivity, the optimal path is the solution under the RCK criterion. Therefore, if the economy starts with a low stock of capital, this criterion promotes the development differently from the first Rawlsian one.

(2) When $x_0 > x^*$, we prove that the optimal accumulation coincides with the capital path under the convex criterion applied in [Ha-Huy and Nguyen \(2022\)](#). Hence, for a sufficiently large equality degree λ , the economy converges in the long run to the *modified golden rule* as in the RCK model, but with a larger discount factor (the reader is still referred to Proposition 2.2 in [Ha-Huy and Nguyen \(2022\)](#)).

These results are formalized and summarized in the next and last proposition.

Proposition 2. *Let $0 < \lambda < 1$.*

(1) *If the economy starts with $0 < x_0 \leq x^*$, then the optimal path under the third Rawlsian criterion coincides with the one under the RCK criterion.*

(2) *If the economy starts with $x_0 > x^*$, then there are two critical values with $0 < \lambda_1 < \lambda_2 < 1$ such that:*

(2.1) *If $0 < \lambda \leq \lambda_1$, the equality part plays no role, and the optimal path is also the one under the RCK criterion, converging to x^* .*

(2.2) *If $\lambda_1 < \lambda < \lambda_2$, then the optimal path is decreasing and converges to a steady state corresponding to a discount rate $\tilde{\delta}$ greater than δ .*

(2.3) *If $\lambda \geq \lambda_2$, then the capital accumulation follows the optimal path of the first Rawlsian criterion.*

4 Conclusion

In this article, we have considered the behavior of the economy under different Rawlsian criteria and their convex combination with the popular utilitarian RCK criterion. We find that, sacrificing the equality by reducing its degree in this convex combination, in the long run, can implement a better situation from a social point of view. We conclude by remarking that, if we replace the convex combination with a new criterion as a strictly increasing function of the standard utilitarian value and the equality value, the qualitative behavior of the economy does not change. However, the shape of the function would affect the optimal degree of equality, that is the efficient sacrifice in terms of equality.

5 Appendix

Proof of Lemma 1

(1) Let π^* the permutation such that $\pi^*(t) = t$ for every t . Since this permutation weights more the lower utility levels, equation (3) follows.

(2) Fix $\varepsilon > 0$. Denote by $(t_n)_{n=0}^\infty$ the subsequence of $(m)_{m=0}^\infty$ such that $u(c_{t_n}) \leq w + (1 - \delta)\varepsilon$. Since the sequence (c_0, c_1, \dots) is bounded from above, we can fix N sufficiently large such that for any $\pi \in \Pi$, we have $\sum_{t=N+1}^\infty \delta^s u(c_{\pi(t)}) < \varepsilon$.

Consider a permutation π^* such that $\pi^*(n) = t_n$ for any $n \in \{0, 1, \dots, N\}$. We have

$$\begin{aligned} \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^s u(c_{\pi(t)}) &\leq \sum_{t=0}^{\infty} \delta^s u(c_{\pi^*(t)}) = \sum_{t=0}^N \delta^s u(c_{t_n}) + \sum_{t=N+1}^{\infty} \delta^s u(c_{\pi^*(t)}) \\ &\leq [w + (1 - \delta)\varepsilon] \sum_{t=0}^N \delta^s + \varepsilon \end{aligned}$$

Let N converge to infinity. Then

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^s u(c_{\pi(t)}) \leq \frac{w}{1-\delta} + 2\varepsilon$$

and, since ε is arbitrary,

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^s u(c_{\pi(t)}) \leq \frac{w}{1-\delta}$$

(3) Choose a permutation π^* such that $\pi^*(0) = \tau$, and $\pi^*(n) = t_n$ for each $n \in \{0, 1, \dots, N\}$ where N is defined as in the proof of part (2). Then

$$\begin{aligned} \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^s u(c_{\pi(t)}) &\leq \sum_{t=0}^{\infty} \delta^s u(c_{\pi^*(t)}) \\ &= u(c_{\tau}) + \sum_{t=1}^N \delta^s u(c_{t_n}) + \sum_{t=N+1}^{\infty} \delta^t u(c_{\pi^*(t)}) \\ &\leq u(c_{\tau}) + [w + (1-\delta)\varepsilon] \sum_{t=1}^N \delta^t + \varepsilon \end{aligned}$$

Since the inequality is satisfied for any N large enough, we can let N converge to infinity to obtain

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^s u(c_{\pi(t)}) \leq u(c_{\tau}) + \frac{\delta w}{1-\delta} + (1+\delta)\varepsilon$$

Since ε is arbitrary, recalling that $u(c_{\tau}) < w$, we obtain the following inequality:

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^s u(c_{\pi(t)}) \leq u(c_{\tau}) + \frac{\delta w}{1-\delta} < \frac{w}{1-\delta}$$

Proof of Proposition 1

For simplicity, with a notational misuse, let us denote

$$U(\chi) \equiv \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}) - x_{\pi(t)+1})$$

for any feasible sequence $\chi \in \Phi(x_0)$.

