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Gaussian Fields through Geometrical Properties

Anne Estrade and Julie Fournier

Abstract The lecture is dedicated to the study of Gaussian random fields through their geometrical properties. In the first section, we introduce general definitions and invariance properties of Gaussian fields indexed by the Euclidean space \mathbb{R}^d . In particular, the crucial role of the covariance function is emphasized and the special case of stationary fields is presented from the Gaussian random wave point of view. Chapter two deals with a geometric feature that is really specific to the multivariate context: anisotropy. We present various models of Gaussian fields whose distributions all share the property of not being invariant under rotations. We try to understand which characteristics of the field are impacted by the anisotropy property and how they are impacted. In the third chapter, we provide Rice formulas and the so-called Gaussian kinematic formulas. They consist in explicitly writing moments of some geometric functionals that depend on the level sets of the Gaussian field. These formulas are perfect tools to study both qualitative and quantitative geometric properties of stationary Gaussian fields. This is done in the last chapter where we visit recent works that aim at inferring features of the considered Gaussian field based on sparse geometrical observations.

1 Introduction to Gaussian Fields

We begin with an introductory section where we set general notions about Gaussian random fields. The Gaussian random fields are particularly used as continuous models for multidimensional structures and a large theory has been built in this framework, where it is often possible to derive explicit formulas under fairly simple

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assumptions. Indeed, as it is explained in the following, the distribution of a Gaussian random field is entirely characterized by its mean and covariance functions. That is why much functional analysis is central in the study of distribution properties of Gaussian random fields. From a modelling point of view, it is sometimes appropriate to consider random fields with certain invariance properties that we present in this section. We also introduce the regularity framework that we will use to describe geometric characteristics of the realizations of Gaussian random fields, that is almost sure regularity. To end with, the Gaussian random wave model is introduced to offer a general analytic representation of the Gaussian random fields, which also provides an easy simulation method.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The space \mathbb{R}^d is endowed with the Borelian σ -algebra $\mathcal{B}(\mathbb{R}^d)$ and the set of functions from \mathbb{R}^d to \mathbb{R} is endowed with the σ -algebra generated by the family \mathcal{C} of cylinders:

$$\mathcal{C} = \bigcup_{n \in \mathbb{N}^*} \bigcup_{(t_1, \dots, t_n) \in (\mathbb{R}^d)^n} \bigcup_{B \in \mathcal{B}(\mathbb{R}^d)} \{f : \mathbb{R}^d \rightarrow \mathbb{R}; (f(t_1), \dots, f(t_n)) \in B\},$$

where \mathbb{N}^* is the set of positive integers.

1.1 Setting

Roughly speaking, a d -dimensional random field is a random function defined on \mathbb{R}^d . If $d = 1$, the term of stochastic process is preferred to the one of random field. Taking $d = 2$, a random field on \mathbb{R}^2 can be used for the modelling of a grey level image. To model biological tissues or porous media, one can use a random field defined on \mathbb{R}^3 , such that $X(x)$ is the occupation density at point x . Space-time phenomena models can be obtained with random fields defined on $\mathbb{R}^d \times \mathbb{R}$.

Definition 1 A random field $X : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ is a collection of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by \mathbb{R}^d , such that $\omega \in \Omega \mapsto X(\omega, \cdot)$ is measurable for the σ -algebra on $\mathbb{R}^{\mathbb{R}^d}$ generated by the family \mathcal{C} of cylinders.

Following a common abuse on notation, we will not note the dependence on ω of a random field X , writing simply $X = \{X(t) : t \in \mathbb{R}^d\}$.

According to Kolmogorov's consistency theorem (see for instance [55] Proposition 4.2 or [10] Theorem 1.1), the law of a random field X is determined by its finite-dimensional distributions, that is to say by the set of the laws of the random vectors

$$(X(t_1), \dots, X(t_n)) : n \in \mathbb{N}^*, (t_1, \dots, t_n) \in (\mathbb{R}^d)^n.$$

We will soon handle Gaussian random fields, which have finite moments of any order. Generally speaking, for any random field X that admits a finite second moment at each point $t \in \mathbb{R}^d$, we can define its expectation function m and its covariance function C as

$$\begin{cases} m : \mathbb{R}^d \rightarrow \mathbb{R}, t \mapsto m(t) = \mathbb{E}[X(t)] \\ C : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}, (s, t) \mapsto C(s, t) = \mathbb{E}[(X(s) - m(s))(X(t) - m(t))] . \end{cases}$$

The covariance function C is a symmetric function of non-negative type, that is

- symmetry : $\forall (s, t) \in (\mathbb{R}^d)^2, C(s, t) = C(t, s)$,
- non-negativeness : $\forall n \in \mathbb{N}^*, \forall (t_i)_{1 \leq i \leq n} \in (\mathbb{R}^d)^n, \forall (\alpha_i)_{1 \leq i \leq n} \in \mathbb{R}^n$,

$$\sum_{1 \leq i, j \leq n} \alpha_i \alpha_j C(t_i, t_j) \geq 0 .$$

In the second point, note that the sum is equal to $\text{Var}(\sum_{1 \leq i \leq n} \alpha_i X(t_i))$.

These two properties are also sufficient for a function $C : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ to be the covariance function of a certain random field. We will see this just after introducing Gaussian random fields.

Definition 2 The random field X is said to be Gaussian if for all $n \in \mathbb{N}^*$, for all $(t_i)_{1 \leq i \leq n} \in (\mathbb{R}^d)^n$, $(X(t_1), \dots, X(t_n))$ is a Gaussian vector.

In this case, the law of one of them $(X(t_1), \dots, X(t_n))$ is determined by its covariance matrix $(C(t_i, t_j))_{1 \leq i, j \leq n}$ and by its expectation vector $(\mathbb{E}[X(t_i)])_{1 \leq i \leq n}$. Therefore, Kolmogorov's theorem yields the following result.

Theorem 1 (Existence theorem)

For any function $m : \mathbb{R}^d \rightarrow \mathbb{R}$, for any symmetric and non-negative covariance function $C : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, there exists a Gaussian random field $X : \mathbb{R}^d \rightarrow \mathbb{R}$ with expectation function m and covariance function C . Moreover, the distribution of X is unique.

To end with, we present a regression result that will come into play when we need to express conditional expectations involving joint Gaussian variables. These conditional expectations arise in Rice formulas (see Section 3).

In the following, we denote by $\mathcal{N}_p(m, \Sigma)$ the p -dimensional Gaussian distribution with mean m and covariance matrix Σ . We recall that the non-degeneracy of a Gaussian random vector in \mathbb{R}^p means that its covariance matrix is invertible or equivalently that it admits a probability density function. We also denote by M^T the transpose matrix of any matrix M .

Proposition 1 (Gaussian regression)

Let $(p, q) \in \mathbb{N}^ \times \mathbb{N}^*$ and (X^1, X^2) be a centered Gaussian vector in $\mathbb{R}^p \times \mathbb{R}^q$. We assume that X^2 is not degenerate. We write C_{11} and C_{22} the covariance matrices of X^1 and X^2 , respectively, and C_{12} the $p \times q$ matrix of the covariances between the coordinates of X^1 and X^2 , such that the covariance matrix of (X^1, X^2) is $\begin{pmatrix} C_{11} & C_{12}^T \\ C_{12} & C_{22} \end{pmatrix}$.*

Then the conditional law of X^1 given X^2 is Gaussian, prescribed by

$$\mathcal{N}_p(C_{12}C_{22}^{-1}X_2, C_{11} - C_{12}C_{22}^{-1}C_{12}^T).$$

Proof There exists a $p \times q$ matrix M , a $p \times p$ covariance matrix Γ and a $\mathcal{N}_p(0, \Gamma)$ random variable ε independent from X^2 such that $X^1 = MX^2 + \varepsilon$. Expressing C_{11} and C_{12} from this equation allows to get an invertible system of two linear equations with unknowns M and Γ . \square

1.2 Law Invariance Properties

We present here some classes of random fields with distribution invariance properties (isotropy, stationarity, stationarity of increments). We focus on the case of Gaussian fields.

From now on, we write $X \stackrel{\mathcal{L}}{\sim} Y$ if X and Y have the same law.

1.2.1 Isotropy

Informally, the isotropy of a field means that it has stochastically the same behaviour on any direction of \mathbb{R}^d . It can also be understood as the lack of preferred directions for the field.

We write $SO(d)$ the group of rotations defined on the Euclidean space \mathbb{R}^d , where the Euclidean norm is denoted by $\|\cdot\|$.

Definition 3 Let X be a random field defined on \mathbb{R}^d . It is said to be isotropic if its law is invariant under any rotation ρ of the parameter space:

$$\forall \rho \in SO(d), \quad X \circ \rho \stackrel{\mathcal{L}}{\sim} X.$$

As a consequence, the expectation function m is radial as well as the variance. Furthermore, thanks to Theorem 1, the next proposition holds.

Proposition 2 Let X be a random field defined on \mathbb{R}^d . If X is isotropic then,

- its expectation function is radial: $\mathbb{E}[X(s)]$ only depends on s through $\|s\|$,
- its covariance function C satisfies

$$\forall \rho \in SO(d), \forall (s, t) \in (\mathbb{R}^d)^2, C(\rho(s), \rho(t)) = C(s, t). \quad (1)$$

If a centered Gaussian random field satisfies (1) then it is isotropic.

Condition (1) on the covariance function is called weak isotropy or second-order isotropy, as opposed to isotropy of Definition 3, sometimes called strong isotropy. Proposition 2 states that strong and weak isotropy are equivalent for centered Gaussian random fields.

1.2.2 Stationarity

Informally, a field is stationary if it has stochastically the same behaviour around any point of \mathbb{R}^d .

Definition 4 Let X be a random field defined on \mathbb{R}^d . It is said to be stationary (or homogeneous) if its law is invariant under any translation of the parameter space:

$$\forall a \in \mathbb{R}^d, \quad X(\cdot + a) \stackrel{\mathcal{L}}{\sim} X .$$

The stationarity property implies strong constraints on the moments of the random field.

Proposition 3 Let X be a random field defined on \mathbb{R}^d . If X is stationary then,

- its expectation and variance functions are constant,
- its covariance function C satisfies

$$\forall (s, t) \in (\mathbb{R}^d)^2, \quad C(s, t) = C(s - t, 0) . \quad (2)$$

If X is a centered Gaussian random field satisfying (2) then it is stationary.

As for isotropy, the above property (2) on a covariance function is called second-order stationarity or weak stationarity as opposed to strong stationarity defined above. Both are equivalent for centered Gaussian random fields.

It is common to assume a stationary random field X as centered, because if not, we only have to subtract to X its constant expectation m .

Definition 5 Let X be a stationary random field defined on \mathbb{R}^d . By abusing the vocabulary, function $r: \mathbb{R}^d \rightarrow \mathbb{R}$ defined by $r(t) = \text{Cov}(X(t), X(0))$ is still called the covariance function of X . It is such that

$$\forall (s, t) \in (\mathbb{R}^d)^2, \quad \text{Cov}(X(s), X(t)) = r(s - t) . \quad (3)$$

The properties of r are simply derived from those of C . For instance, r is even because of the symmetry of C : $r(t) = C(X(t), X(0)) = C(X(0), X(t)) = r(-t)$. Moreover, the covariance function r is of non-negative type, in the sense that

$$\forall n \in \mathbb{N}^*, \forall (t_1, \dots, t_n) \in (\mathbb{R}^d)^n, \forall (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, \quad \sum_{1 \leq i, j \leq n} \alpha_i \alpha_j r(t_i - t_j) \geq 0 .$$

Reciprocally, for any function $m: \mathbb{R}^d \rightarrow \mathbb{R}$ and any even function $r: \mathbb{R}^d \rightarrow \mathbb{R}$ of non-negative type, Theorem 1 allows to state that there exists a unique (in distribution) Gaussian stationary random field with expectation function m and covariance function r .

Many models combine stationarity and isotropy. In such models, the random field has stochastically the same behaviour in any direction and around any point of the

space. In the next section, we will introduce examples of random fields that are not stationary or not isotropic.

The following result simply follows from Propositions 2 and 3.

Proposition 4 *Let X be a centered Gaussian random field defined on \mathbb{R}^d .*

- *If X is stationary and isotropic then there exists a function $R : \mathbb{R}^+ \rightarrow \mathbb{R}$, such that $\forall (s, t) \in (\mathbb{R}^d)^2$, $\text{Cov}(X(s), X(t)) = R(\|s - t\|)$.*
- *Conversely, if $\text{Cov}(X(s), X(t))$ only depends on s and t through $\|s - t\|$ then X is stationary and isotropic.*

A theorem due to Bochner and dating back from 1933 (see [28]) provides an integral expression of the covariance function of a stationary random field. Originally designed in the setting of Fourier analysis, it applies to any complex-valued positive semidefinite function on \mathbb{R}^d , hence exactly to covariance functions of complex-valued, weakly stationary random fields. Since we will only be interested in the real-valued random fields in the following sections, we give below a version that is adapted to our specific context. A statement and a proof of the original version of Bochner's theorem can be found in the lecture notes of Hermine Biermé, Theorem 4.2 in [22].

Theorem 2 (Bochner Theorem)

A continuous function $r : \mathbb{R}^d \rightarrow \mathbb{R}$ is of non-negative type if and only if there exists a finite measure F on the Borel σ -algebra $\mathcal{B}(\mathbb{R}^d)$ such that

- *F is symmetric, i.e. $F(-B) = F(B)$ for all $B \in \mathcal{B}(\mathbb{R}^d)$,*
- *$\forall t \in \mathbb{R}^d$,*

$$r(t) = \int_{\mathbb{R}^d} \exp(i\xi \cdot t) dF(\xi) = \int_{\mathbb{R}^d} \cos(\xi \cdot t) dF(\xi) . \quad (4)$$

Moreover, if F exists, it is unique.

When function r is the covariance function of a stationary random field X then F is called the spectral measure of X . Note that $r(0) = F(\mathbb{R}^d)$. Moreover, if F admits a density $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with respect to the Lebesgue measure in \mathbb{R}^d then f is necessarily even and f is called the spectral density of X .

The spectral moments are parameters of the random field arising with the spectral representation. Writing \mathbb{N}_0 the set of natural integers including 0, we take $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{N}_0^d$ and write $|\mathbf{j}| = \sum_{\ell=1}^d j_\ell$. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we also write $x^{\mathbf{j}} = \prod_{1 \leq \ell \leq d} x_\ell^{j_\ell}$. Then, if $\xi \mapsto \xi^{\mathbf{j}}$ is integrable with respect to measure F on \mathbb{R}^d , the following quantity

$$\lambda_{\mathbf{j}} := \int_{\mathbb{R}^d} \xi^{\mathbf{j}} dF(\xi) \quad (5)$$

defines a spectral moment of X . Its order is $|\mathbf{j}|$. Because of the symmetry of F , if an odd-order spectral moment exists, it is necessarily zero.

If X is a weakly isotropic random field, then F is invariant under rotations, thus we may define the second spectral moment as

$$\lambda_2 := \int_{\mathbb{R}^d} \xi_j^2 dF(\xi), \quad \forall j \in \{1, \dots, d\}, \quad (6)$$

the moments $\int_{\mathbb{R}^d} \xi_i \xi_j dF(\xi)$ being equal to zero if $i \neq j$. In this case, the matrix of the second-order spectral moments of X , $\left(\int_{\mathbb{R}^d} \xi_i \xi_j dF(\xi) \right)_{1 \leq i, j \leq d}$ is simply equal to $\lambda_2 I_d$.

In Section 1.4, we use the integral representation of stationary covariance functions contained in Bochner Theorem to present a general representation of stationary Gaussian fields through what we call Gaussian random waves.

For various examples of stationary covariance functions, we refer for instance to [71] Chapter 4.

1.2.3 Stationarity of Increments

Informally, one says that a field has stationary increments if its variations around any point in the space are stochastically the same.

Definition 6 A random field X on \mathbb{R}^d is said to have stationary increments if

$$\forall a \in \mathbb{R}^d, \quad X(\cdot + a) - X(a) \stackrel{\mathcal{L}}{\sim} X(\cdot) - X(0).$$

Proposition 5 *If X is stationary then X has stationary increments.*

Proof Let X a stationary random field and let $x \in \mathbb{R}^d$. We can prove that $X(x + \cdot) - X(x)$ and $X(\cdot) - X(0)$ have the same law considering all their finite-dimensional distributions. For the sake of simplicity, we focus on the bi-dimensional distributions. Because of stationarity, for any $t_1, t_2 \in \mathbb{R}^d$, the random vectors $((X(x), X(x + t_1), X(x + t_2))$ and $(X(0), X(t_1), X(t_2))$ are identically distributed. Hence, it also applies to $(X(x + t_1) - X(x), X(x + t_2) - X(x))$ and $(X(t_1) - X(0), X(t_2) - X(0))$. \square

Instead of the covariance function, another functional named as the variogram is more appropriate to characterize the dependence structure of a random field with stationary increments.

Definition 7 Let $X : \mathbb{R}^d \rightarrow \mathbb{R}$ be a random field with stationary increments. The function

$$v : \mathbb{R}^d \rightarrow [0; +\infty[, \quad t \mapsto v(t) = \text{Var}(X(t) - X(0)) = \text{Var}(X(a + t) - X(a)),$$

with any $a \in \mathbb{R}^d$, is called the variogram of X .

Note that $v(0) = 0$. Developing $v(s - t) = \text{Var}(X(s) - X(t))$, we get that

$$\text{Cov}(X(s) - X(0), X(t) - X(0)) = \frac{1}{2} (v(s) + v(t) - v(s - t)). \quad (7)$$

Reciprocally, if one is interested in an existence theorem, one can find for instance in [22] sufficient conditions on v such that the right-hand side of (7) prescribes an admissible covariance function.

Since the law of a Gaussian random field is defined by its expectation and covariance functions, the following result is clear.

Proposition 6 *Let $X : \mathbb{R}^d \rightarrow \mathbb{R}$ be a centered Gaussian random field. Then, X has stationary increments if and only if there exists a function $v : \mathbb{R}^d \rightarrow \mathbb{R}$ such that*

$$\text{Cov}(X(s) - X(0), X(t) - X(0)) = \frac{1}{2} (v(s) + v(t) - v(s - t)) .$$

In such case, the variogram of X is equal to v .

In Section 2.3, we introduce a fractional type Gaussian field with stationary increments whose anisotropy properties can be read in its variogram.

1.3 Regularity

We focus on smooth random fields, for which the realizations of the set of critical points, of the level sets and of the excursion sets are non-degenerate geometric objects. Therefore, the kind of regularity that we need is almost sure regularity.