(1) Let $x_0 \leq x^*$. Let χ be the optimal path of the program under the RCK criterion. One of the well-known results in dynamic programming literature is that the utility sequence $(u(f(x_t) - x_{t+1}))_{t=0}^\infty$ is non-decreasing.⁹ Part (1) of Lemma 1 applies and the claim is proven.

(2) The most challenging case is $x^* < x_0 \leq \bar{x}$. We know that, since $x_0 \leq \bar{x}$,

$$\max_{\chi \in \Phi(x_0)} \inf_{t \geq 0} u(f(x_t) - x_{t+1}) = u(f(x_0) - x_0)$$

with unique solution $\chi^* = (x_0, x_0, \dots)$.

Let $\chi^* = (x_t^*)_{t=0}^\infty$ be the optimal path of the initial problem $\max_{\chi \in \Phi(x_0)} U(\chi)$ with $\chi^* \neq (x_0, x_0, \dots)$. Then the following inequality is satisfied:

$$\inf_{t \geq 0} u(f(x_t^*) - x_{t+1}^*) < u(f(x_0) - x_0) \quad (5)$$

We observe that there are finitely many t such that $u(f(x_t^*) - x_{t+1}^*) < u(f(x_0) - x_0)$ (and, at least, one). Indeed, in the contrary case, with an infinite number of t satisfying this inequality, applying the part (3) of Lemma 1, we have $U(\chi^*) < u(f(x_0) - x_0)$ a contradiction.

Let T the largest time index t such that $u(f(x_t^*) - x_{t+1}^*) < u(f(x_0) - x_0)$.

For any $t > T$, $u(f(x_t^*) - x_{t+1}^*) > u(f(x_0) - x_0)$. We observe that

$$\inf_{t > 0} u(f(x_{T+t}^*) - x_{T+t+1}^*) > u(f(x_0) - x_0)$$

On the contrary, for any $\varepsilon > 0$, there are infinitely many t such that

$$u(f(x_t^*) - x_{t+1}^*) < u(f(x_0) - x_0) + \varepsilon$$

By part (3) of Lemma 1, we obtain $U(\chi^*) < u(f(x_0) - x_0) / (1 - \delta)$, a contradiction.

⁹The reader interested in the dynamic properties of the optimal path under the RCK criterion, is referred to chapter 2, section 2.4.4, in [Le Van and Dana \(2002\)](#).

Hence, $\inf_{t>0} u(f(x_{T+t}^*) - x_{T+t+1}^*) > u(f(x_0) - x_0)$, which implies the existence of $a > 0$ such that $u(f(x_t^*) - x_{t+1}^*) > u(f(x_0) - x_0) + a$ for any $t > T$.

Observe that $x_{T+1}^* \geq x_0$. Indeed, assume the contrary, $x_{T+1}^* < x_0$. Since $u(f(x_{T+1}^*) - x_{T+2}^*) > u(f(x_0) - x_0)$, this implies $x_{T+2}^* < x_0$. By induction, for every $t > 0$, $x_{T+t}^* < x_0$. By applying the part (3) of Lemma 1, we obtain $U(\chi^*) < u(f(x_0) - x_0)$, a contradiction.

Fix an $\varepsilon > 0$ sufficiently small such that

$$u(f(x_{T+1}^* - \varepsilon) - x_{T+2}^*) > u(f(x_0) - x_0) + a > u(f(x_T^*) - (x_{T+1}^* - \varepsilon))$$

Denote by $\hat{\Pi}$ the subset of Π such that $\pi^{-1}(T) \leq T$ and $\pi^{-1}(T) < \pi^{-1}(T+1)$.

Let us prove that

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) = \inf_{\pi \in \hat{\Pi}} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*)$$

Indeed, take any $\pi \in \Pi$ which does not satisfy one of the two conditions in the definition of $\hat{\Pi}$, that is: (a) $\pi^{-1}(T) > T$ or (b) $\pi^{-1}(T) \geq \pi^{-1}(T+1)$.

(a) First, consider the case $\pi^{-1}(T) > T$. This means that the place T is now occupied by an element which was in a strictly higher place before the permutation. This implies the existence of some $\tilde{T} > T$ such that $\pi^{-1}(\tilde{T}) \leq T$ because T is finite. Recall that for every $t > T$, we have $f(x_t^*) - x_{t+1}^* > f(x_0) - x_0$. Hence $f(x_{\tilde{T}}^*) - x_{\tilde{T}+1}^* > f(x_0) - x_0$. Let $\hat{T} \equiv \min \{\tilde{T}\}$.

Let $T_1 \equiv \pi^{-1}(T)$ and $T_2 \equiv \pi^{-1}(\hat{T})$. Then, $\pi(T_1) = T$ and $\pi(T_2) = \hat{T}$. Consider an auxiliary permutation $\hat{\pi}$ such that $\hat{\pi}(t) = \pi(t)$ for any $t \neq T_1, T_2$, $\hat{\pi}(T_1) = \hat{T}$ and $\hat{\pi}(T_2) = T$. We have $\pi^{-1}(T) = T_1 = \hat{\pi}^{-1}(\hat{T})$ and, symmetrically, $\pi^{-1}(\hat{T}) = T_2 = \hat{\pi}^{-1}(T)$.