Let us first recall how it is defined. The topology on \mathbb{R}^d is the one given by the Euclidean metric. For the proofs of the results stated in this section, we refer to [5] and [10].

Definition 8 (almost sure regularity) We say that X is a.s. continuous on \mathbb{R}^d if, for almost any $\omega \in \Omega$, $X(\omega, \cdot)$ is continuous on \mathbb{R}^d . For $k \in \mathbb{N}^*$, we say that X is a.s. of class \mathcal{C}^k if, for almost any $\omega \in \Omega$, $X(\omega, \cdot)$ is of class \mathcal{C}^k on \mathbb{R}^d .

If X is a.s. of class \mathcal{C}^1 , we introduce the vector-valued random field $X' = (X'_j)_{1 \leq j \leq d} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ corresponding to the gradient of X in the canonical basis of \mathbb{R}^d , denoted by (e_1, \dots, e_d) . If X is a.s. of class \mathcal{C}^2 , the vector-valued random field $X'' = (X''_{ij})_{1 \leq i, j \leq d} : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ corresponds to the Hessian matrix of X in the canonical basis of \mathbb{R}^d . Note that if X is a Gaussian random field, its almost sure partial derivatives are also Gaussian random fields.

There is a link between almost sure regularity of a random field and the regularity of its covariance function.

Proposition 7 *Let $k \in \mathbb{N}_0$. Let us assume that X is a centered Gaussian random field with realizations almost surely of class \mathcal{C}^k on \mathbb{R}^d . Then its covariance function C is of class \mathcal{C}^{2k} on \mathbb{R}^d .*

Proof We use the fact that the almost sure convergence entails the convergence in distribution, which itself can be related to a convergence result of the characteristic function, according to Lévy Theorem. This allows us to conclude on the regularity of the covariance function when the considered random field is Gaussian. \square

To a certain extent, it is also possible to derive regularity properties of the field from the regularity properties of its covariance function. More precisely, regularity of the covariance function does not imply in general almost sure regularity of the field but it implies another type of regularity, that is regularity in quadratic mean, which we will not discuss here. We refer to [10] Proposition 1.13 for a precise statement about the link between the regularity in quadratic mean of the random field and the regularity of the covariance function.

The result of Proposition 7 can be supplemented with relationships between partial derivatives of the covariance function C and the covariances between partial derivatives of X . For the sake of simplicity, we state here these relationships in the case of a (weakly) stationary random field.

Let us introduce another notation to be able to write derivatives of order higher than two. For $\mathbf{j} = (j_1, \dots, j_d) \in \mathbb{N}_0^d$, we write $|\mathbf{j}| = \sum_{\ell=1}^d j_\ell$ and if f is a map from \mathbb{R}^d to \mathbb{R} with adequate regularity, we write

$$\partial^{\mathbf{j}} f = \frac{\partial^{|\mathbf{j}|} f}{\partial t_1^{j_1} \dots \partial t_d^{j_d}}.$$

Let $(\mathbf{j}, \mathbf{k}) \in \mathbb{N}_0^d \times \mathbb{N}_0^d$, let us assume that X is centered, weakly stationary and a.s. of class $\mathcal{C}^{\max(|\mathbf{j}|, |\mathbf{k}|)}$, then according to Proposition 7, its covariance function r admits a partial derivative $\partial^{\mathbf{j}+\mathbf{k}} r$ on \mathbb{R}^d . Furthermore, for any $(s, t) \in (\mathbb{R}^d)^2$,

$$\mathbb{E} [\partial^{\mathbf{j}} X(s) \partial^{\mathbf{k}} X(t)] = (-1)^{|\mathbf{k}|} \partial^{\mathbf{j}+\mathbf{k}} r(s-t). \quad (8)$$

The fact that r is an even function explains the apparent asymmetry of the right-hand term of this equality. Let us prove this relationship in the simplest case, if $\mathbf{j} = (0, \dots, 0)$ and $\mathbf{k} = (1, 0, \dots, 0)$. Let $(h_n)_{n \in \mathbb{N}^*}$ a real sequence converging to zero. We write $\partial^{\mathbf{k}} X(t) = X'_1(t) = \lim_{h \rightarrow 0} \frac{X(t+he_1) - X(t)}{h}$ and we use the dominated convergence theorem to invert the expectation and the limit sign to get

$$\begin{aligned} \mathbb{E} [X(s) X'_1(t)] &= \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} [X(s) (X(t+he_1) - X(t))] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (r(t-s+he_1) - r(t-s)). \end{aligned}$$

On the right-hand side of this equality, we get $\frac{\partial r}{\partial e_1}(t-s)$, which is $-\frac{\partial r}{\partial e_1}(s-t)$ because a first partial derivative of an even function is odd.

Formula (8) has simple interesting consequences. At each of the following points, we assume that X is a **centered and stationary** random field with the required a.s. regularity so that the partial derivatives of r that are at stake are well-defined.

- For $(i, j) \in \{1, \dots, d\}^2$, the random field X'_i admits the covariance function $-r''_{i,i}$, and $X''_{i,j}$ admits the covariance function $r^{(4)}_{i,i,j,j}$.
- Since r is an even function, $r'_i(0) = r^{(3)}_{i,j,m}(0) = 0$, which entails

$$\forall t \in \mathbb{R}^d, \quad \mathbb{E}[X'_i(t) X(t)] = \mathbb{E}[X''_{i,j}(t) X'_m(t)] = 0. \quad (9)$$

Thus, in the **Gaussian** framework, at any $t \in \mathbb{R}^d$, $X(t)$ is independent of any of its first-order partial derivatives at point t , and the same holds for a first-order partial derivative at point t with any second-order partial derivative at point t . This makes computations a lot easier in the stationary and Gaussian framework.

- The spectral moments (see (5)) may be expressed as some partial derivatives at zero of the covariance function r and thus their existence can be linked to the regularity of X . More precisely, for $\mathbf{j} \in \mathbb{N}_0^d$, Equation (4) yields

$$\partial^{\mathbf{j}} r(0) = i^{|\mathbf{j}|} \lambda_{\mathbf{j}}, \quad (10)$$

the two terms being equal to zero if $|\mathbf{j}|$ is odd. According to Formula (8), these spectral moments also correspond, up to a factor ± 1 , to the covariances between partial derivatives of the random field at a fixed point. For instance, for $\mathbf{j} = (1, 0, \dots, 0)$, we have $\lambda_{\mathbf{j}} = \mathbb{E}[X(0)X'_1(0)]$.

- If X is **not only stationary but also isotropic**, the covariance structure between its partial derivatives is even simpler than in the stationary case, because of the cancellation of some partial derivatives of r at zero, due to its radial property. In fact, the only partial derivatives of r at zero which can be non-zero are the ones computed along one only direction. In particular, according to Equations (8) and (10), for any $(i, j) \in \{1, \dots, d\}^2$,

$$-\mathbb{E}[X''_{i,j}(t) X(t)] = \mathbb{E}[X'_i(t) X'_j(t)] = -r''_{i,j}(0) = \lambda_2 \delta_i^j, \quad (11)$$

where δ is here the Kronecker delta and $\lambda_2 \geq 0$ has been defined in (6) as the second spectral moment. Taking $i = j$, we highlight the fact that the second spectral moment λ_2 is also the variance of any first-order partial derivative of X :

$$\lambda_2 = \text{Var}((X'_i(t))). \quad (12)$$

Thus λ_2 describes the variability of the velocity of change of X in the neighborhood of any point in \mathbb{R}^d . Taking $i \neq j$ in (11) brings that two different first-order partial derivatives at a fixed point are decorrelated and hence that they are independent in the **Gaussian** framework.

In Section 2.3, we will deal with another notion of regularity, intermediate between a.s. continuity and a.s. differentiability, namely Hölder regularity.

1.4 Gaussian Random Wave

Random waves were first studied by physicists who wanted to add variability or uncertainty to deterministic waves models. In his pioneer work [62], Longuet-Higgins studied sea waves by modelling them as a random moving surface. Since then, ran-

dom waves have been an object of interest for physical models or for their own, see in particular [18], [17], [39], [70] and [38].

Let $X : \mathbb{R}^d \rightarrow \mathbb{R}$ be a centered stationary Gaussian random field with unit variance. Then, according to Bochner Theorem (Theorem 2), there exists a symmetric finite measure F on \mathbb{R}^d such that the covariance function of X is given by

$$\forall t \in \mathbb{R}^d, \quad r(t) = \mathbb{E}[X(0)X(t)] = \int_{\mathbb{R}^d} \cos(\xi \cdot t) dF(\xi),$$

where " \cdot " stands for the scalar product in \mathbb{R}^d . Moreover, since $F(\mathbb{R}^d) = r(0) = \text{Var}(X(0)) = 1$, F is a probability measure. It allows us to introduce a random variable \mathbf{k} in \mathbb{R}^d with distribution F . Hence, \mathbf{k} and $-\mathbf{k}$ are identically distributed and the covariance function of X writes

$$\forall t \in \mathbb{R}^d, \quad r(t) = \mathbb{E}[\cos(\mathbf{k} \cdot t)]. \quad (13)$$

The so-called single random wave $\{\sqrt{2} \cos(\mathbf{k} \cdot t + \eta) ; t \in \mathbb{R}^d\}$, where η is uniformly distributed in $[0, 2\pi]$, independent of \mathbf{k} , defines a stationary centered random field whose covariance function is given by (13) but that is obviously not Gaussian.

This fact provides a simulation method of a Gaussian planar random field as soon as its covariance function can be written as (13) with \mathbf{k} a certain random vector in \mathbb{R}^2 that can be simulated. For instance, this is the case of the squared exponential covariance function $r(t) = \exp(-\|t\|^2)$, providing the so-called Bargmann-Fock field (see [71] Chapter 4). It is obtained with the random vector $\mathbf{k} \sim \mathcal{N}_2(0, I_2)$.

This simulation method, which is described from a practical point of view just below, is precisely the one used to obtain Figures 1, 2, 3 and 4.

Methodology

- Chose N to be a large positive integer.
- Generate $(\eta_j)_{1 \leq j \leq N}$ independent and identically uniformly distributed in $[0, 2\pi]$.
- Generate $(\mathbf{k}_j)_{j \in \mathbb{N}}$ independent and identically distributed with the same law as \mathbf{k} .
- Take $\{X_N = \sqrt{\frac{2}{N}} \sum_{j=1}^N \cos(\mathbf{k}_j \cdot t + \eta_j) : t \in \mathbb{R}^d\}$.

According to the central limit theorem applied to finite-dimensional distributions, the distribution of $(X_N)_{N \in \mathbb{N}^*}$ converges as N tends to infinity towards a Gaussian random field with covariance function (13).

Conversely, for any probability measure F on \mathbb{R}^d , and for any random variable \mathbf{k} with law F , the map $t \in \mathbb{R}^d \mapsto \mathbb{E}[\cos(\mathbf{k} \cdot t)]$ is a covariance function as the covariance of the associated single random wave. According to Bochner theorem (uniqueness result), its associated spectral measure is not F but the symmetrized measure associated with F : $\frac{1}{2}(F + \check{F})$, where \check{F} is the probability distribution of the

random vector $-\mathbf{k}$. Hence, there is no uniqueness of the distribution of \mathbf{k} satisfying (13), unless we demand that it is symmetric.

Definition 9 Let X be a Gaussian field defined on \mathbb{R}^d . We call random wavevector associated with X any symmetrically distributed random vector \mathbf{k} in \mathbb{R}^d such that the covariance function of X satisfies (13).

Reciprocally, for any symmetrically distributed random vector \mathbf{k} in \mathbb{R}^d , we call Gaussian Random Wave (GRW) associated with \mathbf{k} any centered Gaussian field whose covariance function satisfies (13).

The next proposition states invariance and regularity properties of the covariance function defined by (13), resulting from this very expression.

Proposition 8 Let \mathbf{k} be a symmetrically distributed random vector in \mathbb{R}^d and let r be defined by $r(t) = E[\cos(\mathbf{k} \cdot t)]$, $t \in \mathbb{R}^d$.

1. A GRW with associated random wavevector \mathbf{k} is isotropic if and only if the law of \mathbf{k} is invariant under rotations.
2. The covariance function r admits derivatives up to order m ($m \in \mathbb{N}^*$) if and only if \mathbf{k} admits moments of order m . In this case, for any $\mathbf{j} \in \mathbb{N}_0^d$ such that $0 < |\mathbf{j}| \leq m$, we have

$$\partial^{\mathbf{j}} r(0) = 0 \text{ if } |\mathbf{j}| \text{ is odd ; } \partial^{\mathbf{j}} r(0) = (-1)^{|\mathbf{j}|/2} \mathbb{E}[\mathbf{k}^{\mathbf{j}}] \text{ if } |\mathbf{j}| \text{ is even .}$$

In particular, the Hessian matrix of r at zero is

$$r''(0) = -\mathbb{E}[\mathbf{k}\mathbf{k}^T] = -(\mathbb{E}[\mathbf{k}_i \mathbf{k}_j])_{1 \leq i, j \leq d} .$$

1.4.1 Examples

We now give different examples of GRW models obtained by prescribing the distribution of their random wavevector. Among our examples, the planar models ($d = 2$) are illustrated in Figures 1, 2, 3 with Python simulations based on the methodology explained above, with $N = 100$. We have chosen to discretize the values of the simulated fields for the representation. Later in Section 3.2.2 we will examine the geometrical properties of these planar models.

Example 1 (Toy model)

Our first example, where $d = 2$, is studied in [56]. Let X be a planar GRW whose random wavevector is prescribed by $\mathbf{k} = (\cos \Theta, \sin \Theta)$ with Θ a random variable with support in $\mathbb{R}/2\pi\mathbb{Z}$ such that, for a fixed $\alpha \geq 0$, the density of Θ with respect to Lebesgue measure on $[-\pi, \pi]$ is given by

$$\theta \mapsto C_\alpha |\cos \theta|^\alpha, \text{ with } C_\alpha = \frac{\Gamma(1 + \alpha/2)}{2\sqrt{\pi}\Gamma(1/2 + \alpha/2)}, \quad (14)$$

where Γ is the usual Gamma function. Parameter α can be considered as an anisotropy parameter. Indeed, taking $\alpha = 0$, one gets an isotropic GRW, whereas, at the opposite, the case $\alpha \rightarrow +\infty$ corresponds with a totally anisotropic GRW where \mathbf{k} is *a.s.* along the x -axis.

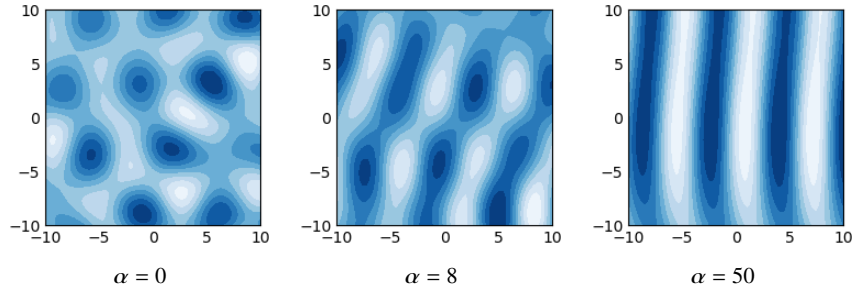


Fig. 1 Toy model of GRW (Example 1) with three different values of α .

Example 2 (Elementary model)

Another example in dimension $d = 2$ is studied in [27] and [70] for instance. The random wavevector is prescribed by $\mathbf{k} = (\cos \Theta, \sin \Theta)$ with Θ a random variable uniformly distributed on $[-\delta, \delta] \cup [\pi - \delta, \pi + \delta]$ with $0 \leq \delta \leq \pi/2$. The parameter δ quantifies the anisotropy. Actually, $\delta = 0$ corresponds with a totally anisotropic GRW model, $\delta \approx 0$ corresponds with a model that is sometimes named as narrow spectrum, and $\delta = \pi/2$ corresponds with an isotropic GRW.

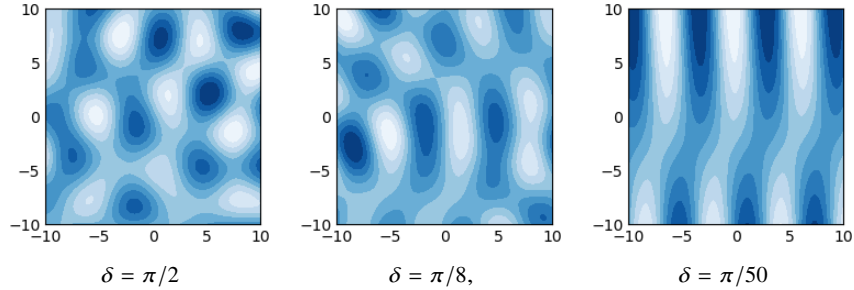


Fig. 2 Elementary model of GRW (Example 2) with three different values of δ .

Example 3 (Sea waves)

In this example where $d = 3$, \mathbb{R}^3 represents the space-time space $\mathbb{R}^2 \times \mathbb{R}$. The random wavevector \mathbf{k} is supported by the Airy surface, namely $\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 - z^4 = 0\}$, and is not necessarily symmetric. The associated GRW is related to the space-time model used for sea waves modelling, assuming that the depth of the sea is infinite (see [62] for the original idea, [11] and [10] for more recent developments).

Example 4 (Berry's random wave)

For $d \geq 2$, when choosing the wavevector \mathbf{k} to be isotropically distributed on $\kappa\mathbb{S}^{d-1}$, the Euclidean sphere of radius $\kappa > 0$, the associated Gaussian Random Wave is known as Berry's monochromatic random wave model. It has originally been presented in [16] and intensively studied in the last years (see [18, 17, 39, 67, 32] and references herein). One can prove that the covariance function of the Berry's monochromatic random wave is given by $r(t) = J_0(\kappa\|t\|)$ in the two-dimensional case, with J_0 the Bessel function of order 0, and by $r(t) = \text{sinc}(\kappa\|t\|)$ in the three-dimensional case, with sinc the cardinal sine. The so-called wavenumber κ can be seen as a frequency and the asymptotics as κ goes to infinity are considered as "high-energy" limits. In Figure 3, a realization of a two-dimensional isotropic Berry's monochromatic random wave (also called Berry's plane wave) has been synthesised with three different values of κ ($\kappa = 0.5$, $\kappa = 1$ and $\kappa = 10$), illustrating the higher energy behavior as κ is getting larger. When dropping the isotropy assumption on \mathbf{k} but keeping the a.s. constant Euclidean norm, the associated GRW is an anisotropic generalization of Berry's random wave that has been studied from a physical point of view in [18] for instance. Let us remark that the toy model and the elementary model in the above Examples 1 and 2 actually provide anisotropic Berry's monochromatic random waves since their corresponding wavevectors are constrained to be unit vectors.