Let us show that $\hat{\pi} \in \hat{\Pi}$. Clearly, $\hat{\pi}^{-1}(T) = \pi^{-1}(\hat{T}) \leq T$.

Moreover, if $\hat{T} = T + 1$, we have $\hat{\pi}^{-1}(T + 1) = \hat{\pi}^{-1}(\hat{T}) = \pi^{-1}(T) > T \geq \pi^{-1}(\hat{T}) = \hat{\pi}^{-1}(T)$.

If $\hat{T} > T + 1$, we have $\pi^{-1}(T + 1) > T$, otherwise $T + 1 < \hat{T}$ will be smallest time index $\tilde{T} > T$ such that $\pi^{-1}(\tilde{T}) \leq T$, a contradiction. Then $\hat{\pi}^{-1}(T + 1) = \pi^{-1}(T + 1) > T \geq \pi^{-1}(\hat{T}) = \hat{\pi}^{-1}(T)$.

In this case, we obtain

$$\begin{aligned}
& \sum_{t=0}^{\infty} \delta^t u(f(x_{\hat{\pi}(t)}^*) - x_{\hat{\pi}(t+1)}^*) - \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t+1)}^*) \\
&= \delta^{\hat{\pi}^{-1}(T)} u(f(x_T^*) - x_{T+1}^*) + \delta^{\hat{\pi}^{-1}(\hat{T})} u(f(x_{\hat{T}}^*) - x_{\hat{T}+1}^*) \\
&- \delta^{\pi^{-1}(T)} u(f(x_T^*) - x_{T+1}^*) - \delta^{\pi^{-1}(\hat{T})} u(f(x_{\hat{T}}^*) - x_{\hat{T}+1}^*) \\
&= \delta^{\pi^{-1}(\hat{T})} u(f(x_T^*) - x_{T+1}^*) + \delta^{\pi^{-1}(T)} u(f(x_{\hat{T}}^*) - x_{\hat{T}+1}^*) \\
&- \delta^{\pi^{-1}(T)} u(f(x_T^*) - x_{T+1}^*) - \delta^{\pi^{-1}(\hat{T})} u(f(x_{\hat{T}}^*) - x_{\hat{T}+1}^*) \\
&= \left[\delta^{\pi^{-1}(\hat{T})} - \delta^{\pi^{-1}(T)} \right] \left[u(f(x_T^*) - x_{T+1}^*) - u(f(x_{\hat{T}}^*) - x_{\hat{T}+1}^*) \right] < 0
\end{aligned}$$

since $0 < \delta < 1$, $\pi^{-1}(\hat{T}) \leq T < \pi^{-1}(T)$ and $f(x_T^*) - x_{T+1}^* < f(x_{\hat{T}}^*) - x_{\hat{T}+1}^*$.

(b) $\pi^{-1}(T) \geq \pi^{-1}(T + 1)$ is equivalent to $\pi^{-1}(T) > \pi^{-1}(T + 1)$ because π^{-1} is a permutation and, then, $\pi^{-1}(T) \neq \pi^{-1}(T + 1)$.

The case $\pi^{-1}(T) > \pi^{-1}(T + 1)$ can be similarly treated.

Let $T_1 \equiv \pi^{-1}(T)$ and $T_2 \equiv \pi^{-1}(T + 1)$. Consider an auxiliary permutation $\hat{\pi}$ such that $\hat{\pi}(t) = \pi(t)$ for any $t \neq T_1, T_2$, $\hat{\pi}(T_1) = T + 1$ and $\hat{\pi}(T_2) = T$. We have $\pi^{-1}(T) = T_1 = \hat{\pi}^{-1}(T + 1)$ and, symmetrically, $\pi^{-1}(T + 1) = T_2 = \hat{\pi}^{-1}(T)$.

Let us show that $\hat{\pi} \in \hat{\Pi}$, that is $\hat{\pi}^{-1}(T) < \hat{\pi}^{-1}(T + 1)$ and $\hat{\pi}^{-1}(T) \leq T$.

Clearly, $\hat{\pi}^{-1}(T) = \pi^{-1}(T + 1) < \pi^{-1}(T) = \hat{\pi}^{-1}(T + 1)$.

(b.1) If $\pi^{-1}(T) \leq T$, we have $\hat{\pi}^{-1}(T) = \pi^{-1}(T + 1) < \pi^{-1}(T) \leq T$.