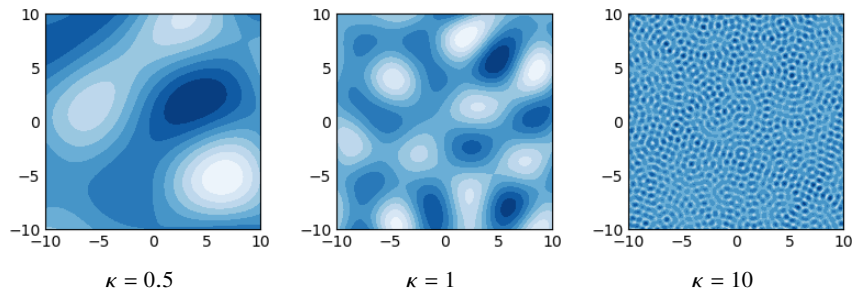


Fig. 3 Isotropic Berry's plane wave (Example 4) with three different wavenumbers.

1.4.2 Link with Partial Differential Equations

We end this section by studying the link between a multivariate polynomial equation *a.s.* satisfied by the coordinates of \mathbf{k} and a partial differential equation *a.s.* satisfied by the GRW associated with random wavevector \mathbf{k} . One can refer to [51] for a recent review on Gaussian solutions of partial differential equations.

Let P be an even d -multivariate polynomial written as

$$\forall x \in \mathbb{R}^d, \quad P(x) = \sum_{\mathbf{j} \in \mathbb{N}_0^d; |\mathbf{j}| \text{ even}} \alpha_{\mathbf{j}} x^{\mathbf{j}},$$

where the sequence of real numbers $(\alpha_{\mathbf{j}})_{\mathbf{j} \in \mathbb{N}_0^d}$ has only finitely many non-zero terms. We associate with P the differential operator

$$\mathcal{L}_P(X) = \sum_{\mathbf{j} \in \mathbb{N}_0^d; |\mathbf{j}| \text{ even}} (-1)^{|\mathbf{j}|/2} \alpha_{\mathbf{j}} \partial^{\mathbf{j}} X.$$

Theorem 3 *Let P be an even multivariate polynomial and let Y be a Gaussian random field defined on \mathbb{R}^d that is centered, stationary, with unit variance and almost surely of class \mathcal{C}^∞ . The following propositions are equivalent.*

1. *The random field Y almost surely satisfies the partial differential equation*

$$\forall t \in \mathbb{R}^d, \quad \mathcal{L}_P(Y)(t) = 0.$$

2. *The random wavevector \mathbf{k}_Y associated with Y almost surely satisfies $P(\mathbf{k}_Y) = 0$.*

Proof Since Y is centered and stationary, so are all its derivatives and also the random field $\mathcal{L}_P(Y)$. Therefore, $\mathcal{L}_P(Y)$ is almost surely identically zero if and only if its variance at each point is zero. Thanks to (8), $\text{Var}(\mathcal{L}_P(Y)(t))$ can be expressed as a linear combination of derivatives of the covariance function r_Y of Y . Hence Y is an *a.s.* solution of the partial differential equation $\mathcal{L}_P(Y) = 0$ if and only if its covariance function r_Y satisfies

$$\sum_{\mathbf{j}, \mathbf{k} \in \mathbb{N}_0^d; |\mathbf{j}|, |\mathbf{k}| \text{ even}} (-1)^{(|\mathbf{j}|+|\mathbf{k}|)/2} \alpha_{\mathbf{j}} \alpha_{\mathbf{k}} \partial^{\mathbf{j}+\mathbf{k}} r_Y(0) = 0. \quad (15)$$

Besides, as it is the covariance function of a stationary centered field, r_Y satisfies Bochner Theorem (Theorem 2): there exists a finite symmetric measure F on \mathbb{R}^d such that $r_Y(t) = \int_{\mathbb{R}^d} \cos(t \cdot x) dF(x)$. Then r_Y satisfies (15) if and only if

$$0 = \int_{\mathbb{R}^d} \left(\sum_{\mathbf{j}, \mathbf{k} \in \mathbb{N}_0^d; |\mathbf{j}|, |\mathbf{k}| \text{ even}} (-1)^{|\mathbf{j}|+|\mathbf{k}|} \alpha_{\mathbf{j}} \alpha_{\mathbf{k}} x^{\mathbf{j}} x^{\mathbf{k}} \right) dF(x) = \int_{\mathbb{R}^d} P(x)^2 dF(x).$$

The above integral vanishes if and only if the measure F is supported by $P^{-1}(\{0\}) = \{x \in \mathbb{R}^d : P(x) = 0\}$. The probability measure F being nothing but the distribution

of the random wavevector \mathbf{k}_Y , saying that F is supported by $P^{-1}(\{0\})$ is equivalent to the assertion $P(\mathbf{k}_Y) = 0$, *a.s.* \square

We insist on the fact that the above theorem provides all the Gaussian *a.s.* solutions, isotropic or not, of the partial differential equation $\mathcal{L}_P(Y) = 0$. Moreover, the equation gives information on the localization of the random wavevector \mathbf{k}_Y associated with any such solution.

To end this section, we apply Theorem 3 to some examples given in Section 1.4.1. This allows us to exhibit random and, generally speaking, anisotropic solutions of some famous partial differential equations.

- The GRW introduced in Example 3 (Sea waves) is an almost sure solution of the partial differential equation

$$\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} + \frac{\partial^4 X}{\partial t^4} = 0 .$$

Indeed, the random wavevector \mathbf{k} associated with this GRW almost surely satisfies Airy equation $P(\mathbf{k}) = 0$ with $P(x) = (x_1)^2 + (x_2)^2 - (x_3)^4$, which up to physical parameters is commonly used to describe the sea waves height.

- In the same vein, the GRW associated with Examples 1 and 2 (Toy and Elementary models) and 4 (anisotropic Berry's random wave) are almost sure solutions of Helmholtz equation

$$\Delta X + \kappa^2 X = 0 ,$$

where Δ is the Euclidean Laplacian operator in dimension 2 for Examples 1 and 2 (with $\kappa = 1$) and in any dimension d for Example 4 (with any $\kappa > 0$). Indeed, its random wavevector \mathbf{k} satisfies $\|\mathbf{k}\|^2 - \kappa^2 = 0$, *a.s.*, which can be written as $P(\mathbf{k}) = 0$, *a.s.* with $P(x) = \sum_{1 \leq j \leq d} (x_j)^2 - \kappa^2$. Note that these GRWs are not necessarily isotropic. However, if one requires isotropy then we can state that there is a unique isotropic stationary standard Gaussian field that *a.s.* solves the above Helmholtz equation. Its covariance function is uniquely prescribed by (13) with random wavevector \mathbf{k} isotropically distributed on the sphere of radius κ .

2 From Isotropy to Anisotropy

The assumptions of stationarity and/or isotropy are sometimes justified by modelling purposes. For instance, in image analysis and synthesis, the homogeneous nature of texture entails that stationary isotropic random fields are commonly used to model them. However, many real world phenomena require the introduction of anisotropy. In [7] for instance, an exhaustive overview of all the anisotropy models that are used in geosciences is presented, starting with the "geometric anisotropy" that corresponds to a linear deformation of the Euclidean space \mathbb{R}^d , going through "zonal anisotropy", which corresponds to the degeneracy of the covariance structure in some directions, and ending with "tensorial anisotropy" where the covariance structure can be written

as a tensorial product of one-dimensional covariances in each of the d canonical directions. In the present section, we go beyond these models and we focus on three different types of Gaussian fields that present specific geometric features with respect to anisotropy: anisotropic random waves in Section 2.1, deformed random fields in Section 2.2 and anisotropic fractional Brownian fields in Section 2.3. The main references for the results presented on each model are respectively [45], [47] and [29]. We will concentrate on the analysis of these models and on the way to quantify or infer anisotropy parameters.

2.1 Anisotropic Random Waves

We come back to the Gaussian Random Wave (GRW) model introduced in Section 1.4 with a focus on its possible anisotropy properties.

We are now interested in directional statistics concerning the GRW. We need to introduce two specific items, the favorite direction of a random vector and the principal direction of a random field, that we define below. These notions are usual tools in directional statistics theory, see [65] or [61] as references in this field. The notion of principal direction has been introduced by Longuet-Higgins in [62] in his study of a planar random wave model for sea waves.

Definition 10 (Favorite direction) Let Z be a random vector with a finite second moment.

We call *favorite direction of Z* any unitary vector $u \in \mathbb{R}^d$ that maximizes the quantity $\mathbb{E}[(Z \cdot u)^2]$. In other words, the set of favorite directions of Z is defined as $\operatorname{Argmax}_{u \in \mathbb{S}^{d-1}} \mathbb{E}[(Z \cdot u)^2]$.

Let us remark that for any $u \in \mathbb{S}^{d-1}$, we have $(Z \cdot u)^2 = u \cdot (ZZ^T)u$. Consequently, the set of favorite directions is nothing but the set of unitary vectors that belong to the eigensubspace of \mathbb{R}^d associated with the largest eigenvalue of the symmetric positive matrix $\mathbb{E}[ZZ^T]$. Obviously, when Z is isotropically distributed, then $\mathbb{E}[ZZ^T] = \alpha I_d$ with $\alpha \geq 0$ and so $\operatorname{Argmax}_{u \in \mathbb{S}^{d-1}} \mathbb{E}[(Z \cdot u)^2] = \mathbb{S}^{d-1}$.

We now turn to random fields and introduce remarkable directions depending on the directional property of the distribution. Let $X : \mathbb{R}^d \rightarrow \mathbb{R}$ be a stationary random field that is *a.s.* differentiable and satisfies $\mathbb{E}[\|X'(0)\|^2] < +\infty$. For $u \in \mathbb{S}^{d-1}$, we denote by $X^u = \{X(tu) ; t \in \mathbb{R}\}$ the one-dimensional stationary process obtained by restricting X to the line $\mathbb{R}u$.

Definition 11 (Principal direction) We call *principal direction of X* any unitary vector $u \in \mathbb{R}^d$ that maximizes the quantity $\mathbb{E}[(X^u)'(0)^2]$. In other words, the set of principal directions of X is defined as $\operatorname{Argmax}_{u \in \mathbb{S}^{d-1}} \mathbb{E}[(X^u)'(0)^2]$.

Note that restricting X to a certain line of \mathbb{R}^d or to any parallel line does not change the law of the obtained process because X is stationary. Note also that

$$\mathbb{E}[(X^u)'(0)^2] = \mathbb{E}[(X'(\mathbf{0}) \cdot u)^2] = \mathbb{E}[(X'(t) \cdot u)^2],$$

for any $t \in \mathbb{R}^d$ by stationarity. Consequently, for any $t \in \mathbb{R}^d$, the principal directions of X coincide with the favorite directions of $X'(t)$.

Once introduced the above notions, we now study the GRW from a directional point of view.

Proposition 9 *Let X be a GRW defined on \mathbb{R}^d and let \mathbf{k}_X be its associated random wavevector, as defined in Definition 9. Then, the next subsets of \mathbb{S}^{d-1} coincide:*

- (i) *the set of favorite directions of \mathbf{k}_X ,*
- (ii) *the set of principal directions of X , or equivalently the set of favorite directions of $X'(t)$ for any $t \in \mathbb{R}^d$.*

They are given as the set of unitary vectors that belong to the eigensubspace of \mathbb{R}^d associated with the largest eigenvalue of matrix $\mathbb{E}[\mathbf{k}_X \mathbf{k}_X^T]$.

Proof Let $u \in \mathbb{S}^{d-1}$. The covariance function of the univariate process $X^u = \{X(tu); t \in \mathbb{R}\}$ is given for any t in \mathbb{R} by $r_u(t) = r(tu) = \mathbb{E}[\cos(tu \cdot \mathbf{k}_X)]$. Hence $E[(X^u)'(0)^2] = -r_u''(0) = \mathbb{E}[(\mathbf{k}_X \cdot u)^2]$, which clearly yields the announced statement. \square

We now turn to the directional study of the level sets of the GRW X with associated random wavevector \mathbf{k}_X . For a fixed level $a \in \mathbb{R}$, the level set

$$X^{-1}(a) = \{t \in \mathbb{R}^d : X(t) = a\}$$

is a $(d-1)$ -dimensional submanifold of \mathbb{R}^d whose tangent space $T_t X^{-1}(a)$ at point $t \in X^{-1}(a)$ is orthogonal to the vector $X'(t)$. Applying Proposition 9 in the case $d=2$ yields the next statement, that sounds physically intuitive and is enunciated in [62] Section 2.3 with a specific vocabulary.

Corollary 1 *Let X be a GRW defined on \mathbb{R}^2 and let \mathbf{k}_X be its associated random wavevector, as defined in Definition 9. Let Z_a be a two-dimensional vector field defined on the level set $X^{-1}(a)$ such that, at any point t , $Z_a(t)$ is tangent to $X^{-1}(a)$ at t . Then, for any t in $X^{-1}(a)$, the favorite direction of $Z_a(t)$ is orthogonal to the favorite direction of \mathbf{k}_X .*

Have a look at the anisotropic cases of Figures 1 and 2 to observe that the level lines have a vertical preferred direction, whereas the favorite direction of the wavevector is horizontal.

2.2 Deformed Random Fields

This section deals with a class of non-stationary and non-isotropic fields called *deformed random fields*. They are obtained by deforming a fixed stationary and

isotropic random field thanks to a deterministic function that transforms bijectively the index set. Most studies on the deformed field model deal with a two-dimensional framework, we will also restrict ourselves to this context. Note that it still covers many applications, particularly in image analysis.

Definition 12 Let $X: \mathbb{R}^2 \rightarrow \mathbb{R}$ a Gaussian, stationary and isotropic field and $\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a bijective deterministic function. We define the random field $X_\theta = X \circ \theta$ called the deformed field with underlying field X and deformation θ .

Even though the underlying random field X is stationary and isotropic, the deformation θ can transform the index space \mathbb{R}^2 in such a way that the stationarity and/or the isotropy are lost when it comes to the deformed field as can be seen in Figure 4.

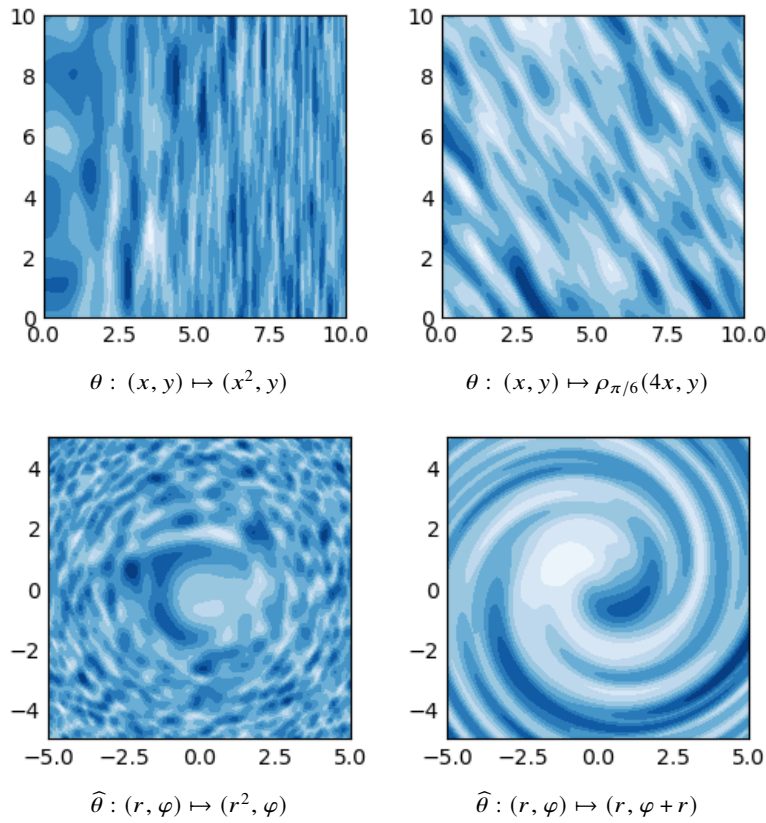


Fig. 4 Deformed Gaussian fields $X \circ \theta$ with X stationary Gaussian with covariance function $r(t) = e^{-\|t\|^2/2}$ and various deformations θ . On the first line, second picture, ρ_α stands for the rotation of center 0 and angle α . On the second line, θ is described through its polar representation $\hat{\theta}$.

The model of deformed fields was introduced in 1992 in a spatial statistics framework by Sampson and Guttorp in [76], with only a stationarity assumption on X and the estimation of the covariance function as an objective. In [68] and in [69], the model is also studied through its covariance function. In the domain of image analysis, deformed fields are involved in the “shape from texture” issue, that is, the problem of recovering a 3-dimensional textured surface thanks to a 2-dimensional projection [33]. Deformed random fields are also widely used in cosmology to model the cosmic microwave background (CMB) deformed anisotropically by the gravitational lensing effect, with mass reconstruction as an objective [52]. In some works, the deformations studied are linear, which entails that the deformed field is still stationary. This kind of models produces “geometric anisotropy” (see [7], [31] and [19]). Besides, the deformed field model can also be studied as a particular case of a larger model, such as in [34] where the main model consists in a deterministic deformation operator applied to a stationary field X . In these different works, the estimation of the deformation θ is generally at stake. We will adopt our geometric viewpoint to deal with this estimation matter in Section 4.2.3.

In the following, we answer a natural question : which are the deformations θ that preserve isotropy, for any underlying field X , or for a fixed one ? Theorem 4 below will give an explicit form for this kind of deformations, which we call spiral deformations. Before stating this result, we introduce some notations and definitions.

We denote by C_θ the covariance function of the deformed field X_θ . Because the field X is stationary, for any $(x, y) \in (\mathbb{R}^2)^2$,

$$C_\theta(x, y) = \text{Cov}(X_\theta(x), X_\theta(y)) = C(\theta(x) - \theta(y)). \quad (16)$$

Without loss of generality regarding the law of the deformed field X_θ , it is possible to make some simplifying assumptions on X and θ . First, we assume that $\theta(0) = 0$. If θ is differentiable, we shall also assume that for any $x \in \mathbb{R}^2$, $\det(J_\theta(x))$ is positive or, in other words, that θ preserves orientation. Indeed, function $x \mapsto \det(J_\theta(x))$ is continuous on \mathbb{R}^2 and does not vanish since θ is invertible, hence it takes either only positive or only negative values. If for all $x \in \mathbb{R}^2$, $\det(J_\theta(x)) < 0$, we can replace θ by $\sigma \circ \theta$, where σ is the symmetry with respect to the axis of abscissas; then for any $x \in \mathbb{R}^2$, $J_{\sigma \circ \theta}(x) = \sigma \circ J_\theta(x)$ and so the Jacobian determinant of $\sigma \circ \theta$ is positive on \mathbb{R}^2 .