Thus, if $\pi^{-1}(T) \leq T$, we have that $\hat{\pi} \in \hat{\Pi}$. In this case, since

$$\begin{aligned}
& \sum_{t=0}^{\infty} \delta^t u(f(x_{\hat{\pi}(t)}^*) - x_{\hat{\pi}(t+1)}^*) - \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t+1)}^*) \\
&= \delta^{\hat{\pi}^{-1}(T)} u(f(x_T^*) - x_{T+1}^*) + \delta^{\hat{\pi}^{-1}(T+1)} u(f(x_{T+1}^*) - x_{T+2}^*) \\
&\quad - \delta^{\pi^{-1}(T)} u(f(x_T^*) - x_{T+1}^*) - \delta^{\pi^{-1}(T+1)} u(f(x_{T+1}^*) - x_{T+2}^*) \\
&= \delta^{\pi^{-1}(T+1)} u(f(x_T^*) - x_{T+1}^*) + \delta^{\pi^{-1}(T)} u(f(x_{T+1}^*) - x_{T+2}^*) \\
&\quad - \delta^{\pi^{-1}(T)} u(f(x_T^*) - x_{T+1}^*) - \delta^{\pi^{-1}(T+1)} u(f(x_{T+1}^*) - x_{T+2}^*) \\
&= \left[\delta^{\pi^{-1}(T+1)} - \delta^{\pi^{-1}(T)} \right] \left[u(f(x_T^*) - x_{T+1}^*) - u(f(x_{T+1}^*) - x_{T+2}^*) \right] < 0
\end{aligned}$$

(b.2) If $\pi^{-1}(T) > T$, we recover the case (a).

Therefore, for any $\pi \notin \hat{\Pi}$, it is always possible to define an auxiliary permutation $\hat{\pi} \in \hat{\Pi}$ such that

$$\sum_{t=0}^{\infty} \delta^t u(f(x_{\hat{\pi}(t)}^*) - x_{\hat{\pi}(t+1)}^*) < \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t+1)}^*).$$

Hence

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t+1)}^*) = \inf_{\pi \in \hat{\Pi}} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t+1)}^*)$$

Fix $\varepsilon > 0$ sufficiently small such that $0 < \varepsilon < x_{T+1}^*$, $0 < x_{T+2}^* < f(x_{T+1}^*) - \varepsilon$, $u(f(x_T^*) - (x_{T+1}^* - \varepsilon)) < u(f(x_0) - x_0)$ and

$$u(f(x_0) - x_0) + a < u(f(x_{T+1}^* - \varepsilon) - x_{T+2}^*)$$

Consider the sequence $\hat{\chi} \in \Phi(x_0)$ such that $\hat{x}_t = x_t^*$ for any $t \neq T+1$ and $\hat{x}_{T+1} = x_{T+1}^* - \varepsilon$. We have $u(f(\hat{x}_T) - \hat{x}_{T+1}) < u(f(\hat{x}_{T+1}) - \hat{x}_{T+2})$.

The following equality is satisfied for $\varepsilon > 0$ sufficiently small:

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t+1)}) = \inf_{\pi \in \hat{\Pi}} \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t+1)})$$

Let $\varphi(x_{T+1}^*) \equiv u(f(x_T^*) - x_{T+1}^*)$ and $\psi(x_{T+1}^*) \equiv u(f(x_{T+1}^*) - x_{T+2}^*)$, where φ is strictly decreasing and strictly concave and ψ is strictly increasing and strictly concave. By the definition of derivative,

$$\begin{aligned}\varphi(x_{T+1}^* - \varepsilon) - \varphi(x_{T+1}^*) &= \varphi'(x_{T+1}^*)(-\varepsilon) - R_1(\varepsilon) \\ \psi(x_{T+1}^* - \varepsilon) - \psi(x_{T+1}^*) &= \psi'(x_{T+1}^*)(-\varepsilon) - R_2(\varepsilon)\end{aligned}$$

with $R_i(\varepsilon) > 0$ and $\lim_{\varepsilon \rightarrow 0} [R_i(\varepsilon)/\varepsilon] = 0$ for $i = 1, 2$. Then,

$$\begin{aligned}& u(f(x_T^*) - (x_{T+1}^* - \varepsilon)) - u(f(x_T^*) - x_{T+1}^*) \\ &= u'(f(x_T^*) - x_{T+1}^*)\varepsilon - R_1(\varepsilon) \\ & u(f(x_{T+1}^* - \varepsilon) - x_{T+2}^*) - u(f(x_{T+1}^*) - x_{T+2}^*) \\ &= u'(f(x_{T+1}^*) - x_{T+2}^*)f'(x_{T+1}^*)(-\varepsilon) - R_2(\varepsilon)\end{aligned}$$

We obtain

$$\begin{aligned}& \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t)+1}) - \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) \\ &= \delta^{\pi^{-1}(T)} [u(f(x_T^*) - (x_{T+1}^* - \varepsilon)) - u(f(x_T^*) - x_{T+1}^*)] \\ & \quad + \delta^{\pi^{-1}(T+1)} [u(f(x_{T+1}^* - \varepsilon) - x_{T+2}^*) - u(f(x_{T+1}^*) - x_{T+2}^*)] \\ &= \delta^{\pi^{-1}(T)} [u'(f(x_T^*) - x_{T+1}^*)\varepsilon - R_1(\varepsilon)] \\ & \quad + \delta^{\pi^{-1}(T+1)} [u'(f(x_{T+1}^*) - x_{T+2}^*)f'(x_{T+1}^*)(-\varepsilon) - R_2(\varepsilon)] \\ &= \delta^{\pi^{-1}(T)} u'(f(x_T^*) - x_{T+1}^*)\varepsilon \\ & \quad - \delta^{\pi^{-1}(T+1)} u'(f(x_{T+1}^*) - x_{T+2}^*)f'(x_{T+1}^*)\varepsilon - R(\varepsilon)\end{aligned}$$

with $R(\varepsilon) \equiv \delta^{\pi^{-1}(T)} R_1(\varepsilon) + \delta^{\pi^{-1}(T+1)} R_2(\varepsilon)$.

Recall that, for any $\pi \in \hat{\Pi}$, we have $\pi^{-1}(T+1) > \pi^{-1}(T)$, then $\pi^{-1}(T+1) \geq$

$\pi^{-1}(T) + 1$ and

$$\begin{aligned}
& \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t)+1}) - \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) \\
& \geq \delta^{\pi^{-1}(T)} u'(f(x_T^*) - x_{T+1}^*) \varepsilon \\
& \quad - \delta^{\pi^{-1}(T)+1} u'(f(x_{T+1}^*) - x_{T+2}^*) f'(x_{T+1}^*) \varepsilon - R(\varepsilon) \\
& = \varepsilon \delta^{\pi^{-1}(T)} [u'(f(x_T^*) - x_{T+1}^*) - \delta u'(f(x_{T+1}^*) - x_{T+2}^*) f'(x_{T+1}^*)] - R(\varepsilon)
\end{aligned}$$

By definition of T , we have

$$u(f(x_T^*) - x_{T+1}^*) < u(f(x_0) - x_0) < u(f(x_{T+1}^*) - x_{T+2}^*)$$

Since u is strictly increasing, we obtain $f(x_T^*) - x_{T+1}^* < f(x_{T+1}^*) - x_{T+2}^*$ and, then, $u'(f(x_T^*) - x_{T+1}^*) > u'(f(x_{T+1}^*) - x_{T+2}^*)$. Therefore,

$$\begin{aligned}
& \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t)+1}) - \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) \\
& \geq \varepsilon \left(\delta^{\pi^{-1}(T)} [u'(f(x_T^*) - x_{T+1}^*) - \delta u'(f(x_{T+1}^*) - x_{T+2}^*) f'(x_{T+1}^*)] - \frac{R(\varepsilon)}{\varepsilon} \right) \\
& \geq \varepsilon \left(\delta^T [u'(f(x_T^*) - x_{T+1}^*) - \delta u'(f(x_{T+1}^*) - x_{T+2}^*) f'(x_{T+1}^*)] - \frac{R(\varepsilon)}{\varepsilon} \right)
\end{aligned}$$

Since $x_{T+1}^* \geq x_0$ and $u'(f(x_T^*) - x_{T+1}^*) > u'(f(x_{T+1}^*) - x_{T+2}^*)$, we have

$$\begin{aligned}
& \frac{1}{\varepsilon} \left[\sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t)+1}) - \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) \right] \\
& \geq \delta^T u'(f(x_T^*) - x_{T+1}^*) [1 - \delta f'(x_0)] - \frac{R(\varepsilon)}{\varepsilon}
\end{aligned}$$

Let ε converge to zero, the RHS converges to a strictly positive value because $\delta f'(x_0) < \delta f'(x^*) = 1$. Moreover, for $\varepsilon > 0$ sufficiently small, the right-hand side is strictly superior to $\delta^T u'(f(x_T^*) - x_{T+1}^*) [1 - \delta f'(x_0)] / 2$.

Hence, if ε is sufficiently small, we obtain

$$\begin{aligned} & \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t)+1}) \\ > \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) + \frac{1}{2} \varepsilon \delta^T u'(f(x_T^*) - x_{T+1}^*) [1 - \delta f'(x_0)] \end{aligned}$$

We observe that ε is independent on $\pi \in \hat{\Pi}$. Thus, we have

$$\begin{aligned} & \inf_{\pi \in \hat{\Pi}} \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t)+1}) \\ = & \inf_{\pi \in \hat{\Pi}} \sum_{t=0}^{\infty} \delta^t u(f(\hat{x}_{\pi(t)}) - \hat{x}_{\pi(t)+1}) \\ \geq & \inf_{\pi \in \hat{\Pi}} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) + \frac{1}{2} \varepsilon \delta^T u'(f(x_T^*) - x_{T+1}^*) [1 - \delta f'(x_0)] \\ > & \inf_{\pi \in \hat{\Pi}} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) = \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}^*) - x_{\pi(t)+1}^*) \end{aligned}$$

a contradiction. Hence, the unique optimal path is $\chi = (x_0, x_0, \dots)$ and

$$\max_{\chi \in \Phi(x_0)} \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}) - x_{\pi(t)+1}) = \frac{u(f(x_0) - x_0)}{1 - \delta}$$