For any set E , we note $\mathcal{D}^0(E)$ the set of bicontinuous bijective functions from E to E taking value 0 at 0, and $\mathcal{D}^1(E)$ the subset of $\mathcal{D}^0(E)$ composed of \mathcal{C}^1 -diffeomorphisms. To exhibit the deformations θ in $\mathcal{D}^1(\mathbb{R}^2)$ preserving the isotropy when they are used in a deformed field model, we will take advantage of the polar representation.

We denote by S the transformation of polar coordinates to cartesian coordinates in the plane deprived of the origin:

$$S: (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^2 \setminus \{0\} \quad (r, \varphi) \mapsto (r \cos \varphi, r \sin \varphi).$$

For any deformation $\theta \in \mathcal{D}^0(\mathbb{R}^2)$, we write $\theta_0 = \theta|_{\mathbb{R}^2 \setminus \{0\}}$ and we define the deformation $\hat{\theta} \in \mathcal{D}^0((0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z})$ by $\hat{\theta} = S^{-1} \circ \theta_0 \circ S$:

$$\begin{array}{ccc}
 \mathbb{R}^2 \setminus \{0\} & \xrightarrow{\theta_0} & \mathbb{R}^2 \setminus \{0\} \\
 S \nearrow & & \searrow S^{-1} \\
 (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} & \xrightarrow{\hat{\theta}} & (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}
 \end{array}$$

Each deformation in $\mathcal{D}^0(\mathbb{R}^2)$ can be associated bijectively to its polar representation defined in $\mathcal{D}^0((0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z})$. Now we use this polar representation to introduce a subset of deformations.

Definition 13 A deformation $\theta \in \mathcal{D}^0(\mathbb{R}^2)$ is a spiral deformation if there exist $f: (0, +\infty) \rightarrow (0, +\infty)$ strictly increasing and surjective, $g: (0, +\infty) \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$ such that θ satisfies

$$\forall (r, \varphi) \in (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}, \quad \hat{\theta}(r, \varphi) = (f(r), g(r) + \varepsilon\varphi). \quad (17)$$

Remark on Definition 13

The choice of f strictly increasing is due to the conditions of continuity and invertibility on θ and to the fact that $\theta(0) = 0$. The 2π -periodicity of the second component of $\hat{\theta}$ entails that the coefficient ε in the angular part of (17) is an integer and the invertibility of θ implies that ε belongs to $\{\pm 1\}$. If we only consider deformations with positive Jacobian determinants, then we can set $\varepsilon = 1$. Indeed, through the relation $\theta_0 = S \circ \hat{\theta} \circ S^{-1}$, one can show that the positivity of the Jacobian determinant of θ_0 is equivalent to the positivity of the one of $\hat{\theta}$.

Let us notice that the linear spiral deformations are the deformations with polar representation either $(r, \varphi) \mapsto (\lambda r, \varphi + \alpha)$ or $(r, \varphi) \mapsto (\lambda r, -\varphi + \alpha)$, with $\lambda > 0$ and $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, that is to say it is of the form $\theta = \lambda\rho$ or $\theta = \lambda\rho \circ \sigma$, with $\lambda > 0$, $\rho \in SO(2)$ and σ the symmetry with respect to the axis of abscissas.

Theorem 4 Let θ be a deformation in $\mathcal{D}^1(\mathbb{R}^2)$. The following propositions are equivalent :

- (a) For any stationary and isotropic Gaussian field X , the field X_θ is isotropic.
- (b) For one stationary and isotropic Gaussian field X such that the covariance function of X writes $\text{Cov}(X(x), X(0)) = R(\|x\|)$ with a map R that is injective, the field X_θ is isotropic.
- (c) θ is a spiral deformation.

Proof We do not write a complete proof, see [47] Theorem 1 for details.

Obviously, (a) implies (b). Let us prove that (b) implies (c). Let us assume that for a specific stationary and isotropic field X , the field X_θ is isotropic. Hence its covariance function, given by (16) is invariant under the action of any rotation:

$$\forall \rho \in SO(2), \forall (x, y) \in (\mathbb{R}^2)^2, \quad C_\theta(\rho(x), \rho(y)) = C_\theta(x, y) .$$

Using the fact that $C(x) = R(\|x\|)$ with R injective, we obtain

$$\forall \rho \in SO(2), \forall (x, y) \in (\mathbb{R}^2)^2, \quad \|\theta(\rho(x)) - \theta(\rho(y))\| = \|\theta(x) - \theta(y)\| . \quad (18)$$

We now use the polar representation of θ , denoting by $\hat{\theta}^1 : (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}_+^*$ and $\hat{\theta}^2 : (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ its coordinate functions (radial part and angular part). Taking $y = 0$, we deduce from (18) that $\hat{\theta}^1$ is radial. We set for any $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ and for any $r > 0$, $\hat{\theta}^1(r, \varphi) = f(r)$. Since θ is bijective, continuous and $\theta(0) = 0$, f is necessarily strictly increasing with $\lim_{r \rightarrow 0} f(r) = 0$ and $\lim_{r \rightarrow +\infty} f(r) = +\infty$. To deduce that $\hat{\theta}^2$ has necessarily the form of the angle coordinate of a spiral transform, and hence to conclude the proof of (b) \Rightarrow (c), we use pure analysis arguments (see [47] for details).

Finally, in order to prove that (c) implies (a), we assume that θ is a spiral deformation with polar representation (17). Let $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, we recall that ρ_α stands for the rotation of angle α in \mathbb{R}^2 . Then, for all $(r, \varphi) \in (0, +\infty) \times \mathbb{R}/2\pi\mathbb{Z}$,

$$\hat{\theta} \circ \hat{\rho}_\alpha(r, \varphi) = (f(r), g(r) + \varepsilon(\varphi + \alpha)) = \hat{\rho}_{\varepsilon\alpha} \circ \hat{\theta}(r, \varphi) .$$

Therefore, θ satisfies the following property:

$$\forall \rho \in SO(2), \quad \exists \rho' \in SO(2) / \theta \circ \rho = \rho' \circ \theta .$$

This entails that $X_\theta \circ \rho = X \circ \rho' \circ \theta$. The isotropy of X implies that $X \circ \rho'$ has the same law as X . Consequently, $X_\theta \circ \rho$ has the same law as X_θ . Thus the isotropy of X_θ is proved. \square

Remark on the proof of Theorem 4

Looking back to the proof of this result, we see that it does not use the whole Gaussian hypothesis on X . In fact, we could replace the former by the assumption that X has a finite second moment.

In Section 4.2.3, we will show that the knowledge of the expected Euler characteristic of different excursion sets of a deformed random field allows one to identify the deformation.

2.3 Anisotropic Fractional Brownian Fields

In this section, we introduce anisotropic Gaussian models where anisotropy is expressed through a regularity parameter that changes with directions. More precisely we focus on Anisotropic Fractional Brownian Fields (AFBF) as generic models for random fields with prescribed Hölder regularity. The anisotropic FBF that we present here is a special case of the anisotropic Gaussian models that have been introduced in [29]. We define it as a Gaussian field with stationary increments, as

defined in Section 1.2.3, whose variogram has a specific shape described through a non isotropic spectral density.

Let us first start with the isotropic version which is nothing but the famous Fractional Brownian Field with Hurst parameter H denoted as B_H . It is a multidimensional extension of Fractional Brownian Motion introduced in the one dimensional case by Kolmogorov in 1941 and popularized by Mandelbrot in [64], see [75] Section 7, [35] or [22] for more recent references. For $H \in (0, 1)$, the Fractional Brownian Field B_H is defined as the (unique) Gaussian field that is centered, vanishes at the origin, has stationary increments and whose variogram is prescribed by

$$\text{Var}(B_H(x)) = 4 \int_{\mathbb{R}^d} \sin^2(x \cdot \xi / 2) \frac{d\xi}{\|\xi\|^{2H+d}} = c_d(H) \|x\|^{2H},$$

with $c_d(H)$ a positive constant depending on d and H . The above second equality follows by a single homogeneity argument, whereas the integral expression allows one to check that the variogram actually yields an admissible covariance function. Starting from its variogram, it is easy to see that B_H is isotropic and is self-similar of order H , which means that for all $\alpha > 0$,

$$B_H(\alpha \cdot) \stackrel{\mathcal{L}}{=} \alpha^H B_H(\cdot).$$

Moreover, a well known result states that there exists a continuous modification of B_H which has Hölder critical exponent equal to H , where the notion of Hölder critical exponent is defined below.

Definition 14 Let $\beta \in (0, 1)$. A process $X = \{X(t) : t \in \mathbb{R}^d\}$ is said to have Hölder critical exponent β whenever it satisfies the two following properties:
- for any $\alpha \in (0, \beta)$, the sample paths of X satisfy a.s. a uniform Hölder condition of order α on any compact set, i.e. for any compact set K in \mathbb{R}^d , there exists a positive random variable A such that a.s.

$$|X(t) - X(s)| \leq A \|s - t\|^\alpha, \quad \forall s, t \in K;$$

- for any $\alpha \in (\beta, 1)$, a.s. the sample paths of X fail to satisfy any uniform Hölder condition of order α .

As already pointed out, B_H is isotropic, H -self-similar and admits H as Hölder critical exponent. We now introduce an anisotropic Gaussian field with stationary increments that is not self-similar nor isotropic.

Definition 15 Let us consider an even positive function h defined on the unit sphere \mathbb{S}^{d-1} . We assume that h takes its values inside the interval $[m, M] \subset (0, 1)$, with $m = \text{essinf } h$ and $M = \text{esssup } h$.

The Anisotropic Fractional Brownian Field (AFBF) with directional Hurst function h is the unique Gaussian field $B_{(h)} = \{B_{(h)}(x) : x \in \mathbb{R}^d\}$ that is centered, vanishes at the origin, has stationary increments and whose variogram is given by

$$v_{(h)}(x) := \text{Var}(B_{(h)}(x)) = 4 \int_{\mathbb{R}^d} \sin^2(x \cdot \xi / 2) \frac{d\xi}{\|\xi\|^{2h(\xi/\|\xi\|)+d}}.$$

The existence of AFBF is ensured by showing that the candidate to be the covariance function of $B_{(h)}$, which is given by

$$\Gamma_{(h)}(x, y) = \frac{1}{2} (v_{(h)}(x) + v_{(h)}(y) - v_{(h)}(x - y)),$$

is clearly a non-negative type function.

We will see that $B_{(h)}$ satisfies a property similar to Hölder regularity but in a relaxed way. More precisely, next proposition states that under a mild assumption on h , the Hölder critical exponent of $B_{(h)}$ considered along different directions is constant, given by the infimum of h .

Proposition 10 *Let us consider an AFBF, $B_{(h)}$ as defined in Definition 15 with a directional Hurst function h such that, for $m = \text{essinf}(h)$, the set $\{\xi \in \mathbb{R}^d ; h(\xi/\|\xi\|) = m\}$ has positive Lebesgue measure. Then, for any $u \in \mathbb{S}^{d-1}$, the one-dimensional process $\{B_{(h)}(tu) : t \in \mathbb{R}\}$ has Hölder critical exponent m .*

Proof We only give a sketch of proof, for more details see [29].

We first prove that the variogram of $\alpha^{-m} B_{(h)}(\alpha \cdot)$ converges to a positive limit as α goes to 0^+ . Actually, for any $\alpha > 0$,

$$\text{Var}\left(\frac{B_{(h)}(\alpha x)}{\alpha^m}\right) = \alpha^{-2m} \int_{\mathbb{R}^d} 4 \sin^2(\alpha x \cdot \xi / 2) \frac{d\xi}{\|\xi\|^{2h(\xi/\|\xi\|)+d}}.$$

We split the integral into $\{\|\xi\| \leq A\}$ and $\{\|\xi\| > A\}$ for some arbitrary positive number A . The first part can be upper bounded by

$$\alpha^{-2m+2} \|x\|^2 \int_{\|\xi\| \leq A} \|\xi\|^2 \frac{d\xi}{\|\xi\|^{2h(\xi/\|\xi\|)+d}},$$

which goes to 0 as α goes to 0 since $m < 1$. With the change of variable $\xi \mapsto \alpha \xi$, we get for the second term,

$$4 \int_{\mathbb{R}^d} \mathbf{1}_{\{\|\xi\| > \alpha A\}} \alpha^{2(h(\xi/\|\xi\|)-m)} \sin^2(x \cdot \xi / 2) \frac{d\xi}{\|\xi\|^{2h(\xi/\|\xi\|)+d}},$$

where $\mathbf{1}_{\{\|\xi\| > \alpha A\}} \alpha^{2(h(\xi/\|\xi\|)-m)} \rightarrow \mathbf{1}_{h(\xi/\|\xi\|)=m}$ as α goes to 0. Hence, by Lebesgue convergence theorem, $\alpha^{-2m} \text{Var}(B_{(h)}(\alpha x))$ tends to a positive non-vanishing integral since $\{\xi \in \mathbb{R}^d ; h(\xi/\|\xi\|) = m\}$ has positive measure.

The Hölder critical exponent can be derived from the asymptotics of the variogram of $B_{(h)}$ by using the Gaussianity assumption. \square

Following previous proposition, considering $B_{(h)}$ along different directions fails to reveal its anisotropy: $B_{(h)}$ seems to be isotropic even with an "anisotropic" Hurst

index h . Actually, only the infimum of h seems to be reachable. However, in the next proposition we present a procedure based on the Radon transform that gives access to the value of h in a prescribed direction. In order to estimate $h(u)$ for any $u \in \mathbb{S}^{d-1}$, it is then enough to proceed by rotation before performing the Radon transform. Let us note that Radon transform is a usual tool in signal theory, image analysis or tomography (see [50]). The simplified definition of Radon transform that we use here is rough enough to get the main idea without technical difficulties.

Proposition 11 *Let us consider an AFBF $B_{(h)}$ as defined in Definition 15 with a directional Hurst function h . Let us introduce the Radon transform of $B_{(h)}$ as*

$$RB_{(h)} = \left\{ \int_0^1 B_{(h)}(x, t) dt : x \in \mathbb{R}^{d-1} \right\},$$

and let us denote by $v_{RB_{(h)}}$ its variogram. Then, there exists a non-negative constant C such that for all $x \in \mathbb{R}^{d-1}$, as α goes to 0,

$$v_{RB_{(h)}}(\alpha x) \sim C \alpha^{2h(e_d)+1},$$

and $RB_{(h)}$ has Hölder critical exponent $h(e_d) + 1/2$.

Proof We only give a sketch of proof considering the simple case where the function h is constant equal to H . For a complete proof within a more general framework, see [29] Theorem 1.

We use the spectral representation of B_H (see [75], Section 7) to write the increments of RB_H . Actually, for x_0 and x in \mathbb{R}^{d-1} and $\alpha > 0$, with $\text{sinc}(z) = \frac{\sin z}{z}$, we have

$$\begin{aligned} v_{RB_H}(\alpha x) &= \text{Var}(RB_H(x_0 + \alpha x) - RB_H(x_0)) \\ &= 4 \int_{\mathbb{R}^{d-1}} \sin^2(\alpha \bar{\xi} \cdot x/2) \left(\int_{\mathbb{R}} \frac{\text{sinc}^2(\zeta/2)}{\|(\bar{\xi}, \zeta)\|^{2H+d}} d\zeta \right) d\bar{\xi} \\ &= \alpha^{2H+1} \int_{\mathbb{R}^{d-1}} 4 \sin^2(\bar{\xi} \cdot x/2) \left(\int_{\mathbb{R}} \frac{\text{sinc}^2(\zeta/2)}{\|(\bar{\xi}, \alpha \zeta)\|^{2H+d}} d\zeta \right) d\bar{\xi}, \end{aligned}$$

where the change of variable $\bar{\xi} \mapsto \alpha \bar{\xi}$ has been performed to get the last line. Hence, as $\alpha \rightarrow 0^+$, we get the following asymptotics,

$$v_{RB_H}(\alpha x) \sim \alpha^{2H+1} \int_{\mathbb{R}^{d-1}} 4 \frac{\sin^2(\bar{\xi} \cdot x/2)}{\|\bar{\xi}\|^{2H+d}} \left(\int_{\mathbb{R}} \text{sinc}^2(\zeta/2) d\zeta \right) d\bar{\xi} \sim \alpha^{2H+1} V_H,$$

where V_H is a constant that does not depend on α . The asymptotic self-similarity of the variogram is enough to conclude for the Hölder critical exponent thanks to the Gaussianity assumption. \square

We now give an illustrative example in dimension $d = 3$ of how to take advantage of Proposition 11 (for more details, see [21]). Suppose that $(B_{(h)}(x))_{x \in \mathbb{R}^3}$ is a model

for a 3D-material whose directional Hurst function h is unknown and has to be inferred in order to reveal the anisotropic structure.

Methodology

- Take a "horizontal" slice of thickness 1 of this material, which can be represented by

$$\{B_{(h)}(x, t); (x, t) \in \mathbb{R}^2 \times [0, 1]\} .$$

- Perform an X-ray image of this slice through the "vertical" Radon transform

$$RB_{(h)} = \left\{ \int_0^1 B_{(h)}(x, t) dt; x \in \mathbb{R}^2 \right\} .$$

- Estimate the asymptotic self-similarity parameter, say $\widehat{\beta}$, of the X-ray image through a log-log regression of the variogram.
- Following Proposition 11, $\widehat{\beta} - 1/2$ is a consistent estimator for the value of h in the "vertical" direction.

Applications of this methodology on medical images, such as mammograms or bones X-ray images, are described in [74] and [53]. Similar ideas with more general models have been developed in [21], [74] or [23]. See also [27], [22], [43] for simulations and parametric inference of AFBF using a spectral method named as the turning-band method. In Figure 5, we show two images of simulated AFBF with prescribed Hurst function h , the first one being actually isotropic ($h = H$ constant). These simulations are obtained thanks to the Python PyAFBF library [73].

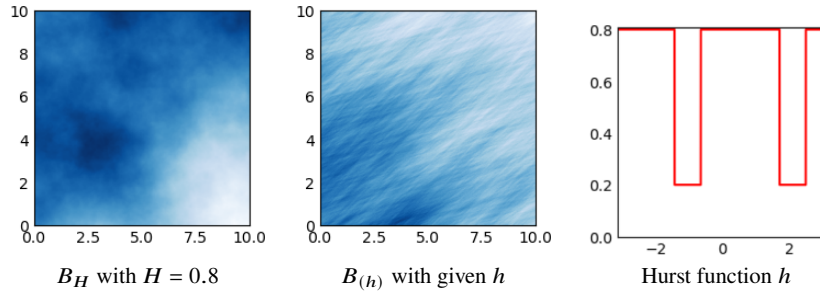


Fig. 5 Left panel: simulation of a FBF with Hurst parameter $H = 0.8$. Center panel: simulation of an AFBF $B_{(h)}$ with Hurst function h prescribed in the left panel.

3 Tools for Geometrical Features

Although the two first sections were dedicated to real-valued random fields, the present section will deal with vector-valued random fields from \mathbb{R}^d to $\mathbb{R}^{d'}$ with $1 \leq d' \leq d$. One of the interests of this approach is to be able to present some geometrical tools that are adapted to both a real-valued random field $X : \mathbb{R}^d \rightarrow \mathbb{R}$ and its derivative $X' : \mathbb{R}^d \rightarrow \mathbb{R}^d$.

Let $X : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ be a random field, where $1 \leq d' \leq d$, and let $a \in \mathbb{R}^{d'}$. We will focus on two types of subsets of \mathbb{R}^d : $X^{-1}(\{a\})$, the a -level of X and in the case $d' = 1$, $X^{-1}([a, +\infty))$, the excursion set of X above level a . Our goal is to extract geometrical or topological knowledge from these subsets that will characterize the geometry of the random field X .

Our first tool will be the family of Rice formulas that provide expressions for the moments of the Hausdorff measure of $X^{-1}(\{a\}) \cap T$, for T a compact subset of \mathbb{R}^d . Subsection 3.1 concerns the case $d = d'$, Subsection 3.2 allows to handle the case $d > d'$.

The second tool that we explore is designed to describe the geometry of the excursion sets $X^{-1}([a, +\infty))$ when $d' = 1$ and X is Gaussian. It is summarized in the so-called Gaussian Kinematic Formula (GKF) and is presented in Subsection 3.3.

3.1 Crossings

Let $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a random field that is a.s. of class \mathcal{C}^1 and let $a \in \mathbb{R}^d$. Under the condition that a.s. for any $t \in \mathbb{R}^d$, the $d \times d$ Jacobian matrix $X'(t)$ is invertible, $X^{-1}(\{a\})$ is a.s. a discrete set and $X^{-1}(\{a\}) \cap T$ is finite when T is a compact set in \mathbb{R}^d . We denote by $N(X, a, T)$ its cardinal. We are also interested in the number of points in the level set satisfying an additional constraint, expressed in term of another random field Y defined on \mathbb{R}^d , taking values in \mathbb{R}^k for some $k \geq 1$. Denoting by B a Borelian of \mathbb{R}^k , we write

$$N(X, a, Y, B, T) = \#\{t \in T : X(t) = a, Y(t) \in B\},$$

and in the non constrained case

$$N(X, a, T) = \#\{t \in T : X(t) = a\}.$$

3.1.1 Rice Formula for Crossings

In the next theorem, we state formulas that are known as Rice or Kac-Rice formulas. This kind of formula was developed by Rice and Kac independently in the forties ([54] and [72] were their first contributions) in dimension one which explains the use of the terminology "crossings" for the level set. In the fifties, Longuet-Higgins

([62]) brought the first contribution to the multiparameter case in the domain of physical oceanography. A general presentation of Rice formulas in the Gaussian random fields setting was done by Adler in [2]. The formulas were also extended to higher moments and to some non-Gaussian models, see [5], [10] or [20], [8] for thorough and recent presentations, [32] for the more general context of manifolds.

In what follows, we use the notation p_Z for the probability density function of a random variable Z , provided that the latter exists.

Theorem 5 (Rice formula, case $d = d'$)

Let T a compact set in \mathbb{R}^d , $a \in \mathbb{R}^d$ and $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$ a Gaussian random field such that

- (i) X is a.s. of class \mathcal{C}^2 ,
- (ii) for any $t \in \mathbb{R}^d$, the random vector $X(t)$ is not degenerate.

Then, $N(X, a, T)$ admits a first moment and

$$\mathbb{E}[N(X, a, T)] = \int_T \mathbb{E}[|\det(X'(t))| \mid X(t) = a] p_{X(t)}(a) dt, \quad (19)$$

where $p_{X(t)}$ denotes the probability density function of $X(t)$.

Moreover, if $Y : \mathbb{R}^d \rightarrow \mathbb{R}^k$ is an a.s. continuous random field such that (X, Y) is a Gaussian random field then, for any $B \in \mathcal{B}(\mathbb{R}^k)$,

$$\mathbb{E}[N(X, a, Y, B, T)] = \int_T \mathbb{E}[\mathbb{1}_B(Y(t)) \mid \det(X'(t)) \mid X(t) = a] p_{X(t)}(a) dt. \quad (20)$$

Furthermore, if the next condition holds,

- (iii) for any (s, t) in $(\mathbb{R}^d)^2$ with $s \neq t$, the random vector $(X(s), X(t))$ is not degenerate,
- then

$$\begin{aligned} & \mathbb{E}[N(X, a, T)(N(X, a, T) - 1)] \\ &= \int_{T \times T} \mathbb{E}[|\det(X'(s))| |\det(X'(t))| \mid X(s) = X(t) = a] p_{X(s), X(t)}(a, a) ds dt, \end{aligned} \quad (21)$$

which is finite whenever the integral on the right hand side is convergent.

Remarks on the assumptions

- This statement is adapted from [10], Theorem 6.2, Theorem 6.4 and Proposition 6.5, where we have reinforced the assumptions in a view of simplification. Note that the second derivatives of X do not come into play in Rice formula. As a matter of fact, the formula can also hold without the a.s. \mathcal{C}^2 regularity assumption, with “a little more than” a.s. \mathcal{C}^1 regularity.
- Besides, the Gaussian assumption allows to simplify a lot the regularity assumptions but we could get rid of it too (see [10], Theorem 6.7 for Rice formula in the non-Gaussian case).

Proof (of Theorem 5) We only give a simplified idea of the proofs of (19) and (21), we refer to [10] for the complete proof of the announced formulas. They are based on Kac's counting formula, a deterministic expression of the number of points in the level set as an integral limit.

Kac's counting formula.

Let us assume that X is a \mathcal{C}^1 -function on \mathbb{R}^d . We assume that for any t in the boundary ∂T of T , $X(t) \neq 0$ and we write $\{x_1, \dots, x_N\} = X^{-1}(a) \cap \overset{\circ}{T}$. Denoting by $X'(x)$ the $d \times d$ Jacobian matrix of X at point x , we assume that for all x_i above, $\det(X'(x_i)) \neq 0$. We write $B(x, r)$ the open d -dimensional ball centered at x and with radius r . We chose $\eta > 0$ such that

- for any $1 \leq i \leq N$, $B(x_i, \eta) \subset T$
- for any $i \neq j$, $B(x_i, \eta) \cap B(x_j, \eta) = \emptyset$
- for any $1 \leq i \leq N$, $X|_{B(x_i, \eta)}$ is invertible (this condition is made possible by the assumption on the Jacobian determinant).

At points x_i , $1 \leq i \leq N$, X is locally continuously invertible, hence there exists $\epsilon > 0$ such that $X^{-1}(B(a, \epsilon)) \subset \bigcup_{i=1}^N B(x_i, \eta)$. Hence we can write

$$\begin{aligned} \int_T \mathbb{1}_{B(a, \epsilon)}(X(t)) |\det(X'(t))| dt &= \int_{\bigcup_{i=1}^N B(x_i, \eta)} \mathbb{1}_{B(a, \epsilon)}(X(t)) |\det(X'(t))| dt \\ &= \sum_{i=1}^N \int_{B(x_i, \eta)} \mathbb{1}_{B(a, \epsilon)}(X(t)) |\det(X'(t))| dt \\ &= \sum_{i=1}^N \int_{\mathbb{R}^d} \mathbb{1}_{B(a, \epsilon)}(u) du \\ &= N |B(a, \epsilon)|_d, \end{aligned}$$

where, here and from now on, $|\cdot|_d$ stands for the d -dimensional Lebesgue measure. For the second equality, we use the fact that the balls $B(x_i, \eta)$ do not intersect. For the third one, we perform a change of variable $u = X(t)$ in each integral.

Finally, since the above equality stands also if we replace ϵ by any positive lower value, we get the famous Kac's counting formula,

$$N = \lim_{\epsilon \rightarrow 0} \int_T \frac{1}{|B(a, \epsilon)|_d} \mathbb{1}_{B(a, \epsilon)}(X(t)) |\det X'(t)| dt.$$

From Kac's counting formula, it is possible to derive an expectation formula in the stochastic case when X is Gaussian and satisfies Assumptions (i) and (ii). In that case, $\mathbb{P}(\exists t \in T : X(t) = a, \det(X'(t)) = 0) = 0$ and for each $t \in \mathbb{R}^d$, $X(t)$ admits a probability density function denoted by $p_{X(t)}$. It only remains to permute the limit as ϵ goes to 0 and the integral over T to get the conditional expectation in the right-hand side of (19). The difficulty of the rigorous demonstration of this equation, that we have explicitly avoided here, lies in the justification of these steps.

In a very aesthetic way, Rice formula for the first moment of $N(X, a, T)$, namely Formula (19), allows one to derive Formula (21), a formula for the second-order factorial moment of this number. To this aim, we define $\tilde{X} = (X, X) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$, $(s, t) \mapsto (X(s), X(t))$ and the set $\tilde{T} = T \times T$. We remark that $N(\tilde{X}, (a, a), \tilde{T}) = N(X, a, T)^2 - N(X, a, T) = N(X, a, T)(N(X, a, T) - 1)$, and hence we are in position to use Formula (19) to express the expectation of $N(\tilde{X}, (a, a), \tilde{T})$. But \tilde{X} does not satisfy the non-degeneracy condition on the diagonal $\{(s, s) \in \tilde{T}\}$. To overcome this obstacle, we introduce for $\delta > 0$ the subset $\tilde{T}^\delta = \{(s, t) \in \tilde{T} : \|s - t\| > \delta\}$ and we apply the first order formula to \tilde{X} on \tilde{T}^δ . Letting δ tend to 0 allows us to get (21). \square

In the same way, one can derive formulas for any higher factorial moment under suited non-degeneracy assumptions, see [10] Theorem 6.3.

3.1.2 Crossings and Critical Points of a Stationary Gaussian Field

When the field X is stationary, Rice formulas in Theorem 5 can be simplified. Indeed, in the expectation formula (19), the integrated function does not depend on point t , while in the second moment formula (21), it only depends on the increment $t - s$.

Number of crossings.

We compute the expectation of the number of zeros of a Gaussian, stationary, a.s. of class \mathcal{C}^2 random field $X : \mathbb{R}^d \rightarrow \mathbb{R}^d$. We furthermore assume that $X(0)$ is not degenerate. Stationarity implies that for any $t \in \mathbb{R}^d$, $X(t) \stackrel{\mathcal{L}}{\sim} X(0)$ and that $X(t)$ and $X'(t)$ are independent. Hence, Rice formula (19) writes

$$\mathbb{E}[N(X, a, T)] = |T|_d \mathbb{E}[|\det(X'(0))|] p_{X(0)}(a) .$$

As a consequence, let us place ourselves in the case where $d = 1$, X is centered and with unit variance, *i.e.* $X(0) \sim \mathcal{N}(0, 1)$, and let us write $\lambda_2 = -r''(0)$, the second spectral moment of X , such that $X'(0) \sim \mathcal{N}(0, \lambda_2)$. Then, $\mathbb{E}(|X'(0)|) = \sqrt{2\lambda_2/\pi}$ and so we get the classical formula for the mean number of crossings at level a :

$$\mathbb{E}[N(X, a, T)] = \frac{\sqrt{\lambda_2}}{\pi} e^{-a^2/2} |T|_1 .$$

Without any surprise, this expectation tends to 0 as a tends to $\pm\infty$.

Number of critical points.

Let $X : \mathbb{R}^d \rightarrow \mathbb{R}$ a Gaussian stationary random field of class \mathcal{C}^3 , such that the random vector $X'(0)$ is not degenerate. We apply Rice formula to $X' : \mathbb{R}^d \rightarrow \mathbb{R}$, the gradient of X . Stationarity again entails simplifications in Equation (19) to get the first moment of $N(X', 0, T)$:

$$\mathbb{E}[N(X', 0, T)] = |T|_d \mathbb{E}[|\det(X''(0))|] p_{X'(0)}(0) .$$

Prescribing the expectation and the covariance function of X would allow to develop this formula.

If one is interested in the second moment of $N(X', 0, T)$, Equation (21) is particularly designed and in the stationary case it yields,

$$\begin{aligned} & \mathbb{E}[N(X', 0, T)(N(X', 0, T) - 1)] \\ &= \int_{\mathbb{R}^d} |T \cap (T - s)|_d \mathbb{E}[|\det(X''(0))| |\det(X''(s))| |X'(0) = X'(s) = a] \end{aligned}$$

both sides being simultaneously finite or infinite.

Exhibiting a necessary and sufficient condition on the covariance function r of X such that $N(X', 0, T)$ has finite variance is an old and still open problem. In dimension one, a simple condition requires that the fourth derivative of r satisfies an integrability condition in a neighbourhood of zero:

$$\exists \delta > 0, \quad \int_0^\delta \frac{r^{(4)}(0) - r^{(4)}(t)}{t} dt < +\infty.$$

Cramér and Leadbetter were the first to propose it in [36] as a sufficient condition, then Geman proved that it was also necessary in [49]. Afterwards, many other papers generalized Geman's result, such as [15, 37, 63, 58] for instance and references therein. In higher dimensions the problem is not yet solved but we can cite the following breakthrough papers, [42, 44] and very recently [8] and [48].

Different types of critical points and repulsion in dimension 2.

In the planar case ($d = 2$), using the constraint Rice formula (20) allows one to establish simple relationships linking the mean number of critical points with the mean number of local minima, local maxima, local extrema or saddle points in T . Actually, let us denote by N_T^c the number of critical points of X restricted to T , *i.e.* with our previous notation $N_T^c = N(X', 0, T)$, and respectively N_T^e , N_T^m , N_T^M and N_T^s , for the number of local extrema, local minima, local maxima and saddle points.

Proposition 12 *Let $X : \mathbb{R}^2 \rightarrow \mathbb{R}$ a centered, Gaussian, stationary random field of class \mathcal{C}^3 , such that $X'(0)$ is not degenerate. Then,*

$$4\mathbb{E}[N_T^m] = 4\mathbb{E}[N_T^M] = 2\mathbb{E}[N_T^e] = 2\mathbb{E}[N_T^s] = \mathbb{E}[N_T^c] .$$

Proof We only give a sketch of proof highlighting the use of Rice formula. For a detailed proof, see for instance [60].

Since X and $-X$ are equal in law, the first equality is obvious. The second one simply follows since $N^e = N^m + N^M$ and the last equality since $N_T^c = N_T^e + N_T^s$. The only point that remains to establish is the third equality.

We use Rice formula to express the expectation of the number of critical points with a constraint on the sign of the Hessian determinant. Doing so, the number of

local extrema can be evaluated as follows,

$$N_T^e = \#\{t \in T : X'(t) = 0, \det(X''(t)) > 0\},$$

and the same holds for N_T^s , the number of saddle points, with $\det(X''(t)) < 0$. Formula (20) in Theorem 5 applies under our assumptions on X and so, $\mathbb{E}[N_T^e] - \mathbb{E}[N_T^s] = \mathbb{E}[\det(X''(0))] p_{X'(0)}(0) v_d(T)$. We express $\mathbb{E}[\det(X''(0))]$ thanks to Equation (8) in Section 1:

$$\mathbb{E}[\det(X''(0))] = \mathbb{E}[X''_{1,1}(0)X''_{2,2}(0) - X''_{1,2}(0,0)X''_{2,1}(0)] = r_{1,1,2,2}^{(4)}(0) - r_{1,2,2,1}^{(4)}(0).$$

According to Schwarz theorem, these two last partial derivatives are equal, so we get $\mathbb{E}[N_T^e] = \mathbb{E}[N_T^s]$. \square

Let us turn to another interesting characteristic of the critical points: their repulsion ability. During the last decade, an increasing interest has been payed to repulsion properties of certain point processes. Actually, the use of repulsive point processes, for instance determinantal instead of Poisson point processes, is proved to be more efficient in sample theory or data analysis. We will not enter into more details about the advantages of repulsive point processes. Let us only quote that a way of characterizing repulsion is given by the probability of finding two points in an arbitrary small ball $B(0, \rho)$, $\rho > 0$ and that it is deeply linked with the second factorial moment of the number of points lying in $B(0, \rho)$. Hence, following [12] and [60], we say that a point process Φ is repulsive if the second factorial moment of $\#(\Phi \cap B(0, \rho))$ divided by the volume of $B(0, \rho)$ tends to 0 as ρ tends to 0. In [12, 9, 60], sharp applications of Rice formula produce the following estimates of the second factorial moment of $N_\rho^c := N(X', 0, B(0, \rho))$ and similarly of $N_\rho^e, N_\rho^{max}, \dots$ for the other types of critical points : as ρ tends to 0,

$$\mathbb{E}[N_\rho^c(N_\rho^c - 1)] \asymp \rho^4; \mathbb{E}[N_\rho^e(N_\rho^e - 1)] \asymp \rho^7; \mathbb{E}[N_\rho^{max} N_\rho^{min}] \asymp \rho^{12},$$

where we write $f(\rho) \asymp g(\rho)$ for the existence of constants c and c' such that $0 < c \leq c' < +\infty$ and $cf(\rho) \leq g(\rho) \leq c'f(\rho)$ for ρ small enough.

Those estimates allows one to state that the critical points of a stationary isotropic Gaussian field are not repulsive whereas the extremal points are repulsive, and maximal and minimal points are even more each other repulsive. Such statements are potentially very useful for applied purposes since simulating stationary Gaussian fields is an easy task compared to simulating determinantal processes.

3.2 Level Sets

We now concentrate on a second type of Rice formula which allows us to deal with a vector-valued random field $X : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ in the case $d > d'$.

Recall that for $a \in \mathbb{R}^{d'}$, the level set of X at level a is defined as the set of all points t in \mathbb{R}^d such that $X(t) = a$, denoted by $X^{-1}(\{a\})$. Moreover, under regularity

assumptions, $X^{-1}(\{a\})$ is a \mathcal{C}^1 -manifold of dimension $d - d'$ in \mathbb{R}^d . When $d' = 1$ and $a = 0$, the associated level set $X^{-1}(0)$ is called the nodal set of X .

3.2.1 Rice formula for level sets

In this section, we consider the $(d - d')$ -Hausdorff measure of the excursion set $X^{-1}(\{a\})$ restricted to a compact subset T of \mathbb{R}^d , which we denote by $\ell(X, a, T)$.