(3) According to (2), for any t , $\inf_{s \geq 0} u(f(x_{t+s}) - x_{t+s+1}) \leq u(f(\bar{x}) - \bar{x})$.¹⁰

Hence, for any $\varepsilon > 0$, there exists a subsequence $(t_n)_{n=0}^{\infty}$ such that

$$u(f(x_{t_n}) - x_{t_n+1}) \leq u(f(\bar{x}) - \bar{x}) + \varepsilon$$

Applying the part (2) of Lemma 1, we obtain

$$U(\chi) \leq \frac{u(f(\bar{x}) - \bar{x})}{1 - \delta} \tag{6}$$

for every $\chi \in \phi(x_0)$.

¹⁰See Proposition A.2, part (ii), in [Ha-Huy and Nguyen \(2022\)](#).

In the proof of Proposition A.2, part (ii), [Ha-Huy and Nguyen \(2022\)](#) show that there are infinitely many optimal paths $\chi = (x_0, x_1, \dots)$ such that

$$\inf_{t \geq 0} u(f(x_t) - x_{t+1}) = u(f(\bar{x}) - \bar{x})$$

Let χ be one of this paths. We have

$$\begin{aligned} U(\chi) &\equiv \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} \delta^t u(f(x_{\pi(t)}) - x_{\pi(t)+1}) \geq \sum_{t=0}^{\infty} \delta^t \inf_{t \geq 0} u(f(x_t) - x_{t+1}) \\ &\geq \sum_{t=0}^{\infty} \delta^t u(f(\bar{x}) - \bar{x}) = \frac{u(f(\bar{x}) - \bar{x})}{1 - \delta} \end{aligned}$$

According to (6), we obtain $U(\chi) = u(f(\bar{x}) - \bar{x}) / (1 - \delta)$. Thus, every optimal path are decreasing and converges to \bar{x} .

Proof of Proposition 2

To simplify the presentation, RZA will denote the convex combination between the RCK criterion and the third Rawlsian criterion introduced by [Zuber and Asheim \(2012\)](#).

(1) Focus on the case $0 < x_0 \leq x^*$. Let $(x_t^*)_{t=0}^{\infty}$ be the optimal path under the RCK criterion. This sequence is increasing and converges to the *modified golden rule* x^* , solution to $\delta f'(x) = 1$ (Proposition 2.4.4 in [Le Van and Dana \(2002\)](#)).

Moreover, the corresponding consumption sequence $c_t^* = f(x_t^*) - x_{t+1}^*$ is increasing and converges to $c^* = f(x^*) - x^*$ (see the Euler equation in Proposition 2.4.4 in [Le Van and Dana \(2002\)](#)). This implies:

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}^*) = \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t^*)$$

Hence, for every feasible sequence $(x_t)_{t=0}^{\infty}$ with the corresponding consumption

sequence $c_t = f(x_t) - x_{t+1}$, we have

$$\begin{aligned}
& (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t^*) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}^*) \\
= & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t^*) + \lambda \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t^*) \\
\geq & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) \\
\geq & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})
\end{aligned}$$

Therefore, the optimal path under the RCK criterion is also the optimal path under the RZA one.

(2) Now, consider the nontrivial case: $x_0 > x^*$. We will prove that the optimal path coincides with the one in [Ha-Huy and Nguyen \(2022\)](#).

Let $(x_t)_{t=0}^{\infty}$ be the optimal path and $c_t = f(x_t) - x_{t+1}$. First, we will prove that $x_t > x^*$ for any $t \geq 0$.

Assume the contrary, that is the existence of T such that $x_T \leq x^*$. It is known that there exists some $t \geq T$ such that $c_t \leq f(x_T) - x_T \leq f(x^*) - x^* = c^*$. Moreover, there are only finitely many t such that $c_t < c^* = f(x^*) - x^*$. Indeed, by part (2) of [Lemma 1](#), the existence of infinitely many t such that $c_t < c^*$ implies

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) \leq u(c^*)$$

Recall that $(x_t^*)_{t=0}^{\infty}$ is the optimal capital path under the RCK criterion. Moreover, since $x_0 > x^*$ (see [Proposition 2.4.4](#) in [Le Van and Dana \(2002\)](#)), the corresponding

consumption sequence $(c_t^*)_{t=0}^\infty$ is decreasing and converges to c^* . We obtain:

$$\begin{aligned}
& (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t^*) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}^*) \\
\geq & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t^*) + \lambda u(c^*) \\
> & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})
\end{aligned}$$

a contradiction with the hypothesis that $(x_t)_{t=0}^\infty$ is the optimal path under the RZA criterion.