Theorem 6 (Rice formula, case $d > d'$)

Let $d > d'$. Let $X : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ be a Gaussian random field. We assume that

- (i) X is almost surely of class \mathcal{C}^2 ,
- (ii) for any $t \in \mathbb{R}^d$, the Gaussian vector $(X(t), X'(t))$ is not degenerate.

Then, for any $a \in \mathbb{R}^{d'}$ and for any $T \in \mathcal{B}(\mathbb{R}^d)$, the expected Hausdorff measure of $X^{-1}(\{a\}) \cap T$ is given by

$$\mathbb{E}[\ell(X, a, T)] = \int_T \mathbb{E}[(\det(X'(t)X'(t)^T))^{1/2} | X(t) = a] p_{X(t)}(a) dt, \quad (22)$$

where both sides are finite if T is a compact set.

In the above equation, $X'(t)$ has to be understood as the matrix $(\partial_j X^i(t))_{1 \leq i \leq d', 1 \leq j \leq d}$. In the case $d' = 1$, $(\det(X'(t)X'(t)^T))^{1/2}$ is nothing but the Euclidean norm $\|X'(t)\|$ of the d -dimensional vector $X'(t)$ and in the case $d' = d$, $(\det(X'(t)X'(t)^T))^{1/2}$ is equal to $|\det(X'(t))|$ with $X'(t)$ a square matrix of size d .

The proof of Theorem (6) can be found in [10] (see Theorem 6.8 and Proposition 6.12). Note that the above assumptions are more restrictive than in the case $d = d'$. However, as in the case $d = d'$, similar formulas hold for higher moments or including a constraint of type " $Y \in B$ ". We again refer to the books [10] and [20] or to the recent review [8] for more Rice type formulas.

Nodal set of a d -dimensional real Gaussian field.

Let us consider an a.s. of class \mathcal{C}^2 , Gaussian random field $X: \mathbb{R}^d \rightarrow \mathbb{R}$ and its associated nodal set $X^{-1}(\{0\})$. Provided that X satisfies the assumptions of Theorem 6, we obtain the following expectation formula,

$$\mathbb{E}[\ell(X, 0, T)] = \int_T \mathbb{E}[\|X'(t)\| | X(t) = 0] p_{X(t)}(0) dt.$$

When X is furthermore stationary, $X(t)$ and $X'(t)$ are independent random variables for any t and their distributions don't depend on t . So, the above equation becomes

$$\mathbb{E}[\ell(X, 0, T)] = |T|_d \mathbb{E}[\|X'(0)\|] p_{X(0)}(0),$$

and assuming moreover that X is isotropic with second spectral moment λ_2 (see (6) in Section 1) then, $(\lambda_2)^{-1/2}X'(0)$ is a standard d -dimensional Gaussian vector

and so $\|(\lambda_2)^{-1/2} X'(0)\|$ is distributed as $\chi(d)$ (square root of a $\chi^2(d)$ distribution), which yields $\mathbb{E}[\|X'(0)\|] = (2\lambda_2)^{1/2} \Gamma((d+1)/2) / \Gamma(d/2)$. The computation of $\mathbb{E}[\|X'(0)\|]$ can be rather tricky when X is no more isotropic. We give more details in the anisotropic Gaussian random waves examples that we describe in the next subsection.

3.2.2 Level Lines of Gaussian Random Waves

We now explore some particular situations where the formula in Theorem 6 applies. We pay a particular attention to anisotropic stationary Gaussian fields where we try to catch the effect of anisotropy on the expected measure of their nodal sets.

Planar random wave.

Recall the Gaussian Random Wave (GRW) models introduced in Section 1.4. We are now interested in the length of the level lines of a two-dimensional GRW. The following results are part of the paper [45].

Let $X_{\mathbf{k}}$ be a Gaussian random wave defined on \mathbb{R}^2 with wavevector \mathbf{k} . Applying Theorem 6 and keeping the same notations yield, for any compact set T in \mathbb{R}^2 and any $a \in \mathbb{R}$,

$$\mathbb{E}[l(X_{\mathbf{k}}, a, T)] = |T|_2 E[\|X'_{\mathbf{k}}(0)\|] p_{X_{\mathbf{k}}(0)}(a),$$

where $X'_{\mathbf{k}}(0)$ is nothing but a two-dimensional centered Gaussian vector with covariance matrix $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$. To compute its expected Euclidean norm, we use the following well known fact, that can be proved with simple algebra.

If M is a symmetric positive definite matrix with eigenvalues γ_- and γ_+ such that $0 < \gamma_- \leq \gamma_+$, then

$$\int_{\mathbb{R}^2} (Mx \cdot x)^{1/2} \frac{e^{-\|x\|^2/2}}{2\pi} dx = (2\gamma_+/\pi)^{1/2} \mathcal{E}\left((1 - \gamma_-/\gamma_+)^{1/2}\right), \quad (23)$$

where \mathcal{E} stands for the elliptic integral given by $\mathcal{E}(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 \theta)^{1/2} d\theta$, for $x \in [0, 1]$.

In our case, we set $M = \mathbb{E}[\mathbf{k}\mathbf{k}^T]$ and so $\gamma_- + \gamma_+ = \text{trace}(\mathbb{E}[\mathbf{k}\mathbf{k}^T]) = \mathbb{E}[\|\mathbf{k}\|^2]$. Hence, introducing the coherency index of M , $c = (\gamma_+ - \gamma_-)/(\gamma_+ + \gamma_-)$, which is commonly used in spatial statistics (see [65]), the following proposition holds.

Proposition 13 *Let \mathbf{k} be a random wavevector in \mathbb{R}^2 and $X_{\mathbf{k}}$ its associated Gaussian random wave. Let us assume that $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ is invertible and let us denote by $c(\mathbf{k})$ its coherency index. Then,*

$$\mathbb{E}[l(X_{\mathbf{k}}, a, T)] = |T|_2 \frac{e^{-a^2/2}}{\pi\sqrt{2}} \mathbb{E}[\|\mathbf{k}\|^2]^{1/2} \mathcal{F}(c(\mathbf{k})),$$

where the map $\mathcal{F} : c \in [0, 1] \mapsto (1+c)^{1/2} \mathcal{E}((2c/(1+c))^{1/2})$ is strictly decreasing.

One can refer to Lemma A.3 in [45] for the proof of the decrease of mapping \mathcal{F} . Another expression for the same expectation can be found in [62] Formula (2.3.13). However the above formulation allows to highlight the effect of the wavevector's distribution on the mean length of level sets.

Indeed, regarding the coherency index $c(\mathbf{k})$ as a parameter that positively quantifies anisotropy, the above formula clearly indicates that the mean length of level curves is decreasing as the anisotropy of \mathbf{k} increases. Moreover, one can note that $c(\mathbf{k}) = c(\tilde{\mathbf{k}})$ when $\tilde{\mathbf{k}} := \mathbf{k}/\|\mathbf{k}\|$ and $\|\mathbf{k}\|$ are independent, which is the case when $\|\mathbf{k}\|$ is constant a.s. In that case, the respective influences of $\|\mathbf{k}\|$ and $\tilde{\mathbf{k}}$ distributions in the previous formula are easily distinguishable.

We now apply Proposition 13 to our favorite examples of GRW introduced in Section 1.4, prescribing the directional distribution of the wavevector \mathbf{k} .

Example 1 (Toy model).

Take \mathbf{k} distributed on \mathbb{S}^1 with probability density function given by (14) for some positive α . A straightforward computation yields $\mathbb{E}[\mathbf{k}\mathbf{k}^T] = 1/(\alpha+2) \begin{pmatrix} \alpha+1 & 0 \\ 0 & 1 \end{pmatrix}$ and consequently, $c(\mathbf{k}) = \alpha/(\alpha+2)$, which is an increasing function of parameter α . As one can observe on Figure 1, the more anisotropic the model, the smaller the expected length of level lines.

Example 2 (Elementary model).

We choose \mathbf{k} to be uniformly distributed on $[-\delta, \delta] \cup [\pi - \delta, \pi + \delta]$ for some $0 < \delta \leq \pi/2$. In that case, $\mathbb{E}[\mathbf{k}\mathbf{k}^T] = 1/(4\delta) \begin{pmatrix} 2\delta + \sin(2\delta) & 0 \\ 0 & 2\delta - \sin(2\delta) \end{pmatrix}$ and hence, $c(\mathbf{k}) = \sin(2\delta)/(2\delta)$, which is decreasing on $(0, \pi/2]$. Again, the mean length of level sets is decreasing with anisotropy, *i.e.* as δ is decreasing, as observed on Figure 2.

Example 4 (Berry's random plane wave).

Recall that we have introduced the Berry's random wave model in Section 1.4. We now concentrate on the Berry's random plane wave, which is defined, for any fixed wavenumber $\kappa > 0$, as the unique standard stationary isotropic Gaussian field $\{b_\kappa(x) : x \in \mathbb{R}^2\}$ that solves the Helmholtz equation $\Delta X + \kappa^2 X = 0$ a.s. Therefore, it appears as an almost sure eigenfunction of the Laplacian Δ on \mathbb{R}^2 associated with the eigenvalue $-\kappa^2$. A classical question turns around the asymptotic behaviour of b_κ as κ tends to infinity, in particular concerning the nodal length $\ell(b_\kappa, 0, T)$. It is shown in [67] (see Theorem 1.1) for instance that $\mathbb{E}[\ell(b_\kappa, 0, T)] = \kappa |T|_2 / (2\sqrt{2})$ and that, as $\kappa \rightarrow \infty$,

$$\text{Var}[\ell(b_\kappa, 0, T)] \sim \frac{|T|_2}{256\pi} \log(\kappa) \quad \text{and} \quad \ell(b_\kappa, 0, T) \text{ satisfies a central limit theorem.}$$

The unexpected logarithmic factor in the variance was already mentioned and discussed in a precursor paper of Berry [17]. The phenomena is well known under the name of Berry's cancellation. It is worth to remark that it only appears in the two-dimensional case. Actually, the next example shows a distinct behaviour in dimension three.

Length of silence lines in \mathbb{R}^3 .

Let us deal with a centered Gaussian field defined on the 3-dimensional Euclidean space and with complex values, that can be summarized in $X : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. It can serve as a model for 3-dimensional acoustic wave, see [38]. The lines of silence or dislocation lines, as named by Berry in [18, 39], are nothing but the one-dimensional submanifold of \mathbb{R}^3 defined as $X^{-1}(\{(0, 0)\}) = \{x \in \mathbb{R}^3 : |X(t)| = 0\}$ when the random field X is assumed to be *a.s.* of class \mathcal{C}^2 . Then, it is proved in [38] that for any compact T in \mathbb{R}^3 , $\mathbb{E}[\ell(X, 0, T)] = \lambda |T|_3 / \pi$ and that, as $T \nearrow \mathbb{R}^3$,

$$\text{Var}[\ell(X, 0, T)] \sim v |T|_3 \text{ and } \ell(X, 0, T) \text{ satisfies a central limit theorem,}$$

with $v \in (0, +\infty)$, as soon as the covariance function of X is either square integrable on \mathbb{R}^3 or equal to $\text{sinc}(\|\cdot\|)$, the latter corresponding to the 3-dimensional Berry's monochromatic random wave (see Section 1.4, Example 4). Note that, in this 3-dimensional realm, no logarithmic factor is involved in the asymptotic behaviour.

3.3 Excursion Sets

Let X be a random field from \mathbb{R}^d to \mathbb{R} . Let T be a compact subset in \mathbb{R}^d , for instance a rectangle, and u a fixed real number. We call excursion set - or exceedance region - of X above level u within the subset T , the following set

$$A(X, u, T) = \{t \in T : X(t) \geq u\}.$$

In this section we are interested in geometrical features of the excursion sets summarized in the $d + 1$ Lipschitz-Killing curvatures, also called intrinsic volumes or Minkovski functionals in the literature.

3.3.1 Lipschitz-Killing Curvatures

For any convex compact set M in \mathbb{R}^d , the Minkovski functionals of M , namely $\mathcal{L}_j(M)$ for $j = 0, 1, \dots, d$, are defined through Steiner's formula (see for instance [5], Section 6.3) that prescribes the volume of the tube $M \oplus B_d(0, \rho)$ around M . Indeed, by defining $M \oplus B_d(0, \rho) = \{x + y : x \in M, y \in B_d(0, \rho)\} = \{x \in \mathbb{R}^d : \text{dist}(x, M) \leq \rho\}$, the Lebesgue measure of this set can be expanded as a polynomial

function of ρ , whose coefficients are given as follows:

$$|M \oplus B_d(0, \rho)|_d = \sum_{j=0}^d \omega_{d-j} \mathcal{L}_j(M) \rho^{d-j}, \quad (24)$$

with ω_j the volume of the j -dimensional unit Euclidean ball, $\omega_j = |B_j(0, 1)|_j = \pi^{j/2}/\Gamma(1 + j/2)$.

Note that taking $\rho = 0$ in (24) yields that $\mathcal{L}_d(M)$ is equal to $|M|_d$, the d -dimensional Lebesgue measure of M , that $\mathcal{L}_{d-1}(M)$ is equal to half the Lebesgue measure of the boundary of M , and that $\mathcal{L}_0(M)$ is equal to 1.

In the case of simple domains like cubes or Euclidean balls in \mathbb{R}^d , we have

$$\mathcal{L}_j([0, a]^d) = \binom{d}{j} a^j \quad \text{and} \quad \mathcal{L}_j(B_d(0, a)) = \binom{d}{j} \frac{\omega_d}{\omega_{d-j}} a^j.$$

For more complex domains, even not convex, the notion of Lipschitz-Killing curvatures (LKC) extends the notion of Minkovski functionals, see [78] for a precise definition. The highest-index characteristics \mathcal{L}_d and \mathcal{L}_{d-1} still respectively measure the d -dimensional Lebesgue measure of the set and half the $(d - 1)$ -dimensional measure of its boundary. The first LKC, \mathcal{L}_0 , is the Euler or Euler-Poincaré characteristic. This integer-valued topological functional provides information on the structure of a set. It satisfies several properties, such as additivity and homotopy invariance.

We give the following heuristic to compute the Euler characteristic of a sufficiently simple set $A \subset \mathbb{R}^d$ in the small dimensions ($\#$ stands for the cardinal) :

- if $d = 1$, $\mathcal{L}_0(A) = \#(\text{disjoint intervals in } A)$;
- if $d = 2$, $\mathcal{L}_0(A) = \#(\text{connected components of } A) - \#(\text{holes in } A)$;
- if $d = 3$, $\mathcal{L}_0(A) = \#(\text{connected components of } A) - \#(\text{handles of } A) + \#(\text{interior hollows in } A)$.

Furthermore, it is possible to use the LKC for the measurements of excursion sets of stationary Gaussian fields that are smooth enough (see [26], Proposition 2.3).

3.3.2 Mean Euler Characteristic of an Excursion Set

Before presenting a general expectation formula for all the Lipschitz-Killing curvatures of an excursion set, we give an idea of how it is possible to obtain such a result by limiting ourselves to the first LKC, the Euler characteristic, and by taking advantage of Rice formulas.

The premises of this formula are due to Adler and can be found in [3], see also [2].

Let $X : \mathbb{R}^d \rightarrow \mathbb{R}$ be a stationary Gaussian field, T be a rectangle in \mathbb{R}^d and $u \in \mathbb{R}$. We are interested in the mean Euler characteristic of the excursion set $A(X, u, T)$. The very definition of the Euler characteristic as one of the Lipschitz-

Killing curvatures is really tricky and we will continue to avoid giving it. However, for a deterministic function f , an expression of the Euler characteristic of any excursion set of f is provided by Morse formula (see for instance the original treatise [66] or [5] Eq.(9.4.1)). Here we will not use the original version of Morse formula but a modification of it. Roughly speaking, we focus on the terms which concern the behavior of the considered function f on the interior of the compact set T . The other terms only depend on the behaviour of f on the border of the set T . That leads to define a new quantity that we called the modified Euler characteristic of the excursion set. Note that the modified Euler characteristic is only defined for excursion sets. It was introduced in [46] (where [5] is acknowledged as a source of inspiration) in order to make easier the study of the Euler characteristic of excursion sets; it was then used in [40]. The definition below is taken from [6], Formula (4.2.9).

Definition 16 Let T a d -dimensional compact rectangle in \mathbb{R}^d , let $u \in \mathbb{R}$ and let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a function of class \mathcal{C}^2 . The modified Euler characteristic of $A(u, f, T)$ is defined as

$$\chi(A(f, u, T)) = \sum_{k=0}^d (-1)^k \mu_k(u, f, T), \quad (25)$$

where for any $k \in \{0, \dots, d\}$,

$$\mu_k(u, f, T) = \#\{t \in T : f(t) \geq u, f'(t) = 0, \text{index}(f''(t)) = d - k\}, \quad (26)$$

where the 'index' stands for the number of negative eigenvalues of any symmetric matrix.

We can see that the modified Euler characteristic is simply an alternate sum of the number of different types of critical points of f restricted to T . For instance, in dimension one ($d = 1$), we get

$$\chi(A(f, u, T)) = \#\{\text{local maxima above } u\} - \#\{\text{local minima above } u\}.$$

Let us come back to the Gaussian field X . To compute the expected modified Euler characteristic of $A(X, u, T)$, we will apply Rice formula to each term in the alternate sum (25). Actually, denoting by \mathfrak{S}_d^{d-k} the set of symmetric matrices of size $d \times d$ with $d - k$ negative eigenvalues, Rice Formula (20) in Theorem 5 can be applied for the number of critical points with the constraint $(X, X'') \in [u, +\infty) \cap \mathfrak{S}_d^{d-k}$. Under the additional assumptions that for any $t \in \mathbb{R}^d$, $X'(t)$ is not degenerate and X is of class \mathcal{C}^3 , we have

$$\begin{aligned} & \mathbb{E}[\chi(A(X, u, T))] \\ &= \sum_{k=0}^d (-1)^k \int_T \mathbb{E}[|\det(X''(t))| \mathbb{1}_{[u, +\infty)}(X(t)) \mathbb{1}_{\mathfrak{S}_d^{d-k}}(X''(t)) \mid X'(t) = 0] p_{X'(t)}(0) dt \\ &= (-1)^d \int_T \mathbb{E}[\det(X''(t)) \mathbb{1}_{[u, +\infty)}(X(t)) \mid X'(t) = 0] p_{X'(t)}(0) dt, \end{aligned}$$

and using the fact that X is stationary, we get

$$\mathbb{E}[\chi(A(X, u, T))] = (-1)^d |T|_d p_{X'(0)}(0) \mathbb{E}[\det(X''(0)) \mathbb{1}_{[u, +\infty)}(X(0)) | X'(0) = 0].$$

The conditional expectation can be expressed thanks to the Gaussian regression result enunciated in Proposition 1, allowing to express the parameters of the Gaussian distribution of $(X(0), X''(0))$ conditioned by $X'(0)$. Since X is furthermore isotropic and has unit variance, one can get a closed formula for $\mathbb{E}[\det(X''(0)) \mathbb{1}_{[u, +\infty)}(X(0)) | X'(0) = 0]$, see [5], Lemma 11.7.1 or [6], Lemma 4.2.2 where some computational tricks are at work, under the additional assumption that $X''(0)$ is not degenerate. Hence, we get the following proposition.