We have proven that there is a finite number of t such that $c_t < c^*$. Let T be the bigger period satisfying this inequality. Therefore, $x_{T+1} \geq x^*$. Indeed, if $x_{T+1} < x^*$, following the same argument we used to solve the optimization problem under the Rawls criterion (see inequality (5), we obtain:

$$\inf_{t>0} u(f(x_{T+t}) - x_{T+t+1}) < u(c^*)$$

This implies the existence of $t > 0$ such that $c_{T+t} < c^*$, a contradiction with the definition of T .

Fix any ε such that $0 < \varepsilon < x_{T+1}$ and $0 < x_{T+2} < f(x_{T+1}) - \varepsilon$. Define the capital

sequence $(\hat{x}_t)_{t=0}^\infty$ such that $\hat{x}_t = x_t$ for any $t \neq T + 1$ and $\hat{x}_{T+1} = x_{T+1} - \varepsilon$. Hence,

$$\begin{aligned}
& \sum_{t=0}^{\infty} \delta^t u(\hat{c}_t) - \sum_{t=0}^{\infty} \delta^t u(c_t) \\
&= \delta^T u(\hat{c}_T) + \delta^{T+1} u(\hat{c}_{T+1}) - \delta^T u(c_T) - \delta^{T+1} u(c_{T+1}) \\
&\geq \delta^T u'(\hat{c}_T)(\hat{c}_T - c_T) + \delta^{T+1} u'(\hat{c}_{T+1})(\hat{c}_{T+1} - c_{T+1}) \\
&= \delta^T u'(\hat{c}_T)(f(x_T) - (x_{T+1} - \varepsilon) - [f(x_T) - x_{T+1}]) \\
&\quad + \delta^{T+1} u'(\hat{c}_{T+1})[f(x_{T+1} - \varepsilon) - x_{T+2} - f(x_{T+1}) + x_{T+2}] \\
&\geq \delta^T u'(\hat{c}_T)\varepsilon - \delta^{T+1} u'(\hat{c}_{T+1})f'(x_{T+1} - \varepsilon)\varepsilon \\
&= \varepsilon \delta^T [u'(\hat{c}_T) - \delta u'(\hat{c}_{T+1})f'(x_{T+1} - \varepsilon)] \tag{7}
\end{aligned}$$

because of the concavity of f and u . We obtain:

$$\frac{1}{\varepsilon \delta^T} \left[\sum_{t=0}^{\infty} \delta^t u(\hat{c}_t) - \sum_{t=0}^{\infty} \delta^t u(c_t) \right] \geq u'(\hat{c}_T) - \delta u'(\hat{c}_{T+1})f'(x_{T+1} - \varepsilon)$$

Since $u'(c_T) > u'(c_{T+1})$ and $\delta f'(x_{T+1}) \leq \delta f'(x^*) = 1$, for ε small enough, the RHS is strictly positive and, thus,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[\sum_{t=0}^{\infty} \delta^t u(\hat{c}_t) - \sum_{t=0}^{\infty} \delta^t u(c_t) \right] > 0$$

Then, for ε sufficiently small, $\sum_{t=0}^{\infty} \delta^t u(\hat{c}_t) > \sum_{t=0}^{\infty} \delta^t u(c_t)$.

Let Π_T be the set of permutations π such that $\pi^{-1}(T) < \pi^{-1}(T + 1)$. Since $c_T < c^* \leq c_{T+1}$, we obtain:

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) = \inf_{\pi \in \Pi_T} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})$$

Now, fix any permutation $\pi \in \Pi^T$ with $\pi^{-1}(T) < \pi^{-1}(T + 1)$. Applying similar

arguments as in (7), we obtain:

$$\begin{aligned}
& \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\hat{c}_{\pi(t)}) - \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) \\
&= \delta^{\pi^{-1}(T)} u(\hat{c}_T) + \delta^{\pi^{-1}(T+1)} u(\hat{c}_{T+1}) - \delta^{\pi^{-1}(T)} u(c_T) - \delta^{\pi^{-1}(T+1)} u(c_{T+1}) \\
&= \delta^{\pi^{-1}(T)} \left([u(\hat{c}_T) - u(c_T)] + \delta^{\pi^{-1}(T+1) - \pi^{-1}(T)} [u(\hat{c}_{T+1}) - u(c_{T+1})] \right) \\
&\geq \varepsilon \delta^{\pi^{-1}(T)} \left[u'(\hat{c}_T) - \delta^{\pi^{-1}(T+1) - \pi^{-1}(T)} u'(\hat{c}_{T+1}) f'(x_{T+1} - \varepsilon) \right] \\
&\geq \varepsilon \delta^{\pi^{-1}(T)} [u'(\hat{c}_T) - \delta u'(\hat{c}_{T+1}) f'(x_{T+1} - \varepsilon)]
\end{aligned}$$

with $\hat{x}_{T+1} = x_{T+1} - \varepsilon$.

Therefore, for any $\pi \in \Pi_T$,

$$\begin{aligned}
& \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\hat{c}_{\pi(t)}) \\
&\geq \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) + \varepsilon \delta^{\pi^{-1}(T)} [u'(\hat{c}_T) - \delta u'(\hat{c}_{T+1}) f'(x_{T+1} - \varepsilon)] \\
&\geq \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})
\end{aligned}$$

for ε small enough. Then, for ε small enough, for any $\pi \in \Pi_T$,

$$\begin{aligned}
\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\hat{c}_{\pi(t)}) &= \inf_{\pi \in \Pi_T} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\hat{c}_{\pi(t)}) \\
&\geq \inf_{\pi \in \Pi_T} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) \\
&\geq \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})
\end{aligned}$$

Therefore, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
& (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\hat{c}_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\hat{c}_{\pi(t)}) \\
&> (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})
\end{aligned}$$

a contradiction.

This contradiction comes from the hypothesis that there exists t such that $x_t \leq x^*$. Hence $x_t > x^*$ for any $t \geq 0$.

Let us prove that $c_t \geq c_{t+1}$ for any $t \geq 0$. Assume the existence of some T such that $c_T < c_{T+1}$. Fix $\varepsilon > 0$ such that $0 < \varepsilon < x_{T+1}$ and $f(x_T) - x_{T+1} + \varepsilon < f(x_{T+1} - \varepsilon) - x_{T+2}$. Consider the sequence $(\hat{x}_t)_{t=0}^\infty$ as in the first part of point (2) of the current proof. Using the same arguments and considering that $x_{T+1} > x^*$ implies $\delta f'(x_{T+1}) < 1$, we can prove that the sequence $(x_t)_{t=0}^\infty$ is not optimal, a contradiction.

This contradiction comes from the hypothesis there exists T such that $c_T < c_{T+1}$. Hence, the sequence $(c_t)_{t=0}^\infty$ is non-increasing: $c_t \geq c_{t+1}$ for any $t \geq 0$. Since the utility function u is strictly increasing, we have $\lim_{t \rightarrow \infty} u(c_t) = \inf_{t \geq 0} u(c_t)$.

Hence, for every $\varepsilon > 0$, there exists an infinite number t such that $u(c_t) < \inf_{t \geq 0} u(c_t) + \varepsilon$. Applying part (2) of Lemma 1, we obtain

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) \leq \inf_{t \geq 0} u(c_t)$$

Clearly,

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) \geq \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t \inf_{t \geq 0} u(c_t) = \inf_{t \geq 0} u(c_t)$$

and, thus,

$$\inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) = \inf_{t \geq 0} u(c_t)$$

Hence, we have

$$\begin{aligned} & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) \\ &= (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{t \geq 0} u(c_t) \end{aligned}$$

Consider the sequence $(\bar{x}_t)_{t=0}^\infty$ which is the optimal path under the hybrid criterion presented in section 2 of [Ha-Huy and Nguyen \(2022\)](#), and the corresponding consumption sequence $(\bar{c}_t)_{t=0}^\infty$. We have

$$\begin{aligned}
& (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)}) \\
= & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{t \geq 0} u(c_t) \\
\leq & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\bar{c}_t) + \lambda \inf_{t \geq 0} u(\bar{c}_t) \\
\leq & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\bar{c}_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(\bar{c}_{\pi(t)}) \\
\leq & (1 - \lambda) \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_t) + \lambda \inf_{\pi \in \Pi} \sum_{t=0}^{\infty} (1 - \delta) \delta^t u(c_{\pi(t)})
\end{aligned}$$

Hence, all these inequalities hold with equality and, finally, the two sequences $(x_t)_{t=0}^\infty$ and $(\bar{x}_t)_{t=0}^\infty$ coincide.

In [Ha-Huy and Nguyen \(2022\)](#), the authors proved that the optimal path under the criterion that is the combination between RCK and the first Rawlsian is non-increasing when $x_0 \geq x^*$. Therefore, this path coincides with the optimal path under the mixed criterion presented in [Ha-Huy and Nguyen \(2022\)](#), with $a = \lambda / (1 - \lambda)$.

For $x_0 \geq x^*$, in [Ha-Huy and Nguyen \(2022\)](#), the optimal path can be described as follows:

(2.1) If $0 \leq a \leq a_1$, the equality part plays no role and the optimal path coincides the optimal path of the RCK criterion.

(2.2) When $a_1 < a < a_2$, the economy converges to the *modified golden rule* as in the RCK model, but with a larger discount factor $\tilde{\delta} > \delta$.

(2.3) If $a \geq a_2$, the equality part fully dominates and we recover the optimal path of the first Rawlsian criterion.

Let $\lambda_1 \equiv a_1/(1 + a_1)$ and $\lambda_2 \equiv a_2/(1 + a_2)$, we have $0 \leq \lambda_1 \leq \lambda_2 \leq 1$ with $a_2 > 0$. Moreover, $\lambda \leq \lambda_1$ if and only if $a \leq a_1$ and $\lambda \geq \lambda_2$ if and only if $a \geq a_2$. Hence, we obtain the results using Proposition 2.2 in [Ha-Huy and Nguyen \(2022\)](#).

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