Proposition 14 *Let T a d -dimensional compact rectangle in \mathbb{R}^d , let $u \in \mathbb{R}$ and let X be a stationary isotropic Gaussian field a.s. of class \mathcal{C}^3 with second spectral moment λ_2 , such that $X'(0)$ as well as $X''(0)$ are non-degenerate Gaussian vectors. Then,*

$$\mathbb{E}[\chi(A(X, u, T))] = (2\pi)^{-(d+1)/2} |T|_d e^{-u^2/2} H_{d-1}(u) \lambda_2^{d/2}.$$

Following a similar approach, one can also derive an explicit integral expression for the second moment of $\chi(A(X, u, T))$ (see [40], Proposition 1, for instance). Let us remark that Lemma 11.7.1 in [5] allows one to apply previous result with more general domains than rectangles. In Section 4.2.3 we will actually consider domains in \mathbb{R}^2 obtained by a smooth deformation of a two-dimensional rectangle.

3.3.3 Gaussian Kinematic Formula

Here we state the Gaussian Kinematic Formula, the general expectation formula for all the LKCs of an excursion set of a random field satisfying the following assumptions.

Assumption A0

$X : \mathbb{R}^d \rightarrow \mathbb{R}$ is a stationary isotropic Gaussian field such that

- it is standard, *i.e.* $\mathbb{E}[X(0)] = 0$ and $\mathbb{E}[X(0)^2] = 1$,
- its sample paths are almost surely \mathcal{C}^3 ,
- $X'(0)$ and $X''(0)$ are non degenerate Gaussian vectors.

Remark on Assumption A0

The non-degeneracy assumption that is required in **A0** means that the covariance matrix of the mentioned Gaussian vector is invertible. Concerning the second derivative of X at point $0 \in \mathbb{R}^d$, one has to write it carefully as a $(d(d+1)/2)$ -dimensional Gaussian vector $(X''_{jk}, 1 \leq j \leq k \leq d)$ in order to avoid an obvious degeneracy due to the natural symmetry of the Hessian matrix. The third point is equivalent to the non-degeneracy of the vector $(X'(0), X''(0))$ because of the stationarity assumption

on X , which implies the independency between $X'(0)$ and $X''(0)$. Finally, let us point out that our regularity hypothesis could be weakened.

We are now in position to introduce the so-called Gaussian Kinematic Formula. We state it in the next theorem under Assumption **A0**, which implies the existence of a non-vanishing second spectral moment λ_2 . The required assumptions are then fulfilled in order to apply Theorem 13.2.1 in [5] or Theorem 4.8.1 in [6].

In the next lines, we use the notation Φ for the standard normal cumulative distribution function, and $\bar{\Phi} = 1 - \Phi$ for its survival function.

Theorem 7 (Gaussian Kinematic Formula)

Let X be a Gaussian field on \mathbb{R}^d satisfying Assumption **A0** with positive second spectral moment $\lambda_2 > 0$. Then, for any $u \in \mathbb{R}$ and any $k = 0, 1, \dots, d$,

$$\mathbb{E}[\mathcal{L}_k(A(X, u, T))] = \sum_{j=0}^{d-k} \binom{j+k}{j} \frac{\omega_{j+k}}{\omega_j \omega_k} \lambda_2^{j/2} \mathcal{L}_{j+k}(T) \rho_j(u), \quad (27)$$

where

$$\rho_0(u) = \bar{\Phi}(u) \text{ and for } j \geq 1, \rho_j(u) = (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2}, \quad (28)$$

with $(H_j)_{j \geq 0}$ the Hermite polynomials, i.e. $H_j(x) = (-1)^j e^{x^2/2} \frac{d^j}{dx^j} (e^{-x^2/2})$.

Let us mention that the Gaussian Kinematic Formula can be extended to more general fields (sub-Gaussian [4], shot-noise [24], Gaussian mixtures [41]) yielding to many applications and many statistical studies. We provide some of them in the next section.

As a remark to conclude this section, let us show how the Gaussian Kinematic Formula concerning the Euler characteristic ($k = 0$) is linked to the expectation formula on the modified Euler characteristic enunciated in Section 3.3.2. We consider a d -dimensional rectangle that grows to \mathbb{R}^d such that $\mathcal{L}_j(T)/\mathcal{L}_d(T) \rightarrow 0$ for all $j < d$. For instance, we can consider $T = T_n$ a d -dimensional cube $[-n; n]^d$, with $n \rightarrow +\infty$. Then, the leader term in the expectation formula (27), which corresponds to $j = d$ in the sum and writes

$$\lambda_2^{d/2} \mathcal{L}_d(T) (2\pi)^{-(d+1)/2} H_{d-1}(u) e^{-u^2/2}, \quad (29)$$

provides an asymptotic for $\mathbb{E}[\mathcal{L}_0(A(X, u, T))]$ as T tends to \mathbb{R}^d . Note that this term precisely equals the expectation of the modified Euler characteristic of the excursion set, which was derived in Proposition 14. This result is not surprising if one remembers that the modified Euler characteristic of $A(X, u, T)$ is defined by dropping the terms concerned with the border of T in the formula that defines the Euler characteristic (see Section 3.3.2). However, this asymptotic reveals how relevant can be the modified Euler characteristic.

4 Inference Based on Geometrical Features

In this last section, we will explore how the closed expectation formulas stated in the previous section, in particular Proposition 14 and Theorem 7, can be used for inference analysis on the random field X . In order to be in the framework of these formulas, we will assume all along the section that the field X satisfies Assumption **A0**, which means in particular that it is stationary and isotropic.

We will focus on the two-dimensional framework that is particularly designed for image analysis. Let X be a Gaussian stationary random field defined on \mathbb{R}^2 that is smooth enough. For a fixed level u and a fixed window T , we start with observing the excursion set $A(X, u, T)$, that is a single black-and-white image, through the rather easy tractable quantities $\chi(A(X, u, T))$ or $\mathcal{L}_k(A(X, u, T))$ for $k = 0, 1, 2$. Those features bring information on the geometry or topology of the excursion set and beyond on the geometry of the image of the field X itself, \mathcal{L}_2 being linked with the occupation density, \mathcal{L}_1 with the regularity and χ and \mathcal{L}_0 with the connectivity. Note that the observation of a single excursion set is really sparse compared to the observation of the whole distribution of the field X . Based on this sparse observation, quantitative features characterizing X could be inferred, such as variance, second spectral moment, anisotropy index, deformation, and qualitative features could be tested, such as Gaussianity, isotropy, extreme values analysis. In the following, we will present a review of some results implementing this approach, mainly taken from our own works [1, 26, 40, 41, 47].

Let us mention one more feature that we have not studied so far and that we will not consider in the rest of the lecture, the number of connected components of the excursion sets. Actually, dealing with this topological feature is a much more complicated task than dealing with the LKCs since the involved functional is no more a local functional, see for instance [13, 14] and references therein for some insights on it.

4.1 Central Limit Theorems

We concentrate on the LKCs of the excursion sets of X on a bounded rectangle T in \mathbb{R}^2 , i.e. $\mathcal{L}_k(A(X, u, T))$ for $k = 0, 1, 2$, $u \in \mathbb{R}$. The probability distribution of those random quantities are totally unknown, only the first moments are theoretically available through formulas like Rice formulas or the Gaussian kinematic ones (see Section 3). We now give a central limit theorem (CLT) concerning $\mathcal{L}_k(A(X, u, T))$ as the domain T grows to \mathbb{R}^2 . We precisely focus on the following limit in distribution, for $k = 0, 1, 2$,

$$\frac{\mathcal{L}_k(A(X, u, T)) - \mathbb{E}[\mathcal{L}_k(A(X, u, T))]}{|T|_d^{1/2}} \xrightarrow{T \nearrow \mathbb{R}^2} \mathcal{N}(0, v_k(u)), \quad (30)$$

with $v_k(u) < +\infty$. The above limit is actually established for instance in [59], for any dimension d and any $0 \leq k \leq d$ under the assumption that X is *a.s.* of class \mathcal{C}^3 , that the Gaussian vector $(X(0), X'(0), X''(0))$ is of full rank and the covariance function of X as well as its derivatives up to order three decrease sufficiently fast at infinity.

Besides the cited result of [59], let us mention that this subject has been the purpose of many papers, see for instance [57, 19] for the length of the level lines ($k = 1$), [30] for the area of the excursion sets ($k = 2$), [46, 40] for the Euler characteristic of the excursion sets ($k = 0$) or [67] in the case of Berry's random planar wave, to cite few of them. Actually, there are still many open questions around the asymptotic behaviour of the excursions LKCs as for instance the conditions that ensure the non-degeneracy of the limit ($v_k(u) > 0$?), the exact computation of the asymptotic variance $v_k(u)$ or the specific cases where $\text{Var} \mathcal{L}_k(A(X, u, T))$ has not the same order of magnitude as $|T|_d^{1/2}$ (hyperuniform behavior). Even more challenging is the question of the correlation between different levels or between the different LKCs.

Such CLT results are essential for establishing rigorous statistical procedures when one aims at using the LKCs of excursion sets as observations. They provide consistent estimators from observations that are given either on a large domain, or on a fixed domain with a dense grid for an infill statistics framework.

Figure 6 is extracted from [26]. It provides an illustration of the consistent estimation of the expected LKCs of excursion sets through a Monte Carlo method. Actually, 100 trajectories of a stationary standard Gaussian field $X : \mathbb{R}^2 \rightarrow \mathbb{R}$ have been synthesised. For varying levels u , the $\mathcal{L}_k(A(X, u, T))$'s have been computed with respective Matlab functions `bweuler`, `bwperim` and `bwarea`, and empirical averages have been performed.

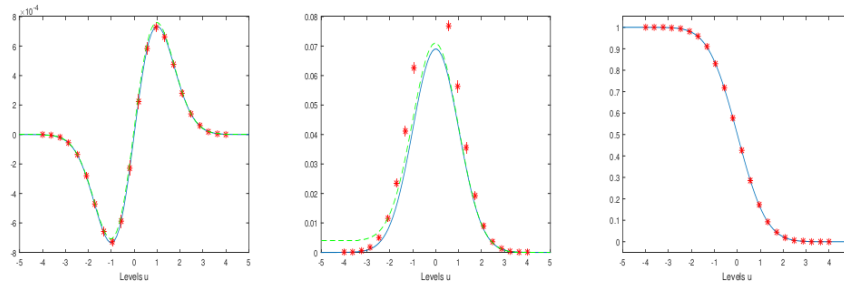


Fig. 6 Estimation of $\mathbb{E}[\mathcal{L}_k(A(X, u, T))]$ for $k = 0$ (left), $k = 1$ (center), $k = 2$ (right), $T = 2^{10} \times 2^{10}$ and X stationary Gaussian with covariance function $e^{-\kappa\|s\|^2}$ for $\kappa = 100/2^{10}$. The blue curves represent the theoretical maps $u \mapsto \mathbb{E}[\mathcal{L}_k(A(X, u, T))]$; the green dashed curves include a bias correction; the red crosses are for the Monte Carlo empirical averages.

Going beyond the illustrative figure and the use of Matlab functions, let us comment how one can get $\mathcal{L}_k(A(X, u, T))$, $k = 0, 1, 2$ when dealing with real data images that have already been segmented at level u . For $k = 2$, it is an easy task by taking the measure of the occupied region (or counting the black pixels). For

$k = 0$, one can use that $\mathcal{L}_0(A(X, u, T)) \approx \#(\max \geq u \text{ in } T) - \#(\min \geq u \text{ in } T)$ (see the definition of the modified Euler characteristic in Section 3.3.2). For $k = 1$, it is a difficult task since bias is coming from the discretization procedure and intrinsic drawbacks (see [25] or [1] for more details). Actually, one can remark the rather bad performance of the Matlab function `bwperim` in the estimation of $\mathbb{E}[\mathcal{L}_1(A(X, u, T))]$ in Figure 6.

4.2 Examples of Parameters Identification

We still consider a two-dimensional stationary isotropic Gaussian field X observed on a rectangular domain T . For any level u , following [26], we define the LKC densities as

$$C_k(u) := \lim_{T \nearrow \mathbb{R}^2} \frac{\mathbb{E} \mathcal{L}_k(A(X, u, T))}{|T|_2}, \quad k = 0, 1, 2.$$

By combining the equations in the Gaussian Kinematic Formula (see Theorem 7) for $k = 0, 1, 2$, we are able to build unbiased estimators for $C_k(u)$ based on the observation on a large domain T , denoted by $\widehat{C}_k(u, T)$ in what follows. Using an appropriate Central Limit Theorem yields confidence intervals for the LKC densities. Then, heuristically, starting from the three observations $\mathcal{L}_k(A(X, u, T))$, $k = 0, 1, 2$ for a fixed u and a fixed large T , one can get a good estimation of three different structure parameters, or a good prevision on three different features. We give hereafter some details on such methodological procedures implemented in specific contexts. In the following cited papers, the statistical crucial point of estimating the expectation and the variance of the excursions LKC is usually treated by empirical methods; we do not go into the details of it in the following results, focusing on the role of the expectation formulas.

4.2.1 Estimation of the Second Spectral Moment λ_2

Let X be a stationary isotropic Gaussian field, standard, with unknown second spectral moment λ_2 . From the definition of $C_0(u)$ and the Gaussian Kinematic Formula stated in Theorem 7 we know that $C_0(u) = \lambda_2 (2\pi)^{-3/2} u e^{-u^2/2}$. Introducing

$$\widehat{\lambda}_2(u, T) := (2\pi)^{3/2} \frac{e^{u^2/2}}{u} \widehat{C}_0(u, T),$$

the next convergence in distribution is proved in [26] (Proposition 2.6),

$$|T|_2^{1/2} \left(\widehat{\lambda}_2(u, T) - \lambda_2 \right) \xrightarrow{T \nearrow \mathbb{R}^2} N(0, \Sigma(u)), \quad \text{with } \Sigma(u) < +\infty.$$

Figure 7 is taken from [26]. It illustrates the previous CLT with an estimation of λ_2 performed on a sample of 100 trajectories of a standard stationary Gaussian field.

The high variability near the level $u = 0$ is explained by the u -term appearing in the denominator of the expression defining $\widehat{\lambda}_2(u, T)$.

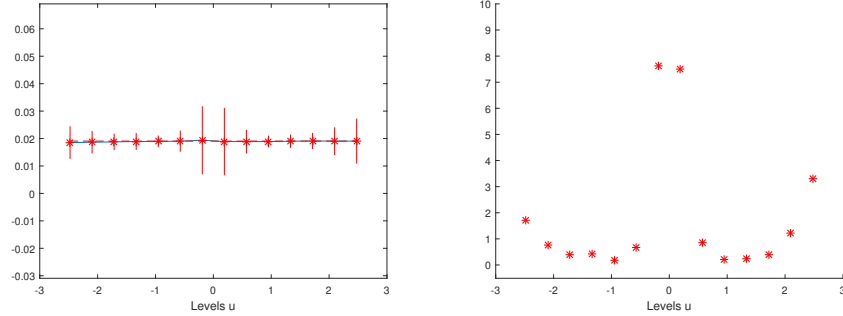


Fig. 7 Left panel : estimate $\widehat{\lambda}_2(u, T)$ with associated confidence intervals for $M = 100$ samples simulations for different values of u , with $T = 2^{10} \times 2^{10}$ and X stationary Gaussian with covariance function $e^{-\kappa \|s\|^2}$ for $\kappa = 100/2^{10}$. Theoretical value $\lambda_2 = 0.019$ is represented by the horizontal line. Right panel : the empirically estimated variance $\widehat{\Sigma}(u)$ for different values of u .

4.2.2 Extreme Values Analysis

We now provide a quick insight in extreme values analysis. The premises of exploring closed equations like the GKF's with the aim of rigorously establishing the so-called Euler characteristic heuristic can be found in Adler's early works [3] and [2]. One can refer to [5, 6] for more recent discussions on the subject. The Euler characteristic heuristic claims that, for a large variety of fields, the next asymptotic holds as u goes to infinity,

$$|\mathbb{P}(\sup_{s \in T} X(s) \geq u) - \mathbb{E}[\mathcal{L}_0(A(X, u, T))]| \leq \text{error}(u), \quad (31)$$

with $\text{error}(u)$ negligible with respect to both terms of the left-hand side part.

Indeed, one can heuristically say that for a large level u , the excursion set above u is either empty, or non-empty but then restricted to only one connected component. Therefore, $\mathcal{L}_0(A(u, X, T))$ appears as a Bernoulli random variable with parameter $\mathbb{P}(A(u, X, T) \neq \emptyset)$, which makes (31) an obvious equality. Concerning Gaussian and related fields, the asymptotic is actually proved for instance in [77].

In view of this heuristic, having an expression for $\mathbb{E}[\mathcal{L}_0(A(X, u, T))]$ given by Theorem 7 yields an accurate estimation for the exceedance Gaussian probability,

$$\mathbb{P}(\sup_{s \in T} X(s) \geq u) \sim \lambda_2 (2\pi)^{-3/2} u e^{-u^2/2}, \text{ as } u \rightarrow +\infty.$$

Following a similar approach, [4] and [41] study the extreme values behavior of non-Gaussian fields, such as stable or Pareto-type random fields. In both papers, establishing a closed equation for the expected LKC of high levels excursion sets permits the exhibition of tight asymptotics for the exceedance probability.

4.2.3 Deformation Identification

Back to the deformed field model introduced in Section 4.2.3, we present a deformation identification method based on observations of some excursion sets of X_θ above fixed levels and the use of their mean modified Euler characteristic. The reference for the following results is [47].

We consider a deformed field $X \circ \theta$, where X is a Gaussian, stationary and isotropic random field satisfying Assumption **A0**. As we have already explained in Section 2.2, we can assume that $\theta(0) = 0$ and that the Jacobian determinant of θ is positive, without any loss of generality regarding the law of the deformed field. We add the assumption that λ_2 , the second spectral moment of X , equals 1. If $\lambda_2 \neq 1$, X_θ is nevertheless equal to $\tilde{X}_{\tilde{\theta}}$ with $\tilde{\theta} = \lambda_2^{1/2}\theta$ and with $\tilde{X}(\cdot) = X(\lambda_2^{-1/2}\cdot)$, whose second spectral moment is equal to 1.

Using that $A(X_\theta, u, T) = \theta^{-1}(A(X, u, \theta(T)))$, we get

$$\chi(A(X_\theta, u, T)) = \chi(A(X, u, \theta(T))).$$

We are then in position to use Proposition 14 on $\theta(T)$ considering two types of domains T in \mathbb{R}^2 :

- T is a rectangle, which yields a two-dimensional domain $\theta(T)$ and so

$$\mathbb{E}[\chi(A(X_\theta, u, T))] = (2\pi)^{-3/2} e^{-u^2/2} u |\theta(T)|_2; \quad (32)$$

- T is a segment, which yields a one-dimensional domain $\theta(T)$ and so

$$\mathbb{E}[\chi(A(X_\theta, u, T))] = (2\pi)^{-1} e^{-u^2/2} |\theta(T)|_1. \quad (33)$$

We will focus on the simple case of a linear deformation which is a first step toward the general case. We write matricially the deformation in a fixed orthonormal basis of \mathbb{R}^2 as $\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$.

We fix $(s, t) \in (\mathbb{R} \setminus \{0\})^2$, $u \neq 0$ and we assume that we know $\mathbb{E}[\chi(A(X_\theta, u, T))]$ for three specific domains T : $[0, s] \times \{0\}$, $\{0\} \times [0, t]$ and $T(s, t) := [0, s] \times [0, t]$. The three real numbers

$$a = \sqrt{\theta_{11}^2 + \theta_{21}^2}, \quad b = \sqrt{\theta_{12}^2 + \theta_{22}^2}, \quad c = \theta_{11}\theta_{22} - \theta_{21}\theta_{12}$$

satisfy

$$|\theta([0, s] \times \{0\})|_1 = |s|a, \quad |\theta(\{0\} \times [0, t])|_1 = |t|b, \quad |\theta(T(s, t))|_2 = |st|c.$$

Therefore, they are solutions of equations given by Formulas (32) and (33) and they can be used to write another expression of θ : there exists $(\alpha, \beta) \in (\mathbb{R}/2\pi\mathbb{Z})^2$ such that $\theta = \begin{pmatrix} a \cos(\alpha) & b \cos(\beta) \\ a \sin(\alpha) & b \sin(\beta) \end{pmatrix}$. Let $\delta = \beta - \alpha$ be the angle between the two column vectors of matrix θ . It satisfies $c = ab \sin(\delta)$, whence

$$\delta \in \{\delta_0, \delta_1\}, \text{ where } \begin{cases} \delta_0 = \arcsin(c/ab) \in (0, \pi/2] \\ \delta_1 = \pi - \arcsin(c/ab) \in [\pi/2, \pi). \end{cases}$$

Consequently, we are able to determine θ up to an unknown rotation: θ belongs to the set $\mathcal{M}(a, b, c)$ defined by

$$\mathcal{M}(a, b, c) = \left\{ \rho_\alpha \begin{pmatrix} a & \sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix}, \rho_\alpha \begin{pmatrix} a & -\sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix}, \alpha \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

In the general case when θ is no more linear, the formulas for the modified Euler characteristic of excursion sets above a rectangle and segments involve the determinant and norms of the Jacobian matrix of θ . The method in the linear case described above allows to show that the Jacobian matrix of θ at a certain point belongs to a set of the form $\mathcal{M}(a, b, c)$. The complete method is explained in [47].

4.3 Examples of Testing Procedures

4.3.1 Test of Gaussianity.

Using again the LKC density of order 0, namely $C_0(u)$, we build in [26] a test of Gaussianity for the field X . Let us mention a previous test of Gaussianity in [46] and [40] that was less relevant; a prior knowledge of the second spectral moment of X was required for instance. We now give a detailed presentation of the testing procedure for the null hypothesis H_0 : X is Gaussian (X being a standard stationary isotropic field).

From the GKF given by Theorem 7, as already said, we know that $C_0(u) = \lambda_2 (2\pi)^{-3/2} u e^{-u^2/2}$ and so, for any $\alpha > 0$, $\frac{C_0(\alpha u)}{C_0(u)} = \alpha e^{u^2(1-\alpha^2)/2}$. Using the unbiased empirical estimators $\widehat{C}_0(u, T_1)$ and $\widehat{C}_0(u, T_2)$ over two disjoint domains T_1 and T_2 , we are able to provide the following tests statistic for fixed α and u ,

$$R(T_1, T_2) = \frac{\widehat{C}_0(\alpha u, T_1)}{\widehat{C}_0(u, T_2)},$$

whose asymptotic Gaussian distribution is known under H_0 as T_1 and T_2 both grow to \mathbb{R}^d . We are then in position to test the null hypothesis H_0 versus H_1 : X is not

Gaussian. Moreover, the power of the test can be evaluated for specific alternatives H_1 such as X is a chi-square field, X is a Student field or X is a shot-noise, since similar formulas as the GKF can be established for these types of fields (see again [26] for the details).

4.3.2 Test of Isotropy

Turning now to the length of level sets or equivalently to the perimeter of excursion sets, *i.e.* $\mathcal{L}_1(A(u, X, T))$, and assuming that the observed field X is isotropic, one can get an explicit expression for $\mathbb{E}[\mathcal{L}_1(A(u, X, T))]$, by applying the GKF for instance. It allows one to test the null hypothesis H_0 : X is isotropic (X being a standard stationary smooth Gaussian field) against X is not isotropic. This idea is exploited in [19] with the particular alternative hypothesis H_1 : X is deformed by a linear transformation matricially given by $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, with $a \neq b$ both non-negative and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. In this study, the expected level length is computed both under the null hypothesis and under the alternative hypothesis, and the test precisely concentrates on whether $a = b$ or not. A similar idea serves as a guideline in [1] which deals with discrete images. Therein, the expected so-called horizontal and vertical perimeters are computed and the test consists in comparing to one the ratio of those perimeters. Although implemented in very different contexts, the isotropy statistical tests presented in both cited papers are based on the same type of observation: the length of some contours, and the same theoretical background: a closed formula for its expected value. We would like to emphasize that this geometrical feature rests on a much sparser information than the whole covariance structure, which is commonly exploited to test isotropy.

References

1. Abaach, M., Biermé, H., Di Bernardino, E., Estrade, A.: Local pixel isotropy test based on oriented perimeter for digital images. Preprint HAL [hal-04037394](#) (2023)
2. Adler, R.: The Geometry of Random Fields. Wiley (1981)
3. Adler, R., Hasofer, A.: Level crossings for random fields. *Ann. Probab.* **4**, 1–12 (1976)
4. Adler, R., Samorodnitsky, G., Taylor, J.: Excursion sets of three classes of stable random fields. *Adv. Appl. Prob.* **42**(2), 293–318 (2010)
5. Adler, R., Taylor, J.: Random Fields and Geometry, 2nd edition. Springer (2009)
6. Adler, R., Taylor, J.: Topological complexity of smooth random functions. Springer (2011)
7. Allard, D., Senoussi, R., Porcu, E.: Anisotropy models for spatial data. *Mathematical Geosciences* **48**(3), 305–328 (2016)
8. Armentano, D., Azaïs, J.M., León, J.: On a general Kac-Rice formula for the measure of a level set. preprint [ArXiv-2304.07424](#) (2023)
9. Azaïs, J.M., Delmas, C.: Mean number and correlation function of critical points of isotropic Gaussian fields. *Stoc. Proc. Appl.* **150**, 411–445 (2022)
10. Azaïs, J.M., Wschebor, M.: Level sets and extrema of random processes and fields. Wiley & Sons (2009)

11. Azaïs, J.M., León, J.R., Ortega, J.: Geometrical characteristics of Gaussian sea waves. *Journal of Applied Probability* **42**(2), 407–425 (2005)
12. Beliaev, D., Cammarota, V., Wigman, I.: No repulsion between critical points for planar Gaussian random fields. *Electronic Communication in Probability* **25**(82), 1–13 (2020)
13. Beliaev, D., McAuley, M., Muirhead, S.: On the number of excursion sets of planar Gaussian fields. *Probability Theory and related Fields* **178**, 655–698 (2020)
14. Beliaev, D., McAuley, M., Muirhead, S.: A central limit theorem for the number of excursion set components of Gaussian fields. *Annals of Probability* (2023+). (In press)
15. Belyaev, Y.: On the number of crossings of a level by a Gaussian random process. i. *Theory of Probability & Its Applications* **11**, 106–113 (1966)
16. Berry, M.: Regular and irregular semiclassical wavefunctions. *Journal of Physics A: Mathematical and General* **10**(12), 2083 (1977)
17. Berry, M.: Statistics of nodal lines and pints in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. *Journal of Physics* **35**, 3025–3038 (2002)
18. Berry, M., Dennis, M.: Phase singularities in isotropic random waves. *Proc. R. Soc. London A: Mathematical, Physical and Engineering Sciences* **456**, 2059–2079 (2000)
19. Berzin, C.: Estimation of local Anisotropy based on level sets. *Electronic Journal of Probability* **26**, 1–72 (2021)
20. Berzin, C., Latour, A., León, J.: Kac-Rice formulas for random fields and their applications in random geometry, roots of random polynomials and some engineering problems. Ediciones IVIC, Instituto Venezolano de Investigaciones Científicas (2017)
21. Biermé, H., Richard, F.: Estimation of anisotropic Gaussian fields through Radon transform. *ESAIM Probab. Stat.* **12**(1), 30–50 (2008)
22. Biermé, H.: Introduction to random fields and scale invariance. In: D. Coupier (ed.) *Stochastic Geometry, Lecture Notes in Mathematics*, vol. 2237, pp. 129–180. Springer (2019)
23. Biermé, H., Bonami, A., León, J.: Central limit theorems and quadratic variations in term of spectral density. *Electronic Journal of Probability* **16**(3), 362–395 (2011)
24. Biermé, H., Desolneux, A.: Mean geometry for 2d random fields: level perimeter and level total curvature integrals. *The Annals of Applied Probability* **30**(2), 561–607 (2020)
25. Biermé, H., Desolneux, A.: The effect of discretization on the mean geometry of a 2d random field. *Annales Henri Lebesgue* **4**, 1295–1345 (2021)
26. Biermé, H., Di Bernardino, E., Duval, C., Estrade, A.: Lipschitz-Killing curvatures of excursion sets for two dimensional random fields. *Electron. J. Statist.* **13**(1), 536–581 (2019)
27. Biermé, H., Moisan, L., Richard, F.: A turning-band method for the simulation of anisotropic fractional Brownian fields. *Journal of Computational and Graphical Statistics* **24**(3) (2015)
28. Bochner, S.: Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse. *Math. Ann* **108**, 378–410 (1933)
29. Bonami, A., Estrade, A.: Anisotropic analysis of some Gaussian models. *Journal of Fourier Analysis and Applications* **9**(3), 215–236 (2003)
30. Bulinski, A., Spodarev, E., Timmermann, F.: Central limit theorems for the excursion set volumes of weakly dependent random fields. *Bernoulli* **18**(1), 100–118 (2012)
31. Cabaña, E.M.: Affine processes: A test of isotropy based on level sets. *SIAM Journal on Applied Mathematics* **47**(4), 886–891 (1987)
32. Canzani, Y., Hanin, B.: Local universality for zeros and critical points of monochromatic random waves. *Communications in Mathematical Physics* **378**(3), 1677–1712 (2020)
33. Clerc, M., Mallat, S.: The texture gradient equation for recovering shape from texture. *IEEE Trans. Pattern Anal. Mach. Intell.* **24**(4), 536–549 (2002)
34. Clerc, M., Mallat, S.: Estimating deformations of stationary processes. *The Annals of Statistics* **31**(6), pp. 1772–1821 (2003)
35. Cohen, S., Istas, J.: *Fractional Fields and Applications, Mathematics and Applications*, vol. 73. Springer (2013)
36. Cramér, H., Leabetter, M.: *Stationary and Related Stochastic Processes*. Wiley (1967)
37. Cuzick, J.: Conditions for finite moments of the number of zero crossings for Gaussian processes. *Ann. Probab.* **3**(5), 849–858 (1975)

38. Dalmao, F., Estrade, A., León, J.: On 3-dimensional Berry's model. *ALEA Lat. Am. J. Probab. Math. Stat.* **18**, 379–399 (2021)
39. Dennis, M.: Nodal densities of planar Gaussian random waves. *The European Physical Journal Special Topics* **145**(1), 191–210 (2007)
40. Di Bernardino, E., Estrade, A., León, J.R.: A test of Gaussianity based on the Euler characteristic of excursion sets. *Electronic Journal of Statistics* **11**(1), 843–890 (2017)
41. Di Bernardino, E., Estrade, A., Opitz, T.: Spatial extremes and stochastic geometry for Gaussian-based peaks-over-threshold processes. preprint HAL **hal-03825701** (2022)
42. Elizarov, A.I.: On the variance of the number of stationary points of a homogeneous Gaussian field. *Theory of Probability & Its Applications* **29**(3), 569–570 (1985)
43. Escande, P., Richard, F.: Full inference for the anisotropic fractional Brownian field. *Theory of Probability and Mathematical Statistics* (2023). (In press)
44. Estrade, A., Fournier, J.: Number of critical points of a Gaussian random field: condition for a finite variance. *Statistics & Probability Letters* **118**, 94 – 99 (2016)
45. Estrade, A., Fournier, J.: Anisotropic random wave models. *ALEA Lat. Am. J. Probab. Math. Stat.* **17**, 329–353 (2020)
46. Estrade, A., León, J.: A central limit theorem for the Euler characteristic of a Gaussian excursion set. *Annals of Probability* **44**(6), 3849–3878 (2016)
47. Fournier, J.: Identification and isotropy characterization of deformed random fields through excursion sets. *Advances in Applied Probability* **50**(3), 706–725 (2019)
48. Gass, L., Stecconi, M.: The number of critical points of a Gaussian field: finiteness of moments. preprint **ArXiv-2305.17586** (2023)
49. Geman, D.: On the variance of the number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* **43**(3), 977–982 (1972)
50. Helgason, S.: *Integral Geometry and Radon Transforms*. Springer (2011)
51. Henderson, I., Noble, P., Roustant, O.: Characterization of the second order random fields subject to linear distributional PDE constraints. *Bernoulli* **29**(4), 3396–3422 (2023)
52. Hu, W., Okamoto, T.: Mass reconstruction with cosmic microwave background polarization. *The Astrophysical Journal* **574**(2), 566–574 (2002)
53. Jennane, R., Harba, R., Lemineur, G., Bretteil, S., Estrade, A., Benhamou, C.: Estimation of the 3d self-similarity parameter of trabecular bone from its 2d projection. *Medical Image Analysis* **11**, 91–98 (2007)
54. Kac, M.: On the average number of real roots of a random algebraic equation. *Bull. Am. Math. Soc.* **27**(1), 83–96 (1943)
55. Kallenberg, O.: *Foundations of Modern Probability*. Springer (1997)
56. Klatt, M.: *Morphometry of random spatial structures in physics*. Ph.D. thesis, FAU Forschungen, Reihe B, Medizin, Naturwissenschaft, Technik (2016)
57. Kratz, M., León, J.R.: Central Limit Theorems for Level Functionals of Stationary Gaussian Processes and Fields. *Journal of Theoretical Probability* **14**(3), 639–672 (2001)
58. Kratz, M., León, J.R.: On the second moment of the number of crossings by a stationary Gaussian process. *Annals of Probability* **34**(4), 1601–1607 (2006)
59. Kratz, M., Vadlamani, S.: Central limit theorem for Lipschitz–Killing curvatures of excursion sets of Gaussian random fields. *J. Theor. Probab.* **31**(3), 1729–1758 (2018)
60. Ladgham, S., Lachièze-Rey, R.: Local repulsion of planar Gaussian critical points. *Stochastic Processes and their Applications* **166**, 104221 (2023)
61. Ley, C., Verdebout, T.: *Modern Directional Statistics*. CRC Press (2017)
62. Longuet-Higgins, M.S.: The statistical analysis of a random, moving surface. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* **249**(966), 321–387 (1957)
63. Malevich, T.L.: On conditions for finiteness of the factorial moments of the number of zeros of Gaussian stationary processes. *Theory of Probability & Its Applications* **29**(3), 534–545 (1985)
64. Mandelbrot, B., Van Ness, J.: Fractional Brownian motion, fractional noises and applications. *SIAM Rev.* **10**, 422–437 (1968)

65. Mardia, K., Jupp, P.: *Directional Statistics*. John Wiley & Sons (2009)
66. Morse, M., Cairn, S.: *Critical Point Theory in Global Analysis and Differential Topology: An Introduction*. Academic Press (1969)
67. Nourdin, I., Peccati, G., Rossi, M.: Nodal statistics of planar random waves. *Communications in Mathematical Physics* **369**(1), 99–151 (2019)
68. Perrin, O., Meiring, W.: Identifiability for non-stationary spatial structure. *Journal of Applied Probability* **36**(4), 1244–1250 (1999)
69. Perrin, O., Senoussi, R.: Reducing non-stationary random fields to stationarity and isotropy using a space deformation. *Statistics & Probability Letters* **48**(1), 23 – 32 (2000)
70. Polisano, K., Clausel, M., Perrier, V., Condat, L.: Riesz-based orientation of localizable Gaussian fields. *Applied and Computational Harmonic Analysis* (2019)
71. Rasmussen, C., Williams, C.: *Gaussian Processes for Machine Learning*. MIT press (2006)
72. Rice, S.: Mathematical analysis of random noise. *Bell Syst. Tech. J.* **23**, 282–332 (1944)
73. Richard, F.: PyAFBF. <https://fjprichard.github.io/PyAFBF/> (2021)
74. Richard, F., Biermé, H.: Statistical tests of anisotropy for fractional Brownian textures. application to full-field digital mammography. *Journal of Mathematical Imaging and Vision* **36**(3), 227–240 (2010)
75. Samorodnitsky, G., Taqqu, M.: *Stable Non-Gaussian Random Processes*. Chapman & Hal (1994)
76. Sampson, P.D., Guttorp, P.: Nonparametric estimation of nonstationary spatial covariance structure. *Journal of the American Statistical Association* **87**(417), 108–119 (1992)
77. Taylor, J.E., Takemura, A., Adler, R.J.: Validity of the expected Euler characteristic heuristic. *Ann. Prob.* **33**(4), 357–368 (2005)
78. Thäle, C.: 50 years sets with positive reach - a survey. *Surveys in Mathematics and its Applications* **3**, 123–165 (2008)